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SEPARATION OF VARIABLES FOR THE HAMILTON–JACOBI EQUATION ON COMPLEX PROJECTIVE SPACES*

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Abstract. The additive separation of variables in the Hamilton–Jacobi equation and the multiplicative separation of variables in the Laplace–Beltrami equation are studied for the complex projective space $\mathbb{C}P^n$ considered as a Riemannian Einstein space with the standard Fubini–Study metric. The isometry group of $\mathbb{C}P^n$ is $SU(n+1)$ and its Cartan subgroup is used to generate n ignorable variables (variables not figuring in the metric tensor). A one-to-one correspondence is established between separable coordinate systems on S^n and separable systems with n ignorable variables on $\mathbb{C}P^n$. The separable coordinates in $\mathbb{C}P^n$ are characterized by $2n$ integrals of motion in involution: n of them are elements of the Cartan subalgebra of $SU(n+1)$ and the remaining n are linear combinations of the Casimir operators of $n(n+1)/2$ different $su(2)$ subalgebras of $su(n+1)$. Each system of $2n$ integrals of motion in involution, and hence each separable system of coordinates on CP^n, thus provides a completely integrable Hamiltonian system. For $n=2$ it is shown that only two separable systems on CP^2 exist, both nonorthogonal with two ignorable variables, coming from spherical and elliptic coordinates on S^2, respectively.

1. Introduction. The purpose of this paper is to discuss the problem of separation of variables for the Hamilton–Jacobi equation

$$g^{ij}S_{x^i}S_{x^j}=E \tag{1.1}$$

on a complex projective space $\mathbb{C}P^n$ with respect to the standard Fubini–Study metric. Indeed we will prove that every separable coordinate system on $\mathbb{C}P^n$ with n ignorable coordinates comes from a separable coordinate system on the real projective space $\mathbb{R}P^n$ or equivalently on the real sphere S^n. Conversely, every orthogonal separable coordinate system on S^n induces a nonorthogonal separable coordinate system on $\mathbb{C}P^n$ with n ignorable coordinates. Moreover, we prove that for $\mathbb{C}P^2$ these are all possible separable coordinates.

Also of interest is the separation of variables problem for the Laplace–Beltrami equation

$$\begin{aligned}&\Delta\psi=E\psi,\\ &\Delta=g^{-1/2}\frac{\partial}{\partial x^i}g^{1/2}g^{ij}\frac{\partial}{\partial x^j},\qquad g=\det(g_{ij}).\end{aligned} \tag{1.2}$$

Since $\mathbb{C}P^n$ (with the standard metric) is a Riemannian Einstein space, the separation of variables problem for (1.2) is equivalent [1] to that for (1.1).

Recall [1], [2] that on any pseudo-Riemannian manifold M a local coordinate system $\{x^i\}$ is said to be a *separable coordinate system* for (1.1) if it is possible to find a solution W of (1.1) depending on n-parameters $\lambda_1,\cdots,\lambda_n$ satisfying

$$W=\sum_{i=1}^{n}W_i(x^i,\lambda_1,\cdots,\lambda_n),\qquad \det\left(\frac{\partial^2 W}{\partial x^i\partial\lambda_j}\right)\neq 0 \tag{1.3}$$

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where the function W_i is independent of x^j for $i \neq j$. This is sometimes referred to as *additive separation* as opposed to *multiplicative separation* [3]–[5] of (1.2).

There are two natural types of coordinates [1]–[5]: (1) Ignorable coordinates x^α for which $\partial_{x^\alpha} g_{ij} = 0$. (2) Essential coordinates x^a for which $\partial_{x^a} g_{ij} \neq 0$. Ignorable coordinates are naturally associated with abelian infinitesimal isometries [6]. In terms of these two types of variables additive separation can be characterized by the equations

$$\begin{aligned} &\partial_{x^a x^b} S = 0 \quad \text{for } a \neq b, \\ &\partial_{x^a x^\alpha} S = 0 \quad \text{for all } a, \alpha, \\ &\partial_{x^\alpha x^\beta} S = 0 \quad \text{for all } \alpha, \beta, \end{aligned} \tag{1.4}$$

where a, b indicate essential coordinates and α, β indicate ignorable coordinates. The pseudogroup $\mathcal{P}$ of coordinate transformations which leaves invariant this system of equations is given by

$$x'^a = X^a(x^a), \qquad x'^\alpha = \sum_\beta A^\alpha_\beta x^\beta + \sum_a f^\alpha_a(x^a) \tag{1.5}$$

where $\det(A^\alpha_\beta) \neq 0$. Such transformations thus preserve separation; hence we say that two coordinate systems $\{x'^i\}$ and $\{x^i\}$ are *equivalent* if they can be related by a transformation in $\mathcal{P}$. By abuse of language a separable coordinate system will mean an equivalence class of separable coordinate systems.

On a Riemannian manifold every separable coordinate system $\{x^i\}$ can be brought to the canonical form [2]

$$(g^{ij}) = \begin{bmatrix} \delta^{ab} H_a^{-2} & 0 \\ 0 & g^{\alpha\beta} \end{bmatrix} \tag{1.6}$$

by a member of $\mathcal{P}$, where δ^{ab} is the Kronecker delta and the functions H_a and $g^{\alpha\beta}$ are specified as follows: (1) The quadratic form $Q = \Sigma_a H_a^2 (dx^a)^2$ is in *Stäckel form*, i.e. H_a satisfies

$$\partial_{x^j x^k} \ln H_i^2 - \partial_{x^j} \ln H_i^2 \partial_{x^k} \ln H_i^2 + \partial_{x^j} \ln H_i^2 \partial_{x^k} \ln H_j^2 + \partial_{x^k} \ln H_i^2 \partial_{x^j} \ln H_k^2 = 0$$

for $j \neq k$. (2) Each function $g^{\alpha\beta}$ is a *Stäckel multiplier*, i.e. the quadratic form $g^{\alpha\beta}Q$ is also in Stäckel form for all α, β. The subpseudogroup $\mathcal{P}_C$ which preserves canonical forms is given by the transformations (1.5) with f^α_a constant. All our coordinate systems will be in canonical form.

Suppose G acts on M as a group of isometries with action $\phi: G \times M \to M$. Then if $\{x^i\}$ is a coordinate system about $p \in M$, then $\phi^*\{x^i\}$ is a coordinate system about $\phi^{-1}(p)$. Furthermore, if $\{x^i\}$ is a separable, so is $\phi^*\{x^i\}$. In fact $\{x^i\}$ and $\phi^*\{x^i\}$ are conjugate and we deal with conjugacy classes under G. Thus separable coordinate systems are classified up to equivalence under $\mathcal{P} \times G$.

It is classical [7] that associated with every orthogonal separable coordinate system there is an n-dimensional vector space of locally defined second order contravariant symmetric C^∞ tensor fields $A_1, \cdots, A_n$ one of which is the metric itself and which mutually commute with respect to the induced Lie bracket [3]. Recently [1, 2] this has been extended to nonorthogonal separable coordinate systems where both first and second order tensor fields must be allowed. Furthermore, practical criteria have been given to determine precisely which tensor fields give rise to separation of variables [2], [8]. Up to now this is purely local; however, we will say that a separable coordinate system $\{x^i\}$ is *globally admissible* if the locally defined tensor fields $A_1, \cdots, A_n$ extend to global C^∞ tensor fields on M.

For the purpose of separation of variables it is convenient to consider the cotangent bundle $T^*(M)$ with its canonical symplectic structure [9]. Let $S^p(M)$ denote the vector space of pth-order symmetric contravariant C^∞ tensor fields on M and let $S(M)$ denote the direct sum $S(M) = \oplus_p S^p(M)$. Every tensor field in $S(M)$ defines a unique C^∞ function on $T^*(M)$ which is a polynomial in the canonical coordinates p_i associated with a coordinate system $\{x^i\}$ on M and vice-versa. The Lie bracket operation in $S(M)$ goes over to the Poisson bracket operation in $C^\infty(T^*(M))$. The contravariant metric g goes over to the Hamiltonian H_0 for geodesic motion or what we call the *free Hamiltonian*, and tensor fields which commute with H_0 become *constants of the motion*. Constants of the motion which commute under Poisson bracket are said to be in involution. Since constants of the motion will be globally defined functions in $C^\infty(T^*(M))$ if and only if the corresponding tensor fields are globally defined on M, a globally admissible separable coordinate system gives a completely integrable Hamiltonian system [9], [10].

Now let the Lie group G act on M as isometries and denote by $\mathcal{G}$ the Lie algebra of G. Let $\mathcal{U}(\mathcal{G})$ denote the universal enveloping algebra [11] of $\mathcal{G}$. $U(\mathcal{G})$ has a canonical filtration $\cdots \supset \mathcal{U}^{(2)} \supset \mathcal{U}^{(1)} \supset \mathcal{U}^0$. Denote by $\operatorname{gr}\mathcal{U}$ its associated graded algebra and by $\mathcal{U}_p = \mathcal{U}^{(p)}/\mathcal{U}^{(p-1)}$ the elements of degree p. $\mathcal{U}_p$ is naturally identified with the symmetric tensors $S^p(\mathcal{G})$. The Lie algebra homomorphism of $\mathcal{G}$ into the C^∞ vector fields on M induces a homomorphism of $\operatorname{gr}\mathcal{U}$ into $S(M)$. We will be particularly interested in $\mathcal{U}_2 \to S^2(M)$. Recall [12] that a separable coordinate system $\{x^i\}$ is *class one* if the corresponding tensor fields $A_1, \cdots, A_n$ lie in the image of $\mathcal{U}_2(\mathcal{G})$, and *class two* otherwise. It follows that every class one coordinate system is globally admissible. It is known [8] that all coordinate systems on the n-sphere S^n are class one and hence globally admissible. Similarly since the isometry group $SO(n+1)$ of S^n passes to the quotient—real projective space $\mathbb{R}P^n$, all coordinate systems on $\mathbb{R}P^n$ are class one and globally admissible.

An example of a space with coordinate systems which are not globally admissible is given by the torus T_n obtained as the quotient space of $\mathbb{R}^n$ by the integer lattice. Here the tensor fields associated with, for example, spherical coordinates on $\mathbb{R}^n$, do not pass to globally defined tensor fields on T_n and thus spherical coordinates are not globally admissible on T_n. Likewise $SO(n)$ does not define global isometries on T_n.

There are two motivations for our work. First a study of separable coordinate systems on Riemannian manifolds with a large group G of isometries leads through the intimate connection [12], [13] between these coordinates and second order elements of $\mathcal{U}(\mathcal{G})$ to an algebraic understanding of special function theory. Our article constitutes a first step towards this understanding in spaces of nonconstant curvature.

Second, $\mathbb{C}P^n$ is of considerable interest in physics. For $n=2$ $\mathbb{C}P^2$ has recently been used as a model for a gravitational instanton [14], [15].

2. The geometry of complex projective spaces. Let us begin by considering the complex manifold $\mathbb{C}^{n+1}$ and the standard complex coordinates $\{\omega^\mu\}$ $\mu = 1, \cdots, n+1$. On $\mathbb{C}^{n+1}$ there is a flat hermitian metric $\tilde{h}$ given in local coordinates by

$$\tilde{h} = \sum_{\mu=1}^{n+1} d\omega^\mu d\bar{\omega}^\mu. \tag{2.1}$$

The real part $\tilde{g} = \operatorname{Re}\tilde{h}$ of $\tilde{h}$ is just the standard flat Riemannian metric on $\mathbb{C}^{n+1} \sim \mathbb{R}^{2n+2}$ and the imaginary part $\operatorname{Im}\tilde{h}$ of $\tilde{h}$ is just the standard Kählerian 2-form on $\mathbb{C}^{n+1}$.

Consider the sphere S^{2n+1} in $\mathbb{C}^{n+1}$ defined by

$$\sum_{\mu} |\omega^{\mu}|^2 = 1. \tag{2.2}$$

If ω is the complex vector in $\mathbb{C}^{n+1}$ whose components are ω^{μ}, and we identify the tangent space to $\mathbb{C}^{n+1}$ at ω with $\mathbb{C}^{n+1}$ itself, then the tangent space $T_{\omega}(S^{2n+1})$ to S^{2n+1} at ω can be identified with the set $\{\xi \in T_{\omega}(\mathbb{C}^{n+1})\colon \tilde{g}(\omega,\xi)=0\}$. Moreover, the flat Riemannian metric $\tilde{g}$ on $\mathbb{R}^{2n+2}$ pulls back to the standard metric on S^{2n+1}.

Recall that the complex projective plane $\mathbb{C}P^n$ is the set of complex lines through the origin $\{0\}$ in $\mathbb{C}^{n+1}$. Two points $z, z' \in \mathbb{C}^{n+1} - \{0\}$ are equivalent if there is a $\lambda \in \mathbb{C} - \{0\}$ such that $z' = \lambda z$. Then $\mathbb{C}P^n$ is the quotient manifold $\mathbb{C}^{n+1} - \{0\} \xrightarrow{p} \mathbb{C}P^n$. Every complex line through 0 intersects the sphere S^{2n+1} in a great circle. These circles can be obtained as the orbits of the free circle group action on S^{2n+1} by

$$\omega \to e^{i\theta}\omega \tag{2.3}$$

and the space of orbits is just $\mathbb{C}P^n$. This gives the well-known Hopf fibration

$$S^1 \to S^{2n+1} \xrightarrow{\pi} \mathbb{C}P^n \tag{2.4}$$

giving S^{2n+1} as a principal bundle over $\mathbb{C}P^n$ with group $U(1) \sim S^1$.

Just as the tangent space to the sphere at a point can be determined by the condition $\tilde{g}(\omega,\xi)=0$, the tangent space to $\mathbb{C}P^n$ at a point $[\omega] \in \mathbb{C}P^n$ (here $[\omega]$ denotes the equivalence class determined by $\omega \in \mathbb{C}^{n+1} - \{0\}$) can be identified with the set

$$\left\{\xi \in T_{\omega}\left(\mathbb{C}^{n+1} - \{0\}\right) : \tilde{h}(\omega,\xi)=0\right\} \tag{2.5}$$

Alternatively this is the set of $\xi \in T_{\omega}(S^{2n+1})$ such that

$$\tilde{g}(i\omega,\xi)=0.$$

But $i\omega$ is a vector tangent to the great circle determined as the intersection of the projective line $[\omega]$ with S^{2n+1}. Thus the tangent space $T_{[\omega]}(\mathbb{C}P^n)$ is precisely the set of vectors tangent to S^{2n+1} and orthogonal to the great circle determined by $[\omega]$.

We can now put a metric on $\mathbb{C}P^n$ by requiring that the distance between two points on $\mathbb{C}P^n$ be measured by the corresponding distance between two great circles on S^{2n+1}. That is, for any tangent vectors $\xi_i \in T_{\omega}(S^{2n+1})$, we put

$$h\left(\pi_*\xi_1, \pi_*\xi_2\right) = \tilde{h}(\xi_1^{\perp}, \xi_2^{\perp}) \tag{2.6}$$

where $\xi^{\perp}$ is the component of ξ which is hermitian orthogonal to ω. It is easily verified that (2.6) depends only on $\pi_*\xi_i$, $i=1,2$. In local coordinates if we put

$$\omega_{n+1} = \left(1+|z|^2\right)^{-1/2}, \qquad \omega_i = z_i\left(1+|z|^2\right)^{-1/2} \tag{2.7}$$

where $|z|^2 = |z_1|^2 + \cdots + |z_n|^2$, we obtain the usual Fubini–Study metric [16] on $\mathbb{C}P^n$, viz.

$$h = \left(1+|z|^2\right)^{-2}\left[|dz|^2\left(1+|z|^2\right) - |\bar{z}\cdot dz|^2\right] \tag{2.8}$$

where we are employing standard vector notation.

Consider again the Hopf fibration (2.4). The isometry group of S^{2n+1} (with the usual metric) is the orthogonal group $O(2n+2)$. A necessary condition for an isometry

ϕ of S^{2n+1} to project to an isometry on $\mathbb{C}P^n$ is that ϕ lie in the centralizer $C(U(1))$ in $O(2n+2)$. But a straightforward computation using local coordinates ω on S^{2n+1} shows that

$$C(U(1)) \simeq U(n+1) \simeq U(1) \times SU(n+1).$$

As is well known $SU(n+1)$ is the isometry group for the metric (2.8) on $\mathbb{C}P^n$. (In fact an effective action is given only by the group $SU(n+1)/\mathbb{Z}_{n+1}$, but we will usually suppress the $\mathbb{Z}_{n+1}$ and consider $SU(n+1)$ as the isometry group of $\mathbb{C}P^n$.) We will be interested in the maximal torus $T^{n+1} \subset U(n+1) \subset O(2n+2)$. On the coordinates ω the action of T^{n+1} is given by

$$\omega^i \to e^{i\theta_i}\omega^i, \tag{2.9}$$

$i = 1, \cdots, n+1$. Notice that T^{n+1} is a maximal torus both for $U(n+1)$ and $O(2n+2)$.

Let us introduce polyspherical coordinates on S^{2n+1} by writing

$$\omega^i = s^i e^{i\alpha_i}, \qquad 0 < s_i < \infty, \quad 0 < \alpha_i < 2\pi. \tag{2.10}$$

The surface defined by $\alpha_i = 0$ is an n-sphere S^n given by

$$\sum_{i=0}^{n} (s^i)^2 = 1. \tag{2.11}$$

Furthermore, we recover the whole coordinate domain by the action of T^{n+1} on S^n. Now consider the circle group action given by (2.3). Its induced action on S^n is the discrete group $\mathbb{Z}_2$. Thus the Hopf fibration of S^{2n+1} induces the fibration of S^n over the real projective space $\mathbb{Z}_2 \to S^n \to \mathbb{R}P^n$ and we get the commutative diagram

$$\begin{array}{ccc} \mathbb{Z}_2 & \to & S^1 \\ \downarrow & & \downarrow \\ S^n & \overset{i}{\to} & S^{2n+1} \\ \tilde{\pi}\downarrow & & \downarrow \pi \\ \mathbb{R}P^n & \overset{\tilde{i}}{\to} & \mathbb{C}P^n \end{array} \tag{2.12}$$

This diagram is fundamental in understanding the underlying geometry. The Fubini–Study metric on $\mathbb{C}P^n$ pulls back under $(i \circ \pi)^*$ to the standard metric on S^n. However, by (2.6) h pulls back under π not to the standard metric on S^{2n+1} but to a degenerate symmetric two form on S^{2n+1} whose null space consists precisely of those tangent vectors on S^{2n+1} that are tangent to the great circles that are the orbits of S^1 in (2.12). This degenerate two form then pulls back under i to the standard metric on S^n.

3. Hamiltonian systems. In this section we discuss the relation between the free Hamiltonian on $\mathbb{C}P^n$ and a singular Hamiltonian on $\mathbb{R}P^n$ with a certain inverse square potential. This relation is an example of a general procedure in classical mechanics known as reduction of the phase space. Although this procedure is classical, it has only recently been understood in its proper context in the work of Marsden and Weinstein [17] and Kazhdan, Kostant, and Sternberg [18]. In the latter work this technique was used to obtain completely integrable Hamiltonian systems. Our interest is the classical method of reduction of the phase space by ignorable coordinates and then using separation of variables in the reduced system to give certain completely integrable Hamiltonian systems.

We briefly outline the reduction technique. For a more detailed treatment we refer to the literature [9], [17], [18]. Let P be a symplectic manifold and Ω its closed nondegenerate 2-form. Let a Lie group G act on P by symplectic diffeomorphisms, i.e. G leaves Ω invariant. Suppose further that this action is Hamiltonian, that is that for every $\xi \in \mathcal{G}$, the Lie algebra of G, the corresponding vector field $\xi^{\#}$ on P satisfies

$$\xi^{\#} \lrcorner \Omega = -d\phi^{\xi} \tag{3.1}$$

for some globally defined C^{∞} functions ϕ^{ξ} on P. Now define the *moment map*: $\Phi : P \to \mathcal{G}^*$ (the dual of $\mathcal{G}$) by $\langle \Phi(x), \xi \rangle = \phi^{\xi}(x)$ where $\langle \, , \rangle$ is the pairing between $\mathcal{G}$ and $\mathcal{G}^*$. Now the map Φ is G-equivariant, i.e. Φ intertwines the G-action on P with the co-adjoint action of G on $\mathcal{G}^*$. Pick a point $u \in \mathcal{G}^*$ and assume that $\Phi^{-1}(u)$ is an embedded submanifold of P. Denote by G_u the isotropy subgroup of G under the co-adjoint action. Suppose that G_u acts freely and properly on $\Phi^{-1}(u)$ so that the quotient map $\pi_u : \Phi^{-1}(u) \to \Phi^{-1}(u)/G_u$ is a submersion. Then $P_u = \Phi^{-1}(u)/G_u$ is a symplectic manifold in a natural way. It is called the *reduced phase space*. Furthermore if H is a Hamiltonian on P which is invariant under G, then the reduced Hamiltonian H_u on P_u is obtained by

$$\pi_u^* H_u = i_u^* H \tag{3.2}$$

where $i_u : \Phi^{-1}(u) \to P$ is the inclusion map. This brief description of reduction is that of Marsden and Weinstein [17], whereas the reduction technique of [18], is more general in that one chooses a co-adjoint orbit $\mathcal{O}$ of $\mathcal{G}^*$ rather than a point u. In the case considered below the two techniques coincide since the group G is abelian.

Let us see how reduction applies in our case. The symplectic manifold in question is $P = T^*(\mathbb{C}P^n)$, the contangent bundle of $\mathbb{C}P^n$. The symplectic 2-form is $\Omega = d\theta$ where θ is the canonical 1-form on $T^*(\mathbb{C}P^n)$, $G = T_n$ the maximal torus in $SU(n+1)$ and its action on P is Hamiltonian since P is a cotangent bundle. We can characterize $T^*(\mathbb{C}P^n)$ in homogeneous coordinates $(\omega^1, \cdots, \omega^{n+1})$ by using the Hopf fibration. Identifying $T^*_{[\omega]}(\mathbb{C}P^n)$ and $T_{[\omega]}(\mathbb{C}P^n)$ canonically by using the metric on $\mathbb{C}P^n$, $T^*(\mathbb{C}P^n)$ can be identified with set of points $([\omega], p)$ satisfying

$$|\omega|^2 = 1, \quad \omega \cdot p + \bar{\omega} \cdot \bar{p} = 0, \quad \omega \cdot p - \bar{\omega} \cdot \bar{p} = 0. \tag{3.3}$$

The moment map Φ: $T^*(\mathbb{C}P^n) \to t_n^*$ is given in homogeneous coordinates by

$$\Phi_i([\omega], p) = \omega_i p_i - \bar{\omega}_i \bar{p}_i \quad \text{(no sum)} \tag{3.4}$$

where $t_n^* \sim i\mathbb{R}^n$ is the dual of the Lie algebra t_n of T_n. Notice that $\Sigma_i \Phi_i = 0$, so $\Phi(x)$ is in $t_n^* \subset t_{n+1}^*$ and we can check that Φ is G-equivariant. Now pick a regular point ($d\Phi$ has rank n) $u = (iu_1, \cdots, iu_{n+1}) \in t_n^*$, $u_i \in \mathbb{R}$, with $\Sigma_{i=1}^{n+1} u_i = 0$ and look at $\Phi^{-1}(u)$. We may choose $u_i \neq 0$ for all $i = 1, \cdots, n+1$ so on $\Phi^{-1}(u)$ we have $\omega_i \neq 0$. $\Phi^{-1}(u)$ is a $3n$-dimensional manifold. The isotropy subgroup G_u of $G = T_n$ is T_n itself since $G = T_n$ is a maximal torus. Moreover, since $\omega_i \neq 0$ on $\Phi^{-1}(u)$, T_n acts freely and properly there, so $P_u = \Phi^{-1}(u)/T_n$ is a manifold. A point of P_u can be represented by

$$\begin{aligned} &\omega_i = s_i, \quad \text{real} \quad \mathbf{s} \cdot \mathbf{s} = 1, \quad \mathbf{s} \sim -\mathbf{s}, \\ &p_i = y_i + iu_i \qquad (u_i \text{ fixed}), \\ &\mathbf{s} \cdot \mathbf{y} = 0. \end{aligned} \tag{3.5}$$

But this is just the cotangent bundle of the real projective space $\mathbb{R}P^n$ minus the $n+1$ copies of $\mathbb{R}P^{n-1}$ obtained by putting $s_i = 0$.

Consider the free Hamiltonian on $T^*(\mathbb{C}P^n)$ given in local projective coordinates $z_i=\omega_i/\omega_{n+1}$, $\omega_{n+1}\neq 0$, by

$$H=4\left(1+|z|^2\right)\left[|p|^2+|z\cdot p|^2\right]. \tag{3.6}$$

Let us write the reduced Hamiltonian H_u in terms of the coordinates (s_i,y_i) on $\mathbb{R}^{2n+2}$. From (3.5) P_u is the immersed submanifold of $\mathbb{R}^{2n+2}$ given by the set $([\mathbf{s}],y_i)$ which satisfies

$$\mathbf{s}\cdot\mathbf{s}=1,\quad \mathbf{s}\cdot\mathbf{y}=0,\quad s_i\neq 0$$

where $[s]$ denotes the equivalence class under the equivalence relation $\mathbf{s}\sim\mathbf{s}'$ if and only if $\mathbf{s}'=\pm\mathbf{s}$. The reduced Hamiltonian is easily found to be

$$H_u=\sum_{i=1}^{n+1} y_i^2+\sum_{i=1}^{n+1}\frac{u_i^2}{s_i^2},\qquad \sum_{i=1}^{n+1} u_i=0. \tag{3.7}$$

The fact that this Hamiltonian does not extend to a regular Hamiltonian on all of $T^*(\mathbb{R}P^n)$ reflects the fact that the moment map (3.4) is singular along the surface $\omega_i=p_i=0$ as well as the fact that the coordinates (2.10) break down at $s_i=0$. These, of course, are general features of reduction by angular ignorable coordinates.

A completely analogous argument can be given to find the reduction for $T^*(S^{2n+1})$ and we will again obtain the Hamiltonian (3.7) on the reduced manifold P_u but *without* the constraint $\Sigma u_i=0$ by starting with the free Hamiltonian on S^{2n+1}.

Let us now consider the reduction corresponding to the Hopf fibration. On $T^*(S^{2n+1})$ the moment map is

$$\Phi(\omega,p)=\omega\cdot p-\bar{\omega}\cdot\bar{p}. \tag{3.8}$$

So $\Phi^{-1}(0)$ is just the set of $(\omega,p)\in\mathbb{C}^{2n}$ which satisfy (3.3). But the circle group S^1 acts freely and properly on $\Phi^{-1}(0)$. Moreover, $\Phi^{-1}(0)/S^1$ is just $T^*(\mathbb{C}P^n)$ and the free Hamiltonians on S^{2n+1} and $\mathbb{C}P^n$ are related by (3.2). This completes the our discussion of the reduction technique applied to the commutative diagram (2.12).

4. Constants of the motion. We are interested in all elements of $S^2(su(n+1))$ which commute (with respect to the induced Lie bracket) with the maximal torus. For any $x,y\in S^2(su(n+1))$ put $x\sim y$ if $x\equiv y \bmod S^2(t_n)$. Let $\bar{x}$ denote the equivalence class of x. Define $\tilde{C}=\{x\in S^2(su(n+1)):[x,h]=0,\ h\in t_n\}$. Since t_n is abelian, $[x,h]=0$ implies $[y,h]=0$ if $x\sim y$. Moreover, $S^2(t_n)\subset\tilde{C}$. Set

$$\bar{C}=\tilde{C}/S^2(t_n). \tag{4.1}$$

We wish to determine $\bar{C}$. To do so consider the root space decomposition [11] of $su(n+1)$, viz. $A_n=t_n\oplus r^+\oplus r^-$. It is more convenient to work with $u(n+1)\otimes\mathbb{C}$ and then construct the corresponding real form $su(n+1)$ by considering traceless skew-Hermitian matrices. Consider the matrices E^i_j defined by putting a 1 in the ith row and jth column, and zeros elsewhere. A basis for t_n is given by iE^j_j, $j=1,\cdots,n+1$ with the one relation $E^1_1+\cdots+E^{n+1}_{n+1}=0$. There are precisely $n(n+1)/2$ A_1 subalgebras generated by $E^i_i-E^j_j$, E^i_j, E^j_i, $1\le i<j\le n+1$. Let A^{ij}_1 denote these subalgebras. We are interested in the real forms $su(2)_{ij}$. Let c_{ij} denote the corresponding Casimir invariant [11] of $su(2)_{ij}$ and $\bar{c}_{ij}$ its class in $\bar{C}$. Denote by C the free vector space spanned by c_{ij}, $1\le i<j\le n+1$. Let A^{ij}_1 denote these subalgebras. We are interested in the real forms $su(2)_{ij}$. Let c_{ij} denote the corresponding Casimir invariant [11] of $su(2)_{ij}$ and $\bar{c}_{ij}$ its

class in $\overline{\mathbb{C}}$. Denote by C the free vector space spanned by c_{ij}, $1 \le i < j \le n+1$. Let $su(k+1)$ be the compact real form corresponding to the A_k algebra generated by E_j^i, $1 \le i < j \le k+1$, and let $c_2(su(k+1))$ denote its Casimir invariant. Then

$$c_2(su(k+1)) \equiv \sum_{1 \le i < j \le k+1} c_{ij} \mod S^2(t_n). \tag{4.2}$$

LEMMA 1. *As a vector space C has dimension $n(n+1)/2$ and the set $\{\bar{c}_{ij}\}$ is a basis for C. Thus $C \sim \overline{C}$.*

Proof. Since $c_{ij} \equiv E_j^i \odot E_i^j \mod S^2(t_n)$ where $\odot$ denotes symmetric tensor product, it is enough to show that $E_j^i \odot E_i^j$ span $\overline{C}$ and that there are no relations. The last statement is clear since as vector spaces $C \subset U_2(su(n+1))$. Now let $\overline{X} \in \overline{C}$ and choose a lifting X of the form

$$X = \sum_{\substack{j \ne k \\ l \ne m}} \alpha_{jl}^{km} E_k^j \odot E_m^l + \sum_{\substack{j \ne k \\ l}} \beta_{jl}^k E_k^j \odot E_l^l.$$

X must satisfy $[E_i^i, X] = 0$, $i = 1, \cdots, n$, and this is independent of the choice of lifting. The Lie bracket relations are given by

$$\left[E_j^i, E_l^k\right] = \delta_{jk} E_l^i - \delta_{il} E_j^k. \tag{4.3}$$

One readily sees that $\beta_{jl}^k = 0$ and that the only nonvanishing α's are α_{jk}^{kj}. This gives the desired result.

Now consider the Lie algebra homomorphism $su(n+1) \to C^\infty(T^*(\mathbb{C}P^n))$ sending $\xi \in su(n+1) \to \phi^\xi \in C^\infty(T^*(\mathbb{C}P^n))$. Lie brackets in $su(n+1)$ go over to Poisson brackets on $C^\infty(T^*(\mathbb{C}P^n))$. This induces a homomorphism of $U(su(n+1)) \to C^\infty(T^*(\mathbb{C}P^n))$ with multiplication in U going over to symmetric multiplication in $C^\infty(T^*(\mathbb{C}P^n))$. In particular, we are interested in the image of $U_2(su(n+1)) \sim S^2(su(n+1))$. Let $C_2^\infty \subset C^\infty(T^*(\mathbb{C}P^n))$ denote the subspace consisting of homogeneous polynomials in the p's of degree 2. Denote by $S_2(su(n+1))$ the image of $S^2(su(n+1))$ in C_2^∞. We can check that this map is injective.

Denote by $\hat{c}_{ij}$ the images of c_{ij} in $S_2(su(n+1))$ and by $\hat{C}$ the image of C. On real projective space $\mathbb{R}P^n$ consider the isometry group $SO(n+1)$ and its Lie algebra $so(n+1)$. Let $\{I_{ij}\}$ with $1 \le i < j \le n+1$, be the basis for the Lie algebra $so(n+1)$ of functions on $T^*(\mathbb{R}P^n)$ given by

$$I_{ij} = s_i y_j - s_j y_i \tag{4.4}$$

where (s_i, y_i) are given by (3.5). Using the notation of (3.2) and (3.5), we have

LEMMA 2. *For every $\hat{A} \in \hat{C}$ there is a $A_u \in C^\infty(P_u)$ such that $i_u^* \hat{A} = \pi_u^* A_u$. Furthermore,*

$$i_u^* c_{ij} = \pi_u^* \left[I_{ij}^2 + \left(1 + \frac{s_j^2}{s_i^2}\right) u_i^2 + \left(1 + \frac{s_i^2}{s_j^2}\right) u_j^2 \right] \tag{4.5}$$

where $\sum_{i=1}^{n+1} u_i = 0$.

Proof. By Lemma 1 any $\hat{A} \in \hat{C}$ is invariant under the action of the maximal torus. Thus $i_u^* A(q)$ depends only on $\pi(q)$; hence there is an $A \in C^\infty(P_u)$ such that $i_u^* A = \pi_u^* A$. To verify (4.5) let $(\omega^1, \cdots, \omega^{n+1})$ be homogeneous coordinates on $\mathbb{C}^{n+1}$. The Lie algebra $u(n+1)$ of functions on $T^*(\mathbb{C}^{n+1})$ is spanned by

$$\tilde{T}_{\mu\nu} = \omega^\mu P_\nu - \omega^\nu P_\mu + c.c, \qquad \tilde{S}_{\mu\nu} = i\left(\omega^\mu P_\nu + \omega^\nu P_\mu\right) + c.c$$

where $c.c$ denotes complex conjugate and (ω^{μ}, P_{ν}) are the standard coordinates on $T^*(\mathbb{C}^{n+1})$. On $\mathbb{C}P^n$ choose projective coordinates which without loss of generality we take as $z^i = \omega^i/\omega^{n+1}$, $i=1,\cdots,n$. Then for $1 \le i < j \le n$ fixed we get functions on $T^*(\mathbb{C}P^n)$ given by

$$T_{ij} = z^i P_j - z^j P_i + c.c., \qquad S_{ij} = i\left(z^i P_j + z^j P_i\right) + c.c$$
$$\frac{S_{ii} - S_{jj}}{2} = i\left(z^i P_i - z^j P_j\right) + c.c.$$

These generate an $su(2)$ subalgebra for each $i \ne j = 1,\cdots,n$. In terms of this basis the Casimir operator is

$$\hat{c}_{ij} = T_{ij}^2 + S_{ij}^2 + \left(\frac{S_{ii} - S_{jj}}{2}\right)^2 \tag{4.6}$$

Writing $z^j = (s^j/s^{n+1})e^{i\alpha_j}$, $j=1,\cdots,n$ and $(s^1)^2 + \cdots + (s^{n+1})^2 = 1$, a short computation gives

$$z^j P_i = \frac{e^{i(\alpha_j - \alpha_i)}}{2}\left(s_j y_i - i\frac{s_j}{s_i} P_{\alpha^i}\right).$$

Restricting to $\Phi^{-1}(u)$ by setting $P_{\alpha^i} = u_i$ and by performing a straightforward computation using the formulas above gives the desired result.

Notice that the free Hamiltonian on S^n is

$$H_0 = \sum_{1 \le i < j \le n+1} I_{ij}^2$$

whereas the free Hamiltonian on $\mathbb{C}P^n$ is just

$$H = c_2(su(n+1)) = \sum_{1 \le i < j \le n+1} \hat{c}_{ij} - 2(n-1)\left(\sum_{i=1}^{n} P_{\alpha_i}^2 + \sum_{i<j} P_{\alpha_i} P_{\alpha_j}\right)$$

Thus performing the double sum over $1 \le i < j \le n+1$ in the formula of Lemma 2 gives precisely (3.2) with the Hamiltonian (3.7).

Assuming the previous notation we have:

LEMMA 3. *$\hat{A}$ is a constant of the motion with respect to the Hamiltonian H if and only if A_u is a constant of the motion with respect to the reduced Hamiltonian H_u. Furthermore, two such constants of the motion $\hat{A}, \hat{B}$ are in involution if and only if the corresponding pair A_u, B_u are in involution.*

Proof. This follows directly from

$$i_u^*\{A, B\} = \pi_u^*\{A_u, B_u\}.$$

5. Basic theorems about separable coordinates. It is clear that an understanding of the separable coordinate systems on $\mathbb{C}P^n$ entails an understanding of the separable coordinates on $\mathbb{R}P^n$ or equivalently S^n. A study of all separable coordinate system on S^n is currently in progress and we will here state and use a theorem whose proof will appear elsewhere.

On $T^*(S^n)$ the subset of functions spanned (over $\mathbb{R}$) by the functions (4.4) form a subalgebra of $C^\infty(T^*(S^n))$ under Poisson bracket isomorphic with $o(n+1)$. By abuse of notation we will denote this subalgebra by $o(n+1)$. Since the manifold S^n is class one [8], all second order constants of the motion are in $S^2(o(n+1))$. Let $\mathfrak{D}\subset S^2(o(n+1))$ be the subspace spanned by diagonal elements I_{ij}^2, $1\leq i<j\leq n+1$. $\mathfrak{D}$ has dimension $n(n+1)/2$. The free Hamiltonian $H=\sum_{i<j}I_{ij}^2\in\mathfrak{D}$ defines a nondegenerate definite bilinear form on $o(n+1)$. The point is that for separation of variables on S^n, it is enough to study $\mathfrak{D}$. On S^n we can strengthen Theorem 6 of [8]:

THEOREM 1. *Necessary and sufficient conditions for the existence of an orthogonal separable coordinate system* $\{x^i\}$ *for the H. J. equation* (1.1) *on* S^n *are that there are n functions* $A_1,\cdots,A_n\in\mathfrak{D}$, *one of which, say,* A_1, *is the free Hamiltonian* H, *which are*

(1) *linearly independent* (*locally*);

(2) *in involution.*

Remarks. (1) The conditions (4) and (5) of [8, Thm. 6] are automatically satisfied on S^n. (2) This theorem is not valid on the complex sphere nor on real hyperboloids.

Let us now use the reduction of §2 to formulate a theorem relating the separation of variables on $\mathbb{C}P^n$ and S^{2n+1} with respect to the free Hamiltonians with the separation on S^n with respect to the reduced Hamiltonian H_u. More precisely, the separation takes place on the open set $U\subset\mathbb{R}P^n$ defined by taking $s^i>0$. Since the Lie algebras of infinitesimal isometries on $\mathbb{C}P^n$ and S^{2n+1} are the compact forms $su(n+1)$ and $o(2n+2)$, respectively, the maximal abelian subalgebras are unique up to conjugacy and have dimensions n and $n+1$, respectively. Thus we apply the reduction technique of section 2 to arrive at

THEOREM 2. *The Hamilton–Jacobi equation* (1,1) *on* $\mathbb{C}P^n(S^{2n+1})$ *with the free Hamiltonian admits a separable coordinate system* $\{x^i,\alpha^j\}$, $i=1,\cdots,n$, $j=1,\cdots,n$ (*respectively* $n+1$) *with n* (*respectively* $n+1$) *ignorable coordinates* $\{\alpha_i\}$ *if and only if the corresponding coordinates* $\{x^i\}$ *on U separate the H. J.* (1.1) *on U with the reduced Hamiltonian* (3.7) (*with the relation* $\sum_{i=1}^{n+1}u_i=0$ *in the case of* CP^n).

Remarks. If the separable coordinates on U are orthogonal, then the corresponding separable coordinates are orthogonal on S^{2n+1} but never on $\mathbb{C}P^n$.

We now state our main result.

THEOREM 3. *Necessary and sufficient conditions for the existence of a separable coordinate system* $\{x^i\}$ *on* $\mathbb{C}P^n$ *with n ignorable coordinates are that there are n-functions* $\hat{A}_1,\cdots,\hat{A}_n\in\hat{C}$ *one of which, say,* $\hat{A}_1$, *is the free Hamiltonian that are*

1) *linearly independent* (*locally*);

2) *in involution.*

Furthermore, there is a bijective correspondence between orthogonal separable coordinate systems on U and separable coordinate systems on $\mathbb{C}P^n$ *with n ignorable coordinates.*

Before giving the proof of this theorem we will give some geometric background.

Let x^i be an orthogonal separable coordinate system for the free Hamiltonian H_0 on S^n. Then all constants of the motion $A_1,\cdots,A_n$ are in $\mathfrak{D}$. The condition that $A_1,\cdots,A_n$ be linearly independent (locally) means that $A_1,\cdots,A_n$ span an n-plane in $n(n+1)/2$-space. Clearly, changing the A_i's by any $GL(n,\mathbb{R})$ transformation does not alter the coordinate system. We have thus determined a point of the Grassmanian $G(n,n(n+1)/2)$ of n-planes in $n(n+1)/2$ space. Using I_{ij}^2 as a basis for $\mathfrak{D}$, we write

$$A_a=\sum_{i<j}\alpha_a^{ij}I_{ij}^2 \tag{5.1}$$

where the sums run over $1\leq i<j\leq n+1$, $a=1,\cdots,n$. We will view $\{\alpha_a^{ij}\}$ as coordinates in $\mathbb{R}^{(n^2(n+1))/2}$.

Since we are dealing with the free Hamiltonian $H_0=c_2(o(n+1))$, we fix $A_1=H_0$. In terms of the coordinates on $R^{n^2(n+1)/2}$, this is given by the linear equations

$$\alpha_1^{ij}=1 \quad \text{for all } 1\leq i<j\leq n+1. \tag{5.2a}$$

In order to determine a separable coordinate system the A_i's must be in involution under Poisson bracket. Since A_1 is just the Casimir operator of $o(n+1)$, $\{A_a,A_1\}=0$ for all $a=2,\cdots,n$. The remaining conditions $\{A_a,A_b\}=0$, $a,b=2,\cdots,n$ are equivalent to a system of first order partial differential equations which upon using (5.1) can be expressed as the quadratic equations

$$\Sigma\left(\alpha_a^{ij}\alpha_b^{ik}-\alpha_b^{ij}\alpha_a^{jk}\right)=0. \tag{5.2b}$$

where the sum is taken over the cyclic permutations on (i,j,k), and $1\leq i<j<k\leq n+1$.

Proof of Theorem 3. Let $\{\hat{A}_1=\hat{H},\hat{A}_2,\cdots,\hat{A}_n\}$ be n functions in $\hat{C}$ which satisfy 1) and 2) of the theorem. Let us write

$$\hat{A}_r=\sum_{i<j}\alpha_{(r)}^{ij}c_{ij}$$

where $\alpha_{(r)}^{ij}$ has rank n. By Lemmas 2 and 3 there are n functions $A_r(u)=i_u^*\hat{A}_r$, $r=1,\cdots,n$ which are in involution and $A_1(u)=H_u$ for all $u\in\mathbb{R}^{n+1}$ satisfying $\Sigma_{i=1}^{n+1}u_i=0$. Moreover, since $\alpha_{(r)}^{ij}$ has rank n, they are locally linearly independent for all u, in particular at $u=0$. But then $A_1(0)=H_0$, the free Hamiltonian on $U\subset S^n$, and $A_r(0)\in\mathfrak{D}$ by (4.5). So by Theorem 1, there corresponds an orthogonal separable coordinate system $\{x^i\}$. But we claim that $\{x^i\}$ also separates the Hamiltonian H_u. To see this we change our point of view and consider $A_r(u)$ as functions on $T^*(S^{2n+1})$ with $n+1$ ignorable coordinates α^j, and $P_{\alpha^j}^2=u_j^2$ and drop the traceless condition on u. Again applying Theorem 1 there is a separable coordinate system $\{x^i,\alpha^j\}$, $i=1,\cdots,n$, $j=1,\cdots,n+1$ on S^{2n+1}. By Theorem 2 the $\{x^i\}$ then separate H_u on U. Once more by Theorem 2 there is a separable coordinate system $\{x^i,\alpha^j\}$, $i,j=1,\cdots,n$, on $\mathbb{C}P^n$ with n ignorable coordinates.

Conversely, given a separable coordinate system $\{x^i,\alpha^j\}$ on $\mathbb{C}P^n$ with n ignorable coordinates, the corresponding coordinates $\{x^i\}$ on U separate H_u for all $u\in\mathbb{R}^n$, and in particular they separate H_0. Thus by Theorem 1 there are n linearly independent elements A_r of $\mathfrak{D}$ which are in involution. We write

$$A_r=\sum_{i<j}\alpha_r^{ij}I_{ij}^2,\qquad \operatorname{rank}\alpha_r^{ij}=n.$$

Define $A_r(u)$ by

$$A_r(u)=\sum\alpha_r^{ij}\left(I_{ij}^2+V_{ij}\right),$$

where

$$V_{ij}=\left(1+\frac{s_j^2}{s_i^2}\right)u_i^2+\left(1+\frac{s_i^2}{s_j^2}\right)u_j^2,\qquad \sum u_i=0.$$

If we can show that the $A_r(u)$'s are in involution, then we can use Lemmas 2 and 3 to get n linearly independent elements of $\hat{C}$ which are in involution and thus prove the theorem (including the last statement). We formulate this as a lemma.

LEMMA 4. *The set* $\{A_r(u)\}$, $r=1,\cdots,n$ *is in involution for all* $u\in\mathbb{R}^{n+1}$ *if and only if the set* $\{A_r(0)\}$ *is in involution.*

Proof. The "only if" part is trivial. As before we consider $U \subset \mathbb{R}^{n+1}$ and use cartesian coordinates $\{s_i\}$, $i=1,\cdots,n+1$ in $\mathbb{R}^{n+1}$ and $\{s_i, y_j\}$ in $T^*(\mathbb{R}^{n+1})$. We must show that

$$\sum \alpha_r^{ij}\alpha_s^{kl}\left(\left[I_{ij}^2, V_{kl}\right]+\left[V_{ij}, I_{kl}^2\right]\right)=0.$$

Now a straightforward computation gives

$$\frac{1}{4}\left[V_{ij}, I_{kl}^2\right]=t_{ijkl}\delta_{il}+t_{jikl}\delta_{jl}+t_{ijlk}\delta_{ik}+t_{jilk}\delta_{jk}$$

where

$$t_{ijkl}=\left(\frac{s_i}{s_j^2}u_j^2-\frac{s_j^2}{s_i^3}u_i^2\right)\left(s_k^2 y_i - s_k s_i y_k\right).$$

Defining $u_{ijki}=t_{ijki}-t_{ikji}$, we are reduced to showing that

(5.3)
$$\sum\left(\alpha_r^{ij}\alpha_s^{ki}u_{ijki}+\alpha_r^{ij}\alpha_s^{jk}u_{jikj}\right)=0.$$

But an explicit computation shows that

$$u_{jikj}=\frac{s_i s_k}{s_j^2}u_j^2 I_{ik}-\frac{s_i s_j}{s_k^2}u_k^2 I_{ij}-\frac{s_j s_k}{s_i^2}u_i^2 I_{jk}$$

which satisfies

(5.4)
$$u_{ijki}+u_{jikj}=0.$$

That $\{A_r(0)\}$ are in involution implies that $\{\alpha_r^{ij}\}$ satisfies equations (5.2). Combining equations (5.4) with (5.2) implies the equality (5.3) and proves the lemma.

We end this section with two corollaries to Theorem 3.

COROLLARY. *Every separable coordinate system on* $\mathbb{C}P^n$ *with n ignorable coordinates is class one and thus globally admissible.*

COROLLARY. *For every orthogonal separable coordinate system on the sphere* S^n *the locally defined functions* $1/s_i^2$ *are Stäckel multipliers for* $i=1,\cdots,n+1$.

Remark. As mentioned in §1, by a separable coordinate system we actually mean an equivalence class of separable systems, equivalent under $P\times G$ (where G is the isometry group). However the statement of Theorem 1 and its subsequent applications require a specific choice of representative, namely one for which $A_r \in \mathfrak{D}$.

6. Explicit examples of separable coordinates.

A. General *n*. In this case we discuss two examples; the most and the least degenerate coordinate systems. The most degenerate is given by spherical coordinates on S^n, viz.

(6.1)
$$\begin{aligned}
s_1&=\sin\phi_1\cdots\sin\phi_{n-1}\sin\phi_n,\\
s_2&=\sin\phi_1\cdots\cos\phi_{n-1}\cos\phi_n,\\
s_3&=\sin\phi_1\cdots\cos\phi_{n-1},\\
&\vdots\\
s_n&=\sin\phi_1\cos\phi_2,\\
s_{n+1}&=\cos\phi_1.
\end{aligned}$$

The separated equations on $\mathbb{C}P^n$ are

$$
(6.2)\qquad
\begin{aligned}
&P_{\alpha_i}=u_i, \qquad 1\le i\le n\\
&P^2_{\phi_n}+\frac{u_1^2}{\sin^2\phi_n}+\frac{u_2^2}{\cos^2\phi_n}=\lambda_n,\\
&P^2_{\phi_{n-1}}+\frac{\lambda_n}{\sin^2\phi_{n-1}}+\frac{u_3^2}{\cos^2\phi_{n-1}}=\lambda_{n-1},\\
&\vdots\\
&P^2_{\phi_1}+\frac{\lambda_2}{\sin^2\phi_1}+\frac{(\Sigma u_i)^2}{\cos^2\phi_1}=\lambda_1.
\end{aligned}
$$

This corresponds to the group reduction $SU(2)\subset\cdots\subset SU(n)\subset SU(n+1)$, and the corresponding Casimir invariants (4.2) give the relevant constants of the motion;

$$
(6.3)\qquad \hat{c}_2(su(k))=\lambda_{n-k+2}.
$$

The solutions of (5.2b) are given by

$$
\alpha^{ij}_{n+2-k}=\begin{cases}1, & i\le j\le k,\\ 0, & \text{otherwise.}\end{cases}
$$

The least degenerate system is given by the general Jacobi elliptic coordinates on S^n, viz.

$$
(6.4)\qquad s_i^2=\frac{\Pi_{j=1}^n(x^j-e^i)}{\Pi_{j\neq i}(e^j-e^i)}, \qquad 1\le i\le n+1
$$

where the constants e^i satisfy $e^1<x^1<e^2<\cdots<x^n<e^{n+1}$. These are known to separate variables [7] on S^n.

B. $\boldsymbol{n=2}$. It is well known [19], [20] that there are precisely two separable coordinate systems on S^2, spherical and elliptical coordinates. Thus by Theorem 3 we get two separable coordinate systems on $\mathbb{C}P^2$. We will now show that these are all the separable coordinate systems on $\mathbb{C}P^2$. In fact we will prove a more general result relevant to the study of selfdual gravitational instantons. We will make use of the classification of canonical forms for four dimensional manifolds given in [5]. Since we are dealing with a Riemannian (positive definite) metric, the only relevant types are B, C, F and H of [5].

Before stating and proving our result we give some background. Let V_4 be a four dimensional Riemannian manifold. Due to the local isomorphism between the groups $SO(4)$ and $SU(2)\times SU(2)$, we can describe local four dimensional Riemannian geometry equally well in terms of local orthogonal or local spinorial moving frames. For example, if Ω denotes the curvature two-form on V_4, then with respect to an orthonormal moving coframe $\{\theta^a\}$ we have

$$
(6.5)\qquad \Omega^a_b=R^a{}_{bcd}\theta^c\wedge\theta^d, \qquad a,b,c,d=1,\cdots,4
$$

whereas, with respect to a local spinor coframe $\theta^{A\dot{A}}$ with $\theta^{1\dot{1}}=\theta^1+i\theta^3$, $\theta^{1\dot{2}}=\theta^2+i\theta^4$, $\theta^{2\dot{2}}=-\overline{\theta^{1\dot{1}}}$, and $\theta^{2\dot{1}}=\overline{\theta^{1\dot{2}}}$, we have

$$\Omega^A{}_B=C^A{}_{BCD}S^{CD}+\frac{R}{12}S^A{}_B+C^A{}_{B\dot{C}\dot{D}}S^{\dot{C}\dot{D}},\tag{6.6}$$

$$\Omega^{\dot{A}}{}_{\dot{B}}=C^{\dot{A}}{}_{\dot{B}\dot{C}\dot{D}}S^{\dot{C}\dot{D}}+\frac{R}{12}S^{\dot{A}}{}_{\dot{B}}+C^{\dot{A}}{}_{\dot{B}CD}S^{CD}$$

$A, B, \dot{A}, \dot{B}=1,2$; and

$$S^{AB}=\frac{1}{2}\varepsilon_{\dot{A}\dot{B}}\theta^{A\dot{A}}\wedge\theta^{B\dot{B}},\tag{6.7}$$

$$S^{\dot{A}\dot{B}}=\frac{1}{2}\varepsilon_{AB}\theta^{A\dot{A}}\wedge\theta^{B\dot{B}},$$

$1=\varepsilon_{12}=-\varepsilon_{21}$, $\varepsilon_{11}=\varepsilon_{22}=0$. We mention that $S^{\dot{A}\dot{B}}$ (S^{AB}) is self-dual (anti self-dual) with respect to the Hodge star operator $*$ on exterior differential forms. The decomposition (6.6) is convenient because it realizes Ω in terms of its irreducible components with respect to the group $SO(4)$. Here $C_{\dot{A}\dot{B}\dot{C}\dot{D}}$ (C_{ABCD}) are the self-dual (anti self-dual) components of the Weyl conformal tensor, $C_{AB\dot{C}\dot{D}}$ is the traceless part of the Ricci tensor, and R is the scalar curvature. V_4 is said to be *self-dual* (*anti self-dual*) if $\Omega^A{}_B=0$ ($\Omega^{\dot{A}}{}_{\dot{B}}=0$), and *conformally self-dual* (*conformally anti self-dual*) if $C_{ABCD}=0$ ($C_{\dot{A}\dot{B}\dot{C}\dot{D}}=0$).

THEOREM 4. *Let V_4 be a conformally anti self-dual Riemannian space. Suppose further that in V_4 the Laplace–Beltrami equation* (1.2) *is separable in the local coordinate system $\{x^i\}$. Then either V_4 is conformally flat or $\{x^i\}$ is type C and nonorthogonal.*

Proof. From equations (6.5)–(6.7) we find

$$C_{ABCD}=S^{ab}{}_{(AB}S^{cd}{}_{CD)}R_{abcd},$$

$$C_{\dot{A}\dot{B}\dot{C}\dot{D}}=S^{ab}{}_{(\dot{A}\dot{B}}S^{cd}{}_{\dot{C}\dot{D})}R_{abcd},$$

where the parentheses denote symmetrization and $S^{ab}{}_{AB}$($S^{ab}{}_{\dot{A}\dot{B}}$) are the components of S_{AB}($S_{\dot{A}\dot{B}}$) with respect to $\theta^a\wedge\theta^b$. Explicitly we have

$$\begin{aligned}
&C_{\dot{1}\dot{1}\dot{1}\dot{1}}-C_{1111}=4(R_{1234}-R_{2314})+4i(R_{1214}+R_{2334}),\\
&C_{\dot{1}\dot{1}\dot{1}\dot{2}}-C_{1112}=2(R_{1413}+R_{2324})+2i(R_{1242}+R_{3431}),\\
&C_{\dot{1}\dot{1}\dot{2}\dot{2}}-C_{1122}=-8R_{1324}+4R_{1234}+4R_{1423}
\end{aligned}\tag{6.8}$$

and $C_{ABCD}=\overline{C^{ABCD}}$ and the same for dotted components. Now suppose $\{x^i\}$ is type H; then by [5, Lemma 5] the only nonvanishing components of Ω are R_{abba}. Thus from (6.8) $C_{ABCD}=0$ implies $C_{\dot{A}\dot{B}\dot{C}\dot{D}}=0$. Suppose now that $\{x^i\}$ is type F. Without loss of generality (by making an $SO(4)$ gauge transformation if necessary) we can choose x^4 as the ignorable coordinate. Then [21, eq. (37.4)] implies $R_{abb4}=0$, $a,b=1,2,3$ and the result follows as before. Similarly if $\{x^i\}$ is type B, we can use results of Petrov [22, pp. 174–175] to show that V_4 is conformally flat. Now suppose $\{x^i\}$ is type C and orthogonal, then again [21, eq. (37.4)] and (6.8) imply that V_4 is conformally flat. It follows that $\{x^i\}$ is necessarily type C and nonorthogonal.

Since in an Einstein space, Hamilton–Jacobi separability implies Laplace–Beltrami separability [1], [5], it follows that Theorem 4 holds when Laplace–Beltrami separability is replaced by Hamilton–Jacobi separability, and V_4 is an Einstein space. Furthermore, since $\mathbb{C}P^2$ with the Fubini–Study metric (2.8) is a conformally anti self-dual Einstein space, we can combine Theorems 3 and 4 with the well-known fact [19], [20] that there are precisely two separable coordinate systems on S^2 to obtain:

COROLLARY. *There are precisely two separable coordinate systems on* $\mathbb{C}P^2$. *They are the two induced by Theorem* 3 *from spherical and elliptic coordinates on* S^2.

These two coordinate systems are given by equations (6.1) and (6.4) with $n=2$. Furthermore, the constants of the motion are given by $c_2(su(3))$, P_{α^1}, P_{α^2} and for

(i) spherical coordinates by $c_{12}=c_2(su(2))$

(ii) elliptic coordinates by $c_{23}+ac_{13}$

where in (6.4) we have taken $e_1=0$, $e_2=1$, $e_3=a$, and c_{ij} is given by (4.6).

C. $n=3$ and 4. For $\mathbb{C}P^3$ there are precisely six classes of separable systems with three ignorable coordinates. These come from [23, systems (1), (3), (6), (13) and (17)] In the real case there are two inequivalent classes of type (13), see [24, Table 1]. For $\mathbb{C}P^4$ there are 14 systems on S^4 which are inequivalent under $SO(5,\mathbb{C})$. These are given by [25, classes I, V(i), VI, VII(i), VIII and X]. The inequivalent types under the real group $SO(5,\mathbb{R})$ can be worked out from these. For all of the systems mentioned above the constants of the motion on S^3 and S^4 can be read off and transformed by (4.5) to constants of the motion on $\mathbb{C}P^3$ and $\mathbb{C}P^4$, respectively. It is then a straightforward task to write down the separated equations in each case.

7. Conclusions. The main result of this paper can be formulated as an algorithm. In order to find all conjugacy classes of coordinate systems in $\mathbb{C}P^n$ having an additive separation of variables in the Hamilton–Jacobi equation (1.1) (or multiplicative separation in the Laplace–Beltrami equation (1.2)), proceed as follows:

1. Introduce n complex coordinates z_k and put

$$z_k=\frac{s_k}{s_{n+1}}e^{i\alpha_k}, \qquad 1\le k\le n \tag{7.1}$$

where

$$s_1^2+\cdots+s_{n+1}^2=1, \tag{7.2}$$

i.e., s_i $(i=1,\cdots,n+1)$ are cartesian coordinates in $\mathbb{R}^{n+1}$.

2. Find all separable coordinate systems $\{\theta_1,\cdots,\theta_n\}$ on the real sphere S^n for which the free Hamilton–Jacobi (or free Laplace–Beltrami) equation on the sphere allows a separation of variables and express s_i $(1\le i\le n+1)$ in terms of these separable coordinates. The corresponding equations on the sphere with the potential induced from $\mathbb{C}P^n$ (see (3.7)) will, as we have shown, also separate. Substitute

$$s_i=s_i(\theta_1\cdots\theta_n), \qquad 1\le i\le n+1 \tag{7.3}$$

back into (7.1). Then the sets

$$(\theta_1,\cdots,\theta_n,\alpha_1,\cdots,\alpha_n) \tag{7.4}$$

provide a complete list of representatives of all conjugacy classes of separable coordinates on $\mathbb{C}P^n$ with n ignorable coordinates.

Several comments are in order here.

1. For $n=2$, i.e. the complex projective plane $\mathbb{C}P^2$ we have proven that all separable coordinate systems have precisely 2 ignorable coordinates, i.e. the maximum possible number equal to the rank n of $su(n+1)$. Thus there exist precisely 2 separable

coordinate systems in $\mathbb{C}P^2$, induced by spherical and elliptic coordinates on S^2, respectively.

2. The $2n$ integrals of motion in involution characterizing each separable system are obtained as follows. The first n of them correspond to the ignorable coordinates α_i; they form a basis for the Cartan subalgebra of $su(n+1)$ and can be identified with the canonical momenta conjugated to α_i:

$$P_{\alpha_i}, \qquad 1 \le i \le n. \tag{7.5}$$

The remaining n constants of motion A_r (including the Hamiltonian (1.1)) can be interpreted as second order operators in the enveloping algebra of $su(n+1)$. Writing the infinitesimal generators of $su(n+1)$ as (again by an abuse of notation)

$$T_{ik}=E_{ik}-E_{ki}, \quad S_{ik}=i(E_{ik}+E_{ki}), \quad H_{ik}=i(E_{ii}-E_{kk}) \; 1 \le i \le k \le n+1, \tag{7.6}$$

in the defining representation of $su(n+1)$, we can write the quadratic constants of motion as n independent linear combinations of the $n(n+1)/2$ Casimir invariants

$$C_{ik}=T_{ik}^2+S_{ik}^2+H_{ik}^2, \qquad 1 \le i < k \le n+1, \tag{7.7}$$

(i, k fixed) of the $su(2)$ algebras (7.6). Thus

$$A_r = \sum_{1 \le i < k \le n+1} \alpha_r^{ik} C_{ik}, \qquad 1 \le r \le n. \tag{7.8}$$

The operators A_r can easily be restricted to $\mathbb{C}P^n$ or to S^n. Upon restriction to S^n the Casimir operators C_{ik} reduce to the form (4.5). The classification of coordinate systems on S^n then reduces to a classification of sets of n operators in involution, all of them being linear combinations of the squares of the generators of $o(n+1)$.

Several problems suggested by this paper are under active consideration:

1. The first concerns special function theory and the separation of variables on $\mathbb{C}P^n$ in spherical coordinates (see §6A of this article). If we separate variables in the Laplace–Beltrami equation, then (6.2) reduces to a system of $2n$ ordinary linear equations. The eigenfunctions of the Laplace–Beltrami equations are then expressed as products of Jacobi functions (and exponentials $e^{iu_k\alpha_k}$). The role of Jacobi polynomials as basis functions of $SU(n+1)$ representations in a basis corresponding to the subgroup reduction $SU(n+1) \supset U(n) \supset U(n-1) \supset \cdots \supset U(2) \supset U(1)$ makes it possible to obtain relations for special functions, in particular addition formulas [26]. But now more poweful methods are at our disposal. We can use the techniques of [12] by constructing a simple model of $SU(n+1)$ acting on the sections of certain holomorphic line bundles over $\mathbb{C}P^n$ and relate this action to the action on harmonic polynomials—namely the Jacobi polynomials. Furthermore, we have many more sets of bases than that given by spherical coordinates. A detailed study of tractable coordinates should give a wealth of special function identities.

2. The approach of this article has been to Hermitian hyperbolic spaces $HH(n)$. The noncompact group $SU(n,1)$ then plays the role that $SU(n+1)$ plays for $\mathbb{C}P(n)$. The results are much richer for $HH(n)$ mainly because $su(n,1)$ has $n+2$ different mutually nonconjugated maximal abelian subalgebras [27]–[28], each of them being of dimension n and leading to different types of coordinate systems with n ignorable variables [29].

3. Separation of variables on a sphere S^n is being studied for arbitrary n (the results are at present known only for $n=2, 3$, and 4) [30].

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