THE UNIVERSITY OF WAIKATO
Te Whare Wānanga o Waikato
Research Commons

https://researchcommons.waikato.ac.nz/

Research Commons at the University of Waikato

Copyright Statement:

The digital copy of this thesis is protected by the Copyright Act 1994 (New Zealand).

The thesis may be consulted by you, provided you comply with the provisions of the Act and the following conditions of use:

- Any use you make of these documents or images must be for research or private study purposes only, and you may not make them available to any other person.
- Authors control the copyright of their thesis. You will recognise the author's right to be identified as the author of the thesis, and due acknowledgement will be made to the author where appropriate.
- You will obtain the author's permission before publishing any material from the thesis.

Iwasawa theory for tensor products of Hilbert modular forms

A thesis
submitted in partial fulfilment
of the requirements for the Degree
of
Doctor of Philosophy in Mathematics
at
The University of Waikato
by

Raiza Miranda Corpuz

2026

Abstract

The main conjecture of Iwasawa theory bridges two seemingly disjoint areas of mathematics: arithmetic and analysis. In particular, it provides a deep connection between the p-adic L-function which interpolates critical values of the complex L-series, and the Selmer group which is an important object used to control the growth of arithmetic data. In this thesis, we explore some questions that arise organically from the Iwasawa Main Conjecture, applied to Hilbert modular forms and their tensor products.

Greenberg and Vatsal developed an approach to study the main conjecture for a large class of elliptic curves simultaneously. They showed that if a given pair of elliptic curves share the same residual Galois representation, then the main conjecture holds for one if and only if it does for the other. The first part of this thesis investigates whether the ideas of Greenberg and Vatsal work for elliptic curves twisted by a CM-Hecke character. The second part of this thesis then extends the method to treat non-ordinary classical modular forms (without any twist). The former permits one to study rational elliptic curves base-changed to an arbitrary number field, whilst the latter requires techniques crafted by Pollack and Kobayashi in the early 2000s.

Finally, the third part of this thesis concerns Euler systems and their applications to the arithmetic of motives arising from modular forms—these objects are indispensable tools which can be used to prove half of the Iwasawa main conjecture. However they often give rise to additional “junk” error terms, as well as causing the p-adic L-function to be stripped away of certain bad Euler factors. For modular forms and their tensor products we devise a method to dispose of the error terms and to replenish the missing factors, allowing one to genuinely obtain half of the main conjecture free from any discrepancies.

Declaration

I declare that all the results in this thesis are original, and are either published or else submitted to a mathematical journal. Results made by other authors are properly cited and are stated in this thesis solely to supplement discussions. Should the reader wish to check, the original results mentioned above can be found in the following papers:

- Corpuz, Raiza and Delbourgo, Daniel. Congruent elliptic curves and the τ-component in the Iwasawa main conjecture. *Mathematical Proceedings of the Cambridge Philosophical Society*, 179(2):373–408, 2025.
- Corpuz, Raiza and Lei, Antonio. Congruences of p-adic L-functions of modular forms at non-ordinary primes, arXiv:2508.09733.
- Corpuz, Raiza and Delbourgo, Daniel. Replenishing p-adic L-functions and Euler systems at their bad primes. *Journal of Number Theory*, 279:527–563, 2026.

Acknowledgements

The writing of this thesis from the first down to the last page was by no means smooth-sailing. This page is dedicated to the numerous people whose combined support have made my doctoral experience remarkable and enjoyable.

Firstly, I would like to offer my utmost gratitude to Daniel Delbourgo. As my supervisor, he has been unsparing with his time and tireless in giving me advice. He has always had my best interest in mind and has expertly navigated me to several opportunities, all of which have enriched me as a mathematician.

I am immensely grateful to Antonio Lei for taking me in as one of his students. I am fortunate to have learned directly from him the side of Iwasawa theory that is not at all ordinary.

I am extremely thankful to Tim Stokes for his steady guidance and for stepping in during a very turbulent time.

My sincere appreciation to Maria Admiraal who is always ready to extend a helping hand, and to the staff in the Mathematics & Statistics Department of the University of Waikato who have all been equally accommodating. I am very lucky to have been a part of the fold.

I also acknowledge the generous financial assistance bestowed upon me by the Waikato Doctoral Scholarship and the Zulauf Trust Scholarship.

My time on the other side of the world as a visiting research student was productive thanks to the excellent environment cultivated at the University of Ottawa and the weekly seminars held by the Number Theory group.

I have also greatly benefited from the companionship of William and Aryan. Our musings over coffee, weekly brunch and short excursions have given me much-needed respite.

I am eternally grateful to my family who has always provided me with unwavering encouragement. The thought of making them proud fuels me to keep going.

Finally, I would like to give my heartfelt gratitude to my partner whose unrelenting support has sustained me throughout.

Contents

Chapter 1

Introduction

Iwasawa theory is a powerful tool for studying the arithmetic of objects over a tower of number fields exhibiting a p-adic analytic structure. When applied to a modular form f, the main conjecture of Iwasawa theory predicts an equality (over the Iwasawa algebra Λ) between the ideal generated by the p-adic L-function attached to f and the characteristic ideal of the p-primary Selmer group attached to f, up to a unit.

It was proposed in the work of Greenberg and Vatsal [GV00] that given two rational elliptic curves sharing the same residual representation, if the Iwasawa Main Conjecture (abbreviated as IMC) holds for one elliptic curve then it does so for the other elliptic curve. Whilst this theory does not establish the validity of the main conjecture, it serves as a way to propagate the equality in the main conjecture to a class of p-congruent elliptic curves. The authors themselves have shown that this concept works when both elliptic curves are *p-ordinary*, meaning that the Fourier coefficient $a_p(E_i)$ is a p-adic unit for each $i = 1, 2$.

The first objective of this thesis is to extend the result of Greenberg and Vatsal to the case of p-ordinary elliptic curves twisted by a Hecke character.

Theorem A. *Let E_1 and E_2 be a pair of p-ordinary rational elliptic curves which are congruent modulo p, and let $\vartheta_K : K \to \mathbb{C}^\times$ be a Hecke character over a CM-extension K/K^+. Under suitable hypotheses there is an equivalence:*

$$\textit{IMC holds for } E_1 \otimes \vartheta_K \iff \textit{IMC holds for } E_2 \otimes \vartheta_K .$$

Chapters 4 and 5 are devoted to the discussion of this topic. We also mention that the motivation behind considering twists of elliptic curves is the identity

$$L(E/K,s) = \prod_{\tau} L(E/\mathbb{Q},\tau,s)^{\dim \tau}$$

where τ ranges over the irreducible representations in $\operatorname{Reg}(K/\mathbb{Q})$, which reduces the analytic properties of a rational elliptic curve E over an arbitrary Galois extension $K/\mathbb{Q}$ to the study of twists by irreducible representations.

Another avenue one can explore is to ask whether or not the proposed idea of Greenberg and Vatsal still applies when one takes instead a pair of *p-non-ordinary* modular forms f_1 and f_2 so that p divides both $a_p(f_1)$ and $a_p(f_2)$. As it turns out, two crucial aspects break down when p-ordinarity is not imposed: first, the p-adic L-function $L_p(f)$ is no longer a bounded power series; second, the p-Selmer group $\operatorname{Sel}_p(f)$ ceases to be a torsion Λ-module. The first issue is dealt with by decomposing the original $L_p(f)$ into a pair of signed p-adic L-functions $L_p^{\pm}(f)$ which are p-integral. This was done by Pollack in the case that $a_p(f_i) = 0$ for $i = 1, 2$ [Pol03] and extended by Sprung for elliptic curves by removing the assumption that $a_p(f_i) = 0$ [Spr12].

As for the second issue, Kobayashi treated the case of elliptic curves by defining a pair of signed p-Selmer groups $\operatorname{Sel}_p^{\pm}(f)$ which are Λ-torsion [Kob03]. This definition hinges on the signed Coleman maps he constructed sending the localisation of Kato's zeta-elements to Pollack's signed p-adic L-functions. In this way, one can identify the signed p-Selmer groups as the algebraic counterparts of the signed p-adic L-functions. The method of Kobayashi has subsequently been generalized in [Lei11; LLZ10; LLZ11] to higher weight forms; this allowed for the synthesis of a signed Iwasawa Main Conjecture which asserts that $L_p^{\natural}(f)$ generates the characteristic ideal of $\operatorname{Sel}_p^{\natural}(f)$ for $\natural \in \{-,+\}$.

It seems then that in the p-non-ordinary case, Greenberg and Vatsal's principle must be verified not once but twice—one for each choice of sign! To harness the techniques in [GV00], one first needs to extend the construction of Sprung to higher weight modular forms and define signed p-adic L-functions as a limit of p-adic polynomials arising from modular symbols. Performing this task along with proving a signed analogue of Greenberg–Vatsal's result for p-non-ordinary modular forms is the second objective of this thesis, and is written up in Chapter 6.

Theorem B. *Let f_1 and f_2 be a pair of p-non-ordinary modular forms which are congruent modulo p. Under various hypotheses, there is an equivalence:*

$$\mathrm{IMC}^{\natural} \textit{ holds for } f_1 \iff \mathrm{IMC}^{\natural} \textit{ holds for } f_2.$$

Here $\mathrm{IMC}^{\natural}$ *refers to the signed main conjecture above with* $\natural \in \{-,+\}$.

Theorems A and B appear in the thesis as Theorems 5.3 and 6.26, respectively.

Both theorems given thus far use the following idea applied to a pure motive M defined over $\mathbb{Q}$. Provided a system of norm-compatible zeta-elements exists and is mapped via Perrin-Riou's big logarithm map to the appropriate p-adic L-function $L_p(M)$, then one obtains an inclusion

$$\big(\Xi_p(M) \cdot L_{p,(S)}(M)\big) \subset (\text{characteristic ideal of } p\text{-Selmer group over } \Lambda).$$

In the expression directly above, $L_{p,(S)}(M)$ denotes the usual p-adic L-function stripped of its local Euler factors $\mathscr{L}_{\mathfrak{q}}(M)$ at the primes $\mathfrak{q}$ inside a finite set S.

The last objective of this thesis is to get rid of the junk factor $\Xi_p(M)$ and to replenish the Euler factors in $L_{p,(S)}(M)$, resulting in a sharper containment

$$\big(L_p(M)\big) \subset (\text{characteristic ideal of } p\text{-Selmer group over } \Lambda) \tag{1.1}$$

thereby giving us half of the main conjecture. The pivotal step is to identify a suitable finite set of primes S and then to show that $L_p^{\phi}(M)$ can be written as a linear combination of the S-depleted function $L_{p,(S)}^{\phi}(M)$ and its dual $L_{p,(S)}^{\phi^{-1}}(M)$, for each fixed branch character $\phi : (\mathbb{Z}/p\mathbb{Z})^{\times} \to \overline{\mathbb{Q}}_p^{\times}$.

Theorem C. *Let ϕ be a fixed branch character. Provided the product of local Euler factors $\prod_{\mathfrak{q}\in S} \mathscr{L}_{\mathfrak{q}}^{\phi}(M)$ is relatively prime to the product $\prod_{\mathfrak{q}\in S} \mathscr{L}_{\mathfrak{q}}^{\phi^{-1}}(M)$ of its dual counterparts, one has a decomposition*

$$L_p^{\phi}(M) = \mathcal{W}^{\pm,\phi}(M) \cdot \mathcal{Y}_1^{\phi}(M) \times L_{p,(S)}^{\phi}(M) + \mathcal{Y}_2^{\phi}(M) \times L_{p,(S)}^{\phi^{-1}}(M)$$

for some p-integral power series $\mathcal{W}^{\pm,\phi}(M), \mathcal{Y}_1^{\phi}(M)$ and $\mathcal{Y}_2^{\phi}(M)$.

Chapter 7 is devoted to verifying that for a suitable subset S of bad primes of M, the products $\prod_{\mathfrak{q}\in S}\mathscr{L}^{\phi}_{\mathfrak{q}}(M)$ and $\prod_{\mathfrak{q}\in S}\mathscr{L}^{\phi^{-1}}_{\mathfrak{q}}(M)$ are relatively prime for several types of motives, namely those attached to: a Hilbert modular form $\mathbf{f}$, the symmetric square representation $\mathrm{Sym}^2(\mathbf{f})$, and the tensor products $\mathbf{f}_1\times\mathbf{f}_2$ and $\mathbf{f}_1\times\mathbf{f}_2\times\mathbf{f}_3$ of Hilbert modular forms. In addition, we have two corollaries which we state below. The first is an immediate consequence of Theorem C.

Corollary D. *If both $L^{\phi}_{p,(S)}(M)$ and its dual $L^{\phi^{-1}}_{p,(S)}(M)$ are holomorphic and p-integral, then the full p-adic L-function $L_p(M)$ must also be holomorphic and p-integral.*

The second application requires us to modify the construction of families of norm-compatible elements to mirror the decomposition in Theorem C, thereby sending these modified zeta-elements to the full p-adic L-function. The details can be found in Chapter 8.

Corollary E. *Let M be a pure motive defined over $\mathbb{Q}$ which is self-dual and for which an Euler system exists. Assuming the hypothesis of Theorem C holds, then one obtains the inclusion*

$$\big(L_p(M)\big)\subset(\textit{characteristic ideal of } p\textit{-Selmer group over } \Lambda).$$

We mention that Theorem C appears in this thesis as Proposition 7.2. Corollaries D and E on the other hand are discussed on a case-by-case basis for each example considered and can be found in Theorem 8.2, Corollary 7.8, Corollary 7.6 and Theorem 8.3.

Finally, Chapters 2 and 3 are devoted to equipping the reader with a sufficient amount of background. More precisely, in Chapter 2 we provide a brief exposition about Hilbert modular forms, and in Chapter 3 we give several statements of the Iwasawa Main Conjecture. Whilst our results pertaining to the main conjecture only hold when the base field is $\mathbb{Q}$ (i.e. for elliptic modular forms), we are able to prove more general results for Hilbert modular forms.

Chapter 2

Hilbert modular forms

In this chapter we present Hilbert modular forms (HMFs) as a generalisation of classical modular forms into several variables. Our discussion shall closely follow that of [Shi78] with slight deviation in notation.

Roughly speaking, a modular form is a function on several copies of the upper-half plane $\mathfrak{H}$ while an automorphic form is a function on some reductive algebraic group G. To a modular form f, one may associate an automorphic form $\mathbf{f}$ using an approximation result that relates $\mathfrak{H}$ to a quotient of G and then taking $\mathbf{f}$ to be the pull-back of f. This is the underlying philosophy behind the adèlic construction in Section 2.1, which uses the approximation result in [Gee88, Proposition 7.2]. In other words, the Hilbert modular form presented in Definition 2.2 is an automorphic form by construction. In the next few sections, we shall enumerate several operators acting on HMFs (Section 2.2) and we explain how the space of Hilbert modular cuspforms splits into an "old" and a "new" subspace (Section 2.3).

The space of automorphic forms decomposes into a direct sum of irreducible invariant subspaces which in turn induce irreducible admissible representations on a given algebraic group. One then defines the *automorphic representation* $\pi(\mathbf{f})$ attached to $\mathbf{f}$ as the irreducible admissible constituent for which the projection of $\mathbf{f}$ is nontrivial (see [JL70] for a more detailed discussion in the case of $G = \mathrm{GL}_2$ and [Bum97, §3.3] in the case of $G = \mathrm{GL}_n$ with $n > 2$). Moreover by the tensor product theorem, one can write $\pi(\mathbf{f})$ as a restricted tensor product $\bigotimes'_{\mathfrak{v}} \pi(\mathbf{f})_{\mathfrak{v}}$ of local representations on $\mathrm{GL}_2(F_{\mathfrak{v}})$ [Bum97, Theorem 3.3.3].

As a consequence of the Jacquet-Langlands theory for GL_2 [JL70, §16], one can attach an L-function $L(\pi(\mathbf{f}), s)$ to $\pi(\mathbf{f})$ which decomposes as a product of local L-factors $L(\pi(\mathbf{f}), s) = \prod_{\mathfrak{v}} L_{\mathfrak{v}}(\pi_{\mathfrak{v}}(\mathbf{f}), s)$, and which satisfies various analytic properties listed in the beginning of Section 2.4. This is where the relationship between Hilbert modular forms and automorphic representations proves to be beneficial. In fact, by explicit computation of each local integral representation, it can be shown that $\Psi(\pi(\mathbf{f}), s) = \Psi\big(\mathbf{f}, s + \frac{k-1}{2}\big)$ where $\Psi(\mathbf{f}, s)$ is defined in Equation (2.34) below. Finally, we close the chapter with a discussion of the Rankin-integral representation which is pertinent to Chapter 4.

2.1 Adèlic framework

Let F be a totally real number field of degree n, and let D_F and $\mathfrak{d}_F$ be the discriminant and different of its ring of algebraic integers $\mathcal{O}_F$. The symbol $\mathcal{N}_{F/\mathbb{Q}}(a)$ or $\mathcal{N}_{F/\mathbb{Q}}(\mathfrak{a})$ denotes the norm of an element a or an ideal $\mathfrak{a}$ in F, thus $\mathcal{N}_{F/\mathbb{Q}}(\mathfrak{d}_F) = D_F$. When the context is clear, we simply write D, $\mathfrak{d}$, $\mathcal{O}$ and $\mathcal{N}$.

Let us write $\widehat{\mathcal{O}}_F$ for the profinite completion of F and $F_{\mathfrak{v}}$ for the completion of F at a place $\mathfrak{v}$. For an Archimedean place $\mathfrak{v}$, either $F_{\mathfrak{v}} = \mathbb{R}$ or $F_{\mathfrak{v}} = \mathbb{C}$, whereas for a non-Archimedean place $\mathfrak{v}$, $F_{\mathfrak{v}}$ is a finite extension of the field $\mathbb{Q}_\ell$ for a rational prime $\ell \mid \mathfrak{v}$. We denote the corresponding integer ring as $\mathcal{O}_{F_{\mathfrak{v}}}$. Recall that there is a one-to-one correspondence between finite places and prime ideals of F so we use these terms interchangeably.

Let $\mathbb{A}_F$ be the ring of adèles of F and let $\mathbb{A}_F^\times$ be the group of idèles of F. Denote by $\mathbb{A}_{F,\infty} \cong F \otimes_{\mathbb{Q}} \mathbb{R}$ and $\mathbb{A}_{F,\mathrm{fin}} \cong F \otimes_{\mathcal{O}_F} \widehat{\mathcal{O}_F}$ the infinite and finite parts of $\mathbb{A}_F$, respectively. One therefore has $\mathbb{A}_F = \mathbb{A}_{F,\infty} \cdot \mathbb{A}_{F,\mathrm{fin}}$, and given an element $a \in \mathbb{A}_F$, one may write it as $a = (a_\infty \mid a_v)$ or $a_\infty \cdot a_{\mathrm{fin}}$, where $a_\infty \in \mathbb{A}_{F,\infty}$ and $a_{\mathrm{fin}} \in \mathbb{A}_{F,\mathrm{fin}}$. For example, $a \in \mathbb{A}_{\mathbb{Q}}$ may be written as $(a_\infty \mid a_2, a_3, a_5, \ldots)$. We write $a_\infty = 1$ to signify that the infinite part of a equals $\mathbf{1} = (1, \ldots, 1)$.

The group $\mathrm{GL}_2(\mathbb{R})$ of invertible 2×2 matrices over $\mathbb{R}$ acts on the upper-half plane $\mathfrak{H} = \left\{z = x + y\sqrt{-1} \in \mathbb{C} : y > 0\right\}$ via the Möbius transformation: for $\alpha = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ and $z \in \mathfrak{H}$,

$$\alpha z = \frac{az+b}{cz+d}\,.$$

This action naturally extends to $\mathfrak{H}^n$: given $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathrm{GL}_2(\mathbb{R})^n$ and $z = (z_1, \ldots, z_n) \in \mathfrak{H}^n$, one has $\alpha z = (\alpha_1(z_1), \ldots, \alpha_n(z_n))$.

Fix an embedding $(\sigma_1, \ldots, \sigma_n) : F \hookrightarrow \mathbb{R}^n$, so that an element $x \in F$ is sent to $(x^{\sigma_1}, \ldots, x^{\sigma_n})$. The subgroup $\mathrm{GL}_2^+(\mathbb{R}) \subset \mathrm{GL}_2(\mathbb{R})$ of elements with positive determinant acts on $\mathfrak{H}^n$ by first mapping $\alpha \in \mathrm{GL}_2^+(\mathbb{R})$ to the n-tuple $(\alpha^{\sigma_1}, \ldots, \alpha^{\sigma_n}) \in \mathrm{GL}_2^+(\mathbb{R})^n$, where α^{σ_j} is the matrix α with σ_j acting on each entry. The element $(\alpha^{\sigma_1}, \ldots, \alpha^{\sigma_n})$ can then be made to act on $z \in \mathfrak{H}^n$ via the Möbius transformation.

For $k = (k_1, \ldots, k_n) \in \mathbb{Z}^n$ and $z = (z_1, \ldots, z_n) \in \mathbb{C}^n$, we use the notations $z^k = \prod_{j=1}^n z_j^{k_j}$ and $\mathrm{Tr}(z) = \sum_{j=1}^n z_j$. Moreover, we recall the exponential function $e_F(z) = e^{2\pi\sqrt{-1}\cdot\mathrm{Tr}(z)}$ over the field F. Given an element $z \in \mathfrak{H}^n$ and a matrix $\alpha = \left(\left(\begin{smallmatrix} a_i & b_i \\ c_i & d_i \end{smallmatrix}\right)\right)_{1\le i\le n} \in \mathrm{GL}_2^+(\mathbb{R})^n$, the *factor of automorphy* is given by

$$j(\alpha, z) = \prod_{j=1}^n (c_i z_i + d_i), \tag{2.1}$$

and the *slash operator* is defined as

$$\left(f\,\big|_k\,\alpha\right)(z) = (\det\alpha)^{k/2} j(\alpha, z)^{-k} f(\alpha(z)). \tag{2.2}$$

Let $G \subset \mathrm{GL}_{2n}$ be a reductive algebraic group. Viewing $\mathrm{GL}_2(F)$ as the subgroup $G(\mathbb{Q})$ of $\mathbb{Q}$-rational points of G allows one to identify $\mathrm{GL}_2(\mathbb{A}_F)$ with the adèlisation $\mathbb{A}_G$ of G. Under this identification along with the fixed embeddings $(\sigma_1, \ldots, \sigma_n) : F \hookrightarrow \mathbb{R}^n$, one sees that $\mathbb{R}^n$ and $\mathrm{GL}_2(\mathbb{R})^n$ correspond to $\mathbb{A}_{F,\infty}$ and $\mathbb{A}_{G,\infty}$, respectively.

Put $\mathbb{A}_G^+ = \left\{a \in \mathbb{A}_G : a_\infty \in \mathrm{GL}_2^+(\mathbb{R})^n\right\}$ and $G(\mathbb{Q})^+ = G(\mathbb{Q}) \cap \mathbb{A}_G^+$. Note that $G(\mathbb{Q})$ embeds into $\mathbb{A}_G$ via the mapping $\alpha \mapsto (\alpha, \ldots, \alpha \mid \alpha, \ldots, \alpha, \ldots)$, and therefore one can think of $G(\mathbb{Q})^+$ as elements of $G(\mathbb{Q})$ with positive determinant. We call $\Gamma \subset G(\mathbb{Q})^+$ a *congruence subgroup* if it contains the subset

$$\Gamma(\mathfrak{n}) = \big\{\alpha \in \mathrm{SL}_2(\mathcal{O}_F) : \alpha - I_2 \in \mathfrak{n} \cdot M_2(\mathcal{O}_F)\big\},$$

for some integral ideal $\mathfrak{n}$ of F.

Definition 2.1. Let $k \in \mathbb{Z}^n$ and let Γ be a congruence subgroup. A *classical Hilbert modular form* of weight k with respect to Γ is a holomorphic function $f : \mathfrak{H}^n \to \mathbb{C}$ satisfying the following:

1. $f\big|_k \alpha = f$, for all $\alpha \in \Gamma$; and
2. f is holomorphic at all the cusps of Γ.

Remark 1. When $F = \mathbb{Q}$, the above definition describes a *classical (or elliptic) modular form.*

The existence of a translation matrix $\left(\begin{smallmatrix}1 & b\\ 0 & 1\end{smallmatrix}\right)$ in Γ tells us that any such f is a periodic function to which one can attach a Fourier expansion

$$f(z) = \sum_{\xi} a(f,\xi) \cdot e_F(\xi z)\,. \tag{2.3}$$

Here, the index ξ runs over 0 and all totally positive elements of a lattice in F. We denote by $\mathcal{M}_k(\Gamma)$ the $\mathbb{C}$-vector space of classical Hilbert modular forms. If the constant term of $f\big|_k \alpha$ vanishes for every $\alpha \in G(\mathbb{Q})^+$, we say that f is a *cusp form* and we denote the subspace of cusp forms by $\mathcal{S}_k(\Gamma)$. Finally, put

$$\mathcal{M}_k = \bigcup_{\mathfrak{n}} \mathcal{M}_k(\Gamma(\mathfrak{n})) \quad \text{and} \quad \mathcal{S}_k = \bigcup_{\mathfrak{n}} \mathcal{S}_k(\Gamma(\mathfrak{n})),$$

with both indices running over integral ideals $\mathfrak{n}$ of F.

Next we demonstrate how to lift classical Hilbert modular forms to functions on the adèlic ring $\mathbb{A}_G$. For a fractional ideal $\mathfrak{a}$ and a prime ideal $\mathfrak{q}$ of F, we regard $\mathfrak{a}_\mathfrak{q}$ as a submodule of the completion $F_\mathfrak{q}$. Given an integral ideal $\mathfrak{c}$ of F, we define the following subsets of $\mathbb{A}_G$:

$$\begin{aligned} Y := Y(\mathfrak{c}) &= \mathbb{A}_G \cap \left(\mathrm{GL}_2^+(\mathbb{R})^n \times \prod_{\mathfrak{q}} Y_\mathfrak{q}\right) \\ \text{and } W := W(\mathfrak{c}) &= \mathrm{GL}_2^+(\mathbb{R})^n \times \prod_{\mathfrak{q}} W_\mathfrak{q} \end{aligned} \tag{2.4}$$

where for each prime ideal $\mathfrak{q}$,

$$Y_{\mathfrak{q}} = \left\{ \begin{pmatrix} a_{\mathfrak{q}} & b_{\mathfrak{q}} \\ c_{\mathfrak{q}} & d_{\mathfrak{q}} \end{pmatrix} \in \mathrm{GL}_2(F_{\mathfrak{q}}) : a_{\mathfrak{q}}\mathcal{O}_{F_{\mathfrak{q}}} + \mathfrak{c}_{\mathfrak{q}} = \mathcal{O}_{F_{\mathfrak{q}}}, b_{\mathfrak{q}} \in \mathfrak{d}_{\mathfrak{q}}^{-1}, c_{\mathfrak{q}} \in \mathfrak{c}_{\mathfrak{q}}\mathfrak{d}_{\mathfrak{q}}, d_{\mathfrak{q}} \in \mathcal{O}_{F_{\mathfrak{q}}} \right\}$$

$$\text{and } W_{\mathfrak{q}} = \{\alpha_{\mathfrak{q}} \in Y_{\mathfrak{q}} : \det \alpha_{\mathfrak{q}} \in \mathcal{O}_{F_{\mathfrak{q}}}^{\times}\}. \tag{2.5}$$

Given a multiplicative character $\eta_0 : (\mathcal{O}_F/\mathfrak{c})^{\times} \to \mathbb{C}^{\times}$ satisfying $\eta_0(x) = \mathrm{sgn}(x)^k$ for every $x \in \mathcal{O}_F^{\times}$, one can define another character $\eta_Y : Y \to \mathbb{C}^{\times}$ as follows:

$$\eta_Y\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right) = \eta_0\left(\pi_{\mathfrak{c}}\left(\prod_{\mathfrak{q}|\mathfrak{c}} a_{\mathfrak{q}}\right)\right). \tag{2.6}$$

where $\pi_{\mathfrak{c}}$ denotes the canonical mapping $\prod_{\mathfrak{q}|\mathfrak{c}} \mathcal{O}_{\mathfrak{q}}^{\times} \to (\mathcal{O}/\mathfrak{c})^{\times}$.

Before we proceed, we state the following class field theoretic result. Let J_F be the group of fractional ideals of F and P_F its subset of principal ideals.

Proposition 2.1. *Let* $\mathrm{Cl}(F) = J_F/P_F$ *be the ideal class group of* F.

(a) There is a natural isomorphism $\mathrm{Cl}(F) \cong F^{\times}\backslash\mathbb{A}_F^{\times}/\mathbb{A}_{F,\infty} \cdot \prod_v \mathcal{O}_v^{\times}$.

(b) (Strong approximation) Let K_0 *be a compact subgroup of* $\mathbb{A}_{G,\mathrm{fin}}$ *such that its image under the determinant map is* $\prod_v \mathcal{O}_v^{\times}$*. There is an isomorphism* $\mathrm{Cl}(F) \cong \mathrm{GL}_2(F)\backslash \mathrm{GL}_2(\mathbb{A}_F)/ \mathrm{GL}_2(\mathbb{A}_{F,\infty})K_0$.

Proof. To prove (a), note that there is a homomorphism $\mathbb{A}_F^{\times} \to J_F$ given by

$$(t_{\mathfrak{v}})_{\mathfrak{v}} \mapsto \prod_{\mathfrak{q}} \mathfrak{q}^{\mathrm{ord}_{\mathfrak{q}}(t_{\mathfrak{q}})} \tag{2.7}$$

which runs through all prime ideals $\mathfrak{q}$ of F. This map is surjective due to the unique factorisation property of ideals, and its kernel is the subgroup of elements $a \in \mathbb{A}_F^{\times}$ for which $a_{\mathrm{fin}} \in \prod_{\mathfrak{v}} \mathcal{O}_{\mathfrak{v}}^{\times}$. The isomorphism follows from the fact that $F^{\times}$ maps to principal ideals.

Strong approximation was proven in [Hum80] for the case $K_0 = \prod_{\mathfrak{v}} \mathrm{GL}_2(\mathcal{O}_{\mathfrak{v}})$. To obtain the general case, we bring the discussion down to (a) by composing with the determinant map

$$\mathrm{GL}_2(F)\backslash \mathrm{GL}_2(\mathbb{A}_F)/ \mathrm{GL}_2(\mathbb{A}_{F,\infty})K_0 \longrightarrow F^{\times}\backslash\mathbb{A}_F^{\times}/ \det(\mathrm{GL}_2(\mathbb{A}_{F,\infty})K_0).$$

Since the image of K_0 under the determinant map is $\prod_{\mathfrak{v}} \mathcal{O}_{\mathfrak{v}}^{\times}$, the situation reduces back to the case considered in *op. cit.* ■

Convention 1. Given a fractional ideal $\mathfrak{a}$, we denote by a the corresponding idèle under the map (2.7). On the other hand, if we are given the idèle a first, we write $\mathfrak{a}$ or $\tilde{a}$ to denote its corresponding fractional ideal.

Let h be the narrow class number of F. Take $\{\tilde{t}_1, \ldots, \tilde{t}_h\}$ to be the complete set of ideal representatives. Then we have the associated idèle representatives $\{t_1, \ldots, t_h\}$ which can be chosen so that $(t_\lambda)_\infty = 1$ for $1 \leq \lambda \leq h$. Next we set

$$x_\lambda = \begin{pmatrix} 1 & 0 \\ 0 & t_\lambda \end{pmatrix} \quad \text{and} \quad x_\lambda^{-\iota} = \begin{pmatrix} t_\lambda^{-1} & 0 \\ 0 & 1 \end{pmatrix}, \tag{2.8}$$

where ι denotes the involution $M^\iota = \det(M) \cdot M^{-1}$ for a matrix $M \in M_2(\mathbb{A}_F)$. We define the *congruence modular group* as

$$\Gamma_\lambda = \Gamma(t_\lambda \mathfrak{d}, \mathfrak{c}) = x_\lambda W x_\lambda^{-1} \cap G(\mathbb{Q})$$

which can be described more explicitly as

$$\left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in G(\mathbb{Q})^+ : a, d \in \mathcal{O}_F, b \in (t_\lambda \mathfrak{d})^{-1}, c \in t_\lambda \mathfrak{d}\mathfrak{c}, ad - bc \in \mathcal{O}_F^\times \right\}. \tag{2.9}$$

We then define the $\mathbb{C}$-vector spaces

$$\mathcal{M}_k(\Gamma_\lambda, \eta_0) = \left\{ f \in \mathcal{M}_k : f \big|_k \alpha = \eta_Y(x_\lambda^{-1} \alpha x_\lambda) \cdot f \text{ for all } \alpha \in \Gamma_\lambda \right\}, \text{ and}$$

$$\mathcal{M}_k(\mathfrak{c}, \eta_0) = \prod_{\lambda=1}^{h} \mathcal{M}_k(\Gamma_\lambda, \eta_0),$$

and the corresponding spaces of cusp forms

$$\mathcal{S}_k(\Gamma_\lambda, \eta_0) = \mathcal{M}_k(\Gamma_\lambda, \eta_0) \cap \mathcal{S}_k \quad \text{and} \quad \mathcal{S}_k(\mathfrak{c}, \eta_0) = \prod_{\lambda=1}^{h} \mathcal{S}_k(\Gamma_\lambda, \eta_0).$$

To an element $(f_1, \ldots, f_h) \in \mathcal{M}_k(\mathfrak{c}, \eta_0)$ with $f_\lambda \in \mathcal{M}_k(\Gamma_\lambda, \eta_0)$ for each λ, we can associate a complex-valued function $\mathbf{f}$ given by

$$\mathbf{f}\big(\alpha x_\lambda^{-\iota} w\big) = \eta_Y(w^\iota) \cdot \big(f_\lambda \big|_k w_\infty\big)(\boldsymbol{i}), \quad \alpha \in G(\mathbb{Q}), w \in W, \tag{2.10}$$

where $\boldsymbol{i} = \big(\sqrt{-1}, \ldots, \sqrt{-1}\big)$. Proposition 2.1(b) with $K_0 = W$ (see (2.4)) implies that we have a decomposition

$$\mathbb{A}_G = \bigsqcup_{\lambda=1}^{h} G(\mathbb{Q}) x_\lambda W = \bigsqcup_{\lambda=1}^{h} G(\mathbb{Q}) x_\lambda^{-\iota} W \tag{2.11}$$

which tells us that $\mathbf{f}$ is a function over $\mathbb{A}_G$.

Remark 2. Throughout this chapter, one will encounter multiple guises of the multiplicative character $\eta_0 : (\mathcal{O}/\mathfrak{c})^\times \to \mathbb{C}^\times$. We summarise them here.

1. A character over $Y \subset \mathbb{A}_G$ given by (2.6).
2. An idèle Hecke character $\eta : \mathbb{A}_F^\times \to \mathbb{C}^\times$ defined analogously with (1): $\eta(u) = \prod_{\mathfrak{v}|\mathfrak{c}} \eta_0(u_\mathfrak{v} \bmod \mathfrak{c})$ for $u \in \prod_\mathfrak{v} \mathcal{O}_\mathfrak{v}^\times$ and $\eta(x) = \operatorname{sgn}(x)^k$ for $x \in \mathbb{A}_{F,\infty}^\times$.
3. An ideal Hecke character $\tilde{\eta} : (J_F/\mathfrak{c})^\times \to \mathbb{C}^\times$ given by $\tilde{\eta} = \tilde{\eta}_{\text{fin}} \cdot \tilde{\eta}_\infty$, where the finite-type is determined by $\tilde{\eta}_{\text{fin}}(a) = \prod_v \eta(\pi_v)^{e_v}$ given an ideal $(a) = \prod_v a_v^{e_v} \in J_F$ and π_v denotes a uniformiser in $\mathcal{O}_v^\times$; the infinite-type is defined by $\tilde{\eta}_\infty(x) = \eta(a \otimes 1) = \operatorname{sgn}(x)^k$ for $x \in \mathbb{R}^n \cong F \otimes_\mathbb{Q} \mathbb{R}$.

We note that the formulation of $\tilde{\eta}$ here depends on η but in general it may be induced directly from the multiplicative character η_0. For more details, we refer the reader to [Neu99, Chapter VII, §6].

We are finally ready to define adèlic Hilbert modular forms.

Definition 2.2. Let $k \in \mathbb{Z}^n$, $\mathfrak{c}$ be an integral ideal of F, and let $\eta : \mathbb{A}_F^\times \to \mathbb{C}^\times$ be a Hecke character. An *adèlic Hilbert modular form* or simply a *Hilbert modular form* of weight k, level $\mathfrak{c}$ and nebentypus η is a function $\mathbf{f} : \mathbb{A}_G \to \mathbb{C}$ satisfying:

(i) $\mathbf{f}(s\alpha x w) = \eta(s)\eta_Y(w^\iota) \cdot \mathbf{f}(x)$, for $s \in \mathbb{A}_F^\times$, $\alpha \in G(\mathbb{Q}), x \in \mathbb{A}_G, w \in W$ with $w_\infty = 1$ and η_Y as in (2.6); and

(ii) For each $1 \leq \lambda \leq h$, there exists $f_\lambda \in \mathcal{M}_k$ such that $\mathbf{f}\big(x_\lambda^{-\iota} y\big) = \big(f_\lambda \big|_k y\big)(\boldsymbol{i})$ for all $y \in \mathrm{GL}_2^+(\mathbb{R})^n$. In other words, $\mathbf{f}$ corresponds to an h-tuple $(f_1, \ldots, f_h)$ of classical Hilbert modular forms, where $f_\lambda \in \mathcal{M}_k(\Gamma_\lambda, \eta_0)$.

If in addition, each f_λ in (ii) is a cusp form over $\mathcal{M}_k(\Gamma_\lambda, \eta_0)$, then we say that $\mathbf{f}$ is a *cusp form* over $\mathcal{M}_k(\mathfrak{c}, \eta)$.

The decomposition in (2.11) thus induces an isomorphism of $\mathbb{C}$-vector spaces:

$$\mathcal{M}_k(\mathfrak{c}, \eta) \cong \bigoplus_{\lambda=1}^{h} \mathcal{M}_k(\Gamma_\lambda, \eta).$$

Remark 3. Throughout this thesis we assume that all our Hilbert modular forms have parallel weight $k := (k, \dots, k)$. Without this assumption, some of the results that we will come across in the future may no longer be true.

Let $\mathbf{f} \in \mathcal{M}_k(\mathfrak{c}, \eta)$ with $f_\lambda \in \mathcal{M}_k(\Gamma_\lambda, \eta_0)$ for each λ. The collection of ideals $\{\tilde{t}_1, \dots, \tilde{t}_h\}$ generates a lattice of F. We recall from (2.3) that each f_λ has a Fourier expansion

$$f_\lambda(z) = a(f_\lambda, 0) + \sum_{\substack{\xi \in \tilde{t}_\lambda \\ \xi \gg 0}} a(f_\lambda, \xi) \cdot e_F(\xi z).$$

Using the fact that every integral ideal $\mathfrak{m}$ of F can be written as $\xi \tilde{t}_\lambda^{-1}$ for a unique λ and a totally positive element $\xi \in \tilde{t}_\lambda$ we define the constants

$$\mathrm{c}(\mathbf{f}, \mathfrak{m}) = \begin{cases} a(f_\lambda, \xi) \cdot \xi^{-k/2}, & \text{if } \mathfrak{m} \text{ is integral and } \mathfrak{m} = \xi \tilde{t}_\lambda^{-1} \\ 0, & \text{if } \mathfrak{m} \text{ is not integral.} \end{cases} \tag{2.12}$$

and

$$\mathrm{c}_0(\mathbf{f}, \mathfrak{m}) = \begin{cases} a(f_\lambda, 0) \cdot \mathcal{N}(\tilde{t}_\lambda)^{-k/2}, & \text{if } \mathfrak{m} \text{ is integral and } \mathfrak{m} = \xi \widetilde{t_\lambda}^{-1} \\ 0, & \text{if } \mathfrak{m} \text{ is not integral.} \end{cases} \tag{2.13}$$

Finally, we define the Fourier expansion of $\mathbf{f}$ in the same manner as [Shi78, (2.18)]: for $x \in \mathbb{A}_F$ and $y \in \mathbb{A}_F^\times$,

$$\mathbf{f}\left(\begin{pmatrix} y & x \\ 0 & 1 \end{pmatrix}\right) = \sum_{\xi \in F^+} \mathrm{c}(\mathbf{f}, \xi y \mathcal{O})(\xi y_\infty)^{k/2} e_F(\xi \boldsymbol{i} y_\infty) \chi_F(\xi x) + |y|_{\mathbb{A}_F}^{k/2} \mathrm{c}_0(\mathbf{f}, y\mathcal{O}), \tag{2.14}$$

where $\chi_F : \mathbb{A}_F / F \to \mathbb{C}^\times$ is Tate's standard additive character in [Tat67, §2.2].

2.2 Operators acting on Hilbert modular forms

Let $\mathfrak{c}$ be a fixed integral ideal of F and let $\eta_0 : (\mathcal{O}_F/\mathfrak{c})^\times \to \mathbb{C}^\times$ be a multiplicative character. We introduce some operators that act on the $\mathbb{C}$-vector space $\mathcal{M}_k(\mathfrak{c}, \eta_0)$.

2.2.1 Double coset operator

Let Γ_λ and Γ_μ be congruence modular groups and take $\alpha \in G(\mathbb{Q}) \cup x_\lambda Y x_\mu^{-1}$. The action of Γ_λ on $\Gamma_\lambda \alpha \Gamma_\mu$ by left-multiplication gives us a finite disjoint union

$$\Gamma_\lambda \alpha \Gamma_\mu = \bigsqcup_j \Gamma_\lambda \alpha_j, \tag{2.15}$$

for some representatives α_j of $\Gamma_\lambda \alpha \Gamma_\mu$. The *weight-$k$ double coset operator* is defined as the operator taking $f_\lambda \in \mathcal{M}_k(\Gamma_\lambda, \eta_0)$ to

$$f_\lambda \,\big|\, \Gamma_\lambda \alpha \Gamma_\mu = \sum_j \eta_Y \left(x_\lambda^{-1} \alpha_j x_\mu\right)^{-1} \cdot \left(f_\lambda \,\big|_k\, \alpha_j\right) \in \mathcal{M}_k(\Gamma_\mu, \eta_0). \tag{2.16}$$

To extend this operator to Hilbert modular forms one observes that for an element $y \in Y$, there is a disjoint union similar to (2.15) given by:

$$WyW = \bigsqcup_j W y_j$$

for coset representatives y_j that are chosen so that $(y_j)_\infty = 1$. One then defines the weight-k double coset operator acting on $\mathbf{f} = (f_1, \ldots, f_h) \in \mathcal{M}_k(\mathfrak{c}, \eta)$ by

$$\left(\mathbf{f} \,\big|\, WyW\right)(x) = \sum_j \eta_Y(y_j)^{-1} \cdot \mathbf{f}(x y_j^{\iota}) \quad \text{for } x \in \mathbb{A}_G. \tag{2.17}$$

Additionally, we note that for each $1 \leq \lambda \leq h$, there is a unique index μ and an element $\alpha_\lambda \in G(\mathbb{Q}) \cap x_\lambda Y x_\mu^{-1}$ such that $WyW = W x_\lambda^{-1} \alpha_\lambda x_\mu W$. Property (ii) in Definition 2.2 therefore tells us that

$$\mathbf{f} \,\big|\, WyW = (g_1, \ldots, g_h), \quad \text{with } g_\mu = f_\lambda \,\big|\, \Gamma_\lambda \alpha_\lambda \Gamma_\mu.$$

2.2.2 Diamond operator

Let $\mathfrak{n}$ be an integral ideal of F. The *diamond operator* $S_\mathfrak{n}(\mathfrak{c})$ is defined as

$$S_\mathfrak{n}(\mathfrak{c}) = \begin{cases} WaW, & \text{if } \mathfrak{n} \text{ is prime to } \mathfrak{c} \text{ and } \mathfrak{n} = a\mathcal{O}_F \text{ for some } a \in \mathbb{A}_F^\times \\ 0, & \text{if } \mathfrak{n} \text{ is not prime to } \mathfrak{c}. \end{cases} \tag{2.18}$$

The level $\mathfrak{c}$ on the left-hand side comes with the definition of W in (2.4).

We describe the effect of this operator on a nontrivial element $\mathbf{f} \in \mathcal{M}_k(\mathfrak{c}, \eta)$. A prime ideal $\mathfrak{q}$ of F corresponds under the map (2.7) to the idèle q with 1 on each component except for the $\mathfrak{q}$-th component which contains the uniformiser $\pi_{\mathfrak{q}}$ of $\mathcal{O}_{F_{\mathfrak{q}}}$. Therefore, WqW has the unique coset representative

$$\left(I_2, \dots, I_2 \;\middle|\; I_2, \dots, I_2, \begin{pmatrix} \pi_{\mathfrak{q}} & 0 \\ 0 & \pi_{\mathfrak{q}} \end{pmatrix}, I_2, \dots \right).$$

Going back to the definition of η_Y in (2.6), we see that (2.17) turns out to be

$$\big(\mathbf{f} \,|\, S_{\mathfrak{q}}(\mathfrak{c})\big)(x) = \begin{cases} \tilde{\eta}(\mathfrak{q}) \cdot \mathbf{f}(x), & \text{if } \mathfrak{q} \nmid \mathfrak{c}, \\ 0, & \text{otherwise} \end{cases}$$

and this readily extends to integral ideals $\mathfrak{n}$ by multiplicativity.

When there is no risk of confusion with the level, we simply write $\langle \mathfrak{q} \rangle$ to denote $S_{\mathfrak{q}}(\mathfrak{c})$, hence the name diamond operator.

2.2.3 Hecke operator

We define the $\mathfrak{n}$-th *Hecke operator* $T_{\mathfrak{n}}(\mathfrak{c})$ as the sum

$$T'_{\mathfrak{n}}(\mathfrak{c}) = \sum_{\substack{y \in Y \\ \det(y)\mathcal{O}_F = \mathfrak{n}}} WyW. \tag{2.19}$$

When the context is clear, we drop the symbol $(\mathfrak{c})$ and simply write $T_{\mathfrak{n}}$.

For a prime ideal $\mathfrak{q} \subset F$, one has $T'_{\mathfrak{q}} = W \left(\begin{smallmatrix} 1 & 0 \\ 0 & q \end{smallmatrix}\right) W$. This follows from the fact that one can find a pair $\tilde{l}$ and $\tilde{m}$ of ideals with $\tilde{m} \subset \tilde{l}$, uniquely determined by y, satisfying $\tilde{l} + \mathfrak{c} = \mathcal{O}$ and $lm = \det(y)$ such that

$$WyW = W \begin{pmatrix} l & 0 \\ 0 & m \end{pmatrix} W =: T'(\tilde{l}, \tilde{m})$$

(c.f. [Shi62, (2)]). Therefore Equation 2.19 can be rewritten as

$$T'_{\mathfrak{n}} = \sum_{\tilde{l}\tilde{m} = \mathfrak{n}} T'(\tilde{l}, \tilde{m}). \tag{2.20}$$

In order to describe the action of $T'_{\mathfrak{q}}$ on a Hilbert modular form $\mathbf{f} \in \mathcal{M}_k(\mathfrak{c}, \eta)$, we need to know what the coset representatives look like (c.f [Hid06, p. 106]).

Proposition 2.2. *Let $\mathfrak{q}$ be a prime ideal of F and let $q \in \mathbb{A}_F^{\times}$ such that $\tilde{q} = \mathfrak{q}$ under the map (2.7). A complete set of representatives of*

$$W \setminus W \begin{pmatrix} 1 & 0 \\ 0 & q^e \end{pmatrix} W$$

is given by

$$\begin{cases} \left\{ \begin{pmatrix} q^{1-f} & u \\ 0 & q^{f} \end{pmatrix} : 0 \leq f \leq 1, \tilde{u} \bmod \tilde{q}^{f}, \tilde{u} + \tilde{q} + \tilde{q}^{1-f} = \mathcal{O} \right\}, & \textit{if } \mathfrak{q} \nmid \mathfrak{c}, \\ \left\{ \begin{pmatrix} 1 & u \\ 0 & q \end{pmatrix} : \tilde{u} \bmod \tilde{q} \right\}, & \textit{otherwise.} \end{cases} \tag{2.21}$$

One can then employ the above result to explicitly compute that

$$\mathrm{c}\big(\mathbf{f} \,\big|\, T'_{\mathfrak{q}}, \mathfrak{m}\big) = \begin{cases} \mathcal{N}(\mathfrak{q}) \cdot \mathrm{c}(\mathbf{f}, \mathfrak{q}\mathfrak{m}), & \mathfrak{q} \mid \mathfrak{c} \\ \mathcal{N}(\mathfrak{q}) \cdot \mathrm{c}(\mathbf{f}, \mathfrak{q}\mathfrak{m}) + \tilde{\eta}(\mathfrak{q}) \cdot \mathrm{c}(\mathbf{f}, \mathfrak{q}^{-1}\mathfrak{m}), & \mathfrak{q} \nmid \mathfrak{c}. \end{cases} \tag{2.22}$$

For each integral ideal $\mathfrak{m}$ of F, we rescale the Hecke operator and the coefficients (2.12) and (2.13) as follows:

$$\begin{aligned} T_{\mathfrak{n}} &= \mathcal{N}(\mathfrak{n})^{(k-2)/2} \cdot T'_{\mathfrak{n}} \quad \text{and} \\ \mathrm{C}(\mathbf{f}, \mathfrak{m}) &= \mathcal{N}(\mathfrak{m})^{k/2} \cdot \mathrm{c}(\mathbf{f}, \mathfrak{m}). \end{aligned} \tag{2.23}$$

As a result, we are able to rewrite (2.22) as:

$$\mathrm{C}\big(\mathbf{f} \,\big|\, T_{\mathfrak{q}}, \mathfrak{m}\big) = \begin{cases} \mathrm{C}(\mathbf{f}, \mathfrak{q}\mathfrak{m}), & \mathfrak{q} \mid \mathfrak{c} \\ \mathrm{C}(\mathbf{f}, \mathfrak{q}\mathfrak{m}) + \tilde{\eta}(\mathfrak{q})\mathcal{N}(\mathfrak{q})^{k-1} \cdot \mathrm{C}(\mathbf{f}, \mathfrak{q}^{-1}\mathfrak{m}), & \mathfrak{q} \nmid \mathfrak{c}. \end{cases} \tag{2.24}$$

The above formula in fact extends to a more general version for integral ideals $\mathfrak{n}$:

$$\mathrm{C}\big(\mathbf{f} \,\big|\, T_{\mathfrak{n}}, \mathfrak{m}\big) = \sum_{\mathfrak{m}+\mathfrak{n} \subset \mathfrak{a}} \tilde{\eta}(\mathfrak{a})\mathcal{N}(\mathfrak{a})^{k-1}\mathrm{C}(\mathfrak{a}^{-2}\mathfrak{m}\mathfrak{n}, \mathbf{f})\,. \tag{2.25}$$

To show this, one first establishes the identity on $T_{\mathfrak{q}^e}$ with $e \geq 2$ using the relation $T_{\mathfrak{q}^{e-1}}T_{\mathfrak{q}} = T_{\mathfrak{q}^e} + \mathcal{N}(\mathfrak{q})^{k-1}S_{\mathfrak{q}}T_{\mathfrak{q}^{e-2}}$ found in [Shi61, Proposition 1.10] and then performing induction on e. One obtains the desired result by exploiting the following multiplicativity result found in [Shi61, Proposition 1.9].

Lemma 2.3. *Given a pair of coprime integral ideals* $\mathfrak{n}_1$ *and* $\mathfrak{n}_2$ *of* F*, we have*

$$T_{\mathfrak{n}}T_{\mathfrak{n}'} = T_{\mathfrak{n}\cdot\mathfrak{n}'}.$$

Convention 2. To easily distinguish between the two cases, we write $U_{\mathfrak{q}}$ to denote the Hecke operator $T_{\mathfrak{q}}$ whenever $\mathfrak{q}$ is prime to $\mathfrak{c}$.

Definition 2.3. The *Hecke algebra* $R(W,Y)$ is defined as the commutative algebra generated by $S_{\mathfrak{q}}$ and $T_{\mathfrak{q}}$, for all prime ideals $\mathfrak{q}$ of $\mathcal{O}_F$.

We end the discussion on Hecke operators with a well known analytic result.

Proposition 2.4. *The Dirichlet series* $\sum_{\mathfrak{m}} T_{\mathfrak{m}} \cdot \mathcal{N}(\mathfrak{m})^{-s}$ *can be written as the formal Euler product*

$$\sum_{\mathfrak{m}} T_{\mathfrak{m}} \mathcal{N}(\mathfrak{m})^{-s} = \prod_{\mathfrak{q}} \left(1 - T_{\mathfrak{q}} \mathcal{N}(\mathfrak{q})^{-s} + S_{\mathfrak{q}} \mathcal{N}(\mathfrak{q})^{k-1-2s}\right)^{-1}$$

where $\mathfrak{q}$ *runs over the prime ideals of* F*.*

Proof. The proof follows from unique factorisation of ideals in a Dedekind domain and the multiplicativity of $T_{\mathfrak{n}}$. ■

2.2.4 J-operator

For an element $\mathbf{f} \in \mathcal{M}_k(\mathfrak{c}, \eta)$, the *$J$-operator*, oftentimes referred to as the Atkin-Lehner operator, is defined by the rule

$$\left(\mathbf{f} \mid J_{\mathfrak{c}}\right)(x) = \mathbf{f}\left(x^{-\iota} b\right) \in \mathcal{M}_k(\mathfrak{c}, \eta^{-1}) \tag{2.26}$$

where $b = b_\infty \cdot b_{\mathrm{fin}} \in \mathbb{A}_G$ with $b_\infty = 1$ and

$$b_{\mathrm{fin}} = \begin{pmatrix} 0 & 1 \\ c_0 & 0 \end{pmatrix} \quad \text{for } c_0 \in \mathbb{A}_{F,\mathrm{fin}} \text{ satisfying } \tilde{c_0} = \mathfrak{c}\mathfrak{d}^2.$$

When the weight k is even, the J-operator turns out to be an involution due to the equality

$$\mathbf{f} \mid J_{\mathfrak{c}}^2 = (-1)^{[F:\mathbb{Q}]\cdot k} \cdot \mathbf{f}.$$

The following result shows how J-operators interact with Hecke operators. (c.f. Proof of [Shi78, Proposition 2.10]).

Proposition 2.5. *Let* $\mathbf{f} \in \mathcal{M}_k(\mathfrak{c}, \eta)$ *be a Hilbert modular form. Then one has the equality*

$$\left(\mathbf{f} \mid J_{\mathfrak{c}}\right) \mid T_{\mathfrak{n}} = \tilde{\eta}(\mathfrak{n})^{-1} \cdot \left(\mathbf{f} \mid T_{\mathfrak{n}}\right) \mid J_{\mathfrak{c}}.$$

The proof of this proposition is straightforward and shall therefore be omitted. We do note however that one only needs to prove the relation for $T_{\mathfrak{q}}$ where $\mathfrak{q}$ is a prime ideal and the general version follows from multiplicativity.

2.2.5 Trace operator

The *trace operator* is a variant of WyW, where instead we consider $W(\mathfrak{c})yW(\mathfrak{c}')$ with $\mathfrak{c} \subset \mathfrak{c}'$, and y is the identity matrix. This operator projects modular forms in $\mathcal{M}_k(\mathfrak{c}, \psi)$ onto the vector subspace $\mathcal{M}_k(\mathfrak{c}', \psi)$. It comes as no suprise that the trace operator has a very similar formulation to Hecke operators:

$$\mathrm{Tr}^{\mathfrak{c}}_{\mathfrak{c}'}(\mathbf{f})(x) := \mathcal{N}(\mathfrak{c} \cdot \mathfrak{c}'^{-1})^{1-k/2} \cdot \sum_{y \in W(\mathfrak{c}) \backslash W(\mathfrak{c}')} \psi(y)^{-1} \cdot \mathbf{f}(xy^{\iota}). \tag{2.27}$$

The following result determines the coset representatives of $W(\mathfrak{c}) \setminus W(\mathfrak{c}')$ allowing us to see explicitly the action of the trace operator on $\mathbf{f}$. We shall omit its proof as it is standard and rather tedious.

Proposition 2.6. *Let $\mathfrak{q}$ be a prime ideal and let $\mathfrak{a}$ be an integral ideal of F. A complete set of representatives of $W(\mathfrak{a}\mathfrak{q}) \setminus W(\mathfrak{a})$ is given by*

$$\left\{ \begin{pmatrix} 1 & 0 \\ ru & 1 \end{pmatrix} \in W(\mathfrak{a}) \;\middle|\; \tilde{u} \bmod \mathfrak{q},\ \tilde{r} = \mathfrak{a}\mathfrak{d}^2 \right\}$$

when $\mathfrak{q} \mid \mathfrak{a}$. The same holds when $\mathfrak{q} \nmid \mathfrak{a}$ with the additional coset representative

$$\begin{pmatrix} * & * \\ a & q' \end{pmatrix}, \quad \tilde{a} = \mathfrak{a}\mathfrak{d}^2 \text{ and } \widetilde{q'} \subseteq \mathfrak{q}.$$

N.B. In both cases, we ensure that the top left entries are chosen so that each determinant in $W_{\mathfrak{q}}$ for $\mathfrak{q} \mid \mathfrak{a}$ is equal to 1.

The following proposition shows us the explicit relation between trace operators and Hecke operators. This is pertinent to the results of Chapter 4.

Proposition 2.7. *Let $\mathfrak{q}$ and $\mathfrak{a}$ be as in Proposition 2.6. Let $\mathbf{f} \in \mathcal{M}_k(\mathfrak{a}\mathfrak{q}, \eta)$ be a Hilbert modular form of weight $k \geq 2$. Then*

$$\mathrm{Tr}^{\mathfrak{a}\mathfrak{q}}_{\mathfrak{a}}\left(\mathbf{f} \,\middle|\, J_{\mathfrak{a}\mathfrak{q}}\right) = \begin{cases} \mathcal{N}(\mathfrak{q})^{1-k/2} \cdot \mathbf{f} \,\big|\, U_{\mathfrak{q}} \circ J_{\mathfrak{a}}, & \text{if } \mathfrak{q} \mid \mathfrak{a} \\ \mathcal{N}(\mathfrak{q})^{1-k/2} \cdot \mathbf{f} \,\big|\, \left(U_{\mathfrak{q}} + \mathcal{N}(\mathfrak{q})^{k-1} \cdot \langle \mathfrak{q} \rangle \circ \mathfrak{q}\right) \circ J_{\mathfrak{a}}, & \text{if } \mathfrak{q} \nmid \mathfrak{a}. \end{cases}$$

Before we prove the proposition we introduce a scaled diamond operator $U'_{\mathfrak{q}}$ via the formula

$$\left(\mathbf{f} \,\middle|\, U'_{\mathfrak{q}}\right)(x) := \mathcal{N}(\mathfrak{q})^{-k/2} \cdot \mathbf{f}\left(x \begin{pmatrix} q & 0 \\ 0 & 1 \end{pmatrix}^{\iota} \begin{pmatrix} \alpha & \beta \\ a & q \end{pmatrix}^{\iota} \right)$$

provided $\mathfrak{q} + \mathfrak{a} = \mathcal{O}_F$. The intuition behind this is the familiar identity $T_q = U_q + q^{k-1} \cdot U'_q$ for elliptic modular forms.

Proof. In the first case where $\mathfrak{q} \mid \mathfrak{a}$, choose $w_1, w_2 \in \mathring{A}_F$ so that $\widetilde{w}_1 = \mathfrak{a}\mathfrak{q}\mathfrak{d}^2$ and $\widetilde{w}_2 = \mathfrak{a}\mathfrak{d}^2$. One then computes:

$$\begin{aligned}
\frac{\mathbf{f} \,\big|\, (J_{\mathfrak{a}\mathfrak{q}} \circ U_{\mathfrak{q}} \circ J_{\mathfrak{a}})(x)}{\mathcal{N}(\mathfrak{q})^{k/2-1}} &= \mathbf{f}\left(x^{-\iota} \begin{pmatrix} 0 & 1 \\ w_1 & 0 \end{pmatrix} \right) \Bigg| (U_{\mathfrak{q}} \circ J_{\mathfrak{a}}) \\
&= \sum_{\widetilde{u} \bmod \mathfrak{q}} \eta_Y \left(\begin{pmatrix} 1 & u \\ 0 & q \end{pmatrix} \right)^{-1} \cdot \mathbf{f}\left(x^{-\iota} \begin{pmatrix} 0 & 1 \\ w_1 & 0 \end{pmatrix} \begin{pmatrix} 1 & u \\ 0 & q \end{pmatrix}^{\iota} \right) \Bigg| J_{\mathfrak{a}} \\
&= \sum_{\widetilde{u} \bmod \mathfrak{q}} \mathbf{f}\left(\left(x^{-\iota} \begin{pmatrix} 0 & 1 \\ w_1 & 0 \end{pmatrix} \begin{pmatrix} 1 & u \\ 0 & q \end{pmatrix}^{\iota} \right)^{-\iota} \begin{pmatrix} 0 & 1 \\ w_2 & 0 \end{pmatrix} \right) \\
&= \sum_{\widetilde{u} \bmod \mathfrak{q}} \mathbf{f}\left(x \begin{pmatrix} 0 & 1 \\ w_1 & 0 \end{pmatrix}^{-\iota} \begin{pmatrix} 1 & u \\ 0 & q \end{pmatrix}^{-1} \begin{pmatrix} 0 & 1 \\ w_2 & 0 \end{pmatrix} \right) \\
&= \sum_{\widetilde{u} \bmod \mathfrak{q}} \mathbf{f}\left(\frac{-1}{qw_1} \cdot x \begin{pmatrix} 0 & 1 \\ w_1 & 0 \end{pmatrix} \begin{pmatrix} q & -u \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ w_2 & 0 \end{pmatrix} \right) \\
&= \sum_{\widetilde{u} \bmod \mathfrak{q}} \mathbf{f}\left(\frac{-1}{q} \cdot \frac{w_2}{w_1} \cdot x \begin{pmatrix} 1 & 0 \\ -w_1 u & q\left(\frac{w_1}{w_2}\right) \end{pmatrix} \right) \\
&= \sum_{\widetilde{u} \bmod \mathfrak{q}} \mathbf{f}\left(\frac{-1}{q} \cdot \frac{w_2}{w_1} \cdot x \begin{pmatrix} 1 & 0 \\ 0 & \frac{w_1}{w_2} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -w_2 u & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & q \end{pmatrix} \right) \\
&= \sum_{\widetilde{u} \bmod \mathfrak{q}} \mathbf{f}\left(-x \cdot \frac{w_2}{w_1} \begin{pmatrix} 1 & 0 \\ 0 & \frac{w_1}{w_2} \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 \\ -w_2 u & 1 \end{pmatrix} \cdot \frac{1}{q} \begin{pmatrix} 1 & 0 \\ 0 & q \end{pmatrix} \right) \\
&= (-1)^k \cdot \sum_{\widetilde{u} \bmod \mathfrak{q}} \mathbf{f}\left(x \begin{pmatrix} 1 & 0 \\ w_2 u & 1 \end{pmatrix}^{\iota} \right) \\
&= (-1)^k \cdot \mathrm{Tr}_{\mathfrak{q}}^{\mathfrak{a}\mathfrak{q}} \big(\mathbf{f}\big)(x).
\end{aligned}$$

Towards the end, note that we can choose the coset representatives for $U_{\mathfrak{q}}$ to be $\begin{pmatrix} 1 & 0 \\ 0 & q \end{pmatrix}^{-1} \begin{pmatrix} 1 & u \\ 0 & q \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & \frac{w_2}{w_1} \end{pmatrix}$ (c.f. [Shi78, §2]), eliminating the extra factors in the third from last line. Thus $\mathbf{f} \,\big|\, (J_{\mathfrak{a}\mathfrak{q}} \circ U_{\mathfrak{q}} \circ J_{\mathfrak{q}}) = (-1)^k \cdot \mathcal{N}(\mathfrak{q})^{k/2-1} \cdot \big(\mathbf{f} \,\big|\, \mathrm{Tr}_{\mathfrak{a}}^{\mathfrak{a}\mathfrak{q}}\big)$ and replacing the Hilbert modular form $\mathbf{f}$ by $\mathbf{f} \,\big|\, J_{\mathfrak{a}\mathfrak{q}}$ yields the desired identity.

For the second case where $\mathfrak{q} \nmid \mathfrak{a}$, we let $w_1, w_2 \in \mathring{A}_F$ be as before and choose $a \in \mathring{A}_F^{\times}$ so that $\widetilde{a} = \mathfrak{a}\mathfrak{d}^2$. The first half of the computation (for the operator $U_{\mathfrak{q}}$) is already dealt with above, and we only need to compute the second half as follows:

$$
\begin{aligned}
\frac{\mathbf{f} \,\big|\, (J_{\mathfrak{a}\mathfrak{q}} \circ U'_{\mathfrak{q}} \circ J_{\mathfrak{a}})(x)}{\mathcal{N}(\mathfrak{q})^{k/2-1}} &= \mathbf{f}\left(x^{-\iota} \begin{pmatrix} 0 & 1 \\ w_1 & 0 \end{pmatrix} \right) \Bigg| (U'_{\mathfrak{q}} \circ J_{\mathfrak{a}}) \\
&= \eta_Y \left(\begin{pmatrix} 1 & 0 \\ 0 & q \end{pmatrix}^{-1} \begin{pmatrix} 1 & 0 \\ 0 & q \end{pmatrix} \begin{pmatrix} \alpha q & \beta \\ a & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & \frac{w_2}{w_1} \end{pmatrix} \right)^{-1} \cdot \\
&\qquad \mathbf{f}\left(x^{-\iota} \begin{pmatrix} 0 & 1 \\ w_1 & 0 \end{pmatrix} \left(\begin{pmatrix} 1 & 0 \\ 0 & q \end{pmatrix}^{-1} \begin{pmatrix} 1 & 0 \\ 0 & q \end{pmatrix} \begin{pmatrix} \alpha q & \beta \\ a & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & \frac{w_2}{w_1} \end{pmatrix} \right)^{\iota} \right) \Bigg| J_{\mathfrak{a}} \\
&= \eta(\alpha q)^{-1} \cdot \mathbf{f}\left(x^{-\iota} \begin{pmatrix} 0 & 1 \\ w_1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & \frac{w_2}{w_1} \end{pmatrix}^{-\iota} \begin{pmatrix} \alpha q & \beta \\ a & 1 \end{pmatrix}^{\iota} \right) \Bigg| J_{\mathfrak{a}} \\
&= \eta(\alpha q)^{-1} \cdot \mathbf{f}\left(\left(x^{-\iota} \begin{pmatrix} 0 & 1 \\ w_1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & \frac{w_2}{w_1} \end{pmatrix}^{-\iota} \begin{pmatrix} \alpha q & \beta \\ a & 1 \end{pmatrix}^{\iota} \right)^{-\iota} \begin{pmatrix} 0 & 1 \\ w_2 & 0 \end{pmatrix} \right) \\
&= \eta(\alpha q)^{-1} \cdot \mathbf{f}\left(x \begin{pmatrix} 0 & 1 \\ w_1 & 0 \end{pmatrix}^{-\iota} \begin{pmatrix} 1 & 0 \\ 0 & \frac{w_2}{w_1} \end{pmatrix} \begin{pmatrix} \alpha & \beta \\ a & q \end{pmatrix}^{-1} \begin{pmatrix} 0 & 1 \\ w_2 & 0 \end{pmatrix} \right) \\
&= \eta(\alpha q)^{-1} \cdot \mathbf{f}\left(\frac{-1}{w_1} \cdot \frac{1}{\alpha q - \beta a} \cdot x \begin{pmatrix} \alpha \cdot q \cdot w_1 & -a \cdot \frac{w_1}{w_2} \\ -\beta \cdot w_1 \cdot w_2 & w_1 \end{pmatrix} \right) \\
&= \eta(\alpha q)^{-1} \cdot \eta(-1) \cdot \eta(\alpha q - \beta a)^{-1} \cdot \mathbf{f}\left(x \begin{pmatrix} \alpha \cdot q & -a \cdot \frac{1}{w_2} \\ -\beta \cdot w_2 & 1 \end{pmatrix} \right) \\
&= (-1)^k \cdot \eta(\alpha q)^{-1} \cdot \mathbf{f}\left(x \begin{pmatrix} \alpha \cdot q & -a \cdot \frac{1}{w_2} \\ -\beta \cdot w_2 & 1 \end{pmatrix} \right) \\
&= (-1)^k \cdot \eta(\alpha q)^{-1} \cdot \mathbf{f}\left(x \begin{pmatrix} 1 & a \cdot \frac{1}{w_2} \\ \beta \cdot w_2 & \alpha \cdot q \end{pmatrix}^{\iota} \right).
\end{aligned}
$$

The above matrix

$$
\begin{pmatrix} 1 & a \cdot \frac{1}{w_2} \\ \beta \cdot w_2 & \alpha \cdot q \end{pmatrix}
$$

may be taken as the last choice of matrix in the coset representatives given in the second part of Proposition 2.6, which completes the calculation for $\mathfrak{q} \nmid \mathfrak{a}$. ■

2.3 Eigenforms

In this section we introduce an inner product over $\mathcal{M}_k(\mathfrak{c}, \eta)$. We then show that as an inner product space, $\mathcal{S}_k(\mathfrak{c}, \eta)$ decomposes into "old" and "new" subspaces.

Let $f, g \in \mathcal{M}_k(\Gamma)$ for some congruence subgroup $\Gamma = \Gamma_\lambda(t_\lambda \mathfrak{d}, \mathfrak{c})$. We define an inner product over the vector space $\mathcal{M}_k(\Gamma)$ by the integral

$$\langle f, g\rangle_{\mathfrak{c}} = \mu(\Gamma \setminus \mathfrak{H}^n)^{-1} \int_{\Gamma\setminus\mathfrak{H}^n} \overline{f(z)} g(z) y^k \, \mathrm{d}\mu(z), \tag{2.28}$$

where $\mathrm{d}\mu(z)$ denotes the Haar measure

$$d\mu(z) = \prod_{j=1}^{n} y_j^{-2} \mathrm{d}x_j \mathrm{d}y_j, \quad \text{for } z^{\sigma_j} = x_j + iy_j$$

and the symbol

$$\mu(\Gamma \setminus \mathfrak{H}^n) = \int_{\Gamma\setminus\mathfrak{H}^n} d\mu(z)$$

denotes the volume of a fundamental domain for $\Gamma \setminus \mathfrak{H}^n$ with respect to $\mathrm{d}\mu(z)$. This inner product over $\mathcal{M}_k(\Gamma)$ converges when fg is a cusp form, meaning at least one of f or g has to be a cusp form (c.f. [Gar90, p.50]). Furthermore, one can clearly observe that the integrand of (2.28) is independent of the choice of Γ. This fact combined with the existence of a normalising factor $\mu(\Gamma \setminus \mathfrak{H}^n)^{-1}$ tells us that the whole inner product $\langle f, g\rangle_{\mathfrak{c}}$ is independent of the choice of Γ.

We define the inner product operation between two adèlic Hilbert modular forms $\mathbf{f} = (f_1, \ldots, f_h)$ and $\mathbf{g} = (g_1, \ldots, g_h)$ belonging to $\mathcal{M}_k(\mathfrak{c}, \eta)$ as the sum

$$\langle \mathbf{f}, \mathbf{g}\rangle_{\mathfrak{c}} = \sum_{\lambda=1}^{h} \langle f_\lambda, g_\lambda\rangle. \tag{2.29}$$

This is of course only meaningful if each summand exists, and so we make the additional assumption that at least one of $\mathbf{f}$ or $\mathbf{g}$ is a cusp form. We also remark that the volume of a fundamental domain for an arbitrary congruence modular subgroup $\Gamma_\lambda \setminus \mathfrak{H}^n$ can be calculated as follows (c.f. [Shi78, (2.31)]):

$$\mu(\Gamma_\lambda \setminus \mathfrak{H}^n) = \mu(\mathrm{SL}_2(\mathcal{O}_F) \setminus \mathfrak{H}^n) \times \left[(\mathcal{O}_F^\times)^+ : (\mathcal{O}_F^\times)^2\right]^{-1} \times \mathcal{N}(\mathfrak{c}) \prod_{\mathfrak{q}|\mathfrak{c}} (1 + \mathcal{N}(\mathfrak{q})^{-1}). \tag{2.30}$$

The following result introduces an operator that takes a Hilbert modular form of lower level to one of a higher level.

Proposition 2.8. *For every integral ideal* $\mathfrak{n}$ *and every* $\mathbf{f} \in \mathcal{M}_k(\mathfrak{c}, \eta)$*, there is a unique element* $\mathbf{f} \,\big|\, \mathfrak{n}$ *of* $\mathcal{M}_k(\mathfrak{n}\mathfrak{c}, \eta)$ *whose Fourier coefficients satisfy*

$$\mathrm{C}\big(\mathfrak{m}, \mathbf{f} \,\big|\, \mathfrak{n}\big) = \mathrm{C}(\mathfrak{n}^{-1}\mathfrak{m}, \mathbf{f}).$$

Proof. Let $n \in \mathbb{A}_F$ such that $\tilde{n} = \mathfrak{n}$. Let us write $\mathbf{f} = (f_1, \ldots, f_h)$. We note that for each index λ, one can find a unique index μ and a totally positive element a_μ such that $qt_\lambda = a_\mu t_\mu$. Put $f'_\lambda = f_\mu \,\big|\, \left(\begin{smallmatrix} a_\mu & 0 \\ 0 & 1 \end{smallmatrix}\right)$ and consider the Hilbert modular form $\mathbf{f}' = (f'_1, \ldots, f'_h)$. It satisfies $\mathrm{C}(\mathbf{f}', \mathfrak{m}) = \mathcal{N}(\mathfrak{n})^{k/2} \cdot \mathrm{C}(\mathbf{f}, \mathfrak{n}^{-1}\mathfrak{m})$. Taking $\mathbf{f} \,\big|\, \mathfrak{n} = \mathcal{N}(\mathfrak{n})^{-k/2} \cdot \mathbf{f}'$ completes the proof of the proposition. ■

Let $\mathcal{S}_k^{\mathrm{old}}(\mathfrak{c}, \eta)$ be the subspace of $\mathcal{S}_k(\mathfrak{c}, \eta)$ spanned by all $\mathbf{f} \,\big|\, \mathfrak{n}$ obtained by applying the previous proposition to $\mathbf{f} \in \mathcal{S}_k(\mathfrak{b}, \eta)$, for every nontrivial integral ideal $\mathfrak{b} \mid \mathfrak{c}$ divisible by the finite part of the conductor of η and $\mathfrak{n}$ running through all ideals dividing $\mathfrak{b}^{-1}\mathfrak{c}$. On the other hand, let $\mathcal{S}_k^{\mathrm{new}}(\mathfrak{c}, \eta)$ be the orthogonal complement of $\mathcal{S}_k^{\mathrm{old}}(\mathfrak{c}, \eta)$ in $\mathcal{S}_k(\mathfrak{c}, \eta)$ with respect to the inner product (2.28). We call $\mathcal{S}^{\mathrm{old}}(\mathfrak{c}, \eta)$ (resp. $\mathcal{S}^{\mathrm{new}}(\mathfrak{c}, \eta)$) the space of oldforms (resp. newforms) of weight k, level $\mathfrak{c}$ and Nebentype η.

Definition 2.4. An element $\mathbf{f} \in \mathcal{M}_k(\mathfrak{c}, \eta)$ is said to be an *eigenform* if it is a common eigenfunction of $T_\mathfrak{n}$ for all integral ideals $\mathfrak{n}$. We say that $\mathbf{f}$ is *normalised* if $\mathrm{C}(\mathbf{f}, \mathcal{O}) = 1$.

Given an eigenform $\mathbf{f}$, one can write $\mathbf{f} \,\big|\, T_\mathfrak{n} = v(\mathfrak{n}, \mathbf{f}) \cdot \mathbf{f}$ for each $\mathfrak{n}$. From equation (2.25), we know that $\mathrm{C}\big(\mathbf{f} \,\big|\, T_\mathfrak{n}, \mathcal{O}\big) = \mathrm{C}(\mathbf{f}, \mathfrak{n})$. Thus if $\mathbf{f}$ is normalised, then the $\mathfrak{n}$-th eigenvalue $v(\mathfrak{n}, \mathbf{f})$ is precisely the $\mathfrak{n}$-th coefficient $\mathrm{C}(\mathbf{f}, \mathfrak{n})$.

Definition 2.5. An element $\mathbf{f} \in \mathcal{S}_k^{\mathrm{new}}(\mathfrak{c}, \eta)$ is said to be a *primitive cusp form* if it is a normalised common eigenfunction for all $T_\mathfrak{n}$. We call $\mathfrak{c}$ the *conductor* of $\mathbf{f}$. When there are multiple HMFs involved, we use the notation $\mathfrak{c}(\mathbf{f})$ instead.

To end this section, we remark that $\mathcal{M}_k(\mathfrak{c}, \eta)$ is stable under the action of Hecke operators. This follows directly from equation (2.25). In fact, both subspaces $\mathcal{S}_k^{\mathrm{new}}(\mathfrak{c}, \eta)$ and $\mathcal{S}_k^{\mathrm{old}}(\mathfrak{c}, \eta)$ are stable under $T_\mathfrak{q}(\mathfrak{c})$ for $\mathfrak{q}$ prime to $\mathfrak{c}$. The former can be deduced from the fact that Hecke operators commute with

$\mathfrak{n}$-operators; the latter follows easily from the following result which we state without proof (c.f. [Shi78, Proposition 2.4]).

Proposition 2.9. *Let $y \in Y$ such that $\det(y)\mathcal{O}$ is prime to $\mathfrak{c}$. Then for any $\mathbf{f}, \mathbf{g} \in \mathcal{M}_k(\mathfrak{c}, \eta)$,*

$$\tilde{\eta}\big(\det(y)\mathcal{O}\big) \cdot \big\langle \mathbf{f} \,\big|WyW, \mathbf{g}\big\rangle = \big\langle \mathbf{f}, \mathbf{g} \,\big|\, WyW\big\rangle.$$

And finally, we state the following immediate result.

Corollary 2.10. *Both $\mathcal{S}_k^{\text{old}}(\mathfrak{c}, \eta)$ and $\mathcal{S}_k^{\text{new}}(\mathfrak{c}, \eta)$ are stable under the action of the Hecke algebra $R(W, Y)$.*

2.4 L-functions attached to Hilbert modular forms

Throughout this thesis, one will come across multiple types of L-functions. The aim of this section is to discuss the complex imprimitive and the primitive L-functions attached to Hilbert modular forms.

An *imprimitive L-function* is given by a Dirichlet series and it decomposes as an Euler product. On the other hand, a *primitive L-function* arises from the Deligne–Taylor representation [Tay89], which can also be written as an infinite product of local Euler factors.

In most cases, these L-functions can be 'completed', i.e., multiplied with a factor at infinity so that the resulting function possesses desirable analytic properties such as: convergence for sufficiently large $\mathrm{Re}(s)$, holomorphicity on the whole complex plane, and the existence of a functional equation. We shall show this in the case of a single Hilbert modular form $\mathbf{f}$ and some products of HMFs including: the double product $\mathbf{f}_1 \times \mathbf{f}_2$, the symmetric square representation $\mathrm{Sym}^2(\mathbf{f})$ and the triple product $\mathbf{f}_1 \times \mathbf{f}_2 \times \mathbf{f}_3$.

2.4.1 A single HMF

Let $\mathbf{f} \in \mathcal{M}_k(\mathfrak{c}, \eta)$ be a Hilbert modular newform of weight k, conductor $\mathfrak{c}$ and Nebentype η. As a consequence of Proposition 2.4, the Dirichlet series $\sum_{\mathfrak{m}} \mathrm{C}(\mathbf{f}, \mathfrak{m}) \cdot \mathcal{N}(\mathfrak{m})^{-s}$ attached to $\mathbf{f}$ has the following infinite product expansion

$$\prod_{\mathfrak{q} \nmid \mathfrak{c}} \left(1 - \mathrm{C}(\mathbf{f}, \mathfrak{q})\mathcal{N}(\mathfrak{q})^{-s} + \tilde{\eta}(\mathfrak{q})\mathcal{N}(\mathfrak{q})^{k-1-2s}\right)^{-1} \times \prod_{\mathfrak{q} \mid \mathfrak{c}} \left(1 - \mathrm{C}(\mathbf{f}, \mathfrak{q})\mathcal{N}(\mathfrak{q})^{-s}\right)^{-1}; \tag{2.31}$$

we call this the *imprimitive L-function* of $\mathbf{f}$ denoted $D(\mathbf{f}, s)$. For each prime $\mathfrak{q} \in \mathrm{Spec}(\mathcal{O}_F)$, the local Euler factor $D_{\mathfrak{q}}(\mathbf{f}, s)$ of $D(\mathbf{f}, s)$ factorises into

$$\left(1 - \alpha_{\mathfrak{q}}(\mathbf{f})\mathcal{N}(\mathfrak{q})^{-s}\right)\left(1 - \beta_{\mathfrak{q}}(\mathbf{f})\mathcal{N}(\mathfrak{q})^{-s}\right) \tag{2.32}$$

where it is understood that for $\mathfrak{q} \mid \mathfrak{c}$, we take $\alpha_{\mathfrak{q}}(\mathbf{f}) = \mathrm{C}(\mathbf{f}, \mathfrak{q})$ and $\beta_{\mathfrak{q}}(\mathbf{f}) = 0$.

On the other hand, let us denote by $\left\{V_\ell(\mathbf{f})\right\}_\ell$ the compatible system of two-dimensional ℓ-adic representations attached to $\mathbf{f}$ attributed to Deligne and Taylor. We define the *primitive L-function* of $\mathbf{f}$ as

$$L(\mathbf{f}, s) = \prod_{\mathfrak{q}} \det\left(\mathrm{Id} - \mathrm{Frob}_{\mathfrak{q}}^{-1} \,\middle|\, V_\ell(\mathbf{f})^{I_{\mathfrak{q}}} \cdot \mathcal{N}(\mathfrak{q})^{-s}\right)^{-1}, \quad \ell \neq \mathfrak{q}. \tag{2.33}$$

Here $\mathrm{Frob}_{\mathfrak{q}}$ is the arithmetic Frobenius element at $\mathfrak{q}$ and $I_{\mathfrak{q}} \subset \mathrm{Gal}(\overline{F}_{\mathfrak{q}}/F_{\mathfrak{q}})$ is the inertia subgroup.

One can easily verify that the imprimitive and primitive L-functions of $\mathbf{f}$ coincide: $D(\mathbf{f}, s) = L(\mathbf{f}, s)$. This is not true in general, and if that is the case, one needs to consider an 'incomplete' version of the L-function. For S a finite set of places in F, the *S-depleted L-function* $L_{(S)}(\mathbf{f}, s)$ of $L(\mathbf{f}, s)$ is obtained by removing the local Euler factors at $\mathfrak{v} \in S$. So as not to be confused with the notation $L_{\mathfrak{q}}$ for local factors, we enclose S in parentheses as a reminder that we are omitting factors.

Lastly, we define the *complex L-function* of $\mathbf{f}$ as the 'completed' product

$$\Psi(\mathbf{f}, s) = L_\infty(\mathbf{f}, s) \times L(\mathbf{f}, s) \tag{2.34}$$

where $L_\infty(\mathbf{f}, s) = (2\pi)^{-ns}\Gamma(s)^n$. Shimura proved that this complex L-function converges for $\mathrm{Re}(s) \gg 0$, it is meromorphic on $\mathbb{C}$ with at most simple poles that may only occur at $s = k$ and $s = 0$, and it satisfies the functional equation

$$\Psi(\mathbf{f}, s) = \epsilon(\mathbf{f}, s) \times \Psi(\mathbf{f}^{\rho}, k - s), \tag{2.35}$$

identifying the critical strip as $s \in \{1, 2, \ldots, k-1\}$ (c.f. [Shi78, (2.48)]). Here, $\mathbf{f}^{\rho}$ represents the Hilbert modular form with coefficients $\mathrm{C}(\mathbf{f}^{\rho}, \mathfrak{m}) = \overline{\mathrm{C}(\mathbf{f}, \mathfrak{m})}$ and $\epsilon(\mathbf{f}, s) = \prod_{\mathfrak{v}} \epsilon_{\mathfrak{v}}(\mathbf{f}, s)$ is the product of local epsilon-factors in [Tat79, §3].

2.4.2 Double product

Let $\mathbf{f}_1 \in \mathcal{M}_{k_1}(\mathfrak{c}_1, \eta_1)$ and $\mathbf{f}_2 \in \mathcal{M}_{k_2}(\mathfrak{c}_2, \eta_2)$ be a pair of Hilbert modular newforms satisfying $k_2 < k_1$. As in the preceding case, the imprimitive L-function of the double product $\mathbf{f}_1 \times \mathbf{f}_2$ is derived from the convoluted Dirichlet L-series $\sum_{\mathfrak{m}} \mathrm{C}(\mathbf{f}_1, \mathfrak{m})\mathrm{C}(\mathbf{f}_2, \mathfrak{m}) \cdot \mathcal{N}(\mathfrak{m})^{-s}$ and is defined as:

$$D(\mathbf{f}_1, \mathbf{f}_2, s) = \zeta_{(\mathfrak{c}_1\mathfrak{c}_2)}(\tilde{\eta}_1\tilde{\eta}_2, 2s + 2 - k_1 - k_2) \times \sum_{\mathfrak{m}} \mathrm{C}(\mathbf{f}_1, \mathfrak{m})\mathrm{C}(\mathbf{f}_2, \mathfrak{m}) \cdot \mathcal{N}(\mathfrak{m})^{-s}. \tag{2.36}$$

Here, $\zeta_{(\mathfrak{c}_1\mathfrak{c}_2)}(\tilde{\eta}_1\tilde{\eta}_2, s) = \sum_{\mathfrak{m}} \tilde{\eta}_1\tilde{\eta}_2(\mathfrak{m}) \cdot \mathcal{N}(\mathfrak{m})^{-s}$ where $\mathfrak{m}$ runs over integral ideals prime to $\mathfrak{c}_1\mathfrak{c}_2$. The imprimitive L-function has an infinite product expansion

$$\prod_{\mathfrak{q}} D_{\mathfrak{q}}(\mathbf{f}_1, \mathbf{f}_2, s) = \prod_{\mathfrak{q}} \Big(\big(1 - \alpha_{\mathfrak{q}}(\mathbf{f}_1)\alpha_{\mathfrak{q}}(\mathbf{f}_2)\mathcal{N}(\mathfrak{q})^{-s}\big)\big(1 - \beta_{\mathfrak{q}}(\mathbf{f}_1)\beta_{\mathfrak{q}}(\mathbf{f}_2)\mathcal{N}(\mathfrak{q})^{-s}\big)$$
$$\big(1 - \alpha_{\mathfrak{q}}(\mathbf{f}_1)\beta_{\mathfrak{q}}(\mathbf{f}_2)\mathcal{N}(\mathfrak{q})^{-s}\big)\big(1 - \beta_{\mathfrak{q}}(\mathbf{f}_1)\alpha_{\mathfrak{q}}(\mathbf{f}_2)\mathcal{N}(\mathfrak{q})^{-s}\big)\Big)^{-1}$$

where $\alpha_{\mathfrak{q}}(\mathbf{f}_i)$ and $\beta_{\mathfrak{q}}(\mathbf{f}_i)$ are the roots of $X^2 - \mathrm{C}(\mathbf{f}, \mathfrak{q})X + \tilde{\eta}(\mathfrak{q})\mathcal{N}(\mathfrak{q})^{k_i - 1}$ for $i = 1, 2$, following the convention in place at (2.32). Meanwhile, the *primitive L-function* of $\mathbf{f}_1 \times \mathbf{f}_2$ is given by the infinite Euler product

$$L(\mathbf{f}_1 \otimes \mathbf{f}_2, s) = \prod_{\mathfrak{q}} \det\Big(1 - \mathrm{Frob}_{\mathfrak{q}}^{-1} \Big| \big(V_{\ell}(\mathbf{f}_1) \otimes V_{\ell}(\mathbf{f}_2)\big)^{I_{\mathfrak{q}}} \cdot \mathcal{N}(\mathfrak{q})^{-s}\Big)^{-1}, \quad \ell \neq \mathfrak{q}. \tag{2.37}$$

and we denote the local L-factor at $\mathfrak{q}$ by $L_{\mathfrak{q}}(\mathbf{f}_1 \otimes \mathbf{f}_2, s)$.

The primes $\mathfrak{q}$ that do not divide $\mathfrak{c}_i$ shall be referred to as the good primes of $\mathbf{f}_i$ while the primes that do divide the conductor shall be called bad primes. Since $V_{\ell}(\mathbf{f}_i)$ is unramified outside $\ell\mathfrak{c}_i$, the good primes are precisely the primes $\mathfrak{q}$ for which $I_{\mathfrak{q}}$ acts trivially on $V_{\ell}(\mathbf{f}_i)$. One can then deduce the equality

$L_{\mathfrak{q}}(\mathbf{f}_1 \otimes \mathbf{f}_2, s) = D_{\mathfrak{q}}(\mathbf{f}_1, \mathbf{f}_2, s)$ for $\mathfrak{q} \nmid \mathfrak{c}_1\mathfrak{c}_2$. Unfortunately, this is not the case for local factors at bad primes which we shall compute explicitly in Chapter 7.

The completion of the primitive L-function is obtained by multiplying the Archimedean L-factor $L_\infty(\mathbf{f}_1 \otimes \mathbf{f}_2, s) = (2\pi)^{-2ns}\Gamma(s)^n\Gamma(s - k_2 + 1)^n$ so that:

$$\Psi(\mathbf{f}_1 \otimes \mathbf{f}_2, s) = L_\infty(\mathbf{f}_1 \otimes \mathbf{f}_2, s) \times L(\mathbf{f}_1 \otimes \mathbf{f}_2, s). \tag{2.38}$$

It converges when $\mathrm{Re}(s) \gg 0$ and it is meromorphic on $\mathbb{C}$. In particular, poles arise only when $k_1 = k_2$, in which case they are simple and may only occur at $s = k$ and $s = k - 1$. Furthermore, its complex functional equation is given by

$$\Psi(\mathbf{f}_1 \otimes \mathbf{f}_2, s) = \epsilon(\mathbf{f}_1 \otimes \mathbf{f}_2, s) \times \Psi(\mathbf{f}_1^\rho \otimes \mathbf{f}_2^\rho, k_1 + k_2 - 1 - s) \tag{2.39}$$

with the critical strip at $\{k_2, k_2 + 1, \ldots, k_1 - 1\}$. These properties mentioned above were proven in [Shi78, §4] using Rankin-Selberg theory and in [JL70, §17] via representation theory.

In Chapter 7 we shall see a few more examples of tensor product of Hilbert modular forms namely: the symmetric square representation and the triple product. We will delay the discussion of their analytic properties until then.

2.5 The Rankin-integral representation

In this section we define the Hilbert-Eisenstein series or simply Eisenstein series. We shall see that the analytic properties exhibited by the L-functions in the previous section turn out to be satisfied by Eisenstein series as well. We also introduce the Rankin-Selberg method, which is a technique for analytically continuing a given L-series by passing off desirable analytic properties that Eisenstein series satisfy on to the given Hilbert modular form.

We first introduce a few notations. Let $\tilde{\psi}$ be a Hecke character of finite order with conductor $\mathfrak{c}(\tilde{\psi})$. Let $r = (r_1, \ldots, r_n) \in (\mathbb{Z}/2\mathbb{Z})^n$ so that $\psi(x_\infty) = \mathrm{sgn}(x_\infty)^r$. We extend $\tilde{\psi}$ to a character of the group of ideals of F by setting $\tilde{\psi}(\mathfrak{a}) = 0$ for each integral ideal $\mathfrak{a}$ not prime to $\mathfrak{c}(\tilde{\psi})$. Define the *Gauss sum* as

$$G(\psi) = \sum_x \mathrm{sgn}(x)^r\, \tilde{\psi}\big(x\mathfrak{c}(\tilde{\psi})\mathfrak{d}\big) \cdot e_F(x). \tag{2.40}$$

where x runs over a complete set of representatives of $\mathfrak{c}(\tilde{\psi})^{-1}\mathfrak{d}^{-1}/\mathfrak{d}^{-1}$ and e_F is the standard additive character in Section 2.1. It is a well-known property of the Gauss sum that $\left|G(\tilde{\psi})\right|^2 = N\big(\mathfrak{c}(\tilde{\psi})\big)$.

2.5.1 Hilbert-Eisenstein series

Let $\mathfrak{a}$ and $\mathfrak{b}$ be fractional ideals while $\mathfrak{c}$ is an integral ideal of F. Let κ be an integer and let $\tilde{\psi}$ be a Hecke character of finite order modulo $\mathfrak{c}(\tilde{\psi})$ such that $\tilde{\psi}(x\mathcal{O}_F) = \operatorname{sgn}(x)^\kappa$ for an element $x \equiv 1 \pmod{\mathfrak{c}}$ of $\mathcal{O}_F^\times$. Following [Pan89], we define for $\operatorname{Re}(s) > 2 - \kappa$ the *Eisenstein series* and its dual as

$$\begin{aligned} K_\kappa(z, s; \mathfrak{a}, \mathfrak{b}; \psi) &= \sum_{c,d} \operatorname{sgn}(\mathcal{N}(d)^\kappa)\, \tilde{\psi}(d\mathfrak{b}^{-1})(cz+d)^{-\kappa\mathbf{1}}|cz+d|^{-2s\mathbf{1}} \text{ and} \\ L_\kappa(z, s; \mathfrak{a}, \mathfrak{b}; \psi) &= \sum_{c,d} \operatorname{sgn}(\mathcal{N}(c)^\kappa)\, \tilde{\psi}(c\mathfrak{a}^{-1})(cz+d)^{-\kappa\mathbf{1}}|cz+d|^{-2s\mathbf{1}}, \end{aligned} \tag{2.41}$$

respectively (κ here is m in *op. cit.*). Each sum is taken over the $\mathcal{O}_F^\times$-equivalence classes of nonzero pairs $(c, d) \in \mathfrak{a} \times \mathfrak{b}$ where the equivalence relation is given by $(c_1, d_1) \sim (c_2, d_2)$ if and only if $c_2 = uc_1$ and $d_2 = ud_1$ for some $u \in \mathcal{O}_F^\times$. This is the equivalent to Shimura's formulation in [Shi78, p.672] with r taken to be $(0, \ldots, 0)$.

For each $1 \le \lambda \le h$, where h is the narrow class number of F, one defines

$$\begin{aligned} K_\kappa(z, s; \mathfrak{a}, \mathfrak{b}; \psi)_\lambda &= \mathcal{N}(\tilde{t}_\lambda)^{s+(\kappa/2)}\mathcal{N}(y)^s K_\kappa(z, s; \tilde{t}_\lambda\mathfrak{d}\mathfrak{a}, \mathfrak{b}; \psi), \text{ and} \\ L_\kappa(z, s; \mathfrak{a}, \mathfrak{b}; \psi)_\lambda &= \mathcal{N}(\tilde{t}_\lambda)^{-s-(\kappa/2)}\mathcal{N}(y)^s L_\kappa(z, s; \mathfrak{a}, \tilde{t}_\lambda^{-1}\mathfrak{d}^{-1}\mathfrak{b}; \psi). \end{aligned} \tag{2.42}$$

Proposition 2.11. *The Eisenstein series in* (2.41) *converges for sufficiently large* $\operatorname{Re}(s)$ *and it has a meromorphic continuation on* $\mathbb{C}$. *In fact, it is entire when* $\kappa > 0$*; it has a simple pole at* $s = 1$ *and possibly another simple pole at* $s = 0$ *otherwise. Furthermore, if* ψ *is primitive, one has a functional equation*

$$\begin{aligned} &\Delta_\kappa(1-\kappa-s)K_\kappa(1-\kappa-s; \mathfrak{a}, \mathfrak{b}; \psi)_\lambda \\ &\qquad\qquad = G(\psi)\mathcal{N}(\mathfrak{d}\mathfrak{a}\mathfrak{b}\mathfrak{c})^{\kappa+2s-1}\Delta_\kappa(s)L_m(z, s; \mathfrak{a}, \mathfrak{b}\mathfrak{c}; \overline{\psi})_\lambda \end{aligned} \tag{2.43}$$

where $\Delta_\kappa(s) = \pi^{-ns}\Gamma(s+\kappa)^n$ *and* $\overline{(-)}$ *signifies complex conjugation.*

Proof. The Eisenstein series $K_\kappa^r(-)$ and $L_\kappa^r(-)$ are linear combinations of the infinite sum $E_{\kappa,U}^r$ defined in [Shi78, (3.8)]. The first two claims follow from the properties of the completed function $\Gamma(s+\kappa+r)^n \cdot E^r_{\kappa,U}$ detailed in [Shi78, p.656]. The functional equation can be found in p.672 of *op. cit.* and is stated in the theorem using the notation in [Pan89]. This proof works for $r \geq 0$. ■

The Eisenstein series (2.41) then extend naturally into functions on $\mathbb{A}_G$ by setting $\mathbf{K}_\kappa(z,s;\mathfrak{a},\mathfrak{b};\eta) := \big(K_\kappa(-)_\lambda\big)_\lambda$ and $\mathbf{L}_\kappa(z,s;\mathfrak{a},\mathfrak{b};\eta) := \big(L_\kappa(-)_\lambda\big)_\lambda$.

2.5.2 The Rankin-Selberg method

The Rankin-Selberg method is a way to write the complex L-function of an HMF as an integral of its associated automorphic form multiplied by an Eisenstein series. This integral is called the Rankin integral representation. The method originated from the independent works of Robert Rankin and Atle Selberg [Ran39; Sel40] who studied the Dirichlet series of the double product of classical modular forms. Shimura applied this to the L-series

$$\mathscr{D}(\mathbf{f},\mathbf{g},s) = \sum_{\mathfrak{m}} \mathrm{C}(\mathbf{f},\mathfrak{m})\mathrm{C}(\mathbf{g},\mathfrak{m}) \cdot \mathcal{N}(\mathfrak{m})^{-s}$$

of Hilbert modular forms $\mathbf{f} \in \mathcal{S}_k(\mathfrak{c}(\mathbf{f}),\eta_{\mathbf{f}})$ and $\mathbf{g} \in \mathcal{M}_l(\mathfrak{c}(\mathbf{g}),\eta_{\mathbf{g}})$ satisfying $k \geq l$ and $\mathfrak{c} := \mathfrak{c}(\mathbf{f}) = \mathfrak{c}(\mathbf{g})$; $\eta_{\mathbf{f}}$ and $\eta_{\mathbf{g}}$ are Hecke characters over $\mathbb{A}_F$ [Shi78].

Our strategy in Chapter 4 uses the following Rankin integral representation of $\mathscr{D}(\mathbf{f},\mathbf{g},s)$ which is given in §4 of *op. cit.* (see also [Pan89, (4.8)])

$$\begin{aligned}\Psi_{(\mathfrak{c})}(\mathbf{f},\mathbf{g},s) = \frac{D_F^{1/2}\Gamma(s+1-l)^n}{\pi^{-ns}} \times \sum_{\lambda=1}^{h} \Bigg(\mathcal{N}(\tilde{t}_\lambda)^{s+1-(k+l)/2} \\ \int_{\Gamma_\lambda(\mathfrak{c})\backslash\mathfrak{H}^n} \overline{f}^\rho_\lambda \cdot g_\lambda \cdot K_{k-l}\big(z, s-k+1; t_\lambda\mathfrak{d}, \mathcal{O}; \eta_{\mathbf{f}}\eta_{\mathbf{g}}^{-1}\big)_\lambda \cdot \mathcal{N}(y)^{s-1}\,\mathrm{d}x\,\mathrm{d}y\Bigg).\end{aligned} \tag{2.44}$$

Here, $\Psi_{(\mathfrak{c})}(\mathbf{f},\mathbf{g},s) = (2\pi)^{-2ns}\Gamma(s)^n\Gamma(s-l+1)^n\zeta_{(\mathfrak{c})}(2s+2-k-l,\psi\varphi)\mathscr{D}(\mathbf{f},\mathbf{g},s)$, which (despite being derived from a Dirichlet series) coincides with the primitive L-function multiplied by the appropriate factor at infinity and whose factors at primes dividing $\mathfrak{c}$ are removed. The fact that $\Psi_{(\mathfrak{c})}$ is entire except when $k = l$ where it has a simple pole at $s = k$ and possibly another simple pole at $s = k-1$ follows from the fact that K_{k-l} is a linear combination of the $E^r_{\kappa,U}$'s as in the proof of Proposition 2.11.

Combining (2.44) with Equation (2.43) allowed Shimura to prove that for integers $l \leq m \leq k-1$,

$$\frac{\zeta_{\mathfrak{c}}(2m+2-k-l, \psi\varphi)\mathscr{D}(m, \mathbf{f}, \mathbf{g})}{\left(2\pi\sqrt{-1}\right)^{n(k-l+2+2m)}\langle \mathbf{f}, \mathbf{f}\rangle_{\mathfrak{c}}}$$

is an algebraic number (c.f. [Shi78, Theorem 4.2]).

To conclude this chapter, we mention that Panchishkin considered the case that both **f** and **g** are Hilbert modular cuspforms with possibly distinct conductors [Pan89, §4]. The main issue that arises with HMFs of unbalanced conductors is that its inner product may no longer be logically sound. To address this problem, one constructs new Hilbert modular forms **F** and **G** from **f** and **g** respectively, so that their conductors divide a common ideal. We discuss the construction of **F** and **G** and a workaround to Panchishkin's additional condition $\mathrm{supp}(\mathfrak{c}(\mathbf{f})) \cap \mathrm{supp}(\mathfrak{c}(\mathbf{g})) = \emptyset$ as we apply it in Chapter 4.

Chapter 3

The Iwasawa Main Conjecture (IMC)

In this chapter, we state several versions of the main conjecture of Iwasawa theory. Section 3.1 starts the chapter off with the classical version for class groups. We also give the statement of the main conjecture for modular forms in Section 3.4. In order to do so, we must first recall in Section 3.3 the p-adic L-function attached to a modular form, the construction of which can be summarized nicely using the theory of motives contained in Section 3.2. We end the chapter by describing in Section 3.5 how p-congruence can be used to study the main conjecture for a class of modular forms.

3.1 Classical IMC

Throughout this chapter we work with a fixed prime $p \geq 3$. An extension of a number field F is called a $\mathbb{Z}_p$-extension if its Galois group over F is isomorphic to $\mathbb{Z}_p$. Let μ_{p^n} be the multiplicative group of the p^n-th roots of unity and recall that $\mathrm{Gal}(\mathbb{Q}(\mu_{p^{n+1}})/\mathbb{Q}) \cong (\mathbb{Z}/p\mathbb{Z})^\times \times (\mathbb{Z}/p^n\mathbb{Z})$. Let $\mathbb{Q}_n \subset \mathbb{Q}(\mu_{p^{n+1}})$ be the fixed field under $(\mathbb{Z}/p\mathbb{Z})^\times$ so that $\mathrm{Gal}(\mathbb{Q}_n/\mathbb{Q}) \cong \mathbb{Z}/p^n\mathbb{Z}$. As n increases, one gets

$$\mathbb{Q} = \mathbb{Q}_0 \subset \mathbb{Q}_1 \subset \ldots \subset \mathbb{Q}^{\mathrm{cyc}} := \bigcup_n \mathbb{Q}_n \tag{3.1}$$

where $\mathrm{Gal}(\mathbb{Q}^{\mathrm{cyc}}/\mathbb{Q}) \cong \varprojlim (\mathbb{Z}/p^n\mathbb{Z}) \cong \mathbb{Z}_p$. We call $\mathbb{Q}^{\mathrm{cyc}}$ the (unique) cyclotomic $\mathbb{Z}_p$-extension of $\mathbb{Q}$.

Let F be a totally real number field and assume that $p \nmid D_F$. The above construction can naturally be extended to F by considering the fields $F(\mu_{p^n})$ with $\mathcal{G}_n := \mathrm{Gal}(F(\mu_{p^{n+1}})/F)$ for $n \geq 1$. These fields form a tower that build up to $F(\mu_{p^\infty}) := \bigcup_n F(\mu_{p^n})$ with Galois group

$$\mathcal{G}_\infty := \mathrm{Gal}(F(\mu_{p^\infty})/F) = \Delta_p \times G_\infty.$$

Here $\Delta_p = \mathrm{Gal}(F(\mu_p)/F) \cong (\mathbb{Z}/p\mathbb{Z})^\times$ and $G_\infty = \mathrm{Gal}(F(\mu_{p^\infty})/F(\mu_p)) \cong \mathbb{Z}_p$. The Galois group of $F^{\mathrm{cyc}} := F\mathbb{Q}^{\mathrm{cyc}} \subset F(\mu_{p^\infty})$ is isomorphic to G_∞ so it is a cyclotomic $\mathbb{Z}_p$-extension of F. The fact that F^{cyc} is the unique $\mathbb{Z}_p$-extension of F follows from Leopoldt's conjecture for totally real abelian fields [Bru67].

Iwasawa theory is the study of arithmetic objects over a $\mathbb{Z}_p$-extension of a given number field. It was first applied to p-class groups where Iwasawa observed that over the tower (3.1), the class numbers behave regularly [Iwa59].

Theorem 3.1. *Let p be a prime and let h_n be the p-part of the class number of $\mathbb{Q}(\mu_{p^n})$. Then there exist positive integers λ and μ, and an integer ν such that for n sufficiently large,*

$$h_n = p^{\mu p^n + \lambda n + \nu}. \tag{3.2}$$

Let G_n be the unique cyclic subgroup of G_∞ with degree p^n. We define the *Iwasawa algebra* of G_∞ as the projective limit

$$\Lambda_{\mathbb{Z}_p} := \mathbb{Z}_p[\![G_\infty]\!] = \varprojlim_n \mathbb{Z}_p[G_n]$$

with respect to the restriction map $G_{n+1} \to G_n$. Let γ be a topological generator of G_∞. One can view the Iwasawa algebra as a power series ring $\mathbb{Z}_p[\![X]\!]$ by sending γ to $1 + X$. This induces a non-canonical isomorphism

$$\mathbb{Z}_p[G_n] \cong \mathbb{Z}_p[\![X]\!] \big/ \big((1+X)^{p^n} - 1\big).$$

The Iwasawa algebra of $\mathcal{G}_\infty$ is defined similarly and is identified with $\mathbb{Z}_p[\Delta_p][\![X]\!]$.

Let H_n be the p-Hilbert class field of $F_n := F\mathbb{Q}_n$ so that $[H_n : F_n] = h_n$ and $H_\infty := \bigcup_n H_n$ is the maximal abelian extension of F^{cyc}. Observe that each $A_n := \mathrm{Gal}(L_n/F_n)$ is a module for the group ring $\mathbb{Z}_p[G_n]$ and so it follows that $\Lambda_{\mathbb{Z}_p}$ acts on the projective limit $A_\infty := \varprojlim_n A_n$. Iwasawa proved that A_∞ is a finitely-generated torsion $\Lambda_{\mathbb{Z}_p}$-module [Iwa59]. The Structure Theorem for finitely-generated modules gives us a natural choice of characteristic polynomial $\mathrm{char}_\Lambda(A_\infty)$ of A_∞ over the Iwasawa algebra $\Lambda_{\mathbb{Z}_p}$ (c.f. *op. cit.* and [Ser60]).

Theorem 3.2. *Let M be a compact finitely-generated Λ-module. Then there is a pseudo-isomorphism (i.e., a Λ-homomorphism with finite kernel and cokernel)*

$$M \xrightarrow{\sim} \Lambda^r \oplus \left(\bigoplus_{i=1}^{c} \Lambda \big/ p^{\mu_i}\Lambda \right) \oplus \left(\bigoplus_{j=1}^{d} \Lambda \big/ \mathscr{F}_j(X)^{\lambda_j}\Lambda \right) \tag{3.3}$$

for some nonnegative integers μ_i, λ_j, and $r = \mathrm{rank}_\Lambda(M)$, and each $\mathscr{F}_j(X)$ is a distinguished polynomial over Λ.

Indeed, one can read off from Equation (3.3) that

$$\mathrm{char}_{\Lambda_{\mathbb{Z}_p}}(A_\infty) = \prod_{i=1}^{c} p^{\mu_i} \times \prod_{j=1}^{d} f_j^{\lambda_j}.$$

Let $\omega : \Delta_p \to \mathbb{Z}_p^\times$ be the Teichmüller character, i.e., the character sending $x \in \Delta_p$ to the unique $(p-1)$-st root of unity in $\mathbb{Z}_p$ congruent to $x \pmod p$. Then the group $\mathrm{Hom}(\Delta_p, \mathbb{Z}_p^\times)$ consists of ω^t for $0 \le t \le p-2$. Let u be the image of γ under the cyclotomic character $\kappa_{\mathrm{cyc}} : \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to \mathbb{Z}_p^\times$. We state the (classical) Iwasawa main conjecture for class groups.

Conjecture 3.3. *For odd integers t not congruent to 1 modulo $p-1$, there is an equality*

$$\mathrm{char}_\Lambda \left(A_\infty^{[t]} \right) = L_p\big(\omega^{1-t}, T\big) \tag{3.4}$$

up to a unit factor over the Iwasawa algebra $\Lambda_{\mathbb{Z}_p}$.

Here $A^{[t]} := A^{[\omega^t]} = \{a \in A : \sigma(a) = \omega^t(\sigma)(a) \text{ for all } \sigma \in \Delta_p\}$ is the ω^t-isotypic component of A. Moreover, $L_p(\omega^t, T)$ is the unique p-adic function which satisfies the relation $L_p(\omega^t, (1+p)^s - 1) = \zeta_p(\omega^t, s)$ where $\zeta_p(\omega^t, s)$ denotes the Kubota-Leopoldt p-adic L-function associated to the character ω^t.

We briefly outline the progress made towards the proof of Conjecture 3.3. Mazur and Wiles have succeeded in proving the main conjecture for primes $p \geq 3$ over the field $F = \mathbb{Q}$ using Jacobians of modular curves [MW84]. Shortly after, Wiles supplied the missing case of $p = 2$ over the field of rational numbers along with a proof of the main conjecture for totally real fields [Wil90]. Rubin gave an alternative proof to the result of [MW84] using the theory of Euler systems [Rub91]. With this method, he was also able to extend the validity of the main conjecture for imaginary quadratic fields.

Before we proceed with the rest of the chapter, we recall the decomposition $\mathbb{Z}_p^\times = \Delta_p \times (1 + p\mathbb{Z}_p)$ where $1 + p\mathbb{Z}_p \cong (\mathbb{Z}_p, +)$. Then an element $x \in \mathbb{Z}_p^\times$ may be written as $x = \omega(x) \cdot \langle x \rangle$. Here, the Teichmüller character ω is regarded as a character over $\mathbb{Z}_p^\times$ via the map $\mathbb{Z}_p^\times \overset{\bmod p}{\longrightarrow} \Delta_p \longrightarrow \mathbb{Z}_p^\times$ and $\langle x \rangle = x \cdot \omega(x)^{-1}$ is viewed as the projection of x onto the multiplicative group $1 + p\mathbb{Z}_p$.

We also recall the usual p-adic logarithmic function $\log_p$ which induces an isomorphism $(1 + p^n\mathbb{Z}_p, \times) \to (p^n\mathbb{Z}_p, +)$ for $n \geq 1$. Recall that γ is taken to be a topological generator of $1 + p\mathbb{Z}_p$ and define $\log_\gamma(y) := \log_p(y)/\log_\gamma(p)$. If $x \in \mathbb{Z}_p^\times$ is a lift of an element $\alpha \in (\mathbb{Z}/p^{n+1}\mathbb{Z})^\times$, then one can write $\alpha \equiv \omega(x) \cdot \langle x \rangle \equiv \omega(x) \cdot \gamma^{\log_\gamma(x)} \mod p^{n+1}$.

3.2 Motives over totally real fields

Motives or “motifs” originated from Grothendieck’s 1965 paper. The proposed idea is that motives are the “building blocks” of smooth projective algebraic varieties and that one can build a universal cohomology theory that underlies all pre-existing cohomologies. Grothendieck himself never published these ideas but he communicated them to Serre in the form of letters.

Among the many conjectures put forth by Langlands is one that predicts that a motive has an associated automorphic representation. In general, most properties of motives are still conjectural but we place ourselves in the motivic setting anyway to gain access to the uniformity that this framework offers (*e.g.*

the factor at infinity (see (3.5) and the p-adic multiplier (see Definition 3.1)).

Let M denote a pure motive over a number field F of degree $n = [F : \mathbb{Q}]$ whose coefficients lie in another number field T, and suppose M has rank $d = d(M)$. In other words, one has the following arithmetic data:

- $\mathcal{J}_F = \{\sigma : F \hookrightarrow \mathbb{C}\}$ – the set of complex embeddings of the number field;
- $M_{\mathrm{B},\sigma}$ – the Betti realisation of M (with respect to an embedding $\sigma \in \mathcal{J}_F$) which is a d-dimensional T-vector space with a T-rational involution ρ_σ;
- M_{dR} – the de Rham realisation of M, a free $T \otimes F$-module of rank d equipped with a decreasing filtration $\mathrm{Fil}^i_{\mathrm{dR}}(M) \subset M_{\mathrm{dR}}$ of $T \otimes F$-modules;
- $\{V_\ell^{\text{ét}}(M)\}$ – a compatible system of ℓ-adic realisations at finite places of T equipped with an action of the absolute Galois group $G_F := \mathrm{Gal}(\overline{F}/F)$.

Moreover, there is a complex comparison isomorphism of $T \otimes \mathbb{C}$-modules

$$I_{\infty,\sigma} : M_{\mathrm{B},\sigma} \otimes_{\mathbb{Q}} \mathbb{C} \overset{\sim}{\longrightarrow} M_{\mathrm{dR}} \otimes_{F,\sigma} \mathbb{C} \quad \text{at every } \sigma \in \mathcal{J}_F\,;$$

similarly, for each finite place ℓ there exists an ℓ-adic comparison isomorphism

$$I_{\ell,\sigma} : M_{\mathrm{B},\sigma} \otimes_T T_\ell \overset{\sim}{\longrightarrow} V_\ell^{\text{ét}}(M) \quad \text{of } T_\ell\text{-vector spaces.}$$

The Betti realisation admits a Hodge decomposition into $\mathbb{C}$-vector subspaces

$$M_{\mathrm{B},\sigma} \otimes_{\mathbb{Q}} \mathbb{C} = \bigoplus_{i,j} M_\sigma^{i,j}$$

for which $\rho_\sigma\big(M_\sigma^{i,j}\big) \subset M_\sigma^{i,j}$ at real embeddings $\sigma \in \mathcal{J}_F$, and the Hodge numbers $h(i,j) = \dim_{\mathbb{C}}\big(M_\sigma^{i,j}\big)$ are independent of σ. We have the further compatibility

$$I_{\infty,\sigma}\left(\bigoplus_{i' \geq i} M_\sigma^{i',j}\right) = \mathrm{Fil}^i_{\mathrm{dR}}(M) \otimes_{F,\sigma} \mathbb{C}.$$

Throughout, we assume M is a pure motive of weight w (and thus $i + j = w$).

One next introduces a pair of integers (m_*, m^*) by setting

$$m_* := \max\big\{i \in \mathbb{Z} \,\big|\, \text{there exists } j \text{ with } i < j \text{ such that } h(i,j) \neq 0\big\} + 1$$

$$m^* := \min\left\{ i \in \mathbb{Z} \;\middle|\; \text{there exists } j \text{ with } i > j \text{ such that } h(i,j) \neq 0 \right\}$$

so that $m^* - m_* + 2$ is the width of the critical strip for the complex L-function.

We shall also require the various operations on motives listed below:

(i) the dual motive $M^\vee$, whose ℓ-adic realisations satisfy the isomorphisms

$$V_\ell^{\text{ét}}(M^\vee) \cong V_\ell^{\text{ét}}(M)^* := \operatorname{Hom}\big(V_\ell^{\text{ét}}(M), T_\ell\big)$$

the right-hand side being the contragradient representation of $V_\ell^{\text{ét}}(M)$;

(ii) the tensor product $M_1 \otimes_F M_2$, whose ℓ-adic realisations are given by

$$V_\ell^{\text{ét}}(M_1 \otimes_F M_2) \cong V_\ell^{\text{ét}}(M_1) \otimes_{T_\ell} V_\ell^{\text{ét}}(M_2);$$

(iii) the restriction of scalars motive $R_{F/F'}M$, whose ℓ-adic realisations equal $V_\ell^{\text{ét}}(R_{F/F'}M) \cong \operatorname{Ind}_{G_{F'}}^{G_F}\big(V_\ell^{\text{ét}}(M)\big)$ where F' denotes a fixed subfield of F;

(iv) for a Hecke character χ of finite order, one considers the Artin motive $[\chi]$, whose l-adic realisations $[\chi]_\ell$ arise from the map $G_F^{\text{ab}} \xrightarrow{\text{rec}_F^{-1}} \mathbb{A}_F^\times/F^\times \xrightarrow{\chi} \overline{\mathbb{Q}}^\times$ where rec_F denotes the Artin reciprocity map;

(v) the cyclotomic Tate motive $F(1)$, whose ℓ-adic realisations are defined by the G_F-action on the l-power roots of unity (l: characteristic of ℓ);

(vi) lastly, for each $m \in \mathbb{Z}$ one has the motive $F(m)$, with ℓ-adic realisations

$$V_\ell^{\text{ét}}\big(F(m)\big) = \begin{cases} T \otimes \big(\mathbb{Z}_l(1)\big)^{\otimes m} & \text{if } m \geq 0 \\ T \otimes \big(\operatorname{Hom}(\mathbb{Z}_l(1), \mathbb{Z}_l)\big)^{\otimes -m} & \text{if } m < 0. \end{cases}$$

Henceforth assume that F is a totally real number field. For each embedding $\sigma : F \hookrightarrow \mathbb{R}$ one obtains a decomposition $M_{B,\sigma} = M_{B,\sigma}^+ \oplus M_{B,\sigma}^-$ into a direct sum of ± 1-eigenspaces, with corresponding dimensions $d^\pm(M)$ which are known to be independent of σ. One defines $\text{Fil}_{\text{dR}}^\pm(M) \subset M_{\text{dR}}$ as in [CPR89, p33] so that $d^\pm(M) = \dim\big(M_{\text{dR}}/\text{Fil}_{\text{dR}}^\pm(M)\big)$; in particular, $I_{\infty,\sigma}$ induces isomorphisms

$$I_{\infty,\sigma}^\pm : M_{\text{B},\sigma}^\pm \otimes_{\mathbb{Q}} \mathbb{C} \xrightarrow{\sim} M_{\text{dR}}/\text{Fil}_{\text{dR}}^\pm(M) \otimes_{F,\sigma} \mathbb{C} \quad \text{at any (every) } \sigma \in J_F.$$

Similar to Section 2.4, there is a fairly standard recipe to attach a zeta-function to a given motive $M_{/F}$. Provided $\mathrm{Re}(s) \gg 0$, the complex L-function can be defined via an Euler product

$$L(M,s) \ := \prod_{\mathfrak{v}\in\mathrm{Spec}(\mathcal{O}_F)} \det\Big(1 - \mathrm{Frob}_{\mathfrak{v}}^{-1}\Big| \ \big(V_\ell^{\text{ét}}(M)\big)^{I_{F_{\mathfrak{v}}}} \cdot \mathcal{N}(\mathfrak{v})^{-s}\Big)^{-1} \ \text{for any } \ell \neq \mathfrak{v};$$

One next defines the completed L-function

$$\Psi(M,s) := L_{\infty,\mathbb{R}}(M,s) \times L_{\infty,\mathbb{C}}(M,s) \times L(M,s) \tag{3.5}$$

where $L_{\infty,\mathbb{R}}(M,s) = \Gamma_{\mathbb{R}}(s-w/2)^{d^+(M)}\Gamma_{\mathbb{R}}(s+1-w/2)^{d^-(M)}$ and $L_{\infty,\mathbb{C}}(M,s) = \prod_{i<j}\Gamma_{\mathbb{C}}(s-i)^{h_{i,j}}$ with $h_{i,j}$ denoting the dimension of $(R_{F/\mathbb{Q}}M)_\sigma^{i,j}$.

The following conjecture by Deligne[Del73] predicts that motives exhibit exactly the same analytic properties as those discussed in Section 2.4.

Conjecture 3.4. *The function $\Psi(M,s)$ can be meromorphically continued over $\mathbb{C}$, and it should satisfy a functional equation*

$$\Psi(M,s) = \epsilon(M,s) \times \Psi(M^\vee, 1-s) \tag{3.6}$$

where $\epsilon(M,s) = \prod_{\mathfrak{v}} \epsilon_{\mathfrak{v}}(M,s)$ is a product of local factors. Furthermore, if $M_\sigma^{w/2,w/2} = 0$, then $\Lambda_\infty(M,s)$ has an analytic continuation to $\mathbb{C}$.

For example, if $M = F(m)[\chi] = F(m) \otimes_F [\chi]$ for some $m \in \mathbb{Z}$ and some Hecke character χ, then the element $\mathrm{Frob}_{\mathfrak{v}} \in G_{F_{\mathfrak{v}}}$ acts through multiplication by $\chi\mathcal{N}_{F/\mathbb{Q}}(\mathfrak{v})^m$ on $V_\ell^{\text{ét}}\big(F(m)\big)$. In particular, one has an identification of L-functions

$$L\big(F(m)[\chi],s\big) \ = \prod_{\mathfrak{v}\in\mathrm{Spec}(\mathcal{O}_F)} \frac{1}{1-\chi^{-1}(\mathfrak{v})\mathcal{N}_{F/\mathbb{Q}}(\mathfrak{v})^{-(s+m)}} \ = \ \zeta_F\big(\chi^{-1}, s+m\big)$$

which is the χ^{-1}-twisted Dedekind zeta-function, right-shifted by m. The factor at infinity equals $\Gamma_{\mathbb{R}}(s+m)^n$, while $L\big(F(m)[\chi],s\big)$ has a simple pole at $s = 1-m$ (with residue given by the analytic class number formula for F).

3.2.1 Conjecture of Coates, Perrin-Riou & Panchishkin

A usual approach for attaching a p-adic L-function to a motive M involves first establishing a version of Deligne's period conjecture [Del79]; as there are several formulations of this statement, we state a version that includes twists by finite order characters (c.f. [Pan94, §2]). Let F be totally real and denote by $\mathrm{Sgn}_F \subset F_\infty^\times$ the group of signs of F (i.e. the elements of order 2 in $F_\infty^\times$).

Conjecture 3.5. *For each choice* $\varepsilon_0 = \{\varepsilon_{0,\sigma}\} \in \mathrm{Sgn}_F$ *there exists a nonzero motivic period* $\Omega(\varepsilon_0, M) = \big(2\pi\sqrt{-1}\big)^{r(R_{F/\mathbb{Q}}M)} c^{\varepsilon_0}(R_{F/\mathbb{Q}}M) \in (T\otimes\mathbb{C})^\times$ *such that*

$$\Big(G(\chi)^{-1}\Big(1\otimes D_F^{1/2}\Big)\Big)^{d^{\varepsilon_0}(M)} \times \frac{\Psi(M(\chi),m)}{\Omega(\varepsilon_0,M)} \in T(\chi) \subset T\otimes\mathbb{C}$$

for all finite order Hecke characters χ *and every Tate twist* $m\in\mathbb{Z}$ *for which* $M(\chi)(m)$ *is critical at the point* $s=0$ *(here* $G(\chi)$ *is the Gauss sum for* χ*).*

Let us assume that $p \nmid 2D_F$ and that the motive $M_{/F}$ is ordinary at all $\mathfrak{p}|p$. One may refer to [CPR89, Definition 4.1] for the definition of p-ordinary motives, but suffice to say that a p-ordinary motive should correspond via the Langlands conjecture to a p-ordinary modular form (see Section 3.4.1). We fix embeddings $\iota_\infty : \overline{\mathbb{Q}} \hookrightarrow \mathbb{C}$ and $\iota_p : \overline{\mathbb{Q}}_p \hookrightarrow \mathbb{C}_p$, i.e. into the Tate field which is endowed with a p-adic norm $|\cdot|_p$ normalised so that $|p|_p = p^{-1}$.

Consider the $\mathbb{C}_p$-analytic Lie group $\mathfrak{X}_p = \mathrm{Hom}_{\mathrm{cont}}\big(\mathrm{Gal}\big(F_{p,\infty}^{\mathrm{ab}}/F\big), \mathbb{C}_p^\times\big)$ where $F_{p,\infty}^{\mathrm{ab}}$ is the maximal abelian extension of F unramified outside the primes over p and ∞. The elements of finite order $\chi \in \mathfrak{X}_p^{\mathrm{tors}}$ may be regarded as Hecke characters (whose conductor $\mathfrak{c}(\chi)$ contains only primes of F lying above p), upon composing the maps

$$\chi : \mathbb{A}_F^\times/F^\times \overset{\mathrm{C.F.T}}{\longrightarrow} \mathrm{Gal}\big(F_{p,\infty}^{\mathrm{ab}}/F\big) \longrightarrow \overline{\mathbb{Q}}^\times \overset{\iota_\infty}{\hookrightarrow} \mathbb{C}^\times.$$

One also has a Tate twist homomorphism $\mathcal{N}x_p \in \mathfrak{X}_p$ given as the norm character

$$\mathcal{N}x_p : \mathrm{Gal}\big(F_{p,\infty}^{\mathrm{ab}}/F\big) \longrightarrow \mathrm{Gal}\big(\mathbb{Q}_{p,\infty}^{\mathrm{ab}}/\mathbb{Q}\big) \overset{\sim}{\longrightarrow} \mathbb{Z}_p^\times \hookrightarrow \mathbb{C}_p^\times. \tag{3.7}$$

Panchishkin details the properties of p-adic measures defined on all of $\mathfrak{X}_p$ [Pan94]; in this chapter we restrict ourselves exclusively to the cyclotomic line

$$\mathrm{Hom}_{\mathrm{cont}}\big(\mathrm{Gal}\big(F_{p,\infty}^{\mathrm{cyc}}/F\big), \mathbb{C}_p^\times\big)$$

which we henceforth assign as the new $\mathfrak{X}_p$. Here $F_{p,\infty}^{\mathrm{cyc}}$ denotes the maximal cyclotomic extension of F lying inside of $F_{p,\infty}^{\mathrm{ab}}$. In particular, there is a short exact sequence of abelian groups

$$1 \to \mathrm{Gal}\big(F_{p,\infty}^{\mathrm{cyc}}/F\big)_0 \to \mathrm{Gal}\big(F_{p,\infty}^{\mathrm{cyc}}/F\big) \to \mathrm{Gal}\big(\big(F_{p,\infty}^{\mathrm{cyc}} \cap F_{\infty}^{\mathrm{ab}}\big)/F\big) \to 1 \tag{3.8}$$

where $\mathrm{Gal}\big(F_{p,\infty}^{\mathrm{cyc}}/F\big)_0 \cong \Delta_p \times (1 + p\mathbb{Z}_p)$.

Definition 3.1. For a finite order Hecke character $\mathfrak{X}_p$ of conductor $\mathfrak{c}(\chi)$, the $\mathfrak{p}$-*adic multiplier term* is given by the formula

$$\mathcal{A}_{\mathfrak{p}}(M(\chi), s) := \begin{cases} \displaystyle\prod_{i=1}^{d^+}\left(1 - \frac{\overline{\chi}(\mathfrak{p})}{\alpha^{(i)}(\mathfrak{p})}\mathcal{N}(\mathfrak{p})^{s-1}\right) & \\ \displaystyle\times \prod_{i=d^++1}^{d}\left(1 - \chi(\mathfrak{p})\alpha^{(i)}(\mathfrak{p})\mathcal{N}(\mathfrak{p})^{-s}\right) & \text{if } \mathfrak{p} \nmid \mathfrak{c}(\chi) \\ \displaystyle\prod_{i=1}^{d^+}\left(\frac{\mathcal{N}(\mathfrak{p})^s}{\alpha^{(i)}(\mathfrak{p})}\right)^{\mathrm{ord}_{\mathfrak{p}}\mathfrak{c}(\chi)} & \text{otherwise,} \end{cases}$$

for each prime ideal $\mathfrak{p} \in \mathrm{Spec}(\mathcal{O}_F)$ lying over the original prime p and the $\alpha^{(i)}(\mathfrak{p})$ are reciprocal roots of the polynomial $\det\Big(1 - \mathrm{Frob}_{\mathfrak{p}}^{-1}\,\Big|\,\big(V_\ell^{\text{ét}}(M)\big)^{I_{F_{\mathfrak{p}}}} \cdot X\Big)$.

The conjecture below was formulated by Coates and Perrin-Riou [CPR89; Coa91] over $F = \mathbb{Q}$, and generalised a few years later by Panchiskin [Pan94] to totally real number fields.

Conjecture 3.6. *For each choice $\varepsilon_0 = \{\varepsilon_{0,\sigma}\} \in \mathrm{Sgn}_F$ there exists a unique $\mathbb{C}_p$-analytic function $\mathbf{L}_p^{(\varepsilon_0)}(M)$ defined on $\mathfrak{X}_p$, interpolating at all but finitely many pairs (m, χ) for which the motive $M(\chi)(m)$ is critical at $s = 0$, the following data:*

$$\begin{aligned}&\chi\mathcal{N}x_p^m\Big(\mathbf{L}_p^{(\varepsilon_0)}(M)\Big) \\ &\qquad = \iota_p \circ \iota_\infty^{-1}\left(\frac{D_F^{md^{\varepsilon_0}(M)/2}}{G(\chi)^{d^{\varepsilon_0}(M)}} \times \prod_{\mathfrak{p}|p}\mathcal{A}_{\mathfrak{p}}\big(M(\chi), m\big) \times \frac{\Psi(M(\chi), m)}{\Omega(\varepsilon_0, M)}\right).\end{aligned} \tag{3.9}$$

If $M_\sigma^{w/2,w/2} = 0$, then $\mathbf{L}_p^{(\varepsilon_0)}$ is a bounded holomorphic function on $\mathfrak{X}_p$, otherwise it has only a finite number of poles (and hence it corresponds to a bounded pseudo-measure).

Remark 4.

1. One can replace the condition that the motive M is p-ordinary with a weaker condition on the Newton and Hodge polygons of M, namely that
$$P_{\mathrm{Newton},\sigma}\big(d^+(M)\big) = P_{\mathrm{Hodge},\sigma}\big(d^+(M)\big) \quad \text{for all } \sigma \in \mathcal{J}_F \ ,$$
and the conjecture above then coincides with Conjecture 6.2 in *op. cit.*
2. In the general case, the $\mathbb{C}_p$-analytic function $\mathbf{L}_p^{(\varepsilon_0)}(-)$ should be the Mellin transform of an h-admissible measure on $\mathfrak{X}_p$, i.e. a function of type $o\big(\log_p^h\big)$ where
$$h = h_p(M) \ := \ \max_{\sigma \in \mathcal{J}_F} \big[P_{\mathrm{Newton},\sigma}\big(d^\pm(M)\big) - P_{\mathrm{Hodge},\sigma}\big(d^\pm(M)\big)\big] + 1.$$
3. Provided that $h_p(M) \leq m^* - m_* + 1$ holds, the p-adic L-function $\mathbf{L}_p^{(\varepsilon_0)}(-)$ is uniquely determined by its critical values at those characters of the form $\chi \mathcal{N} x_p^m$.

3.3 p-adic L-functions

Let h be a nonnegative integer and let $\mathfrak{p} \in \mathrm{Spec}(\mathcal{O}_F)$ lying over p. An *h-admissible measure* μ is a linear map from the space of $\mathbb{C}_p$-valued functions that are locally polynomials of degree $\leq h$ to $\mathbb{C}_p$, satisfying the growth condition

$$\sup_{a \in \mathrm{Gal}_p} \left|\mu(x-a)^j \cdot \mathbb{1}_{a+(\mathfrak{p}^n)}\right| = o\big(|\mathfrak{p}^n|^{(j-h)}\big), \quad \text{for all } 0 \leq j < h \tag{3.10}$$

where $a + (\mathfrak{p}^n) = \{x \in \mathrm{Gal}_p : x \equiv a \bmod \mathfrak{p}^n\}$ and $\mathbb{1}_{a+(\mathfrak{p}^n)}$ denotes the characteristic function on the given set. When $h = 1$, μ is a bounded measure.

Amice-Vélu and Višik [AV75; Viš76] introduced h-admissible measures as a means to construct p-adic L-functions associated to classical modular forms. More precisely, a p-adic L-function is a function taking a continuous p-adic character $\chi \in \mathfrak{X}_p$ and integrating it against a given h-admissible measure μ:

$$\begin{aligned} L_p : \mathfrak{X}_p &\longrightarrow \mathbb{C}_p^\times \\ \chi &\longmapsto \int_{\mathrm{Gal}_p} \chi \, \mathrm{d}\mu. \end{aligned}$$

The growth condition (3.10) ensures that L_p is analytic and $o\big(\log_p(1+T)^h\big)$ (c.f. [Viš76, Theorem 2.3]. Moreover, L_p is completely determined by its values on $\chi\mathcal{N}x_p^j$ for finite order characters $\chi \in \mathfrak{X}_p^{\mathrm{tors}}$ and $0 \leq j \leq h-1$.

We recall that finite order character may be written as $\chi = \phi \cdot \psi$ for some $\phi : \Delta_p \to \mathbb{C}_p^\times$ and some $\psi : \mathrm{Gal}_p \to (1+p\mathbb{Z}_p, \times) \to \mathbb{C}_p^\times$. The character ϕ is referred to as the *tame* or *branch character* and it can always be written as ω^t for some integer $0 \leq t \leq p-2$ and where ω is the Teichmüller character. The character ψ on the other hand is called the *wild character.* The Tate twist homomorphism $\mathcal{N}x_p$ in (3.7) corresponds to the character $\omega^{-1}\kappa_{\mathrm{cyc}}\big|_{\mathrm{Gal}_p}$ restricted to Gal_p.

In Section 2.4 we showed instances wherein the functional equation in Conjecture 3.4 is satisfied by tensor products of Hilbert modular forms. For the remainder of this section, we discuss the corresponding (conjectural) p-adic functional equation. Let M be a pure motive and $\varepsilon_0 \in \mathrm{Sgn}_F$ as in Section 3.2. In what follows, we consider tuples $\mathbf{L}_p^{(\varepsilon_0)}(M) = \big\{\mathbf{L}_p^{(\varepsilon_0),\phi}(M)\big\}_\phi$ where each $\mathbf{L}_p^{(\varepsilon_0),\phi}(M) : 1+p\mathbb{Z}_p \to \mathbb{C}_p$ indicates a branch of the p-adic L-function. We begin by stating the hypothetical p-adic functional equation in the language of motives and then we give a concrete example featuring Hilbert modular forms.

Let $\Sigma^{\mathrm{bad}}(M)$ be the primes of $\mathcal{O}_F$ dividing the conductor of M, excluding those that divide p. For a given subset $S \subset \Sigma^{\mathrm{bad}}(M)$, we recall that the S-imprimitive complex L-function $L_{(S)}(M,s)$ is related to the usual L-function by $L_{(S)}(M,s) = L(M,s) \times \prod_{\mathfrak{v}\in S} L_{\mathfrak{v}}(M,s)$ where $L_{\mathfrak{v}}(M,s)$ denotes the local L-factor at $\mathfrak{v}$. We also consider the depleted p-adic L-function $\mathbf{L}_{p,(S)}^{(\varepsilon_0)}(M)$ as interpolating exactly the same values in Conjecture 3.6, except $\Psi(M(\chi),m)$ is now replaced by

$$\Psi(M(\chi),m) \times \prod_{\mathfrak{v}\in S} L_{\mathfrak{v}}(M(\chi),m)$$

(the existence and holomorphy of $\mathbf{L}_{p,(S)}^{(\varepsilon_0)}(M)$ is implied by Conjecture 3.6).

Henceforth assume that Conjectures 3.4, 3.5, 3.6 hold for the motive $M_{/F}$ i.e. the holomorphy and complex functional equation for $\Psi(M(\chi),s)$, Deligne's period conjecture, and the existence of $\mathbf{L}_p^{(\varepsilon_0)}(M)$ each hold true. We write out the complex functional equation (3.6) at the critical point $s = m$:

$$\Psi(M(\chi), m) = \epsilon(M(\chi), m) \times \Psi\big(M^\vee(\chi^{-1}), 1-m\big)\,.$$

This equation induces a non-trivial p-adic relation between the special values

$$\chi\mathcal{N}x_p^m\Big(\mathbf{L}_p^{(\varepsilon_0)}(M)\Big) = w_p(M(\chi), m) \times \chi^{-1}\mathcal{N}x_p^{1-m}\Big(\mathbf{L}_p^{(-\varepsilon_0)}(M^\vee)\Big) \tag{3.11}$$

at all but finitely many (m, χ) for which the motive $M(\chi)(m)$ is critical at $s = 0$. By the formula given in Conjecture 3.6, the factor $w_p(M(\chi), m)$ equals

$$\iota_p\left(\epsilon(M(\chi), m) \times \frac{\Omega(-\varepsilon_0, M^\vee)}{\Omega(\varepsilon_0, M)} \times \frac{D_F^{md^{\varepsilon_0}(M)/2}}{D_F^{(1-m)d^{\varepsilon_0}(M^\vee)/2}} \times \prod_{\mathfrak{p}|p} \frac{A_\mathfrak{p}(M(\chi), m)}{A_\mathfrak{p}\big(M^\vee(\chi^{-1}), 1-m\big)}\right).$$

3.3.1 A single HMF

Let $\mathbf{f} \in \mathcal{S}_k(\mathfrak{c}, \eta)$ be a newform as in Section 2.4.1. The motive $M(\mathbf{f})$ attached to $\mathbf{f}$ (due to Scholl [Sch90] over $F = \mathbb{Q}$ and various others if $F \neq \mathbb{Q}$) has rank two, weight $w = k-1$ and coefficient field $T = F\big(\mathrm{C}(\mathfrak{m}, \mathbf{f}) : \mathfrak{m}\big)$, i.e. the minimal extension of F containing the Fourier coefficients of $\mathbf{f}$. The complex L-functions of $\mathbf{f}$ and $M(\mathbf{f})$ are related by $L(M(\mathbf{f}), s) = L(\mathbf{f}, s + \frac{w}{2})$.

By generalising the ideas in [AV75; Viš76] to Jacquet-Langlands L-functions associated to automorphic representations in GL_2, Manin constructed a unique element $L_p(\mathbf{f}) \in \mathcal{O}_{T,p}[\Delta][\![\gamma - 1]\!] \otimes \mathbb{Q}$ interpolating

$$\chi\mathcal{N}x_p^m(L_p(\mathbf{f})) = \mathcal{B}(\mathbf{f}, \chi, m) \times \prod_{\mathfrak{p}|p} \mathcal{A}_\mathfrak{p}\big(M(\mathbf{f}), \chi, 1+m\big) \times \frac{\Psi(\mathbf{f}, \chi, 1+m)}{\Omega^{\mathrm{sgn}(\chi)}(\mathbf{f})} \tag{3.12}$$

at every Tate twist $m \in \{0, 1, \ldots, k-2\}$ and for each $\chi \in \mathfrak{X}_p^{\mathrm{tors}}$ [Man76]. Here, $\Psi(\mathbf{f}, \chi, s)$ is the completed L-function (2.34) of the the χ-twist of $\mathbf{f}$ while the exact forms of $\mathcal{B}(-)$, $\prod_{\mathfrak{p}|p} \mathcal{A}_\mathfrak{p}(-)$ and $\Omega^{\mathrm{sgn}(\chi)}(\mathbf{f})$ shall be supplied in Section 3.4.1 (see also [Pan94, Theorem 8.2]). Since the dual motive $M(\mathbf{f})^\vee$ can be identified with $M(\mathbf{f}^\rho)$, the p-adic functional equation becomes

$$\chi\mathcal{N}x_p^m\big(L_p(\mathbf{f})\big) = w_p\big(M(\mathbf{f})(\chi), m\big) \times \chi^{-1}\mathcal{N}x_p^{k-2-m}\big(L_p(\mathbf{f}^\rho)\big). \tag{3.13}$$

at all such critical twists (m, χ).

Remark 5. We replace $\Psi(\mathbf{f},\chi,m)$ in [Pan94] by $\Psi(\mathbf{f},\chi,1+m)$ in (3.12). This is to be consistent with the fact that $L_p(\mathbf{f})$ is the limit of p-adic polynomials interpolating the modular symbols associated to $\mathbf{f}$, and evaluating at $\chi\mathcal{N}x_p^m$ gives the m-th coefficient of the modular symbol which encodes the special value $\Psi(\mathbf{f},\chi,1+m)$. This will be discussed thoroughly in Chapter 6.

An analogues for the the symmetric square representation $\mathrm{Sym}(\mathbf{f})$ and the tensor products $\mathbf{f}_1\times\mathbf{f}_2$ and $\mathbf{f}_1\times\mathbf{f}_2\times\mathbf{f}_3$ of Hilbert modular forms can be found in Chapter 7 where we will be requiring knowledge of their corresponding p-adic functional equations. We shall postpone the discussion until then.

3.4 IMC for modular forms

The version of Iwasawa Main Conjecture we shall now introduce below is synthesised by Greenberg in a string of papers (see for example [Gre94]). This is the version that we shall need in Chapters 4 and 5. It is worthwhile to mention that between this version and the classical one stated at the beginning of the chapter, another variant of the main conjecture for elliptic curves was formulated by Mazur and Swinnerton-Dyer (see [MSD74] for details).

3.4.1 Ordinary case

Let p be the odd prime we fixed in the beginning of this chapter. In this section, we shall consider a Hilbert modular form $\mathbf{f}\in\mathcal{S}_k(\mathfrak{c},\eta)$ with Fourier expansion $\sum_{\mathfrak{m}}\mathrm{C}(\mathbf{f},\mathfrak{m})\cdot\mathcal{N}(\mathfrak{m})^{-s}$. Let K be the minimal extension of $\mathbb{Q}_p$ containing the image of $\mathrm{C}(\mathbf{f},\mathfrak{v})$ under the fixed embedding ι_p for all $\mathfrak{v}\in\mathrm{Spec}(\mathcal{O}_F)$ and let $\mathcal{O}_K$ be its discrete valuation ring. We choose a uniformizer ϖ of the maximal ideal $\mathfrak{m}_K$ of $\mathcal{O}_K$ and we denote by $\mathbb{F}_K:=\mathcal{O}_K/\mathfrak{m}_K$ the residue field.

Let us recall a few other notations. For any discrete G-module M, one writes $H^i(G,M)$ for its i-th cohomology group. If L/K denotes an algebraic extension of fields and $G=\mathrm{Gal}(L/K)$, we abbreviate this cohomology group by $H^i(L/K,M)$. Furthermore, one defines $H^i(K,M):=H^i\big(G_K,M\big)$, where as usual G_K denotes the absolute Galois group $\mathrm{Gal}(\overline{K}/K)$.

Suppose $\mathbf{f}$ satisfies the following conditions:

(a) $\mathbf{f}$ is p-ordinary, i.e., $\mathrm{C}(\mathbf{f}, p)$ is a p-adic unit; and

(b) the residual representation $\overline{\rho}_{\mathbf{f}} : G_{\mathbb{Q}} \to \mathrm{GL}_2(\mathbb{F}_K)$ is absolutely irreducible.

Let us denote by $V_{\mathbf{f}}$ the Deligne–Taylor p-adic representation of $\mathbf{f}$. Condition (b) implies that there is a unique lattice $T_{\mathbf{f}} \subset V_{\mathbf{f}}$ (up to isomorphism) which is $G_{F_{\mathfrak{p}}}$-stable for each prime ideal $\mathfrak{p} \mid p$. We write $W_{\mathbf{f}} = \mathrm{Hom}(T_{\mathbf{f}}, K/\mathcal{O}_K)(1)$.

Let us begin setting up the algebraic half of the main conjecture. The p-ordinary condition ensures the existence of a $G_{F_{\mathfrak{p}}}$-stable one-dimensional subspace $V_{\mathbf{f}}^+ \subset V_{\mathbf{f}}$. The corresponding lattice $T_{\mathbf{f}}^+ = T_{\mathbf{f}} \cap V_{\mathbf{f}}^+$ induces a short exact sequence of $G_{F_{\mathfrak{p}}}$-modules (c.f. [Gro90, Proposition 12.1]):

$$0 \longrightarrow W_{\mathbf{f}}^+ \longrightarrow W_{\mathbf{f}} \longrightarrow W_{\mathbf{f}}^- \longrightarrow 0. \tag{3.14}$$

where $W_{\mathbf{f}}^+ = \mathrm{Hom}(T_{\mathbf{f}}^+, K/\mathcal{O}_K)(1)$ and $W_{\mathbf{f}}^-$ is the ètale quotient $W_{\mathbf{f}}/W_{\mathbf{f}}^+$ of negative Hodge-Tate weight. Greenberg's *p-primary Selmer group* attached to a p-ordinary modular form $\mathbf{f}$ is defined as (c.f. [Gre89, §2]):

$$\mathrm{Sel}_{\mathrm{Gr}}(F, \mathbf{f}) := \ker\left(H^1(F, W_{\mathbf{f}}) \longrightarrow \prod_{v \in \mathrm{Spec}(\mathcal{O}_F)} \frac{H^1(F_v, W_{\mathbf{f}})}{H^1_{f,\mathrm{Gr}}(F_v, W_{\mathbf{f}})} \right) \tag{3.15}$$

where the local conditions are given by

$$H^1_{f,\mathrm{Gr}}(F_v, W) = \begin{cases} \ker\left(H^1(F_v, W_{\mathbf{f}}) \longrightarrow H^1(I_{F_v}, W_{\mathbf{f}})\right) & \text{for } v \nmid p, \\ \ker\left(H^1(F_v, W_{\mathbf{f}}) \longrightarrow H^1(I_{F_v}, W_{\mathbf{f}}^-)\right) & \text{for } v \mid p. \end{cases}$$

Recall the n-th layers $\{F_n\}_{n \geq 1}$ of the cyclotomic $\mathbb{Z}_p$-extension F^{cyc} at the beginning of this chapter and write $H^1(F^{\mathrm{cyc}}, W_{\mathbf{f}}) = \varinjlim_n H^1(F_n, W_{\mathbf{f}})$. The p-primary Selmer group of the p-ordinary form $\mathbf{f}$ over F^{cyc} is given by

$$\mathrm{Sel}_{\mathrm{Gr}}(F^{\mathrm{cyc}}, \mathbf{f}) := \ker\left(H^1(F^{\mathrm{cyc}}, W_{\mathbf{f}}) \longrightarrow \prod_{v \in \mathrm{Spec}(\mathcal{O}_F)} \frac{H^1(F_v^{\mathrm{cyc}}, W_{\mathbf{f}})}{H^1_{f,\mathrm{Gr}}(F_v^{\mathrm{cyc}}, W_{\mathbf{f}})} \right) \tag{3.16}$$

We write $\mathrm{Sel}_{\mathrm{Gr}}(F^{\mathrm{cyc}}, \mathbf{f})^{[t]}$ for the ω^t-isotypic component of $\mathrm{Sel}_{\mathrm{Gr}}(F^{\mathrm{cyc}}, \mathbf{f})$. The cyclotomic Selmer group is a discrete cofinitely-generated $\Lambda_{\mathcal{O}_K}$-module (c.f.

[Gre89, Proposition 6]) which means that its Pontryagin dual

$$\mathcal{X}_{\mathrm{Gr}}(\mathbf{f})^{[t]} := \mathrm{Sel}_{\mathrm{Gr}}(F^{\mathrm{cyc}}, \mathbf{f})^{\vee,[t]} = \mathrm{Hom}\left(\mathrm{Sel}_{\mathrm{Gr}}(F^{\mathrm{cyc}}, \mathbf{f})^{[t]}, \mathbb{Q}_p/\mathbb{Z}_p\right)$$

has the structure of a compact finitely-generated $\Lambda_{\mathcal{O}_K}$-module.

We recall the definition of Iwasawa invariants.

Definition 3.2. For a nonzero element $\mathscr{G}(X) = \sum_{m\geq 0} c_m X^m \in \mathcal{O}[\![X]\!]$, the μ-invariant and λ-invariant of $\mathscr{G}$ are defined as

$$\mu(\mathscr{G}) := \min_m \left\{ \mathrm{ord}_p(c_m)\right\} \quad \text{and}$$
$$\lambda(\mathscr{G}) := \min\left\{m : \mathrm{ord}_p(c_m) = \mu(\mathscr{G})\right\}.$$

The cyclotomic p-primary Selmer group in the case of ordinary modular forms is always $\Lambda_{\mathcal{O}_K}$-cotorsion (c.f. [Kat04]). Applying Structure Theorem 3.2 to $\mathcal{X}_{\mathrm{Gr}}(\mathbf{f})^{[t]}$ gives

$$\mathrm{char}_{\Lambda_{\mathcal{O}_K}}\left(\mathcal{X}_{\mathrm{Gr}}(\mathbf{f})^{[t]}\right) = \prod_{i=1}^{c} p^{\mu_i} \times \prod_{j=1}^{d} \mathscr{F}_j(X)^{\lambda_j} \tag{3.17}$$

which allows one to extract the following algebraic Iwasawa invariants.

Definition 3.3. Let f be a p-ordinary modular form. For $0 \leq t \leq p-2$ we define the *algebraic* μ*- and* λ*-invariants* of $\mathbf{f}$ over the branch character ω^t as

$$\mu(\mathbf{f}, \omega^t)_{\mathrm{alg}} := \sum_{i=1}^{c} \mu_i \quad \text{and}$$
$$\lambda(\mathbf{f}, \omega^t)_{\mathrm{alg}} := \sum_{j=1}^{d} \lambda_j \cdot \deg(\mathscr{F}_j),$$

respectively.

Next, we give some background on the analytic half of the main conjecture. The construction of Amice-Vélu and Višik for the field $F = \mathbb{Q}$ tells us that given an allowable root Υ of $X^2 - a_p(\mathbf{f})X + \epsilon(p)p^{k-1}$ (i.e. a root Υ with $\mathrm{ord}_p(\Upsilon) < k-1$) then one can construct an h-admissible measure from which the p-adic L-function is uniquely defined (c.f. [Viš76, Lemma 3.8]). This idea was subsequently extended by Manin for Hilbert modular forms [Man76].

Since $\mathbf{f}$ is p-ordinary, there is a unique p-adic root $\Upsilon = \alpha_p(\mathbf{f})$ of the Hecke polynomial. The resulting p-adic L-function $L_p(\mathbf{f})$ lies inside $\mathcal{O}_K[\Delta_p][\![X]\!] \otimes K$ and has the following interpolation property: for χ a finite order Hecke character of conductor $\mathfrak{p}^v$ (with $\mathfrak{p} \mid p$) one obtains

$$\chi\mathcal{N}x_p^j\big(L_p(\mathbf{f})\big) = \frac{D_F^j i^{nj}}{G(\chi)} \times \prod_{\mathfrak{p}|p} \mathcal{A}_{\mathfrak{p}}\big(\mathbf{f}(\chi), 1+j\big) \times \frac{\Psi\big(\mathbf{f}(\chi), j+1\big)}{\Omega(\varepsilon_0, M)} \tag{3.18}$$

where the p-adic multiplier term is given by

$$\mathcal{A}_{\mathfrak{p}}\big(\mathbf{f}(\chi), 1+j\big) = \left(\frac{\mathcal{N}(\mathfrak{p})^{1+j}}{\alpha_{\mathfrak{p}}(\mathbf{f})}\right)^v \left(1 - \frac{\chi^{-1}(\mathfrak{p})\mathcal{N}(\mathfrak{p})^{k-2-j}}{\alpha_{\mathfrak{p}}(\mathbf{f})}\right) \left(1 - \frac{\chi(\mathfrak{p})\mathcal{N}(\mathfrak{p})^j}{\alpha_{\mathfrak{p}}(\mathbf{f})}\right).$$

We remind the reader that Equation (3.18) agrees with the formula in [Pan94, Theorem 8.2(i)] after applying a shift $m = j+1$ to the right-hand side of the equation (see Remark 5).

The element $L_p(\mathbf{f})$ in Equation 3.18 is treated as a p-adic distribution rather than as a power series. The power series representation in $\mathcal{O}_K[\Delta][\![X]\!]$ arises upon applying the Amice transform to $L_p(\mathbf{f})$. Let us take $\chi\mathcal{N}x_p^j$ where once more $\chi = \omega^t \cdot \psi$: evaluating at the fixed branch character ω^t gives us $L_p\big(\mathbf{f}, \omega^t, X\big) \in \mathcal{O}_K[\![X]\!]$; evaluating next at $\psi\mathcal{N}x_p^j$ where ψ is an arbitrary character and $0 \le j \le k-2$ results in $L_p\big(\mathbf{f}, \omega^{t-j}, u^j(1+X)-1\big) \in \mathcal{O}_K[\![X]\!]$.

Definition 3.4. Let f be a p-ordinary modular form. For $0 \le t \le p-2$ we define the *analytic* μ- and λ-*invariants* of $\mathbf{f}$ over the branch character ω^t as

$$\mu(\mathbf{f}, \omega^i)_{\text{an}} := \mu\big(L_p(\mathbf{f}, \omega^t, X)\big) \quad \text{and}$$
$$\lambda(\mathbf{f}, \omega^i)_{\text{an}} := \lambda\big(L_p(\mathbf{f}, \omega^t, X)\big),$$

respectively

We are now ready to state the Iwasawa main conjecture for p-ordinary Hilbert modular forms.

Conjecture 3.7. *Let* $\mathbf{f} \in \mathcal{S}_k(\mathfrak{c}, \eta)$ *be a Hilbert modular form which satisfies Conditions (a) and (b) at the beginning of Section 3.4.1. For each integer* $0 \le t \le p-2$, *there is an equality*

$$\text{char}_{\Lambda_{\mathcal{O}_K}}\big(\mathcal{X}_{\text{Gr}}(\mathbf{f})^{[t]}\big) = L_p\big(\mathbf{f}, \omega^t, X\big),$$

up to a unit factor over the Iwasawa algebra $\Lambda_{\mathcal{O}_K}$.

The above conjecture is now a theorem for a large group of classical modular forms f thanks to the works of Kato and Skinner-Urban. More precisely, Kato formulated an alternative version of the Iwasawa main conjecture without p-adic L-functions and demonstrated one-divisibility, i.e. the characteristic polynomial of $\mathrm{Sel}_{\mathrm{BK}}(\mathbb{Q}^{\mathrm{cyc}}, f)^{\vee,[t]}$ (see (3.19) below) divides into $L_p\big(f, \omega^t, X\big)$ [Kat04]. A decade later, Skinner and Urban proved the reverse-divisibility [SU14] with Greenberg's Selmer group. Even with two different Selmer groups, there should be no discrepancy between [Kat04] and [SU14] since $\mathrm{Sel}_{\mathrm{BK}}(\mathbb{Q}^{\mathrm{cyc}}, f)$ is a finite index subgroup of $\mathrm{Sel}_{\mathrm{Gr}}(\mathbb{Q}^{\mathrm{cyc}}, f)$ (c.f. [Och00, Proposition 4.2]) and therefore they generate the same characteristic ideal over $\Lambda_{\mathcal{O}_K}$.

3.4.2 The main conjecture of Kato and Perrin-Riou

As mentioned above, Kato presented an equivalent version of Conjecture 3.7 without p-adic L-functions in [Kat04]. This completely algebraic formulation attributed to Kato and Perrin-Riou was done with the use of Euler systems.

Simply put, an *Euler system* for a motive M refers to a norm-compatible system of zeta-elements

$$\underline{\mathbf{z}}(M) = \big\{\mathbf{z}_{p^m}(M)\big\}_{m\geq 1} \in \varprojlim_{m} H^1_{\text{ét}}\big(\mathbb{Z}[\mu_{p^m}, 1/Np], T\big)$$

where $T \subset V_\ell^{\text{ét}}(M)$ denotes a Galois stable lattice, N divides the conductor of M, and the inverse limit is taken with respect to the natural corestriction maps [Rub00, Chapter II.1].

In order to state Kato's main conjecture, we first specialize to modular forms. Let V_f be the two-dimensional Deligne representation of $G_{\mathbb{Q}}$ attached to f. We take a $G_{\mathbb{Q}}$-stable lattice $T_f \subset V_f$ and we write $W_f = \mathrm{Hom}(T_f, K/\mathcal{O}_K)(1)$ for the p-divisible version. Let us consider

$$\mathbb{H}^i(T) := \varprojlim_{m} H^i_{\text{ét}}\big(\mathbb{Z}[\zeta_{p^m}, 1/p], T\big) \quad \text{and} \quad \mathbb{H}^i(V) := \mathbb{H}^i(T) \otimes_{\mathbb{Z}} \mathbb{Q}$$

where the inverse limit is taken over corestriction maps.

According to Theorem 12.5(1) in [Kat04], there exists a unique K_f-linear map $V_f \longrightarrow \mathbb{H}^1(V_f)$ sending $v \mapsto \mathbf{z}_v$. Let us fix an element $v \in V_f$ so that its image $v^{\pm}$ in the $\pm$-eigenspace $V_f^{\pm}$ is nonzero. We denote by $\mathbb{Z}(T_f) \subset \mathbb{H}^1(V_f)$ the Λ-module generated by $\mathbf{z}_{v^{\pm}} \in T_f$. The following conjecture is a restatement of Conjecture 12.10 in *op. cit.*.

Conjecture 3.8. *The inclusion* $\mathbb{Z}(T_f) \subset \mathbb{H}^1(T_f)$ *holds and for a branch character* $\phi : \Delta_p \to \mathbb{C}_p^{\times}$, *there is an equality*

$$\mathrm{char}_{\Lambda_{\mathcal{O}_K}}\left(\frac{\mathbb{H}^1(T_f)^{\phi}}{\mathbb{Z}(T_f)^{\phi}}\right) = \mathrm{char}_{\Lambda_{\mathcal{O}_K}}\big(\mathbb{H}^2(T_f)^{\phi}\big)$$

up to a unit factor over the Iwasawa algebra $\Lambda_{\mathcal{O}_K}$.

We recall the definition of the *Bloch-Kato p-Selmer group* in [BK90, §3]:

$$\mathrm{Sel}_{\mathrm{BK}}(\mathbb{Q}, f) := \ker\left(H^1(\mathbb{Q}, W_f) \longrightarrow \prod_{v \text{ prime}} \frac{H^1(\mathbb{Q}_v, W_f)}{H^1_{f,\mathrm{BK}}(\mathbb{Q}_v, W_f)}\right) \tag{3.19}$$

where the local conditions are given by

$$H^1_{f,\mathrm{BK}}(\mathbb{Q}_v, W_f) = \begin{cases} \ker\left(H^1(\mathbb{Q}_v, W_f) \longrightarrow H^1(I_{\mathbb{Q}_v}, W_f)\right) & \text{for } v \nmid p, \\ \ker\big(H^1_{f,\mathrm{BK}}(\mathbb{Q}_v, V_f) \longrightarrow H^1(I_{\mathbb{Q}_v}, W_f)\big) & \text{for } v \mid p. \end{cases} \tag{3.20}$$

and $H^1_{f,\mathrm{BK}}(\mathbb{Q}_v, V) = \ker\big(H^1(\mathbb{Q}_v, V) \longrightarrow H^1(\mathbb{Q}_v, V_f \otimes_{\mathbb{Q}_p} B_{\mathrm{cris}})\big)$. We denote by $\mathcal{X}_{p,\mathrm{BK}}(\mathbf{f})$ the corresponding Pontryagin dual of $\mathrm{Sel}_{\mathrm{BK}}(\mathbb{Q}, f)$. As before, we add the supercript $[t]$ to indicate that we are only considering the ω^t-isotypic component.

Fix a branch character ϕ. The Poitou-Tate duality yields an exact sequence

$$\begin{aligned} 0 \longrightarrow \mathbb{H}^1(T_{\bar{f}}(k-1))^{\phi} \longrightarrow & \frac{H^1_{\mathrm{Iw}}(\mathbb{Q}_p, T_{\bar{f}}(k-1))^{\phi}}{\varprojlim_n H^1_f(\mathbb{Q}_p(\zeta_{p^n}), T_f(k-1))^{\phi}} \\ & \longrightarrow \mathcal{X}_{\mathrm{BK}}(f)^{\phi} \longrightarrow \mathbb{H}^2(T_{\bar{f}}(k-1))^{\phi} \longrightarrow 0. \end{aligned} \tag{3.21}$$

We briefly explain how the above conjecture is equivalent to Conjecture 3.7. A combination of Theorems 17.11 and 16.6 in [Kat04] yields a non-canonical isomorphism

$$\Lambda_{\mathcal{O}_K} \cong H^1_{\mathrm{Iw}}(\mathbb{Q}_p, T_{\bar{f}}(k-1))^{\phi} / \varprojlim_n H^1_f(\mathbb{Q}_p(\zeta_{p^n}), T_f(k-1))^{\phi}$$

which sends the image of $\mathbb{Z}(T_{\bar{f}}(k-1))^{\phi}$ to $\Lambda \cdot L_p(f,\phi)$ via Perrin-Riou's regulator map (see Definition 8.1). The equivalence between Conjectures 3.7 and 3.8 follows shortly.

The fact that Kato's main conjecture does not assume p-ordinarity became the impetus for a string of results in the p-non-ordinary direction. We discuss this in greater detail in Chapter 6.

3.5 Equivalence between IMCs

Fix an odd prime $p \geq 3$ and let $\mathbf{f} \in \mathcal{S}_k(\mathfrak{c}(\mathbf{f}), \eta)$ be a Hilbert modular eigenform over a totally real number field F. Recall our notation $K/\mathbb{Q}_p$ for the coefficient field of $\mathbf{f}$ and $\mathcal{O}_K$ for the corresponding discrete valuation ring. Assume further that $\mathbf{f}$ is normalized so that the representation

$$\rho_{\mathbf{f}} : G_F \to \mathrm{GL}_2(\mathcal{O}_K)$$

is unramified outside $p \cdot \mathfrak{c}(\mathbf{f})$ such that for each prime of F not dividing $p \cdot \mathfrak{c}(\mathbf{f})$,

$$\begin{aligned} \mathrm{Tr}(\rho(\mathrm{Frob}_{\mathfrak{v}}^{-1})) &= \mathrm{C}(\mathbf{f}, \mathfrak{v}) \quad \text{and} \\ \det(\rho(\mathrm{Frob}_{\mathfrak{v}}^{-1})) &= \eta(\mathfrak{v}) \cdot \mathcal{N}(\mathfrak{v})^{k-1}. \end{aligned} \tag{3.22}$$

The Hilbert modular eigenform $\mathbf{f}$ is said to be *p-congruent* to another Hilbert modular eigenform $\mathbf{g}$ if there is an isomorphism $\overline{\rho}_{\mathbf{f}} \cong \overline{\rho}_{\mathbf{g}}$ between their residual Galois representations. It follows from (3.22) that this is equivalent to

$$\mathrm{C}(\mathbf{f}, \mathfrak{m}) \equiv \mathrm{C}(\mathbf{g}, \mathfrak{m}) \mod p \quad \text{for each } \mathfrak{m} \subset \mathcal{O}_F \text{ with } \mathfrak{m} + \mathfrak{c}(\mathbf{f}) \cdot \mathfrak{c}(\mathbf{g}) = \mathcal{O}_F. \tag{3.23}$$

It was shown in the seminal work of Vatsal [Vat98] that given a pair of p-ordinary (classical) Hecke eigenforms satisfying the stronger condition that congruence (3.23) holds for *all* positive integers m, their corresponding p-adic L-functions are congruent modulo p. A natural question that arises is whether this property is also exhibited by the corresponding Selmer groups. In the case of rational elliptic curves, this was answered in the affirmative by Greenberg and Vatsal and they showed that the algebraic and analytic Iwasawa invariants behave in parallel manner [GV00]. With Kato's one-divisibility result, one may then deduce that if IMC holds for one elliptic curve then it holds for the other. In this case we say that there is an "equivalence of Iwasawa main conjectures".

These ideas have been extended over the intervening years by many authors [EPW06; CKL17; CK17; Del20; DL20; Del21; Del24; RSV23] to allow variation in the weight of the newforms, as well as including more exotic fields e.g. anticyclotomic $\mathbb{Z}_p^d$-extensions, or non-abelian p-adic Lie extensions.

Chapter 4

The τ-component in the IMC: Analytic theory

Fix a prime number $p \neq 2$ and let E_1 and E_2 be a pair of modular elliptic curves defined over a totally real number field F. We write $\rho_{E_i} : G_F \to \mathrm{GL}_2(\mathbb{Z}_p)$ for the two-dimensional representation of the absolute Galois group G_F arising from its action on the p-adic Tate module of E_i for $i = 1, 2$. The goal of Chapters 4 and 5 is to prove an analogue of Greenberg and Vatsal's result when ρ_{E_1} and ρ_{E_2} are twisted by a fixed Artin representation $\tau : G_F \to \mathrm{GL}_2(\overline{\mathbb{Q}})$.

4.1 Main result

Let us retain the fixed embeddings ι_∞ and ι_p from Chapter 3. For this chapter and the next, our main hypothesis shall be as follows.

Main Hypothesis 4.1. *E_1 and E_2 are p-congruent, i.e. the corresponding weight two Hilbert modular newforms $\mathbf{f}_1$ and $\mathbf{f}_2$ of trivial Nebentype satisfy* (3.23).

As mentioned in Section 3.5, E_1 and E_2 are p-congruent precisely if ρ_{E_1} and ρ_{E_2} have isomorphic residual representations, namely $\overline{\rho} = \overline{\rho}_E : G_F \to \mathrm{GL}_2(\mathbb{F}_p)$. In other words, $E_1[p] \cong E_2[p]$ as G_F-modules. We administer three additional running assumptions:

Hypothesis 4.1.

(i) the Fourier coefficients $\mathrm{C}(\mathbf{f}_1,\mathfrak{p})$ *and* $\mathrm{C}(\mathbf{f}_2,\mathfrak{p})$ *are p-adic units at all prime ideals* $\mathfrak{p}\mid p$*;*

(ii) $\overline{\rho}_E : G_F \to \mathrm{GL}_2(\mathbb{F}_p)$ *is an absolutely irreducible G_F-representation; and*

(iii) the semi-simplification of $\overline{\rho}_E\big|_{G_{F_\mathfrak{p}}}$ *is $\mathfrak{p}$-distinguished at all primes* $\mathfrak{p}\big|p$*.*

The first condition is the assumption of p-ordinarity for both E_1 and E_2 while the second condition is automatically satisfied when $\mathrm{im}(\rho_{E_1}) = \mathrm{GL}_2(\mathbb{Z}_p) = \mathrm{im}(\rho_{E_2})$. The third condition is true if the restriction $\omega\big|_{G_F}$ is nontrivial.

We shall also consider a finite order Hecke character ϑ_K defined over a CM-extension K/F, and then set

$$\tau := \mathrm{Ind}_F^K(\vartheta_K) : G_F \to \mathrm{GL}_2(\overline{\mathbb{Q}}).$$

Henceforth we choose a finite integral extension $R/\mathbb{Z}_p$ so that $\mathrm{im}(\tau) \overset{\iota_p}{\hookrightarrow} \mathrm{GL}_2(R)$.

For simplicity, let us suppose the prime p is inert in F, so that $p\cdot\mathcal{O}_F = \mathfrak{p}$ say. Recall from Section 2.4.2 that the complex L-function attached to $\rho_E\otimes\tau$ over the ω^t-branch can be defined via the Euler product

$$L(E,\tau,\omega^t,s) := \prod_{\mathfrak{v}\in\mathrm{Spec}(\mathcal{O}_F)} \det\Big(\mathrm{Id} - \mathrm{Frob}_\mathfrak{v}^{-1}\,\Big|\, \big(V_{\rho_E\otimes\tau\otimes\omega^t}\big)^{I_{F_\mathfrak{v}}}\cdot\mathcal{N}(\mathfrak{v})^{-s}\Big)$$

which converges for $\mathrm{Re}(s)\gg 0$ and has an analytic continuation to all of $\mathbb{C}$.

For each choice $E = E_1$ or $E = E_2$ and any branch $t\in\{0,\dots,p-2\}$, the analytic p-adic L-function $L_p(E,\tau\otimes\omega^t)$ is an element of the Iwasawa algebra $\Lambda_R = R[\![G_\infty]\!]$ interpolating twists of the complex L-function [Hid91; Pan89]. Specifically, at any character χ of $\mathfrak{p}$-power conductor with $\chi\big|_{\mathbb{F}_p^\times} = \omega^t$, one has

$$\chi\Big(L_p\big(E,\tau\otimes\omega^t\big)\Big) = \iota_p\circ\iota_\infty^{-1}\Bigg(\mathfrak{h}_E \times \frac{\epsilon_F(\tau\otimes\chi,0)}{\alpha_\mathfrak{p}(E)^{\mathrm{ord}_\mathfrak{p}(\mathrm{cond}(\tau\otimes\chi))}} \times \mathcal{A}_\mathfrak{p}(E,\tau\otimes\chi,1)\times\frac{L\big(E,\tau^*\otimes\chi^{-1},1\big)}{(2\pi)^{[K:\mathbb{Q}]}\cdot\Omega_{E/F}^{\mathrm{Pet}}}\Bigg) \tag{4.1}$$

where $\mathfrak{h}_E$ is a congruence number (see Definition 7.2), $\epsilon_F(\tau\otimes\theta,s)$ is the ϵ-factor associated to $\tau\otimes\theta$, $\alpha_\mathfrak{p}(E)$ is the unit eigenvalue of $\mathrm{Frob}_\mathfrak{p}$, and $\mathcal{A}_\mathfrak{p}(E,\tau\otimes\theta,s)$

is the p-adic multiplier term in Definition 3.1 and made explicit in (4.4). The eventual 'correct choice' of transcendental period $\Omega_{E/F}^{\mathrm{Pet}}$ is quite complicated (see Sections 4.3 and 4.6), but for now it suffices to say that we make a choice that ensures both $L_p(E_1, \tau \otimes \omega^t)$ and $L_p(E_2, \tau \otimes \omega^t)$ are p-integral power series.

Throughout this chapter, we define the set of bad primes for (E_1, E_2, τ) as

$$\Sigma^{\mathrm{bad}} := \big\{\mathfrak{q} \in \mathrm{Spec}(\mathcal{O}_F) \text{ such that } \mathfrak{q} \mid \mathrm{cond}(\tau) \cdot \mathrm{cond}(E_1) \cdot \mathrm{cond}(E_2) \text{ and } \mathfrak{q} \nmid p\big\}.$$

As in Definition 3.2, we denote by $\mu(E, \tau, \omega^t)_{\mathrm{an}}$ the largest power of p dividing the coefficients of $L_p\big(E, \tau \otimes \omega^t\big)$, and by $\lambda(E, \tau, \omega^t)_{\mathrm{an}}$ the number of zeroes (counting multiplicity) of the p-adic L-function on the open unit disk in $\mathbb{C}_p$. The main result of this chapter (Corollary 4.10) supplies the analytic half of the analogue of Greenberg–Vatsal's result for p-ordinary τ-twisted elliptic curves.

Theorem 4.1. *Assume (i) p is inert in F, (ii) the version of Ihara's Lemma in Section 4.6 holds, (iii) the non-square-free part of* $\mathrm{cond}(E_i)$ *divides* $\mathrm{cond}(\tau)$*, and (iv) at all $\mathfrak{q} \in \Sigma^{\mathrm{bad}}$:*

- $\mathrm{ord}_{\mathfrak{q}}\big(\mathrm{cond}(E_1)\big) = \mathrm{ord}_{\mathfrak{q}}\big(\mathrm{cond}(E_2)\big) = \mathrm{ord}_{\mathfrak{q}}\big(\mathrm{cond}(\overline{\rho})\big)$*, or*
- $\mathrm{ord}_{\mathfrak{q}}\big(\mathrm{cond}(\overline{\rho})\big) = 0$*,* $\mathrm{ord}_{\mathfrak{q}}\big(\mathrm{cond}(E_1)\big) \leq 2$ *and* $\mathrm{ord}_{\mathfrak{q}}\big(\mathrm{cond}(E_2)\big) \leq 2$*, or*
- $1 = \mathrm{ord}_{\mathfrak{q}}\big(\mathrm{cond}(\overline{\rho})\big) \leq \mathrm{ord}_{\mathfrak{q}}\big(\mathrm{cond}(E_i)\big) \leq \mathrm{ord}_{\mathfrak{q}}\big(\mathrm{cond}(\overline{\rho})\big) + 1$ *for $i \in \{1, 2\}$.*

Then the (cyclotomic) analytic μ-invariant of $\rho_{E_1} \otimes \tau \otimes \omega^t$ vanishes if and only if the (cyclotomic) analytic μ-invariant of $\rho_{E_2} \otimes \tau \otimes \omega^t$ vanishes, in which case

$$\lambda(E_1, \tau, \omega^t)_{\mathrm{an}} = \lambda(E_2, \tau, \omega^t)_{\mathrm{an}} + \sum_{\mathfrak{q} \in \Sigma^{\mathrm{bad}}} \mathbf{e}_{\mathfrak{q}}(E_2, \tau, \omega^t) - \mathbf{e}_{\mathfrak{q}}(E_1, \tau, \omega^t)$$

where $\mathbf{e}_{\mathfrak{q}}(E, \tau, \omega^t)$ for $E = E_1$ or $E = E_2$ denotes the (cyclotomic) analytic λ-invariant of the power series interpolating the factor $L_{\mathfrak{q}}\big(E \otimes \tau \otimes \psi\omega^t, 1\big)$ at characters $\psi : G_\infty \to \overline{\mathbb{Q}}_p^\times$.

These conditions above on the conductor are required in order to generalise the modified Hida pairings constructed for elliptic cusp forms by Ray, Sujatha and Vatsal [RSV23] to the Hilbert modular setting. In particular, if $F = \mathbb{Q}$

then these conditions on the conductor are automatically satisfied by utilising a fundamental result of Gouvêa. If $F \neq \mathbb{Q}$, it is still open as to whether the conductors of elliptic curves sharing the same residual Galois representation $\overline{\rho}$ always satisfy these conditions.

The proof of the analytic λ-invariant formula exploits a congruence modulo p between $L_p\big(E_1, \tau \otimes \omega^t\big)$ and $L_p\big(E_2, \tau \otimes \omega^t\big)$, or more accurately a congruence between their Σ^{bad}-depleted versions (Theorem 4.9 and Corollary 4.10). Once we perform depletion and level-raising however, the automorphic periods may change, and in this scenario we need a version of Ihara's lemma which will allows us to relate the two periods up to a unit. The precise form of Ihara's lemma we are considering is described at length in Section 4.6, and represents a natural extension of the statement given in *op. cit.* If $F = \mathbb{Q}$ then Ihara's lemma is known [MS21] so in this case we omit it as a condition.

Here is a brief plan of this chapter. In Section 4.2 we give a careful resynthesis of the construction in [Del20, §3] of the explicit Rankin-integral expansion for the double product of HMFs. In Section 4.3 we extend the pairings in [RSV23] to cover the case of Hilbert modular cusp forms, and apply them to the Rankin-Selberg convolution $L(E, \tau, s)$. It is at this point that one requires Ihara's lemma, as well as our hypotheses on the conductor which are unavoidable if $F \neq \mathbb{Q}$. Our resulting choice of periods then allows us to deduce a mod p congruence between the depleted Rankin-Selberg p-adic L-functions in Section 4.4, which we specialise to a pair of elliptic curves over a totally real field tensored by a weight one theta series in Section 4.5. Lastly, in Section 4.6 we explore the validity of Ihara's Lemma in a more general context.

4.2 Rankin-Selberg L-functions

Let $\mathbf{f} \in \mathcal{S}_2(\mathfrak{c}(\mathbf{f}), \mathbf{1}_F)$ be an ordinary Hilbert modular cusp form of parallel weight two, level $\mathfrak{c}(\mathbf{f})$ and trivial Nebentype, with complex L-series given by

$$L(\mathbf{f}, s) = \sum_{\mathfrak{m} \lhd \mathcal{O}_F} C(\mathbf{f}, \mathfrak{m}) \cdot \mathcal{N}(\mathfrak{m})^{-s}.$$

We assume that $\mathbf{f}$ is $\mathfrak{p}$-stabilised, meaning $\mathfrak{c}(\mathbf{f})$ is divisible precisely by the first power of $\mathfrak{p}$ so that one can write $\mathfrak{c}(\mathbf{f}) = \mathfrak{p} \cdot \mathfrak{c}(\mathbf{f})^{\mathrm{tame}}$ with $\mathfrak{p} + \mathfrak{c}(\mathbf{f})^{\mathrm{tame}} = \mathcal{O}_F$. Then the above summation has an Euler product expansion

$$L(\mathbf{f}, s) = \prod_{\mathfrak{q} \nmid \mathfrak{p}\mathfrak{c}(\mathbf{f})} \left(1 - \mathrm{C}(\mathbf{f}, \mathfrak{q})\mathcal{N}(\mathfrak{q})^{-s} + \mathcal{N}(\mathfrak{q})^{1-2s}\right)^{-1} \times \prod_{\mathfrak{q} \mid \mathfrak{p}\mathfrak{c}(\mathbf{f})} \left(1 - \mathrm{C}(\mathbf{f}, \mathfrak{q})\mathcal{N}(\mathfrak{q})^{-s}\right)^{-1}.$$

The local factor $L_{\mathfrak{q}}(\mathbf{f}, s)$ at each $\mathfrak{q} \in \mathrm{Spec}(\mathcal{O}_F)$ with $\mathfrak{q} \nmid \mathfrak{c}(\mathbf{f})$ factorises into

$$L_{\mathfrak{q}}(\mathbf{f}, s) = (1 - \alpha_{\mathfrak{q}}(\mathbf{f})\mathcal{N}(\mathfrak{q})^{-s})(1 - \beta_{\mathfrak{q}}(\mathbf{f})\mathcal{N}(\mathfrak{q})^{-s}) \tag{4.2}$$

and if $\mathfrak{q} \mid \mathfrak{c}(\mathbf{f})$, we will always choose $\alpha_{\mathfrak{q}}(\mathbf{f}) = \mathrm{C}(\mathbf{f}, \mathfrak{q})$ and $\beta_{\mathfrak{q}}(\mathbf{f}) = 0$. We will extend α multiplicatively to the group of nonzero fractional ideals of $\mathcal{O}_F$ so that under the decomposition $\mathfrak{a} = \prod_i \mathfrak{q}_i^{n_i}$, one has $\alpha_{\mathfrak{a}} := \prod_i \alpha_{\mathfrak{q}_i}^{n_i}$.

Suppose we are given another Hilbert eigenform $\mathbf{g}$ of parallel weight one, level $\mathfrak{c}(\mathbf{g})$ and Nebentype $\eta_{\mathbf{g}}$. The local Euler factor at $\mathfrak{q}$ similarly splits into

$$L_{\mathfrak{q}}(\mathbf{g}, s) = (1 - \alpha_{\mathfrak{q}}(\mathbf{g})\mathcal{N}(\mathfrak{q})^{-s})(1 - \beta_{\mathfrak{q}}(\mathbf{g})\mathcal{N}(\mathfrak{q})^{-s}).$$

where $\alpha_{\mathfrak{q}}(\mathbf{g})$ and $\beta_{\mathfrak{q}}(\mathbf{g})$ are algebraic numbers following the convention in (4.2). Let us choose a finite set Σ of places of F not containing the prime $\mathfrak{p}$ such that $\mathrm{supp}\big(\mathfrak{c}(\mathbf{g})\big) \subset \Sigma \cup \{\mathfrak{p}\}$ and factorise the conductor of $\mathbf{f} \in \mathcal{S}_2\big(\mathfrak{c}(\mathbf{f}), \mathbf{1}_{\mathbf{f}}\big)$ as:

$$\mathfrak{c}(\mathbf{f}) = \mathfrak{p} \cdot \mathfrak{c}(\mathbf{f})^{\mathrm{tame}} = \mathfrak{p} \cdot \mathfrak{c}(\mathbf{f})^{\sharp} \cdot \mathfrak{c}(\mathbf{f})^{\flat}$$

with $\mathrm{supp}\big(\mathfrak{c}(\mathbf{f})^{\sharp}\big) \subset \Sigma$ and $\mathrm{supp}\big(\mathfrak{c}(\mathbf{f})^{\flat}\big) \cap \Sigma = \emptyset$.

Towards the end of Section 2.5 we remarked that $\mathbf{f}$ and $\mathbf{g}$ may need to be replaced so that the Rankin-integral representation of the convoluted L-series

$$\mathscr{D}(\mathbf{f}, \mathbf{g}, s) = \sum_{\mathfrak{m} \subset \mathcal{O}_F} \mathrm{C}(\mathfrak{m}, \mathbf{f})\mathrm{C}(\mathfrak{m}, \mathbf{g}) \cdot \mathcal{N}(\mathfrak{m})^{-s}$$

makes sense. More precisely, one needs to derive new HMFs out of $\mathbf{f}$ and $\mathbf{g}$ so that the new levels divide a common ideal $\mathfrak{n}$. We detail this process below following the technique applied in [Pan89, §4], and allowing for a potential non-empty intersection between $\mathrm{supp}\big(\mathfrak{c}(\mathbf{f})\big)$ and $\mathrm{supp}\big(\mathfrak{c}(\mathbf{g})\big)$.

Definition 4.1. Let us define $\mathfrak{n}_\Sigma := \prod_{\mathfrak{q}\in\Sigma}\mathfrak{q}$ and $\mathfrak{n}_0 := \mathrm{lcm}\big(\mathfrak{p}\mathfrak{n}_\Sigma, \mathfrak{c}(\mathbf{g})\big)$.

(a) One constructs $\mathbf{f}_\Sigma \in \mathcal{S}_2\big(\mathfrak{p}\cdot\mathrm{lcm}\big(\mathfrak{n}_\Sigma, \mathfrak{c}(\mathbf{f})^\sharp\big)\cdot\mathfrak{c}(\mathbf{f})^\flat, \mathbf{1}_F\big)$ via the summation

$$\mathbf{f}_\Sigma := \sum_{\mathfrak{a}|\mathfrak{n}_\Sigma}\mu_F(\mathfrak{a})\cdot\beta_{\mathfrak{a}}(\mathbf{f})\cdot\big(\mathbf{f}\,\big|\,\mathfrak{a}\big)$$

where μ_F denotes the Möbius function on the ideals of $\mathcal{O}_F$.

(b) Define the *depleted cusp form* $\mathbf{g}_{\mathfrak{n}_0} \in \mathcal{S}_1\big(\,\mathrm{lcm}\big(\mathfrak{c}(\mathbf{g}), (\mathfrak{p}\cdot\mathfrak{n}_\Sigma)^2\big), \eta_{\mathbf{g}}\big)$ by taking

$$\mathbf{g}_{\mathfrak{n}_0} := \sum_{\mathfrak{a}|\mathfrak{n}_0}\mu_F(\mathfrak{a})\cdot\big(\mathbf{g}\,\big|\,U_{\mathfrak{a}}\circ\mathfrak{a}\big)\,,$$

hence $\mathbf{g}_{\mathfrak{n}_0}$ is the unique HMF which satisfies $\mathrm{C}\big(\mathbf{g}_{\mathfrak{n}_0}, \mathfrak{m}\big) = \mathrm{C}(\mathbf{g}, \mathfrak{m})$ if $\mathfrak{m}+\mathfrak{n}_0 = \mathcal{O}_F$ and $\mathrm{C}\big(\mathbf{g}_{\mathfrak{n}_0}, \mathfrak{m}\big) = 0$ otherwise.

(c) Setting $\mathfrak{n}' := \mathfrak{p}^{\mathbf{e}}\cdot\mathrm{lcm}\big(\mathfrak{c}(\mathbf{f})^\sharp, \mathfrak{c}(\mathbf{g})^{\mathrm{tame}}, \mathfrak{n}_\Sigma^2\big)$ with $\mathbf{e} > 1$ chosen sufficiently large, the levels of both $\mathbf{f}_\Sigma$ and $\mathbf{g}_{\mathfrak{n}_0}$ divide the common ideal $\mathfrak{n}'\cdot\mathfrak{c}(\mathbf{f})^\flat$.

If we specialise the discussion in Section 2.4.2 to the Hilbert modular cusp forms $\mathbf{F} \in \mathcal{S}_2\big(\mathfrak{c}(\mathbf{F}), \eta_{\mathbf{F}}\big)$ and $\mathbf{G} \in \mathcal{S}_1\big(\mathfrak{c}(\mathbf{G}), \eta_{\mathbf{G}}\big)$, then for $\mathrm{Re}(s) \gg 0$, the completed L-function

$$\Psi(s, \mathbf{F}, \mathbf{G}) := \frac{\Gamma(s)^{2d}}{(2\pi)^{2ds}} \times \zeta_{(\mathfrak{c}(\mathbf{F})\mathfrak{c}(\mathbf{G}))}(2s-1, \eta_{\mathbf{G}}) \times \mathscr{D}(\mathbf{F}, \mathbf{G}, s)$$

can be holomorphically continued to the whole complex plane $\mathbb{C}$. It further satisfies a functional equation relating its value at s to its value at $k+l-1-s$. Shimura proved in [Shi78, Equation (4.32)] that if the levels of $\mathbf{F}$ and $\mathbf{G}$ divide a common ideal $\mathfrak{c}$, then

$$\Psi(\mathbf{F}, \mathbf{G}, s) = \frac{\sqrt{D_F}\cdot\Gamma(s-l+1)^n}{\pi^{ns}} \times \Big\langle \mathbf{F}^\rho, \mathbf{G}\cdot\mathbf{K}_1\big(s-1; \mathfrak{c}; \eta_{\mathbf{G}}^{-1}\big)\Big\rangle_{\mathfrak{c}}$$

where $\mathbf{F}^\rho$ is obtained by replacing each Fourier coefficient by its corresponding complex conjugate and $\mathbf{K}_1(s-1; \mathfrak{c}; \eta_{\mathbf{G}}^{-1})$ is shorthand for the Eisenstein series $\mathbf{K}_\kappa(z, s-1; \mathfrak{c}, \mathcal{O}; \eta_{\mathbf{G}}^{-1})$ introduced in (2.41). We also define the *normalised Eisenstein series* $\mathbf{E}_m(s; \mathfrak{c}; \eta)$ as in [Pan89, Section 4.5]:

$$\frac{2^{-n}\,D^{1/2}\,\Gamma(m+r)^n}{(-4\pi)^{rn}\,(-2\pi i)^{mn}} \times (-1)^{mn}\mathcal{N}(\mathfrak{c}\mathfrak{d}^2)^{s+m/2}\cdot\big(\mathbf{K}_m(s; \mathfrak{c}, \mathfrak{b}; \eta)\,\big|\,J_{\mathfrak{c}}\big) \qquad (4.3)$$

where $J_{\mathfrak{c}}$ denotes the J-operator for HMFs with level $\mathfrak{c}$ in Section 2.2.4.

Before we proceed, we remark that the symbols $\mathbf{F}, \mathbf{G}, \mathbf{H}, \mathbf{F}_1$ and $\mathbf{F}_2$ are merely used as placeholders to recall general properties of HMFs that is needed in this chapter. In particular, $\mathbf{F}$ and $\mathbf{G}$ are often paired together for the purpose of discussing the double product $\rho_E \otimes \tau$, while $\mathbf{F}_1$ and $\mathbf{F}_2$ are paired in place of the newforms associated to E_1 and E_2.

To construct a p-adic measure, one first makes a shrewd choice of $\mathbf{F}$ and $\mathbf{G}$. Following [Pan89] and [Del20] let us therefore choose

$$\mathfrak{c} = \mathfrak{n}' \cdot \mathfrak{c}(\mathbf{f})^{\flat}, \qquad \mathbf{F} = \mathbf{f}_{\Sigma} \quad \text{and} \quad \mathbf{G} = \mathbf{g}_{\mathfrak{n}_0} \,\big|\, J_{\mathfrak{n}'\cdot\mathfrak{c}(\mathbf{f})^{\flat}} \, .$$

Henceforth we set the $\mathcal{O}_F$-ideal $\mathfrak{n}_1 := \mathrm{lcm}\big(\mathfrak{c}(\mathbf{f})^{\sharp}, \mathfrak{c}(\mathbf{g})^{\mathrm{tame}}, \mathfrak{p}\mathfrak{n}_{\Sigma}^2\big)$ so that $\mathfrak{n}_1 \big| \mathfrak{n}'$. In particular, from Definition 4.1(c) the quotient ideal $\frac{\mathfrak{n}'}{\mathfrak{n}_1} = \mathfrak{p}^{\mathrm{e}-1}$ belongs to $\mathfrak{p}^{\mathbb{N}}$. In order to obtain a nontrivial algebraic relation between $\Psi\big(\mathbf{F}, \mathbf{G}, 1\big)$ and the inner product $\big\langle \mathbf{F}^{\rho}, \mathbf{G} \cdot \mathbf{E}_1(0; \mathfrak{c}; \eta_{\mathbf{G}}^{-1}) \,\big|\, J_{\mathfrak{c}} \big\rangle_{\mathfrak{c}}$ we must introduce further assumptions.

Hypothesis 4.2.

(i) the coefficient $\mathrm{C}\big(\mathbf{f}, \mathfrak{c}(\mathbf{f})^{\flat}\big) \neq 0$*;*

(ii) $\mathbf{g}$ *is a primitive Hecke eigenform;*

(iii) the conductor $\mathfrak{c}(\mathbf{g}) \subset \mathfrak{p}^2$*;*

(iv) $\mathrm{lcm}\big(\mathfrak{c}(\mathbf{g})^{\mathrm{tame}}, \mathfrak{n}_{\Sigma}\big) \subset \mathfrak{c}(\mathbf{f})^{\sharp}$.

One can easily dispatch with conditions (iii) and (iv) by starting out with twists of both $\mathbf{f}$ and $\mathbf{g}$ by suitable Hecke characters. On the other hand, (i) is a crucial assumption that prevents the p-adic measure constructed in [Del20] from becoming identically zero. Note that if $\mathbf{g}$ is a primitive eigenform of conductor $\mathfrak{c}(\mathbf{g})$, then $\mathbf{g} \,\big|\, J_{\mathfrak{c}(\mathbf{g})} = \upsilon(\mathbf{g}) \cdot \mathbf{g}^{\rho}$ where the pseudo-eigenvalue $\upsilon(\mathbf{g})$ of the J-operator lies on the complex unit circle. Moreover $\mathbf{g}^{\rho}$ is also a primitive form, and its local L-factor at $\mathfrak{q}$ factorises into

$$1 - \mathrm{C}(\mathbf{g}^{\rho}, \mathfrak{q})\mathcal{N}(\mathfrak{q})^{-s} + \eta_{\mathbf{g}}^{-1}(\mathfrak{q})\mathcal{N}(\mathfrak{q})^{-2s} = \big(1 - \alpha_{\mathfrak{q}}(\mathbf{g}^{\rho})\mathcal{N}(\mathfrak{q})^{-s}\big)\big(1 - \beta_{\mathfrak{q}}(\mathbf{g}^{\rho})\mathcal{N}(\mathfrak{q})^{-s}\big) \, .$$

with $\alpha_{\mathfrak{q}}(\mathbf{g}^{\rho}) = \overline{\eta}_{\mathbf{g}}(\mathfrak{q}) \cdot \alpha_{\mathfrak{q}}(\mathbf{g})$ and $\beta_{\mathfrak{q}}(\mathbf{g}^{\rho}) = \overline{\eta}_{\mathbf{g}}(\mathfrak{q}) \cdot \beta_{\mathfrak{q}}(\mathbf{g})$. The main example we will consider is when $\mathbf{g}$ is a theta-lift of a Hecke character.

The following is an explicit expansion of equation (2.44). It is a special case of Corollary 3.7 developed in [Del20] for HMFs of parallel weight $k \geq 2$ which we shall only state for parallel weight $k = 2$. *Assuming Hypotheses 4.2(i)–(iv) hold, if* $s = 1$ *then*

$$\frac{\upsilon(\mathbf{g}) \cdot \mathcal{N}\big(\mathfrak{c}(\mathbf{g})\mathfrak{d}^2\big)^{s-\frac{1}{2}}}{\alpha_{\mathfrak{p}}(\mathbf{f})^{\mathrm{ord}_{\mathfrak{p}}\,\mathfrak{c}(\mathbf{g})}} \times \mathcal{A}_{\Sigma}(\mathbf{f}, \mathbf{g}^{\rho}, s) \times \Psi(\mathbf{f}, \mathbf{g}^{\rho}, s) \;=\; \frac{\big(-4\sqrt{-1}\big)^{d}}{\mathrm{C}(\mathbf{f}, \mathfrak{c}(\mathbf{f})^{\flat}) \cdot \alpha_{\frac{\mathfrak{n}'}{\mathfrak{c}(\mathbf{g})^{\mathrm{tame}}}}(\mathbf{f})}$$

$$\times \left\langle \mathbf{f}^{\rho}_{\Sigma}\,, \Big(\mathbf{g}_{\mathfrak{n}_0} \cdot \mathbf{E}_1\big(s-1; \mathfrak{n}' \cdot \mathfrak{c}(\mathbf{f})^{\flat}; \eta_{\mathbf{g}}^{-1}\big)\Big) \,\Big|\, U_{\frac{\mathfrak{n}'}{\mathfrak{n}_1}} \circ J_{\mathfrak{n}_1 \cdot \mathfrak{c}(\mathbf{f})^{\flat}} \right\rangle_{\mathfrak{n}_1 \cdot \mathfrak{c}(\mathbf{f})^{\flat}}$$

where the p-adic multiplier in this case is given by the finite product

$$\begin{aligned} \mathcal{A}_{\Sigma}(\mathbf{f}, \mathbf{g}^{\rho}, s) = \prod_{\mathfrak{q} \in \Sigma^{\mathrm{ps}}} &\big(1 - \beta_{\mathfrak{q}}(\mathbf{f})\alpha_{\mathfrak{q}}(\mathbf{g}^{\rho})\mathcal{N}(\mathfrak{q})^{-s}\big)\big(1 - \beta_{\mathfrak{q}}(\mathbf{f})\beta_{\mathfrak{q}}(\mathbf{g}^{\rho})\mathcal{N}(\mathfrak{q})^{-s}\big) \\ \times \prod_{\substack{\mathfrak{q} \in \Sigma^{\mathrm{ps}} \cup \Sigma^{\mathrm{sp}} \\ \cup \{\mathfrak{p}\}}} &\big(1 - \alpha_{\mathfrak{q}}(\mathbf{f})^{-1}\alpha_{\mathfrak{q}}(\mathbf{g})\mathcal{N}(\mathfrak{q})^{s-1}\big)\big(1 - \alpha_{\mathfrak{q}}(\mathbf{f})^{-1}\beta_{\mathfrak{q}}(\mathbf{g})\mathcal{N}(\mathfrak{q})^{s-1}\big) \end{aligned} \tag{4.4}$$

with Σ^{ps} *consisting of the subset of places* $\mathfrak{q} \in \Sigma$ *at which* $\mathbf{f}$ *is principal series, and* Σ^{sp} *consisting of the subset of places* $\mathfrak{q} \in \Sigma - \Sigma^{\mathrm{ps}}$ *at which* $C\big(\mathbf{f}, \mathfrak{q}\big) \neq 0$.

The reader will notice the presence of unwanted factors $\mathcal{A}_{\Sigma}(\cdot)$ which have the potential to lengthen our calculation of the λ-invariant of $\mathbf{f} \otimes \mathbf{g}$ since they will have to be removed at some point. To simplify matters considerably, we shall now introduce the Σ-depletion of $\mathbf{f}$ by taking

$$\mathbf{f}^{\mathrm{dep}}_{\Sigma} := \sum_{\mathfrak{a} | \mathfrak{n}_{\Sigma}} \mu_F(\mathfrak{a}) \cdot \big(\mathbf{f} \,\big|\, U_{\mathfrak{a}} \circ \mathfrak{a}\big) \;\in\; \mathcal{S}_2\Big(\, \mathrm{lcm}\big(\mathfrak{c}(\mathbf{f}), \mathfrak{n}_{\Sigma}^2\big), \mathbf{1}_F\Big). \tag{4.5}$$

For reasons that will become apparent in the next section, we shall need to work at a smaller level than $\mathfrak{n}_1 \cdot \mathfrak{c}(\mathbf{f})^{\flat}$ in general; we will thus choose

$$\mathfrak{n}_2 := \mathrm{lcm}\big(\mathfrak{c}(\mathbf{f})^{\sharp}, \mathfrak{p}\mathfrak{n}_{\Sigma}^2\big). \tag{4.6}$$

Recall the trace map $\mathrm{Tr}^{\mathfrak{n}_1 \cdot \mathfrak{c}(\mathbf{f})^{\flat}}_{\mathfrak{n}_2 \cdot \mathfrak{c}(\mathbf{f})^{\flat}} : \mathcal{M}_2\big(\mathfrak{n}_1 \cdot \mathfrak{c}(\mathbf{f})^{\flat}, \mathbf{1}_F\big) \to \mathcal{M}_2\big(\mathfrak{n}_2 \cdot \mathfrak{c}(\mathbf{f})^{\flat}, \mathbf{1}_F\big)$ introduced in Section 2.2.5 which satisfies

$$\big\langle \mathbf{F}_1, \mathbf{F}_2 \big\rangle_{\mathfrak{n}_1 \cdot \mathfrak{c}(\mathcal{F})^{\flat}} = \big\langle \mathbf{F}_1, \mathrm{Tr}^{\mathfrak{n}_1 \cdot \mathfrak{c}(\mathbf{f})^{\flat}}_{\mathfrak{n}_2 \cdot \mathfrak{c}(\mathbf{f})^{\flat}}\, \mathbf{F}_2 \big\rangle_{\mathfrak{n}_2 \cdot \mathfrak{c}(\mathcal{F})^{\flat}}$$

for any pair $\mathbf{F}_1 \in \mathcal{S}_2\big(\mathfrak{n}_2 \cdot \mathfrak{c}(\mathbf{F})^{\flat}, \mathbf{1}_F\big)$ and $\mathbf{F}_2 \in \mathcal{M}_2\big(\mathfrak{n}_1 \cdot \mathfrak{c}(\mathbf{F})^{\flat}, \mathbf{1}_F\big)$ of HMFs. Repeating the Rankin integral calculation with $\mathbf{f}^{\mathrm{dep}}_{\Sigma}$ in place of $\mathbf{f}_{\Sigma}$, one obtains:

Proposition 4.2. *If the Hypotheses 4.2(i)–(iv) hold, then one has an equality*

$$\begin{aligned}\frac{\upsilon(\mathbf{g}) \times \mathcal{N}\big(\mathfrak{c}(\mathbf{g})\mathfrak{d}^2\big)^{\frac{1}{2}}}{\alpha_{\mathfrak{p}}(\mathbf{f})^{\mathrm{ord}_{\mathfrak{p}}\, \mathfrak{c}(\mathbf{g})}} \times \Psi_\Sigma\big(\mathbf{f}, \mathbf{g}^\rho, 1\big) \;&=\; \frac{\big(-4\sqrt{-1}\big)^n}{\mathrm{C}\big(\mathbf{f}, \mathfrak{c}(\mathbf{f})^\flat\big) \cdot \alpha_{\frac{\mathfrak{n}'}{\mathfrak{c}(\mathbf{g})^{\mathrm{tame}}}}(\mathbf{f})} \\ &\times \left\langle (\mathbf{f}_\Sigma^{\mathrm{dep}})^\rho \,, \tilde{T}_{\mathbf{g}}\Big(\mathbf{g}_{\mathfrak{n}_0} \cdot \mathbf{E}_1\big(0; \mathfrak{n}' \cdot \mathfrak{c}(\mathbf{f})^\flat; \eta_{\mathbf{g}}^{-1}\big) \,\Big|\, U_{\frac{\mathfrak{n}'}{\mathfrak{n}_1}}\Big) \,\Big|\, J_{\mathfrak{n}_2 \cdot \mathfrak{c}(\mathbf{f})^\flat} \right\rangle_{\mathfrak{n}_2 \cdot \mathfrak{c}(\mathbf{f})^\flat}\end{aligned} \tag{4.7}$$

where the symbol $\Psi_\Sigma\big(\mathbf{f}, \mathbf{g}^\rho, s\big)$ *denotes the completed L-function* $\Psi\big(\mathbf{f}, \mathbf{g}^\rho, s\big)$ *with the Euler factors at primes dividing* Σ *stripped away, and the twisted trace mapping* $\tilde{T}_{\mathbf{g}} : \mathcal{M}_2\big(\mathfrak{n}_1 \cdot \mathfrak{c}(\mathbf{f})^\flat, \mathbf{1}_F\big) \to \mathcal{M}_2\big(\mathfrak{n}_2 \cdot \mathfrak{c}(\mathbf{f})^\flat, \mathbf{1}_F\big)$ *is defined by*

$$\tilde{T}_{\mathbf{g}}(-) := \mathrm{Tr}^{\mathfrak{n}_1}_{\mathfrak{n}_2}\big(- \;\big|\, J_{\mathfrak{n}_1 \cdot \mathfrak{c}(\mathbf{f})^\flat}\big) \,\big|\, J^{-1}_{\mathfrak{n}_2 \cdot \mathfrak{c}(\mathbf{f})^\flat} \,. \tag{4.8}$$

The left-hand side of Equation (4.7) is independent of the choice of ideals $\mathfrak{n}_1$, $\mathfrak{n}_2$ and $\mathfrak{n}'$, hence so is the right-hand side. Note that the pseudo-eigenvalue $\upsilon(\mathbf{g}) \neq 0$ since $\big|\upsilon(\mathbf{g})\big|_\infty = 1$ under Hypothesis 4.2(ii), while $\mathrm{C}\big(\mathbf{f}, \mathfrak{c}(\mathbf{f})^\flat\big) \neq 0$ under Hypothesis 4.2(i). Therefore, Equation (4.7) is a non-trivial identity.

Proof. From the special case of [Del20, Corollary 3.7] given earlier, one deduces

$$\begin{aligned}\frac{\upsilon(\mathbf{g}) \cdot \mathcal{N}\big(\mathfrak{c}(\mathbf{g})\mathfrak{d}^2\big)^{\frac{1}{2}}}{\alpha_{\mathfrak{p}}(\mathbf{f})^{\mathrm{ord}_{\mathfrak{p}}\, \mathfrak{c}(\mathbf{g})}} \times \Psi_\Sigma\big(\mathbf{f}, \mathbf{g}^\rho, 1\big) \;&=\; \frac{\big(-4\sqrt{-1}\big)^n}{\mathrm{C}\big(\mathbf{f}, \mathfrak{c}(\mathbf{f})^\flat\big) \cdot \alpha_{\frac{\mathfrak{n}'}{\mathfrak{c}(\mathbf{g})^{\mathrm{tame}}}}(\mathbf{f})} \\ &\times \left\langle (\mathbf{f}_\Sigma^{\mathrm{dep}})^\rho \,, \Big(\mathbf{g}_{\mathfrak{n}_0} \cdot \mathbf{E}_1\big(0; \mathfrak{n}' \cdot \mathfrak{c}(\mathbf{f})^\flat; \eta_{\mathbf{g}}^{-1}\big)\Big) \,\Big|\, U_{\frac{\mathfrak{n}'}{\mathfrak{n}_1}} \circ J_{\mathfrak{n}_1 \cdot \mathfrak{c}(\mathbf{f})^\flat} \right\rangle_{\mathfrak{n}_1 \cdot \mathfrak{c}(\mathbf{f})^\flat} \,.\end{aligned} \tag{4.9}$$

Tracing down via $\mathrm{Tr}^{\mathfrak{n}_1 \cdot \mathfrak{c}(\mathbf{f})^\flat}_{\mathfrak{n}_2 \cdot \mathfrak{c}(\mathbf{f})^\flat}$ and cleaning up J-operators, the result follows. ■

4.3 The Ray-Sujatha-Vatsal Pairing

In order to obtain a congruence modulo p, one needs to work in a setting with an underlying integral structure. Following the work of Ray, Sujatha and Vatsal in [RSV23, §2] for elliptic modular forms, we now extend their algebraic pairings to include Hilbert modular forms of parallel weight $k \geq 2$ and arbitrary central character η.

For each ideal $\mathfrak{n} \lhd \mathcal{O}_F$ and a fixed integral extension $R/\mathbb{Z}_p$, let $\mathbf{T}(\mathfrak{n}; R)$ be the algebra generated by the Hecke operators $T_{\mathfrak{q}}$, $U_{\mathfrak{q}}$ and $U_{\mathfrak{p}}$ acting upon $\mathcal{S}_k(\mathfrak{n}; R)$. We recall Hida's ordinary idempotent $\mathfrak{e}^{\mathrm{ord}} := \lim_{n\to\infty} U_{\mathfrak{p}}^{n!}$ (c.f. [Hid91, p. 335]) and let us put $\mathcal{M}_k^{\mathrm{ord}}(\mathfrak{n};R) = \mathcal{M}_k(\mathfrak{n};R)\,\big|\,\mathfrak{e}^{\mathrm{ord}}$ and $\mathcal{S}_k^{\mathrm{ord}}(\mathfrak{n};R) = \mathcal{S}_k(\mathfrak{n};R)\,\big|\,\mathfrak{e}^{\mathrm{ord}}$. If R contains the Fourier coefficients of some Hecke eigenform $\mathbf{F} \in \mathcal{S}_k^{\mathrm{ord}}(\mathfrak{n};R)$ then one has a ring homomorphism $\lambda_{\mathbf{F}} : \mathbf{T}(\mathfrak{n};R) \to R$, and there is a unique maximal ideal $\mathfrak{M}$ of $\mathbf{T}(\mathfrak{n};R)$ containing both the kernel of $\lambda_{\mathbf{F}}$ and the ideal $\mathfrak{m}_R{\cdot}\mathbf{T}(\mathfrak{n};R)$ (here $\mathfrak{m}_R$ is the maximal ideal of R so that $R/\mathfrak{m}_R = \mathbb{F}_R$). Localising at the maximal ideal $\mathfrak{M}$, one obtains a perfect pairing

$$\begin{array}{ccc} \mathcal{S}_k(\mathfrak{n};R)_{\mathfrak{M}} \times \mathbf{T}(\mathfrak{n};R)_{\mathfrak{M}} & \longrightarrow & R \\ (\mathbf{H},T) & \longmapsto & \mathrm{C}(\mathbf{H} \mid T, \mathcal{O}_F) \end{array} \tag{4.10}$$

which induces a canonical ring isomorphism $\mathbf{T}(\mathfrak{n};R)_{\mathfrak{M}} \cong \mathrm{Hom}_R\big(\mathcal{S}_k(\mathfrak{n};R)_{\mathfrak{M}}, R\big)$. (In the sequel, we will choose $\mathfrak{n}$ to be a minimal common level for both $\mathbf{F}_1$ and $\mathbf{F}_2$, we always take the parallel weight to be two, and choose $\mathbf{F}_i = \mathbf{f}_{i,\Sigma}^{\mathrm{dep}}$ as in Section 4.2.)

Hypothesis 4.3.

(i) p is inert in the extension $F/\mathbb{Q}$, so that $p \cdot \mathcal{O}_F = \mathfrak{p}$ and $p \nmid D_F$;

(ii) the Galois representation $\rho_{\mathbf{f}} : G_F \to \mathrm{GL}_2(R)$ attached to the eigenform $\mathbf{f}$ is absolutely irreducible, $\overline{\rho}_{\mathbf{f}}\big|_{G_{F_{\mathfrak{p}}}}$ is $\mathfrak{p}$-ordinary, and the semisimplification $\overline{\rho}_{\mathbf{f}}^{\mathrm{ss}}\big|_{G_{F_{\mathfrak{p}}}}$ is non-scalar;

(iii) at each prime ideal $\mathfrak{q} \in \Sigma$, the oldform $\mathbf{f}_{\Sigma}^{\mathrm{dep}}\,\big|\,\mathfrak{q}$ does not occur at level $\mathfrak{n}$.

The motivation for imposing these assumptions is identical to that in [RSV23], i.e, without them there is no way to ensure that the p-adic L-function is integral. Conditions (i) and (ii) imply that $\mathbf{T}(\mathfrak{n};R)_{\mathfrak{M}}$ is a Gorenstein ring using [Dim05, Theorem 6.7], in which case $\mathbf{T}(\mathfrak{n};R)_{\mathfrak{M}} \cong \mathrm{Hom}\big(\mathbf{T}(\mathfrak{n};R)_{\mathfrak{M}}, R\big)$ as left Hecke modules. The right-hand module is isomorphic to $\mathcal{S}_k(\mathfrak{n};R)_{\mathfrak{M}}$ from the pairing (4.10), and the left-action of $\mathbf{T}(\mathfrak{n};R)_{\mathfrak{M}}$ on $\mathrm{Hom}\big(\mathbf{T}(\mathfrak{n};R)_{\mathfrak{M}}, R\big) \cong \mathcal{S}_k(\mathfrak{n};R)_{\mathfrak{M}}$ coincides with the standard right-action of the Hecke algebra. We therefore obtain a duality

$$(-,-)_{\mathfrak{n}} : \mathcal{S}_k(\mathfrak{n};R)_{\mathfrak{M}} \times \mathcal{S}_k(\mathfrak{n};R)_{\mathfrak{M}} \longrightarrow R \tag{4.11}$$

which is a Hecke-equivariant bilinear pairing of localised modules [Hid93, §7].

The following proposition has been proven in [RSV23] for the field $F = \mathbb{Q}$ (c.f. Lemma 2.14 and Corollary 2.15); we illustrate its natural extension to totally real fields.

Proposition 4.3. *Assume that each of the Hypotheses 4.3(i)-(iii) holds true.*

(a) There is an isomorphism $\big(\mathbf{T}(\mathfrak{n}; R)_{\mathfrak{M}}\big)_{\mathcal{P}_{\mathbf{F}}} \cong \mathrm{Frac}(R)$ *with* $\mathcal{P}_{\mathbf{F}} = \mathrm{Ker}(\lambda_{\mathbf{F}})_{\mathfrak{M}}$.

(b) The pairing of $\mathbf{F} = \mathbf{f}_{\Sigma}^{\mathrm{dep}}$ *with itself is non-trivial, i.e.,* $\big(\mathbf{f}_{\Sigma}^{\mathrm{dep}}, \mathbf{f}_{\Sigma}^{\mathrm{dep}}\big)_{\mathfrak{n}} \neq 0$.

Proof of (a). Set $\mathbf{T}_{p,\mathfrak{M}} := \mathbf{T}(\mathfrak{n}; R)_{\mathfrak{M}} \otimes \mathbb{Q}_p$. This is a finite-dimensional algebra over $\mathrm{Frac}(R)$ as implied by [Hid88, Theorem 3.1], and therefore $\mathbf{T}_{p,\mathfrak{M}} = \prod_i K_i$ where each K_i represents a finite-dimensional (over $\mathrm{Frac}(R)$) localisation of $\mathbf{T}_{p,\mathfrak{M}}$ at a maximal ideal. Suppose K_1 is the image of $\big(\mathbf{T}(\mathfrak{n}; R)_{\mathfrak{M}}\big)_{\mathcal{P}_{\mathbf{F}}}$ under this identification. Thus in order to prove (a), one needs to show that K_1 is a field.

Let $\mathcal{S}_k^{\mathrm{new}}(\mathfrak{n}; R)$ be the subspace of newforms spanned by common eigenforms $\mathbf{f}_j$ of $\mathbf{T}(\mathfrak{n}; R)$ as defined in Section 2.3. The subalgebra $\mathbf{T}_{0,\mathfrak{M}} \subset \mathbf{T}(\mathfrak{n}; R)_{\mathfrak{M}} \otimes \mathbb{Q}$ generated by $\mathfrak{m}$-th Hecke operators for which $\mathfrak{m} + \mathfrak{n} = \mathcal{O}_F$ acts semisimply on $\mathcal{S}_k^{\mathrm{new}}(\mathfrak{n}; R)_{\mathfrak{M}}$ (c.f. [Miy71, Theorem 1, §2]). As a consequence, one can write $\mathbf{T}_{0,\mathfrak{M}} = \prod_j L_j$ where each L_j is the corresponding field of definition of the newform $\mathbf{f}_j$. If, say $L_1 = \mathrm{Frac}(R)$, then one obtains a homomorphism $\mathbf{T}_{0,\mathfrak{M}} \to L_1 \hookrightarrow K_1$ whose kernel $\mathcal{P}'$ lies below $\mathcal{P}_{\mathbf{F}}$.

The duality of $\mathcal{S}_k(\mathfrak{n}; R)_{\mathfrak{M}}$ and $\mathbf{T}(\mathfrak{n}; R)_{\mathfrak{M}}$ translates the problem into proving that the subspace $\mathcal{S}_k(\mathfrak{n}; R)_{\mathfrak{M}} \otimes \mathbb{Q}_p$ annihilated by $\mathcal{P}'$ has dimension one over $\mathrm{Frac}(R)$. This is automatically true by multiplicity one provided $\mathbf{F}$ is a newform, however we wish to apply this on the depleted form $\mathbf{F} = \mathbf{f}_{\Sigma}^{\mathrm{dep}}$. Note that the conductor $\mathfrak{n}$ of $\mathbf{f}_{\Sigma}^{\mathrm{dep}}$ differs from the conductor $\mathfrak{c}(\mathbf{f}^{\dagger})$ of the newform $\mathbf{f}^{\dagger}$ associated to $\mathbf{f}$ at the primes $\{\mathfrak{q}_1 \dots, \mathfrak{q}_s\} = \Sigma \cup \{\mathfrak{p}\}$. We use the superscript $\mathfrak{a}^{(\mathfrak{q})}$ to denote the exact power of $\mathfrak{q}$ dividing the ideal $\mathfrak{a}$ and write $\mathfrak{n}^{(\mathfrak{q})} = \mathfrak{q}^{e_{\mathfrak{q}}} \cdot \mathfrak{c}(\mathbf{f}^{\dagger})^{(\mathfrak{q})}$ with $e_{\mathfrak{q}} \geq 0$. Starting with $\mathcal{S}_k\big(\mathfrak{c}(\mathbf{f}^{\dagger}); R\big)_{\mathfrak{M}} \otimes \mathbb{Q}_p$ we show that $\mathcal{S}_k\big(\mathfrak{c}(\mathbf{f}^{\dagger}) \cdot \mathfrak{q}_1^{e_{\mathfrak{q}_1}}; R)_{\mathfrak{M}} \otimes \mathbb{Q}_p$ has dimension equal to one and we repeat this process iteratively until we obtain $\mathcal{S}_k\big(\mathfrak{n}; R\big)_{\mathfrak{M}} \otimes \mathbb{Q}_p$.

In order to carry out the idea outlined above, we define for every prime ideal $\mathfrak{q} \in \Sigma \cup \{\mathfrak{p}\}$ a subspace $\mathcal{S}_k(\mathfrak{q}) \subset \mathcal{S}_k(\mathfrak{n}; R)$ as follows:

(i) if $\mathfrak{q} = \mathfrak{p}$ and $\mathfrak{p} \nmid \mathfrak{c}(\mathbf{f}^\dagger)$, then $\mathcal{S}_k(\mathfrak{q})$ is the space generated by $\mathbf{f}^\dagger$ and $\mathbf{f}^\dagger \mid \mathfrak{p}$;

(ii) if $\mathfrak{q} = \mathfrak{p}$ and $\mathfrak{p} \,\|\, \mathfrak{c}(\mathbf{f}^\dagger)$, then $\mathcal{S}_k(\mathfrak{q})$ is the line generated by $\mathbf{f}^\dagger$;

(iii) if $\mathfrak{q} \neq \mathfrak{p}$ and $\mathfrak{q} \nmid \mathfrak{c}(\mathbf{f})^\sharp$, then $\mathcal{S}_k(\mathfrak{q})$ is the space generated by $\mathbf{f}^\dagger$, $\mathbf{f}^\dagger \mid \mathfrak{q}$ and $\mathbf{f}^\dagger \mid \mathfrak{q}^2$;

(iv) if $\mathfrak{q} \neq \mathfrak{p}$ and $\mathfrak{q} \,\|\, \mathfrak{c}(\mathbf{f})^\sharp$, then $\mathcal{S}_k(\mathfrak{q})$ is the space generated by $\mathbf{f}^\dagger$ and $\mathbf{f}^\dagger \mid \mathfrak{q}$;

(v) if $\mathfrak{q} \neq \mathfrak{p}$ and $\mathfrak{q}^2 \,\|\, \mathfrak{c}(\mathbf{f})^\sharp$, then $\mathcal{S}_k(\mathfrak{q})$ is the line generated by $\mathbf{f}^\dagger$.

We claim that each localised component $\mathcal{S}_k(\mathfrak{q})_{\mathfrak{M}}$ is one-dimensional. If this is indeed the case, then as $\mathcal{S}_k\big(\mathfrak{c}(\mathbf{f}^\dagger) \cdot \mathfrak{q}_1^{e_{\mathfrak{q}_1}}; R\big)_{\mathfrak{M}} \otimes \mathbb{Q}_p = \mathcal{S}_k\big(\mathfrak{c}(\mathbf{f}^\dagger); R\big)_{\mathfrak{M}} \otimes \mathcal{S}_k(\mathfrak{q})_{\mathfrak{M}}$, the result follows.

Let us prove the claim we made in the above paragraph. First we note that each subspace $\mathcal{S}_k(\mathfrak{q})$ is annihilated by $\mathcal{P}'$ and is stable under the full Hecke algebra $\mathbf{T}(\mathfrak{n}; R)$. Using [SW93, Theorem 3.3] which is a generalization of Atkin-Lehner theory to Hilbert modular forms, one is able to deduce that the $U_{\mathfrak{q}}$-eigenvalues of each space $\mathcal{S}_k(\mathfrak{q})$ are distinct. Under Hypothesis 4.3(iii), we easily see that $\mathbf{f}_\Sigma^{\mathrm{dep}}$ is the unique Hecke eigenform of level $\mathfrak{n}$ (in the $\mathbf{f}^\dagger$-isotypic component) which has eigenvalues equal to zero at all prime ideals $\mathfrak{q} \in \Sigma$. Each localised component $\mathcal{S}_k(\mathfrak{q})_{\mathfrak{M}}$ is therefore one-dimensional as required. ■

Remark 6. Item (b) in Proposition 4.3 readily follows the the non-degeneracy of the pairing $(-,-)_{\mathfrak{n}}$ extended to $\mathcal{S}_k(\mathfrak{n}; R)_{\mathfrak{M}} \otimes \mathbb{Q}_p$ and the fact that the subspace of $\mathcal{S}_k(\mathfrak{n}; R)_{\mathfrak{M}} \otimes \mathbb{Q}_p$ annihilated by $\mathcal{P}'$ is one-dimensional over $\mathrm{Frac}(R)$.

Let us define the twisted Petersson inner product as

$$\big\langle \mathbf{F}_1, \mathbf{F}_2 \big\rangle_{\mathfrak{n}}' := \big\langle \mathbf{F}_1^\rho, \mathbf{F}_2 \,\big|\, J_{\mathfrak{n}} \big\rangle_{\mathfrak{n}} \tag{4.12}$$

for all $\mathbf{F}_1, \mathbf{F}_2 \in \mathcal{S}_k(\mathfrak{n}, \eta; \mathbb{C})$. This pairing is Hecke-equivariant under the right-action of $\mathbf{T}(\mathfrak{n}; \mathbb{C})$. We remark that $\mathbf{F}_2 \,\big|\, J_{\mathfrak{n}}$ is a constant factor of $\mathbf{F}_2^\rho$

following Proposition 2.5 and the fact that for integral ideals $\mathfrak{m}$ prime to $\mathfrak{n}$, the eigenvalues satisfy $v(\mathfrak{m}) = \overline{\eta}(\mathfrak{m}) \cdot v(\mathfrak{m})$. As a consequence, one gets $\big\langle \mathbf{f}_\Sigma^{\mathrm{dep}}, \mathbf{f}_\Sigma^{\mathrm{dep}} \big\rangle_{\mathfrak{n}}' \neq 0$. Identifying $\mathbb{C} \cong \mathbb{C}_p$ as fields, we now have two $\mathbb{C}_p$-valued linear functionals

$$\phi_{\mathfrak{n},\mathbf{F}}^{\mathrm{alg}}(-) = \big(\mathbf{f}_\Sigma^{\mathrm{dep}}, -\big)_{\mathfrak{n}}\,,\ \phi_{\mathfrak{n},\mathbf{F}}^{\mathrm{Pet}}(-) = \big\langle \mathbf{f}_\Sigma^{\mathrm{dep}}, -\big\rangle_{\mathfrak{n}}' :\ \mathcal{S}_k(\mathfrak{n}, \eta; \mathbb{C}_p)_{\mathfrak{M}} \longrightarrow \mathbb{C}_p$$

which are both nonzero on the Σ-depleted Hilbert modular eigenform $\mathbf{F} = \mathbf{f}_\Sigma^{\mathrm{dep}}$.

Definition 4.2. We define a *p-adic period* by

$$\Omega_{\mathfrak{n},\Sigma}(\mathbf{f}) := \frac{\phi_{\mathfrak{n},\mathbf{F}}^{\mathrm{Pet}}\big(\mathbf{f}_\Sigma^{\mathrm{dep}}\big)}{\phi_{\mathfrak{n},\mathbf{F}}^{\mathrm{alg}}\big(\mathbf{f}_\Sigma^{\mathrm{dep}}\big)} \in \mathbb{C}_p^\times. \tag{4.13}$$

Of course, one can redo the construction of these pairings by working at level $\mathfrak{c}(\mathbf{f})$ and without Σ-depleting $\mathbf{f}$ first, in which case one obtains a period $\Omega_{\mathfrak{c}(\mathbf{f}),\emptyset}(\mathbf{f})$. We conjecture that if p is unramified in F, then $\Omega_{\mathfrak{n},\Sigma}(\mathbf{f})$ coincides with $\Omega_{\mathfrak{n}(\mathbf{f}),\emptyset}(\mathbf{f})$ up to a p-adic unit (and some other simple factors). Indeed if $F = \mathbb{Q}$ then Ray, Sujatha and Vatsal [RSV23, Lemma 2.20] proved this statement using an appropriate version of Ihara's lemma—we will return to this question again in Section 4.6.

4.4 Congruences between algebraic L-values

We shall now combine together the results in the previous two sections to obtain our p-adic congruences. Suppose we are given two $\mathfrak{p}$-stabilised ordinary eigenforms $\mathbf{f}_1 \in \mathcal{S}_2\big(\mathfrak{c}(\mathbf{f}_1), \mathbf{1}_F; R\big)$ and $\mathbf{f}_2 \in \mathcal{S}_2\big(\mathfrak{c}(\mathbf{f}_2), \mathbf{1}_F; R\big)$, where R is a finite extension of $\mathbb{Z}_p$ with uniformiser ϖ. *Henceforth, one assumes that for some fixed positive integer r:*

$$\mathrm{C}\big(\mathbf{f}_{1,\Sigma}^{\mathrm{dep}}, \mathfrak{m}\big) \equiv \mathrm{C}\big(\mathbf{f}_{2,\Sigma}^{\mathrm{dep}}, \mathfrak{m}\big) \bmod \varpi^r \cdot R \quad \textit{at every ideal } \mathfrak{m}. \tag{4.14}$$

In the next section, we will discuss in greater detail how this situation arises for a pair of congruent modular elliptic curves E_1 and E_2 defined over F.

We once again fix a primitive Hecke eigenform $\mathbf{g}$ of parallel weight one which satisfies Hypothesis 4.2 for both $\mathbf{f}_1$ and $\mathbf{f}_2$, where $\Sigma \subset \mathrm{Spec}(\mathcal{O}_F) - \{\mathfrak{p}\}$

is a finite set consisting of prime $\mathcal{O}_F$-ideals dividing $\mathfrak{c}(\mathbf{f}_1)^\sharp \cdot \mathfrak{c}(\mathbf{f}_2)^\sharp \cdot \mathfrak{c}(\mathbf{g})$ but not $\mathfrak{c}(\mathbf{f}_1)^\flat \cdot \mathfrak{c}(\mathbf{f}_2)^\flat$. In order to express the algebraic L-values for $\mathbf{f}_i \otimes \mathbf{g}^\rho$ at each $i \in \{1, 2\}$ using a common inner product, the calculations are done at a level with stable $\mathfrak{p}$-part, namely

$$\mathfrak{n} := \mathfrak{n}_2 \cdot \mathrm{lcm}\big(\mathfrak{c}(\mathbf{f}_1)^\flat, \mathfrak{c}(\mathbf{f}_2)^\flat\big). \tag{4.15}$$

Here one modifies $\mathfrak{n}_2$ from (4.6) to $\mathfrak{n}_2 := \mathrm{lcm}\big(\mathfrak{c}(\mathbf{f}_1)^\sharp, \mathfrak{c}(\mathbf{f}_2)^\sharp, \mathfrak{p}\cdot\mathfrak{n}_\Sigma^2\big)$ to involve both $\mathbf{f}_1$ and $\mathbf{f}_2$; this new $\mathfrak{n}_2$ clearly divides $\mathfrak{n}' := \mathfrak{p}^{\mathbf{e}} \cdot \mathrm{lcm}\big(\mathfrak{c}(\mathbf{f}_1)^\sharp, \mathfrak{c}(\mathbf{f}_2)^\sharp, \mathfrak{c}(\mathbf{g})^{\mathrm{tame}}, \mathfrak{n}_\Sigma^2\big)$.

Lemma 4.4. *If we put $\overline{\rho} = \overline{\rho}_{\mathbf{f}_1} = \overline{\rho}_{\mathbf{f}_2}$, and if either of the following conditions:*

- $\mathrm{ord}_{\mathfrak{q}}\big(\mathfrak{c}(\mathbf{f}_1)^\sharp\big) = \mathrm{ord}_{\mathfrak{q}}\big(\mathfrak{c}(\mathbf{f}_2)^\sharp\big) = \mathrm{ord}_{\mathfrak{q}}\big(\mathrm{cond}(\overline{\rho})\big)$, *or*
- $\mathrm{ord}_{\mathfrak{q}}\big(\mathrm{cond}(\overline{\rho})\big) = 0$, $\mathrm{ord}_{\mathfrak{q}}\big(\mathfrak{c}(\mathbf{f}_1)^\sharp\big) \leq 2$ *and* $\mathrm{ord}_{\mathfrak{q}}\big(\mathfrak{c}(\mathbf{f}_2)^\sharp\big) \leq 2$, *or*
- $1 = \mathrm{ord}_{\mathfrak{q}}\big(\mathrm{cond}(\overline{\rho})\big) \leq \mathrm{ord}_{\mathfrak{q}}\big(\mathfrak{c}(\mathbf{f}_i)^\sharp\big) \leq \mathrm{ord}_{\mathfrak{q}}\big(\mathrm{cond}(\overline{\rho})\big) + 1$ *for* $i \in \{1, 2\}$

holds at every prime $\mathfrak{q} \in \Sigma$, then Hypothesis 4.3(iii) is automatically satisfied.

Proof. Note that $\mathfrak{n} = \mathrm{lcm}\big(\mathfrak{c}(\mathbf{f}_1), \mathfrak{c}(\mathbf{f}_2), \mathfrak{p} \cdot \mathfrak{n}_\Sigma^2\big)$, and so we must show that the $\mathfrak{q}$-part of the level of $\mathbf{f}_{i,\Sigma}^{\mathrm{dep}} \big| \mathfrak{q}$ is bigger than $\mathfrak{n}^{(\mathfrak{q})} := \mathfrak{q}^{\mathrm{ord}_{\mathfrak{q}}(\mathfrak{n})}$ for each $\mathfrak{q} \in \Sigma$.

(i) *Suppose that* $\mathrm{ord}_{\mathfrak{q}}\big(\mathfrak{c}(\mathbf{f}_1)^\sharp\big) = \mathrm{ord}_{\mathfrak{q}}\big(\mathfrak{c}(\mathbf{f}_2)^\sharp\big) = \mathrm{ord}_{\mathfrak{q}}\big(\mathrm{cond}(\overline{\rho})\big)$*:* then

- *either* $\mathrm{ord}_{\mathfrak{q}}(\mathfrak{n}) = \mathrm{ord}_{\mathfrak{q}}\big(\mathrm{cond}(\overline{\rho})\big) \geq 2$
- *or* $\mathrm{ord}_{\mathfrak{q}}(\mathfrak{n}) = 2 > \mathrm{ord}_{\mathfrak{q}}\big(\mathrm{cond}(\overline{\rho})\big)$.

In both cases $\mathbf{f}_{i,\Sigma}^{\mathrm{dep}}$ has exact $\mathfrak{q}$-level $\mathfrak{n}^{(\mathfrak{q})}$ so the $\mathfrak{q}$-level of $\mathbf{f}_{i,\Sigma}^{\mathrm{dep}} \big| \mathfrak{q}$ is too big.

(ii) *Suppose that* $\mathrm{ord}_{\mathfrak{q}}\big(\mathrm{cond}(\overline{\rho})\big) = 0$, $\mathrm{ord}_{\mathfrak{q}}\big(\mathfrak{c}(\mathbf{f}_1)^\sharp\big) \leq 2$ *and* $\mathrm{ord}_{\mathfrak{q}}\big(\mathfrak{c}(\mathbf{f}_2)^\sharp\big) \leq 2$*:* then it is easy to see that the $\mathfrak{q}$-level of both forms $\mathbf{f}_{1,\Sigma}^{\mathrm{dep}} \big| \mathfrak{q}$ and $\mathbf{f}_{2,\Sigma}^{\mathrm{dep}} \big| \mathfrak{q}$ exceeds $\mathfrak{n}^{(\mathfrak{q})} = \mathfrak{q}^2$.

(iii) *Suppose that* $1 = \mathrm{ord}_{\mathfrak{q}}\big(\mathrm{cond}(\overline{\rho})\big) \leq \mathrm{ord}_{\mathfrak{q}}\big(\mathfrak{c}(\mathbf{f}_i)^\sharp\big) \leq \mathrm{ord}_{\mathfrak{q}}\big(\mathrm{cond}(\overline{\rho})\big) + 1$*:* then

- *either* $\mathrm{ord}_{\mathfrak{q}}\big(\mathfrak{c}(\mathbf{f}_1)^\sharp\big) = \mathrm{ord}_{\mathfrak{q}}\big(\mathfrak{c}(\mathbf{f}_2)^\sharp\big) = \mathrm{ord}_{\mathfrak{q}}\big(\mathrm{cond}(\overline{\rho})\big)$, which is done!

– *or* $\quad \mathrm{ord}_{\mathfrak{q}}\big(\mathfrak{c}(\mathbf{f}_1)^{\sharp}\big) = \mathrm{ord}_{\mathfrak{q}}\big(\mathfrak{c}(\mathbf{f}_2)^{\sharp}\big) = \mathrm{ord}_{\mathfrak{q}}\big(\mathrm{cond}(\overline{\rho})\big) + 1 = 2$

– *or* $\quad \mathrm{ord}_{\mathfrak{q}}\big(\mathfrak{c}(\mathbf{f}_1)^{\sharp}\big) - 1 = \mathrm{ord}_{\mathfrak{q}}\big(\mathfrak{c}(\mathbf{f}_2)^{\sharp}\big) = \mathrm{ord}_{\mathfrak{q}}\big(\mathrm{cond}(\overline{\rho})\big) = 1$

– *or* $\quad \mathrm{ord}_{\mathfrak{q}}\big(\mathfrak{c}(\mathbf{f}_1)^{\sharp}\big) = \mathrm{ord}_{\mathfrak{q}}\big(\mathfrak{c}(\mathbf{f}_2)^{\sharp}\big) - 1 = \mathrm{ord}_{\mathfrak{q}}\big(\mathrm{cond}(\overline{\rho})\big) = 1.$

In the last three cases $\mathrm{ord}_{\mathfrak{q}}(\mathfrak{n}) = \mathrm{ord}_{\mathfrak{q}}\big(\mathrm{cond}(\overline{\rho})\big) + 1 = 2$, hence the $\mathfrak{q}$-level of both Σ-depleted Hilbert modular forms $\mathbf{f}_{1,\Sigma}^{\mathrm{dep}} \,\big|\, \mathfrak{q}$ and $\mathbf{f}_{2,\Sigma}^{\mathrm{dep}} \,\big|\, \mathfrak{q}$ exceeds $\mathfrak{n}^{(\mathfrak{q})} = \mathfrak{q}^2$ again.

We have exhausted all possible scenarios hence the proof is complete. ■

Corollary 4.5. *If $F = \mathbb{Q}$ then Hypothesis 4.3(iii) always holds true.*

Proof. The corollary follows immediately upon combining together each of [Gou90, Proposition 6], [RSV23, Lemma 2.26], and finally Lemma 4.4 above.

■

Proposition 4.6.

(a) The twisted trace operator $\tilde{T}_{\mathbf{g}}$ in (4.8) *preserves p-integral q-expansions.*

(b) The form $\mathbf{g}_{\mathfrak{n}_0} \cdot \mathbf{E}_1(0; \mathfrak{n}'\mathfrak{c}(\mathbf{f})^{\flat}; \eta_{\mathbf{g}}^{-1}) \,\Big|\, U_{\frac{\mathfrak{n}'}{\mathfrak{n}_1}}$ has an algebraic integral q-expansion.

Proof. Beginning with assertion (a), we note that $\frac{\mathfrak{n}_1}{\mathfrak{n}_2} = \prod_{\mathfrak{q}|\mathfrak{c}(\mathbf{g})^{\mathrm{tame}}} \mathfrak{q}^{e_{\mathfrak{q}}}$ say with each exponent $e_{\mathfrak{q}} \geq 0$. If we define an auxiliary level $\mathfrak{n}_2^{\dagger} := \prod_{e_{\mathfrak{q}} \geq 2} \mathfrak{q}^{e_{\mathfrak{q}}-1}$, then the twisted trace operator $\tilde{T}_{\mathbf{g}}(-)$ can be factorised into $\tilde{T}_{\mathbf{g}} = \tilde{T}_{\mathbf{g}}^{\dagger\dagger} \circ \tilde{T}_{\mathbf{g}}^{\dagger}$, where

$$\begin{aligned} \tilde{T}_{\mathbf{g}}^{\dagger}(-) &:= \mathrm{Tr}^{\mathfrak{n}_1 \cdot \mathfrak{c}(\mathbf{f})^{\flat}}_{\mathfrak{n}_2^{\dagger} \cdot \mathfrak{c}(\mathbf{f})^{\flat}}\big(- \,\big|\, J_{\mathfrak{n}_1 \cdot \mathfrak{c}(\mathbf{f})^{\flat}}\big) \,\big|\, J^{-1}_{\mathfrak{n}_2^{\dagger} \cdot \mathfrak{c}(\mathbf{f})^{\flat}} \quad \text{and} \\ \tilde{T}_{\mathbf{g}}^{\dagger\dagger}(-) &:= \mathrm{Tr}^{\mathfrak{n}_2^{\dagger} \cdot \mathfrak{c}(\mathbf{f})^{\flat}}_{\mathfrak{n}_2 \cdot \mathfrak{c}(\mathbf{f})^{\flat}}\big(- \,\big|\, J_{\mathfrak{n}_2^{\dagger} \cdot \mathfrak{c}(\mathbf{f})^{\flat}}\big) \,\big|\, J^{-1}_{\mathfrak{n}_2 \cdot \mathfrak{c}(\mathbf{f})^{\flat}} \,. \end{aligned}$$

Let us make the important observation that $\mathrm{supp}(\mathfrak{n}_2^{\dagger}) = \mathrm{supp}(\mathfrak{n}_1)$, whilst tracing down from level $\mathfrak{n}_2^{\dagger}$ to level $\mathfrak{n}_2$ consists exclusively of removing primes with $e_{\mathfrak{q}} \geq 1$. Let $\mathfrak{a}$ be an $\mathcal{O}_F$-ideal. We recall the trace mapping relation in Proposition 2.7: for a Hilbert modular form $\mathbf{H} \in \mathcal{M}_k(\mathfrak{a}\mathfrak{q}, \epsilon)$ of weight $k \geq 2$,

$$\mathrm{Tr}^{\mathfrak{a}\mathfrak{q}}_{\mathfrak{a}}\big(\mathbf{H} \,\big|\, J_{\mathfrak{a}\mathfrak{q}}\big) = \begin{cases} \mathcal{N}(\mathfrak{q})^{1-k/2} \cdot \mathbf{H} \,\big|\, U_{\mathfrak{q}} \circ J_{\mathfrak{a}}, & \text{if } \mathfrak{q} \mid \mathfrak{a} \\ \mathcal{N}(\mathfrak{q})^{1-k/2} \cdot \mathbf{H} \,\big|\, \big(U_{\mathfrak{q}} + \mathcal{N}(\mathfrak{q})^{k-1} \cdot \langle \mathfrak{q} \rangle \circ \mathfrak{q}\big) \circ J_{\mathfrak{a}}, & \text{if } \mathfrak{q} \nmid \mathfrak{a}. \end{cases}$$

A simple induction argument that exclusively utilises the first case gives us

$$\mathbf{H}^{\dagger} := \tilde{T}^{\dagger}_{\mathbf{g}}(\mathbf{H}) = \mathcal{N}\big(\mathfrak{n}_1/\mathfrak{n}_2^{\dagger}\big)^{1-k/2} \cdot \mathbf{H} \,\Big|\, \prod_{e_{\mathfrak{q}}\geq 2} U_{\mathfrak{q}}^{e_{\mathfrak{q}}-1}.$$

Next, employing induction and using instead the second case yields the identity

$$\tilde{T}_{\mathbf{g}}(\mathbf{H}) = \tilde{T}^{\dagger\dagger}_{\mathbf{g}}(\mathbf{H}^{\dagger}) = \mathcal{N}\big(\mathfrak{n}_2^{\dagger}/\mathfrak{n}_2\big)^{1-k/2} \cdot \mathbf{H}^{\dagger} \,\Big|\, \prod_{e_{\mathfrak{q}}\geq 1} \big(U_{\mathfrak{q}} + \mathcal{N}(\mathfrak{q})^{k-1} \cdot \langle \mathfrak{q} \rangle \circ \mathfrak{q}\big).$$

At weight $k = 2$, one has

$$\tilde{T}_{\mathbf{g}}(\mathbf{H}) = \Big(\mathbf{H} \,\Big|\, \prod_{e_{\mathfrak{q}}\geq 2} U_{\mathfrak{q}}^{e_{\mathfrak{q}}-1}\Big) \,\Big|\, \prod_{e_{\mathfrak{q}}\geq 1} \big(U_{\mathfrak{q}} + \mathcal{N}(\mathfrak{q}) \cdot \langle \mathfrak{q} \rangle \circ \mathfrak{q}\big)\,;$$

because the Hecke operators $U_{\mathfrak{q}}$, the diamond operators $\langle \mathfrak{q} \rangle$, and the $\mathfrak{q}$-operators preserve p-integrality, part (a) follows.

To prove (b), note that the primitive form $\mathbf{g}$ has an algebraic integer Fourier expansion and so does $\mathbf{g}_{\mathfrak{n}_0}$, while by the formula in [Pan89, Proposition 4.2]

$$\mathbf{E}_1(0;\eta_{\mathbf{g}}^{-1};\mathfrak{n}' \cdot \mathfrak{c}(\mathbf{f})^{\flat})_{\lambda} = \mathcal{N}(\tilde{t}_{\lambda})^{-3/2} \cdot \sum_{0\ll \xi \in \tilde{t}_{\lambda}} \sum_{\tilde{\xi}=\tilde{b}\tilde{c}} \eta^{*}_{\mathbf{g}}(\tilde{c})^{-1} \cdot e_F(\xi z).$$

Therefore the product $\mathbf{g}_{\mathfrak{n}_0} \cdot \mathbf{E}_1(0;\eta_{\mathbf{g}}^{-1};\mathfrak{n}' \cdot \mathfrak{c}(\mathbf{f})^{\flat})$ also has an algebraic integer Fourier expansion, and the $U_{\frac{\mathfrak{n}'}{\mathfrak{n}_1}}$-operator will preserve integrality (at the very least on holomorphic HMFs). ■

At our current set up, the levels of the Σ-depleted forms $\mathbf{f}^{\mathrm{dep}}_{1,\Sigma}$ and $\mathbf{f}^{\mathrm{dep}}_{2,\Sigma}$ divide into our previously fixed $\mathfrak{n}$ in (4.15) (in fact, equality occurs if Σ is chosen as in Lemma 4.8). This is crucial for the following result which establishes the congruence modulo p of the special values of the Σ-depleted L-functions attached to $\mathbf{f}_1 \otimes \mathbf{g}^{\rho}$ and $\mathbf{f}_2 \otimes \mathbf{g}^{\rho}$.

Theorem 4.7. *Assume that the Hypotheses 4.3(i)-(iii) hold for* $\mathbf{f}_1$ *and* $\mathbf{f}_2$. *Then following the notation of Section 4.2, one obtains*

$$\begin{aligned} &\mathfrak{u}_{\mathfrak{n}'}(\mathbf{f}_1) \times \upsilon(\mathbf{g}) \times \frac{\mathcal{N}\big(\mathfrak{c}(\mathbf{g})\mathfrak{d}^2\big)^{\frac{1}{2}}}{\alpha_{\mathfrak{p}}(\mathbf{f}_1)^{\mathrm{ord}_{\mathfrak{p}}\,\mathfrak{c}(\mathbf{g})}} \times \frac{\Psi_{\Sigma}\big(\mathbf{f}_1, \mathbf{g}^{\rho}, 1\big)}{\Omega_{\mathfrak{n},\Sigma}(\mathbf{f}_1)} \\ \equiv\; &\mathfrak{u}_{\mathfrak{n}'}(\mathbf{f}_2) \times \upsilon(\mathbf{g}) \times \frac{\mathcal{N}\big(\mathfrak{c}(\mathbf{g})\mathfrak{d}^2\big)^{\frac{1}{2}}}{\alpha_{\mathfrak{p}}(\mathbf{f}_2)^{\mathrm{ord}_{\mathfrak{p}}\,\mathfrak{c}(\mathbf{g})}} \times \frac{\Psi_{\Sigma}\big(\mathbf{f}_2, \mathbf{g}^{\rho}, 1\big)}{\Omega_{\mathfrak{n},\Sigma}(\mathbf{f}_2)} \quad \bmod \varpi^r \cdot \mathcal{O}_{\mathbb{C}_p} \end{aligned}$$

where the scalar terms $\mathfrak{u}_{\mathfrak{n}'}(\mathbf{f}_i) := \mathrm{C}\big(\mathbf{f}_i, \mathfrak{c}(\mathbf{f}_i)^{\flat}\big) \cdot \alpha_{\frac{\mathfrak{n}'}{\mathfrak{c}(\mathbf{g})^{\mathrm{tame}}}}(\mathbf{f}_i) \cdot \big(-4\sqrt{-1}\big)^{-n} \in \mathcal{O}_{\mathbb{C}_p}^{\times}$.

Proof. We first observe that by Proposition 4.6(a) and (b),

$$\Phi\big(\mathbf{g}\cdot\mathbf{E}_1\big) := \tilde{T}_{\mathbf{g}}\Big(\mathbf{g}_{\mathfrak{n}_0}\cdot\mathbf{E}_1(0;\eta_{\mathbf{g}}^{-1};\mathfrak{p}^{\mathbf{e}-1}\mathfrak{n})\,\Big|\,U_{\frac{\mathfrak{n}'}{\mathfrak{n}_1}}\Big)$$

is holomorphic of level $\mathfrak{n}$ and character $\mathbf{1}_F$, and moreover $\Phi\big(\mathbf{g}\cdot\mathbf{E}_1\big)\in\mathcal{O}_{\mathbb{C}_p}[\![q]\!]$; enlarging R if necessary, one may then assume that $\Phi\big(\mathbf{g}\cdot\mathbf{E}_1\big)\in\mathcal{M}_2\big(\mathfrak{n},\mathbf{1}_F;R\big)$. One clearly deduces that the ordinary projection $\Phi\big(\mathbf{g}\cdot\mathbf{E}_1\big)^{\mathrm{ord}} := \Phi\big(\mathbf{g}\cdot\mathbf{E}_1\big)\,\Big|\,\mathfrak{e}^{\mathrm{ord}}$ also has p-integral Fourier coefficients, although it might not end up being a cusp form in general.

Writing $\mathbf{H}^{\mathrm{ord}}(\mathfrak{n};R)$ for the algebra generated by the Hecke operators over F acting on the space $\mathcal{M}_2^{\mathrm{ord}}\big(\mathfrak{n},\mathbf{1}_F;R\big)$, one observes a surjection of R-algebras $\mathcal{I}:\mathbf{H}^{\mathrm{ord}}(\mathfrak{n};R)\twoheadrightarrow\mathbf{T}^{\mathrm{ord}}(\mathfrak{n};R)$. Moreover, $\mathbf{T}^{\mathrm{ord}}(\mathfrak{n};R)\overset{\sim}{\to}\bigoplus_i\mathbf{T}(\mathfrak{n};R)_{\mathfrak{M}_i}$ where the direct sum ranges over a finite set of maximal ideals $\mathfrak{M}_i\lhd\mathbf{T}(\mathfrak{n};R)$, namely the associated primes of the Hecke algebra. The congruence (4.14) between the weight two Hilbert modular forms ensures that the specialisation maps $\lambda_{\mathbf{F}}:\mathbf{T}(\mathfrak{n};R)\twoheadrightarrow\mathbf{T}^{\mathrm{ord}}(\mathfrak{n};R)\to R$ for both of $\mathbf{F}=\mathbf{f}_{1,\Sigma}^{\mathrm{dep}}$ and $\mathbf{F}=\mathbf{f}_{2,\Sigma}^{\mathrm{dep}}$ are congruent mod ϖ^r, hence they share the same maximal ideal $\mathfrak{M}$.

We define $\mathfrak{M}^{\dagger}:=\mathcal{I}^{-1}(\mathfrak{M})$, which is a maximal ideal of $\mathbf{H}^{\mathrm{ord}}(\mathfrak{n};R)$. One may then write $\mathfrak{e}_{\mathfrak{M}}^{\dagger}\in\mathbf{H}^{\mathrm{ord}}(\mathfrak{n};R)$ and $\mathfrak{e}_{\mathfrak{M}}\in\mathbf{T}^{\mathrm{ord}}(\mathfrak{n};R)$ for the corresponding projectors. If we choose $\mathbf{F}=\mathbf{f}_{1,\Sigma}^{\mathrm{dep}}$ or $\mathbf{F}=\mathbf{f}_{2,\Sigma}^{\mathrm{dep}}$, then a simple series of calculations reveals that

$$\begin{aligned}\phi_{\mathfrak{n},\mathbf{F}}^{\mathrm{Pet}}\big(\Phi\big(\mathbf{g}\cdot\mathbf{E}_1\big)\big) &= \big\langle\mathbf{F},\Phi(\mathbf{g}\cdot\mathbf{E}_1)\big\rangle_{\mathfrak{n}}' = \big\langle\mathbf{F},\Phi(\mathbf{g}\cdot\mathbf{E}_1)^{\mathrm{ord}}\big\rangle_{\mathfrak{n}}' \\ &= \big\langle\mathbf{F}\,\big|\,\mathfrak{e}_{\mathfrak{M}},\Phi(\mathbf{g}\cdot\mathbf{E}_1)^{\mathrm{ord}}\big\rangle_{\mathfrak{n}}' = \big\langle\mathbf{F},\Phi(\mathbf{g}\cdot\mathbf{E}_1)^{\mathrm{ord}}\,\big|\,\mathfrak{e}_{\mathfrak{M}}^{\dagger}\big\rangle_{\mathfrak{n}}'\,.\end{aligned}$$

Using the period from Definition 4.2, if $\mathbf{F}=\mathbf{f}_{i,\Sigma}^{\mathrm{dep}}$ then one has a compatibility

$$\phi_{\mathfrak{n},\mathbf{F}}^{\mathrm{Pet}}\big(\Phi\big(\mathbf{g}\cdot\mathbf{E}_1\big)^{\mathrm{ord}}\,\big|\,\mathfrak{e}_{\mathfrak{M}}^{\dagger}\big) \;=\; \phi_{\mathfrak{n},\mathbf{F}}^{\mathrm{alg}}\big(\Phi\big(\mathbf{g}\cdot\mathbf{E}_1\big)^{\mathrm{ord}}\,\big|\,\mathfrak{e}_{\mathfrak{M}}^{\dagger}\big)\times\Omega_{\mathfrak{n},\Sigma}(\mathbf{f}_i).$$

Put $\mathbf{H}=\Phi\big(\mathbf{g}\cdot\mathbf{E}_1\big)^{\mathrm{ord}}\,\big|\,\mathfrak{e}_{\mathfrak{M}}^{\dagger}\in\mathcal{S}_2(\mathfrak{n},\mathbf{1}_F;R)_{\mathfrak{M}}$. By definition of the twisted Petersson inner product (4.12), one has

$$\phi_{\mathfrak{n},\mathbf{F}}^{\mathrm{alg}}(\mathbf{H}) \;=\; \Omega_{\mathfrak{n},\Sigma}(\mathbf{f}_i)^{-1}\times\Big\langle\mathbf{f}_{i,\Sigma}^{\mathrm{dep}},\Phi(\mathbf{g}\cdot\mathbf{E}_1)\Big\rangle_{\mathfrak{n}}' \;=\; \frac{\big\langle\big(\mathbf{f}_{i,\Sigma}^{\mathrm{dep}}\big)^{\rho},\Phi(\mathbf{g}\cdot\mathbf{E}_1)\,\big|\,J_{\mathfrak{n}}\big\rangle_{\mathfrak{n}}}{\Omega_{\mathfrak{n},\Sigma}(\mathbf{f}_i)}\,.$$

The numerator can be readily evaluated using Proposition 4.2, in which case

$$\phi^{\mathrm{alg}}_{\mathfrak{n},\mathbf{F}}(\mathbf{H}) \;=\; \mathfrak{u}_{\mathfrak{n}'}(\mathbf{f}_i) \times \frac{\upsilon(\mathbf{g}) \cdot \mathcal{N}\big(\mathfrak{c}(\mathbf{g})\mathfrak{d}^2\big)^{\frac{1}{2}}}{\alpha_{\mathfrak{p}}(\mathbf{f}_i)^{\mathrm{ord}_{\mathfrak{p}}\,\mathfrak{c}(\mathbf{g})}} \times \frac{\Psi_\Sigma\big(\mathbf{f}_i, \mathbf{g}^\rho, 1\big)}{\Omega_{\mathfrak{n},\Sigma}(\mathbf{f}_i)} \,.$$

By assumption, $\mathrm{C}\big(\mathbf{f}^{\mathrm{dep}}_{1,\Sigma}, \mathfrak{n}\big) \equiv \mathrm{C}\big(\mathbf{f}^{\mathrm{dep}}_{2,\Sigma}, \mathfrak{n}\big) \bmod \varpi^r \cdot R$ at every $\mathcal{O}_F$-ideal $\mathfrak{n}$. Therefore the algebraic pairing $\big(\mathbf{f}^{\mathrm{dep}}_{i,\Sigma}, -\big)_{\mathfrak{n}}$ must preserve congruences modulo ϖ^r as we switch between indices $i = 1$ and $i = 2$. The result follows easily. ■

Remark 7. As we will see, one can also consider a family of twists $\mathbf{g} \otimes \chi$ where χ ranges over characters factoring through the cyclotomic $\mathbb{Z}_p$-extension F^{cyc}/F. One may then interpret Theorem 4.7 as the congruence mod ϖ^r of the Σ-depleted p-adic L-functions attached to $\mathbf{f}_1 \otimes \mathbf{g}^\rho$ and $\mathbf{f}_2 \otimes \mathbf{g}^\rho$, respectively. In fact this congruence will be deduced by simply reading off the special values.

4.5 Elliptic curves over totally real fields

We recall our fixed prime $p > 2$ assumed to be inert in F, so that $p \cdot \mathcal{O}_F = \mathfrak{p}$. Let E_1 and E_2 be two modular elliptic curves defined over the totally real field F, and assume they are $\mathfrak{p}$-congruent. Lastly, we ensure that both elliptic curves E_1 and E_2 either have good ordinary or bad multiplicative reduction over $F_{\mathfrak{p}}$, hence $a_{\mathfrak{p}}(E_1)$ and $a_{\mathfrak{p}}(E_2)$ are both p-adic units. We shall write $\mathbf{f}_1^\dagger$ and $\mathbf{f}_2^\dagger$ to denote the Hilbert modular cusp forms of parallel weight 2 and trivial nebentypus, associated to the curves E_1 and E_2 respectively; to be consistent, their $\mathfrak{p}$-stabilisations will be denoted by $\mathbf{f}_1$ and $\mathbf{f}_2$ as before.

Suppose $\tau : G_F \to \mathrm{GL}_2(\overline{\mathbb{Q}})$ is a two-dimensional Artin representation induced from a primitive Hecke character over K, where K/F is a CM field extension. One can consider the weight one theta-series, $\mathbf{g}_\tau$, arising from τ in [Shi78, §5], so that

$$L\big(\mathbf{g}_\tau, s\big) = \prod_{\mathfrak{q} \in \mathrm{Spec}(\mathcal{O}_F)} \det\big(\mathrm{Id} - \tau(\mathrm{Frob}_{\mathfrak{q}}) \cdot \mathcal{N}(\mathfrak{q})^{-s}\big)$$

for all $s \in \mathbb{C}$ with $\mathrm{Re}(s) \gg 0$. In particular, one has $\mathbf{g}_\tau \in \mathcal{M}_1\big(\mathfrak{c}(\mathbf{g}_\tau), \eta_\tau\big)$ where $\eta_\tau = \det(\tau)$ and $\mathfrak{c}(\mathbf{g}_\tau) = \mathrm{cond}(\tau)$. To make any further progress, we must restrict the allowable levels of the forms $\mathbf{f}_1$, $\mathbf{f}_2$ and $\mathbf{g}_\tau$ appropriately, in order to apply our preceding congruence formulae.

Hypothesis 4.4. *For every prime ideal* $\mathfrak{q} \in \Sigma^{\mathrm{bad}}$*, either*

- $\mathrm{ord}_{\mathfrak{q}}\big(\mathfrak{c}(\mathbf{f}_1)^\sharp\big) = \mathrm{ord}_{\mathfrak{q}}\big(\mathfrak{c}(\mathbf{f}_2)^\sharp\big) = \mathrm{ord}_{\mathfrak{q}}\big(\,\mathrm{cond}(\overline{\rho})\big)$*, or*
- $\mathrm{ord}_{\mathfrak{q}}\big(\,\mathrm{cond}(\overline{\rho})\big) = 0$*,* $\mathrm{ord}_{\mathfrak{q}}\big(\mathfrak{c}(\mathbf{f}_1)^\sharp\big) \leq 2$ *and* $\mathrm{ord}_{\mathfrak{q}}\big(\mathfrak{c}(\mathbf{f}_2)^\sharp\big) \leq 2$*, or*
- $1 = \mathrm{ord}_{\mathfrak{q}}\big(\,\mathrm{cond}(\overline{\rho})\big) \leq \mathrm{ord}_{\mathfrak{q}}\big(\mathfrak{c}(\mathbf{f}_i)^\sharp\big) \leq \mathrm{ord}_{\mathfrak{q}}\big(\,\mathrm{cond}(\overline{\rho})\big) + 1$ *for* $i \in \{1, 2\}$*.*

In addition, if $\mathfrak{q}^2 \mid \mathfrak{c}(\mathbf{f}_i)$*, then* $\mathrm{ord}_{\mathfrak{q}}\big(\mathfrak{c}(\mathbf{f}_i)\big) \leq \mathrm{ord}_{\mathfrak{q}}\big(\,\mathrm{cond}(\tau)\big)$ *for* $i \in \{1, 2\}$*.*

The first three conditions are symmetric in E_1 and E_2, and imply that the levels of the eigenforms $\mathbf{f}_i$ are close to each other, as well as to the Artin conductor of $\overline{\rho}$. The fourth condition will be satisfied whenever the Hecke character ϑ_K that induces τ has conductor sufficiently divisible by the primes dividing $\mathfrak{c}(\mathbf{f}_1) \cdot \mathfrak{c}(\mathbf{f}_2) \cdot \mathcal{O}_K$. *Henceforth we assume Hypothesis 4.4 holds for the elliptic curves* E_1 *and* E_2*, and the fixed Artin representation* $\tau : G_F \to \mathrm{GL}_2(\overline{\mathbb{Q}})$. The following technical result guarantees that Theorem 4.7 is applicable in the scenario that we have set up.

Lemma 4.8. *If* $\chi \neq \mathbf{1}_F$ *is a finite order character factoring through* $F(\mu_{p^\infty})/F$ *and if we adopt the following choice of finite places*

$$\Sigma := \Big\{\mathfrak{q} \in \mathrm{Spec}(\mathcal{O}_F) : \mathfrak{q} \mid \mathfrak{c}(\mathbf{f}_1) \cdot \mathfrak{c}(\mathbf{f}_2) \cdot \mathfrak{c}(\mathbf{g}_\tau) \ \textit{and}\ \mathfrak{q} \neq \mathfrak{p}\Big\},$$

then Hypotheses 4.2(i)-(iv) and 4.3(i)-(iii) hold for $(\mathbf{f}_1, \mathbf{g}_\tau \otimes \chi)$ *and* $(\mathbf{f}_2, \mathbf{g}_\tau \otimes \chi)$*.*

Proof. First, we check through the conditions listed in Hypothesis 4.2 in order:

(i) our choice of Σ ensures that $\mathfrak{c}(\mathbf{f}_1)^\flat = \mathfrak{c}(\mathbf{f}_2)^\flat = \mathcal{O}_F$, hence $\mathrm{C}\big(\mathbf{f}_i, \mathfrak{c}(\mathbf{f}_i)^\flat\big) = 1$;

(ii) $\mathbf{g}_\tau$ is a primitive form whenever the Hecke character over K is primitive;

(iii) the level of $\mathbf{g}_\tau \otimes \chi$ is divisible by $\mathfrak{p}^2$ provided χ is a non-trivial character;

(iv) Hypothesis 4.4 and our choice of Σ imply that $\mathfrak{c}(\mathbf{f}_i)^\sharp \,\big|\, \mathrm{lcm}\,\big(\mathfrak{c}(\mathbf{g}_\tau \otimes \chi), \mathfrak{n}_\Sigma\big)$.

Second, we check through the various conditions in Hypothesis 4.3 in order:

(i) throughout this section, p is assumed to be inert in the totally real field;

(ii) the absolute irreducibility and $\mathfrak{p}$-ordinarity follows from Hypotheses 4.1, while $\overline{\rho}^{\,\mathrm{ss}}_{E_i}\big|_{G_{F_\mathfrak{p}}} \sim \begin{pmatrix} \omega\phi_\alpha^{-1} & 0 \\ 0 & \phi_\alpha \end{pmatrix}$ and $\omega\big|_{G_{F_\mathfrak{p}}} \neq \mathbf{1}_{F_\mathfrak{p}}$ since $F_\mathfrak{p} \cap \mathbb{Q}_p(\mu_p) = \mathbb{Q}_p$;

(iii) Hypothesis 4.4 in tandem with Lemma 4.4 implies Hypothesis 4.3(iii).

All seven required conditions have been verified hence the proof is finished. ■

With the above choice of Σ, the Σ-depleted forms $\mathbf{f}_{1,\Sigma}^{\mathrm{dep}}$ and $\mathbf{f}_{2,\Sigma}^{\mathrm{dep}}$ therefore satisfy condition (4.14), and using Lemma 4.4 one can show that their levels are equal to our fixed $\mathfrak{n}$. We now present a p-adic analogue of Theorem 4.7.

Theorem 4.9. *Under Hypothesis 4.4 and for the same choice of Σ as in Lemma 4.8, one obtains a congruence of imprimitive p-adic L-functions*

$$L_{p,(\Sigma)}^{\Omega_\mathfrak{n}}\big(E_1, \tau \otimes \omega^t\big) \equiv L_{p,(\Sigma)}^{\Omega_\mathfrak{n}}\big(E_2, \tau \otimes \omega^t\big) \mod \mathfrak{p}$$

at each separate branch $t \in \{0, \ldots, p-2\}$.

Proof. We begin by choosing the integer $r \geq 1$ such that $\mathfrak{p} \cdot \mathcal{O}_{\mathbb{C}_p} = \varpi^r \cdot \mathcal{O}_{\mathbb{C}_p}$, which means $\mathrm{C}\big(\mathbf{f}_{1,\Sigma}^{\mathrm{dep}}, \mathfrak{m}\big) \equiv \mathrm{C}\big(\mathbf{f}_{2,\Sigma}^{\mathrm{dep}}, \mathfrak{m}\big) \mod \varpi^r$ for *every* ideal $\mathfrak{m} \lhd \mathcal{O}_F$. Now at any non-trivial [1] Hecke character χ of $\mathfrak{p}$-power conductor with $\chi\big|_{\mathbb{F}_p^\times} = \omega^t$,

$$\begin{aligned} &\chi\Big(L_{p,(\Sigma)}^{\Omega_\mathfrak{n}}\big(E_i, \tau \otimes \omega^t\big)\Big) \\ &= \frac{\epsilon_F(\tau \otimes \chi, 0)}{\alpha_\mathfrak{p}(\mathbf{f}_i)^{\mathrm{ord}_\mathfrak{p}(\mathrm{cond}(\tau\otimes\chi))}} \times \mathcal{A}_\mathfrak{p}\big(E_i, \tau \otimes \chi\big) \times \frac{\Psi_\Sigma\big(E_i, \tau^* \otimes \chi^{-1}, 1\big)}{\Omega_{\mathfrak{n},\Sigma}(\mathbf{f}_i)} \\ &= \sqrt{-1}^{[F:\mathbb{Q}]} \times \upsilon(\mathbf{g}_\tau \otimes \chi) \times \frac{\mathcal{N}\big(\mathfrak{c}(\mathbf{g}_\tau \otimes \chi) \times \mathfrak{d}^2\big)^{\frac{1}{2}}}{\alpha_\mathfrak{p}(\mathbf{f}_i)^{\mathrm{ord}_\mathfrak{p}\mathfrak{c}(\mathbf{g}_\tau\otimes\chi)}} \times \frac{\Psi_\Sigma\big(\mathbf{f}_i, (\mathbf{g}_\tau \otimes \chi)^\rho, 1\big)}{\Omega_{\mathfrak{n},\Sigma}(\mathbf{f}_i)} \end{aligned}$$

where $A_\mathfrak{p}\big(E_i, \tau\otimes\chi\big) = \Big(1 - \frac{\chi^{-1}(\mathfrak{p})\cdot\alpha_\mathfrak{p}(\mathbf{g})}{\alpha_\mathfrak{p}(\mathbf{f}_i)}\Big)\Big(1 - \frac{\chi^{-1}(\mathfrak{p})\cdot\beta_\mathfrak{p}(\mathbf{g})}{\alpha_\mathfrak{p}(\mathbf{f}_i)}\Big)$ is the $\mathfrak{p}$-adic multiplier. The second equality follows from a combination of a standard ϵ-factor identity

$$\epsilon_F(\tau \otimes \chi, s)\Big|_{s=0} = \sqrt{-1}^{[F:\mathbb{Q}]} \cdot D_F \cdot \sqrt{\mathcal{N}_{F/\mathbb{Q}}\big(\mathfrak{c}(\mathbf{g}_\tau \otimes \chi)\big)} \cdot \upsilon(\mathbf{g}_\tau \otimes \chi)$$

---

[1] If $\chi = \mathbf{1}_F$ then one needs to introduce an extra Euler factor at $\mathfrak{p}$ to obtain a congruence, which relates back to why the condition $\mathfrak{p}^2 \mid \mathfrak{c}(\mathbf{g}_\tau \otimes \chi)$ was necessary in Hypothesis 4.2(iii).

(c.f. [Del16, Lemma B.1]) together with the property that $A_{\mathfrak{p}}\big(E_i, \tau\otimes\chi\big) = 1$ whenever one twists by $\chi \neq \mathbf{1}_F$.

Lemma 4.8 confirms that the congruence in Theorem 4.7 is valid for the family of weight one twisted HMFs $\mathbf{g}_\tau\otimes\chi$ with $\chi\neq\mathbf{1}_F$. It follows readily that

$$\mathfrak{u}_{\mathfrak{n}'}(\mathbf{f}_1)\cdot\chi\left(L^{\Omega_{\mathfrak{n}}}_{p,(\Sigma)}\big(E_1,\tau\otimes\omega^t\big)\right) \equiv \mathfrak{u}_{\mathfrak{n}'}(\mathbf{f}_2)\cdot\chi\left(L^{\Omega_{\mathfrak{n}}}_{p,(\Sigma)}\big(E_2,\tau\otimes\omega^t\big)\right) \mod \varpi^r\cdot\mathcal{O}_{\mathbb{C}_p}.$$

Of course, we have been forced to omit the twist by the trivial character $\chi = \mathbf{1}_F$. Fortunately the (punctured) set of finite order characters

$$\mathrm{Hom}_{\mathrm{cont}}\Big(\mathrm{Gal}\left(F(\mu_{p^\infty})/F\right), \overline{\mathbb{Q}_p}^{\times}\Big)_{\mathrm{tors}} - \big\{\mathbf{1}_F\big\}$$

is p-adically dense inside the space parameterising the cyclotomic deformations of the double product $\mathbf{f}_i\otimes(\mathbf{g}_\tau\otimes\chi)^\rho$. Using properties of continuity, one can therefore extend the mod ϖ^r congruence above to include the p-adic L-functions themselves.

Lastly, the scalar terms from Theorem 4.7 satisfy $\mathfrak{u}_{\mathfrak{n}'}(\mathbf{f}_1)\equiv\mathfrak{u}_{\mathfrak{n}'}(\mathbf{f}_2) \mod \mathfrak{p}$ due to the congruence $\alpha_{\mathfrak{p}}(\mathbf{f}_1)\equiv\alpha_{\mathfrak{p}}(\mathbf{f}_2) \mod \mathfrak{p}$, and the result follows. ■

Corollary 4.10. *Assume that Hypothesis 4.4 holds for* E_1*,* E_2 *and* τ*, and let* Σ *be as in Lemma 4.8. We also recall from Section 4.1 that* $\mu(E_i,\tau,\omega^t)_{\mathrm{an}}$ *and* $\lambda(E_i,\tau,\omega^t)_{\mathrm{an}}$ *represents the* μ*-invariant and* λ*-invariant of the full* p*-adic* L*-function* $L^{\Omega_{\mathfrak{n}}}_{p,\emptyset}(E_1,\tau\otimes\omega^t)$*, respectively.*

(a) The invariant $\mu(E_1,\tau,\omega^t)_{\mathrm{an}} = 0$ *if and only if* $\mu(E_2,\tau,\omega^t)_{\mathrm{an}} = 0$*.*

(b) If one (and therefore both) of these μ*-invariants equals zero, then*

$$\lambda(E_1,\tau,\omega^t)_{\mathrm{an}} = \lambda(E_2,\tau,\omega^t)_{\mathrm{an}} + \sum_{\mathfrak{q}\in\Sigma}\mathbf{e}_{\mathfrak{q}}(E_2,\tau,\omega^t) - \mathbf{e}_{\mathfrak{q}}(E_1,\tau,\omega^t).$$

Proof. One begins by reintroducing the local Euler factors removed in the process of depleting the eigenforms $\mathbf{f}_i$, for each $i = 1,2$. Let $\mathbf{e}_{\mathfrak{q}}(E_i,\tau,\omega^t)$ be the cyclotomic λ-invariant of the p-adic polynomial interpolating the local L-factor $L_{\mathfrak{q}}(E_i,\tau\otimes\psi\omega^t,1)$ at characters $\psi: G_\infty\to\mathbb{Q}_p^\times$. The assertions follow from the nontriviality of the congruence obtained in Theorem 4.9 and the fact that $\mu\big(\mathbf{L}^{\Omega_{\mathfrak{n}}}_{p,\emptyset}(E_i,\tau\otimes\omega^t)\big) = \mu\big(\mathbf{L}^{\Omega_{\mathfrak{n}}}_{p,(\Sigma)}(E_i,\tau\otimes\omega^t)\big)$. ■

4.6 Canonical periods and Ihara's Lemma

The reader may have noticed that the canonical periods $\Omega_{\mathfrak{n},\Sigma}(\mathbf{f}_i)$ differ from the automorphic periods associated to $E_{i/F}$. We recall from Definition 4.2 that

$$\Omega_{\mathfrak{c}(\mathbf{f}_i),\emptyset}(\mathbf{f}_i) \;=\; \frac{\langle \mathbf{f}_i, \mathbf{f}_i\rangle'_{\mathfrak{c}(\mathbf{f}_i)}}{(\mathbf{f}_i, \mathbf{f}_i)_{\mathfrak{c}(\mathbf{f}_i)}} \;=\; \frac{\big\langle \mathbf{f}_i, \mathbf{f}_i \,\big|\, J_{\mathfrak{c}(\mathbf{f}_i)}\big\rangle_{\mathfrak{c}(\mathbf{f}_i)}}{(\mathbf{f}_i, \mathbf{f}_i)_{\mathfrak{c}(\mathbf{f}_i)}} \;.$$

The numerator is the twisted Petersson inner product of $\mathbf{f}_i^\dagger$ with itself (at weight two), and the denominator is the congruence number $\mathfrak{h}_{E_i} = (\mathbf{f}_i, \mathbf{f}_i)_{\mathfrak{c}(\mathbf{f}_i)}$ for the HMF. Therefore, if we divide through by this value, the p-adic L-function for $E_i \otimes \tau$ has $\mathfrak{h}_{E_i}$ in its numerator and $\Omega^{\mathrm{Pet}}_{E_i/F} := \langle \mathbf{f}_i^\dagger, \mathbf{f}_i^\dagger\rangle_{\mathfrak{c}(\mathbf{f}_i)}$ in its denominator.

Question. *Is it always the case that* $\Omega_{\mathfrak{n},\Sigma}(\mathbf{f}_i) = \big(p\text{-adic unit}\big) \times \Omega_{\mathfrak{c}(\mathbf{f}_i),\emptyset}(\mathbf{f}_i)$*?*

To work in more generality, let $\mathbf{F}$ be a Hilbert modular eigenform of parallel weight $k \geq 2$, level $\mathfrak{c}$ and Nebentype η, along with a maximal ideal $\mathfrak{M} \lhd \mathbf{T}(\mathfrak{c}; R)$. Applying the Eichler-Shimura-Harder isomorphism, an R-valued Hecke eigenform $\mathbf{F}$ corresponds to a d-cocycle in $\delta(\mathbf{F}) \in H^n_{\mathrm{cusp}}\big(Y_1(\mathfrak{c})^{\mathrm{an}}, V_{k-2}(R)\big)$ where $Y_1(\mathfrak{c})^{\mathrm{an}}$ is the (analytic) Hilbert modular variety of level $K_1(\mathfrak{c})$, the symbol $V_{k-2}(R)$ represents the sheaf of homogeneous polynomials of degree $k-2$, and $n = [F : \mathbb{Q}]$. In fact, if $k > 2$ then $H^n_{\mathrm{cusp}}\big(Y_1(\mathfrak{c})^{\mathrm{an}}, V_{k-2}(R)\big) \cong H^n_!\big(Y_1(\mathfrak{c})^{\mathrm{an}}, V_{k-2}(R)\big)$ where the latter R-module is the image of the cuspidal cohomology in the full H^n.

Hida proved for $k > 2$ that δ induces an isomorphism [2]

$$\bigoplus_{\mathfrak{c}'\mathfrak{c}''|\mathfrak{c}} \bigoplus_{\psi} \mathcal{S}_k^{\mathrm{new}}\big(\mathfrak{c}', \eta; R\big) \,\big|\, \mathfrak{c}'' \;\xrightarrow{\sim}\; H^n_!\big(Y_1(\mathfrak{c})^{\mathrm{an}}, V_{k-2}(R)\big)$$

of $\mathbf{T}(\mathfrak{c}; R)$-modules, where η runs over primitive finite order Hecke characters of conductor dividing $\mathfrak{c}$ [Hid94]. Note also that the twisted Poincaré pairing

$$[-,-]_{\mathfrak{c}} : H^n_!\big(Y_1(\mathfrak{c})^{\mathrm{an}}, V_{k-2}(R)\big) \times H^n_!\big(Y_1(\mathfrak{c})^{\mathrm{an}}, V_{k-2}(R)\big) \;\longrightarrow\; R$$

---

[2] If the weight $k = 2$ then the n-form $\big(2\pi\sqrt{-1}\big)^n \cdot \mathbf{F}(z_1, \ldots, z_d) \cdot \mathrm{d}z_1 \cdots \mathrm{d}z_n$ defines an element of $H^n\big(X_1(\mathfrak{c})^{\mathrm{an}}, \mathbb{C}\big)$ for the compactification $X_1(\mathfrak{c}) = \overline{Y_1(\mathfrak{c})}$, and we should work with this cohomology.

is compatible with the algebraic pairing $(-,-)_{\mathfrak{c}} : \mathcal{S}_k(\mathfrak{c}; R)^{\times 2}_{\mathfrak{M}} \to R$ under $\delta \times \delta$ (see [Dim05, (6)] for details). Moreover, $H^n_!\big(Y_1(\mathfrak{c})^{\mathrm{an}}, V_{k-2}(R)\big)_{\mathfrak{M}}$ is free of rank 2^n over $\mathbf{T}(\mathfrak{c}; R)_{\mathfrak{M}}$ by [Dim05, Theorem 6.7].

Remark 8.

(a) We consider the R-module $L(\mathfrak{c}, R) \subset H^n_!\big(Y_1(\mathfrak{c})^{\mathrm{an}}, V_{k-2}(R)\big)_{\mathfrak{M}}$ generated by the R-valued HMFs, as well as its dual lattice $L(\mathfrak{c}, R)^*$ under $[-,-]_{\mathfrak{c}}$. For each $\mathfrak{q} \in \Sigma$ with $\mathfrak{cq} \mid \mathfrak{n}$, there are two separate degeneration maps

$$\mathrm{id}^*,\ \big(-\ \big|\, \mathfrak{q}\big)^* : H^n_!\big(Y_1(\mathfrak{c})^{\mathrm{an}}, V_{k-2}(R)\big) \longrightarrow H^n_!\big(Y_1(\mathfrak{cq})^{\mathrm{an}}, V_{k-2}(R)\big)\,,$$

both sending the lattice $L(\mathfrak{c}, R)$ into $L(\mathfrak{cq}, R)$ at the higher level $K_1(\mathfrak{cq})$.

(b) If one writes $\mathrm{dep}^*_{\mathfrak{q}} : H^n_!\big(Y_1(\mathfrak{c})^{\mathrm{an}}, V_{k-2}(R)\big) \longrightarrow H^n_!\big(Y_1(\mathfrak{cq}^{\mathrm{ord}_{\mathfrak{q}}(\mathfrak{n})})^{\mathrm{an}}, V_{k-2}(R)\big)$ for the (induced) $\mathfrak{q}$-depletion mapping at level $K_1(\mathfrak{c})$ where

$$\mathrm{dep}_{\mathfrak{q}}(x) := \begin{cases} x - x \,\big|\, U_{\mathfrak{q}} \circ \mathfrak{q} & \text{if } \mathfrak{cq}^2 \nmid \mathfrak{n} \\ x - x \,\big|\, T_{\mathfrak{q}} \circ \mathfrak{q} + \mathcal{N}(\mathfrak{q})^{-1} \cdot x \,\big|\, \langle \mathfrak{q} \rangle \circ \mathfrak{q}^2 & \text{if } \mathfrak{cq}^2 \mid \mathfrak{n}, \end{cases}$$

then $\mathrm{dep}^*_{\mathfrak{q}}\big(L(\mathfrak{c}, R)\big) \subset L(\mathfrak{cq}^{\mathrm{ord}_{\mathfrak{q}}(\mathfrak{n})}, R)$ as a direct consequence of point (a).

Using exactly the same argument as described in [RSV23, Section 2.8] over $F = \mathbb{Q}$, one can deduce that both periods $\Omega_{\mathfrak{n},\Sigma}(\mathbf{F})$ and $\Omega_{\mathfrak{c}(\mathbf{F}),\emptyset}(\mathbf{F})$ coincide (up to a unit) if the following two conditions hold:

- for the dual lattices, $\mathrm{dep}^*_{\mathfrak{q}}\big(L(\mathfrak{c}, R)^*\big) \subset L(\mathfrak{cq}^{\mathrm{ord}_{\mathfrak{q}}(\mathfrak{n})}, R)^*$ at all $\mathfrak{q} \in \Sigma$;
- the quotient module $\frac{L(\mathfrak{cq}^{\mathrm{ord}_{\mathfrak{q}}(\mathfrak{n})}, R)^*}{\mathrm{dep}^*_{\mathfrak{q}}(L(\mathfrak{c}, R)^*)}$ is R-torsion free at every $\mathfrak{q} \in \Sigma$.

Whilst the inclusion of $\mathrm{dep}^*_{\mathfrak{q}}\big(L(\mathfrak{c}, R)^*\big) \subset L\big(\mathfrak{cq}^{\mathrm{ord}_{\mathfrak{q}}(\mathfrak{n})}, R\big)^*$ can be established using the Hecke-equivariance of the twisted pairings $[-,-]$ at levels $\mathfrak{c}$ and $\mathfrak{cq}^{\mathrm{ord}_{\mathfrak{q}}(\mathfrak{n})}$, the second bullet above requires a suitable version of Ihara's lemma.

If $\mathfrak{q} \,\|\, \mathfrak{n}$ and $k > 2$, [Dim09, Theorem 3.1] implies that the map

$$\mathrm{id}^* + \big(-\ \big|\, \mathfrak{q}\big)^* : H^n\big(Y_1(\mathfrak{c})^{\mathrm{an}}, V_{k-2}(R)\big)^{\oplus 2} \longrightarrow H^n\big(Y_1(\mathfrak{cq})^{\mathrm{an}}, V_{k-2}(R)\big)$$

is injective with flat cokernel, so after localisation one obtains the second statement. Alternatively, if $\mathfrak{q} \,\|\, \mathfrak{n}$ and $k = 2$, then one can likely deduce the second statement from a mod p version established by Manning and Shotton in [MS21, Theorem 6.8].

However if $\mathfrak{q}^2 \mid \mathfrak{n}$, then it is still open as to whether the second statement is true, to the best of our knowledge. Fortunately, some important new work of Maletto [Mal25] should treat the remaining cases, thereby removing all the Ihara's lemma hypotheses needed in this chapter as well as in [RSV23].

Chapter 5

The τ-component in the IMC: Algebraic theory

We shall now treat the algebraic analogue of the problem we considered in Chapter 4. Let $F^{\rm cyc}$ denote the cyclotomic $\mathbb{Z}_p$-extension of F with associated Galois group $G_\infty := \mathrm{Gal}(F^{\rm cyc}/F) \cong 1 + p\mathbb{Z}_p$. As before, let E_1 and E_2 be two ordinary p-congruent elliptic curves satisfying Hypothesis 4.1. We prove in this chapter that the τ-twisted Selmer groups over $F^{\rm cyc}$ of E_1 and E_2 exhibit the same congruence properties observed in Section 4.2.

5.1 Main results

To any ordinary p-adic representation, Greenberg has attached a p-primary Selmer group, which in our case corresponds to the group

$$\mathrm{Sel}(E,\tau)^{[t]} \subset H^1\Big(\mathrm{Gal}\left(\overline{F}/F^{\rm cyc}\right),\ \left(\rho_E \otimes \tau \otimes \omega^i\right) \otimes \mathbb{Q}_p/\mathbb{Z}_p\Big), \quad t \in \{0, \ldots, p-2\}$$

comprised of one-cocycles which are unramified outside of the primes dividing p, and satisfy an '$H^1_f(F_{\mathfrak{p}}, -)$' local condition at the primes $\mathfrak{p}\big|p$ (see Definition 5.1 and also Section 3.4.1). Let $R/\mathbb{Z}_p$ be the same extension we have set in Chapter 4. The cyclotomic Selmer group has the structure of a discrete cofinitely-generated Λ_R-module where $\Lambda_R := R[\![G_\infty]\!]$, which means that its Pontryagin dual $\mathcal{X}(E,\tau)^{[t]} := \mathrm{Sel}(E,\tau)^{\vee,[t]}$ is compact and finitely-generated as a Λ_R-module. Provided $\mathcal{X}(E,\tau)^{[t]}$ is Λ_R-torsion, we write $\mu(E,\tau,\omega^t)_{\mathrm{alg}}$ for the highest exponent of p dividing $\mathrm{char}_{\Lambda_R}\big(\mathcal{X}(E,\tau)^{[t]}\big)$ and $\lambda(E,\tau,\omega^t)_{\mathrm{alg}}$ for the number of its zeroes (including multiplicity) as dictated by Definition 3.2.

We also recall that Σ^{bad} is the finite set comprised of primes $\mathfrak{q} \in \mathrm{Spec}(\mathcal{O}_F)$ relatively prime to p satisfying $\mathfrak{c} \mid \mathrm{cond}(\tau)\cdot\mathrm{cond}(E_1)\cdot\mathrm{cond}(E_2)$. The main result of this chapter (Theorem 5.8) relates the arithmetic of the four-dimensional representation $\rho_{E_1} \otimes \tau$ to the congruent representation $\rho_{E_2} \otimes \tau$.

Theorem 5.1. *Assume there exists an elliptic curve E^{opt} defined over F which is p-congruent to E_1 and E_2, such that the (tame) conductor of $\rho_{E^{\mathrm{opt}}} \otimes \tau$ is equal to $\mathrm{cond}(\overline{\rho}_E \otimes \overline{\tau})$. Then $\mathrm{Sel}(E_1,\tau)^{[t]}$ will be Λ_R-cotorsion with a trivial μ-invariant if and only if $\mathrm{Sel}(E_2,\tau)^{[t]}$ is Λ_R-cotorsion with a trivial μ-invariant, in which case*

$$\lambda(E_1,\tau,\omega^t)_{\mathrm{alg}} = \lambda(E_2,\tau,\omega^t)_{\mathrm{alg}} + \sum_{\mathfrak{q}\in\Sigma^{\mathrm{bad}}} \mathbf{e}_{\mathfrak{q}}(E_2,\tau,\omega^t) - \mathbf{e}_{\mathfrak{q}}(E_1,\tau,\omega^t)$$

where $\mathbf{e}_{\mathfrak{q}}(E,\tau,\omega^t)$ for $E = E_1$ or $E = E_2$ are the same as in Theorem 4.1.

We should point out that if E_1 and E_2 are defined over $\mathbb{Q}$, the existence of this optimal curve $E^{\mathrm{opt}}_{/F}$ is easily verified from the work in [KW09] on Serre's conjecture (see Proposition 5.5), so that the theorem above holds unconditionally. However we have been unable to establish the existence of this optimal curve $E^{\mathrm{opt}}_{/F}$ in general, as the problem is intimately connected to modularity lifting theorems.

The crucial step in proving the above theorem is to use the optimal elliptic curve $E^{\mathrm{opt}}_{/F}$ to relate the p-torsion in Greenberg's Selmer group to the residual Selmer group $\mathrm{Sel}\big(\overline{\rho}_E \otimes \tau\big)^{[t]}$. This idea will be carefully set up in Sections 5.2 and 5.3, and then applied in Section 5.4. Next, we demonstrate in Section 5.5 a trio of numerical examples of p-congruent elliptic curves E_1 and E_2 defined over $\mathbb{Q}$ with a fixed Artin representation $\tau : G_F \to \mathrm{GL}_2(\mathbb{C})$, which nicely illustrate the theorems presented. We end the chapter with Section 5.6 which displays two tables summarizing our several other examples.

An important application of Theorems 4.1 and 5.1 concerns the Iwasawa Main Conjecture for the double product which we shall state below.

Conjecture 5.2. *For each* $0 \leq t \leq p-2$*, the* p*-Selmer group* $\mathrm{Sel}_p(E,\tau)^{[t]}$ *is a cotorsion* Λ_R*-module. Moreover, there is an equality*

$$\mathrm{char}_{\Lambda_R}\big(\,\mathrm{Sel}(E,\tau)^{\vee,\,[t]}\big) = L_p\big(E,\tau,\omega^t,X\big),$$

up to a unit factor under the identification $\Lambda_R \cong R[\![X]\!]$ *of the Iwasawa algebra. N.B. For any such pair* (E,τ) *and branch* ω^t *we abbreviate this conjecture by* $\mathrm{IMC}_{\omega^t}(E,\tau)$*.*

To realise the application of the above theorems, it is worthwhile to discuss the benefit of having an Euler system of Beilinson-Flach elements over $F(\mu_{p^r})$:

$$\big({}_c\mathbf{BF}_r(E,\tau)\big)_r \in \varprojlim_r H^1_{\text{ét}}\left(\,\mathrm{Spec}\left(\mathcal{O}_F\left[1/\Sigma^{\mathrm{bad}},\mu_{p^r}\right]\right),\, \varprojlim_n E[p^n]\otimes\tau\right)$$

which maps down to the p-adic L-function $L^{\Omega_E}_{p,(\Sigma^{\mathrm{bad}})}(E,\tau)$ in the sense of [KLZ17]. Granted the existence of an Euler system with suitable properties, it is folklore that one can show that $\mathrm{Sel}(E,\tau)^{[t]}$ is Λ_R-cotorsion for modular elliptic curves $E_{/F}$. Moreover, one should obtain the following divisibility of p-adic functions

$$\text{“}\mathrm{char}_{\Lambda_R}\big(\,\mathrm{Sel}_p(E,\tau)^{\vee,\omega^i}\big)\ \text{ divides }\ p^{??}\cdot L_{p,(\Sigma^{\mathrm{bad}})}\big(E,\tau\otimes\omega^i\big)\text{”}$$

where the power of p depends on the structure of the Galois representation, among other things. In any case, if one combines this (hypothetical) divisibility with the transition formulae in Theorems 4.1 and 5.1, it is an easy exercise to show the equivalence of $\mathrm{IMC}_{\omega^t}(E_1,\tau)$ and $\mathrm{IMC}_{\omega^t}(E_2,\tau)$ in the situation where the μ-invariant vanishes.

As nobody has yet constructed these elements $\big({}_c\mathbf{BF}_r(E,\tau)\big)_r$ over a general totally real field F, this discussion is nothing more than idle speculation. The notable exception is the rational field $F=\mathbb{Q}$, where such an Euler system has been found by Lei, Loeffler, Zerbes and Kings [LLZ14; KLZ17]. Combining their Λ-cotorsion and divisibility results with Theorems 5.1 and 4.1, we deduce the following equivalence of Iwasawa Main Conjectures as a by-product.

Theorem 5.3. *For $F = \mathbb{Q}$ and $0 \leq t \leq p-2$, if at each choice $i \in \{1,2\}$ the non-square-free part of the conductor of E_i divides into* $\mathrm{cond}(\tau)$*, then the following statement*

$$\text{“}\mathrm{IMC}_{\omega^t}(E_1,\tau) \text{ is true } \iff \mathrm{IMC}_{\omega^t}(E_2,\tau) \text{ is true”}$$

will always hold, provided that one, and hence both of these μ-invariants vanish.

5.2 Λ-structure of the cyclotomic Selmer group

Let p be a fixed prime. If E is one of our elliptic curves E_1 or E_2 (over the totally real field F) then the p-adic Tate module $\mathrm{Ta}_p(E) = \varprojlim_n E[p^n]$ is given by the projective limit of the p^n torsion points of E. *We relax our assumption that p is inert in F and no longer insist that $p \nmid D_F$.* The following discussion is an echo of Section 3.4.1 specialised to the τ-twist of an elliptic curve.

Provided E has good ordinary or bad multiplicative reduction at each place $\mathfrak{p} \mid p$, there exists a short exact sequence of discrete $\mathrm{Gal}\left(\overline{F}_{\mathfrak{p}}/F_{\mathfrak{p}}\right)$-modules

$$0 \longrightarrow \widehat{E}_{/\mathcal{O}_{F_{\mathfrak{p}}}}[p^\infty] \longrightarrow E[p^\infty] \longrightarrow \mathcal{C}_{\mathfrak{p}}^{\text{ét}} \longrightarrow 0$$

where $\widehat{E}_{/\mathcal{O}_{F_{\mathfrak{p}}}}[p^\infty]$ is the height one formal group, and $\mathcal{C}_{\mathfrak{p}}^{\text{ét}}$ is an étale p-divisible group. For example, if E has good ordinary reduction at $\mathfrak{p}$, then $\mathcal{C}_{\mathfrak{p}}^{\text{ét}} \cong \tilde{E}[p^\infty]$ where $\tilde{E}_{/\mathbb{F}}$ denotes the reduced (non-singular) curve defined over $\mathbb{F} := \mathcal{O}_F/\mathfrak{p}$.

Recall that $\tau : G_F \to \mathrm{GL}_2(\overline{\mathbb{Q}})$ was the two-dimensional Artin representation induced from a Hecke character ϑ_K over the CM-field K, so that $\tau = \mathrm{Ind}_F^K(\vartheta_K)$. By enlarging the finite field $\mathbb{F}$ if necessary, we may assume $\mathrm{im}(\overline{\tau}) \subset \mathrm{GL}_2(\mathbb{F})$. Taking Teichmüller twists by ω^t with $0 \leq t \leq p-2$, the Galois representation

$$\rho_E \otimes \tau \otimes \omega^t : G_F \to \mathrm{GL}_4(\overline{\mathbb{Q}}_p)$$

has an associated free rank four R-module, $T(E)_\tau^{[t]}$, equipped with a G_F-action:

$$T(E)_\tau^{[t]} := \mathrm{Ta}_p(E) \otimes \mathrm{Ind}_F^K(\vartheta_K) \otimes \omega^t.$$

We write $W(E)_{\tau}^{[t]} = T(E)_{\tau}^{[t]} \otimes_{\mathbb{Z}_p} \mathbb{Q}_p/\mathbb{Z}_p$ to denote the corresponding p-primary object. When viewed as a local $G_{F_{\mathfrak{p}}}$-module, at each place $\mathfrak{p} \mid p$ one sets

$$T_{\mathfrak{p}}^{+}(E)_{\tau}^{[t]} := \mathrm{Ta}_p\big(\widehat{E}_{/\mathcal{O}_{F_{\mathfrak{p}}}}\big) \otimes \mathrm{Ind}_F^K(\vartheta_K) \otimes \omega^t \;\text{ and }\; T_{\mathfrak{p}}^{-}(E)_{\tau}^{[t]} := \frac{T(E)_{\tau}^{[t]}}{T_{\mathfrak{p}}^{+}(E)_{\tau}^{[t]}}$$

with analogous notation for the p-divisible versions: $W_{\mathfrak{p}}^{+}(E)_{\tau}^{[t]}$ and $W_{\mathfrak{p}}^{-}(E)_{\tau}^{[t]}$.

Definition 5.1. For a finite subset $\Sigma \subset \mathrm{Spec}(\mathcal{O}_F)$ not containing the places $\mathfrak{p} \mid p$, the *Σ-depleted p-Selmer group* $\mathrm{Sel}_{\Sigma}(F^{\mathrm{cyc}}, E \otimes \tau)^{[t]}$ over the cyclotomic extension F^{cyc} (or $\mathrm{Sel}_{\Sigma}(E,\tau)^{[t]}$ for short) is the kernel of the restriction maps

$$\begin{aligned} H^1\Big(\overline{F}/F^{\mathrm{cyc}}, W(E)_{\tau}^{[t]}\Big) \xrightarrow{\prod \mathrm{res}_{\nu}} & \prod_{\nu \notin \Sigma^{\mathrm{cyc}} \cup \{\mathfrak{p}|p\}} H^1\Big(F_{\nu}^{\mathrm{cyc}}, W(E)_{\tau}^{[t]}\Big) \\ & \times \prod_{\mathfrak{p}|p} H^1\Big(I_{F_{\mathfrak{p}}^{\mathrm{cyc}}}, W_{\mathfrak{p}}^{-}(E)_{\tau}^{[t]}\Big) \end{aligned}$$

which is a module over the Iwasawa algebra $\Lambda_R = R[\![\mathrm{Gal}(F^{\mathrm{cyc}}/F)]\!] \cong R[\![X]\!]$. Here we have written Σ^{cyc} for the (finite) set of places of F^{cyc} lying over Σ.

The Pontryagin dual $\mathcal{X}_{\Sigma}(E,\tau)^{[t]} := \mathrm{Hom}_{\mathrm{cont}}\big(\mathrm{Sel}_{\Sigma}\,(E,\tau)^{[t]}, \mathbb{Q}_p/\mathbb{Z}_p\big)$ is a compact Λ_R-module as the τ-twisted cyclotomic Selmer group above is discrete. One may therefore apply structure theory to analyse $\mathcal{X}_{\Sigma}\big(E,\tau\big)^{[t]}$. In particular, we shall study the irreducible factors in its characteristic power series. Let us recall our fixed uniformiser ϖ for the discrete valuation ring R. The following technical lemma summarises the results we will need later on.

Lemma 5.4.

(a) The module $\mathcal{X}_{\Sigma}(E,\tau)^{[t]}$ is finitely-generated over the Iwasawa algebra Λ_R.

(b) $\mathrm{Sel}_{\Sigma}(E,\tau)^{[t]}[\varpi]$ *is finite iff $\mathcal{X}_{\Sigma}(E,\tau)^{[t]}$ is torsion with trivial μ-invariant.*

(c) If $\mathrm{Sel}_{\Sigma}(E,\tau)^{[t]}[\varpi]$ *is finite then* $\mathrm{Sel}_{\Sigma}(E,\tau)^{[t]}$ *is ϖ-divisible, in which case the λ-invariant of* $\mathrm{char}_{\Lambda_R}\big(\mathcal{X}_{\Sigma}(E,\tau)^{[t]}\big)$ *coincides with the $\mathbb{F}$-dimension of* $\mathrm{Sel}_{\Sigma}(E,\tau)^{[t]}[\varpi]$.

(d) If $\mathrm{ord}_{\mathfrak{q}}\big(\mathrm{cond}(\rho_E \otimes \tau)\big) = \mathrm{ord}_{\mathfrak{q}}\big(\mathrm{cond}(\overline{\rho}_E \otimes \overline{\tau})\big)$ *for $\mathfrak{q} \in$* $\mathrm{Spec}(\mathcal{O}_F)$ *with $\mathfrak{q} \nmid p$, then the cohomology groups $H^0\Big(F_{\nu}^{\mathrm{cyc}}, W(E)_{\tau}^{[t]}\Big)$ are ϖ-divisible at places $\nu \mid \mathfrak{q}$.*

Proof. Statements (a)-(c) follow from [Del24, Proposition 4.3], which is itself a generalisation of Greenberg's argument in [Gre97, Proposition 4.14] for elliptic curves. We should point out that in (c), the vanishing condition $H^0\Big(F, T(E)^{[t]}_{\tau}\otimes\mathbb{Q}\Big) = \{0\}$ is required in order to apply [Del24, 4.3(c)]; in fact if $K' = \overline{\mathbb{Q}}^{\ker(\vartheta_K)}$ then

$$H^0\Big(F, T(E)^{[t]}_{\tau}\otimes\mathbb{Q}\Big) \hookrightarrow H^0\Big(K'(\mu_p), T(E)^{[t]}_{\tau}\otimes\mathbb{Q}\Big) \cong H^0\big(K'(\mu_p), V_{\rho_E}\otimes_{\mathbb{Z}_p}R\big)^{\oplus 2};$$

the right-hand group is zero, as the residual representation $\overline{\rho}_E : G_F \to \mathrm{GL}_2(\mathbb{F}_p)$ arising from the action on the p-torsion points is absolutely irreducible.

To show (d), we use the reduction formula for the Swan conductor:

$$\mathrm{ord}_{\mathfrak{q}}\big(\mathrm{cond}(\rho_E\otimes\tau)\big)+\dim\Big(V^{I_{F_{\mathfrak{q}}}}_{\rho_E\otimes\tau}\Big) = \mathrm{ord}_{\mathfrak{q}}\big(\mathrm{cond}(\overline{\rho}_E\otimes\overline{\tau})\big)+\dim_{\mathbb{F}}\Big(W(E)^{[0]}_{\tau}[\varpi]^{I_{F_{\mathfrak{q}}}}\Big).$$

(c.f. [Liv89, §1]). The condition stated in (d) instantly implies the equality $\dim_{\mathbb{F}}\Big(W(E)^{[0]}_{\tau}[\varpi]^{I_{F_{\mathfrak{q}}}}\Big) = \dim\Big(V^{I_{F_{\mathfrak{q}}}}_{\rho_E\otimes\tau}\Big)$ and consequently the inertia invariants $H^0\Big(I_{F_{\mathfrak{q}}}, W(E)^{[0]}_{\tau}\Big)$ are ϖ-divisible.

Clearly, $H^0\Big(I_{F_{\mathfrak{q}}}, W(E)^{[0]}_{\tau}\Big) \cong H^0\Big(I_{F_{\mathfrak{q}}}, W(E)^{[t]}_{\tau}\Big)$ for all of the ω^t-twists as $\omega\big|_{G_F}$ is unramified at $\mathfrak{q}$. Secondly, as $\mathfrak{q}$ does not ramify in the cyclotomic $\mathbb{Z}_p$-extension, $H^0\Big(I_{F^{\mathrm{cyc}}_{\nu}}, W(E)^{[t]}_{\tau}\Big) = H^0\Big(I_{F_{\mathfrak{q}}}, W(E)^{[t]}_{\tau}\Big)$ must be ϖ-divisible. Lastly, we may conclude that the group

$$H^0\Big(F^{\mathrm{cyc}}_{\nu}, W(E)^{[t]}_{\tau}\Big) = H^0\Big(I_{F^{\mathrm{cyc}}_{\nu}}, W(E)^{[t]}_{\tau}\Big)^{\mathrm{Frob}_{\mathfrak{q}}=1}$$

is also ϖ–divisible, since taking $\mathrm{Frob}_{\mathfrak{q}}$-invariants preserves this property. ∎

The condition stated in (d) on the conductors plays a pivotal role later in this chapter. Indeed, the existence of an elliptic curve that is congruent to E modulo p and which satisfies property (d) at primes $\mathfrak{q} \in \mathrm{Spec}(\mathcal{O}_F)$ not lying above p will prove to be invaluable.

Hypothesis 5.1. *There is a 'τ-optimal curve', $E^{\mathrm{opt}}_{/F}$, which is congruent to $E_{/F}$ modulo p, with* $\mathrm{ord}_{\mathfrak{q}}\big(\mathrm{cond}(\rho_{E^{\mathrm{opt}}}\otimes\tau)\big) = \mathrm{ord}_{\mathfrak{q}}\big(\mathrm{cond}(\overline{\rho}_E\otimes\overline{\tau})\big)$ *for* $\mathfrak{q}\nmid p\cdot\mathrm{cond}(\tau)$.

Whilst we have been unable to establish that this τ-optimal congruent elliptic curve $E^{\mathrm{opt}}_{/F}$ exists over general totally real number fields F, if we consider exclusively the situation from the introduction, then one can prove the existence of such a curve.

Proposition 5.5. *If the elliptic curve E is defined over $\mathbb{Q}$ then Hypothesis 5.1 is true, i.e. a τ-optimal p-congruent elliptic curve $E^{\mathrm{opt}}_{/F}$ exists.*

Proof. Let $\rho_E : G_{\mathbb{Q}} \to \mathrm{Aut}\big(\mathrm{Ta}_p(E)\big) \cong \mathrm{GL}_2(\mathbb{Z}_p)$ denote the p-adic Galois representation of $E_{/\mathbb{Q}}$. By results of Khare and Wintenberger [KW09], there exists a classical newform $f^{\mathrm{opt}} \in S_2\big(\Gamma_0(\mathrm{cond}(\overline{\rho}_E)\cdot p^e)\big)$ with level equal to the conductor of $\overline{\rho}_E : G_{\mathbb{Q}} \to \mathrm{GL}_2(\mathbb{F}_p)$ (except maybe for the power of p), and which shares the same mod p representation. We write $E^{\mathrm{opt}} = E^{\mathrm{opt}}_{/\mathbb{Q}}$ for the rational elliptic curve inside $J_0\big(\mathrm{cond}(\overline{\rho}_E)\cdot p^e\big)$ associated to the f^{opt}-isotypic part, thus E^{opt} is p-congruent to E and is a curve with minimal conductor outside p.

Fix a place $\mathfrak{Q} \in \mathrm{Spec}(\mathcal{O}_K)$ of the CM-field lying above a prime $q \neq p$ and let $\mathfrak{q} \in \mathrm{Spec}(\mathcal{O}_F)$ be the unique prime ideal lying under $\mathfrak{Q}$. If $\mathfrak{q} \nmid \mathrm{cond}(\tau)$ then $\mathfrak{q} \nmid D_K$ and q is unramified in $K/\mathbb{Q}$. Moreover, it follows from [Liv89] that

$$\mathrm{ord}_{\mathfrak{Q}}\Big(\mathrm{cond}\left(\rho_{E^{\mathrm{opt}}}\big|_{G_K}\right)\Big) \geq \mathrm{ord}_{\mathfrak{Q}}\Big(\mathrm{cond}\left(\overline{\rho}_E\big|_{G_K}\right)\Big). \tag{5.1}$$

If (5.1) were a strict inequality, then $\mathrm{ord}_q\left(\mathrm{cond}(\rho_{E^{\mathrm{opt}}})\right) > \mathrm{ord}_q\left(\mathrm{cond}(\overline{\rho}_E)\right)$ because $E^{\mathrm{opt}}_{/\mathbb{Q}}$ has the same reduction type (good, multiplicative or additive) at q as $E^{\mathrm{opt}}_{/K}$ does at $\mathfrak{Q}$, since the extension of local fields $K_{\mathfrak{Q}}/\mathbb{Q}_q$ is unramified. However this contradicts the minimality property of the $\mathbb{Q}$-optimal curve E^{opt} stated in the previous paragraph, hence we deduce that (5.1) is an equality.

Let us next recall that $\tau = \mathrm{Ind}_F^K(\vartheta_K)$ where ϑ_K was a finite order character. Because $\mathfrak{q} \nmid \mathrm{cond}(\tau)$ it follows that ϑ_K is unramified at $\mathfrak{Q}$, whence

$$\mathrm{ord}_{\mathfrak{Q}}\Big(\mathrm{cond}\left(\rho_{E^{\mathrm{opt}}}\big|_{G_K}\otimes\vartheta_K\right)\Big) \overset{(5.1)}{=} \mathrm{ord}_{\mathfrak{Q}}\Big(\mathrm{cond}\left(\overline{\rho}_E\big|_{G_K}\otimes\vartheta_K\right)\Big)$$

as the $\mathfrak{Q}$-part of the conductors does not change [1] if one twists by the character ϑ_K. Taguchi has shown that if one induces from K down to $F = K^+$, the

---

[1] If one extends the definition of a τ-optimal elliptic curve to include primes $\mathfrak{q} \mid \mathrm{cond}(\tau)$, the equality between the $\mathfrak{Q}$-part of the conductor of $\rho_{E^{\mathrm{opt}}}\big|_{G_K} \otimes \vartheta_K$ and its residual representation would *not* hold in general. Moreover one requires the work of Corbett and Saha [CS18] on twisting automorphic representations of GL_n by GL_1, and then one must avoid all the pathological cases.

conductors of both $\rho_{E^{\mathrm{opt}}}\big|_{G_F} \otimes \tau \cong \mathrm{Ind}_F^K\big(\rho_{E^{\mathrm{opt}}}\big|_{G_K} \otimes \vartheta_K\big)$ and its residual version satisfy precisely the same descent formula [Tag02, p2866]. We therefore conclude that $\mathrm{ord}_{\mathfrak{q}}\big(\mathrm{cond}(\rho_{E^{\mathrm{opt}}}\big|_{G_F} \otimes \tau)\big) = \mathrm{ord}_{\mathfrak{q}}\big(\mathrm{cond}(\overline{\rho}_E\big|_{G_F} \otimes \overline{\tau})\big)$ at all primes $\mathfrak{q} \nmid p \cdot \mathrm{cond}(\tau)$. ■

5.3 $\mathbb{F}$-rank of the residual Selmer group

Let us now introduce a residual version of the Λ_R-module $\mathrm{Sel}_\Sigma(E,\tau)^{[t]}$ which has the structure of an $\mathbb{F}$-vector space, where $\mathbb{F}$ is the finite field $\mathcal{O}_F/\mathfrak{p}$. Recall that $E = E_1$, $E = E_2$, and (if it exists) $E = E^{\mathrm{opt}}_{/F}$, have a common mod p representation $\overline{\rho}_E : G_F \to \mathrm{GL}_2(\mathbb{F}_p)$. For $t \in \{0, \dots, p-2\}$, one can then consider the tensor product of Galois representations (in characteristic p),

$$\overline{\rho}_E \otimes \overline{\tau} \otimes \omega^t : G_F \to \mathrm{Aut}_{\mathbb{F}}\big(\overline{\mathcal{V}}^{[t]}\big) \cong \mathrm{GL}_4(\mathbb{F})$$

where $\overline{\mathcal{V}}^{[t]} \cong \frac{1}{p} \cdot \mathbb{Z}_p \otimes_{\mathbb{Z}_p} T(E)_\tau^{[t]}$ denotes the underlying rank four $\mathbb{F}$-vector space. N.B. $\overline{\mathcal{V}}^{[t]}$ is identified as the ϖ-torsion subgroup of $W(E)_\tau^{[t]} = \mathbb{Q}_p/\mathbb{Z}_p \otimes_{\mathbb{Z}_p} T(E)_\tau^{[t]}$.

At each place $\mathfrak{p} \mid p$, one gets an exact sequence of local $\mathrm{Gal}\big(\overline{F}_{\mathfrak{p}}/F_{\mathfrak{p}}\big)$-modules

$$0 \longrightarrow \overline{\mathcal{V}}_{\mathfrak{p}}^{+,[t]} \longrightarrow \overline{\mathcal{V}}^{[t]} \longrightarrow \overline{\mathcal{V}}_{\mathfrak{p}}^{-,[t]} \longrightarrow 0$$

satisfying $\dim_{\mathbb{F}}\big(\overline{\mathcal{V}}_{\mathfrak{p}}^{+,[t]}\big) = \dim_{\mathbb{F}}\big(\overline{\mathcal{V}}_{\mathfrak{p}}^{-,[t]}\big) = 2$, where $\overline{\mathcal{V}}_{\mathfrak{p}}^{\pm,[t]} := \frac{1}{p} \cdot \mathbb{Z}_p \otimes_{\mathbb{Z}_p} T_{\mathfrak{p}}^{\pm}(E)_\tau^{[t]}$.

Definition 5.2. Let $\Sigma \subset \mathrm{Spec}(\mathcal{O}_F)$ be a finite subset not containing any of the primes above p, and write Σ^{cyc} for those places of F^{cyc} lying above Σ. For $t \in \{0, \dots, p-2\}$, we define the residual Selmer group $\mathrm{Sel}_\Sigma(\overline{\rho}_E \otimes \tau)^{[t]}$ to equal

$$\ker\left(H^1\big(F^{\mathrm{cyc}}, \overline{\mathcal{V}}^{[t]}\big) \xrightarrow{\prod \mathrm{res}_\nu} \prod_{\nu \notin \Sigma^{\mathrm{cyc}} \cup \{\mathfrak{p}|p\}} H^1\big(F^{\mathrm{cyc}}_\nu, \overline{\mathcal{V}}^{[t]}\big) \times \prod_{\mathfrak{p}|p} H^1\big(I_{F^{\mathrm{cyc}}_{\mathfrak{p}}}, \overline{\mathcal{V}}_{\mathfrak{p}}^{-,[t]}\big) \right).$$

We can compare the residual Selmer group with the ϖ-torsion in the usual one. For each $i \geq 0$ and closed normal subgroup $G \lhd G_F$, there is a short exact sequence

$$0 \longrightarrow H^i\big(G, W(E)_\tau^{[t]}\big)/\varpi \xrightarrow{\partial_i} H^{i+1}\big(G, \overline{\mathcal{V}}^{[t]}\big) \xrightarrow{\beta_{i+1}} H^{i+1}\big(G, W(E)_\tau^{[t]}\big)[\varpi] \longrightarrow 0. \tag{5.2}$$

In particular, one obtains a large commutative diagram with exact columns:

$$\begin{tikzcd}
0 \arrow[d] & 0 \arrow[d] \\
H^0\big(F^{\mathrm{cyc}}, W^{[t]}\big)/\varpi \arrow[r, "\prod \mathrm{loc}_{\mathfrak{p}}^{(0)}"] \arrow[d, "\partial_0"'] & \displaystyle\prod_{\nu\nmid p, \nu\notin\Sigma^{\mathrm{cyc}}} \{0\} \times \prod_{\mathfrak{p}|p} H^0\big(I_{F_{\mathfrak{p}}^{\mathrm{cyc}}}, W_{\mathfrak{p}}^{-,[t]}\big)/\varpi \arrow[d, "{(0,\partial_0^-)}"'] \\
H^1\big(F^{\mathrm{cyc}}, \overline{\mathcal{V}}^{[t]}\big) \arrow[r, "\prod \mathrm{res}_{\nu}"] \arrow[d, "\beta_1"'] & \displaystyle\prod_{\nu\nmid p, \nu\notin\Sigma^{\mathrm{cyc}}} H^1\big(F_{\nu}^{\mathrm{cyc}}, W^{[t]}\big)[\varpi] \times \prod_{\mathfrak{p}|p} H^1\big(I_{F_{\mathfrak{p}}^{\mathrm{cyc}}}, \overline{\mathcal{V}}_{\mathfrak{p}}^{-,[t]}\big) \arrow[d, "{(\mathrm{Id},\beta_1^-)}"'] \\
H^1\big(F^{\mathrm{cyc}}, W^{[t]}\big)[\varpi] \arrow[r, "\prod \mathrm{res}_{\nu}"] \arrow[d] & \displaystyle\prod_{\nu\nmid p, \nu\notin\Sigma^{\mathrm{cyc}}} H^1\big(F_{\nu}^{\mathrm{cyc}}, W^{[t]}\big)[\varpi] \times \prod_{\mathfrak{p}|p} H^1\big(I_{F_{\mathfrak{p}}^{\mathrm{cyc}}}, W_{\mathfrak{p}}^{-,[t]}\big)[\varpi] \arrow[d] \\
0 & 0
\end{tikzcd}$$

because the context is clear, we have suppressed E and τ in the notation.

From Lemma 5.4(d), if $\mathrm{ord}_{\mathfrak{q}}\big(\mathrm{cond}(\rho_E\otimes\tau)\big) = \mathrm{ord}_{\mathfrak{q}}\big(\mathrm{cond}(\overline{\rho}_E\otimes\overline{\tau})\big)$ for some $\mathfrak{q}\nmid p$, then $H^0\big(F_{\nu}^{\mathrm{cyc}}, W(E)_{\tau}^{[t]}\big)$ is ϖ-divisible at all places $\nu \mid \mathfrak{q}$, and so

$$\beta_1 : H^1\big(F_{\nu}^{\mathrm{cyc}}, \overline{\mathcal{V}}^{[t]}\big) \xrightarrow{\sim} H^1\big(F_{\nu}^{\mathrm{cyc}}, W(E)_{\tau}^{[t]}\big)[\varpi] \qquad (5.3)$$

is an isomorphism (by the exactness of the sequence in (5.2) above at $i = 0$). For example, if $E = E_{/F}^{\mathrm{opt}}$ is an optimal elliptic curve, then this condition on the conductors holds for all $\mathfrak{q}\nmid p\cdot\mathrm{cond}(\tau)$. More generally, if we choose Σ 'sufficiently large', then β_1 will be an isomorphism of local cohomology groups.

Lemma 5.6. *Suppose the finite set of primes* Σ *contains all* $\mathfrak{q}$ *such that* $\mathrm{ord}_{\mathfrak{q}}\big(\mathrm{cond}(\rho_E\otimes\tau)\big) > \mathrm{ord}_{\mathfrak{q}}\big(\mathrm{cond}(\overline{\rho}_E\otimes\overline{\tau})\big)$. *Then the induced mapping*

$$\beta_{1,*} : \mathrm{Sel}_{\Sigma}\big(\overline{\rho}_E\otimes\tau\big)^{[t]} \longrightarrow \mathrm{Sel}_{\Sigma}(E,\tau)^{[t]}[\varpi]$$

yields an isomorphism of vector spaces over $\mathbb{F} = \mathcal{O}_F/\mathfrak{p}$.

Proof. From the previous discussion, at places $\nu \notin \Sigma^{\mathrm{cyc}}$ not lying above p one deduces that $H^1\big(F_\nu^{\mathrm{cyc}}, \overline{\mathcal{V}}^{[t]}\big) \cong H^1\big(F_\nu^{\mathrm{cyc}}, W(E)_\tau^{[t]}\big)[\varpi]$. Applying the Snake Lemma to our large commutative diagram, one obtains the exact sequence

$$0 \to \prod_{\mathfrak{p}|p} \ker\big(\operatorname{loc}_{\mathfrak{p}}^{(0)}\big) \xrightarrow{\partial_{0,*}} \operatorname{Sel}_\Sigma\big(\overline{\rho}_E \otimes \tau\big)^{[t]} \xrightarrow{\beta_{1,*}} \operatorname{Sel}_\Sigma(E,\tau)^{[t]}[\varpi] \to \prod_{\mathfrak{p}|p} \operatorname{coker}\big(\operatorname{loc}_{\mathfrak{p}}^{(0)}\big).$$

The kernel of $\operatorname{loc}_{\mathfrak{p}}^{(0)}$ vanishes at each place $\mathfrak{p} \mid p$ since $H^0\big(F, \overline{\mathcal{V}}^{[t]}\big)$ is identically zero. Moreover, the vanishing of the cokernel of $\operatorname{loc}_{\mathfrak{p}}^{(0)}$ at each $\mathfrak{p} \mid p$ can be shown using the same argument as in [Del24, Proof of 4.7(c)], so we are done. ■

Henceforth we will always choose our finite set of places $\Sigma \subset \operatorname{Spec}(\mathcal{O}_F)$ to include the primes dividing $\operatorname{cond}(\tau)$, excluding the primes lying above p. We also fix an elliptic curve $E_{/F}$, and we pick any branch $t \in \{0, \dots, p-2\}$.

Corollary 5.7. *Assume that Hypothesis 5.1 holds true, i.e. that the optimal elliptic curve* $E_{/F}^{\mathrm{opt}}$ *exists. Then* $\operatorname{Sel}_\Sigma\big(\overline{\rho}_E \otimes \tau\big)^{[t]}$ *is finite if and only if* $\mathcal{X}_\Sigma(E',\tau)^{[t]}$ *is* Λ_R*-torsion with* μ*-invariant equal to zero for some (and therefore every) elliptic curve* $E'_{/F}$ *with the same residual representation* $\overline{\rho}_E : G_F \to \mathrm{GL}_2(\mathbb{F}_p)$.

Proof. We begin by comparing the residual Selmer group with the one for the optimal elliptic curve $E_{/F}^{\mathrm{opt}}$. Since $\operatorname{ord}_{\mathfrak{q}}\big(\operatorname{cond}(\rho_{E^{\mathrm{opt}}} \otimes \tau)\big) = \operatorname{ord}_{\mathfrak{q}}\big(\operatorname{cond}(\overline{\rho}_E \otimes \overline{\tau})\big)$ for all $\mathfrak{q} \nmid p \cdot \operatorname{cond}(\tau)$, clearly Lemma 5.6 implies that there is an isomorphism

$$\beta_{1,*} : \operatorname{Sel}_\Sigma\big(\overline{\rho}_E \otimes \tau\big)^{[t]} \xrightarrow{\sim} \operatorname{Sel}_\Sigma\big(E^{\mathrm{opt}}, \tau\big)^{[t]}[\varpi].$$

In particular, $\operatorname{Sel}_\Sigma\big(\overline{\rho}_E \otimes \tau\big)^{[t]}$ is finite if and only if $\operatorname{Sel}_\Sigma\big(E^{\mathrm{opt}}, \tau\big)^{[t]}[\varpi]$ is finite.

Next, replacing the set Σ by a possibly larger set Σ' containing all primes $\mathfrak{q}' \nmid p$ such that $\operatorname{ord}_{\mathfrak{q}'}\big(\operatorname{cond}(\rho_{E'} \otimes \tau)\big) > \operatorname{ord}_{\mathfrak{q}'}\big(\operatorname{cond}(\overline{\rho}_E \otimes \overline{\tau})\big)$, one calculates:

$$\begin{aligned} \dim_{\mathbb{F}}\Big(\operatorname{Sel}_{\Sigma'}(E',\tau)^{[t]}[\varpi]\Big) &\overset{5.3}{=} \dim_{\mathbb{F}}\Big(\operatorname{Sel}_{\Sigma'}(E^{\mathrm{opt}},\tau)^{[t]}[\varpi]\Big) \\ &= \dim_{\mathbb{F}}\Big(\operatorname{Sel}_\Sigma\big(\overline{\rho}_E \otimes \tau\big)^{[t]}\Big) + \sum_{\nu|\mathfrak{q}\in\Sigma'\setminus\Sigma} \dim_{\mathbb{F}}\Big(H^1\big(F_\nu^{\mathrm{cyc}}, \overline{\mathcal{V}}^{[t]}\big)\Big). \end{aligned}$$

The stated result (for the larger set Σ') now follows directly from Lemma 5.4(b). Finally to descend from Σ' back down to Σ, we remark that the following quotient module satisfies

$$\frac{\mathcal{X}_{\Sigma'}(E',\tau)^{[t]}}{\mathcal{X}_{\Sigma}(E',\tau)^{[t]}} \;\cong\; \prod_{\nu|\mathfrak{q}\in\Sigma'\backslash\Sigma} \mathrm{Hom}_{\mathrm{cont}}\Big(H^1\big(F_\nu^{\mathrm{cyc}}, W(E',\tau)^{[t]}\big), \mathbb{Q}_p/\mathbb{Z}_p\Big)\;;$$

each one of the right-hand groups is Λ_R-torsion of trivial μ-invariant. ■

5.4 Transition formulae for λ-invariants

We remind the reader that E_1 and E_2 denote a pair of p-congruent elliptic curves defined over the totally real number field F, and that $\tau = \mathrm{Ind}_F^K(\vartheta_K)$ was a two-dimensional Artin representation. *Throughout this section we shall assume the finiteness of* $\mathrm{Sel}_\Sigma\left(\overline{\rho}_E \otimes \tau\right)^{[t]}$. As a consequence of Lemma 5.4(b), this condition is equivalent to the cyclotomic Selmer groups attached to E_1 and E_2 being both p-divisible and Λ_R-cotorsion. Here our goal is to show that the algebraic analogue of Corollary 4.10 holds. Most of the technical results have already been established, so all that is left to do is essentially read off the Iwasawa invariants from the dimension formulae in the proof of Corollary 5.7.

Theorem 5.8. *Assume that Hypothesis 5.1 is true, i.e. the curve* $E^{\mathrm{opt}}_{/F}$ *exists. Recall from Section 4.1 that* $\mu(E_i,\tau,\omega^t)_{\mathrm{alg}}$ *and* $\lambda(E_i,\tau,\omega^t)_{\mathrm{alg}}$ *represents the* μ*-invariant and* λ*-invariant of the dual p-Selmer group* $\mathcal{X}(E,\tau)^{[t]}$*, respectively.*

(a) The invariant $\mu(E_1,\tau,\omega^t)_{\mathrm{alg}} = 0$ *if and only if* $\mu(E_2,\tau,\omega^t)_{\mathrm{alg}} = 0$*.*

(b) If one (and therefor both) of these μ*-invariants equals zero, then*

$$\lambda(E_1,\tau,\omega^i)_{\mathrm{alg}} = \lambda(E_2,\tau,\omega^i)_{\mathrm{alg}} + \sum_{\mathfrak{q}\in\Sigma} \mathbf{e}_{\mathfrak{q}}(E_2,\tau,\omega^i) - \mathbf{e}_{\mathfrak{q}}(E_1,\tau,\omega^i)$$

with $\Sigma = \mathrm{supp}\big(\mathrm{cond}(\tau)\cdot\mathrm{cond}(E_1)\cdot\mathrm{cond}(E_2)\big)$ *but excluding primes* $\mathfrak{p} \mid p$*.*

Proof. We should first point out that Corollary 5.7 guarantees that all the Selmer groups we are considering are Λ_R-cotorsion and with trivial μ-invariant. In particular, statement (a) follows immediately from the aforementioned result.

To establish statement (b), an identical argument used in the proof of Corollary 5.7 implies that

$$\dim_{\mathbb{F}}\Big(\mathrm{Sel}_\Sigma(E,\tau)^{[t]}[\varpi]\Big) = \dim_{\mathbb{F}}\Big(\mathrm{Sel}_\emptyset\big(\overline{\rho}_E\otimes\tau\big)^{[t]}\Big) + \sum_{\nu|\mathfrak{q}\in\Sigma} \dim_{\mathbb{F}}\Big(H^1\big(F_\nu^{\mathrm{cyc}}, \overline{\mathcal{V}}^{[t]}\big)\Big)$$

for both the elliptic curves $E = E_1$ and $E = E_2$.

Because the right-hand side is independent of this choice of either elliptic curve, it follows that $\mathrm{Sel}_\Sigma(E_1,\tau)^{[t]}[\varpi]$ and $\mathrm{Sel}_\Sigma(E_2,\tau)^{[t]}[\varpi]$ must have the same $\mathbb{F}$-dimension. Furthermore, Lemma 5.4(c) informs us that this $\mathbb{F}$-dimension coincides with the λ-invariant $\lambda_\Sigma(E,\tau,\omega^t)_{\mathrm{alg}}$ of $\mathrm{char}_{\Lambda_R}\big(\mathcal{X}_\Sigma(E,\tau)^{[t]}\big)$, and hence

$$\lambda_\Sigma(E_1,\tau,\omega^t)_{\mathrm{alg}} = \lambda_\Sigma(E_2,\tau,\omega^t)_{\mathrm{alg}}. \tag{5.4}$$

Choosing either $E = E_1$ or $E = E_2$, a simple calculation reveals that

$$\lambda_\Sigma(E,\tau,\omega^t)_{\mathrm{alg}} = \lambda(E,\tau,\omega^t)_{\mathrm{alg}} + \sum_{\nu|\mathfrak{q}\in\Sigma} \lambda\Big(\mathrm{char}_{\Lambda_R}\Big(H^1\big(F_\nu^{\mathrm{cyc}}, W(E)_\tau^{[t]}\big)^\vee\Big)\Big).$$

However, $\lambda\Big(\mathrm{char}_{\Lambda_R}\Big(H^1\big(F_\nu^{\mathrm{cyc}}, W(E)_\tau^{[t]}\big)^\vee\Big)\Big)$ is equal to $\mathbf{e}_\mathfrak{q}(E,\tau,\omega^t)$ for each prime $\mathfrak{q}\in\Sigma$ by [Del21, Lemma 3.5], and as a direct consequence

$$\lambda_\Sigma(E,\tau,\omega^t)_{\mathrm{alg}} = \lambda(E,\tau,\omega^t)_{\mathrm{alg}} + \sum_{\mathfrak{q}\in\Sigma} \mathbf{e}_\mathfrak{q}(E,\tau,\omega^t). \tag{5.5}$$

Combining together Equations (5.4) and (5.5) proves statement (b). ■

5.5 Numerical examples

We begin with a basic but very useful lemma. Let Σ be a finite collection of primes and once again let γ be a topological generator of G_∞. At every branch ω^t, the p-adic L-function factorises into a product

$$L_{p,(\Sigma)}^{\Omega}\big(E,\tau\otimes\omega^t\big) = L_{p,\emptyset}^{\Omega}\big(E,\tau\otimes\omega^t\big) \times \prod_{\mathfrak{q}\in\Sigma} \mathscr{L}_\mathfrak{q}(E,\tau\otimes\omega^t) \tag{5.6}$$

where $\mathscr{L}_\mathfrak{q}(E,\tau\otimes\omega^t)\in R[\![G_\infty]\!]\cong R[\![X]\!]$ interpolates $\iota_p\left(L_\mathfrak{q}\big(E,\tau^*\otimes\chi^{-1},1\big)\right)$ at characters χ of $\mathfrak{p}$-power conductor satisfying $\chi\big|_{\mathbb{F}_p^\times} = \omega^t$. In particular, one has

$$L_\mathfrak{q}\big(E,\tau^*\otimes\chi^{-1},s\big) = \mathscr{P}_\mathfrak{q}(E,\tau^*;T)\Big|_{T=\chi^{-1}(\mathfrak{q})\cdot\mathcal{N}(\mathfrak{q})^{-s}}$$

where the polynomial $\mathscr{P}_\mathfrak{q}(E,\tau^*;T)$ divides into the $\mathfrak{q}$-th Euler factor term

$$\big(1-\alpha_\mathfrak{q}(\mathbf{f})\alpha_\mathfrak{q}(\mathbf{g})T\big)\big(1-\beta_\mathfrak{q}(\mathbf{f})\alpha_\mathfrak{q}(\mathbf{g})T\big)\big(1-\alpha_\mathfrak{q}(\mathbf{f})\beta_\mathfrak{q}(\mathbf{g})T\big)\big(1-\beta_\mathfrak{q}(\mathbf{f})\beta_\mathfrak{q}(\mathbf{g})T\big).$$

Any one of these four brackets is either equal to 1 or of the form $(1-w_{\mathfrak{q}}T)$, with $w_{\mathfrak{q}}$ a p-adic unit; since $\mathscr{L}_{\mathfrak{q}}(E,\tau\otimes\omega^t)\Big|_{X=\chi(\gamma)-1} = \mathscr{P}_{\mathfrak{q}}\big(E,\tau^*;\chi^{-1}(\mathfrak{q})\cdot\mathcal{N}(\mathfrak{q})^{-1}\big)$ it is then straightforward to establish that

$$\mathscr{L}_{\mathfrak{q}}(E,\tau\otimes\omega^t) \;=\; \prod_{w_{\mathfrak{q}}\in\mathcal{W}_{\mathfrak{q}}}\left(1-w_{\mathfrak{q}}\cdot\omega^{-t}(\mathfrak{q})\cdot(1+X)^{-\frac{\log_p(\mathcal{N}(\mathfrak{q}))}{\log_p(\gamma)}}\right)$$

for some subset $\mathcal{W}_{\mathfrak{q}}\subset\big\{\alpha_{\mathfrak{q}}(\mathbf{f})\alpha_{\mathfrak{q}}(\mathbf{g}),\beta_{\mathfrak{q}}(\mathbf{f})\alpha_{\mathfrak{q}}(\mathbf{g}),\alpha_{\mathfrak{q}}(\mathbf{f})\beta_{\mathfrak{q}}(\mathbf{g}),\beta_{\mathfrak{q}}(\mathbf{f})\beta_{\mathfrak{q}}(\mathbf{g})\big\}$. Since the μ-invariant of $1+X$ is trivial, then the μ-invariant of $\mathscr{L}_{\mathfrak{q}}(E,\tau\otimes\omega^t)$ is also trivial as $\mathfrak{q}\neq\mathfrak{p}$. Hence

$$\mu\big(L^{\Omega}_{p,(\Sigma)}\big(E,\tau\otimes\omega^t\big)\big)=\mu\big(L^{\Omega}_{p,\emptyset}\big(E,\tau\otimes\omega^t\big)\big)$$

using Equation (5.6) together with the additivity property of the μ-invariant.

Lemma 5.9. *If* $p\nmid\mathfrak{h}_E\cdot\epsilon_F(\tau\otimes\omega^t,0)\cdot\mathcal{A}_{\mathfrak{p}}\big(E,\tau\otimes\omega^t\big)\cdot\frac{L(E,\tau^*\otimes\omega^{-t},1)}{(2\pi)^{[K:\mathbb{Q}]}\cdot\Omega^{\mathrm{Pet}}_{E/F}}$, *then*

$$\mu\big(L^{\Omega}_{p,(\Sigma)}\big(E,\tau\otimes\omega^t\big)\big)=\mu\big(L^{\Omega}_{p,\emptyset}\big(E,\tau\otimes\omega^t\big)\big)=0$$

where the $\mathfrak{p}$*-adic multiplier is* $\mathcal{A}_{\mathfrak{p}}\big(E,\tau\otimes\chi\big)=\Big(1-\frac{\omega^{-t}(\mathfrak{p})\cdot\alpha_{\mathfrak{p}}(\mathbf{g})}{\alpha_{\mathfrak{p}}(\mathbf{f})}\Big)\Big(1-\frac{\omega^{-t}(\mathfrak{p})\cdot\beta_{\mathfrak{p}}(\mathbf{g})}{\alpha_{\mathfrak{p}}(\mathbf{f})}\Big)$.

Proof. This follows immediately from the above discussion, and the fact that $L^{\Omega}_{p,\emptyset}\big(E,\tau\otimes\omega^t\big)$ takes the value $\mathfrak{h}_E\times\frac{\epsilon_F(\tau\otimes\omega^t,0)}{\alpha_{\mathfrak{p}}(\mathbf{f})^{\mathrm{ord}_{\mathfrak{p}}(\mathrm{cond}(\tau\otimes\omega^t))}}\times\mathcal{A}_{\mathfrak{p}}\big(E,\tau\otimes\omega^t\big)\cdot\frac{L(E,\tau^*\otimes\omega^{-t},1)}{(2\pi)^{[K:\mathbb{Q}]}\cdot\Omega^{\mathrm{Pet}}_{E/F}}$ when evaluated at the trivial character (by its interpolation properties). ■

The above result provides a simple criterion to check that the "$\mu=0$" condition holds true for both these analytic objects. *Throughout we shall only consider pairs of p-congruent elliptic curve E_1 and E_2 defined over F such that*

$$\mu\big(L^{\Omega}_{p,\emptyset}\big(E_1,\tau\otimes\omega^t\big)\big)=\mu\big(L^{\Omega}_{p,\emptyset}\big(E_2,\tau\otimes\omega^t\big)\big)=0\quad\textit{at the branch } \omega^t=\mathbf{1}_F.$$

However there is one caveat: we replace $(2\pi)^{[K:\mathbb{Q}]}\cdot\Omega^{\mathrm{Pet}}_{E/F}/\mathfrak{h}_E$ with $(\Omega^+_E\Omega^-_E)^{[F:\mathbb{Q}]}$ where $\Omega^{\pm}_E=\int_{E(\mathbb{C})^{\pm}}\varpi_E$ are the real/imaginary Néron periods of the elliptic curve. This swap is fine as long as Ihara's Lemma holds (e.g. if $F=\mathbb{Q}$), otherwise the period ratio might not be a unit, although conjecturally this should not happen.

We shall work exclusively over totally real number fields F of small degree as the calculations become very slow for larger field extensions of $\mathbb{Q}$. The following examples are computed using the Dokchitser brothers' MAGMA routines [BCP; DD07], as well as John Cremona's database in [LMFDB].

Example 5.1. Consider the pair of elliptic curves $E_1 = \texttt{24a4}$ and $E_2 = \texttt{312e1}$. Their minimal Weierstrass equations are given by

$$\begin{aligned} E_1 :& y^2 = x^3 - x^2 + x \\ E_2 :& y^2 = x^3 - x^2 - 651x + 6228. \end{aligned}$$

One can check that these two elliptic curves are congruent mod $p = 3$, meaning

$$a_m(\mathbf{f}_1) \equiv a_m(\mathbf{f}_2) \bmod 3 \quad \text{for all } m \in \mathbb{N} \text{ with } \gcd(m, 2 \cdot 13) = 1.$$

E_1 and E_2 both have a non-split multiplicative reduction over $\mathbb{Q}_3$, hence both these curves are p-ordinary. Let $F = \mathbb{Q}$ and $K = \mathbb{Q}(\sqrt{-3})$, and pick $\Delta > 1$ to be a cube-free integer. We consider the two-dimensional Artin representation

$$\tau^{(\Delta)} := \mathrm{Ind}_F^K(\vartheta)$$

where $\vartheta : \mathrm{Gal}(K(\Delta^{1/3})/K) \xrightarrow{\sim} \mu_3$. Then $\tau^{(\Delta)} : G_{\mathbb{Q}} \to \mathrm{GL}_2(\mathbb{C})$ is realisable over the field of rational numbers. It follows that $\tau^{(\Delta)}$ will be self-dual (that is $\tau = \tau^*$), thereby providing an alternative way to compute $L(E, \tau^{(\Delta)}, s)$:

$$L(E, \tau^{(\Delta)}, s) = \frac{L(E/\mathbb{Q}(\Delta^{1/3}), s)}{L(E/\mathbb{Q}, s)}.$$

Note that the pairs $(E_1, \tau^{(\Delta)})$ and $(E_2, \tau^{(\Delta)})$ may not automatically satisfy condition (iii) of Theorem 4.1 for any choice of Δ. However, if we replace them by twisted pairs $\left(E_1 \otimes \eta^{-1}, \tau^{(\Delta)} \otimes \eta\right)$ and $\left(E_2 \otimes \eta^{-1}, \tau^{(\Delta)} \otimes \eta\right)$ where η is a Hecke character over F whose $\mathfrak{q}$-conductor is large at the primes $\mathfrak{q}$ dividing the non-square-free part of $\mathrm{cond}(E_1)$ and $\mathrm{cond}(E_2)$, then the twisted pairs will satisfy condition (iii), and so one still expects the mod p congruences to hold.

Put $\Delta = 2$ and assign $\Sigma^{\mathrm{bad}} = \{2, 13\}$. Evaluating $L(E_1, \tau^{(2)}, s)$ at $s = 1$ yields

$$L(E_1, \tau^{(2)}, 1) \approx 1.399245719600$$

implying that the algebraic L-value is given by

$$\mathcal{L}^*(E_1, \tau^{(2)}) = \left| \frac{L(E_1, \tau^{(2)}, 1) \cdot \sqrt{\mathrm{disc}(\mathbb{Q}(\Delta^{1/3}))}}{(2\Omega_{E_1}^+ \Omega_{E_1}^-)^{(3-1)/2}} \right| \approx 0.999999999999.$$

Similarly, evaluating $L(E_2, \tau^{(2)}, s)$ at $s = 1$ yields

$$L(E_2, \tau^{(2)}, 1) \approx 1.645733674248$$

which we then use to compute

$$\mathcal{L}^*(E_2, \tau^{(2)}) = \left| \frac{L(E_2, \tau^{(2)}, 1) \cdot \sqrt{\mathrm{disc}(\mathbb{Q}(\Delta^{1/3}))}}{(2\Omega^+_{E_2}\Omega^-_{E_2})^{(3-1)/2}} \right| \approx 24.9999999999879.$$

Remark 9. We take approximations $\mathcal{L}^*(E_1, \tau^{(2)}) \approx 1$ and $\mathcal{L}^*(E_2, \tau^{(2)}) \approx 25$ as we do not expect either of these numbers to be divisible by large primes. In fact for all our numerical examples below, these numbers were integers.

Next we will evaluate the $\tau^{(2)}$-twisted p-adic L-functions of $E = E_1$ and $E = E_2$ at the trivial character, utilising the interpolation formula (here we use $\mathfrak{f}(\tau, \mathfrak{p})$ is used as a shorthand for the value $\mathrm{ord}_{\mathfrak{p}}(\mathrm{cond}(\tau \otimes \mathbf{1}_F))$

$$\mathbf{1}\big(\mathbf{L}_{p,(\Sigma)}(E, \tau)\big) = \iota_3 \circ \iota_\infty^{-1} \left(\frac{\epsilon_F(\tau, 0)_{\mathfrak{p}}}{a_{\mathfrak{p}}(E)^{\mathfrak{f}(\tau,\mathfrak{p})}} \cdot \mathcal{A}_{\mathfrak{p}}(E, \tau) \cdot \prod_{q \in \Sigma^{\mathrm{bad}}} L_q(E, \tau^*, 1) \cdot \frac{L(E, \tau^*, 1)}{(\Omega^+_E \Omega^-_E)^{[F:\mathbb{Q}]}} \right).$$

Applying the Dokchitsers' method for computing local ϵ-factors, we then obtain

$$\epsilon_F(\tau^{(2)})_3 = \big(-1.04520385168448 \times 10^{-14}\big) + 5.19615242270663i \approx 3\sqrt{-3}.$$

Checking the Fourier expansion of E_1 and E_2 shows $a_3(E_1) = -1 = a_3(E_2)$, and as $\mathrm{cond}(\tau^{(2)}) = 108$, one deduces that

$$a_3(E_1)^{\mathfrak{f}(\tau^{(2)},3)} = -1 = a_3(E_2)^{\mathfrak{f}(\tau^{(2)},3)}.$$

The Euler factors that we require can be calculated using a built-in routine from the MAGMA package, and thus we determine that

$$L_2(E_1, \tau^{(2)}, s) = 1,$$
$$L_2(E_2, \tau^{(2)}, s) = 1,$$
$$L_{13}(E_1, \tau^{(2)}, s) = 169 \cdot 13^{-4s} - 26 \cdot 13^{-3s} - 9 \cdot 13^{-2s} - 2 \cdot 13^{-s} + 1, \text{ and}$$
$$L_{13}(E_2, \tau^{(2)}, s) = 13^{-2s} - 13^{-s} + 1.$$

Lastly, note that since we chose $\Delta > 1$ so that $\mathrm{ord}_p(\Delta^{p-1} - 1) - 1 = 0$, we have

$$\mathcal{A}_3(E_1, \tau^{(2)}) = (1 - a_3(E_1)\vartheta(3)) = 1 \quad \text{and} \quad \mathcal{A}_3(E_2, \tau^{(2)}) = 1.$$

Plugging in all the factors that we have computed above, we conclude that

$$\begin{aligned}\mathbf{1}\big(L_{3,(\Sigma)}\big(E_1,\tau^{(2)}\big)\big) &= \iota_3\circ\iota_\infty^{-1}\left(\frac{\epsilon_F(\tau)_3}{a_3(E_1)^{\mathfrak{f}(\tau,3)}}\cdot\frac{2\cdot\mathcal{A}_3(E_1,\tau)}{\sqrt{\operatorname{disc}(\mathbb{Q}(\Delta^{1/3}))}}\cdot\prod_{q\in\Sigma^{\mathrm{bad}}}L_q(E_1,\tau^*,1)\right.\\ &\qquad\left.\cdot\frac{L(E_1,\tau^*,1)\cdot\sqrt{\operatorname{disc}(\mathbb{Q}(\Delta^{1/3}))}}{(2\Omega_{E_1}^+\Omega_{E_1}^-)}\right)\\ &\approx \frac{1}{2}\cdot\frac{2}{-1}\cdot\left(1\cdot\frac{133}{169}\right)\cdot 1\\ &\approx 1358 \;=\; [2,2,0,2,1,2,1,0,\ldots]\quad\text{as a 3-adic number.}\end{aligned}$$

Repeating the process for E_2, one readily obtains

$$\begin{aligned}\mathbf{1}\big(L_{3,(\Sigma)}\big(E_2,\tau^{(2)}\big)\big) &\approx \frac{1}{2}\cdot\frac{2}{-1}\cdot\left(1\cdot\frac{157}{169}\right)\cdot 25\\ &\approx 365 \;=\; [2,1,1,1,1,1,0,0,\ldots]\quad\text{as a 3-adic number.}\end{aligned}$$

These values are indeed congruent modulo 3, as was predicted by Theorem 4.1.

Example 5.2. Consider the elliptic curves $E_1 =$ `52a1` and $E_2 =$ `1612a1` with minimal Weierstrass equations

$$E_1 : y^2 = x^3 + x - 10$$

$$E_2 : y^2 = x^3 - 272x + 80980.$$

Using MAGMA, one can verify that these curves are congruent modulo $p = 3$. We consider the same two-dimensional Artin representation $\tau^{(\Delta)}$ as above with $\Delta = 2$, so that $\Sigma^{\mathrm{bad}} = \{2, 13, 31\}$. Evaluating $L(E_1, \tau^{(2)}, s)$ at $s = 1$, one gets

$$L(E_1,\tau^{(2)},1)\approx 1.823895795706$$

which we then use to calculate the algebraic L-value

$$\mathcal{L}^*(E_1,\tau^{(2)}) = \left|\frac{L(E_1,\tau^{(2)},1)\cdot\sqrt{\operatorname{disc}(\mathbb{Q}(\Delta^{1/3}))}}{(2\Omega_{E_1}^+\Omega_{E_1}^-)^{(3-1)/2}}\right| \approx 3.999999999998.$$

Repeating the process for the congruent elliptic curve E_2, we compute that

$$L(E_2,\tau^{(2)},1)\approx 0.363500752332$$

and as a consequence,

$$\mathcal{L}^*(E_2,\tau^{(2)})\approx 15.999999999980.$$

We make the rational approximations $\mathcal{L}^*(E_1, \tau^{(2)}) \approx 4$ and $\mathcal{L}^*(E_2, \tau^{(2)}) \approx 16$.

Because E_1 and E_2 both have supersingular reduction at $p = 3$, we have $a_3(E_1) = 0 = a_3(E_2)$ and hence $\mathcal{A}_3(E_1, \tau^{(2)}) = 1 = \mathcal{A}_3(E_2, \tau^{(2)})$. Furthermore, we shall fix the same root $\alpha_3(E_1) = \sqrt{-3} = \alpha_3(E_2)$ of the Hecke polynomial $T^2 + 3$. Using the MAGMA package again, one easily determines the following Euler factors:

$$\begin{aligned}
&L_2(E_1, \tau^{(2)}, s) = 1,\\
&L_{13}(E_1, \tau^{(2)}, s) = 13^{-2s} - 13^{-s} + 1,\\
&L_{23}(E_1, \tau^{(2)}, s) = 961 \cdot 31^{-4s} - 620 \cdot 31^{-3s} + 162 \cdot 31^{-2s} - 20 \cdot 13^{-s} + 1,\\
&L_2(E_2, \tau^{(2)}, s) = 1,\\
&L_{13}(E_2, \tau^{(2)}, s) = 13^{-2s} - 13^{-s} + 1, \text{ and}\\
&L_{23}(E_2, \tau^{(2)}, s) = 31^{-2s} + 2 \cdot 31^{-2s} + 1.
\end{aligned}$$

Abusing notation and writing $\mathbf{1}\big(L_{3,(\Sigma)}\big(E_i, \tau^{(2)}\big)\big)$ for the 3-adic L-value of the $\tau^{(2)}$-twist of E_i, one finds

$$\begin{aligned}
\mathbf{1}\big(L_{3,(\Sigma)}\big(E_1, \tau^{(2)}\big)\big) &\approx \iota_3 \circ \iota_\infty^{-1}\left(\frac{1}{2} \cdot \frac{2}{(\sqrt{-3})^3} \cdot \left(1 \cdot \frac{157}{169} \cdot \frac{484}{961}\right) \cdot 4\right)\\
&\approx 1306 \cdot \iota_3 \circ \iota_\infty^{-1}\left(-3^{-3/2}\right).
\end{aligned}$$

and similarly

$$\begin{aligned}
\mathbf{1}\big(L_{3,(\Sigma)}\big(E_2, \tau^{(2)}\big)\big) &\approx \iota_3 \circ \iota_\infty^{-1}\left(\frac{1}{2} \cdot \frac{2}{(\sqrt{-3})^3} \cdot \left(1 \cdot \frac{157}{169} \cdot \frac{1024}{961}\right) \cdot 16\right)\\
&\approx \left(5359 + \mathrm{O}(3^8)\right) \cdot \iota_3 \circ \iota_\infty^{-1}\left(-3^{-3/2}\right).
\end{aligned}$$

Despite both values lying in $\mathbb{Q}_3(\sqrt{-3})$, they are in fact congruent mod $p = 3$.

Example 5.3. The elliptic curves $E_1 =$ `19a3` and $E_2 =$ `323a1` are given by

$$\begin{aligned}
E_1 &: y^2 + xy + y = x^3 + 4x - 6\\
E_2 &: y^2 + xy = x^3 + 7x - 7.
\end{aligned}$$

By checking the q-expansion of the associated newforms, one sees that these two curves are congruent mod $p = 5$. Let $K = \mathbb{Q}(\mu_5)$ and write $F = \mathbb{Q}(\mu_5)^+$ for its totally real subfield. We consider the two-dimensional Artin representation

$$\tau^{(\Delta)} := \mathrm{Ind}_F^K(\vartheta)$$

induced from a Hecke character $\vartheta : \mathrm{Gal}(K(\Delta^{1/5})/K) \xrightarrow{\sim} \mu_5$, where $\Delta > 1$ is an arbitrary fifth-power free integer. Applying the Artin formalism yields

$$L\big(E/F, \tau^{(\Delta)}, s\big) = L\big(E/\mathbb{Q}, \mathrm{Ind}_{\mathbb{Q}}^K(\vartheta), s\big) = \frac{L(E/\mathbb{Q}(\Delta^{1/5}), s)}{L(E/\mathbb{Q}, s)}.$$

For ease of notation, we write $\tau_{\mathbb{Q}}^{(\Delta)}$ for the induced representation $\mathrm{Ind}_{\mathbb{Q}}^K(\vartheta)$. If we set $\Delta = 2$ and $\Sigma^{\mathrm{bad}} = \{2, 7, 13\}$, then using built-in MAGMA functions

$$L\big(E_1, \tau_{\mathbb{Q}}^{(2)}, 1\big) \approx 1.267565010650$$

which allows us to calculate

$$\mathcal{L}^*\big(E_1, \tau_{\mathbb{Q}}^{(2)}\big) = \left| \frac{L\big(E_1, \tau_{\mathbb{Q}}^{(2)}, 1\big) \cdot \sqrt{\mathrm{disc}(\mathbb{Q}(2^{1/5}))}}{(2\Omega_{E_1}^+ \Omega_{E_1}^-)^{(5-1)/2}} \right| \approx 0.999999999999 \approx 1.$$

N.B. As E_2 has a point of infinite order over $\mathbb{Q}(2^{1/5})$, clearly $L\big(E_2, \tau_{\mathbb{Q}}^{(2)}, 1\big) = 0$, in which case it follows that $\mathbf{1}\big(L_5(E_2, \tau^{(2)})\big) = 0$ via the interpolation formula.

Applying the Dokchitsers' method for computing local ϵ-factors, one finds

$$\epsilon_F\big(\tau^{(2)}\big)_5 = -55.90169945 - \big(1.647223047i \times 10^{-8}\big) \approx 25\sqrt{-5}.$$

The Euler factors we require can be calculated using MAGMA, namely

$$\begin{aligned} &L_2(E_1, \tau^{(2)}, s) = 1, \\ &L_{17}(E_1, \tau^{(2)}, s) = 83521 \cdot 17^{-8s} - 47 \cdot 17^{-4s} + 1, \text{ and} \\ &L_{19}(E_1, \tau^{(2)}, s) = 19^{-4s} - 2 \cdot 19^{-2s} + 1. \end{aligned}$$

One readily computes that $\mathcal{A}_5(E_1, \tau^{(2)}) = 1$, and plugging in the factors above:

$$\begin{aligned} \mathbf{1}\big(L_{5,(\Sigma)}\big(E_1, \tau^{(2)}\big)\big) &= \iota_5 \circ \iota_\infty^{-1} \left(\frac{\epsilon_F\big(\tau^{(2)}\big)_5}{\alpha_5(E_1)^{\mathfrak{f}(\tau,5)}} \cdot \frac{2^2 \cdot \mathcal{A}_5(E_1, \tau)}{\sqrt{\mathrm{disc}(\mathbb{Q}(2^{1/5}))}} \cdot \prod_{q \in \Sigma^{\mathrm{bad}}} L_q(E_1, \tau^*, 1) \right. \\ &\qquad \left. \cdot \frac{L(E_1, \tau^*, 1) \cdot \sqrt{\mathrm{disc}(\mathbb{Q}(\Delta^{1/5}))}}{(2\Omega_{E_1}^+ \Omega_{E_1}^-)^2} \right) \\ &\approx \frac{i}{4} \cdot \frac{4}{\alpha_5(E_1)^5} \cdot \left(1 \cdot \frac{83475}{83521} \cdot \frac{129600}{130321} \right) \cdot (-1) \\ &\approx 386250 \ = \ [0, 0, 0, 0, 1, 1, 3, 3, \ldots] \quad \text{as a 5-adic number.} \end{aligned}$$

In particular, $\mathbf{1}\big(L_{5,(\Sigma)}\big(E_1, \tau^{(2)}\big)\big) \equiv \mathbf{1}\big(L_{5,(\Sigma)}\big(E_2, \tau^{(2)}\big)\big) \equiv 0 \mod 5$ as expected.

5.6 Tables of mod p congruences

Listed below are some calculations of congruences between p-adic L-functions evaluated at the trivial character; here we take τ to be the Artin representation $\mathrm{Ind}_{\mathbb{Q}}^{K}(\vartheta)$ where $\vartheta : \mathrm{Gal}(K(\Delta^{1/p})/K) \xrightarrow{\sim} \mu_p$.

Table 5.1: Pairs with good ordinary or multiplicative reduction at $p = 3$.

Δ	EC	$L(E,\tau,1)$	$\mathcal{L}^*(E,\tau,1)$	$\mathbf{1}(\mathbf{L}_p(E,\tau))$	$\mathbf{1}(\mathbf{L}_{p,(\Sigma)}(E,\tau))$
2	11a1	1.781648682364	≈ 5	[1,1,0,2,1,0,0,2]	[0,1,1,1,1,1,1,0]
	77c1	3.657242366397	≈ 8	[1,0,1,0,2,2,1,2]	[0,1,1,2,0,2,0,0]
2	15a1	1.721043980809	≈ 4	[2,1,2,2,2,2,2,2]	[0,1,2,0,1,1,2,0]
	195d1	0.130935136834	≈ 1	[2,2,2,2,2,2,2,2]	[0,1,0,0,1,0,0,1]
2	15a1	1.721043980809	≈ 4	[2,1,2,2,2,2,2,2]	[0,1,2,2,2,0,2,2]
	915a1	0.305093964816	≈ 7	[2,0,2,2,2,2,2,2]	[0,1,1,1,0,2,2,1]
2	195d1	0.130935136834	≈ 1	[2,2,2,2,2,2,2,2]	[0,1,2,0,0,0,1,1]
	915a1	0.305093964816	≈ 7	[2,0,2,2,2,2,2,2]	[0,1,0,0,2,2,1,0]
2	33a1	0.789496453287	≈ 2	[1,2,2,2,2,2,2,2]	[0,1,2,1,2,2,2,0]
	1023a1	2.833889853162	≈ 500	[1,1,1,2,2,0,2,2]	[0,1,1,2,0,2,1,1]
3	11a1	1.187765788242	≈ 5	[**2**,2,0,1,0,1,0,1]	[2,2,2,2,2,2,0,0]
	77c1	0.609540394399	≈ 2	[**2**,1,0,0,1,1,2,2]	[2,0,1,0,1,0,0,0]
3	14a1	2.021689274629	≈ 6	[1,1,2,2,1,1,0,0]	[0,0,0,0,1,1,0,0]
	182b1	2.664859317082	≈ 9	[0,2,0,2,2,2,0,2]	[0,0,0,0,2,2,0,1]
3	14a1	2.021689274629	≈ 6	[1,1,2,2,1,1,0,0]	[0,0,0,0,1,0,2,0]
	434b1	2.3274271572479	≈ 9	[0,2,0,2,2,2,0,2]	[0,0,0,0,2,2,1,0]
3	15a1	1.147362653873	≈ 4	[**1**,0,2,2,2,2,2,2]	[2,1,1,2,2,1,1,2]
	195d1	2.182252280571	≈ 25	[**1**,1,0,1,2,2,2,2]	[2,1,2,0,2,2,0,1]
3	24a4	1.865660959467	≈ 2	[**2**,1,2,2,2,2,2,2]	[**2**,1,0,0,1,1,1,2]
	312e1	2.194311565665	≈ 50	[**2**,2,0,2,1,2,2,2]	[**2**,0,0,0,0,0,2,0]

3	`33a1`	0.526330968858	≈ 2	[**2**,1,2,2,2,2,2,2]	[2,0,1,0,2,0,2,2]
	`1023a1`	0.472314975527	≈ 125	[**2**,0,2,2,2,1,2,2]	[2,1,2,2,0,0,0,1]
3	`195d1`	1.147362653873	≈ 4	[**1**,0,2,2,2,2,2,2]	[2,2,0,0,2,1,0,1]
	`915a1`	0.203395976544	≈ 7	[**1**,1,1,2,2,2,2,2]	[2,2,0,2,1,0,2,1]
5	`11a1`	2.850637891783	≈ 20	[1,0,0,1,0,2,2,0]	[0,1,0,1,2,1,0,0]
	`77c1`	1.462896946559	≈ 8	[1,1,0,0,2,1,2,2]	[0,1,2,2,0,0,2,0]
5	`24a4`	1.119396575680	≈ 2	[1,0,1,2,1,0,1,2]	[1,2,0,2,1,0,1,1]
	`312e1`	0.210653910303	≈ 8	[1,1,1,0,1,2,1,0]	[1,1,1,1,0,2,2,0]
5	`33a1`	0.315798581315	≈ 2	[1,0,1,2,1,0,1,2]	[0,1,1,1,2,2,1,1]
	`1023a1`	1.915709540737	≈ 845	[1,1,1,2,1,1,2,2]	[0,1,0,2,2,0,2,0]
7	`11a1`	0.509042480675	≈ 5	[2,1,0,0,0,1,1,1]	[0,2,1,0,0,1,1,2]
	`77c1`	0.261231597599	≈ 2	[2,0,2,1,0,0,2,1]	[0,2,0,2,1,1,1,0]
7	`14a1`	2.599314781666	≈ 18	[0,0,1,2,2,1,0,0]	[0,0,0,0,0,1,0,0]
	`182b1`	0	0	[0,0,0,0,0,0,0,0]	[0,0,0,0,0,0,0,0]
7	`15a1`	1.966907406639	≈ 16	[1,2,0,0,2,1,2,0]	[0,2,2,0,2,0,0,0]
	`195d1`	0.598560625528	≈ 16	[1,2,0,0,2,1,2,0]	[0,2,0,2,1,1,0,2]
7	`24a4`	0.799568982628	≈ 2	[2,0,1,0,2,1,2,0]	[2,0,2,1,0,0,1,0]
	`312e1`	6.357234078812	≈ 338	[2,1,2,2,0,1,2,0]	[2,2,2,0,2,1,1,2]
7	`33a1`	0.225570415225	≈ 2	[2,0,1,0,2,1,2,0]	[0,2,2,1,2,2,0,0]
	`1023a1`	0.008096828151	≈ 5	[2,1,1,2,0,1,0,2]	[0,2,2,0,2,2,1,2]
11	`14a1`	4.962328219546	≈ 54	[0,0,0,2,0,1,0,0]	[0,0,0,0,0,0,2,2]
	`182b1`	0	0	[0,0,0,0,0,0,0,0]	[0,0,0,0,0,0,0,0]
11	`15a1`	1.251668349679	≈ 16	[2,0,0,2,2,1,1,0]	[0,1,0,1,0,2,0,2]
	`195d1`	2.880573010354	≈ 121	[2,1,0,2,2,2,2,2]	[0,1,1,2,1,1,2,1]

The numbers above are truncated at $O(3^8)$ with the coefficients of 3^m for m negative written in boldface (e.g. $[\mathbf{2}, 1, 0, 2, \dots]$ means $(2 \times 3^{-1}) + (1 \times 3^0) + (0 \times 3^1) + (2 \times 3^2) + \dots$).

Table 5.2: Pairs with good supersingular reduction at $p = 3$.

Δ	EC	$L(E,\tau,1)$	$\mathcal{L}^*(E,\tau,1)$	$\mathbf{1}(\mathbf{L}_p(E,\tau))$	$\mathbf{1}(\mathbf{L}_{p,(\Sigma)}(E,\tau))$
2	`17a1`	1.635008772863	≈ 4	$[\mathbf{2},1,2,2,2,2,2,2]$	$[0,1,2,0,0,0,2,1]$
	`221a1`	4.164948598819	≈ 49	$[\mathbf{2},1,0,1,2,2,2,2]$	$[0,1,1,2,1,1,2,2]$
3	`17a1`	0.654003509145	≈ 4	$[\mathbf{2},0,2,1,0,1,2,1]$	$[0,1,2,2,0,2,1,2]$
	`221a1`	2.175973145505	≈ 64	$[\mathbf{2},1,2,0,0,1,2,1]$	$[0,1,2,1,1,0,2,1]$
7	`17a1`	0.467145363675	≈ 4	$[\mathbf{1},1,2,0,1,0,2,1]$	$[0,2,0,0,0,0,1,1]$
	`221a1`	4.104235062393	≈ 169	$[\mathbf{1},2,2,2,1,0,1,0]$	$[0,2,0,0,0,0,2,0]$
2	`17a1`	1.635008772863	≈ 4	$[\mathbf{2},1,2,2,2,2,2,2]$	$[0,1,2,1,0,1,0,1]$
	`629b1`	1.074505543131	≈ 8	$[\mathbf{2},2,0,0,0,0,0,0]$	$[0,1,1,1,0,1,2,2]$

As in Example 5.2, the values in the last two columns of Table 5.2 belong to the local field $\mathbb{Q}_3(\sqrt{-3})$. It is also noticeable from these tables that the constant term of the p-adic L-function, namely $\mathbf{1}(L_{p,(\Sigma)}(E,\tau))$, is frequently highly divisible by p. This is for the most part attributable to the Iwasawa power series interpolating the Euler factors $L_q(E,\tau^*,s)$ that we are taking out at primes $q \in \Sigma^{\mathrm{bad}}$ having zeroes very close to the trivial character. Nevertheless, provided it is the case that for either $E = E_1$ or $E = E_2$

$$p \text{ does not divide } \left(\mathfrak{h}_E \cdot \epsilon_F(\tau,0) \cdot \mathcal{E}_{\mathfrak{p}}(E,\tau) \cdot \frac{L(E,\tau^*,1)}{(2\pi)^{[K:\mathbb{Q}]} \cdot \Omega_{E/F}^{\mathrm{Pet}}}\right),$$

then the analytic μ-invariant of $E \otimes \tau$ vanishes by Lemma 5.9.

Chapter 6

Variation of Iwasawa invariants of p-non-ordinary modular forms

We once again fix a prime number $p \geq 3$. The principal goal of this chapter is to investigate whether the results of Greenberg and Vatsal apply to p-non-ordinary modular forms. We restrict to the case $F = \mathbb{Q}$ (i.e. classical modular forms) as p-adic L-functions corresponding to non-ordinary modular forms have yet to be constructed when $F \neq \mathbb{Q}$. Let us write $M_k(N, \eta; E)$ to denote the space of modular forms with weight k, level $N \in \mathbb{N}$ and character η defined over a field E, and $S_k(N, \eta; E)$ its corresponding subspace of cusp forms.

There are two points of divergence from the ordinary case: first, the p-adic L-functions constructed in [AV75; Viš76; MTT86] do not give rise to bounded power series; second, the dual p-primary Selmer groups over the cyclotomic $\mathbb{Z}_p$-extension of $\mathbb{Q}$ are not torsion over the corresponding Iwasawa algebra. Once we get past these issues, we show by a similar argument as Chapter 4 that Greenberg-Vatsal's results do indeed hold in the p-non-ordinary case. To avoid confusion we carry over only the notations in Chapters 2 and 3.

The last few decades have seen substantial progress towards treating the two issues arising in the p-non-ordinary scenario which are enumerated above. For a modular form $f = \sum a_n q^n \in S_k(N, \eta)$ of weight $k \geq 2$ and $a_p(f) = 0$, Pollack constructed plus and minus p-adic L-functions $L_p^{\pm}(f)$ that give rise to bounded power series [Pol03]. Let Υ be a root of the Hecke polynomial $X^2 + a_p(f)X + \eta(p)p^{k-1}$ and write $L_p(f, \Upsilon)$ for the p-adic L-function given in [AV75; Viš76; MTT86]. We have

$$L_p(f, \Upsilon) = \log^+_{p,k} \cdot L^+_p(f) + \Upsilon \cdot \log^-_{p,k} \cdot L^-_p(f),$$

where $\log^{\pm}_{p,k}$ are explicit functions defined using cyclotomic polynomials. Sprung generalized this result by introducing a pair of p-adic functions $L_p(f, \sharp)$ and $L_p(f, \flat)$ for modular forms of weight two without the assumption that $a_p(f) = 0$ [Spr12; Spr17].

When f corresponds to an elliptic curve with $a_p(f) = 0$, Kobayashi gave an arithmetic interpretation of Pollack's signed p-adic L-functions [Kob03] by constructing the plus and minus Selmer groups $\mathrm{Sel}^{\pm}_p(f)$. The duals of these groups are torsion over the Iwasawa algebra of the cyclotomic $\mathbb{Z}_p$-extension of $\mathbb{Q}$, allowing us to formulate the plus and minus Iwasawa main conjectures, which assert that there exists $n^{\pm} \in \mathbb{Z}$ such that the characteristic ideal of $\mathrm{Sel}^{\pm}_p(f)^{\vee}$ is generated by $p^{n^{\pm}} \cdot L^{\pm}_p(f)$. One inclusion of this conjecture was proved in [Kob03]. When the elliptic curve has complex multiplication, the full main conjecture was proved by Pollack–Rubin [PR04]. These ideas have been generalized to higher-weight modular forms in [Lei11; LLZ10; LLZ11] using p-adic Hodge theory and Wach modules.

6.1 Main results

Let $p \geq 3$ be our fixed prime and let $\iota_p : \overline{\mathbb{Q}} \hookrightarrow \mathbb{C}_p$ be a fixed embedding. We consider two normalized cuspidal eigenforms $f = \sum_{n\geq 1} a_n \cdot q^n \in S_k(\Gamma_1(N_f), \eta_f)$ and $g = \sum_{n\geq 1} b_n \cdot q^n \in S_k(\Gamma_1(N_g), \eta_g)$ which are both *p-non-ordinary*, meaning the values $\mathrm{ord}_p(a_p)$ and $\mathrm{ord}_p(b_p)$ are positive. Additionally, we set the integer $N := \mathrm{lcm}(N_f, N_g) \geq 4$. Let $E/\mathbb{Q}_p$ be a finite extension containing $\iota_p(a_n)$ and $\iota_p(b_n)$ for all $n \in \mathbb{N}$ and choose a uniformizer ϖ of the maximal ideal of $\mathcal{O}$.

We recall the Teichmüller character $\omega : \mathrm{Gal}(\mathbb{Q}(\mu_p)/\mathbb{Q}) \to \mathbb{Z}_p^{\times}$ and we write $L_p(*, \sharp, \omega^t, X)$ and $L_p(*, \flat, \omega^t, X)$ for the respective ω^t-isotypic component of the signed p-adic L-functions $L_p(*, \sharp)$ and $L_p(*, \flat)$ for $* \in \{f, g\}$ (see Section 6.4 for more details). In this article, we work under the following assumptions.

Hypothesis 6.1. *(analytic congruence) The eigenforms f and g are congruent modulo ϖ^r for some positive integer r, in other words, $a_m \equiv b_m \bmod \varpi^r$ for all integers m that are relatively prime to N.*

Hypothesis 6.2. *(Fontaine–Laffaille) The fixed prime p satisfies $p > k$.*

Hypothesis 6.3. *(unramified) The coefficient field $E/\mathbb{Q}_p$ is unramified.*

Hypothesis 6.4. *(nonzero) The choice of the symbol $\natural \in \{\sharp, \flat\}$ and integer $0 \leq t \leq p-2$ are such that both $L_p(f, \natural, \omega^t, X)$ and $L_p(g, \natural, \omega^t, X)$ are nonzero.*

Hypothesis 6.5. *(algebraic congruence) There is an isomorphism $T_f/\varpi T_f \cong T_g/\varpi T_g$ as $G_{\mathbb{Q}}$-representations.*

In Hypothesis 6.5, T_f and T_g are $G_{\mathbb{Q}}$-stable $\mathcal{O}$-lattices inside Deligne's representations attached to f and g, respectively. As f and g are assumed to be p-non-ordinary, these representations are irreducible and the $G_{\mathbb{Q}}$-stable lattices are unique up to isomorphism. Note that we may take $\varpi = p$ under Hypothesis 6.3. Furthermore, if Hypothesis 6.5 holds, then $a_m \equiv b_m \bmod p$ for all m that are relatively prime to N. In other words, Hypothesis 6.5 implies that Hypothesis 6.1 holds for $r = 1$. Conversely, under Hypotheses 6.1 and 6.2, the residual representations are irreducible (see [Edi92, Theorem 2.6]) which implies that Hypothesis 6.5 holds.

Our strategy is as follows. In Section 6.2, we recall the relationship between modular symbols and special values of the complex L-function associated with modular forms. We then make use of Hypotheses 6.1 and 6.2 to show that the congruence between f and g translates to a congruence between their respective modular symbols after choosing appropriate periods. Towards the end of the section, we recall the definition of Mazur–Tate elements. In Section 6.3, we review how the Mazur–Tate elements for different critical twists are used to construct the p-adic L-functions in [AV75; Viš76; MTT86].

We then explain how to decompose these p-adic L-functions into the signed p-adic L-functions $L_p(f,\sharp,\omega^t,X)$ and $L_p(f,\flat,\omega^t,X)$ using the logarithm matrix introduced in [BL21] and we give a detailed analysis of the integrality of these elements under Hypotheses 6.2 and 6.3. In Section 6.4, we demonstrate how the congruence in Section 6.2 trickles down to a pair of congruences:

$$\begin{aligned} L_{p,(\Sigma_0)}(f,\sharp,\omega^t,X) &\equiv L_{p,(\Sigma_0)}(g,\sharp,\omega^t,X) \mod \varpi^r\cdot\mathcal{O}[\![X]\!], \\ L_{p,(\Sigma_0)}(f,\flat,\omega^t,X) &\equiv L_{p,(\Sigma_0)}(g,\flat,\omega^t,X) \mod \varpi^r\cdot\mathcal{O}[\![X]\!], \end{aligned}$$

where Σ_0 denotes the set of primes dividing N_fN_g and $L_{p,(\Sigma_0)}$ represents the Σ_0-imprimitive p-adic L-function. The reader is referred to Theorem 6.21 for a precise statement. Finally, we show in Section 6.5 how the above congruences allow us to deduce the following theorem (c.f. Theorem 6.22) regarding the behavior of the Iwasawa invariants $\mu(*,\omega^t)^\natural_{\mathrm{an}}$ and $\lambda(*,\omega^t)^\natural_{\mathrm{an}}$ of $L_p(*,\natural,\omega^t,X)$.

Theorem 6.1. *Let f and g be p-non-ordinary modular forms, $t\in\{0,\dots,p-2\}$ and $\natural\in\{\sharp,\flat\}$ satisfying Hypotheses 6.2–6.5. Let Σ_0 be the finite set of primes dividing N_fN_g. Then $\mu(f,\omega^t)^\natural_{\mathrm{an}}=0$ if and only if $\mu(g,\omega^t)^\natural_{\mathrm{an}}=0$, in which case*

$$\lambda(f,\omega^t)^\natural_{\mathrm{an}}=\lambda(g,\omega^t)^\natural_{\mathrm{an}}+\sum_{v\in\Sigma_0}\left(\mathbf{e}_v(g,\omega^t)-\mathbf{e}_v(f,\omega^t)\right),$$

where each $\mathbf{e}_v(*,\omega^t)$ corresponds to the λ-invariant of an Iwasawa function interpolating the Euler factor of the L-function of $*\in\{f,g\}$ at $v\in\Sigma_0$.

The latter half of Section 6.5 is spent recalling the signed Selmer groups of p-non-ordinary modular forms and their respective duals, denoted by $\mathcal{X}^\natural(*)$, for $\natural\in\{\sharp,\flat\}$ and $*\in\{f,g\}$. The Iwasawa invariants of its ω^t-isotypic component are denoted by $\mu(*,\omega^t)^\natural_{\mathrm{alg}}$ and $\lambda(*,\omega^t)^\natural_{\mathrm{alg}}$. We extrapolate an algebraic version of Theorem 6.1 (c.f. Theorem 6.25), which follows from the results in [HL19].

Theorem 6.2. *Let f, g, $\natural$, t and Σ_0 be as in Theorem 6.1. Then, $\mathcal{X}^\natural(f)^{[t]}$ and $\mathcal{X}^\natural(f)^{[t]}$ is $\mathcal{O}[\![X]\!]$-torsion. In addition, $\mu(f,\omega^t)^\natural_{\mathrm{alg}}=0$ if and only if $\mu(g,\omega^t)^\natural_{\mathrm{alg}}=0$, in which case*

$$\lambda(f,\omega^t)^\natural_{\mathrm{alg}}=\lambda(g,\omega^t)^\natural_{\mathrm{alg}}+\sum_{\mathfrak{v}\in\Sigma_0}\left(\mathbf{e}_v(g,\omega^t)-\mathbf{e}_v(f,\omega^t)\right).$$

Together, these two theorems above give the following application (c.f. Theorem 6.26) to the signed Iwasawa main conjectures (see Conjecture 6.23).

Theorem 6.3. *Let f, g, $\natural$, t and Σ_0 be as in Theorem 6.1. If the equality $\mu(*,\omega^t)^\natural_{\mathrm{alg}} = \mu(*,\omega^t)^\natural_{\mathrm{an}} = 0$ holds for $* \in \{f,g\}$, then the signed Iwasawa main conjecture holds for $\mathcal{X}^\natural(f)^{[t]}$ if and only if it holds for $\mathcal{X}^\natural(g)^{[t]}$.*

Similar questions were studied in [KLP19], where the authors showed that modules involving Kato's zeta elements that are used to formulate the Iwasawa main conjecture without p-adic L-functions behave well under congruences. In particular, they proved that the Iwasawa invariants of these modules satisfy similar relations to those given by Theorems 6.1 and 6.2 when f and g have isomorphic residual representations. Furthermore, they obtained applications to the signed Iwasawa main conjectures in a similar manner as Theorem 6.3. Our method is different, and we work under different hypotheses. Finally, we remark that in the special case of elliptic curves, Theorem 6.2 was already established by Kim in [Kim09]. Other similar results were studied in [Kim21; CK17; RS23; Ham23; AL21; Pon20; HLV22] in various contexts.

6.2 Review of the Mazur–Tate elements

The aim of this section is to introduce modular symbols attached to an arbitrary modular form and to show its relation to the critical L-values. As we will see, these elements have associated Mazur-Tate elements which we will need in order to define the non-integral p-adic L-function in the next section.

6.2.1 Modular symbols attached to cusp forms

Let $\mathbf{D} = \mathrm{Div}(\mathbb{P}^1(\mathbb{Q}))$ be the free abelian group of divisors supported on the rational cusps $\mathbb{P}^1(\mathbb{Q})$ and let $\mathbf{D}^0 \subseteq \mathbf{D}$ be its subgroup of degree 0 divisors. Then $\mathbf{D}^0$ is generated by the symbols $[r]-[s]$ where $r,s \in \mathbb{P}^1(\mathbb{Q})$. Given a commutative ring R and a congruence subgroup Γ we take A to be a right $R[\Gamma]$-module. An additive homomorphism $\xi : \mathbf{D}^0 \to A$ that satisfies

$\xi(\gamma D) \,\big|\, \gamma = \xi(D)$ for all $\gamma \in \Gamma$ is called an *A-valued modular symbol* and we denote by $\mathrm{Symb}(\Gamma, A)$ the R-module of modular symbols.

The group of matrices $\mathrm{GL}_2(R)$ acts on the space $V_{k-2}(R)$ of degree $k-2$ homogeneous polynomials in the indeterminates X and Y defined over R as follows: for $P(X,Y) \in V_{k-2}(R)$ and $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{GL}_2(R)$,

$$P(X,Y) \,\big|\, \gamma := P\big((X,Y)\gamma^{\iota}\big) = P(dX - cY, -bX + aY). \tag{6.1}$$

Consider a cusp form $f \in S_k(\Gamma_1(N_f), \eta_f)$. One may view $\{r\} - \{s\}$ as any simple path $s \to r$ in the upper half plane which allows us to construct a $V_{k-2}(\mathbb{C})$-valued modular symbol

$$\xi_f : \{r\} - \{s\} \mapsto 2\pi\sqrt{-1} \int_s^r f(z)(zX + Y)^{k-2}\, \mathrm{d}z \tag{6.2}$$

that lives in $\mathrm{Symb}\big(\Gamma_1(N_f), V_{k-2}(\mathbb{C})\big) := \mathrm{Hom}_\Gamma\big(\mathbf{D}^0, V_{k-2}(\mathbb{C})\big)$.

It is well known that the modular symbol ξ_f encodes the special values of the L-function $L(f,s)$ associated to f. For a positive integer D, we let $\chi : (\mathbb{Z}/D\mathbb{Z})^\times \to \mathbb{C}^\times$ be a Dirichlet character. The L-function of the twisted form $f_{\overline{\chi}} = \sum \overline{\chi}(n) a_n \cdot q^n$ can be written as

$$L(f, \overline{\chi}, s) = \frac{(-2\pi\sqrt{-1})^s}{\Gamma(s) G(\chi)} \sum_{a \in (\mathbb{Z}/D\mathbb{Z})^\times} \chi(a) \int_{a/D}^{\infty} f(z) \left(z - \frac{a}{D}\right)^{s-1} \mathrm{d}z \tag{6.3}$$

where as usual, $\Gamma(s)$ denotes the Gamma function and $G(\chi)$ is the Gauss sum given by $\sum_{a \in (\mathbb{Z}/D\mathbb{Z})^\times} \chi(a) e^{2\pi\sqrt{-1}a/D}$. We apply the action in (6.1) as follows:

$$\xi_f \left| \begin{pmatrix} 1 & a \\ 0 & D \end{pmatrix} \right. (\{\infty\} - \{0\}) = 2\pi\sqrt{-1} \int_{a/D}^{\infty} f(z)(DzX + (-aX + Y))^{k-2}\, \mathrm{d}z.$$

Setting $(X,Y) = (1,1)$ in the above expression, its right-hand side becomes

$$2\pi\sqrt{-1} \sum_{j=0}^{k-2} \binom{k-2}{j} D^j \int_{a/D}^{\infty} f(z) \left(z - \frac{a}{D}\right)^j \mathrm{d}z.$$

For $0 \le j \le k-2$, let $\xi_{f,j}(a, D)$ be the projection of the above expression to the j-th exponent. Equation 6.3 can then be written as

$$L(f, \overline{\chi}, j+1) = \frac{(-2\pi\sqrt{-1})^j}{j!\tau(\chi)} \cdot \frac{1}{D^j} \cdot \binom{k-2}{j}^{-1} \sum_{a \in (\mathbb{Z}/D\mathbb{Z})^\times} \chi(a)\, \xi_{f,j}(a, D). \tag{6.4}$$

6.2.2 The Eichler-Shimura isomorphism

We bring $g \in S_k(\Gamma_1(N_g), \eta_g)$ into the mix and we assume further that both f and g are normalized cuspidal eigenforms. Let $\Gamma := \Gamma_1(N)$ with $N \geq 4$ a common multiple of N_f and N_g, making $\Gamma \subset \mathrm{SL}_2(\mathbb{Z})$ a discrete torsion-free subgroup.

For a Γ-module A, let $H^q(\Gamma, A)$ be the q-th cohomology group and denote by $H^q_c(\Gamma, A)$ those with compact support. We define the parabolic cohomology group $H^1_P(\Gamma, A)$ as the natural image of $H^q_c(\Gamma, A) \to H^q(\Gamma, A)$. Let R be a ring as before and set $A = V_{k-2}(R)$. It follows from [AS86, Proposition 4.2] that there is a canonical isomorphism

$$H^1_c\big(\Gamma, V_{k-2}(R)\big) \cong \mathrm{Symb}\,\big(\Gamma, V_{k-2}(R)\big)$$

resulting in an exact sequence

$$S_k(\Gamma) \xrightarrow{f \mapsto \xi_f} \mathrm{Symb}(\Gamma, V_{k-2}(\mathbb{C})) \xrightarrow{\sim} H^1_c(\Gamma, V_{k-2}(\mathbb{C})) \longrightarrow H^1(\Gamma, V_{k-2}(\mathbb{C}))\,. \quad (6.5)$$

The following theorem, referred to as the Eichler-Shimura isomorphism, relates the $\mathbb{C}$-vector space of modular forms to cohomology groups, allowing us to analyze the former using techniques in cohomology theory.

Theorem 6.4. *There is an isomorphism*

$$H^1_P(\Gamma, V_{k-2}(\mathbb{C})) \cong S_k(\Gamma) \oplus S_k(\Gamma)^c$$

which extends to

$$H^1(\Gamma, V_{k-2}(\mathbb{C})) \cong M_k(\Gamma) \oplus S_k(\Gamma)^c \quad (6.6)$$

where c denotes the complex conjugation operator.

Proof. We first note that $S_k(\Gamma) \otimes_{\mathbb{R}} \mathbb{C} \cong S_k(\Gamma) \oplus S_k(\Gamma)^c$. One can then read off the first isomorphism from the exact sequence (6.5). As for the second isomorphism, we recall the decomposition of $M_k(\Gamma) \cong S_k(\Gamma) \oplus E_k(\Gamma)$ into the space $S_k(\Gamma)$ of cusp forms and the space $E_k(\Gamma)$ of Eisenstein series. The claim follows using a dimension formula argument detailed in [Hid93, §6.2]. ■

The idea is that if f and g both satisfy Hypotheses 6.1 and 6.2, then one should be able to observe a congruence between their associated modular symbols. In order to do so, we first bring the discussion to an integral setting.

Lemma 6.5. *If $A \subset \mathbb{C}$ is an R-flat algebra, then*

$$H^1_P(\Gamma, V_{k-2}(\mathbb{C})) = H^1_P(\Gamma, V_{k-2}(A)) \otimes_R \mathbb{C}\,.$$

The above lemma is an application of a result given in [Hid93, §6.2 (1b)]. Let E be the coefficient field of f and g with integer ring $\mathcal{O}$ as in the introduction. Then Lemma 6.5 combined with Theorem 6.4 yields

$$\begin{aligned} H^1_P(\Gamma, V_{k-2}(E)) \otimes \mathbb{C} &= H^1_P(\Gamma, V_{k-2}(\mathbb{C})) \\ &\cong S_k(\Gamma) \oplus S_k(\Gamma)^c \\ &\cong (S_k(\Gamma; E) \otimes \mathbb{C}) \oplus (S_k(\Gamma; E)^c \otimes \mathbb{C}) \\ &= (S_k(\Gamma; E) \oplus S_k(\Gamma; E)^c) \otimes \mathbb{C}, \end{aligned} \tag{6.7}$$

where the last equality follows from [Hid93, §6.3, Theorem 2].

Remark 10. Let $f \in S_k(\Gamma; \mathcal{O})$. The $V_{k-2}(\mathbb{C})$-valued modular symbol ξ_f can be decomposed as $\xi_f^+ + \xi_f^-$, where $\xi_f^\pm$ belongs to the ± 1-eigenspace of the complex conjugation. Lemma 6.5 implies that we can divide $\xi_f^\pm$ by a complex number $\Omega_f^\pm$ to obtain an $V_{k-2}(K)$-valued modular symbol $\varphi_f^\pm$. In fact, we can choose $\Omega_f^\pm$ so that $\varphi_f^\pm \in H^1_P(\Gamma, V_{k-2}(\mathcal{O}))^\pm$. We refer to these numbers as *cohomological periods*, which are well-defined up to units in $\mathcal{O}$ (see [PW11, Definition 2.1 and Remark 2.2]).

Let $\mathbf{T}_k(\Gamma; \mathcal{O})$ be the $\mathcal{O}$-algebra generated by all Hecke operators T_ℓ acting faithfully on the space of cusp forms $S_k(\Gamma; \mathcal{O})$ defined over $\mathcal{O}$. It is well known that these algebras act on the torsion-free part of $H^1(\Gamma, V_{k-2}(\mathcal{O}))$ and $H^1_P(\Gamma, V_{k-2}(\mathcal{O}))$, respectively. As observed in Section 4.3, the bilinear pairing

$$S_k(\Gamma, \mathcal{O}) \times \mathbf{T}_k(\Gamma; \mathcal{O}) \to \mathcal{O}$$

given by $(T_n, f) = a_1\big(f \,\big|\, T_n\big)$ induces a duality $\mathrm{Hom}_{\mathcal{O}}(\mathbf{T}_k(\Gamma; \mathcal{O}), \mathcal{O}) \cong S_k(\Gamma; \mathcal{O})$. Our eigenforms f and g give rise to the following group homomorphisms

$$\lambda_f, \lambda_g : \mathbf{T}_k(\Gamma; \mathcal{O}) \to \mathcal{O}.$$

If f and g are p-congruent eigenforms, then the compositions of λ_f and λ_g with the reduction map modulo ϖ coincide and we denote their common kernel by $\mathfrak{M}$. This $\mathfrak{M}$ is a maximal ideal in $\mathbf{T}_k(\Gamma;\mathcal{O})$ and it determines an identical semisimple representation for f and g

$$\rho_{\mathfrak{M}} : \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to \mathrm{GL}_2\left(\mathbf{T}_k(\Gamma;\mathcal{O})/\mathfrak{M}\right)$$

that is unramified outside Np, such that for all primes $\ell \nmid Np$,

$$\begin{cases} \mathrm{Tr}(\rho(\mathrm{Frob}_\ell^{-1})) = T_\ell \mod \mathfrak{M} \\ \det(\rho(\mathrm{Frob}_\ell^{-1})) = \ell\langle\ell\rangle \mod \mathfrak{M}. \end{cases}$$

We recall the Fourier expansion of our eigenforms $f = \sum_{n\geq 1} a_n q^n$ and $g = \sum_{n\geq 1} b_n q^n$. Let Σ be any finite set of primes and consider the Σ-depletion of f and g given by:

$$F := f_\Sigma = \sum_{n\geq 1} a'_n q^n \qquad \text{and} \qquad G := g_\Sigma = \sum_{n\geq 1} b'_n q^n,$$

where $a'_n = b'_n = 0$ for all $n \in \mathbb{N}$ divisible by primes in Σ, and $a'_n = a_n$, $b'_n = b_n$ otherwise. We also recall from Section 2.4 that for $\mathrm{Re}(s) \gg 0$ the L-function associated to f can be written as (ℓ here is any prime not dividing v):

$$L(f,s) = \det\Big(1 - X\cdot\mathrm{Frob}_v^{-1}\,\Big|\,V_\ell(f)^{I_{\mathbb{Q}_v}}\Big) = \prod_v L_v(f,v^{-s})^{-1} \qquad (6.8)$$

where each local factor equals $L_v(f,X) = \prod_{v\nmid N_f}\left(1 - a_v(f)v^{-s} + \eta_f(v)v^{k-1-2s}\right)^{-1}$ $\times \prod_{v|N_f}(1-a_v(f)v^{-s})^{-1}$; a similar formulation applies to g. We write $L_{(\Sigma)}(f,s)$ and $L_{(\Sigma)}(g,s)$ for the Σ-imprimitive L-function attached to f and g, respectively.

Let Σ_0 be the set of primes dividing $N_f N_g$. Then the Hecke eigenvalues of the Σ_0-depletions $F = f_{\Sigma_0}$ and $G = g_{\Sigma_0}$ agree on *every* $n \geq 1$. If M is the smallest integer so that F and G are both modular forms of level M then M is divisible only by primes in Σ_0. This is because depleting a modular form, say at a prime q, increases the level by at most q^2. We henceforth set $\Gamma = \Gamma_1(M)$.

The main result of this section is that if f and g satisfy Hypotheses 6.1 and 6.2, that is if f and g are ϖ^r-congruent for some positive integer r and if $p > k$, then the special L-values of F and G are also congruent modulo ϖ^r. Let ξ_F

and ξ_G be the modular symbols attached to F and G via Equation (6.2). We normalise them by their respective cohomological periods (see Remark 10) to define the $V_{k-2}(\mathcal{O})$-valued modular symbols $\varphi_F^{\pm}$ and $\varphi_G^{\pm}$ satisfying

$$\varphi_*^{\pm} = \frac{\xi_*^{\pm}}{\Omega_*^{\pm}}, \quad * \in \{F, G\}.$$

We show that φ_F^{+} and φ_F^{-} are congruent to φ_G^{+} and φ_G^{-} modulo ϖ^r, respectively.

As discussed above, the congruence class of F and G determines a maximal ideal $\mathfrak{M} \subset \mathbf{T}_k(\Gamma; \mathcal{O})$ so that F and G lie in the same class in $S_k(\Gamma; \mathcal{O})_{\mathfrak{M}}$. To prove that $\varphi_F^{\pm} \equiv \varphi_G^{\pm} \bmod \varpi^r$, we need the following multiplicity one result which is a consequence of [FJ95, Theorem 2.1]: if $p \nmid M$ and $p > k$, then

$$H_P^1(\Gamma, V_{k-2}(\mathcal{O}))_{\mathfrak{M}}^{\pm} \cong S_k(\Gamma; \mathcal{O})_{\mathfrak{M}}. \tag{6.9}$$

Applying [GS93, (6.7)] with $A = \mathcal{O}$ and localizing at $\mathfrak{M}$, we deduce the following short exact sequence

$$0 \to \mathrm{B}\big(\Gamma, V_{k-2}(\mathcal{O})\big)_{\mathfrak{M}} \to \mathrm{Symb}\,\big(\Gamma, V_{k-2}(\mathcal{O})\big)_{\mathfrak{M}} \xrightarrow{\delta} H_P^1\big(\Gamma, V_{k-2}(\mathcal{O})\big)_{\mathfrak{M}} \to 0\,.$$

The modular symbols $\varphi_*^{\pm}$ are sent to $\delta(\varphi_*^{\pm}) \in H_P^1(\Gamma, V_{k-2}(\mathcal{O}))_{\mathfrak{M}}^{\pm}$ and since $F \equiv G \bmod \varpi^r S_k(\Gamma; \mathcal{O})_{\mathfrak{M}}$, we get $\delta(\varphi_F^{\pm}) \equiv \delta(\varphi_G^{\pm}) \bmod \varpi^r \cdot H_P^1(\Gamma, V_{k-2}(\mathcal{O}))_{\mathfrak{M}}$. Note that this congruence lifts naturally to $\varphi_*^{\pm}$ in $\mathrm{Symb}(\Gamma, V_{k-2}(\mathcal{O}))_{\mathfrak{M}}$. This is because the action of Hecke operators on the boundary symbols $\mathrm{B}(\Gamma, V_{k-2}(\mathcal{O}))_{\mathfrak{M}}$ are well-known to be Eisenstein, and the irreducibility of $\rho_{\mathfrak{M}}$ guarantees that the maximal ideal $\mathfrak{M}$ is non-Eisenstein.

Therefore, we have the following congruence between modular symbols:

$$\begin{aligned} &\sum_{a \in (\mathbb{Z}/p^n\mathbb{Z})^\times} \varphi_F \,\Bigg|\, \begin{pmatrix} 1 & -a \\ 0 & p^n \end{pmatrix} (\{\infty\} - \{0\}) \\ \equiv &\sum_{a \in (\mathbb{Z}/p^n\mathbb{Z})^\times} \varphi_G \,\Bigg|\, \begin{pmatrix} 1 & -a \\ 0 & p^n \end{pmatrix} (\{\infty\} - \{0\}) \bmod \varpi^r. \end{aligned} \tag{6.10}$$

Since $p \nmid M$ and $p > k$, a version of Ihara's Lemma is available for us to use in order to relate the periods of the modular forms f and g to the periods of their respective Σ_0-depletions F and G via $\Omega_f^{\pm} = u_f^{\pm} \cdot \Omega_F^{\pm}$ and $\Omega_g^{\pm} = u_g^{\pm} \cdot \Omega_G^{\pm}$, where $u_f^{\pm}$, $u_g^{\pm}$ are p-adic units. We refer the reader to [DFG04, Proposition 1.4(c)] for the full statement of Ihara's Lemma. In fact without loss of generality, we can take $u_f^{\pm} = u_g^{\pm} = 1$. This then allows us to prove the following result.

Proposition 6.6. *Let $\chi : (\mathbb{Z}/D\mathbb{Z})^\times \to \mathbb{C}^\times$ be a Dirichlet character and suppose f and g satisfy Hypotheses 6.1 and 6.2. Let Σ_0 be the set of primes dividing $N_f N_g$. Then there exist complex numbers $\Omega_f^{\pm}$ and $\Omega_g^{\pm}$ such that for $0 \leq j \leq k-2$*

$$\begin{aligned} &\frac{j! \cdot G(\chi)}{(-2\pi\sqrt{-1})^j} \cdot \frac{D^j}{\Omega_f^{\pm}} \cdot \binom{k-2}{j} \cdot L_{(\Sigma_0)}(f, \overline{\chi}, j+1) \\ \equiv\ &\frac{j! \cdot G(\chi)}{(-2\pi\sqrt{-1})^j} \cdot \frac{D^j}{\Omega_g^{\pm}} \cdot \binom{k-2}{j} \cdot L_{(\Sigma_0)}(g, \overline{\chi}, j+1) \bmod \varpi^r. \end{aligned}$$

Proof. The proposition follows upon combining Equations (6.4) and (6.10). ■

6.2.3 Mazur–Tate elements

Let us put in place a few more notations. For $n \geq 1$, let $\mathcal{G}_n = \mathrm{Gal}(\mathbb{Q}(\mu_{p^n})/\mathbb{Q})$ and let $\mathcal{G}_\infty = \mathrm{Gal}\left(\bigcup_n \mathbb{Q}(\mu_{p^n})/\mathbb{Q}\right)$ which decomposes into $\mathcal{G}_\infty \cong \Delta \times G_\infty$ where $\Delta = \mathrm{Gal}(\mathbb{Q}(\mu_p)/\mathbb{Q}) \cong (\mathbb{Z}/p\mathbb{Z})^\times$ and $G_\infty = \mathrm{Gal}(\mathbb{Q}(\mu_{p^\infty})/\mathbb{Q}(\mu_p)) \cong \mathbb{Z}_p$. Write $\mathbb{Q}^{\mathrm{cyc}}$ for the cyclotomic $\mathbb{Z}_p$-extension of $\mathbb{Q}$ whose Galois group is isomorphic to G_∞ and define the Iwasawa algebra of G_∞ as the completed group algebra $\Lambda := \mathcal{O}[\![G_\infty]\!] = \varprojlim \mathcal{O}[G_n]$ where $G_n = \mathrm{Gal}(\mathbb{Q}(\mu_{p^{n+1}})/\mathbb{Q}(\mu_p)) \cong \mathbb{Z}/p^n\mathbb{Z}$. Recall that is an isomorphism $\mathcal{G}_n \to (\mathbb{Z}/p^n\mathbb{Z})^\times$ given by $\sigma_a \mapsto a$, where σ_a is the automorphism $\zeta \mapsto \zeta^a$, for any $\zeta \in \mu_{p^n}$.

Definition 6.1. For a modular symbol $\varphi \in H_c^1\big(\Gamma_1(N), V_{k-2}(R)\big)$, we define the associated *Mazur–Tate elements* $\vartheta_n(\varphi)$ of level $n \geq 1$ by

$$\vartheta_n(\varphi) = \sum_{a \in (\mathbb{Z}/p^n\mathbb{Z})^\times} \varphi \Bigg| \begin{pmatrix} 1 & -a \\ 0 & p^n \end{pmatrix} (\{\infty\} - \{0\}) \cdot \sigma_a \in R[X,Y][\mathcal{G}_n].$$

and we write

$$\vartheta_n(\varphi) = \sum_{j=0}^{k-2} \binom{k-2}{j} X^j Y^{k-2-j} \cdot \vartheta_{n,j}(\varphi)$$

where $\vartheta_{n,j}(\varphi) \in R[\mathcal{G}_n]$.

For the modular symbol ξ_f defined in (6.2), one has the explicit formula

$$\vartheta_{n,j}(\xi_f) = 2\pi\sqrt{-1} \int_{-a/p^n}^{\infty} f(z)(p^n z + a)^j \,\mathrm{d}z \cdot \sigma_a \in \mathbb{C}[\mathcal{G}_n].$$

Under the decomposition $\xi_f = \xi_f^+ + \xi_f^-$ implied by Remark 10, one obtains

$$\vartheta_{n,j}(\xi_f^\pm) = \pi\sqrt{-1}\Big(\int_{-a/p^n}^{\infty} f(z)(p^n z + a)^j \,\mathrm{d}z \pm (-1)^{k+j}\int_{a/p^n}^{\infty} f(z)(p^n z - a)^j \,\mathrm{d}z\Big) \cdot \sigma_a \in \mathbb{C}[\mathcal{G}_n].$$

Dividing through by the cohomological periods, we set

$$\theta_{n,j}(\xi_f^\pm) := \frac{\vartheta_{n,j}(\xi_f^\pm)}{\Omega_f^\pm} \in \mathcal{O}[\mathcal{G}_n].$$

Recall that there is a decomposition $\mathcal{G}_{n+1} \cong \Delta \times G_n$ where G_n is a cyclic group of order p^n and recall also that ω is the Teichmüller character on Δ. When R is a $\mathbb{Z}_p$-algebra, we obtain an induced map $\omega^t : R[\mathcal{G}_{n+1}] \to R[G_n]$ for each $0 \le t \le p-2$.

Definition 6.2. For integers $0 \le t \le p-2$, $0 \le j \le k-2$ and $n \ge 0$, we define the *theta element* $\theta_{n,j}(f,\omega^t) \in \mathcal{O}[G_n]$ as the image of $\theta_{n+1,j}\left(\xi_f^{(-1)^t}\right)$ under ω^{t-j}.

More explicitly,

$$\theta_{n,j}(f,\omega^t) = \frac{\pi\sqrt{-1}}{\Omega_f^{(-1)^t}} \sum_{a\in(\mathbb{Z}/p^{n+1}\mathbb{Z})^\times} \left(\int_{-a/p^{n+1}}^{\infty} f(z)(p^{n+1}z + a)^j \,\mathrm{d}z + (-1)^{t+k+j}\int_{a/p^{n+1}}^{\infty} f(z)(p^{n+1}z - a)^j \,\mathrm{d}z\right)\omega^{t-j}(a)\bar{\sigma}_a$$

where $\bar{\sigma}_a$ denotes the image of σ_a under the natural projection $\mathcal{G}_{n+1} \to G_n$. Moreover, if Υ is a root of the Hecke polynomial $X^2 - a_p(f)X + \eta_f(p)p^{k-1}$, we define for $n \ge 1$ the *p-stabilized theta element*

$$\theta_{n,j}(f,\Upsilon,\omega^t) = \frac{1}{\Upsilon^{n+1}} \cdot \theta_{n,j}(f,\omega^t) - \frac{\eta_f(p)p^{k-2}}{\Upsilon^{n+2}} \cdot \nu_{n-1}^n \theta_{n-1,j}(f,\omega^t) \in E(\Upsilon)[G_n],$$

where $\nu_{n-1}^n : \mathcal{O}[G_{n-1}] \to \mathcal{O}[G_n]$ is the norm map that sends $\sigma \in G_{n-1}$ to the sum of the pre-images of σ in G_n under the projection map $\pi_{n-1}^n : G_n \to G_{n-1}$.

These theta elements satisfy the following properties (c.f. [MTT86, §4 and §10]): for $n \ge 1$ and $0 \le t \le p-2$:

1. $\pi_n^{n+1}\theta_{n+1,j}(f,\omega^t) = a_p(f)\theta_{n,j}(f,\omega^t) - \eta_f(p)p^{k-2}\nu_{n-1}^n\theta_{n-1,j}(f,\omega^t)$;
2. $\pi_n^{n+1}\theta_{n+1,j}(f,\Upsilon,\omega^t) = \theta_{n,j}(f,\Upsilon,\omega^t)$.

6.3 Signed p-adic L-functions

Let $E/\mathbb{Q}_p$ be the finite extension with integer ring $\mathcal{O}$ as in Section 6.2.2. We review in this section the construction of signed p-adic L-functions. In [LLZ10; LLZ11; BL21], these elements are defined as the image of Kato's zeta elements under some local Coleman maps. We however outline an alternative construction that relies on decomposing the Mazur–Tate elements using a logarithmic matrix. While we only state our results for f in this section, all results hold for g analogously.

6.3.1 Non-integral p-adic L-functions as limits of polynomials

Let us first introduce a few tools and notations used in p-adic analysis.

Definition 6.3. Let $\mathcal{P}(X) = \sum_{i=0}^{\infty} c_i \cdot X^i \in \mathcal{O}[\![X]\!] \otimes_{\mathcal{O}} E$. We define its norm as $\|\mathcal{P}\| = \sup_i |c_i|_p$, where $|\cdot|_p$ is the p-adic norm, normalized by $|p|_p = p^{-1}$. Given a real number $\rho \leq 1$, we define $\|\mathcal{P}\|_\rho = \sup_{|z|_p<\rho} \big(|\mathcal{P}(z)|_p\big)$.

We set $\Phi_0 = X$ and for $n \geq 1$ we define $\Phi_n = \sum_{j=0}^{p-1}(1+X)^{jp^{n-1}}$ which is the p^n-th cyclotomic polynomial in $X+1$. If we write $\omega_n = (1+X)^{p^n} - 1$ then we have the elementary identity $\Phi_n = \omega_n/\omega_{n-1}$ for $n \geq 1$. We also recall the p-adic logarithm given by

$$\log_p(1+X) = X \prod_{n\geq 1} \frac{\Phi_n(1+X)}{p}.$$

Let γ be a topological generator of $G_\infty = \mathrm{Gal}(\mathbb{Q}^{\mathrm{cyc}}/\mathbb{Q})$ and we identify $\mathcal{O}[\![G_\infty]\!]$ with the power series ring $\mathcal{O}[\![X]\!]$ via $\gamma \mapsto 1+X$. In what follows, u is the image of γ under the cyclotomic character κ_{cyc}. For an integer $h \geq 1$, we write

$$\begin{aligned} \Phi_{n,h} &:= \prod_{j=0}^{h-1} \Phi_n\big(u^{-j}(1+X)-1\big), \\ \omega_{n,h} &:= \prod_{j=0}^{h-1} \omega_n\big(u^{-j}(1+X)-1\big), \text{ and} \\ \log_{p,h} &:= \prod_{j=0}^{h-1} \log_p\big(u^{-j}(1+X)-1\big). \end{aligned}$$

Given an element $\sum_{n\geq 0} c_n \cdot X^n \in E[\![X]\!]$, we say that it is $O(\log_p^r)$ if r is a real number such that

$$\sup_n \frac{|c_n|_p}{n^r} < \infty.$$

Similarly, we say that an element $\sum_{n\geq 0, \sigma\in\Delta} c_{n,\sigma}\sigma \cdot X^n \in E[\Delta][\![X]\!]$ is $O(\log_p^r)$ if

$$\sup \frac{|c_{n,\sigma}|_p}{n^r} < \infty$$

for all $\sigma \in \Delta$.

The following result comes from [BL21, Lemmas 2.2 and 2.3, combined with Remark 2.4], which is based on [PR94, proof of Lemme 1.2.2].

Lemma 6.7. *Let $h \geq 1$ be an integer. For $0 \leq j \leq h-1$, let $(\mathcal{Q}_{n,j})_{n\geq 0}$ be a sequence of polynomials in $E[X]$ such that $\|\mathcal{Q}_{n,j}\| \leq \infty$, and for $n \geq 0$*

$$\left\| p^{-j(n+1)} \sum_{t=0}^{j} (-1)^{j-t} \binom{j}{t} \mathcal{Q}_{n,t}\left(u^{-t}(1+X)-1\right) \right\| \leq d_n$$

for some constant $d_n \in \mathbb{R}$. Denote by $\mathcal{P}_n \in E[X]$ the unique polynomial of degree $< hp^n$ such that

$$\mathcal{P}_n \equiv \mathcal{Q}_{n,j}\big(u^{-j}(1+X)-1\big) \mod \omega_n\big(u^{-j}(1+X)-1\big)\cdot E[X]$$

for all $0 \leq j \leq h-1$ (the existence and uniqueness of $\mathcal{P}_n$ follows from the Chinese remainder theorem). Then we have $\|\mathcal{P}_n\| \leq c_h d_n$, where c_h is a constant that depends only on h. (If $h=1$, we may take $c_h = 1$.)

Suppose further that $\mathcal{Q}_{n+1,j} \equiv \mathcal{Q}_{n,j} \mod \omega_n \cdot E[X]$ for all $n \geq 0$. Then $\mathcal{P}_{n+1} \equiv \mathcal{P}_n \mod \omega_{n,h}\cdot E[X]$. If, in addition, there exists a real number r such that $0 \leq r < h$ and $d_n \leq p^{rn}$ for all $n \geq 0$, then the sequence $(\mathcal{P}_n)_{n\geq 0}$ converges to a power series $\mathcal{P}_\infty \in E[\![X]\!]$ that is $O(\log_p^r)$.

We can make the bound c_h more precise under Hypotheses 6.2 and 6.3 by carefully analyzing [PR94, proof of Lemme 1.2.2]. For $0 \leq \ell \leq h-1$, let

$$\mathscr{H}_\ell(X) = \sum_{i=0}^{\ell} (-1)^{\ell-i}\binom{\ell}{i}\mathcal{Q}_{n,i}\big(u^{-i}(1+X)-1\big).$$

One can verify that

$$H(X,Z) := \sum_{\ell=0}^{h-1}\binom{Z}{\ell}\mathscr{H}_\ell(X) \tag{6.11}$$

is a polynomial of degree $< h$ in Z with coefficients in the ring $\mathcal{O}[X]$, satisfying $H(X,j) = \mathcal{Q}_{n,j}\big(u^{-j}(1+X)-1\big)$ for $0 \le j \le h-1$. On the other hand, define

$$S(X,Y) := \sum_{j=0}^{h-1} \mathcal{Q}_{n,j}\big(u^{-j}(1+X)-1\big) \prod_{\substack{0\le i\le h-1\\ i\neq j}} \frac{u^{-ip^n}(1+Y)-1}{u^{(j-i)p^n}-1}.$$

It is the unique polynomial of degree $< h$ in Y with coefficients in $\mathcal{O}[X]$, satisfying $S\big(X, u^{jp^n}-1\big) = \mathcal{Q}_{n,j}\big(u^{-j}(1+X)-1\big)$. Furthermore, the polynomial $\mathcal{P}_n(X)$ in the statement of Lemma 6.7 is equal to $S\big(X,(1+X)^{p^n}-1\big)$.

In what follows, given a two-variable polynomial $\mathscr{R}(X,Y) \in \mathcal{O}[X,Y]$ and a real number r, we write

$$\|\mathscr{R}(X,Y)\|_r := \sup\big\{|\mathscr{R}(x,y)|_p : |x|_p \le 1, |y|_p \le r\big\}.$$

Let ρ be a real number satisfying the inequality $p^{-1} < \rho < p^{-1/(p-1)}$. One can calculate easily that $\frac{\rho}{|u^{p^n}-1|_p} = \rho p^{n+1} > 1$ for all $n \ge 0$. The homeomorphism between the following open disks

$$\begin{aligned} \big\{y : |y|_p < \rho\big\} &\longrightarrow \big\{z : |z|_p < \rho/|u^{p^n}-1|_p\big\} \\ y &\longmapsto \log(1+y)/\log u^{p^n} \\ u^{zp^n}-1 &\longleftarrow\!\shortmid\, z \end{aligned}$$

gives the following equality of sup-norms:

$$\|S(X,Y)\|_\rho = \|H(X,Z)\|_{\frac{\rho}{|u^{p^n}-1|_p}}. \tag{6.12}$$

This, combined with the expression in (6.11) yields

$$\|H(X,Z)\|_{\frac{\rho}{|u^{p^n}-1|_p}} = \sup_{0\le j\le h-1} \|\mathscr{H}_j(X)\| \left\|\binom{Z}{j}\right\|_{\frac{\rho}{|u^{p^n}-1|_p}}.$$

Assuming $p > h-1$, then one obtains

$$\left\|\binom{Z}{j}\right\|_{\frac{\rho}{|u^{p^n}-1|_p}} = \left\|\frac{Z(Z-1)\dots(Z-j+1)}{j!}\right\|_{\frac{\rho}{|u^{p^n}-1|_p}} = \left(\frac{\rho}{|u^{p^n}-1|_p}\right)^j$$

since $\rho/|u^{p^n}-1|_p > 1$.

Let us write $S(X,Y) = \sum_{\ell=0}^{h-1} S_\ell(X)\cdot Y^\ell$. If $\sup_j \|\mathscr{H}_j(X)\| \le p^{-(n+1)j}$, then Equation (6.12) becomes

$$\sup_{0\le \ell\le h-1} \big(\|S_\ell(X)\|\rho^\ell\big) = \sup_{0\le j\le h-1}\left(\|\mathscr{H}_j(X)\| \cdot \frac{\rho^j}{|u^{p^n}-1|_p^j}\right)$$

$$= \sup_{0 \le j \le h-1} \left(\| p^{-(n+1)j} \cdot \mathscr{H}_j(X) \| \cdot \rho^j \right) \le \sup_{0 \le j \le h-1} \rho^j = 1.$$

Therefore, $\|S_\ell(X)\| \le \rho^{-\ell}$ for each $0 \le \ell \le h-1$, which implies that

$$\|\mathcal{P}_n(X)\| = \left\| \sum_{\ell=0}^{h-1} S_\ell(X) \big((1+X)^{p^n} - 1 \big)^\ell \right\| \le \rho^{-(h-1)}.$$

Since ρ can be arbitrarily close to $p^{-1/(p-1)}$, then supposing $p > h$, one has

$$\|\mathcal{P}_n(X)\| \le p^{(h-1)/(p-1)} < p\,.$$

If in addition $\mathcal{P}_n(X)$ is defined over an unramified extension of $\mathbb{Q}_p$, then $\|\mathcal{P}_n(X)\|$ is an integral power of p. In particular, $\|\mathcal{P}_n(X)\| \le 1$. We can summarize the discussion above with the following refinement of Lemma 6.7.

Proposition 6.8. *Let h and $\mathcal{Q}_{n,j}$ be as in the statement of Lemma 6.7. Assume that $p > h$ and that $E/\mathbb{Q}_p$ is unramified. Then c_h can be taken to equal* 1. *In particular, if $d_n = 1$, we have $\mathcal{P}_n \in \mathcal{O}[X]$.*

Let us write $E' = E(\alpha_p(f)) = E(\beta_p(f))$, where $\alpha_p(f)$ and $\beta_p(f)$ are the roots of $X^2 - a_p(f)X + \eta_f(p)p^{k-1}$. We write $\mathcal{O}'$ for the ring of integers of E'. For simplicity, we shall write $\Upsilon := \Upsilon_f$, for $\Upsilon_f \in \{\alpha_p(f), \beta_p(f)\}$. Recall that there is a natural identification

$$E[G_n] = E[\![\gamma - 1]\!]/\big(\gamma^{p^n} - 1\big) = E[\![X]\!]/(\omega_n)\,. \tag{6.13}$$

The same is true if we replace E by E'. Note that G_n is a cyclic group of order p^n generated by the image of γ in G_n. For $a \in (\mathbb{Z}/p^n\mathbb{Z})^\times$ the element $\bar{\sigma}_a$ in $E[G_n]$ (or $E'[G_n]$) can be regarded as the polynomial $(1+X)^{m(a)}$, where $m(a)$ is the unique integer satisfying $0 \le m(a) \le p^n - 1$ and $\bar{\sigma}_a = \gamma^{m(a)} \bmod \gamma^{p^n}$.

We define $\mathcal{Q}_{n,j}\big(f, \omega^t\big) \in \mathcal{O}[X]$ as the image of the theta element $\theta_{n,j}\big(f, \omega^t\big)$ under this identification and define $\mathcal{Q}_{n,j}\big(f, \Upsilon, \omega^t\big) \in E'[X]$ similarly.

Lemma 6.9. *Suppose that $p > k-1$. For an integer j such that $0 \le j \le k-2$, we have the inequality*

$$\left\| \sum_{i=0}^{j} (-1)^{j-i} \binom{j}{i} \mathcal{Q}_{n,i}(f, \omega^t)(u^{-i}(1+X) - 1) \right\| \le p^{-(n+1)j}.$$

Proof. For an integer $i \in \{0, \ldots, k-2\}$, we have

$$\begin{aligned}
&\mathcal{Q}_{n,i}(f,\omega^t)(u^{-i}(1+X)-1)\\
&= \frac{\pi\sqrt{-1}}{\Omega_f^{(-1)^t}} \sum_a \Bigg(\int_{-a/p^{n+1}}^{\infty} f(z)(p^{n+1}z+a)^i \,\mathrm{d}z \\
&\qquad + (-1)^{t+k+i} \int_{a/p^{n+1}}^{\infty} f(z)(p^{n+1}z-a)^i \,\mathrm{d}z \Bigg) u^{-im(a)}\omega^{t-i}(a)(1+X)^{m(a)}\\
&= \frac{\pi\sqrt{-1}}{\Omega_f^{(-1)^t}} \sum_a \Bigg(\int_{-a/p^{n+1}}^{\infty} f(z) \left(\frac{p^{n+1}z+a}{\omega(a)u^{m(a)}}\right)^i \mathrm{d}z \\
&\qquad + (-1)^{t+k+i} \int_{a/p^{n+1}}^{\infty} f(z) \left(\frac{p^{n+1}z-a}{\omega(a)u^{m(a)}}\right)^i \mathrm{d}z \Bigg) \omega^t(a)(1+X)^{m(a)}.
\end{aligned}$$

Let $\hat{a} = \omega(a)\cdot u^{m(a)}$ so that $\hat{a} \equiv a \mod p^{n+1}\cdot\mathbb{Z}_p$. One therefore obtains

$$\begin{aligned}
&\sum_{i=0}^{j} (-1)^{j-i}\binom{j}{i} \mathcal{Q}_{n,i}(f,\omega^t)\big(u^{-i}(1+X)-1\big)\\
&= \frac{\pi\sqrt{-1}}{\Omega_f^{(-1)^t}} \sum_a \Bigg(\int_{-a/p^{n+1}}^{\infty} f(z) \left(\frac{p^{n+1}z+a}{\hat{a}}-1\right)^j \mathrm{d}z \\
&\qquad + (-1)^{t+k+j} \int_{a/p^{n+1}}^{\infty} f(z) \left(\frac{p^{n+1}z-a}{\hat{a}}+1\right)^j \mathrm{d}z \Bigg) \omega^t(a)(1+X)^{m(a)}\\
&= \frac{\pi\sqrt{-1}}{\Omega_f^{(-1)^t}} \sum_a \Bigg(\int_{-a/p^{n+1}}^{\infty} f(z)z^j \,\mathrm{d}z \\
&\qquad + (-1)^{t+k} \int_{a/p^{n+1}}^{\infty} f(z)(-z)^j \,\mathrm{d}z \Bigg) \frac{p^{(n+1)j}}{\hat{a}^j}\omega^t(a)(1+X)^{m(a)},
\end{aligned}$$

which belongs to $p^{(n+1)j}\cdot\mathcal{O}[X]$ since

$$\begin{aligned}
\xi_f^{(-1)^t}(\{\infty\}-\{a/p^{n+1}\}) = \pi\sqrt{-1}\Bigg(&\int_{-a/p^{n+1}}^{\infty} f(z)(zX+Y)^{k-2}\,\mathrm{d}z \\
&+ (-1)^{t+k} \int_{a/p^{n+1}}^{\infty} f(z)(-zX+Y)^{k-2}\,\mathrm{d}z \Bigg)
\end{aligned}$$

belongs to $\Omega_f^{(-1)^t}\cdot\mathcal{O}[X,Y]$. Hence,

$$\frac{\pi\sqrt{-1}}{\Omega_f^{(-1)^t}} \left(\int_{-a/p^{n+1}}^{\infty} f(z)z^j \mathrm{d}z + (-1)^{t+k} \int_{a/p^{n+1}}^{\infty} f(z)(-z)^j \,\mathrm{d}z \right) \in \mathcal{O},$$

from which the lemma follows. ■

Let $\mathcal{P}_n(f,\omega^t)$ be the unique polynomial of degree $< (k-1)p^n$ such that

$$\mathcal{P}_n(f,\omega^t) \equiv \mathcal{Q}_{n,j}(f,\omega^t)\big(u^{-j}(1+X)-1\big) \mod \omega_n\big(u^{-j}(1+X)-1\big)\cdot E[X]$$

for $0 \le j \le k-2$.

Corollary 6.10. *Suppose $p > k-1$ and that $E/\mathbb{Q}_p$ is unramified. For all integers $n \geq 0$ and $0 \leq t \leq p-2$, we have $\mathcal{P}_n(f, \omega^t) \in \mathcal{O}[X]$.*

Proof. This follows from applying Lemma 6.7 to $\mathcal{Q}_{n,j}(f, \omega^t)$ with $d_n = 1$ from Lemma 6.9. Finally, the assertion is a consequence of Proposition 6.8. ■

We now turn our attention to the p-stabilized version of these polynomials. For each $\Upsilon \in \{\alpha_p(f), \beta_p(f)\}$, define $\mathcal{P}_n(f, \Upsilon, \omega^t)$ as the unique polynomial of degree $< (k-1)p^n$ such that

$$\mathcal{P}_n(f, \Upsilon, \omega^t) \equiv \mathcal{Q}_{n,j}(f, \Upsilon, \omega^t)\big(u^{-j}(1+X)-1\big) \bmod \omega_n\big(u^{-j}(1+X)-1\big) \cdot E'[X]$$

for $0 \leq j \leq k-2$.

Lemma 6.11. *For all $n \geq 1$ and $j \in \{0, \dots, k-2\}$, we have*

$$\mathcal{Q}_{n,j}(f, \Upsilon, \omega^t) = \frac{1}{\Upsilon^{n+1}} \mathcal{Q}_{n,j}(f, \omega^t) - \frac{\eta_f(p) p^{k-2}}{\Upsilon^{n+2}} \Phi_n \cdot \mathcal{Q}_{n-1,j}(f, \omega^t).$$

Proof. This follows from the three-term relation satisfied by the Mazur–Tate elements stated at the end of Section 6.2.3. The map $\nu_{n/n-1}$ sends $(1+X)^i$ to $\sum_{s=0}^{p-1}(1+X)^{i+sp^{n-1}}$. Thus, it is equivalent to the multiplication by Φ_n. ■

Lemma 6.12. *There exists a polynomial $p^{k-2} \cdot \tilde{\Phi}_{n,k-1}$ such that for each $0 \leq j \leq k-2$, we have*

$$p^{k-2} \cdot \tilde{\Phi}_{n,k-1} \equiv p^{k-2} \cdot \Phi_n\big(u^{-j}(1+X)-1\big) \bmod \omega_n\big(u^{-j}(1+X)-1\big)$$

with $\|p^{k-2} \cdot \tilde{\Phi}_{n,k-1}\| \leq c_{k-1}$. If in addition one has $p \geq k-1$, then $p^{k-2} \cdot \tilde{\Phi}_{n,k-1} \in \mathbb{Z}_p[X]$.

Proof. Let $0 \leq j \leq k-2$. We have

$$\begin{aligned}
&p^{k-2} \cdot \sum_{i=0}^{j} (-1)^{j-i} \binom{j}{i} \Phi_n\big(u^{-i}(1+X)-1\big) \\
&= p^{k-2} \cdot \sum_{i=0}^{j} (-1)^{j-i} \binom{j}{i} \sum_{s=0}^{p-1} \big(u^{-i}(1+X)\big)^{sp^{n-1}} \\
&= p^{k-2} \cdot \sum_{s=0}^{p-1} \big(u^{-sp^{n-1}} - 1\big)^j (1+X)^{sp^{n-1}} \in p^{(n+1)j} \cdot \mathbb{Z}_p[X]
\end{aligned}$$

since $u \in 1 + p\mathbb{Z}_p$. Thus, the existence of $p^{k-2} \cdot \tilde{\Phi}_{n,k-1}$ follows from Lemma 6.7. The last assertion is a consequence of Proposition 6.8. ■

A p-stabilised version of Lemma 6.11 follows upon combining it with Lemma 6.12.

Lemma 6.13. *For all $n \geq 1$, $0 \leq t \leq p-2$ and $\Upsilon \in \{\alpha_p(f), \beta_p(f)\}$, we have*

$$\mathcal{P}_n(f, \Upsilon, \omega^t) \equiv \frac{1}{\Upsilon^{n+1}} \mathcal{P}_n(f, \omega^t) - \frac{\eta_f(p) p^{k-2}}{\Upsilon^{n+2}} \tilde{\Phi}_{n,k-1} \cdot \mathcal{P}_{n-1}(f, \omega^t) \mod \omega_{n,k-1} .$$

This congruence allows us to give a bound for $\|\mathcal{P}_n(f, \Upsilon, \omega^t)\|$.

Corollary 6.14. *Suppose $p > k-1$ and that $E/\mathbb{Q}_p$ is unramified. For all $n \geq 0$ and $0 \leq t \leq p-2$, we have*

$$\left\|\mathcal{P}_n(f, \Upsilon, \omega^t)\right\| \leq \max\left(\left| \frac{1}{\Upsilon^{n+1}} \right|_p , \left| \frac{p^{k-2}}{\Upsilon^{n+2}} \right|_p \right) .$$

Proof. The result follows from combining Corollary 6.10 with Lemmas 6.12 and 6.13. ■

In particular, this corollary implies that $\left\|p^{\mathrm{ord}_p(\Upsilon)\cdot n} \mathcal{P}_n(f, \Upsilon, \omega^t)\right\| = O(1)$. This combined with Lemma 6.9 allows us to then apply Lemma 6.7 after setting $h = k-1$, $r = \mathrm{ord}_p(\Upsilon)$ (which is $< k-1$ as f is non-ordinary at p), and $\mathcal{Q}_{n,j} = \mathcal{Q}_{n,j}(f, \Upsilon, \omega^t)$. To be precise, we can define $L_p(f, \Upsilon, \omega^t, X) \in E'[\![X]\!]$ as the limit of the polynomials $\mathcal{P}_n(f, \Upsilon, \omega^t)$. In other words,

$$L_p(f, \Upsilon, \omega^t, X) = \lim_{n \to \infty} \mathcal{P}_n(f, \Upsilon, \omega^t) , \tag{6.14}$$

which is $O\big(\log_p^{\mathrm{ord}_p(\Upsilon)} \big)$. Furthermore, it satisfies the congruence

$$L_p(f, \Upsilon, \omega^t, X) \equiv \mathcal{P}_n(f, \Upsilon, \omega^t) \mod \omega_{n,k-1} . \tag{6.15}$$

Let $L_p(f, \Upsilon, X) \in E'[\Delta][\![X]\!]$ denote the Amice transform of the E'-valued distribution constructed in [AV75; Viš76; MTT86]. The limit $L_p(f, \Upsilon, \omega^t, X)$ in (6.14) is the power series obtained from $L_p(f, \Upsilon, X)$ after applying ω^t to Δ.

6.3.2 Construction of the signed p-adic L-functions

We review the definition of the logarithmic matrix attached to f at p studied in [BL21]. We write $\mathcal{O}[\![\pi]\!]$ for the ring of power series in π, which is equipped with an $\mathcal{O}$-linear operator φ that sends π to $(1+\pi)^p - 1$ and an $\mathcal{O}$-linear action by $\mathcal{G}_\infty$ given by $\sigma \cdot \pi = (1+\pi)^{\chi_{\mathrm{cyc}}(\sigma)} - 1$. The Mellin transform that sends $a \in \mathcal{O}[\![\mathcal{G}_\infty]\!]$ to $a \cdot (1+\pi)$ induces an isomorphism

$$\mathscr{M} : \mathcal{O}[\![\mathcal{G}_\infty]\!] \xrightarrow{\sim} \mathcal{O}[\![\pi]\!]^{\psi=0},$$

where ψ is a left-inverse of the operator φ.

Definition 6.4. Let $q = \varphi(\pi)/\pi \in \mathcal{O}[\![\pi]\!]$ and let $\delta = p/\big(q - \pi^{p-1}\big) \in \mathcal{O}[\![\pi]\!]^\times$. We define the matrix

$$P_f := \begin{pmatrix} 0 & \frac{-1}{\eta_f(p)q^{k-1}} \\ \delta^{k-1} & \frac{a_p(f)}{\eta_f(p)q^{k-1}} \end{pmatrix}.$$

For $n \geq 1$, we define $C_{n,f}$ to be the 2×2 matrix of polynomials of degree $< (k-1)p^n$ that coincide with the image of

$$\mathscr{M}^{-1}\left((1+\pi)\varphi^n(P_f^{-1}) \cdots \varphi(P_f^{-1})\right) \bmod \omega_{n,k-1}\,.$$

Let us abbreviate $\alpha := \alpha_p(f)$ and $\beta := \beta_p(f)$ and define the matrices

$$A_f := \begin{pmatrix} 0 & \frac{-1}{\eta_f(p)p^{k-1}} \\ 1 & \frac{a_p(f)}{\eta_f(p)p^{k-1}} \end{pmatrix}, \quad Q_f := \begin{pmatrix} \alpha & -\beta \\ -\alpha\beta & \alpha\beta \end{pmatrix}.$$

The proposition below summarizes Lemmas 2.6–2.8 of [BL21] detailing the properties of the matrices defined above which we will need in this chapter.

Proposition 6.15. *The determinant of $C_{n,f}$ is equal, up to a unit of $\mathcal{O}[\![X]\!]^\times$, to $\omega_{n,k-1}/\Phi_{0,k-1}$. The sequence of matrices $A_f^{n+1}C_{n,f}$ converges to a matrix $M_{\log,f}$ defined over $E[\![X]\!]$. Furthermore, the entries in the first row of $Q_f^{-1}M_{\log,f}$ are elements of $E'[\![X]\!]$ that are $O\big(\log_p^{\mathrm{ord}_p(\alpha)}\big)$, while those in the second row are $O\big(\log^{\mathrm{ord}_p(\beta)}\big)$. We call $M_{\log,f}$ the "logarithmic matrix" attached to f.*

We now have all the tools we need to define the signed p-adic L-functions.

Theorem 6.16. *For each $0 \leq t \leq p-2$, there exist elments $L_p(f, \sharp, \omega^t, X)$ and $L_p(f, \flat, \omega^t, X)$ of $\mathcal{O}[\![X]\!] \otimes_{\mathcal{O}} E$ such that*

$$\frac{1}{\alpha - \beta} \cdot \begin{pmatrix} L_p(f,\alpha,\omega^t,X) \\ L_p(f,\beta,\omega^t,X) \end{pmatrix} = Q_f^{-1} M_{\log,f} \begin{pmatrix} L_p(f,\sharp,\omega^t,X) \\ L_p(f,\flat,\omega^t,X) \end{pmatrix}.$$

Furthermore, for all $n \geq 1$, we have

$$\begin{pmatrix} \mathcal{P}_n(f,\omega^t) \\ -\eta_f(p)p^{k-2}\tilde{\Phi}_{n,k-1}\mathcal{P}_{n-1}(f,\omega^t) \end{pmatrix} \equiv C_{n,f} \begin{pmatrix} L_p(f,\sharp,\omega^t,X) \\ L_p(f,\flat,\omega^t,X) \end{pmatrix} \bmod \omega_{n,k-1}\,.$$

Proof. Let $F_\Upsilon = \frac{1}{\alpha-\beta} \cdot L_p(f, \Upsilon, \omega^t, X)$. We can apply [BL21, Proposition 2.11] to the pair of elements (F_α, F_β) to deduce that there exist $L_p(f, \sharp, \omega^t, X)$, $L_p(f, \flat, \omega^t, X) \in \mathcal{O}\llbracket X \rrbracket \otimes_{\mathcal{O}} K'$ such that

$$\frac{1}{\alpha-\beta} \cdot \begin{pmatrix} L_p(f,\alpha,\omega^i,X) \\ L_p(f,\beta,\omega^i,X) \end{pmatrix} = Q_f^{-1} M_{\log,f} \begin{pmatrix} L_p(f,\sharp,\omega^i,X) \\ L_p(f,\flat,\omega^i,X) \end{pmatrix}.$$

A direct computation shows that

$$\frac{1}{\alpha-\beta} \cdot A_f^{-n-1} Q_f = \frac{1}{\alpha-\beta} \cdot Q_f \begin{pmatrix} \alpha^{n+1} & 0 \\ 0 & \beta^{n+1} \end{pmatrix} = \frac{1}{\alpha-\beta} \cdot \begin{pmatrix} \alpha^{n+2} & -\beta^{n+2} \\ -\alpha^{n+2}\beta & \alpha\beta^{n+2} \end{pmatrix}.$$

Therefore, combined with (6.15) and Lemma 6.13, we deduce that

$$\begin{aligned} &\frac{1}{\alpha-\beta} \cdot A_f^{-n-1} Q_f \begin{pmatrix} L_p(f,\alpha,\omega^t,X) \\ L_p(f,\beta,\omega^t,X) \end{pmatrix} \\ &\qquad \equiv \frac{1}{\alpha-\beta} \cdot \begin{pmatrix} \alpha^{n+2} & -\beta^{n+2} \\ -\alpha^{n+2}\beta & \alpha\beta^{n+2} \end{pmatrix} \begin{pmatrix} \alpha^{-n-1} & \alpha^{-n-2} \\ \beta^{-n-1} & \beta^{-n-2} \end{pmatrix} \\ &\qquad\quad \begin{pmatrix} \mathcal{P}_n(f,\omega^t) \\ -\eta_f(p) p^{k-2} \tilde{\Phi}_{n,k-1} \mathcal{P}_{n-1}(f,\omega^t) \end{pmatrix} \mod \omega_{n,k-1} \\ &\qquad \equiv \begin{pmatrix} \mathcal{P}_n(f,\omega^t) \\ -\eta_f(p) p^{k-2} \tilde{\Phi}_{n,k-1} \mathcal{P}_{n-1}(f,\omega^t) \end{pmatrix} \mod \omega_{n,k-1}. \end{aligned}$$

Thus, it follows from Proposition 6.15 that

$$\begin{aligned} \begin{pmatrix} \mathcal{P}_n(f,\omega^t) \\ -\eta(p) p^{k-2} \tilde{\Phi}_{n,k-1} \mathcal{P}_{n-1}(f,\omega^t) \end{pmatrix} &\equiv A_f^{-n-1} Q_f Q_f^{-1} M_{\log,f} \begin{pmatrix} L_p(f,\sharp,\omega^t,X) \\ L_p(f,\flat,\omega^t,X) \end{pmatrix} \\ &\equiv C_{n,f} \begin{pmatrix} L_p(f,\sharp,\omega^t,X) \\ L_p(f,\flat,\omega^t,X) \end{pmatrix} \mod \omega_{n,k-1}. \end{aligned}$$

Since the left-hand side consists of elements belonging to $\varpi^{-s} \cdot \mathcal{O}[X]$ for some integer s that is independent of n, we deduce that

$$\begin{pmatrix} \varpi^s L_p(f,\sharp,\omega^t,X) \\ \varpi^s L_p(f,\flat,\omega^t,X) \end{pmatrix} \in \varprojlim \mathcal{O}\llbracket X \rrbracket^{\oplus 2} \big/ \ker h_n = \mathcal{O}\llbracket X \rrbracket^{\oplus 2},$$

where $h_n : \mathcal{O}\llbracket X \rrbracket^{\oplus 2} \to \mathcal{O}\llbracket X \rrbracket^{\oplus 2} / \omega_{n,k-1}$ is the map given by the multiplication by $C_{n,f}$ and the last equality is due to [BL21, Lemma 2.9]. Thus, $L_p(f, \sharp, \omega^t, X)$ and $L_p(f, \flat, \omega^t, X)$ belong to $\mathcal{O}\llbracket X \rrbracket \otimes_{\mathcal{O}} E$ (rather than $\mathcal{O}\llbracket X \rrbracket \otimes_{\mathcal{O}} E'$). ■

We end this section with the following result which asserts the p-integrality of the elements $L_p(f, \natural, \omega^t, X)$ for each choice of sign $\natural \in \{\sharp, \flat\}$.

Theorem 6.17. *Suppose $p > k-1$ and that $E/\mathbb{Q}_p$ is unramified. Then for $0 \leq t \leq p-2$, the elements $L_p(f, \sharp, \omega^t, X)$ and $L_p(f, \flat, \omega^t, X)$ belong in $\mathcal{O}[\![X]\!]$.*

Proof. We consider two separate cases: $\|L_p(f, \sharp, \omega^t, X)\| \geq \|L_p(f, \flat, \omega^t, X)\|$ or $\|L_p(f, \sharp, \omega^t, X)\| \leq \|L_p(f, \flat, \omega^t, X)\|$.

We first consider the case where $\|L_p(f, \sharp, \omega^t, X)\| \geq \|L_p(f, \flat, \omega^t, X)\|$ and suppose for contradiction that $\|L_p(f, \sharp, \omega^t, X)\| > 1$. Let $\mathcal{P}_n(f, \natural, \omega^t)$ be the image of $L_p(f, \natural, \omega^t, X)$ modulo $\omega_{n,k-1}$. This implies that when n is sufficiently large, one has $\|L_p(f, \natural, \omega^t, X)\| = \|\mathcal{P}_n(f, \natural, \omega^t)\|$.

From the definition of P_f and [BL21, Lemma 2.6] it follows that

$$C_{n,f} \equiv \begin{pmatrix} 0 & * \\ *\Phi_{n,k-1} & 0 \end{pmatrix} \cdots \begin{pmatrix} 0 & * \\ *\Phi_{1,k-1} & 0 \end{pmatrix} \mod \varpi$$

where $*$ represents a unit in $\mathcal{O}[\![X]\!]$. Let us set $\Phi^+_{n,k-1} := \prod_{m \leq n, \text{even}} \Phi_{m,k-1}$ and $\Phi^-_{n,k-1} := \prod_{m \leq n, \text{odd}} \Phi_{m,k-1}$. If n is odd,

$$C_{n,f} \equiv \begin{pmatrix} 0 & *\Phi^+_{n,k-1} \\ *\Phi^-_{n,k-1} & 0 \end{pmatrix} \mod \varpi$$

whereas when n is even

$$C_{n,f} \equiv \begin{pmatrix} *\Phi^-_{n,k-1} & 0 \\ 0 & *\Phi^+_{n,k-1} \end{pmatrix} \mod \varpi\,.$$

When n is even and sufficiently large, we have

$$C_{n,f} \begin{pmatrix} \mathcal{P}_n(f, \sharp, \omega^t) \\ \mathcal{P}_n(f, \flat, \omega^t) \end{pmatrix} = \begin{pmatrix} (*\Phi^-_{n,k-1} + \varpi\Theta)\mathcal{P}_n(f, \sharp, \omega^t) + \varpi\Xi\mathcal{P}_n(f, \flat, \omega^t) \\ \Psi \end{pmatrix}$$

for some $\Theta, \Xi \in \mathcal{O}[X]$ and $\Psi \in K[X]$. Since $\Phi^-_{n,k-1}$ is monic, we have

$$\begin{aligned} \|\varpi\Xi\mathcal{P}_n(f, \flat, \omega^t)\| &< \|L_p(f, \flat, \omega^t, X)\| \leq \|L_p(f, \sharp, \omega^t, X)\| \\ &= \|(*\Phi^-_{n,k-1} + \varpi\Theta)\mathcal{P}_n(f, \sharp, \omega^t)\|\,. \end{aligned}$$

Therefore, the strong triangle inequality implies that

$$\begin{aligned} &\|(*\Phi^-_{n,k-1} + \varpi\Theta)\mathcal{P}_n(f, \sharp, \omega^t) + \varpi\Xi\mathcal{P}_n(f, \flat, \omega^t)\| \\ &\quad = \|(*\Phi^-_{n,k-1} + \varpi\Theta)\mathcal{P}_n(f, \sharp, \omega^t)\| = \|L_p(f, \sharp, \omega^t, X)\| > 1. \end{aligned}$$

Combining this with Theorem 6.16 allows us to deduce that $\|P_n(f, \omega^t)\| > 1$. This however contradicts the statement of Corollary 6.10.

On the other hand, if $\|L_p(f,\flat,\omega^t,X)\| \geq \|L_p(f,\sharp,\omega^t,X)\|$, then we instead suppose that $\|L_p(f,\flat,\omega^t,X)\| > 1$ and deduce a similar contradiction by considering an odd n that is sufficiently large. We therefore conclude that in both cases we have $\|L_p(f,\sharp,\omega^t,X)\|$, $\|L_p(f,\flat,\omega^t,X)\| \leq 1$, as required. ■

6.4 Congruence of signed p-adic L-functions

Let f and g be the same pair of ϖ^r-congruent normalized cuspidal eigenforms of equal weight k defined over a finite extension $E/\mathbb{Q}_p$ as in the introduction. In this section we establish an analogous version of Proposition 6.6 modulo p, i.e. the p-adic L-functions attached to f and g are themselves ϖ^r-congruent.

For a prime v, we define

$$\mathscr{L}_v(f) = L_v(f, v^{-1}\cdot\gamma_v) \in \mathcal{O}[\![\mathcal{G}_\infty]\!] = \mathcal{O}[\Delta][\![G_\infty]\!],$$

where $L_v(f,X) \in \mathcal{O}[X]$ is defined as in (6.8) and $\gamma_v \in \mathcal{G}_\infty$ is the element that sends $\zeta \in \mu_{p^\infty}$ to ζ^v. For $t \in \{0,\dots,p-2\}$, we define $\mathscr{L}_v(f,\omega^t) \in \mathcal{O}[\![X]\!]$ to be the image of $\mathscr{L}_v(f)$ after applying ω^t to Δ. Moreover, for an arbitrary finite set of primes Σ, let $\mathcal{Q}_{n,(\Sigma)}(f,\omega^t) \equiv \mathcal{Q}_n(f,\omega^t) \times \prod_{v\in\Sigma} \mathscr{L}_v(f,\omega^t) \bmod \omega_n$ be the Σ-imprimitive version of the polynomial $\mathcal{Q}_n$ introduced in Section 6.3.

Remark 11. Take $\Sigma = \Sigma_0$, the finite set of primes dividing $N_f N_g$. We recall that $\mathcal{Q}_n(f,j)$ is the image of $\theta_{n,j}(f,\omega^t)$ under the identification (6.13). We also recall our notation F and G for the respective Σ_0-depletions of f and g. In the proof of Proposition 6.6, we chose $u_f^\pm, u_g^\pm = 1$ so that $\Omega_f^\pm = \Omega_F^\pm$ and $\Omega_g^\pm = \Omega_G^\pm$. We can therefore infer that $\mathcal{Q}_{n,(\Sigma_0)}(f,\omega^t)$ and $\mathcal{Q}_n(F,\omega^t)$ interpolate the same L-values, so they are congruent to each other modulo ω_n. The uniqueness implied by the Chinese remainder theorem gives us $\mathcal{P}_{n,(\Sigma_0)}(f,\omega^t) = \mathcal{P}_n(F,\omega^t)$.

From this point onward, we take $\Sigma = \Sigma_0$ and suppose that Hypotheses 6.2 and 6.3 hold. Furthermore, since $E/\mathbb{Q}_p$ is unramified, we can take $\varpi = p$.

Proposition 6.18. *Let f and g be modular forms of weight $k \geq 2$ satisfying Hypotheses 6.1–6.3. Then for all $n \geq 1$ and $0 \leq t \leq p-2$, we have*

$$\mathcal{P}_{n,(\Sigma)}(f,\omega^t) \equiv \mathcal{P}_{n,(\Sigma)}(g,\omega^t) \mod p^r \cdot \omega_{n,k-1} \cdot \mathcal{O}[X].$$

Proof. We have shown in (6.10) that for each $n \geq 1$,

$$\sum_{a \in (\mathbb{Z}/p^{n+1}\mathbb{Z})} \varphi_F \left| \begin{pmatrix} 1 & -a \\ 0 & p^n \end{pmatrix} \right. \big(\{\infty\} - \{0\}\big)$$
$$\equiv \sum_{a \in (\mathbb{Z}/p^{n+1}\mathbb{Z})} \varphi_G \left| \begin{pmatrix} 1 & -a \\ 0 & p^n \end{pmatrix} \right. \big(\{\infty\} - \{0\}\big) \mod p^r .$$

If we set $\mathcal{Q}_{n,j} = \mathcal{Q}_{n,j}(F,\omega^t) - \mathcal{Q}_{n,j}(G,\omega^t)$ and replace the modular form f in the proof of Lemma 6.9 by $F - G$, we deduce that

$$\left\| \sum_{i=0}^{j} (-1)^{j-i} \binom{j}{i} \mathcal{Q}_{n,i}\big(u^{-i}(1+X) - 1\big) \right\| \leq p^{-(n+1)j-r} .$$

Therefore, we can apply Proposition 6.8 with $d_n = p^{-r}$ to conclude that $\|\mathcal{P}_n(F,\omega^t) - \mathcal{P}_n(G,\omega^t)\| \leq p^{-r}$. The result then follows from Remark 11. ■

We state two lemmas which we shall be needing shortly.

Lemma 6.19. *If f and g are a pair of p^r-congruent modular forms, then one obtains the congruence $\eta_f(p) \equiv \eta_g(p) \mod p^r$.*

Proof. The hypothesis that f and g are p^r-congruent implies that

$$a_{\ell^2}(f) \equiv a_\ell(f)^2 - \eta_f(\ell)\ell^{k-1} \equiv a_{\ell^2}(g) \equiv a_\ell(g)^2 - \eta_g(\ell)\ell^{k-1} \mod p^r$$

for all $\ell \nmid N_f N_g$. Therefore, for all $\ell \nmid pN_f N_g$, we have

$$\eta_f(\ell) \equiv \eta_g(\ell) \mod p^r.$$

By Dirichlet's theorem, there exists a prime number $\ell \neq p$ that is congruent to p mod $N_f N_g$. For any such ℓ, we have $\eta_f(\ell) = \eta_f(p)$ and $\eta_g(\ell) = \eta_g(p)$. Hence, the lemma follows. ■

Lemma 6.20. *If f and g are p^r-congruent modular forms, then for $n \geq 1$ we have an entry-wise congruence of the matrices $C_{n,f}$ and $C_{n,g}$ defined in Definition 6.4:*

$$C_{n,f} \equiv C_{n,g} \mod p^r \cdot \omega_{n,k-1} .$$

Proof. As $a_p(f) \equiv a_p(g), \eta_f(p) \equiv \eta_g(p) \mod p^r$, the lemma follows immediately from the definition of $C_{n,f}$ and $C_{n,g}$. ■

Next, we define for each $\natural \in \{\sharp, \flat\}$ the Σ-imprimitive signed p-adic L-function

$$L_{p,(\Sigma)}(f, \natural, \omega^t, X) := L_p(f, \natural, \omega^t, X) \times \prod_{v \in \Sigma} \mathscr{L}_v(f, \omega^t). \tag{6.16}$$

Theorem 6.21. *Let f and g be a pair of p^r-congruent modular forms of equal weight $k \geq 2$ satisfying Hypotheses 6.1–6.3. Then for $0 \leq t \leq p-2$, we have a pair of congruences*

$$\begin{aligned} &L_{p,(\Sigma)}(f, \sharp, \omega^t, X) \equiv L_{p,(\Sigma)}(g, \sharp, \omega^t, X) \mod p^r \cdot \mathcal{O}[\![X]\!] \text{ and} \\ &L_{p,(\Sigma)}(f, \flat, \omega^t, X) \equiv L_{p,(\Sigma)}(g, \flat, \omega^t, X) \mod p^r \cdot \mathcal{O}[\![X]\!]. \end{aligned}$$

Proof. The proof resembles that of Theorem 6.17. Let $P_{n,(\Sigma)}(*, \natural, \omega^t)$ be the image modulo $\omega_{n,k-1}$ of $L_{p,(\Sigma)}(f, \natural, \omega^t, X)$, for each $\natural \in \{\sharp, \flat\}$ and $* \in \{f, g\}$. Set

$$\mathbf{L}_\natural := L_{p,(\Sigma)}(f, \natural, \omega^t, X) - L_{p,(\Sigma)}(g, \natural, \omega^t, X).$$

Choose n sufficiently large so that for both choices of $\natural$

$$\|\mathbf{L}_\natural\| = \|\mathcal{P}_{n,(\Sigma)}(f, \natural, \omega^t) - \mathcal{P}_{n,(\Sigma)}(g, \natural, \omega^t)\|.$$

Suppose without loss of generality that $\|\mathbf{L}_\sharp\| \geq \|\mathbf{L}_\flat\|$ and assume $\|\mathbf{L}_\sharp\| > p^{-r}$.

For $n \gg 0$ even and $h \in \{f, g\}$, we can write as in the proof of Theorem 6.17:

$$C_{n,h} \begin{pmatrix} \mathcal{P}_{n,(\Sigma)}(h, \sharp, \omega^t) \\ \mathcal{P}_{n,(\Sigma)}(h, \flat, \omega^t) \end{pmatrix} = \begin{pmatrix} (U_h \Phi^-_{n,k-1} + p\Theta_h)\mathcal{P}_{n,(\Sigma)}(h, \sharp, \omega^t) + p\Xi_h \mathcal{P}_{n,(\Sigma)}(h, \flat, \omega^t) \\ \Psi_h \end{pmatrix}$$

where $U_h \in \mathcal{O}[\![X]\!]^\times$ and $\Theta_h, \Xi_h \in \mathcal{O}[X]$. By Lemma 6.20, $\star_f \equiv \star_g \mod p^r$ for $\star \in \{U, \Theta, \Xi\}$. Thus, the first row of

$$C_{n,f} \begin{pmatrix} \mathcal{P}_{n,(\Sigma)}(f, \sharp, \omega^t) \\ \mathcal{P}_{n,(\Sigma)}(f, \flat, \omega^t) \end{pmatrix} - C_{n,g} \begin{pmatrix} \mathcal{P}_{n,(\Sigma)}(g, \sharp, \omega^t) \\ \mathcal{P}_{n,(\Sigma)}(g, \flat, \omega^t) \end{pmatrix} \tag{6.17}$$

is equal to

$$\begin{aligned} &(U_g \Phi^-_{n,k-1} + p\Theta_g)\left(\mathcal{P}_{n,(\Sigma)}(f, \sharp, \omega^t) - \mathcal{P}_{n,(\Sigma)}(g, \sharp, \omega^t)\right) \\ &\quad + p\Xi_g\left(\mathcal{P}_{n,(\Sigma)}(f, \flat, \omega^t) - \mathcal{P}_{n,(\Sigma)}(g, \flat, \omega^t)\right) \\ &\quad + p^r\left(S\mathcal{P}_{n,(\Sigma)}(f, \sharp, \omega^t) - T\mathcal{P}_{n,(\Sigma)}(f, \flat, \omega^t)\right) \end{aligned} \tag{6.18}$$

for some polynomials $S, T \in \mathcal{O}[X]$. We deduce that

- $\|p^r(S\mathcal{P}_{n,(\Sigma)}(f,\sharp,\omega^t) - T\mathcal{P}_{n,(\Sigma)}(f,\flat,\omega^t))\| \leq p^{-r} < \|\mathbf{L}_\sharp\|$;
- $\|p\Xi_g(\mathcal{P}_{n,(\Sigma)}(f,\flat,\omega^t) - \mathcal{P}_{n,(\Sigma)}(g,\flat,\omega^t))\| < \|\mathbf{L}_\flat\| \leq \|\mathbf{L}_\sharp\|$, by assumption;
- $\|\mathbf{L}_\sharp\| = \|(U_g\Phi^-_{n,k-1} + p\Theta_g)(\mathcal{P}_{n,(\Sigma)}(f,\sharp,\omega^t) - \mathcal{P}_{n,(\Sigma)}(g,\sharp,\omega^t))\|$, since $\Phi^-_{n,k-1}$ is monic.

By the strong triangle inequality, the norm of the expression in (6.18) is equal to $\|\mathbf{L}_\sharp\|$ which is larger than p^{-r} by assumption. As a result of Theorem 6.16, one obtains $\|\mathbf{L}_\sharp\| = \|\mathcal{P}_{n,(\Sigma)}(f,\omega^t) - \mathcal{P}_{n,(\Sigma)}(g,\omega^t)\| > p^{-r}$ which contradicts Proposition 6.18. Therefore, we have $\|\mathbf{L}_\sharp\| \leq p^{-r}$.

If instead we have $\|\mathbf{L}_\sharp\| \leq \|\mathbf{L}_\flat\|$, we assume $\|\mathbf{L}_\flat\| > p^r$; we can repeat the proof above but with $n \gg 0$ odd to deduce a contradiction. ■

6.5 Iwasawa invariants of the signed p-adic L-functions

Let f and g be as in the previous section. We now prove Theorems 6.1–6.3 presented in the beginning of the chapter. Throughout, we assume that Hypotheses 6.2–6.5 hold, which implies that Hypothesis 6.1 holds for $\varpi = p$ and $r = 1$ so that $a_m(f) \equiv a_m(g) \mod p$ for all natural numbers $m \nmid N_f N_g$.

6.5.1 Analytic side

Firstly, we define the analytic Iwasawa invariants. Given a nonnegative integer $0 \leq t \leq p-2$ and a sign $\natural \in \{\sharp, \flat\}$ satisfying Hypothesis 6.4, we set

$$\begin{aligned} \mu(f,\omega^t)^\natural_{\mathrm{an}} &:= \mu\left(L_p(f,\natural,\omega^t,X)\right), \text{ and} \\ \lambda(f,\omega^t)^\natural_{\mathrm{an}} &:= \lambda(L_p(f,\natural,\omega^t,X)) \end{aligned}$$

where $\mu(\mathscr{G})$ and $\lambda(\mathscr{G})$ for a given power series $\mathscr{G}$ is as in Definition 3.2. Let us now prove Theorem 6.1.

Theorem 6.22. *Let f, g, the sign $\natural \in \{\sharp, \flat\}$ and the integer $0 \leq t \leq p-2$ be chosen so that Hypotheses 6.2–6.5 hold. Let Σ_0 be the finite set of primes dividing $N_f N_g$. Then $\mu(f, \omega^t)^{\natural}_{\mathrm{an}} = 0$ if and only if $\mu(g, \omega^t)^{\natural}_{\mathrm{an}} = 0$, whence*

$$\lambda(f, \omega^t)^{\natural}_{\mathrm{an}} = \lambda(g, \omega^t)^{\natural}_{\mathrm{an}} + \sum_{v \in \Sigma_0} \left(\mathbf{e}_v(g, \omega^t) - \mathbf{e}_v(f, \omega^t) \right),$$

where $\mathbf{e}_v(*, \omega^t)$ denotes the λ-invariant of $\mathscr{L}_v(*, \omega^t)$ for $* \in \{f, g\}$.

Proof. The fact that $\mathscr{L}_v(f, \omega^t)$ has trivial μ-invariant is due to Lemma 3.7.4 of [EPW06], so the first assertion of the theorem follows from Theorem 6.21 and (6.16). To prove the second assertion, we note that since $L_{p,(\Sigma_0)}(f, \natural, \omega^t, X)$ is an element of $\mathcal{O}[\![X]\!]$, we can apply the Weierstrass Preparation Theorem which permits us to write it as $p^{\mu(f)^{\natural}_{\mathrm{an}}} \cdot D(X) \cdot U(X)$, where $U(X) \in \mathcal{O}[\![X]\!]^{\times}$ and $D(X)$ represents a distinguished polynomial in $\mathcal{O}[X]$. The lambda-invariant $\lambda(f)^{\natural}_{\mathrm{an}}$ corresponds to the degree of $D(X)$, which is determined by the image of $L_p(f, \natural, \omega^t, X)$ modulo p. Computing the λ-invariant of $L_p(f, \natural, \omega^t, X)$ and of $L_p(g, \natural, \omega^t, X)$ using the identity (6.16), and then relating the two using Theorem 6.21 gives us

$$\lambda(f, \omega^t)^{\natural}_{\mathrm{an}} + \sum_{v \in \Sigma_0} \lambda(\mathscr{L}(f, \omega^t)) = \lambda(g, \omega^t)^{\natural}_{\mathrm{an}} + \sum_{v \in \Sigma_0} \lambda(\mathscr{L}(g, \omega^t)),$$

which concludes the proof. ■

6.5.2 Algebraic side

In order to prove Theorems 6.2 and 6.3, we need to recall the signed Bloch–Kato p-primary Selmer groups. As before, we narrate the discussion centred on the modular form f with the understanding that the same applies to g.

Let V_f be the Deligne p-adic representation attached to a modular form f constructed in [Del69] and let $T_f \subset V_f$ be a $G_{\mathbb{Q}_p}$-stable $\mathcal{O}$-lattice. We write $W_f = \mathrm{Hom}(T_f, E/\mathcal{O})(1)$. Take Σ to be any finite set of primes containing p and ∞, as well as the primes dividing N_f. Let K be a number field and write K_Σ for the maximal extension of K unramified outside of Σ. The *Bloch-Kato*

p-primary Selmer group of f over a number field K is given by the kernel of the local-global map

$$\begin{aligned}\mathrm{Sel}(K,W_f) &= \ker\left(H^1(K,W_f)\to\prod_{\mathfrak{v}}\frac{H^1(K_{\mathfrak{v}},W_f)}{H^1_f(K_{\mathfrak{v}},W_f)}\right)\\ &\cong \ker\left(H^1(K_\Sigma/K,W_f)\to\prod_{\mathfrak{v}\mid v\in\Sigma}\frac{H^1(K_{\mathfrak{v}},W_f)}{H^1_f(K_{\mathfrak{v}},W_f)}\right),\end{aligned}$$

where $H^1_f(K_{\mathfrak{v}},W_f)$ is as in (3.20). If $\mathcal{K}$ is an infinite algebraic extension of $\mathbb{Q}$, we define $\mathrm{Sel}(\mathcal{K},W_f)=\varinjlim_K \mathrm{Sel}(K,W_f)$, where K runs over finite extensions of $\mathbb{Q}$ contained inside $\mathcal{K}$ and the connecting maps are restrictions.

While the Pontryagin dual $\mathrm{Sel}(\mathbb{Q}^{\mathrm{cyc}},W_f)^\vee$ is known to be a finitely generated $\mathcal{O}[\![G_\infty]\!]$-module, it is not torsion when f is non-ordinary at p. For this reason we recall the definition of sharp/flat Selmer groups, whose duals are expected to be torsion over $\mathcal{O}[\![G_\infty]\!]$. These Selmer groups can be regarded as the algebraic counterparts of the signed p-adic L-functions we studied previously.

Before we proceed, we set up a few more notations. Let $\mathbb{Q}_\infty=\bigcup_n\mathbb{Q}(\mu_{p^n})$ be the field extension with Galois group $\mathcal{G}_\infty\cong\mathbb{Z}_p^\times$. For $\theta\in\mathrm{Hom}(\Delta,\mathbb{Z}_p^\times)$, we write $e_\theta=\frac{1}{p-1}\sum_{\tau\in\Delta}\theta(\tau)\cdot\tau^{-1}$ for the corresponding idempotent element of $\mathcal{O}[\Delta]$. Given an $\mathcal{O}[\![\mathcal{G}_\infty]\!]$-module M, we denote by $M^\theta=e_\theta\cdot M$ the θ-isotypic component of M.

Let $\mathfrak{p}$ denote the unique prime of $\mathbb{Q}_\infty$ lying above p. Then $\mathbb{Q}_{\infty,\mathfrak{p}}$ can be identified with $\mathbb{Q}_p(\mu_{p^\infty})$. We write

$$H^1_{\mathrm{Iw}}(\mathbb{Q}_p(\mu_{p^\infty}),T_f)=\varprojlim_n H^1(\mathbb{Q}_p(\mu_{p^n}),T_f)$$

where the connecting maps are corestrictions. As explained in [LLZ10, §3.2] and [BL21, §2.3], the logarithmic matrix $M_{\log,f}$ decomposes the Perrin-Riou map on $H^1_{\mathrm{Iw}}\big(\mathbb{Q}_p(\mu_{p^\infty}),T_f\big)$ into a pair of Coleman maps

$$\mathrm{Col}^\natural : H^1_{\mathrm{Iw}}\big(\mathbb{Q}_p(\mu_{p^\infty}),T_f\big)\longrightarrow\mathcal{O}[\![\mathcal{G}_\infty]\!],$$

where $\natural\in\{\sharp,\flat\}$. Moreover, [BL21, Theorem 2.14] says that for $0\le t\le p-2$, there exists an explicit product of linear polynomials $\xi_{t,\natural}\in\mathcal{O}[X]$ such that

$$e_{\omega^t}\cdot\mathrm{im}(\mathrm{Col}^\natural)\subset\xi_{t,\natural}\cdot\mathcal{O}[\![G_\infty]\!]$$

and the containment is of finite index. For an explicit description of $\xi_{t,\natural}$ (which only depends on k and p), we refer to [LLZ11, §5A] or [HL14, Appendix A].

Definition 6.5. We define the *sharp and flat Selmer groups* of f over $\mathbb{Q}_\infty$ as

$$\mathrm{Sel}^\natural(\mathbb{Q}_\infty, W_f) = \ker\left(\mathrm{Sel}(\mathbb{Q}_\infty, W_f) \to \frac{H^1(\mathbb{Q}_{\infty,\mathfrak{p}}, W_f)}{H^1_f(\mathbb{Q}_{\infty,\mathfrak{p}}, W_f)^\natural}\right),$$

where $H^1_f(\mathbb{Q}_{\infty,\mathfrak{p}}, W_f)^\natural$ is defined to be the orthogonal complement of $\ker(\mathrm{Col}^\natural)$ with respect to the local Tate pairing $H^1_{\mathrm{Iw}}(\mathbb{Q}_{\infty,\mathfrak{p}}, T_f) \times H^1(\mathbb{Q}_{\infty,\mathfrak{p}}, W_f) \to E/\mathcal{O}$ for $\natural \in \{\sharp, \flat\}$. We write $\mathcal{X}^\natural(f)$ to denote the Pontryagin dual of $\mathrm{Sel}^\natural(\mathbb{Q}_\infty, W_f)$. For an integer $0 \leq t \leq p-2$, we write $\mathcal{X}^\natural(f)^{[t]}$ for the ω^t-isotypic component of $\mathcal{X}^\natural(f)$, which is a finitely generated $\mathcal{O}[\![G_\infty]\!]$-module.

Remark 12. From the proof of Theorem 6.5 in [LLZ10], Hypothesis 6.4 implies that $\mathcal{X}^\natural(f)^{[t]}$ is torsion over $\Lambda := \mathcal{O}[\![G_\infty]\!]$ for each $\natural \in \{\sharp, \flat\}$. Note that this assumption holds for any choice of t and $\natural$ provided either $k > 2$ or $a_p(f) = 0$ (see Corollary 3.29 and Remark 3.30 in *op. cit.*).

As discussed in Section 3.4.2, Kato's main conjecture (c.f. Conjecture 3.8) without p-adic zeta function is equivalent to:

Conjecture 6.23. *For $\natural \in \{\sharp, \flat\}$ and $0 \leq t \leq p-2$, there is an equality of Λ-ideals*

$$\mathrm{char}_\Lambda\big(\mathcal{X}^\natural(f)^{[t]}\big) = \left(\frac{L_p(f, \natural, \omega^t, X)}{\xi_{t,\natural}}\right).$$

If Hypothesis 6.4 holds for $0 \leq t \leq p-2$ and $\natural \in \{\sharp, \flat\}$, we define the following algebraic Iwasawa invariants:

$$\mu(f, \omega^t)^\natural_{\mathrm{alg}} := \mu\left(\mathcal{X}^\natural(f)^{[t]}\right), \text{ and}$$
$$\lambda(f, \omega^t)^\natural_{\mathrm{alg}} := \lambda\left(\mathcal{X}^\natural(f)^{[t]}\right).$$

The following result gives us half of Conjecture 6.23 provided Hypothesis 6.4 is true.

Proposition 6.24. *If Hypothesis 6.4 holds and $\mu(f, \omega^t)^\natural_{\mathrm{alg}} = \mu(f, \omega^t)^\natural_{\mathrm{an}} = 0$, then one obtains the following containment*

$$\frac{L_p(f, \natural, \omega^t, X)}{\xi_{t,\natural}} \in \mathrm{char}_\Lambda\big(\mathcal{X}^\natural(f)^{[t]}\big).$$

Proof. The proof of Corollary 6.8 in [LLZ10] shows that there exists an integer $n^{\natural}$ such that

$$p^{n^{\natural}} \cdot \frac{L_p(f, \natural, \omega^t, X)}{\xi_{t,\natural}} \in \mathrm{char}_{\Lambda}\big(\mathcal{X}^{\natural}(f)^{[t]}\big).$$

This is applicable to our case since the p-adic L-function considered in *loc. cit.* corresponds to the one studied in the present chapter up to a constant factor given by the ratio of the respective periods utilized. The assumption on the vanishing of the μ-invariants allows us to remove the constant $p^{n^{\natural}}$. ■

Let Σ_0 be the finite set of primes dividing $N_f N_g$ as before. We consider the Σ_0-imprimitive version of the sharp and flat Selmer groups which is obtained by ignoring the local conditions at primes in Σ_0

$$\mathrm{Sel}^{\natural}_{(\Sigma_0)}(\mathbb{Q}_\infty, W_f) = \ker\left(H^1(\mathbb{Q}_{\Sigma_0}/\mathbb{Q}_\infty, W_f) \to \frac{H^1(\mathbb{Q}_{\infty,\mathfrak{p}}, W_f)}{H^1_f(\mathbb{Q}_{\infty,\mathfrak{p}}, W_f)^{\sharp/\flat}} \right), \ \natural \in \{\sharp, \flat\}$$

and its Pontryagin dual $\mathcal{X}^{\natural}_{(\Sigma_0)}$. As shown in [HL19, §5], we have the isomorphism

$$\mathrm{Sel}^{\natural}_{(\Sigma_0)}(\mathbb{Q}_\infty, W_f)/\,\mathrm{Sel}^{\natural}(\mathbb{Q}_\infty, W_f) \cong \prod_{\mathfrak{v}|v\in\Sigma_0} \frac{H^1(\mathbb{Q}_{\infty,\mathfrak{v}}, W_f)}{H^1_f(\mathbb{Q}_{\infty,\mathfrak{v}}, W_f)}. \tag{6.19}$$

In fact, $H^1_f(\mathbb{Q}_{\infty,\mathfrak{v}}, W_f) = 0$ whenever $\mathfrak{v} \nmid p$ since it is dual to the inverse limit

$$\varprojlim_n \frac{H^1(\mathbb{Q}_{\mathfrak{v}}(\mu_{p^n}), T_f)}{H^1_f(\mathbb{Q}_{\mathfrak{v}}(\mu_{p^n}), T_f)},$$

which is zero by [Lei11, Lemma 6.2]. Therefore, we have the equality

$$\mathrm{char}_{\Lambda}\Big(\mathcal{X}^{\natural}_{(\Sigma_0)}(f)^{[t]}\Big) = \mathrm{char}_{\Lambda}\big(\mathcal{X}^{\natural}(f)^{[t]}\big) \times \prod_{v\in\Sigma_0} \mathscr{E}_v(f, \omega^t) \tag{6.20}$$

for each $\natural \in \{\sharp, \flat\}$, where $\mathscr{E}_v(f, \omega^t)$ represents the characteristic ideal of the ω^t-isotypic component of $\prod_{\mathfrak{v}|v} H^1(\mathbb{Q}_{\infty,\mathfrak{v}}, W_f)^{\vee}$.

We now have all the necessary information to prove Theorem 6.2.

Theorem 6.25. *Let f, g, $t \in \{0, \dots, p-2\}$, $\natural \in \{\sharp, \flat\}$ and Σ_0 be as in Theorem 6.22. Then $\mathcal{X}^{\natural}(f)^{[t]}$ and $\mathcal{X}^{\natural}(g)^{[t]}$ are both Λ-torsion. In addition, we have that $\mu(f, \omega^t)^{\natural}_{\mathrm{alg}} = 0$ if and only if $\mu(g, \omega^t)^{\natural}_{\mathrm{alg}} = 0$, in which case*

$$\lambda(f, \omega^t)^{\natural}_{\mathrm{alg}} = \lambda(g, \omega^t)^{\natural}_{\mathrm{alg}} + \sum_{v\in\Sigma_0} \big(\mathbf{e}_v(g, \omega^t) - \mathbf{e}_v(f, \omega^t)\big),$$

where $\mathbf{e}_v(*, \omega^t)$ for $* \in \{f, g\}$ are defined as in the statement of Theorem 6.22.

Proof. Let us first remark that for $* \in \{f, g\}$, the element $\mathscr{L}_v(*, \omega^t)$ defined in Section 6.4 generates the characteristic ideal of $\mathscr{E}_v(*, \omega^t)$. Furthermore, the $\mathcal{O}_f$-rank of $\prod_{\mathfrak{v}|v} H^1(\mathbb{Q}_{\infty,\mathfrak{v}}, W_f)^\vee$ coincides with the degree of $\mathscr{E}_v(*, \omega^t)$. The preceding statements are derived from the proof of [GV00, Proposition 2.4] carried out on p.37–38 of *op. cit.*, after identifying the ω^t-isotypic component of $H^1(\mathbb{Q}_{\infty,\mathfrak{v}}, W_*)^\vee$ with $H^1(\mathbb{Q}_{\mathfrak{v}}^{\mathrm{cyc}}, W_*(\omega^{-t}))^\vee$. It follows that the μ-invariant of $\mathscr{E}_v(*, \omega^t)$ is zero, whereas its λ-invariant is equal to $\mathbf{e}_v(*, \omega^t)$.

Recall from Remark 12 that Hypothesis 6.4 guarantees that $\mathcal{X}^\natural(f)^{[t]}$ and $\mathcal{X}^\natural(g)^{[t]}$ are torsion over $\Lambda = \mathcal{O}[\![G_\infty]\!]$. The assertion on the μ-invariants follows from [HL19, Theorem 4.6].

By *loc. cit.*, the λ-invariants of the dual Σ_0-imprimitive Selmer groups $\mathcal{X}^{\sharp/\flat}_{(\Sigma_0)}(f)^{[t]}$ and $\mathcal{X}^{\sharp/\flat}_{(\Sigma_0)}(g)^{[t]}$ coincide. Combining this fact with (6.20), we obtain

$$\lambda(f, \omega^t)^\natural_{\mathrm{alg}} + \sum_{v \in \Sigma_0} \lambda(\mathscr{E}_v(f, \omega^t)) = \lambda(g, \omega^t)^\natural_{\mathrm{alg}} + \sum_{v \in \Sigma_0} \lambda(\mathscr{E}_v(g, \omega^t))\,.$$

N.B. It was assumed in *loc. cit.* that the eigenforms f and g have even weight $k \geq 2$. This condition is not used in the proof and carries over verbatim in cases where k is odd. ■

We conclude this chapter with the following application to the Iwasawa main conjecture.

Theorem 6.26. *Let f, g, $t \in \{0, \ldots, p-2\}$ and $\natural \in \{\sharp, \flat\}$ be chosen so that Hypotheses 6.2–6.5 hold. If $\mu(*, \omega^t)^\natural_{\mathrm{alg}} = \mu(*, \omega^t)^\natural_{\mathrm{an}} = 0$ for $* \in \{f, g\}$, then Conjecture 6.23 holds for $\mathcal{X}^\natural(f)^{[t]}$ if and only if it holds for $\mathcal{X}^\natural(g)^{[t]}$.*

Proof. Proposition 6.24 and Hypothesis 6.4 imply that

$$\lambda(*, \omega^t)^\natural_{\mathrm{an}} - \deg \xi_{t,\natural} \geq \lambda(*, \omega^t)^\natural_{\mathrm{alg}}\,.$$

Furthermore, Conjecture 6.23 holds if and only if the equality holds.

After combining Theorems 6.25 and 6.22, one obtains the equality

$$\lambda(f, \omega^t)^\natural_{\mathrm{an}} - \lambda(g, \omega^t)^\natural_{\mathrm{an}} = \lambda(f, \omega^t)^\natural_{\mathrm{alg}} - \lambda(g, \omega^t)^\natural_{\mathrm{alg}}$$

from which the desired equivalence between the main conjectures follows. ■

Chapter 7

Replenishing p-adic L-functions at bad primes

The principal goal of this chapter is to identify a finite set of primes S and a condition which we will call (S^{dep}) so that a given p-adic L-function can be written in terms of the S-depleted version and its dual. This decomposition shall be used in the next chapter to deduce one of the two divisibilities required to prove the Iwasawa Main Conjecture.

7.1 Summary of results

Fix an odd prime p and a totally real number field F in which p does not ramify. Let $\pi_F = \bigotimes'_{\mathfrak{v}} \pi_{F,\mathfrak{v}}$ denote a cuspidal automorphic representation over F whose complex L-function, $L(\pi_F, s)$, has the critical strip $\{k_*, \ldots, k^*\}$ which is non-empty. Provided the Galois representation attached to π_F satisfies an ordinarity condition at all places of F above p, the Iwasawa Main Conjecture for π_F predicts the equality (up to a unit factor over the Iwasawa algebra Λ):

$$\mathrm{char}_{\Lambda}\big(\,\mathrm{Sel}(F(\mu_{p^\infty}), \pi_F)^{\vee}\big) = L_p(\pi_F; X)\,.$$

In other words there should be a connection between the arithmetic of π_F over the $\mathbb{Z}_p$-extension, and the object $L_p(\pi_F; X)$ which interpolates the twists $L(\pi_F \otimes \chi, k_* + j)$ for integers $j \in \{0, \ldots, k^* - k_*\}$.

The properties of Euler systems (assuming they exist for π_F) typically allow us to obtain a weak divisibility of the left-hand side into the right-hand side:

$$\mathrm{char}_\Lambda\big(\,\mathrm{Sel}(F(\mu_{p^\infty}),\pi_F)^\vee\big)\ \text{ divides }\ \Xi_p(\pi_F)\cdot L_{p,(S)}(\pi_F;X)\,.$$

Unfortunately the S-depleted p-adic L-function $L_{p,(S)}(\pi_F;X)$ is missing various bad Euler factors for technical reasons, while the junk factor $\Xi_p(\pi_F)$ is an unwanted by-product of the construction of these elements as cup-products of Siegel units in K-theory. In the subsequent two chapters we demonstrate how one can replace $L_{p,(S)}$ with the full p-adic L-function L_p and remove the factor $\Xi_p(\pi_F)$, allowing for a sharper divisibility in the Iwasawa Main Conjecture.

The strategy that we shall adopt employs the four main steps listed below.

Step 1: Under a certain condition (S^{dep}), one shows at every branch ϕ that

$$L_p^{(\phi)}(\pi;X)\ =\ \frac{\Omega(\widetilde{\pi}_1)}{\Omega(\pi_1)}\times\Big(\mathcal{W}^\phi\cdot\mathcal{Y}_1^\phi\cdot\mathrm{Tw}_{k^*}\Big(L_{p,(S)}^{(\phi^{-1})}\big(\pi^*;\tfrac{-X}{X+1}\big)\Big)+\mathcal{Y}_2^\phi\cdot L_{p,(S)}^{(\phi)}\big(\pi;X\big)\Big)$$

where $L_{p,(S)}(-)$ is an S-depleted version of $L_p(-)$, the integral elements $\mathcal{W}$, $\mathcal{Y}_1$, $\mathcal{Y}_2$ lie in the Iwasawa algebra Λ, and $\frac{\Omega(\widetilde{\pi}_1)}{\Omega(\pi_1)}$ is a ratio of automorphic periods. Here $\widetilde{\pi}_1$ occurs as a minimal twist in a family of decompositions $\pi\cong\pi_1\otimes\pi_2$.

Step 2: Establish that the condition (S^{dep}) holds true for each particular π.

Step 3: Construct the zeta-elements $\underline{\mathbf{z}}_S^{(\phi)}(\pi)$ and $\underline{\mathbf{z}}_S^{\star\,(\phi^{-1})}(\pi^\star)$ which map to $L_{p,(S)}^{(\phi)}\big(\pi;X\big)$ and $\mathrm{Tw}_{k^*}\Big(L_{p,(S)}^{(\phi^{-1})}\big(\pi^*;\tfrac{-X}{X+1}\big)\Big)$ under Perrin-Riou's p-adic regulator, respectively.

Step 4: Define the *primitive* norm-compatible element associated to π by taking

$$\underline{\mathbf{z}}_{\mathrm{prim}}^{(\phi)}(\pi)\ :=\ \frac{\Omega(\widetilde{\pi}_1)}{\Omega(\pi_1)}\times\Big(\mathcal{W}^\phi\cdot\mathcal{Y}_1^\phi\cdot\underline{\mathbf{z}}_S^{\star\,(\phi^{-1})}(\pi^\star)+\mathcal{Y}_2^\phi\cdot\underline{\mathbf{z}}_S^{(\phi)}(\pi)\Big)$$

so it is readily true that $\underline{\mathbf{z}}_{\mathrm{prim}}^{(\phi)}(\pi)$ is sent to $L_p^{(\phi)}(\pi;X)$ via Perrin-Riou's map.

Using the Poitou-Tate sequence and the arithmetic properties of the Euler system, one can then show that the characteristic power series of $H^2_{\mathrm{Iw}}(F,\mathbb{T}_p(\pi))$

divides that of $\frac{\bigcap^r_\Lambda H^1_{\mathrm{Iw}}(F,\mathbb{T}_p(\pi))}{\Lambda\cdot\underline{\mathbf{z}}_{\mathrm{prim}}(\pi)}$, which is 'half' of the Main Conjecture (in the notation of [KS24, §3]). Steps 1 and 2 will be the focus of this chapter and the algebraic construction of primitive zeta-elements will be relegated to Chapter 8.

The main examples we will be considering are as follows: the automorphic representation $\pi(\mathbf{f})$ for GL_2 attached to a parallel weight Hilbert modular form $\mathbf{f}$ over F, the double product $\pi(\mathbf{f}_1)\times\pi(\mathbf{f}_2)$ of such a pair, the triple product $\pi(\mathbf{f}_1)\times\pi(\mathbf{f}_2)\times\pi(\mathbf{f}_3)$ of a balanced triple, and the symmetric square $\mathrm{Sym}^2(\mathbf{f})$ which corresponds to GL_3. As a consequence of this strategy and in no particular order, we can deduce that:

- If $F=\mathbb{Q}$ then the *primitive* Kato-Beilinson zeta-element (which maps to $L_p(\mathbf{f};X)$) has no poles in the cyclotomic direction, even if the image of the residual Galois representation $\overline{\rho}_{\mathbf{f}}: G_{\mathbb{Q}}\to\mathrm{GL}_2(\overline{\mathbb{F}}_p)$ does not contain a conjugate of $\mathrm{SL}_2(\mathbb{F}_p)$[1] (see also the recent work of C.-H. Kim [Kim23]);

- The *primitive* p-adic L-function attached to $\mathrm{Sym}^2(\mathbf{f})$ has no poles if the unitarisation of $\chi\cdot\det(\pi(\mathbf{f}))^{-1}$ is not imaginary quadratic, otherwise it has at most a simple pole—this holomorphy property was previously proven when the ground field $F=\mathbb{Q}$ via the work of Schmidt and others [DD97; Sch88];

- The *primitive* p-adic L-function attached to $\pi(\mathbf{f}_1)\times\pi(\mathbf{f}_2)$ is holomorphic and becomes p-integral after multiplying by the congruence number $\mathfrak{h}_{\mathbf{f}_1}$ (this was only previously known for the imprimitive Rankin-Selberg version by [Hid91]);

- If $F=\mathbb{Q}$ and under the same assumptions as in Kings, Loeffler and Zerbes in [KLZ17, Theorem 11.6.4], there is a divisibility of p-integral

---

[1] What one actually needs is for the image of $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}(\zeta_{p^\infty}))$ under $\overline{\rho}$ to contain (a conjugate of) $\mathrm{SL}_2(\mathbb{F}_p)$ in order to prove the result [Kat04, Theorem 12.4(2)], which is instrumental in demonstrating the integrality of the zeta-element. The condition that $\overline{\rho}$ contains a conjugate of $\mathrm{SL}_2(\mathbb{F}_p)$ is sufficient for all odd primes p except for the case $p=3$ (see [Kim23, Lemma 2.2]).

power series

$$\mathrm{char}_\Lambda\Big(\,\mathrm{Sel}\left(\mathbb{Q}(\mu_{p^\infty}),\pi(\mathbf{f}_1)\times\pi(\mathbf{f}_2)\right)^\vee\Big)^\phi \ \text{ divides } \ \mathfrak{h}_{\mathbf{f}_1}\cdot L_p\big(\pi(\mathbf{f}_1)\times\pi(\mathbf{f}_2);X\big)^\phi$$

at each branch ϕ (essentially the same result was shown in *op. cit.* but the Hida-Panchiskin imprimitive version appears on the right-hand side).

These results shall appear in the ensuing sections as Theorem 8.2, Corollary 7.8, Corollary 7.6, and as an easy to derive consequence of Theorem 8.3. The reader will have noticed that we have said nothing about the triple product $\pi(\mathbf{f}_1)\times\pi(\mathbf{f}_2)\times\pi(\mathbf{f}_3)$. In fact, we shall prove that the condition (S^{dep}) holds for such a representation but the construction of Hsieh and Yamana [HY24] only treats the situation where the levels of the three modular forms are square-free, hence there is no discrepancy between the imprimitive and primitive p-adic L-functions.

We have decided to work within the framework of motives as the conjectures of Coates, Perrin-Riou and Panchiskin [Coa91; CPR89; Pan94] from the 1990s are written in this manner. There are presumably other ways one can replenish the missing Euler factors in these examples whilst preserving either holomorphy or p-integrality, however the advantage of our approach is that it offers a unified method to tackle this problem. This chapter is divided into two parts. In Section 7.2 we state the condition (S^{dep}) precisely and explain how it is used to decompose the primitive p-adic L-function. We establish this condition in Section 7.3 for all the examples we have enumerated above.

7.2 Decomposition of the primitive p-adic L-function

Let us reiterate a few notations that will be used throughout this chapter.

(a) If F is a number field or a finite extension of $\mathbb{Q}_p$, then $\mathrm{Gal}\left(F(\mu_{p^\infty})/F\right)$ is a product of a finite group Δ and a free part $G_\infty\cong 1+p\mathbb{Z}_p$; we shall choose a topological generator γ of the latter group, so that $\gamma^{\mathbb{Z}_p}=G_\infty$.

(b) The Iwasawa algebra $\Lambda := \Lambda(G_\infty)$ is the completed group algebra

$$\Lambda(G_\infty) := \varprojlim_m \mathbb{Z}_p\big[G_\infty/(G_\infty)^{p^m}\big] \;\cong\; \mathbb{Z}_p[\![G_\infty]\!] \;\xrightarrow{\sim}\; \mathbb{Z}_p[\![X]\!]$$

where the right-most (non-canonical) isomorphism arises from sending $\gamma \mapsto X+1$.

(c) Using κ_{cyc} to denote the p-th cyclotomic character, for each integer j there is an isomorphism $\mathrm{Tw}_j : \Lambda(G_\infty) \xrightarrow{\sim} \Lambda(G_\infty)$ sending a generator $\gamma \mapsto \kappa_{\text{cyc}}^j(\gamma)\cdot\gamma$.

(d) If T is a $\mathrm{Gal}(\overline{F}/F)$-module over $\mathbb{Z}_p$ then $H^i\big(\overline{F}/F(\mu_{p^\infty}),T\big)$ for any integer $i \geq 0$ has a natural action of $\mathrm{Gal}\big(F(\mu_{p^\infty})/F\big)$, which extends linearly and continuously to an action of the full Iwasawa algebra $\mathbb{Z}_p[\Delta][\![G_\infty]\!] \cong \mathbb{Z}_p[\Delta][\![X]\!]$ on cohomology.

(e) If we fix a generator $(\zeta_{p^m})_m$ of the Tate module $\mathbb{Z}_p(1)$ where $\zeta_{p^m} \in \mu_{p^m}$ are primitive roots of unity satisfying $(\zeta_{p^{m+1}})^p = \zeta_{p^m}$, there exist twist isomorphisms

$$\begin{aligned}\mathrm{Tw}_j : H^i\big(\overline{F}/F(\mu_{p^\infty}),T\big) &\xrightarrow{\sim} H^i\big(\overline{F}/F(\mu_{p^\infty}),T(j)\big)\\ \sigma &\longmapsto \sigma \otimes (\zeta_{p^m})_m^{\otimes j}.\end{aligned}$$

(f) We write $^\bullet$ for the involution on $\Lambda(G_\infty)$ which sends $\gamma \mapsto \gamma^{-1}$, and if M is a Λ-module then $M^\bullet$ denotes the same module with its G_∞-action flipped by $^\bullet$.

Let M be a pure motive defined over a totally real number field F of degree $n = [F:\mathbb{Q}]$. Denote by $\Sigma^{\text{bad}}(M)$ the primes of $\mathcal{O}_F$ dividing the conductor of M excluding those that divide p. Let us take a subset $S \subset \Sigma^{\text{bad}}(M)$ and choose $\varepsilon_0 \in \mathrm{Sgn}_F$. Furthermore, we assume that Conjectures 3.4, 3.5, 3.6 hold for the motive M, i.e. the holomorphy and complex functional equation for the completed L-function $\Psi(M(\chi),s)$, Deligne's period conjecture, and the existence of $L_p^{(\varepsilon_0)}(M)$ each hold true. We refer the reader to Section 3.2 for a more in-depth discussion on motives.

Our plan for this section is to decompose the p-adic L-function $L_p^{(\varepsilon_0)}(M)$ in terms of its S-depleted versions $L_{p,(S)}^{(\varepsilon_0)}(M)$ and $L_{p,(S)}^{(-\varepsilon_0)}\big(M^\vee(1)\big)$. To begin, we construct an element in $\mathcal{O}_{T,p}[\![X]\!]^\times$ interpolating the factor w_p in the p-adic functional equation (3.11).

Lemma 7.1. *At every branch character $\phi : \Delta \to \overline{\mathbb{Q}}_p^\times$ there exists an invertible power series $\mathcal{W}^{(\varepsilon_0),\phi}(M) \in \mathcal{O}_{T,p}[\![X]\!]^\times$ such that for each finite order character $\chi = \phi \cdot \psi \in \mathcal{X}_p$,*

$$\mathcal{W}^{(\varepsilon_0),\phi}(M)\Big|_{X=\psi(\gamma)\kappa_{\rm cyc}(\gamma)^m-1} = w_p\big(M(\chi), m\big).$$

Proof. Observing that $\mathcal{O}_{T,p}[\![X]\!] \cong \mathcal{O}_{T,p}[\![\gamma-1]\!]$, let us define a power series

$$\mathcal{W}^{(\varepsilon_0),\phi}(M) := \frac{L_p^{(\varepsilon_0),\phi}(M)}{L_p^{(-\varepsilon_0),\phi^{-1}}(M^\vee(1))^\bullet} \in \operatorname{Frac}\Big(\mathcal{O}_{T,p}[\![\gamma-1]\!]\Big)$$

where $\bullet : \mathcal{O}_{T,p}[\![\gamma-1]\!] \xrightarrow{\sim} \mathcal{O}_{T,p}[\![\gamma-1]\!]$ extends the involution $\gamma \mapsto \gamma^{-1}$ linearly. In fact, $\psi\mathcal{N}x_p^m\big(L_p^{(-\varepsilon_0),\phi^{-1}}(M^\vee(1))^\bullet\big) = \psi^{-1}\mathcal{N}x_p^{1-m}\big(L_p^{(-\varepsilon_0),\phi^{-1}}(M^\vee)\big)$ which implies

$$\psi\mathcal{N}x_p^m\big(\mathcal{W}^{(\varepsilon_0),\phi}(M)\big) = w_p\big(M(\chi), m\big) \quad \text{with } \chi = \phi \cdot \psi,$$

at all but finitely many critical (m, χ) via the p-adic functional equation (3.11).

To establish that $\mathcal{W}^{(\varepsilon_0),\phi}(M) \in \mathcal{O}_{T,p}[\![\gamma-1]\!]^\times$, it is enough to show that the interpolated factor $w_p\big(M(\chi), m\big)$ is a p-adic unit at all but finitely many critical twists (m, χ). Following the same notation as that of Coates and Perrin-Riou for the ϵ-factor over $F = \mathbb{Q}$ [Coa91; CPR89], one obtains

$$\Psi_{(\infty,p)}^{(\varepsilon_0)}\big(M(\chi), m\big) = \prod_{\mathfrak{v}\nmid p,\infty} \epsilon_{\mathfrak{v}}\big(M(\chi), m; \varphi, \mathrm{d}x\big) \times \Psi_{(\infty,p)}^{(-\varepsilon_0)}\big(M^\vee(\chi^{-1}), 1-m\big)$$

(for example, see [Coa91, Equations (37) and (25)]). Dividing through by the period $\Omega(\varepsilon_0, M)$ yields

$$\frac{\Psi_{(\infty,p)}^{(\varepsilon_0)}\big(M(\chi), m\big)}{\Omega(\varepsilon_0, M)} = w_p^\dagger\big(M(\chi), m\big) \times \frac{\Psi_{(\infty,p)}^{(-\varepsilon_0)}\big(M^\vee(\chi^{-1}), 1-m\big)}{\Omega(-\varepsilon_0, M^\vee)}$$

with $w_p^\dagger\big(M(\chi), m\big) := \frac{\Omega(-\varepsilon_0, M^\vee)}{\Omega(\varepsilon_0, M)} \cdot \prod_{\mathfrak{v}\nmid p,\infty} \epsilon_{\mathfrak{v}}\big(M(\chi), m; \varphi, \mathrm{d}x\big)$ an algebraic number. It is also important to note that the period ratio is independent of p, while $w_p\big(M(\chi), m\big) = \iota_p\Big(w_p^\dagger\big(M(\chi), m\big)\Big)$.

To determine the behaviour of the local ϵ-factors under both types of twisting, let us recall from Tate that at a finite place $\mathfrak{v} \nmid p$ and with W unramified:

$$\epsilon_{\mathfrak{v}}\big(V \otimes W, m; \varphi, \mathrm{d}x\big) = \epsilon_{\mathfrak{v}}\big(V, m; \varphi, \mathrm{d}x\big)^{\dim W} \times \det W\big(\pi_{\mathfrak{v}}^{a(V)+\dim V \cdot n(\phi)}\big)$$

(c.f. [Tat79, (3.4.6)]). In particular, if W denotes the one-dimensional representation associated to χ then W is unramified at $\mathfrak{v}$ because $\operatorname{supp}\big\{\mathfrak{c}(\chi)\big\}$ consists of primes above p, in which case

$$\epsilon_{\mathfrak{v}}\big(V(\chi), m; \varphi, \mathrm{d}x\big) = \epsilon_{\mathfrak{v}}\big(V, m; \varphi, \mathrm{d}x\big) \times (\text{a root of unity})$$

so that $\Big|\prod_{\mathfrak{v}\nmid p,\infty} \epsilon_{\mathfrak{v}}\big(V(\chi), m; \varphi, \mathrm{d}x\big)\Big|_p = \Big|\prod_{\mathfrak{v}\nmid p,\infty} \epsilon_{\mathfrak{v}}\big(V, m; \varphi, \mathrm{d}x\big)\Big|_p$. Furthermore,

$$\epsilon_{\mathfrak{v}}\big(V, m; \varphi, \mathrm{d}x\big) \;=\; \epsilon_{\mathfrak{v}}\big(V; \varphi, \mathrm{d}x\big) \times \mathfrak{f}(V)^{-m} \cdot \delta(\varphi)^{-m\cdot\mathrm{dim}V}$$

using [Tat79, (3.4.5)], and therefore one obtains the equality of absolute values $\Big|\prod_{\mathfrak{v}\nmid p,\infty} \epsilon_{\mathfrak{v}}\big(V(\chi), m; \varphi, \mathrm{d}x\big)\Big|_p = \Big|\prod_{\mathfrak{v}\nmid p,\infty} \epsilon_{\mathfrak{v}}\big(V; \varphi, \mathrm{d}x\big)\Big|_p$. Applying this formula to the ℓ-adic realisations of the motive $M_{/F}$, one has

$$\Big|w_p^{\dagger}\big(M(\chi), m\big)\Big|_p = \left|\iota_\infty^{-1}\left(\frac{\Omega(-\varepsilon_0, M^\vee)}{\Omega(\varepsilon_0, M)}\right) \times \prod_{\mathfrak{v}\nmid p,\infty} \epsilon_{\mathfrak{v}}\big(M; \varphi, \mathrm{d}x\big)\right|_p = 1$$

where the final equality follows [Coa91, (25)]. Thus $w_p\big(M(\chi), m\big) \in \mathcal{O}_{\mathbb{C}_p}^\times$. ■

Definition 7.1. For each prime $\mathfrak{v} \in \Sigma^{\mathrm{bad}}(M)$ we write $\mathscr{L}_{\mathfrak{v}}^{\phi}(M) \in \mathcal{O}_{T,p}[\![\gamma - 1]\!]$ for the unique element interpolating

$$\psi\mathcal{N}x_p^m\big(\mathscr{L}_{\mathfrak{v}}^{\phi}(M)\big) = \iota_p\big(L_{\mathfrak{v}}(M, \phi\cdot\psi, m)\big)$$

at every finite order character $\psi : \mathrm{Gal}\big(F_{p,\infty}^{\mathrm{cyc}}/F\big)_0 \to \overline{\mathbb{Q}}_p^\times$ which is trivial on Δ and the Galois group $\mathrm{Gal}\big(F_{p,\infty}^{\mathrm{cyc}}/F\big)_0$ is as in the exact sequence (3.8).

We now state the hypothesis we need in order to write $L_p^{(\varepsilon_0),\phi}(M)$ as an $\mathcal{O}_{T,p}[\![\gamma - 1]\!]$-linear combination of its S-depleted counterparts.

Hypothesis 7.1. (S^{dep}) *At every branch ϕ and for a fixed subset $S \subset \Sigma^{\mathrm{bad}}(M)$, the pair of principal ideals generated by the product of the bad Euler factors*

$$\prod_{\mathfrak{v}\in S} \mathscr{L}_{\mathfrak{v}}^{\phi}(M; \gamma) \quad \textit{and} \quad \prod_{\mathfrak{v}\in S} \mathscr{L}_{\mathfrak{v}}^{\phi^{-1}}\big(M^\vee(1); \gamma^{-1}\big)$$

respectively, are relatively prime to each other inside the algebra $\mathcal{O}_{T,p}[\![\gamma - 1]\!]$.

Proposition 7.2. *If Hypothesis* (S^{dep}) *holds for the motive* M *then at each branch* ϕ, *there exist* $\mathcal{Y}_1^{\phi}(M), \mathcal{Y}_2^{\phi}(M) \in \mathcal{O}_{T,p}[\![\gamma-1]\!]$ *and a decomposition*

$$\text{\L}_p^{(\varepsilon_0),\phi}(M) = \mathcal{W}^{(\varepsilon_0),\phi}(M) \cdot \mathcal{Y}_1^{\phi}(M) \times L_{p,(S)}^{(-\varepsilon_0),\phi^{-1}}\big(M^{\vee}(1)\big)^{\bullet} + \mathcal{Y}_2^{\phi}(M) \times L_{p,(S)}^{(\varepsilon_0),\phi}(M).$$

Hence, if both $L_{p,(S)}^{(\varepsilon_0)}(M)$ *and* $L_{p,(S)}^{(-\varepsilon_0)}\big(M^{\vee}(1)\big)$ *are p-integral then so is* $L_p^{(\varepsilon_0)}(M)$.

Proof. Fix a branch $\phi : \Delta \to \overline{\mathbb{Q}}_p^{\times}$. Utilising Equation (3.11) and Lemma 7.1, we know that $L_p^{(\varepsilon_0),\phi}(M) = \mathcal{W}^{(\varepsilon_0),\phi}(M) \times L_p^{(-\varepsilon_0),\phi^{-1}}\big(M^{\vee}(1)\big)^{\bullet}$, which means

$$\prod_{\mathfrak{v}\in S} \mathscr{L}_{\mathfrak{v}}^{\phi^{-1}}\big(M^{\vee}(1);\gamma^{-1}\big) \times L_p^{(\varepsilon_0),\phi}(M) = \mathcal{W}^{(\varepsilon_0),\phi}(M) \times L_{p,(S)}^{(-\varepsilon_0),\phi^{-1}}\big(M^{\vee}(1)\big)^{\bullet} \quad (7.1)$$

in terms of the S-depleted p-adic L-function attached to the dual $M^{\vee}(1)$.

Under Hypothesis (S^{dep}), there exist elements $\mathcal{Y}_1^{\phi}(M), \mathcal{Y}_2^{\phi}(M) \in \mathcal{O}_{T,p}[\![\gamma-1]\!]$ satisfying the identity

$$1 = \mathcal{Y}_1^{\phi}(M) \cdot \prod_{\mathfrak{v}\in S} \mathscr{L}_{\mathfrak{v}}^{\phi^{-1}}\big(M^{\vee}(1);\gamma^{-1}\big) + \mathcal{Y}_2^{\phi}(M) \cdot \prod_{\mathfrak{v}\in S} \mathscr{L}_{\mathfrak{v}}^{\phi}(M;\gamma).$$

Multiplying both sides by $L_p^{(\varepsilon_0),\phi}(M)$, simplifying, and then using Equation (7.1) produces:

$$\begin{aligned} L_p^{(\varepsilon_0),\phi}(M) &= \mathcal{Y}_1^{\phi}(M) \cdot \prod_{\mathfrak{v}\in S} \mathscr{L}_{\mathfrak{v}}^{\phi^{-1}}(M^{\vee}(1);\gamma^{-1}) \times L_p^{(\varepsilon_0),\phi}(M) + \mathcal{Y}_2^{\phi}(M) \cdot L_{p,(S)}^{(\varepsilon_0),\phi}(M) \\ &= \mathcal{Y}_1^{\phi}(M) \cdot \mathcal{W}^{(\varepsilon_0),\phi}(M) \times L_{p,(S)}^{(-\varepsilon_0),\phi^{-1}}\big(M^{\vee}(1)\big)^{\bullet} + \mathcal{Y}_2^{\phi}(M) \cdot L_{p,(S)}^{(\varepsilon_0),\phi}(M) \end{aligned}$$

as required. The second statement follows directly from the first. ■

7.3 Verifying (S^{dep}) for tensor products

To get a decomposition for $L_p^{(\varepsilon_0),\phi}(M)$ that is unconditional (c.f. Proposition 7.2) one must verify that Hypothesis (S^{dep}) holds for each motive $M_{/F}$ being considered. As will become apparent later on, the elements $\mathscr{L}_{\mathfrak{v}}^{\phi}(M;\gamma)$ and $\mathscr{L}_{\mathfrak{v}}^{\phi^{-1}}\big(M^{\vee}(1);\gamma^{-1}\big)$ have trivial μ-invariant at all bad places $\mathfrak{v} \nmid p$, therefore it sufficient to show that the zeros of

$$\prod_{\mathfrak{v}\in S} \mathscr{L}_{\mathfrak{v}}^{\phi}(M;\gamma) \quad \text{and} \quad \prod_{\mathfrak{v}\in S} \mathscr{L}_{\mathfrak{v}}^{\phi^{-1}}\big(M^{\vee}(1);\gamma^{-1}\big)$$

yield disjoint subsets of the open disk $B_{\mathbb{C}_p}(0,1^{-}) = \big\{x \in \mathcal{O}_{\mathbb{C}_p} : \big|x\big|_p < 1\big\}$.

7.3.1 The Manin-Vishik p-adic L-function

Let $\mathbf{f}$ be a Hilbert modular newform in $\mathcal{S}_k(\mathfrak{c}, \eta)$ of parallel weight $k \geq 2$, conductor $\mathfrak{c}$, Nebentypus η, and L-function $L(\mathbf{f}, s) = \sum_{\mathfrak{m} \lhd \mathcal{O}_F} \mathrm{C}(\mathbf{f}, \mathfrak{m}) \cdot \mathcal{N}(\mathfrak{m})^{-s}$ with Euler product expansion given by (2.31). The following result [SW93, Theorem 3.3] describes the Fourier coefficient of $\mathbf{f}$ at the bad primes.

Theorem 7.3. *Suppose that $\mathfrak{v}$ is a prime ideal of $\mathcal{O}_F$ dividing the level $\mathfrak{c} = \mathfrak{c}(\mathbf{f})$.*

(i) If η is not defined modulo $\mathfrak{c}\mathfrak{v}^{-1}$ then we either have $\mathrm{C}(\mathbf{f}, \mathfrak{v}) = 0$ or $\big|\mathrm{C}(\mathbf{f}, \mathfrak{v})\big|_\infty = \mathcal{N}(\mathfrak{v})^{\frac{k-1}{2}}$. Moreover, $\mathrm{C}(\mathbf{f}, \mathfrak{v}) \neq 0$ whenever the inertial degree of $\mathfrak{v}$ equals 1, or when $\mathfrak{v} \mid \mathfrak{c}$.

(ii) If η is a character modulo $\mathfrak{c}\mathfrak{v}^{-1}$ then we either have $\mathrm{C}(\mathbf{f}, \mathfrak{v}) = 0$ given that $\mathfrak{v}^2 \mid \mathfrak{c}$, or instead $\big|\mathrm{C}(\mathbf{f}, \mathfrak{v})\big|_\infty^2 = \mathcal{N}(\mathfrak{v})^{k-2}$ in the situation where $\mathfrak{v}^2 \nmid \mathfrak{c}$.

The interpolation property of the p-adic L-function $L_p(\mathbf{f})$ attached to $\mathbf{f}$ is given by (3.12) along with a p-adic functional equation (3.13). We refer the reader to Section 3.3.1 for further detail.

Proposition 7.4. *Hypothesis (S^{dep}) holds for $M(\mathbf{f})$ and the set $S = \Sigma^{\mathrm{bad}}(\mathbf{f})$.*

Proof. If $\mathfrak{v} \in \Sigma^{\mathrm{bad}}(\mathbf{f})$ then $L_{\mathfrak{v}}(\mathbf{f}, s) = 1$, or else $L_{\mathfrak{v}}(\mathbf{f}, s) = 1 - \alpha_{\mathfrak{v}}(\mathbf{f})\mathcal{N}(\mathfrak{v})^{-s}$ where $\alpha_{\mathfrak{v}}(\mathbf{f}) = \mathcal{N}(\mathfrak{v})^{e_{\mathfrak{v}}} \cdot u_{\mathfrak{v}}$ with $\big|u_{\mathfrak{v}}\big|_\infty = 1$ and $e_{\mathfrak{v}} \in \big\{\frac{k-2}{2}, \frac{k-1}{2}\big\}$ by Theorem 7.3. Given a branch ϕ, the Euler factor $\big(1 - \chi(\mathfrak{v})\alpha_{\mathfrak{v}}(\mathbf{f})\mathcal{N}(\mathfrak{v})^{-r}\big)$ is interpolated by

$$\Big(1 - \phi(\mathfrak{v})\alpha_{\mathfrak{v}}(\mathbf{f}) \cdot (1+X)^{\frac{\log_p(\mathcal{N}(\mathfrak{v}))}{\log_p \kappa_{\mathrm{cyc}}(\gamma)}}\Big) \text{ evaluated at } X = \kappa_{\mathrm{cyc}}(\gamma)^{-r}\chi(\gamma) - 1.$$

Note that $\mathcal{N}(\mathfrak{v})$ coincides with the image of $\omega_F(\mathfrak{v}) \cdot \gamma^{\frac{\log_p(\mathcal{N}(\mathfrak{v}))}{\log_p \kappa_{\mathrm{cyc}}(\gamma)}} \in \mathbb{Z}_p[\![\gamma - 1]\!]$ under κ_{cyc} where ω_F is the restriction of the Teichmüller character to the field F; likewise, $\chi(\mathfrak{v}) = \phi(\mathfrak{v}) \cdot \psi(\mathfrak{v}) = \phi(\mathfrak{v}) \cdot \chi(\gamma)^{\frac{\log_p(\mathcal{N}(\mathfrak{v}))}{\log_p \kappa_{\mathrm{cyc}}(\gamma)}}$ in the same notation as Section 7.2. Therefore, any zeroes of $\mathscr{L}_{\mathfrak{v}}^{\phi}\big(M; \kappa_{\mathrm{cyc}}(\gamma)^{-r} \cdot \gamma\big)$ must have the form

$$\begin{aligned} \mathbf{x}_{\mathfrak{v}}^{\phi} &= \Big(\phi(\mathfrak{v}) \cdot \iota_p\big(\mathcal{N}(\mathfrak{v})^{e_{\mathfrak{v}}} \cdot u_{\mathfrak{v}}\big)\Big)^{-\frac{\log_p \kappa_{\mathrm{cyc}}(\gamma)}{\log_p(\mathcal{N}(\mathfrak{v}))}} \kappa_{\mathrm{cyc}}(\gamma)^{r} - 1 \\ &= \Big(\phi(\mathfrak{v}) \cdot \iota_p\big(\omega_F(\mathfrak{v})^{e_{\mathfrak{v}}} \cdot u_{\mathfrak{v}}\big)\Big)^{-\frac{\log_p \kappa_{\mathrm{cyc}}(\gamma)}{\log_p(\mathcal{N}(\mathfrak{v}))}} \kappa_{\mathrm{cyc}}(\gamma)^{r-e_{\mathfrak{v}}} - 1. \end{aligned} \tag{7.2}$$

If $\mathfrak{q} \in \Sigma^{\mathrm{bad}}(\mathbf{f})$ then $L_{\mathfrak{q}}(\mathbf{f}^{\rho}, s) = 1$, or else $L_{\mathfrak{q}}(\mathbf{f}^{\rho}, s) = 1 - \alpha_{\mathfrak{q}}(\mathbf{f}^{\rho})\mathcal{N}(\mathfrak{q})^{-s}$ where $\alpha_{\mathfrak{q}}(\mathbf{f}^{\rho}) = \overline{\alpha_{\mathfrak{q}}(\mathbf{f})} = \mathcal{N}(\mathfrak{q})^{e_{\mathfrak{q}}} \cdot \overline{u}_{\mathfrak{q}}$ with $\left|u_{\mathfrak{q}}\right|_{\infty} = 1$, $e_{\mathfrak{q}} \in \left\{\frac{k-2}{2}, \frac{k-1}{2}\right\}$ by Theorem 7.3. At any branch ϕ^{-1}, the dual factor $\left(1 - \overline{\chi}(\mathfrak{q})\overline{\alpha_{\mathfrak{q}}(\mathbf{f})}\mathcal{N}(\mathfrak{q})^{r-k}\right)$ is interpolated by $\left(1 - \overline{\phi}(\mathfrak{q})\overline{\alpha_{\mathfrak{q}}(\mathbf{f})}\mathcal{N}(\mathfrak{q})^{-k} \cdot (1+X)^{-\frac{\log_p(\mathcal{N}(\mathfrak{q}))}{\log_p \kappa_{\mathrm{cyc}}(\gamma)}}\right)$ evaluated at $X = \kappa_{\mathrm{cyc}}(\gamma)^{-r}\chi(\gamma) - 1$.

It follows that any zeroes of $\mathscr{L}_{\mathfrak{q}}^{\phi^{-1}}\left(M^{\vee}(1); \kappa_{\mathrm{cyc}}(\gamma)^{r} \cdot \gamma^{-1}\right)$ must be of the form

$$\begin{aligned} \mathbf{x}_{\mathfrak{q}}^{\phi^{-1}} &= \Big(\overline{\phi}(\mathfrak{q}) \cdot \iota_p\big(\mathcal{N}(\mathfrak{q})^{e_{\mathfrak{q}}-k} \cdot \overline{u_{\mathfrak{q}}}\big)\Big)^{\frac{\log_p \kappa_{\mathrm{cyc}}(\gamma)}{\log_p(\mathcal{N}(\mathfrak{q}))}} \kappa_{\mathrm{cyc}}(\gamma)^{r} - 1 \\ &= \Big(\overline{\phi}(\mathfrak{q}) \cdot \iota_p\big(\omega_F(\mathfrak{q})^{e_{\mathfrak{q}}-k} \cdot \overline{u_{\mathfrak{q}}}\big)\Big)^{\frac{\log_p \kappa_{\mathrm{cyc}}(\gamma)}{\log_p(\mathcal{N}(\mathfrak{q}))}} \kappa_{\mathrm{cyc}}(\gamma)^{r+e_{\mathfrak{q}}-k} - 1. \end{aligned} \tag{7.3}$$

In the expressions above, the interior terms of $(-)^{-\frac{\log_p \kappa_{\mathrm{cyc}}(\gamma)}{\log_p(\mathcal{N}(\mathfrak{v}))}}$ and $(-)^{\frac{\log_p \kappa_{\mathrm{cyc}}(\gamma)}{\log_p(\mathcal{N}(\mathfrak{q}))}}$ belong to $\iota_p(\overline{\mathbb{Q}})$ and both have complex absolute value equal to one under the map $\iota_{\infty} \circ \iota_p^{-1}$. We shall next carefully pick the topological generator $\gamma \in \mathrm{Gal}(F_{p,\infty}^{\mathrm{cyc}}/F)_0$ so that $\kappa_{\mathrm{cyc}}(\gamma)$ lies in $\iota_p(\overline{\mathbb{Q}}) \cap (1 + p\mathbb{Z}_p)$ and that the complex number $\iota_{\infty} \circ \iota_p^{-1}\big(\kappa_{\mathrm{cyc}}(\gamma)\big)$ does *not* have absolute value one, e.g. choosing γ with $\kappa_{\mathrm{cyc}}(\gamma) = 1 + p$ works fine. As a direct consequence, one immediately finds $\left|\iota_{\infty} \circ \iota_p^{-1}\big(\kappa_{\mathrm{cyc}}(\gamma)\big)\right|_{\infty} = 1 + p \neq 1$.

Suppose that $\mathbf{x}_{\mathfrak{v}}^{\phi} = \mathbf{x}_{\mathfrak{q}}^{\phi^{-1}}$ *for some* $\mathfrak{v}, \mathfrak{q} \in \Sigma^{\mathrm{bad}}(\mathbf{f})$. Together, (7.2) and (7.3) imply that

$$\kappa_{\mathrm{cyc}}(\gamma)^{r-e_{\mathfrak{v}}} = \kappa_{\mathrm{cyc}}(\gamma)^{r+e_{\mathfrak{q}}-k}, \quad \text{i.e.} \quad e_{\mathfrak{v}} + e_{\mathfrak{q}} = k.$$

The latter equality is impossible as $e_{\mathfrak{v}}, e_{\mathfrak{q}} \in \left\{\frac{k-2}{2}, \frac{k-1}{2}\right\}$ and *hence* $\mathbf{x}_{\mathfrak{v}}^{\phi} \neq \mathbf{x}_{\mathfrak{q}}^{\phi^{-1}}$. We conclude that the zeroes of $\mathscr{L}_{\mathfrak{v}}^{\phi}(M; \gamma)$ and $\mathscr{L}_{\mathfrak{q}}^{\phi^{-1}}\left(M^{\vee}(1); \gamma^{-1}\right)$ are disjoint. Finally, the μ-invariants of both these products of bad Euler factors equal zero hence their principal ideals are relatively prime, and Hypothesis(S^{dep}) holds. ■

7.3.2 Hida's double product p-adic L-function

Let $\mathbf{f}_1 \in \mathcal{S}_{k_1}(\mathfrak{c}_1, \eta_1)$ and $\mathbf{f}_2 \in \mathcal{S}_{k_2}(\mathfrak{c}_2, \eta_2)$ be a pair of Hilbert modular newforms satisfying $k_2 < k_1$. To the double product $\mathbf{f}_1 \times \mathbf{f}_2$ one can associate the motive $M(\mathbf{f}_1 \otimes \mathbf{f}_2)$ equivalent to the tensor product of motives $M(\mathbf{f}_1) \otimes M(\mathbf{f}_2)$ which has rank four and weight $w = k_1 + k_2 - 2$.

Let us recall from Section 2.4.2 that the *imprimitive L-function* of $\mathbf{f}_1 \times \mathbf{f}_2$ is defined as

$$D(\mathbf{f}_1, \mathbf{f}_2, s) = \zeta_{(\mathfrak{c}_1\mathfrak{c}_2)}(\tilde{\eta}_1\tilde{\eta}_2, 2s{+}2{-}k_1{-}k_2) \times \sum_{\mathfrak{m}} \mathrm{C}(\mathbf{f}_1, \mathfrak{m})\mathrm{C}(\mathbf{f}_2, \mathfrak{m}) \cdot \mathcal{N}(\mathfrak{m})^{-s} \quad (7.4)$$

with an infinite product expansion

$$\prod_{\mathfrak{v}} D_{\mathfrak{v}}(\mathbf{f}_1, \mathbf{f}_2, s) = \prod_{\mathfrak{v}} \Big(\big(1 - \alpha_{\mathfrak{v}}(\mathbf{f}_1)\alpha_{\mathfrak{v}}(\mathbf{f}_2)\mathcal{N}(\mathfrak{v})^{-s}\big)\big(1 - \beta_{\mathfrak{v}}(\mathbf{f}_1)\beta_{\mathfrak{v}}(\mathbf{f}_2)\mathcal{N}(\mathfrak{v})^{-s}\big)$$
$$\big(1 - \alpha_{\mathfrak{v}}(\mathbf{f}_1)\beta_{\mathfrak{v}}(\mathbf{f}_2)\mathcal{N}(\mathfrak{v})^{-s}\big)\big(1 - \beta_{\mathfrak{v}}(\mathbf{f}_1)\alpha_{\mathfrak{v}}(\mathbf{f}_2)\mathcal{N}(\mathfrak{v})^{-s}\big)\Big)^{-1}$$

where $\zeta_{(\mathfrak{c}_1\mathfrak{c}_2)}(\tilde{\eta}_1\tilde{\eta}_2, s) = \sum_{\mathfrak{m}} \tilde{\eta}_1\tilde{\eta}_2(\mathfrak{m}) \cdot \mathcal{N}(\mathfrak{m})^{-s}$ over integral ideals $\mathfrak{m}$ prime to $\mathfrak{c}_1\mathfrak{c}_2$. Here, $\alpha_{\mathfrak{v}}(\mathbf{f}_i)$ and $\beta_{\mathfrak{v}}(\mathbf{f}_i)$ represent the roots of $X^2 - \mathrm{C}(\mathbf{f}, \mathfrak{v})X + \tilde{\eta}(\mathfrak{v})\mathcal{N}(\mathfrak{v})^{k_i - 1}$ so that for places $\mathfrak{v} \mid \mathfrak{c}$, we assign $\alpha_{\mathfrak{v}} = \mathrm{C}(\mathbf{f}_i, \mathfrak{v})$ and $\beta_{\mathfrak{v}} = 0$ for $i = 1, 2$. Meanwhile, the *primitive L-function* of $\mathbf{f}_1 \times \mathbf{f}_2$ is given by the infinite product

$$L(\mathbf{f}_1 \otimes \mathbf{f}_2, s) = \prod_{\mathfrak{v}} \det\left(1 - \mathrm{Frob}_{\mathfrak{v}}^{-1} \Big| \big(V_\ell(\mathbf{f}_1) \otimes V_\ell(\mathbf{f}_2)\big)^{I_{\mathfrak{v}}} \cdot \mathcal{N}(\mathfrak{v})^{-s}\right)^{-1}, \ \ \ell \neq \mathfrak{v}. \quad (7.5)$$

where for $i \in \{1, 2\}$, $V_\ell(\mathbf{f}_i)$ is the Deligne-Taylor representation attached to $\mathbf{f}_i$, $\mathrm{Frob}_{\mathfrak{v}}$ denotes the arithmetic Frobenius element at $\mathfrak{v}$ and $I_{\mathfrak{v}} \subset G_{F_{\mathfrak{v}}}$ is the inertia subgroup. We denote the local factor of $L(\mathbf{f}_1 \otimes \mathbf{f}_2, s)$ at $\mathfrak{v}$ by $L_{\mathfrak{v}}(\mathbf{f}_1 \otimes \mathbf{f}_2, s)$.

Let $\pi_F(\mathbf{f}_1) = \bigotimes_{\mathfrak{v}}' \pi_{F_{\mathfrak{v}}}(\mathbf{f}_1)$ and $\pi_F(\mathbf{f}_2) = \bigotimes_{\mathfrak{v}}' \pi_{F_{\mathfrak{v}}}(\mathbf{f}_2)$ be the irreducible cuspidal automorphic representations of $G = \mathrm{GL}_2(F)$ associated to $\mathbf{f}_1$ and $\mathbf{f}_2$, respectively. One also writes $\mathfrak{c}(\pi_F(\mathbf{f}_1)) \lhd \mathcal{O}_F$ and $\mathfrak{c}(\pi_F(\mathbf{f}_2)) \lhd \mathcal{O}_F$ to indicate their F-conductors. The work of Langlands, Jacquet and Gelbart in [GJ78; JL70] yields a description of the double product Euler factors occurring at primes $\mathfrak{v} \in \Sigma^{\mathrm{bad}}(\mathbf{f})$, as we now outline:

(I)' $\pi_{F_{\mathfrak{v}}}(\mathbf{f}_1) = \pi\big(\mu_1, \nu_1\big)$ is an unramified principal series, while $\pi_{F_{\mathfrak{v}}}(\mathbf{f}_2)$ is arbitrary but $\mathfrak{v} \mid \mathfrak{c}(\pi_F(\mathbf{f}_2))$. Here $\big(V_\ell(\mathbf{f}_1) \otimes V_\ell(\mathbf{f}_2)\big)^{I_{F_{\mathfrak{v}}}} = V_\ell(\mathbf{f}_1) \otimes \big(V_\ell(\mathbf{f}_2)\big)^{I_{F_{\mathfrak{v}}}}$ so that $L_{\mathfrak{v}}\big(\mathbf{f}_1 \otimes \mathbf{f}_2, s\big)$ and $D_{\mathfrak{v}}\big(\mathbf{f}_1, \mathbf{f}_2, s\big)$ both equal the (degree 0 or 2) factor

$$\big(1 - \alpha_{\mathfrak{v}}(\mathbf{f}_1)\alpha_{\mathfrak{v}}(\mathbf{f}_2)\mathcal{N}(\mathfrak{v})^{-s}\big)\big(1 - \beta_{\mathfrak{v}}(\mathbf{f}_1)\alpha_{\mathfrak{v}}(\mathbf{f}_2)\mathcal{N}(\mathfrak{v})^{-s}\big).$$

(II)' For some ramified character $\varepsilon_{\mathfrak{v}}$ one has that $\pi_{F_{\mathfrak{v}}}(\mathbf{f}_1)\otimes\varepsilon_{\mathfrak{v}}=\pi\big(\mu_1\varepsilon_{\mathfrak{v}},\nu_1\varepsilon_{\mathfrak{v}}\big)$ is an unramified principal series, and $\pi_{F_{\mathfrak{v}}}(\mathbf{f}_2)$ satisfies $\mathfrak{v}\mid\mathfrak{c}\big(\pi_F(\mathbf{f}_2)\otimes\varepsilon_{\mathfrak{v}}^{-1}\big)$. Because this is just an $\varepsilon_{\mathfrak{v}}$-twist of Case (I)', one may readily deduce that

$$\begin{aligned}L_{\mathfrak{v}}\big(\mathbf{f}_1\otimes\mathbf{f}_2,s\big)&=L_{\mathfrak{v}}\big((\mathbf{f}_1\otimes\varepsilon_{\mathfrak{v}})\otimes(\mathbf{f}_2\otimes\varepsilon_{\mathfrak{v}}^{-1}),s\big)\\&=\big(1-\alpha_{\mathfrak{v}}(\mathbf{f}_1\otimes\varepsilon_{\mathfrak{v}})\alpha_{\mathfrak{v}}(\mathbf{f}_2\otimes\varepsilon_{\mathfrak{v}}^{-1})\cdot\mathcal{N}(\mathfrak{v})^{-s}\big)\\&\quad\times\big(1-\beta_{\mathfrak{v}}(\mathbf{f}_1\otimes\varepsilon_{\mathfrak{v}})\alpha_{\mathfrak{v}}(\mathbf{f}_2\otimes\varepsilon_{\mathfrak{v}}^{-1})\cdot\mathcal{N}(\mathfrak{v})^{-s}\big)\end{aligned}$$

which coincides exactly with the Euler factor $D_{\mathfrak{v}}\big(\mathbf{f}_1\otimes\varepsilon_{\mathfrak{v}},\mathbf{f}_2\otimes\varepsilon_{\mathfrak{v}}^{-1},s\big)$.

(III) For some ramified character $\varepsilon_{\mathfrak{v}}$ both of $\pi_{F_{\mathfrak{v}}}\big(\mathbf{f}_1\otimes\varepsilon_{\mathfrak{v}}\big)$ and $\pi_{F_{\mathfrak{v}}}\big(\mathbf{f}_2\otimes\varepsilon_{\mathfrak{v}}^{-1}\big)$ are principal series and unramified, and it follows immediately that

$$\begin{aligned}L_{\mathfrak{v}}\big(\mathbf{f}_1\otimes\mathbf{f}_2,s\big)&=L_{\mathfrak{v}}\big((\mathbf{f}_1\otimes\varepsilon_{\mathfrak{v}})\otimes(\mathbf{f}_2\otimes\varepsilon_{\mathfrak{v}}^{-1}),s\big)\\&=D_{\mathfrak{v}}\big((\mathbf{f}_1\otimes\varepsilon_{\mathfrak{v}}),(\mathbf{f}_2\otimes\varepsilon_{\mathfrak{v}}^{-1}),s\big)\end{aligned}$$

which are both of degree 4.

(IV)' $\pi_1=\pi_{F_{\mathfrak{v}}}(\mathbf{f}_1)=\sigma\big(\mu_1,\nu_1\big)$ is a special representation with $\mu_1\nu_1^{-1}=\mathcal{N}$ whilst $\pi_2=\pi_{F_{\mathfrak{v}}}(\mathbf{f}_2)$ is *not* a special representation, so that upon applying [GJ78, Equation (1.4.2)] we find $L\big(\pi_1\times\pi_2,s\big)=L\big(s,\mu_1\otimes\pi_2\big)$; thus

$$L_{\mathfrak{v}}\big(\mathbf{f}_1\otimes\mathbf{f}_2,s\big)=\big(1-\alpha_{\mathfrak{v}}(\mathbf{f}_1)\alpha_{\mathfrak{v}}(\mathbf{f}_2)\mathcal{N}(\mathfrak{v})^{-s}\big)\big(1-\alpha_{\mathfrak{v}}(\mathbf{f}_1)\beta_{\mathfrak{v}}(\mathbf{f}_2)\mathcal{N}(\mathfrak{v})^{-s}\big)$$

which coincides again with the imprimitive Euler factor $D_{\mathfrak{v}}\big(\mathbf{f}_1,\mathbf{f}_2,s\big)$.

(V) Both of $\pi_1=\pi_{F_{\mathfrak{v}}}(\mathbf{f}_1)=\sigma\big(\mu_1,\nu_1\big)$ and $\pi_2=\pi_{F_{\mathfrak{v}}}(\mathbf{f}_2)=\sigma\big(\mu_2,\nu_2\big)$ are special series representations satisfying $\mu_j\nu_j^{-1}=\mathcal{N}$, so that by [GJ78, Equation (1.4.3)] one has

$$\begin{aligned}&L\big(\pi_1\times\pi_2,s\big)=L\big(\mu_1\cdot\mu_2,s\big)\times L\big(\nu_1\cdot\mu_2,s\big)\\&=L\big(\mu_1\cdot\mu_2,s\big)\times L\big(\mathcal{N}^{-1}\cdot\mu_1\cdot\mu_2,s\big)=L\big(\mu_1\cdot\mu_2,s\big)\times L\big(\mu_1\cdot\mu_2,s-1\big).\end{aligned}$$

Furthermore, in terms of the Hilbert modular newforms we deduce that

$$\begin{aligned}L_{\mathfrak{v}}\big(\mathbf{f}_1\otimes\mathbf{f}_2,s\big)&=\big(1-\alpha_{\mathfrak{v}}(\mathbf{f}_1)\alpha_{\mathfrak{v}}(\mathbf{f}_2)\mathcal{N}(\mathfrak{v})^{-s}\big)\big(1-\alpha_{\mathfrak{v}}(\mathbf{f}_1)\alpha_{\mathfrak{v}}(\mathbf{f}_2)\mathcal{N}(\mathfrak{v})^{1-s}\big)\\&\neq\big(1-\alpha_{\mathfrak{v}}(\mathbf{f}_1)\alpha_{\mathfrak{v}}(\mathbf{f}_2)\mathcal{N}(\mathfrak{v})^{-s}\big)=D_{\mathfrak{v}}\big(\mathbf{f}_1,\mathbf{f}_2,s\big)\end{aligned}$$

with $\big|\alpha_{\mathfrak{v}}(\mathbf{f}_1)\big|_\infty=\mathcal{N}(\mathfrak{v})^{\frac{k_1}{2}-1}$ and $\big|\alpha_{\mathfrak{v}}(\mathbf{f}_2)\big|_\infty=\mathcal{N}(\mathfrak{v})^{\frac{k_2}{2}-1}$ by Theorem 7.3(ii).

(VI) Both of $\pi_1 = \pi_{F_{\mathfrak{v}}}(\mathbf{f}_1)$ and $\pi_2 = \pi_{F_{\mathfrak{v}}}(\mathbf{f}_2)$ are supercuspidal representations and by [GJ78, Proposition 1.2] the only possible pole of $L\big(\pi_1 \times \pi_2, s\big)^{-1}$ is simple; such a pole occurs at $s = s_0$ precisely when $\mathcal{N}^{2s_0} = \big(\vartheta_1\vartheta_2\big)^{-1}$ where ϑ_i denotes the quasi-central character of π_i for $i = 1, 2$, which means $s_0 = \frac{1}{2}(k_1 + k_2 - 2)$. It follows that $D_{\mathfrak{v}}\big(\mathbf{f}_1, \mathbf{f}_2, s\big) = 1$ whilst either $L_{\mathfrak{v}}\big(\mathbf{f}_1 \otimes \mathbf{f}_2, s\big) = 1$, or instead

$$L_{\mathfrak{v}}\big(\mathbf{f}_1 \otimes \mathbf{f}_2, s\big) = \big(1 - \mathfrak{U}_{\mathfrak{v}} \cdot \mathcal{N}(\mathfrak{v})^{-s}\big) \quad \text{with } \big|\mathfrak{U}_{\mathfrak{v}}\big|_\infty = \mathcal{N}(\mathfrak{v})^{\frac{1}{2}(k_1+k_2-2)}.$$

The reader may have noticed that we apostrophised each of the Cases (I), (II), (IV). In fact, there are complementary Cases (I)”, (II)”, (IV)” obtained by exchanging the roles of $\pi_{F_{\mathfrak{v}}}(\mathbf{f}_1)$ and $\pi_{F_{\mathfrak{v}}}(\mathbf{f}_2)$, however the calculations are otherwise identical.

Notation. For each choice of $\star \in \{\text{I,II,III,IV,V,VI}\}$ one writes $\Sigma^{\text{bad}}_{(\star)}$ for the subset of bad primes corresponding to Case $(\star)$ above, hence there is a partition

$$\Sigma^{\text{bad}}(\mathbf{f}_1, \mathbf{f}_2) \ = \ \Sigma^{\text{bad}}_{(\text{I})} \ \cup \ \Sigma^{\text{bad}}_{(\text{II})} \ \cup \ \Sigma^{\text{bad}}_{(\text{III})} \ \cup \ \Sigma^{\text{bad}}_{(\text{IV})} \ \cup \ \Sigma^{\text{bad}}_{(\text{V})} \ \cup \ \Sigma^{\text{bad}}_{(\text{VI})}\,.$$

If one defines $\tilde{\mathbf{f}}_1 := \mathbf{f}_1 \otimes \varepsilon$ and $\tilde{\mathbf{f}}_2 := \mathbf{f}_2 \otimes \varepsilon^{-1}$ where $\varepsilon = \prod_{\mathfrak{v} \in \Sigma^{\text{bad}}_{(\text{II})} \cup \Sigma^{\text{bad}}_{(\text{III})}} \varepsilon_{\mathfrak{v}}$, then

$$\begin{aligned} L(\mathbf{f}_1 \otimes \mathbf{f}_2, s) &= L_{(S)}\big(\mathbf{f}_1 \otimes \mathbf{f}_2, s\big) \times \prod_{\mathfrak{v} \in S} L_{\mathfrak{v}}\big(\mathbf{f}_1 \otimes \mathbf{f}_2, s\big)^{-1} \\ &= D_{(S)}\big(\tilde{\mathbf{f}}_1, \tilde{\mathbf{f}}_2, s\big) \times \prod_{\mathfrak{v} \in S} L_{\mathfrak{v}}\big(\mathbf{f}_1 \otimes \mathbf{f}_2, s\big)^{-1} \end{aligned}$$

where the finite set $S = S(\mathbf{f}_1, \mathbf{f}_2)$ consists exclusively of primes in $\Sigma^{\text{bad}}_{(\text{V})} \cup \Sigma^{\text{bad}}_{(\text{VI})}$.

Panchishkin, Hida and a few others [Pan89; Hid91; Dis17; Del20] have constructed a p-adic L-function that interpolates the imprimitive complex version (7.4) on the critical strip $\{k_2, \ldots, k_1 - 1\}$. More precisely, if we write $\Psi\big(\mathbf{f}_1, \mathbf{f}_2, \chi, s\big)$ for the completed L-function (2.38) associated to the Dirichlet series $D(\mathbf{f}_1, \mathbf{f}_2, \chi, s)$ in place of $L(\mathbf{f}_1 \otimes \mathbf{f}_2 \otimes \chi, s)$, then

$$\begin{aligned} \chi\mathcal{N}x_p^m\big(D_p(\mathbf{f}_1, \mathbf{f}_2)\big) = \mathcal{B}(\mathbf{f}_1, \mathbf{f}_2, \chi, m) \times \prod_{\mathfrak{p}|p} \mathcal{A}_{\mathfrak{p}}\big(M(\mathbf{f}_1 \otimes \mathbf{f}_2), \chi, 1 + m\big) \\ \times \frac{\Psi(\mathbf{f}_1, \mathbf{f}_2, \chi, 1 + m)}{\Omega(\mathbf{f}_1, \mathbf{f}_2)} \end{aligned} \tag{7.6}$$

at every Tate twist $m \in \{0, 1, \dots, k-2\}$ and for each $\chi \in \mathfrak{X}_p^{\mathrm{tors}}$. Here the automorphic period $\Omega(\mathbf{f}_1, \mathbf{f}_2)$ is equal to the Petersson norm $\langle \mathbf{f}_1, \mathbf{f}_1 \rangle_{\mathfrak{c}_1}$. A more precise version of Equation 7.6 can be found in [Pan89, Theorem 2.1] (see also Equation (4.1)). The p-adic functional equation involves the primitive p-adic L-function $L_p(\mathbf{f}_1 \otimes \mathbf{f}_2)$ which à priori only belongs to the quotient field of $\mathcal{O}_{T,p}[\Delta][\![\gamma - 1]\!]$. In particular one has for every critical twist (m, χ) that

$$\begin{aligned} \chi \mathcal{N} x_p^m \big(L_p(\mathbf{f}_1 \otimes \mathbf{f}_2)\big) = \; & w_p\big(M(\mathbf{f}_1 \otimes \mathbf{f}_2)(\chi), m\big) \\ & \times \chi^{-1} \mathcal{N} x_p^{k_1 - k_2 - 1 - m} \big(L_p(\mathbf{f}_1^{\rho} \otimes \mathbf{f}_2^{\rho})\big) \,. \end{aligned} \tag{7.7}$$

The following result identifies a necessary condition for Hypothesis (S^{dep}) to be valid in the case of double products.

Proposition 7.5. *Hypothesis (S^{dep}) is true for $M(\mathbf{f}_1 \otimes \mathbf{f}_2)$ and $S = \Sigma^{\mathrm{bad}}_{(\mathrm{V})} \cup \Sigma^{\mathrm{bad}}_{(\mathrm{VI})}$ if one replaces the newform pair $(\mathbf{f}_1, \mathbf{f}_2)$ with its twisted counterpart $\big(\tilde{\mathbf{f}}_1, \tilde{\mathbf{f}}_2\big)$.*

Proof. We remark that for $M = M(\mathbf{f}_1 \otimes \mathbf{f}_2)$, there is a pair of factorisations

$$\begin{aligned} L^{\phi}_{p,(S)}(\tilde{\mathbf{f}}_1, \tilde{\mathbf{f}}_2) &= \frac{L_p^{\phi}(\mathbf{f}_1 \otimes \mathbf{f}_2)}{\Omega(\tilde{\mathbf{f}}_1) \cdot \Omega \mathbf{f}_1)^{-1}} \times \prod_{\mathfrak{v} \in \Sigma^{\mathrm{bad}}_{(\mathrm{V})}} \mathscr{L}_{\mathfrak{v}}^{\phi}(M; \gamma) \times \prod_{\mathfrak{v} \in \Sigma^{\mathrm{bad}}_{(\mathrm{VI})}} \mathscr{L}_{\mathfrak{v}}^{\phi}(M; \gamma) \quad \text{and} \\ L^{\phi^{-1}}_{p,(S)}(\tilde{\mathbf{f}}_1^{\rho}, \tilde{\mathbf{f}}_2^{\rho})^{\bullet} &= \frac{L_p^{\phi^{-1}}(\mathbf{f}_1^{\rho} \otimes \mathbf{f}_2^{\rho})^{\bullet}}{\Omega(\tilde{\mathbf{f}}_1^{\rho}) \cdot \Omega(\mathbf{f}_1^{\rho})^{-1}} \times \prod_{\mathfrak{v} \in \Sigma^{\mathrm{bad}}_{(\mathrm{V})}} \mathscr{L}_{\mathfrak{v}}^{\phi^{-1}}\big(M^{\vee}(1); \gamma^{-1}\big) \\ & \qquad \times \prod_{\mathfrak{v} \in \Sigma^{\mathrm{bad}}_{(\mathrm{VI})}} \mathscr{L}_{\mathfrak{v}}^{\phi^{-1}}\big(M^{\vee}(1); \gamma^{-1}\big). \end{aligned}$$

One must then compute the zeroes of these bad factors in Cases (V) and (VI).

Firstly, $L_{\mathfrak{v}}\big(\mathbf{f}_1 \otimes \mathbf{f}_2, s\big) = \big(1 - \alpha_{\mathfrak{v}}(\mathbf{f}_1)\alpha_{\mathfrak{v}}(\mathbf{f}_2)\mathcal{N}(\mathfrak{v})^{-s}\big)\big(1 - \alpha_{\mathfrak{v}}(\mathbf{f}_1)\alpha_{\mathfrak{v}}(\mathbf{f}_2)\mathcal{N}(\mathfrak{v})^{1-s}\big)$ if $\mathfrak{v} \in \Sigma^{\mathrm{bad}}_{(\mathrm{V})}$ with $\big|\alpha_{\mathfrak{v}}(\mathbf{f}_1)\big|_{\infty} = \mathcal{N}(\mathfrak{v})^{\frac{k_1}{2} - 1}$ and $\big|\alpha_{\mathfrak{v}}(\mathbf{f}_2)\big|_{\infty} = \mathcal{N}(\mathfrak{v})^{\frac{k_2}{2} - 1}$, therefore

$$\Big|\alpha_{\mathfrak{v}}(\mathbf{f}_1)\alpha_{\mathfrak{v}}(\mathbf{f}_2)\Big|_{\infty} = \mathcal{N}(\mathfrak{v})^{\frac{k_1 + k_2}{2} - 2} \quad \text{and} \quad \Big|\alpha_{\mathfrak{v}}(\mathbf{f}_1)\alpha_{\mathfrak{v}}(\mathbf{f}_2)\mathcal{N}(\mathfrak{v})^1\Big|_{\infty} = \mathcal{N}(\mathfrak{v})^{\frac{k_1 + k_2}{2} - 1} .$$

Given a branch ϕ, any zeroes of $\mathscr{L}_{\mathfrak{v}}^{\phi}\big(M; \kappa_{\mathrm{cyc}}(\gamma)^{-r} \cdot \gamma\big)$ must then have the form

$$\mathbf{x}_{\mathfrak{v}}^{\phi} = \Big(\phi(\mathfrak{v}) \cdot \iota_p\big(\omega_F(\mathfrak{v})^{e_{\mathfrak{v}}} \cdot u_{\mathfrak{v}}\big)\Big)^{-\frac{\log_p \kappa_{\mathrm{cyc}}(\gamma)}{\log_p(\mathcal{N}(\mathfrak{v}))}} \kappa_{\mathrm{cyc}}(\gamma)^{r - e_{\mathfrak{v}}} - 1 \tag{7.8}$$

where $\big|u_{\mathfrak{v}}\big|_{\infty} = 1$, and either $e_{\mathfrak{v}} = \frac{k_1 + k_2}{2} - 2$ or $e_{\mathfrak{v}} = \frac{k_1 + k_2}{2} - 1$.

If $\mathfrak{v} \in \Sigma^{\mathrm{bad}}_{(\mathrm{VI})}$, either $L_{\mathfrak{v}}\big(\mathbf{f}_1 \otimes \mathbf{f}_2, s\big) = 1$ or $L_{\mathfrak{v}}\big(\mathbf{f}_1 \otimes \mathbf{f}_2, s\big) = \big(1 - \mathfrak{U}_{\mathfrak{v}} \cdot \mathcal{N}(\mathfrak{v})^{-s}\big)$ with $\big|\mathfrak{U}_{\mathfrak{v}}\big|_{\infty} = \mathcal{N}(\mathfrak{v})^{\frac{1}{2}(k_1+k_2-2)}$, whence the zeroes of $\mathscr{L}^{\phi}_{\mathfrak{v}}\big(M; \kappa_{\mathrm{cyc}}(\gamma)^{-r} \cdot \gamma\big)$ assume the form

$$\mathbf{x}^{\phi}_{\mathfrak{v}} = \Big(\phi(\mathfrak{v}) \cdot \iota_p\big(\omega_F(\mathfrak{v})^{e_{\mathfrak{v}}} \cdot u_{\mathfrak{v}}\big)\Big)^{-\frac{\log_p \kappa_{\mathrm{cyc}}(\gamma)}{\log_p(\mathcal{N}(\mathfrak{v}))}} \kappa_{\mathrm{cyc}}(\gamma)^{r-e_{\mathfrak{v}}} - 1 \tag{7.9}$$

where $\big|u_{\mathfrak{v}}\big|_{\infty} = 1$ and $e_{\mathfrak{v}} = \frac{k_1+k_2}{2} - 1$.

Alternatively, if $\mathfrak{q} \in \Sigma^{\mathrm{bad}}_{(\mathrm{V})}$ then by a similar argument, at each branch ϕ^{-1} any zeroes of the dual Euler factor $\mathscr{L}^{\phi^{-1}}_{\mathfrak{q}}\big(M^{\vee}(1); \kappa_{\mathrm{cyc}}(\gamma)^{r} \cdot \gamma^{-1}\big)$ have the form

$$\mathbf{x}^{\phi^{-1}}_{\mathfrak{q}} = \Big(\overline{\phi}(\mathfrak{q}) \cdot \iota_p\big(\omega_F(\mathfrak{q})^{e_{\mathfrak{q}}-2k^{\dagger}} \cdot \overline{u_{\mathfrak{q}}}\big)\Big)^{\frac{\log_p \kappa_{\mathrm{cyc}}(\gamma)}{\log_p(\mathcal{N}(\mathfrak{q}))}} \kappa_{\mathrm{cyc}}(\gamma)^{r+e_{\mathfrak{q}}-2k^{\dagger}} - 1. \tag{7.10}$$

where $k^{\dagger} = \frac{1}{2}(k_1+k_2-1)$ is the symmetry point, and $e_{\mathfrak{q}} \in \big\{\frac{k_1+k_2}{2}-2, \frac{k_1+k_2}{2}-1\big\}$. Likewise, if $\mathfrak{q} \in \Sigma^{\mathrm{bad}}_{(\mathrm{VI})}$ then any zeroes of $\mathscr{L}^{\phi^{-1}}_{\mathfrak{q}}\big(M^{\vee}(1); \kappa_{\mathrm{cyc}}(\gamma)^{r} \cdot \gamma^{-1}\big)$ is of the form

$$\mathbf{x}^{\phi^{-1}}_{\mathfrak{q}} = \Big(\overline{\phi}(\mathfrak{q}) \cdot \iota_p\big(\omega_F(\mathfrak{q})^{e_{\mathfrak{q}}-2k^{\dagger}} \cdot \overline{u_{\mathfrak{q}}}\big)\Big)^{\frac{\log_p \kappa_{\mathrm{cyc}}(\gamma)}{\log_p(\mathcal{N}(\mathfrak{q}))}} \kappa_{\mathrm{cyc}}(\gamma)^{r+e_{\mathfrak{q}}-2k^{\dagger}} - 1. \tag{7.11}$$

Suppose the zeroes are not disjoint, say $\mathbf{x}^{\phi}_{\mathfrak{v}} = \mathbf{x}^{\phi^{-1}}_{\mathfrak{q}}$ *for some* $\mathfrak{v}, \mathfrak{q} \in \Sigma^{\mathrm{bad}}_{(\mathrm{V})} \cup \Sigma^{\mathrm{bad}}_{(\mathrm{VI})}$. Together, Equations (7.8)–(7.11) imply that

$$\kappa_{\mathrm{cyc}}(\gamma)^{r-e_{\mathfrak{v}}} = \kappa_{\mathrm{cyc}}(\gamma)^{r+e_{\mathfrak{q}}-2k^{\dagger}}, \quad \text{i.e.} \quad e_{\mathfrak{v}} + e_{\mathfrak{q}} = 2k^{\dagger} = k_1 + k_2 - 1.$$

But this is impossible as $e_{\mathfrak{v}}, e_{\mathfrak{q}} \in \big\{\frac{k_1+k_2}{2}-2, \frac{k_1+k_2}{2}-1\big\}$, *therefore* $\mathbf{x}^{\phi}_{\mathfrak{v}} \neq \mathbf{x}^{\phi^{-1}}_{\mathfrak{q}}$. ■

In preparation for the following corollary, we provide a definition of the congruence number in the sense of [Dia91; Hid91; Wil95].

Definition 7.2. To a cusp form $\mathbf{f} \in \mathcal{S}_k(\mathfrak{c}; R)$, one can associate the congruence number $\mathfrak{h}_{\mathbf{f}} := \mathrm{Ann}\big(\ker \mathscr{W}_{\mathbf{f}}\big)$ where $\mathscr{W}_{\mathbf{f}} : \mathcal{S}_k(\mathfrak{c}; R) \to R$ denotes the map sending $\mathbf{g} \mapsto (\mathbf{f}, \mathbf{g})_{\mathfrak{c}}$ with $(-,-)_{\mathfrak{c}}$ being Hida's pairing introduced in (4.11).

Corollary 7.6. *The primitive function* $L_p(\mathbf{f}_1 \otimes \mathbf{f}_2)$ *has no poles on* $\mathfrak{X}_p$, *in fact*

$$\mathfrak{h}_{\mathbf{f}_1} \cdot L_p(\mathbf{f}_1 \otimes \mathbf{f}_2) \in \mathcal{O}_{T,p}[\Delta]\llbracket \gamma - 1 \rrbracket\,.$$

Proof. As Hypothesis (S^{dep}) holds for $M = M(\tilde{\mathbf{f}}_1 \otimes \tilde{\mathbf{f}}_2)$ with $S = \Sigma^{\mathrm{bad}}_{(\mathrm{V})} \cup \Sigma^{\mathrm{bad}}_{(\mathrm{VI})}$ it follows from Proposition 7.5 that for elements $\mathcal{Y}^{\phi}_1(M), \mathcal{Y}^{\phi}_2(M) \in \mathcal{O}_{T,p}\llbracket \gamma - 1 \rrbracket$,

$$L^{\phi}_p(\tilde{\mathbf{f}}_1 \otimes \tilde{\mathbf{f}}_2) = \mathcal{W}^{(\varepsilon_0),\phi}(M) \cdot \mathcal{Y}^{\phi}_1(M) \times \mathrm{Tw}_{k_1-k_2-1}\, D^{\phi^{-1}}_{p,(S)}\big(\tilde{\mathbf{f}}^{\rho}_1, \tilde{\mathbf{f}}^{\rho}_2\big)^{\bullet}$$

$$+\,\mathcal{Y}_2^{\phi}(M) \times D^{\phi}_{p,(S)}(\tilde{\mathbf{f}}_1, \tilde{\mathbf{f}}_2)\,.$$

The holomorphy of both $D^{\phi}_{p,(S)}(\tilde{\mathbf{f}}_1, \tilde{\mathbf{f}}_2)$ and $D^{\phi^{-1}}_{p,(S)}\big(\tilde{\mathbf{f}}_1^{\rho}, \tilde{\mathbf{f}}_2^{\rho}\big)$ on $\mathfrak{X}_p$ was established by Hida in [Hid91, Theorems 5.1 and 5.2], which immediately implies the holomorphy of

$$L_p^{\phi}(\mathbf{f}_1 \otimes \mathbf{f}_2) = \iota_p \circ \iota_\infty^{-1}\Big(\Omega(\tilde{\mathbf{f}}_1) \cdot \Omega(\mathbf{f}_1)^{-1}\Big) \times L_p^{\phi}(\tilde{\mathbf{f}}_1 \otimes \tilde{\mathbf{f}}_2) \in \mathbb{Q} \otimes \mathcal{O}_{T,p}[\![\gamma - 1]\!]\,.$$

The p-integrality of $\mathfrak{h}_{\mathbf{f}_1} \cdot L_p^{\phi}(\mathbf{f}_1 \otimes \mathbf{f}_2)$ is an easy consequence of the p-integrality of $\mathfrak{h}_{\mathbf{f}_1} \cdot D_p^{\phi}(\mathbf{f}_1, \mathbf{f}_2)$ (cf. the last line in Theorem 5.1 of *op. cit.*) since the quotient $\frac{L_p^{\phi}(\mathbf{f}_1 \otimes \mathbf{f}_2)}{D_p^{\phi}(\mathbf{f}_1, \mathbf{f}_2)}$ is composed of products of p-adic Euler factors, each of which has a trivial μ-invariant. ■

7.3.3 The symmetric square p-adic L-function

Let $\mathbf{f} \in \mathcal{S}_k(\mathfrak{c}, \eta)$ be a Hilbert modular newform and $\big\{V_\ell(\mathbf{f})\big\}_\ell$ its system of ℓ-adic representations. The symmetric square representation $\mathrm{Sym}^2(\mathbf{f})$ is the three-dimensional component in the tensor product

$$V_\ell(\mathbf{f}) \otimes V_\ell(\mathbf{f}) \cong \mathrm{Sym}^2(\mathbf{f}) \oplus (\tilde{\eta}\kappa_{\mathrm{cyc}}^{k-1})\,.$$

The hypothetical symmetric square motive $M\big(\mathrm{Sym}^2(\mathbf{f})\big)$ has rank three and weight $w = 2k - 2$.

Once again, we define the *primitive L-function* for $\mathrm{Sym}^2(\mathbf{f})$ as the product

$$L(\mathrm{Sym}^2(\mathbf{f}), s) = \prod_{\mathfrak{v}} \det\Big(1 - \mathrm{Frob}_{\mathfrak{v}}^{-1} \,\Big|\, \big(\mathrm{Sym}^2(\mathbf{f})\big)^{I_{\mathfrak{v}}} \cdot \mathcal{N}(\mathfrak{v})^{-s}\Big)^{-1}, \quad \ell \neq \mathfrak{v}\,. \tag{7.12}$$

and the *imprimitive L-function* for $\mathrm{Sym}^2(\mathbf{f})$ as the product of Dirichlet series

$$D(\mathrm{Sym}^2(\mathbf{f}), s) = \zeta_{(\mathfrak{c})}(\tilde{\eta}^2, 2s + 2 - 2k) \times \sum_{\mathfrak{m}} \mathrm{C}(\mathbf{f}, \mathfrak{m}^2) \cdot \mathcal{N}(\mathfrak{m})^{-s} \tag{7.13}$$

with infinite product expansion

$$\prod_{\mathfrak{v}} \Big(\big(1 - \alpha_{\mathfrak{v}}(\mathbf{f})^2 \mathcal{N}(\mathfrak{v})^{-s}\big)\big(1 - \beta_{\mathfrak{v}}(\mathbf{f})^2 \mathcal{N}(\mathfrak{v})^{-s}\big)\big(1 - \tilde{\eta}(\mathfrak{v})\mathcal{N}(\mathfrak{v})^{k-1-s}\big)\Big)^{-1}.$$

Both L-functions converge when $\mathrm{Re}(s) > k$ and the L-factors agree at $\mathfrak{v} \nmid \mathfrak{c}$.

The factor at infinity for the symmetric square representation is given by

$$L_\infty(\mathrm{Sym}^2(\mathbf{f}), s) := \begin{cases} 2(2\pi)^{-ns}\Gamma(s)^n\Gamma(s-k+1) & \text{if } k \text{ is odd} \\ 2(2\pi)^{-ns}\Gamma(s)^n\Gamma(s-k+2) & \text{if } k \text{ is even.} \end{cases} \tag{7.14}$$

and one writes as usual the completed L-function:

$$\Psi\big(\mathrm{Sym}^2(\mathbf{f}), s\big) = L_\infty\big(\mathrm{Sym}^2(\mathbf{f}), s\big) \times L\big(\mathrm{Sym}^2(\mathbf{f}), s\big)\,. \tag{7.15}$$

Gelbart and Jacquet proved the analytic continuation of $\Psi\big(\mathrm{Sym}^2(\mathbf{f}), s\big)$ to $\mathbb{C}$ along with the complex functional equation

$$\Psi\big(\mathrm{Sym}^2(\mathbf{f}), s\big) = \epsilon\big(\mathrm{Sym}^2(\mathbf{f}), s\big) \times \Psi\big(\mathrm{Sym}^2(\mathbf{f}^\rho), 2k-1-s\big)\,. \tag{7.16}$$

using the automorphic representation associated to $\mathrm{Sym}^2(\mathbf{f})$ [GJ78].

The p-adic analogues of these objects were constructed buy Coates–Schmidt, Schmidt and others [CS87; Sch88; Gue95; DD97; Ros16] by utilising Shimura's theory of half-integral weight modular forms. There is a long back-story to the order in which results where obtained, however suffice to say that over $F = \mathbb{Q}$ most of the work is already done. It is important to note that the behavior of the completed L-function on the critical strip $\{1, 2, \ldots, 2k-2\}$ changes at the point of symmetry $s = \frac{2k-1}{2}$, which means there are two separate p-adic L-functions, $D_p^-\big(\mathrm{Sym}^2(\mathbf{f})\big)$ and $D_p^+\big(\mathrm{Sym}^2(\mathbf{f})\big)$, satisfying at all *even* $\chi\mathcal{N}x_p^m$:

$$\begin{aligned}\chi\mathcal{N}x_p^m\Big(D_p^\pm\big(\mathrm{Sym}^2(\mathbf{f})\big)\Big) = \mathcal{B}\big(\mathrm{Sym}^2(\mathbf{f}), \chi, m\big) \times \prod_{\mathfrak{p}|p} \mathcal{A}_\mathfrak{p}\big(\mathrm{Sym}^2(\mathbf{f}), \chi, 1+m\big) \\ \times \frac{\Psi\big(\mathrm{Sym}^2(\mathbf{f}), \chi, 1+m\big)}{\Omega^\pm\big(\mathrm{Sym}^2(\mathbf{f}), m\big)}\end{aligned} \tag{7.17}$$

where we will choose '$-$' if $m \in \{0, \ldots, k-2\}$ and '$+$' if $m \in \{k-1, \ldots, 2k-3\}$ (at odd characters $\chi\mathcal{N}x_p^m$ both p-adic L-functions are identically zero).

Remark 13. The automorphic period in the above equation is given by:

$$\Omega^-\big(\mathrm{Sym}^2(\mathbf{f}), m\big) = \frac{(2\pi)^{n(m+1)}\langle \mathbf{f}, \mathbf{f}\rangle_\mathfrak{c}^2}{\Gamma(m+1)^n} \quad \text{and}$$

$$\Omega^+\big(\mathrm{Sym}^2(\mathbf{f}), m\big) = \frac{(2\pi)^{n(2m+3-k)}\langle \mathbf{f}, \mathbf{f}\rangle_\mathfrak{c}^2}{\big(2\Gamma(m+2-k)\Gamma(m+1)\big)^n}$$

with ± chosen as above (c.f. [DD97, Theorem 3.1.2]).

The p-adic functional equation concerns the primitive versions $L_p^{\pm}\big(\mathrm{Sym}^2(\mathbf{f})\big)$ which à priori only lie inside of $\mathrm{Frac}\big(\mathcal{O}_{T,p}[\Delta^+][\![\gamma-1]\!]\big)$. More exactly, one has

$$\begin{aligned}\chi\mathcal{N}x_p^m\Big(L_p^-\big(\mathrm{Sym}^2(\mathbf{f})\big)\Big) &= w_p\Big(M\big(\mathrm{Sym}^2(\mathbf{f})\big)(\chi), m\Big)\\ &\qquad \times \chi^{-1}\mathcal{N}x_p^{2k-3-m}\Big(L_p^+\big(\mathrm{Sym}^2(\mathbf{f}^{\rho})\big)\Big)\end{aligned} \tag{7.18}$$

where $L_p^{\pm}$ interpolate the same special values as $D_p^{\pm}$, except $D\big(\mathrm{Sym}^2(\mathbf{f}),\chi,-\big)$ is replaced by the primitive L-function $L\big(\mathrm{Sym}^2(\mathbf{f}),\chi,-\big)$ in the formulae.

Proposition 7.7. *Hypothesis (S^{dep}) is true for the motive $M = M\big(\mathrm{Sym}^2(\mathbf{f})\big)$ if one takes S to consist of the primes where $\mathbf{f}$ is supercuspidal, and then replaces $\mathbf{f}$ with a minimal twist $\mathbf{f}^{\natural} = \mathbf{f}\otimes\varepsilon$ amongst all quadratic characters $\varepsilon_{/F}$.*

Proof. The argument closely follows the $F = \mathbb{Q}$ scenario outlined in [DD97; Sch88]. Let $\pi_F(\mathbf{f}) = \bigotimes_{\mathfrak{v}}' \pi_{F_{\mathfrak{v}}}(\mathbf{f})$ be the cuspidal automorphic representation of $\mathrm{GL}_2(F)$ associated to $\mathbf{f}$. Then by [GJ78; Ros16] the following occurs at the finite places $\mathfrak{v}$ of F:

(I) $\pi_{F_{\mathfrak{v}}}(\mathbf{f}) = \pi(\mu,\nu)$ is principal series with μ and ν unramified, whence $\big(\mathrm{Sym}^2 V_\ell(\mathbf{f})\big)^{I_{F_{\mathfrak{v}}}} = \mathrm{Sym}^2 V_\ell(\mathbf{f})$ and $L_{\mathfrak{v}}\big(\mathrm{Sym}^2(\mathbf{f}),s\big) = D_{\mathfrak{v}}\big(\mathrm{Sym}^2(\mathbf{f}),s\big)$ equal

$$\big(1-\alpha_{\mathfrak{v}}^2(\mathbf{f})\mathcal{N}(\mathfrak{v})^{-s}\big)\big(1-\beta_{\mathfrak{v}}^2(\mathbf{f})\mathcal{N}(\mathfrak{v})^{-s}\big)\big(1-\eta(\mathfrak{v})\mathcal{N}(\mathfrak{v})^{k-1-s}\big).$$

(II) $\pi_{F_{\mathfrak{v}}}(\mathbf{f})\otimes\varepsilon_{\mathfrak{v}}$ is unramified principal series with $\varepsilon_{\mathfrak{v}}$ a quadratic character, so that $\big(\mathrm{Sym}^2 V_\ell(\mathbf{f})\big)^{I_{F_{\mathfrak{v}}}} \cong \mathrm{Sym}^2\big(V_\ell(\mathbf{f})\otimes\varepsilon_{\mathfrak{v}}\big)$ and $L_{\mathfrak{v}}\big(\mathrm{Sym}^2(\mathbf{f}),s\big)$ equals

$$\big(1-\alpha_{\mathfrak{v}}^2(\mathbf{f}\otimes\varepsilon_{\mathfrak{v}})\mathcal{N}(\mathfrak{v})^{-s}\big)\big(1-\beta_{\mathfrak{v}}^2(\mathbf{f}\otimes\varepsilon_{\mathfrak{v}})\mathcal{N}(\mathfrak{v})^{-s}\big)\big(1-\eta(\mathfrak{v})\mathcal{N}(\mathfrak{v})^{k-1-s}\big)$$

which coincides with the imprimitive Euler factor $D_{\mathfrak{v}}\big(\mathrm{Sym}^2(\mathbf{f}\otimes\varepsilon_{\mathfrak{v}}),s\big)$.

(III) $\pi_{F_{\mathfrak{v}}}(\mathbf{f}) = \sigma(\mu,\nu)$ is a special representation with $\mu\nu^{-1} = \mathcal{N}$, so that $L_{\mathfrak{v}}\big(\mathrm{Sym}^2(\mathbf{f}),s\big)$ and $D_{\mathfrak{v}}\big(\mathrm{Sym}^2(\mathbf{f}),s\big)$ are both equal to $\big(1-\mathcal{N}(\mathfrak{v})^{k-2-s}\big)$.

(IV) $\pi_{F_{\mathfrak{v}}}(\mathbf{f})$ is a supercuspidal representation so either $L_{\mathfrak{v}}\big(\mathrm{Sym}^2(\mathbf{f}),s\big) = 1$, or

$$L_{\mathfrak{v}}\big(\mathrm{Sym}^2(\mathbf{f}),s\big) = \big(1-\mathfrak{z}_{\mathfrak{v}}\cdot\mathcal{N}(\mathfrak{v})^{k-1-s}\big) \text{ for some root of unity } \mathfrak{z}_{\mathfrak{v}}\in\overline{\mathbb{Q}}^{\times};$$

note however that in both situations, one always has $D_{\mathfrak{v}}\big(\mathrm{Sym}^2(\mathbf{f}),s\big) = 1$ (one can specify the root of unity $\mathfrak{z}_{\mathfrak{v}}$ exactly, e.g., see [Ros16, §4.2]).

For the motive $M = M(\mathrm{Sym}^2(\mathbf{f}))$ and $S \subset \Sigma^{\mathrm{bad}}(M)$ as above, one has factorisations

$$D^{-,\phi}_{p,(S)}\big(\,\mathrm{Sym}^2(\mathbf{f}^\natural)\big) = \frac{L_p^{-,\phi}\big(\,\mathrm{Sym}^2(\mathbf{f})\big)}{\|\mathbf{f}^\natural\|^2_{\mathfrak{c}(\mathbf{f}^\natural)} \cdot \|\mathbf{f}\|^{-2}_{\mathfrak{c}(\mathbf{f})}} \times \prod_{\mathfrak{v}\in S} \mathscr{L}^{\phi}_{\mathfrak{v}}(M;\gamma) \qquad \text{and}$$

$$D^{+,\phi}_{p,(S)}\big(\,\mathrm{Sym}^2(\mathbf{f}^{\natural,\rho})\big)^{\bullet} = \frac{L_p^{+,\phi^{-1}}(\mathrm{Sym}^2(\mathbf{f}^{\rho}))^{\bullet}}{\|\mathbf{f}^{\natural,\rho}\|^2_{\mathfrak{c}(\mathbf{f}^\natural)} \cdot \|\mathbf{f}^{\rho}\|^{-2}_{\mathfrak{c}(\mathbf{f})}} \times \prod_{\mathfrak{v}\in S} \mathscr{L}^{\phi^{-1}}_{\mathfrak{v}}\big(M^{\vee}(1);\gamma^{-1}\big)\,.$$

Based on our analysis of the local factors above, one only observes a discrepancy between the imprimitive and primitive local Euler factors when $\mathfrak{v}$ belongs to Case (IV). Therefore, we take $S = \Sigma^{\mathrm{bad}}_{(\mathrm{IV})}$ and compute for the zeroes of $\mathscr{L}^{\phi}_{\mathfrak{v}}(M;\gamma)$ and $\mathscr{L}^{\phi^{-1}}_{\mathfrak{v}}\big(M^{\vee}(1);\gamma^{-1}\big)$ at the places $\mathfrak{v} \in S$.

Firstly, if $\mathfrak{v} \in S$ then by the same arguments as in the previous two sections, for a given branch ϕ any p-adic zeroes of $\mathscr{L}^{\phi}_{\mathfrak{v}}\big(M;\kappa_{\mathrm{cyc}}(\gamma)^{-r}\cdot\gamma\big)$ takes the form

$$\mathbf{x}^{\phi}_{\mathfrak{v}} = \big(\phi(\mathfrak{v})\cdot\iota_p\big(\omega_F(\mathfrak{v})^{e_{\mathfrak{v}}}\cdot u_{\mathfrak{v}}\big)\big)^{-\frac{\log_p \kappa_{\mathrm{cyc}}(\gamma)}{\log_p(\mathcal{N}(\mathfrak{v}))}}\kappa_{\mathrm{cyc}}(\gamma)^{r-e_{\mathfrak{v}}} - 1 \tag{7.19}$$

where $\big|u_{\mathfrak{v}}\big|_{\infty} = 1$ and $e_{\mathfrak{v}} = k-1$ from the shape of the factors in (IV). Secondly, if $\mathfrak{q} \in S$ then any zeroes of $\mathscr{L}^{\phi^{-1}}_{\mathfrak{q}}\big(M^{\vee}(1);\kappa_{\mathrm{cyc}}(\gamma)^{r}\cdot\gamma^{-1}\big)$ has the form

$$\mathbf{x}^{\phi^{-1}}_{\mathfrak{q}} = \big(\overline{\phi}(\mathfrak{q})\cdot\iota_p\big(\omega_F(\mathfrak{q})^{e_{\mathfrak{q}}+1-2k}\cdot\overline{u_{\mathfrak{q}}}\big)\big)^{\frac{\log_p \kappa_{\mathrm{cyc}}(\gamma)}{\log_p(\mathcal{N}(\mathfrak{q}))}}\kappa_{\mathrm{cyc}}(\gamma)^{r+e_{\mathfrak{q}}+1-2k} - 1. \tag{7.20}$$

Suppose that $\mathbf{x}^{\phi}_{\mathfrak{v}} = \mathbf{x}^{\phi^{-1}}_{\mathfrak{q}}$ *for some* $\mathfrak{v},\mathfrak{q} \in S$*:* Equations (7.19) and (7.20) imply

$$\kappa_{\mathrm{cyc}}(\gamma)^{r-e_{\mathfrak{v}}} = \kappa_{\mathrm{cyc}}(\gamma)^{r+e_{\mathfrak{q}}+1-2k}, \quad \text{i.e.} \quad e_{\mathfrak{v}} + e_{\mathfrak{q}} = 2k-1.$$

Clearly this is impossible as $e_{\mathfrak{v}}, e_{\mathfrak{q}} = k-1$, *in which case* $\mathbf{x}^{\phi}_{\mathfrak{v}} \neq \mathbf{x}^{\phi^{-1}}_{\mathfrak{q}}$. ■

Corollary 7.8. *The element* $L_p^{\pm}\big(\,\mathrm{Sym}^2(\mathbf{f})\big)$ *is either holomorphic on* $\mathfrak{X}_p$*, or has at most a simple pole at those* $\chi \in \mathfrak{X}_p$ *where* $\chi\eta^{-1}$ *is imaginary quadratic. In particular, if* $F = \mathbb{Q}$ *then* $L_p^{\pm}\big(\,\mathrm{Sym}^2(\mathbf{f})\big)$ *is an entire function defined on* $\mathfrak{X}_p$*.*

Proof. Applying both of Propositions 7.2 and 7.7, we have a decomposition

$$\begin{aligned} L_p^{-,\phi}\big(\,\mathrm{Sym}^2(\mathbf{f})\big) = \mathcal{W}^{(\varepsilon_0),\phi}\cdot\mathcal{Y}^{\phi}_1 \times \mathrm{Tw}_{2k-3}\, D^{+,\phi^{-1}}_{p,(S)}\big(\,\mathrm{Sym}^2(\mathbf{f}^{\rho,\natural})\big)^{\bullet} \\ + \mathcal{Y}^{\phi}_2 \times D^{-,\phi}_{p,(S)}\big(\,\mathrm{Sym}^2(\mathbf{f}^{\natural})\big) \end{aligned}$$

where $\mathcal{W}^{(\varepsilon_0),\phi}, \mathcal{Y}^{\phi}_1, \mathcal{Y}^{\phi}_2$ belong to $\mathcal{O}_{T,p}[\![\gamma-1]\!]$. The holomorphy of $D_p^{-,\phi}\big(\,\mathrm{Sym}^2(\mathbf{f}^{\natural})\big)$ follows from the result [Hid91, Theorem 5.1] specialised to the cyclotomic line. Moreover $D_p^{+,\phi^{-1}}\big(\mathrm{Sym}^2(\mathbf{f}^{\rho,\natural})\big)$ is well known to be holomorphic [Ros16, Theorem 7.7] except maybe for simple poles if $\chi\eta^{-1}$ is an imaginary quadratic character over F. The first assertion is now a consequence of the above $\mathcal{O}_{T,p}[\![\gamma-1]\!]$-linear decomposition. The second assertion follows from the fact that if $F=\mathbb{Q}$, then by [DD97, Theorem 3.1.2] the imprimitive p-adic L-function $D_p^{+}\big(\mathrm{Sym}^2(\mathbf{f}^{\rho,\natural})\big)$ is entire on the whole of $\mathfrak{X}_p$, therefore $L_p^{-,\phi}\big(\mathrm{Sym}^2(\mathbf{f})\big)$ must be an entire function too (via our decomposition).

N.B. The corresponding assertions for $L_p^{+}\big(\mathrm{Sym}^2(\mathbf{f})\big)$ are directly implied by the p-adic functional equation because $L_p^{+}\big(\mathrm{Sym}^2(\mathbf{f})\big)$ and $\mathrm{Tw}_{3-2k}\left(L_p^{-}\big(\mathrm{Sym}^2(\mathbf{f})\big)^{\bullet}\right)$ coincide after dividing through by the unit $\mathcal{W}^{(\varepsilon_0)}(M)\in\mathcal{O}_{T,p}[\Delta^+][\![\gamma-1]\!]^{\times}$. $\blacksquare$

Remark 14. If $F=\mathbb{Q}$ then the integrality of $\mathfrak{h}_{\mathbf{f}}\cdot D_p^{\pm}\big(\mathrm{Sym}^2(\mathbf{f})\big)$ has been shown in [RSV23, Theorem 1.4], and therefore one acquires the integrality of $\mathfrak{h}_{\mathbf{f}}\cdot L_p^{\pm}\big(\mathrm{Sym}^2(\mathbf{f})\big)$ for free. If $F\neq\mathbb{Q}$ then it is conjectured that $\mathfrak{h}_{\mathbf{f}}\cdot L_p^{\pm}\big(\mathrm{Sym}^2(\mathbf{f})\big)$ has non-negative μ-invariant.

7.3.4 The triple product p-adic L-function

The final example features another tensor product of Hilbert modular newforms. Let $\mathbf{f}_i\in\mathcal{S}_{k_i}(\mathfrak{c}(\mathbf{f}_i),\eta_i)$ for $i=1,2,3$ satisfying $k_1\geq k_2\geq k_3$. The motive $M=M(\mathbf{f}_1\otimes\mathbf{f}_2\otimes\mathbf{f}_3)$ associated to the triple product is given by the tensor product of motives $M(\mathbf{f}_1)\otimes M(\mathbf{f}_2)\otimes M(\mathbf{f}_3)$ with rank eight and weight $w=k_1+k_2+k_3-3$.

The *imprimitive L-function* for the triple product $\mathbf{f}_1\times\mathbf{f}_2\times\mathbf{f}_3$ is given by

$$D(\mathbf{f}_1,\mathbf{f}_2,\mathbf{f}_3,s)=\prod_{\mathfrak{q}}\prod_{\Upsilon_{\mathfrak{q}}\in\{\alpha_{\mathfrak{q}},\beta_{\mathfrak{q}}\}}\big(1-\Upsilon_{\mathfrak{q}}(\mathbf{f}_1)\Upsilon_{\mathfrak{q}}(\mathbf{f}_2)\Upsilon_{\mathfrak{q}}(\mathbf{f}_3)\mathcal{N}(\mathfrak{q})^{-s}\big)^{-1} \qquad (7.21)$$

where $X^2-\mathrm{c}(\mathbf{f}_i)\mathcal{N}(\mathfrak{v})^{-s}+\eta_i(\mathfrak{v})\mathcal{N}(\mathfrak{v})^{-2s}=(X-\alpha_{\mathfrak{v}}(\mathbf{f}_i))(X-\beta_{\mathfrak{v}}(\mathbf{f}_i))$ for each i with $\alpha_{\mathfrak{v}}(\mathbf{f}_i)$ and $\beta_{\mathfrak{v}}(\mathbf{f}_i)$ following the convention laid down in the previous cases. Meanwhile, the *primitive L-function* is the infinite product given by

$$L(\mathbf{f}_1\otimes\mathbf{f}_2\otimes\mathbf{f}_3,s)=\prod_{\mathfrak{v}}\det\left(1-\mathrm{Frob}_{\mathfrak{v}}^{-1}\,\Bigg|\,\left(\bigotimes_{i=1}^{3}V_\ell(\mathbf{f}_i)\right)^{I_v}\cdot\mathcal{N}(\mathfrak{v})^{-s}\right)^{-1},\quad \ell\neq\mathfrak{v}. \qquad (7.22)$$

We assume in this thesis chapter the triple product $\mathbf{f}_1 \times \mathbf{f}_2 \times \mathbf{f}_3$ *is balanced, meaning* $k_1 + k_2 + k_3 > 2k^*$, *where* $k^* = \max\{k_i : i = 1, 2, 3\}$. This is equivalent to $k_2 + k_3 > k_1$ by our assumption on the weights. In fact, we impose a slightly stronger condition $k_2 + k_3 \geq k_1 + 2$ so that the critical strip $\{k_1, k_1 + 1, \ldots, k_2 + k_3 - 2\}$ is non-empty. In this case, the factor at infinity is given by $L_\infty(\mathbf{f}_1 \otimes \mathbf{f}_2 \otimes \mathbf{f}_3, s) = \Gamma_\mathbb{C}(s + \frac{k_1+k_2+k_3-2}{2})^n \times \prod_{i=1}^{3} \Gamma_\mathbb{C}(s + k_i^*)^n$, where $k_i^* = \frac{k_1+k_2+k_3}{2} - k_i$, for each i and the completed L-function is defined as

$$\Psi(\mathbf{f}_1 \otimes \mathbf{f}_2 \otimes \mathbf{f}_3, s) = L_\infty(\mathbf{f}_1 \otimes \mathbf{f}_2 \otimes \mathbf{f}_3, s) \times L(\mathbf{f}_1 \otimes \mathbf{f}_2 \otimes \mathbf{f}_3, s) \,. \tag{7.23}$$

The analytic properties of the triple tensor product has been studied in [PSR87], namely $\Psi(\mathbf{f}_1 \otimes \mathbf{f}_2 \otimes \mathbf{f}_3, s)$ converges for $\mathrm{Re}(s) \gg 0$, it is meromorphic on $\mathbb{C}$ possessing only a finite number of poles, and it has the functional equation

$$\Psi(\mathbf{f}_1 \otimes \mathbf{f}_2 \otimes \mathbf{f}_3, s) = \epsilon(\mathbf{f}_1 \otimes \mathbf{f}_2 \otimes \mathbf{f}_3, s) \times \Psi\big(\mathbf{f}_1^\rho \otimes \mathbf{f}_2^\rho \otimes \mathbf{f}_3^\rho, k_1 + k_2 + k_3 - 2 - s\big) \,. \tag{7.24}$$

For technical reasons, we also suppose each character η_j is a power of ω_F.

Conjecture 7.9. *There exists a meromorphic function* $L_p(\mathbf{f}_1 \otimes \mathbf{f}_2 \otimes \mathbf{f}_3)$ *defined on the whole of* $\mathfrak{X}_p$, *uniquely determined by the data*

$$\begin{aligned} \chi \mathcal{N} x_p^m \big(L_p(\mathbf{f}_1 \otimes \mathbf{f}_2 \otimes \mathbf{f}_3)\big) &= \mathcal{B}(\mathbf{f}_1, \mathbf{f}_2, \mathbf{f}_3, \chi, m) \times \prod_{\mathfrak{p}|p} \mathcal{A}_\mathfrak{p}(M, \chi, k_1 + m) \\ &\quad \times \frac{\Psi(\mathbf{f}_1 \otimes \mathbf{f}_2 \otimes \mathbf{f}_3, \chi, k_1 + m)}{\Omega(\mathbf{f}_1) \cdot \Omega(\mathbf{f}_2) \cdot \Omega(\mathbf{f}_3)} \end{aligned}$$

at every Tate twist $m \in \{0, \ldots, k_2 + k_3 - k_1 - 2\}$ *and finite order character* $\chi \in \mathcal{X}_p$. *Furthermore, it should satisfy the p-adic functional equation*

$$\begin{aligned} \chi \mathcal{N} x_p^m \big(L_p(\mathbf{f}_1 \otimes \mathbf{f}_2 \otimes \mathbf{f}_3)\big) &= w_p\big(M(\mathbf{f}_1 \otimes \mathbf{f}_2 \otimes \mathbf{f}_3)(\chi), m\big) \\ &\quad \times \chi^{-1} \mathcal{N} x_p^{k_2+k_3-k_1-m-2} \big(L_p(\mathbf{f}_1^\rho \otimes \mathbf{f}_2^\rho \otimes \mathbf{f}_3^\rho)\big) \,. \end{aligned}$$

Remark 15.

1. The above is just a restatement of Conjecture 3.9 for the triple product motive $M = M(\mathbf{f}_1 \otimes \mathbf{f}_2 \otimes \mathbf{f}_3)$ together with a functional equation.

2. Hsieh and Yamana have already proven Conjecture 7.9 when $F = \mathbb{Q}$ and the $\mathrm{lcm}\big(\mathfrak{c}(\mathbf{f}_1), \mathfrak{c}(\mathbf{f}_2), \mathfrak{c}(\mathbf{f}_3)\big)$ is a square-free integer [HY24, §7.6].

3. Provided Conjecture 7.9 holds, if the residual Galois representations attached to each newform $\mathbf{f}_i$ are absolutely irreducible and p-distinguished, it follows from Corollary 7.9 in *op. cit.* that $\mathfrak{h}_{\mathbf{f}_1}\mathfrak{h}_{\mathbf{f}_2}\mathfrak{h}_{\mathbf{f}_3} \cdot L_p(\mathbf{f}_1 \otimes \mathbf{f}_2 \otimes \mathbf{f}_3)$ is both analytic on $\mathfrak{X}_p$ and p-integral (here $\mathfrak{h}_{\mathbf{f}_i}$ is the congruence number associated to $\mathbf{f}_i$).

Let $D_p(\mathbf{f}_1, \mathbf{f}_2, \mathbf{f}_3)$ denote the (conjectural) triple product p-adic L-function interpolating the same arithmetic data as $L_p(\mathbf{f}_1 \otimes \mathbf{f}_2 \otimes \mathbf{f}_3)$, with the caveat that $L\big(\mathbf{f}_1 \otimes \mathbf{f}_2 \otimes \mathbf{f}_3, \chi, -\big)$ is replaced by its imprimitive version $D(\mathbf{f}_1, \mathbf{f}_2, \mathbf{f}_3, \chi, -\big)$. Our goal in this section is to deduce holomorphy and integrality properties of $L_p(\mathbf{f}_1 \otimes \mathbf{f}_2 \otimes \mathbf{f}_3)$ from that of its imprimitive cousin, which means we must verify Hypothesis (S^{dep}) for an appropriate set S. Unfortunately the calculation of the bad Euler factors for the triple product is extremely fiddly, forcing us to make a simplifying assumption.

Definition 7.3. We say that $\big(\mathbf{f}_1, \mathbf{f}_2, \mathbf{f}_3\big)$ is an *M-smooth triple* if for every finite prime $\mathfrak{v} \mid \mathfrak{c}(\mathbf{f}_1) \cdot \mathfrak{c}(\mathbf{f}_2) \cdot \mathfrak{c}(\mathbf{f}_3)$, at least one of the $\pi_{F_\mathfrak{v}}(\mathbf{f}_i)$'s is a $\vartheta_\mathfrak{v}$-twist of an unramified principal series representation, and where the Hecke polynomial of $\mathbf{f}_i \otimes \vartheta_\mathfrak{v}$ has roots $\alpha_\mathfrak{v}(\mathbf{f}_i \otimes \vartheta_\mathfrak{v})$ and $\beta_\mathfrak{v}(\mathbf{f}_i \otimes \vartheta_\mathfrak{v})$ satisfying $\big|\alpha_\mathfrak{v}(\mathbf{f}_i \otimes \vartheta_\mathfrak{v})\big|_\infty = \big|\beta_\mathfrak{v}(\mathbf{f}_i \otimes \vartheta_\mathfrak{v})\big|_\infty = \mathcal{N}\big(\mathfrak{v}^{(k_i-1)/2}\big)$.

Proposition 7.10. *For an M-smooth triple* $(\mathbf{f}_1, \mathbf{f}_2, \mathbf{f}_3)$*, Hypothesis (*$S^{\mathrm{dep}}$*) holds true for the motive* $M = M\big(\mathbf{f}_1 \otimes \mathbf{f}_2 \otimes \mathbf{f}_3\big)$ *provided:*

- *Wwe replace* $\big(\mathbf{f}_1, \mathbf{f}_2, \mathbf{f}_3\big)$ *with a certain twisted triple*$\big(\mathbf{f}_1 \otimes \varepsilon_1', \mathbf{f}_2 \otimes \varepsilon_2', \mathbf{f}_3 \otimes \varepsilon_3'\big)$ *where* $\varepsilon_1'\varepsilon_2'\varepsilon_3'$ *is trivial; and*
- *We take S to be the set consisting of bad primes $\mathfrak{v}$ where one of the Hilbert modular newforms $\mathbf{f}_i$ is a twist of an unramified principal series at $\mathfrak{v}$, and the other HMFs are either special or supercuspidal at $\mathfrak{v}$.*

Proof. Let $\mathfrak{v} \in \Sigma^{\mathrm{bad}}(\mathbf{f})$. As $\big(\mathbf{f}_1, \mathbf{f}_2, \mathbf{f}_3\big)$ is M-smooth, without loss of generality one may assume that $\mathbf{f}_3 \otimes \vartheta_\mathfrak{v}$ is principal series and unramified. Consequently,

$$\big(V_\ell(\mathbf{f}_1) \otimes V_\ell(\mathbf{f}_2) \otimes V_\ell(\mathbf{f}_3)\big)^{I_{F_\mathfrak{v}}} \cong \big(V_\ell(\mathbf{f}_1) \otimes V_\ell(\mathbf{f}_2) \otimes \vartheta_\mathfrak{v}^{-1}\big)^{I_{F_\mathfrak{v}}} \otimes \big(V_\ell(\mathbf{f}_3) \otimes \vartheta_\mathfrak{v}\big)$$

in which case the polynomial $L_{\mathfrak{v}}\big(\mathbf{f}_1\otimes\mathbf{f}_2\otimes\mathbf{f}_3, X\big)$ coincides with the L-factor

$$L_{\mathfrak{v}}\big(\mathbf{f}_1\otimes\mathbf{f}_2\otimes\vartheta_{\mathfrak{v}}^{-1},\ \alpha_{\mathfrak{v}}(\mathbf{f}_3\otimes\vartheta_{\mathfrak{v}})\cdot X\big)\times L_{\mathfrak{v}}\big(\mathbf{f}_1\otimes\mathbf{f}_2\otimes\vartheta_{\mathfrak{v}}^{-1},\ \beta_{\mathfrak{v}}(\mathbf{f}_3\otimes\vartheta_{\mathfrak{v}})\cdot X\big)$$

where $\big|\alpha_{\mathfrak{v}}(\mathbf{f}_3\otimes\vartheta_{\mathfrak{v}})\big|_\infty=\big|\beta_{\mathfrak{v}}(\mathbf{f}_3\otimes\vartheta_{\mathfrak{v}})\big|_\infty=\mathcal{N}\big(\mathfrak{v}^{(k_3-1)/2}\big)$ under our hypothesis.

We adopt the notations in Section 7.3.2 for the twisted pair $\big(\mathbf{f}_1,\mathbf{f}_2\otimes\vartheta_{\mathfrak{v}}^{-1}\big)$. If $\mathfrak{v}\in\Sigma^{\mathrm{bad}}_{(\mathrm{I})}\cup\Sigma^{\mathrm{bad}}_{(\mathrm{II})}\cup\Sigma^{\mathrm{bad}}_{(\mathrm{III})}\cup\Sigma^{\mathrm{bad}}_{(\mathrm{IV})}$ then studying the proof of Proposition 7.5,

$$L_{\mathfrak{v}}\big(\mathbf{f}_1\otimes\mathbf{f}_2\otimes\vartheta_{\mathfrak{v}}^{-1},X\big)=D_{\mathfrak{v}}\big(\mathbf{f}_1\otimes\varepsilon_{\mathfrak{v}},\mathbf{f}_2\otimes(\varepsilon_{\mathfrak{v}}\vartheta_{\mathfrak{v}})^{-1},X\big)$$

therefore $L_{\mathfrak{v}}\big(\mathbf{f}_1\otimes\mathbf{f}_2\otimes\mathbf{f}_3,X\big)=D_{\mathfrak{v}}\big(\mathbf{f}_1\otimes\varepsilon_{\mathfrak{v}},\mathbf{f}_2\otimes(\varepsilon_{\mathfrak{v}}\vartheta_{\mathfrak{v}})^{-1},\mathbf{f}_3\otimes\vartheta_{\mathfrak{v}},X\big)$ and one can ignore these primes entirely (i.e. we do not need to include them within the set S). In this situation, one would define the local characters $\varepsilon'_{1,\mathfrak{v}}:=\varepsilon_{\mathfrak{v}}$, $\varepsilon'_{2,\mathfrak{v}}:=(\varepsilon_{\mathfrak{v}}\vartheta_{\mathfrak{v}})^{-1}$ and $\varepsilon'_{3,\mathfrak{v}}:=\vartheta_{\mathfrak{v}}$, so that the product $\varepsilon'_{1,\mathfrak{v}}\varepsilon'_{2,\mathfrak{v}}\varepsilon'_{3,\mathfrak{v}}=\varepsilon_{\mathfrak{v}}\cdot(\varepsilon_{\mathfrak{v}}\vartheta_{\mathfrak{v}})^{-1}\cdot\vartheta_{\mathfrak{v}}$ is trivial.

Alternatively, if $\mathfrak{v}\in\Sigma^{\mathrm{bad}}_{(\mathrm{V})}\cup\Sigma^{\mathrm{bad}}_{(\mathrm{VI})}$ for the twisted pair $\big(\mathbf{f}_1,\mathbf{f}_2\otimes\vartheta_{\mathfrak{v}}^{-1}\big)$ then either both of the representations $\pi_{F_{\mathfrak{v}}}(\mathbf{f}_1)$ and $\pi_{F_{\mathfrak{v}}}(\mathbf{f}_2)$ are special series so that

$$L_{\mathfrak{v}}\big(\mathbf{f}_1\otimes\mathbf{f}_2\otimes\vartheta_{\mathfrak{v}}^{-1},X\big)=\big(1-\mathfrak{U}'_{\mathfrak{v}}\cdot X\big)\big(1-\mathcal{N}(\mathfrak{v})\cdot\mathfrak{U}'_{\mathfrak{v}}\cdot X\big)$$

with $\big|\mathfrak{U}'_{\mathfrak{v}}\big|_\infty=\mathcal{N}(\mathfrak{v})^{\frac{1}{2}(k_1+k_2)-2}$, or both are supercuspidal at $\mathfrak{v}$, in which case:

(i) $L_{\mathfrak{v}}\big(\mathbf{f}_1\otimes\mathbf{f}_2\otimes\vartheta_{\mathfrak{v}}^{-1},X\big)=1$, or alternatively

(ii) $L_{\mathfrak{v}}\big(\mathbf{f}_1\otimes\mathbf{f}_2\otimes\vartheta_{\mathfrak{v}}^{-1},X\big)=\big(1-\mathfrak{U}'_{\mathfrak{v}}\cdot X\big)$ with $\big|\mathfrak{U}'_{\mathfrak{v}}\big|_\infty=\mathcal{N}(\mathfrak{v})^{\frac{1}{2}(k_1+k_2)-1}$.

It follows that $L_{\mathfrak{v}}\big(\mathbf{f}_1\otimes\mathbf{f}_2\otimes\mathbf{f}_3,X\big)$ is composed of a product of linear terms $\big(1-\mathfrak{U}_{\mathfrak{v}}\cdot X\big)$ where $\big|\mathfrak{U}_{\mathfrak{v}}\big|_\infty$ is equal to $\mathcal{N}(\mathfrak{v})^{\frac{1}{2}(k_1+k_2+k_3-5)}$ or $\mathcal{N}(\mathfrak{v})^{\frac{1}{2}(k_1+k_2+k_3-3)}$. *N.B.* The same reasoning works fine if any one of $\mathbf{f}_1\otimes\vartheta_{\mathfrak{v}}$ or $\mathbf{f}_2\otimes\vartheta_{\mathfrak{v}}$ are principal series and unramified, one just has to permute the indices appropriately.

Let us next define global characters $\varepsilon'_i:=\prod_{\mathfrak{v}\in\Sigma^{\mathrm{bad}}(M)}\varepsilon'_{i,\mathfrak{v}}$ for each $i=1,2,3$. From the above discussion and the choice of set S, one readily computes that

$$L_{(S)}\big(\mathbf{f}_1\otimes\mathbf{f}_2\otimes\mathbf{f}_3,s\big)=D_{(S)}\big(\mathbf{f}_1\otimes\varepsilon'_1,\mathbf{f}_2\otimes\varepsilon'_2,\mathbf{f}_3\otimes\varepsilon'_3,s\big).$$

We are now in a good position to check whether the Euler factors at primes in S arising in the p-adic functional equation (at a fixed branch ϕ) are coprime.

Suppose that a zero of $\mathscr{L}^{\phi}_{\mathfrak{v}}\big(M;\kappa_{\mathrm{cyc}}(\gamma)^{-r}\cdot\gamma\big)$ *is the same as a zero of the dual factor* $\mathscr{L}^{\phi^{-1}}_{\mathfrak{q}}\big(M^{\vee}(1);\kappa_{\mathrm{cyc}}(\gamma)^{r}\cdot\gamma^{-1}\big)$*, i.e.* $\mathbf{x}^{\phi}_{\mathfrak{v}}=\mathbf{x}^{\phi^{-1}}_{\mathfrak{q}}$ *for some* $\mathfrak{v},\mathfrak{q}\in\Sigma^{\mathrm{bad}}_{(\mathrm{V})}\cup\Sigma^{\mathrm{bad}}_{(\mathrm{VI})}$. Then by a similar argument used in Sections 7.3.1–7.3.3, one concludes that

$$\kappa_{\mathrm{cyc}}(\gamma)^{r-e_{\mathfrak{v}}}=\kappa_{\mathrm{cyc}}(\gamma)^{r+e_{\mathfrak{q}}+2-k_1-k_2-k_3},\quad \text{i.e.}\quad e_{\mathfrak{v}}+e_{\mathfrak{q}}=k_1+k_2+k_3-2.$$

However this is not possible since $e_{\mathfrak{v}},e_{\mathfrak{q}}\in\big\{\frac{1}{2}(k_1+k_2+k_3-5),\frac{1}{2}(k_1+k_2+k_3-3)\big\}$ *in which case* $\mathbf{x}^{\phi}_{\mathfrak{v}}\neq\mathbf{x}^{\phi^{-1}}_{\mathfrak{q}}$, and so we conclude that Hypothesis (S^{dep}) holds when $S=\Sigma^{\mathrm{bad}}_{(\mathrm{V})}\cup\Sigma^{\mathrm{bad}}_{(\mathrm{VI})}$. ■

Corollary 7.11. *For an M-smooth triple* $(\mathbf{f}_1,\mathbf{f}_2,\mathbf{f}_3)$ *as above, the holomorphy (resp. integrality) of* $\mathfrak{h}_{\mathbf{f}_1}\mathfrak{h}_{\mathbf{f}_2}\mathfrak{h}_{\mathbf{f}_3}\cdot L_p(\mathbf{f}_1\otimes\mathbf{f}_2\otimes\mathbf{f}_3)$ *follows directly from the holomorphy (resp. integrality) of* $\mathfrak{h}_{\mathbf{f}_1}\mathfrak{h}_{\mathbf{f}_2}\mathfrak{h}_{\mathbf{f}_3}\cdot D_p(\mathbf{f}_1\otimes\varepsilon_1',\mathbf{f}_2\otimes\varepsilon_2',\mathbf{f}_3\otimes\varepsilon_3')$ *in our preceding notation.*

Proof. The argument uses identical reasoning to Corollaries 7.6 and 7.8. ■

This corollary puts us in the rather unusual position of being able to deduce several nice analytic properties of the primitive triple product p-adic L-function (in the balanced case) from the analytic properties of an imprimitive version that has not been constructed, and whose existence is purely conjectural!

Although the definition of an M-smooth triple excludes the possibility all three HMFs are Steinberg at $\mathfrak{v}$, this scenario has been treated by Hsieh and Yamana for $F=\mathbb{Q}$ in their work on the exceptional zero conjecture for elliptic curves [HY24]. While the case where all three Hilbert modular forms are supercuspidal at $\mathfrak{v}$ is currently beyond us, we are able to at least give a description of the local Euler factors in the case where each $\mathbf{f}_i$ arises from a modular elliptic curve E_i defined over a totally real number field.

7.3.5 Euler factors of triple product of elliptic curves

Let us fix a prime number $r\neq p$ and for simplicity let us take E_i to be a rational elliptic curve for each i. Let $\ell\neq p,r$ be another prime and recall the ℓ-adic Galois representation given by $\rho_{E_i,\ell}:G_{\mathbb{Q}}\to\mathrm{Aut}(T_\ell(E_i))\cong\mathrm{GL}_2(\mathbb{Z}_\ell)$. We shall write $V_i := T_\ell(E_i) \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell$ for the corresponding two-dimensional vector space over $\mathbb{Q}_\ell$. The objective of this section is to extend the procedure in [CS87] by Coates and Schmidt in order to compute

$$L_r(E_1 \otimes E_2 \otimes E_3, s) = \det\left(\mathrm{Id} - \mathrm{Frob}_r^{-1} \cdot r^{-s} \,\middle|\, \big(V_1 \otimes V_2 \otimes V_3\big)^{I_r}\right). \tag{7.25}$$

Case I. *At least one elliptic curve has good ordinary reduction at ℓ.*

Say E_1 has good ordinary reduction at ℓ. This amounts to the representation $\rho_{E_1,\ell}$ being unramified at ℓ, implying that $V_1^{I_r} = V_1$. Therefore, one can deduce that $\big(V_1 \otimes V_2 \otimes V_3\big)^{I_r} = V_1 \otimes (V_2 \otimes V_3)^{I_r}$, and one can then use the same technique applied in the single and the double product case in order to compute for (7.25). The same method applies for the rest of the subcases.

Case II. *The j-invariant of each E_i has a non-integral ℓ-adic valuation, i.e., $\mathrm{ord}_\ell(j(E_i)) < 0$ for each i.*

This is the case of *potential multiplicative reduction.* In this scenario there is a unique $q_i \in \mathbb{Q}^\times$ with $|q_i|_\ell < 1$ producing an isomorphism $E_i \cong \mathbb{G}_m/q_i^{\mathbb{Z}} =: E_{q_i}$ over $\overline{\mathbb{Q}}_\ell$ (c.f. [Sil94, Theorem 5.3]). In fact, this isomorphism occurs over $\mathbb{Q}$ if E_i has split multiplicative reduction and over an unramified (resp. ramified) quadratic extension $K/\mathbb{Q}$ if E_i has non-split multiplicative (resp. additive) reduction at ℓ.

Most of Case II has been dealt with by Hsieh and Yamana in [HY24, Remark 3.5] by explicit computation of local zeta-integrals. The remaining scenario is when each of the E_i's are additive. By applying the appropriate quadratic twists however, the situation becomes that of one additive and two split multiplicative elliptic curves. One then uses linear algebra to conclude that the dimension of $\big(V_1 \otimes V_2 \otimes V_3\big)^{I_r}$ is either 0 or 4.

Case III. *Suppose the j-invariant of all three elliptic curves have integral ℓ-adic valuation. This puts us in the case of potential good reduction at ℓ.*

Let $m \geq 3$ be an integer not divisible by p. If E_i has potential good reduction at the prime ℓ, the inertia group I_q acts on $T_\ell(E_i)$ through a finite quotient $\mathrm{Gal}\big(\overline{\mathbb{Q}}_r/\mathbb{Q}_r\big) \to \mathrm{Gal}\big(\mathbb{Q}_r(E_i[m])/\mathbb{Q}_r\big) \to \mathrm{Aut}(T_\ell(E_i))$. Let us denote by Φ_r the inertia subgroup of $\mathrm{Gal}\big(\mathbb{Q}_r(E_i[m])/\mathbb{Q}_r\big)$. The structure of Φ_r does not depend on m and is limited to the following:

- If $r \geq 3$, then Φ_r is cyclic of order 2, 3, 4 or 6;
- If $r = 2$ or 3 and Φ_r is abelian, then it is cyclic of order 2, 3, 4 or 6;
- If $r = 3$ and Φ_r is non-abelian, then it is isomorphic to the semi-direct product $\mathbb{Z}/3\mathbb{Z} \ltimes \mathbb{Z}/4\mathbb{Z}$; and
- If $r = 2$ and Φ_r is non-abelian, then it is either isomorphic to Q_8 the quaternion group of order 8 or $\mathrm{SL}_2(\mathbb{F}_3)$.

In fact, all the enumerated possibilities have been shown to occur [Ser72, §5].

The above list renders three subcases directly below Case III to consider.

(A) *Suppose the inertia subgroup Φ_r acts on $T_\ell(E_i)$ through a group of order 2 for at least one E_i.* This means that one can take an appropriate quadratic twist of E_i so that it has good reduction at ℓ, which takes us back to Case I.

(B) *Suppose Φ_r acts on $T_\ell(E_i)$ through a cyclic group of order $d_i \geq 3$ generated by τ_i, for each E_i.* To tackle this case, we need the following well-known fact.

Lemma 7.12. *Let us suppose that the inertia subgroup Φ_r is cyclic of order d and write $\overline{V_i} = V_i \otimes \overline{\mathbb{Q}}_\ell$. Then one obtains the following decomposition*

$$\overline{V_i} = \overline{V_i}(\zeta_i) \oplus \overline{V_i}(\zeta_i^{-1}) \tag{7.26}$$

satisfying $\dim_{\overline{\mathbb{Q}}_\ell}(\overline{V_i}(\zeta)) = \dim_{\overline{\mathbb{Q}}_\ell}(\overline{V_i}(\zeta^{-1})) = 1$, for ζ a primitive d-th root of unity. The generator τ_i acts on $\overline{V_i}(\zeta)$ and $\overline{V_i}(\zeta^{-1})$ via ζ and ζ^{-1}, respectively.

For each i, let u_i and w_i be a basis for $\overline{V_i}(\zeta_i)$ and $\overline{V_i}(\zeta_i)$, respectively. The action described above naturally extends to the triple tensor product $\overline{V_1} \otimes \overline{V_2} \otimes \overline{V_3}$. For instance, take $\tau \in I_r$. Then one computes

$$\begin{aligned}\tau(u_1 \otimes w_2 \otimes u_3) &= \tau_1(u_1) \otimes \tau_2(w_2) \otimes \tau_3(u_3)\\ &= (\zeta_1 \cdot u_1) \otimes (\zeta_2^{-1} \cdot w_2) \otimes (\zeta_3 \cdot u_3)\\ &= \zeta_1\zeta_2^{-1}\zeta_3 \cdot (u_1 \otimes w_2 \otimes u_3)\,.\end{aligned}$$

In other words, the action of I_r on $\overline{V}_1 \otimes \overline{V}_2 \otimes \overline{V}_3$ is characterised as follows:

$$\overline{\mathbb{Q}}_\ell(\zeta_1\zeta_2\zeta_3) \oplus \overline{\mathbb{Q}}_\ell(\zeta_1\zeta_2\zeta_3^{-1}) \oplus \overline{\mathbb{Q}}_\ell(\zeta_1\zeta_2^{-1}\zeta_3) \oplus \overline{\mathbb{Q}}_\ell(\zeta_1^{-1}\zeta_2\zeta_3)$$
$$\oplus\ \overline{\mathbb{Q}}_\ell(\zeta_1\zeta_2^{-1}\zeta_3^{-1}) \oplus \overline{\mathbb{Q}}_\ell(\zeta_1^{-1}\zeta_2\zeta_3^{-1}) \oplus \overline{\mathbb{Q}}_\ell(\zeta_1^{-1}\zeta_2^{-1}\zeta_3) \oplus \overline{\mathbb{Q}}_\ell(\zeta_1^{-1}\zeta_2^{-1}\zeta_3^{-1})$$

which suggests that the dimension of the I_r-invariant subgroup of the triple tensor product is equal to the number of components above that are isomorphic to $\overline{\mathbb{Q}}_\ell$ as local Galois modules.

One can easily verify that only when $\{d_1, d_2, d_3\} = \{3, 3, 3\}$ or $\{6, 6, 3\}$, up to permutation, do we get a non-trivial I_r-invariant subspace. Moreover, in both cases, the dimension is either 0 or 2. To compute for the Euler factor explicitly, we turn our attention to the larger field $\mathrm{Gal}\big(\mathbb{Q}_r(E[\ell^\infty])/\mathbb{Q}_r\big)$. Let $\sigma_i \in \mathrm{Gal}\big(\mathbb{Q}_r(E_i[\ell^\infty])/\mathbb{Q}_r\big)$ be an element mapping onto the geometric Frobenius element Frob_r^{-1} in the decomposition group D_r. By [ST68, p. 499], there exist $\alpha_r(E_i), \beta_r(E_i) \in \mathbb{C}$ with absolute value $r^{1/2}$ such that

$$\det\big(\mathrm{Id} - \sigma_i X \,\big|\, \overline{V}_i\big) = \big(1 - \alpha_r(E_i)X\big)\big(1 - \beta_r(E_i)X\big)\,.$$

(i) *If* $\mathbb{Q}_r(E_i[\ell])/\mathbb{Q}_r$ *is abelian for all* i, then it follows from [CS87, Lemma 1.7] that $\mathbb{Q}_r(E_i[\ell^\infty])/\mathbb{Q}_r$ is also abelian. Consequently, $\mathrm{Gal}\big(\mathbb{Q}_r(E_i[\ell^\infty])/\mathbb{Q}_r\big)$ is generated by both σ_i and τ_i, and these generators commute by assumption. Hence, σ_i respects decomposition (7.26) and its action described as follows:

$$\sigma_i(u_i) = \alpha_\ell(E_i)\cdot u_i \quad \text{and} \quad \sigma_i(w_i) = \beta_\ell(E_i)\cdot w_i\,.$$

For ease of computation we choose the primitive roots of unity in (7.26) so that τ fixes the generators $u_1 \otimes u_2 \otimes u_3$ and $w_1 \otimes w_2 \otimes w_3$. Then $L_\ell(E_1 \otimes E_2 \otimes E_3, X) = \big(1 - \prod_{i=1}^3 \alpha_\ell(E_i)\cdot X\big)\big(1 - \prod_{i=1}^3 \beta_\ell(E_i)\cdot X\big)$.

(ii) *If* $\mathbb{Q}_r(E_i[\ell])/\mathbb{Q}_r$ *is non-abelian for all* i, then again by [CS87, Lemma 1.7] it follows that the extension $\mathbb{Q}_r(E_i[\ell^\infty])/\mathbb{Q}_r$ is also non-abelian. In this scenario, σ_i and τ_i do not commute. However, the fact that Φ_r is a normal subgroup of $\mathrm{Gal}\big(\mathbb{Q}_r(E_i[\ell^\infty])/\mathbb{Q}\big)$ gives us the relation

$$\sigma_i\tau_i\sigma_i^{-1} = \tau^{-1}$$

and we deduce that σ_i acts by switching the subspaces $\overline{V}_i(\zeta_i)$ and $\overline{V}_i(\zeta_i^{-1})$.

In particular, we can choose

$$\sigma_i(u_i) = \alpha_\ell(E_i) \cdot w_i \quad \text{and} \quad \sigma_i(w_i) = -\beta_\ell(E_i) \cdot u_i.$$

Hence $L_\ell(E_1 \otimes E_2 \otimes E_3, X) = 1 + \big(\prod_{i=1}^3 \alpha_\ell(E_i)\big)\big(\prod_{i=1}^3 \beta_\ell(E_i)\big) \cdot X^2$.

(C) *Suppose Φ_r acts on $T_\ell(E_i)$ as a non-cyclic group for each E_i.* In particular, $\Phi_r \cong \mathbb{Z}/3\mathbb{Z} \ltimes \mathbb{Z}/4\mathbb{Z}$, Q_8 or $\mathrm{SL}_2(\mathbb{F}_3)$, each of which possess a cyclic normal subgroup isomorphic to either $\mathbb{Z}/3\mathbb{Z}$ or $\mathbb{Z}/4\mathbb{Z}$.

Let us for a moment hone in on a given index i. The following table summarises the group properties of $\mathbb{Z}/3\mathbb{Z} \ltimes \mathbb{Z}/4\mathbb{Z}$ and Q_8 provided τ_i is the generator of the respective cyclic normal subgroup.

Group	Generators	Properties
$\mathbb{Z}/3\mathbb{Z} \ltimes \mathbb{Z}/4\mathbb{Z}$	τ_i, λ_i	$\lambda_i\tau_i = \tau_i^2\lambda_i$
Q_8	τ_i, ϱ_i	$\tau_i^4 = 1, \tau_i^2 = \varrho_i^2, \varrho_i\tau_i = \tau_i^3\varrho_i$

Once again we have a decomposition (7.26) of $\overline{V}_1$ induced by the action of τ_i. The table above implies that λ_i and ϱ_i interchange the subspaces $\overline{V}_i(\zeta_i)$ and $\overline{V}_i(\zeta_i^{-1})$, which suggests that

$$\begin{aligned} &\lambda_i(u_i) = w_i, \quad \lambda_i(w_i) = -u_i, \ \text{or} \\ &\varrho_i(u_i) = w_i, \quad \varrho_i(w_i) = -u_i\,. \end{aligned}$$

By a simple (yet tedious) linear algebra argument, one can check that the I_r-invariant subspace in this case is trivial.

Case IV. *Suppose Φ_r acts as a non-cyclic group for at least one E_i.* We summarise our findings in the table below. The symbol PM denotes potential multiplicative reduction. The symbol PG on the other hand denotes potential good reduction, and the subscripts B1, B2, C signify cyclic abelian, cyclic non-abelian and non-cyclic following the labelling above. As above, the notation $\{-,-,-\}$ includes all possible permutations of the triple inside while the second column labelled dimension represents the value $\dim_{\overline{\mathbb{Q}}_\ell}\big((\overline{V}_1 \otimes \overline{V}_2 \otimes \overline{V}_3)^{I_r}\big)$.

Table 7.1: Euler factors of triple product of elliptic curves belonging to Case IV.

$\{-,-,-\}$	dimension	$L_\ell(E_1 \otimes E_2 \otimes E_3, X)$
$\{\mathrm{PG}_{B1}, \mathrm{PM}, \mathrm{PM}\}$	0	1
$\{\mathrm{PG}_{B2}, \mathrm{PM}, \mathrm{PM}\}$	0	1
$\{\mathrm{PG}_{C}, \mathrm{PM}, \mathrm{PM}\}$	0	1
$\{\mathrm{PG}_{B1}, \mathrm{PG}_{B1}, \mathrm{PM}\}$	0 or 2	$\big(1-\alpha_1\alpha_2\beta_3\cdot X\big)\big(1-\beta_1\beta_2\beta_3\cdot X\big)$
$\{\mathrm{PG}_{B2}, \mathrm{PG}_{B2}, \mathrm{PM}\}$	0 or 2	$\big(1-(\alpha_1\alpha_2\beta_3)(\beta_1\beta_2\beta_3)\cdot X^2\big)$
$\{\mathrm{PG}_{C}, \mathrm{PG}_{C}, \mathrm{PM}\}$	0 or 2	$\big(1-\mathfrak{U}_\ell\cdot r^{3/2}\cdot X\big)^2$
$\{\mathrm{PG}_{B1}, \mathrm{PG}_{B2}, \mathrm{PM}\}^\dagger$	0 or 2	$\big(1+\mathfrak{U}_\ell\cdot r^{3}\cdot X^2\big)$
$\{\mathrm{PG}_{B1}, \mathrm{PG}_{C}, \mathrm{PM}\}^\dagger$	0 or 2	$\big(1-\mathfrak{U}_\ell\cdot r^{3/2}\cdot X\big)^2$
$\{\mathrm{PG}_{B2}, \mathrm{PG}_{C}, \mathrm{PM}\}$	0 or 2	$\big(1+\mathfrak{U}_\ell\cdot r^{3}\cdot X^2\big)$
$\{\mathrm{PG}_{B1}, \mathrm{PG}_{B1}, \mathrm{PG}_{B2}\}^\dagger$	0 or 2	$\big(1+\mathfrak{U}_\ell\cdot r^{3}\cdot X^2\big)$
$\{\mathrm{PG}_{B1}, \mathrm{PG}_{B1}, \mathrm{PG}_{C}\}$	0	1
$\{\mathrm{PG}_{B2}, \mathrm{PG}_{B2}, \mathrm{PG}_{B1}\}^\dagger$	0 or 2	$\big(1-\mathfrak{U}_\ell\cdot r^{3/2}\cdot X\big)^2$
$\{\mathrm{PG}_{B2}, \mathrm{PG}_{B2}, \mathrm{PG}_{C}\}$	0	1
$\{\mathrm{PG}_{C}, \mathrm{PG}_{C}, \mathrm{PG}_{B1}\}$	0	1
$\{\mathrm{PG}_{C}, \mathrm{PG}_{C}, \mathrm{PG}_{B2}\}$	0	1
$\{\mathrm{PG}_{B1}, \mathrm{PG}_{B2}, \mathrm{PG}_{C}\}$	0	1

The entries with superscript (†) indicates that the actual Euler factor divides the computed expression on the third column. Moreover, the symbol $\mathfrak{U}_\ell$ above serves a placeholder for a root of unity.

We demonstrate how the table is obtained by going through an example.

Example 7.1. Let us consider the case $\{\mathrm{PG}_{B1}, \mathrm{PG}_{B1}, \mathrm{PM}\}$ taking E_1 and E_2 to be the pair of elliptic curves with potential good reduction. First we note that by replacing E_1 with its twist by a quadratic character, we can assume that the elliptic curve E_3 has semistable reduction. The action of τ on the space $\overline{V}_1 \otimes \overline{V}_2 \otimes \overline{V}_3$ is then characterised by the tensor product

$$\begin{pmatrix} \zeta_1 & 0 \\ 0 & \zeta_1^{-1} \end{pmatrix} \otimes \begin{pmatrix} \zeta_2 & 0 \\ 0 & \zeta_2^{-1} \end{pmatrix} \otimes \begin{pmatrix} 1 & 0 \\ c & 1 \end{pmatrix}$$

where ζ_i is a primitive d_i-th root of unity and $c \neq 0$. The above tensor product is equal to the following matrix:

$$\left(\begin{array}{cccccccc}
\zeta_1\zeta_2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
c\zeta_1\zeta_2 & \zeta_1\zeta_2 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & \zeta_1\zeta_2^{-1} & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & c\zeta_1\zeta_2^{-1} & \zeta_1\zeta_2^{-1} & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & \zeta_1^{-1}\zeta_2 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & c\zeta_1^{-1}\zeta_2 & \zeta_1^{-1}\zeta_2 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & \zeta_1^{-1}\zeta_2^{-1} & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & c\zeta_1^{-1}\zeta_2^{-1} & \zeta_1^{-1}\zeta_2^{-1}
\end{array}\right)
\begin{array}{c}
u_1 \otimes u_2 \otimes u_3 \\
u_1 \otimes u_2 \otimes w_3 \\
u_1 \otimes w_2 \otimes u_3 \\
u_1 \otimes w_2 \otimes w_3 \\
w_1 \otimes u_2 \otimes u_3 \\
w_1 \otimes u_2 \otimes w_3 \\
w_1 \otimes w_2 \otimes u_3 \\
w_1 \otimes w_2 \otimes w_3
\end{array}$$

Here we have written the array of basis elements on the right side simply as a guide, e.g. $\tau(u_1 \otimes u_2 \otimes u_3) = (\zeta_1\zeta_2)(u_1 \otimes u_2 \otimes u_3) + \big(c\zeta_1\zeta_2\big)(u_1 \otimes u_2 \otimes w_3)$.

The next step is to determine the linear combinations of the basis elements of $\overline{V}_i \otimes \overline{V}_2 \otimes \overline{V}_3$ which are invariant under τ. This is done by solving the following system of equations

(1) $\zeta_1\zeta_2 \cdot A + c\zeta_1\zeta_2 \cdot B = A$

(2) $\zeta_1\zeta_2 \cdot B = B$

(3) $\zeta_1\zeta_2^{-1} \cdot C + c\zeta_1\zeta_2^{-1} \cdot D = C$

(4) $\zeta_1\zeta_2^{-1} \cdot D = D$

(5) $\zeta_1^{-1}\zeta_2 \cdot E + c\zeta_1^{-1}\zeta_2 \cdot F = E$

(6) $\zeta_1^{-1}\zeta_2 \cdot F = F$

(7) $\zeta_1^{-1}\zeta_2^{-1} \cdot G + c\zeta_1^{-1}\zeta_2^{-1} \cdot H = G$

(8) $\zeta_1^{-1}\zeta_2^{-1} \cdot H = H$

If one chooses the primitive roots of unity in (7.26) so that $\zeta_1 = \zeta_2^{-1}$, then we obtain two free variables B and H while the rest is equal to 0. This tells us that the I_r-invariant subspace of the triple product has dimension 2.

For simplicity let us set $B = H = 1$. Then τ fixes the basis elements $u_1 \otimes u_2 \otimes u_3$ and $w_1 \otimes w_2 \otimes u_3$. Applying the discussion above, we can deduce that the element $\sigma \in \mathrm{Gal}\big(\mathbb{Q}_r(E_i[\ell^\infty])/\mathbb{Q}_r\big)$ mapping onto the geometric Frobenius acts as follows:

$$\begin{aligned}\sigma(u_1 \otimes u_2 \otimes w_3) &= \alpha_\ell(E_1) \cdot u_1 \otimes \alpha_\ell(E_2) \cdot u_2 \otimes \beta_\ell(E_3) \cdot u_3 \\ \sigma(w_1 \otimes w_2 \otimes w_3) &= \beta_\ell(E_1) \cdot w_1 \otimes \beta_\ell(E_2) \cdot w_2 \otimes \beta_\ell(E_3) \cdot u_3 \,.\end{aligned}$$

Hence the Euler factor at ℓ is given by

$$\begin{aligned}\det\Big(\mathrm{Id} - \sigma X \,\Big|\, \big(\overline{V}_1 \otimes \overline{V}_2 \otimes \overline{V}_3\big)^{I_r}\Big) = &\big(1 - \alpha_\ell(E_1)\alpha_\ell(E_2)\alpha_\ell(E_3) \cdot X\big) \\ &\big(1 - \beta_\ell(E_1)\beta_\ell(E_2)\alpha_\ell(E_3) \cdot X\big) \,.\end{aligned}$$

Finally we note that dimension 2 is obtained only when $\{d_1, d_2\} = \{3{,}3\}$, $\{4{,}4\}$ or $\{6{,}6\}$ (here, d_i is as in Lemma 7.12). Any other assignment of $\{d_1, d_2\}$ would yield a dimension 0 in which case the local Euler factor equals 1.

Remark 16. Using the method of computation employed above, one can verify the dimensions of the cases considered in [HY24, Remark 3.5]. However, an explicit description of the local Euler factors eludes us due to the sheer amount of unknown variables. This is an advantage of the method in [HY24].

Chapter 8

Constructing primitive Euler systems

In this chapter we discuss an arithmetic application of Chapter 7. In particular, we describe in great detail how one can harness the decomposition obtained from Proposition 7.2 to construct a system of *primitive* zeta-elements, by which we mean the elements sent by Perrin-Riou's map to the *full* p-adic L-function.

We assume in this chapter that $F = \mathbb{Q}$ as our Euler systems have $\mathbb{Q}$ as their ground field. In particular, the Hilbert modular forms from the previous chapter will now be replaced by elliptic cusp forms, the conductors of these newforms all belong to $\mathbb{N}$, and their Nebentypes are now simply Dirichlet characters (for a background on Euler systems, we refer the reader to the survey paper [Ber+14]).

Given a pro-system of étale sheaves $\mathcal{F} = \{\mathcal{F}_r\}_{r\geq 1}$ (indexed by r) on a scheme S satisfying the Mittag-Leffler condition [Jan88, Equation (3.1)], there is an isomorphism

$$H^i_{\text{ét}}\big(S, \mathcal{F}\big) \cong \varprojlim_r H^i_{\text{ét}}\big(S, \mathcal{F}_r\big)$$

due to the finiteness of $H^{i-1}_{\text{ét}}\big(S, \mathcal{F}_r\big)$ at each $r \geq 1$. The most common example that we consider is if S is one of the affine schemes Spec $\big(\mathbb{Z}[\mu_{p^m}, 1/Np]\big)$, and $\mathcal{F}_r$ is the $\mathbb{Z}/p^r\mathbb{Z}$-module arising from a newform f, or indeed from a tensor product of them. The zeta-elements that we will study may be viewed as norm-compatible systems

$$\underline{\mathbf{z}}(N) = \big\{\mathbf{z}_{p^m}(N)\big\}_{m\geq 1} \in \varprojlim_m H^1_{\text{ét}}\big(\mathbb{Z}[\mu_{p^m}, 1/pM], T\big)$$

where $T \subset V_\ell^{\text{ét}}(N)$ denotes a Galois stable lattice, M divides the conductor N, and the inverse limit here is taken with respect to natural corestriction maps.

This chapter is divided into two parts. Section 8.1 is spent summarizing the interpolation properties of Perrin-Riou's big logarithm map, which we use in Section 8.2 together with Proposition 7.2 as a guide to construct a system of norm-compatible primitive zeta-elements. One then obtains the sharper one-divisibility in the Iwasawa Main Conjecture that was alluded to in Chapter 7 as a consequence of the existence of these primitive elements.

8.1 Perrin-Riou's big logarithm map

Let K be a field of characteristic zero and we write $\mu_{p^m} \subset \overline{K}^\times$ for its group of p^m-th roots of unity. Throughout we will fix a compatible system $\zeta_{p^m} \in \mu_{p^m}$ such that $(\zeta_{p^{m+1}})^p = \zeta_{p^m}$. Upon passing to the inverse limit under the corestriction maps, one can define the Iwasawa cohomology groups

$$H^i_{\text{Iw}}(K, T) := \varprojlim_m H^i\big(K(\mu_{p^m}), T\big) \quad \text{and} \quad H^i_{\text{Iw}}(K, V) := H^i_{\text{Iw}}(K, T) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$$

where $V = T \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$, both of which have the structure of $\mathbb{Z}_p[\Delta][\![\gamma - 1]\!]$-modules. Let us assume for the rest of this section that K is a finite extension of $\mathbb{Q}_p$.

Remark 17.

1. We write $B_{\text{cris}} \subset B_{\text{dR}}$ to denote the rings of p-adic periods introduced by Fontaine and his colleagues (see for example [FM87; BK90]). For a finite-dimensional continuous p-adic G_K-representation V, one defines

$$\mathbf{D}_{\text{cris},K}(V) := \big(V \otimes_{\mathbb{Q}_p} B_{\text{cris}}\big)^{G_K} \quad \text{and} \quad \mathbf{D}_{\text{dR},K}(V) := \big(V \otimes_{\mathbb{Q}_p} B_{\text{dR}}\big)^{G_K};$$

the former is a vector space over $K_0 := K \cap \mathbb{Q}_p^{\text{ur}}$, and the latter is a K-vector space. Moreover B_{cris} comes equipped with a Frobenius 'φ', which $\mathbf{D}_{\text{cris}}(V)$ also inherits. Furthermore, the K-vector space $\mathbf{D}_{\text{dR}}(V)$ has a decreasing exhaustive filtration $\text{Fil}^i\,\mathbf{D}_{\text{dR}}(V) = \big(V \otimes_{\mathbb{Q}_p} \mathfrak{t}^i \cdot B_{\text{dR}}^+\big)^{G_K}$ with $\mathfrak{t}$ a uniformiser for the ring of integers B_{dR}^+.

2. The well known short exact sequence $0 \to \mathbb{Q}_p \to B_{\text{cris}}^{\varphi=1} \to B_{\text{dR}}/B_{\text{dR}}^+ \to 0$ of G_K-modules induces a long exact sequence on local cohomology groups

$$\begin{aligned} 0 \longrightarrow V^{G_K} \longrightarrow \big(V \otimes_{\mathbb{Q}_p} B_{\text{cris}}^{\varphi=1}\big)^{G_K} \longrightarrow \big(V \otimes_{\mathbb{Q}_p} B_{\text{dR}}/B_{\text{dR}}^+\big)^{G_K} \\ \xrightarrow{\partial} H^1\big(K, V\big) \longrightarrow H^1\big(K, V \otimes_{\mathbb{Q}_p} B_{\text{cris}}^{\varphi=1}\big) \longrightarrow \cdots\ . \end{aligned} \tag{8.1}$$

If V is a de Rham representation so that $\dim_K\big(\mathbf{D}_{\text{dR},K}(V)\big) = \dim_{\mathbb{Q}_p}(V)$, we write

$$\exp_{K,V} : \text{tang}(V/K) \cong \mathbf{D}_{\text{dR},K}(V)\big/\text{Fil}^0\mathbf{D}_{\text{dR},K}(V) \longrightarrow H^1\big(K, V\big)$$

for the exponential map induced from the boundary map ∂ in (8.1).

3. Finally if we identify the cotangent space for V/K with $\text{Fil}^0\,\mathbf{D}_{\text{dR},K}(V)$, then the dual exponential map $\exp^*_{K,V}$ is defined by the commutativity of the following diagram.

$$\begin{tikzcd} \text{cotang}(V/K) & \times & \text{tang}\,\big(V^*(1)/K\big) \arrow[r, "\cup_{\text{dR}}"] \arrow[d, "\exp_{K,V^*}"'] & \mathbf{D}_{\text{dR},K}(\mathbb{Q}_p(1)) \cong K \arrow[r, "\text{Tr}_{K/\mathbb{Q}_p}"] & \mathbb{Q}_p \arrow[d, equal] \\ H^1(K,V) \arrow[u, "\exp^*_{K,V}"] & \times & H^1\big(K, V^*(1)\big) \arrow[r, "\cup"] & H^2\big(K, \mathbb{Q}_p(1)\big) \arrow[r, "\text{inv}_K"] & \mathbb{Q}_p \end{tikzcd}$$

N.B. The kernel of the dual exponential $\exp^*_{K,V}$ coincides with the kernel of the map $H^1(K,V) \to H^1\big(K, V \otimes_{\mathbb{Q}_p} B_{\text{dR}}\big)$.

Building on the work of Coleman [Col79], in the early 1990s Perrin-Riou constructed a machine to convert norm-compatible families into power series [PR94]. There are several formulations of her regulator so we select one that works for us. In fact the version we outline is actually the *dual* of Perrin-Riou's regulator map, and relies intimately on an explicit reciprocity law which has subsequently been established (independently) by Kato-Kurihara-Tsuji, Benois and Colmez [KKT96; Ben97; Col98]. We also remind the reader that V is crystalline if $\dim_{K_0}\big(\mathbf{D}_{\mathrm{cris},K}(V)\big) = \dim_{\mathbb{Q}_p}(V)$.

Let us define a K_0-subspace $\mathbf{D}_{\mathrm{cris},K}(V)^\dagger \subset \mathbf{D}_{\mathrm{cris},K}(V)$ as the K_0-span of those vectors on which the φ-action is quasi-unipotent, and we further assume that $\mathbf{D}_{\mathrm{cris},K}(V)^\dagger \neq 0$. Henceforth let us now suppose that the field K is a finite unramified extension of $\mathbb{Q}_p$, and that the Frobenius φ acting on $\mathbf{D}_{\mathrm{cris}}(V)^\dagger$ has no eigenvalues belonging to $p^{\mathbb{N}}$.

Theorem 8.1. *There is a unique* $\mathbb{Z}_p[\Delta][\![\gamma-1]\!]$*-homomorphism*

$$\mathrm{LOG} = \mathrm{LOG}_{K,V^*} : H^1_{\mathrm{Iw}}(K, V^*) \otimes_{\mathbb{Q}_p} \mathbf{D}_{\mathrm{cris},K}(V)^\dagger \longrightarrow \mathbb{Z}_p[\Delta][\![\gamma-1]\!] \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$$

which interpolates at all finite order characters $\chi \neq \mathbf{1}$ *and integers* $j \geq 0$,

$$\begin{aligned}\chi\mathcal{N}x_p^j\Big(\mathrm{LOG}(\underline{x}\otimes\mathfrak{d})\Big) &= p^{mj}G(\chi^{-1}) \\ &\qquad \times \sum_{\sigma}\chi(\sigma)\left(\exp^*_{V^*(-j)}\left(x_{p^m}\otimes\zeta_{p^m}^{\otimes -j}\right)^{\sigma}\cup\varphi^{-m}\mathfrak{d}\right)\end{aligned} \tag{8.2}$$

where σ *is taken over* $\mathrm{Gal}(K(\mu_{p^m})/K)$, *whilst at the trivial character* $\chi = \mathbf{1}$ *and for integers* $j \geq 0$,

$$\mathcal{N}x_p^j\Big(\mathrm{LOG}(\underline{x}\otimes\mathfrak{d})\Big) = j!\exp^*_{V^*(-j)}(x_1)\cup\left(1-p^j\varphi^{-1}\right)\left(1-p^{-j-1}\varphi\right)^{-1}\mathfrak{d}.$$

Here we should point out that $\exp^*_{K(\mu_{p^m}),V^*(-j)}$ above takes values inside of $\mathrm{Fil}^0\,\mathbf{D}_{\mathrm{dR},K(\mu_{p^m})}\big(V^*(-j)\big)$ and $\mathfrak{d} \in \mathbf{D}_{\mathrm{cris}}(V)^\dagger$, which means that the cup-product satisfies

$$\exp^*_{V^*(-j)}\left(x_{p^m}\otimes\zeta_{p^m}^{\otimes -j}\right)^{\sigma}\cup_{\mathrm{dR}}\mathfrak{d} \in \mathbf{D}_{\mathrm{dR},K(\mu_{p^m})}\big(\mathbb{Q}_p(-j)\big) \cong K(\mu_{p^m}).$$

If one relaxes these conditions and allows $\mathfrak{d} \notin \mathbf{D}_{\mathrm{cris}}(V)^\dagger$, then the image of LOG lies in the space of tempered r-admissible functions '$\mathcal{H}_r$' where r is the slope of $\mathfrak{d}$. It thus remains to use Perrin-Riou's logarithm map (and some low-brow algebra) to modify the imprimitive zeta-elements in [Kat04; KLZ17; LLZ14; LZ19], and hence replenish their missing Euler factors at the primes $\ell \neq p$ dividing the conductor of each motive.

8.2 Application to self-dual motives

In this concluding section we illustrate in detail the last two steps of the procedure outlined in Section 7.1 pertaining to the construction of primitive zeta-elements. We apply this procedure to several examples of self-dual tensor product of elliptic modular forms namely: the single product, the double product and the symmetric square.

The proof of the theorems will be quite lengthy but it follows a general strategy which we provide here. Firstly, one fixes a critical s-value j and a branch character ϕ. One then recalls the existence of zeta-elements mapping to a twist of the S-depleted function $L^{\phi}_{p,(S)}(-)$; these elements are not p-integral in nature and will require certain assumptions to force p-integrality. Next, one constructs the corresponding dual zeta-elements mapping to a twist of the dual function $L^{\phi^{-1}}_{p,(S)}(-)$. Whilst one can apply an identical method to ensure the p-integrality of these elements, one additionally has to guarantee that the global cohomology group where these elements live is a rank one torsion-free module. From here, one pieces together a system of primitive zeta-elements over the fixed branch ϕ using the decomposition in Proposition 7.2 as a blueprint. The finished product is obtained by taking the aggregate of each ϕ-component.

8.2.1 The Kato–Beilinson Euler system

Let f be a newform of weight k, central character η and level $N_f \in \mathbb{N}$, with the q-expansion $f(q) = \sum_{n=1}^{\infty} a_n(f) \cdot q^n$. We write K_f for the extension of $\mathbb{Q}_p$ containing all the Fourier coefficients $a_n(f)$. Recall that if V_f is Deligne's p-adic representation attached to f, one has isomorphisms

$$V_{f^\rho} \cong V_f^*(1-k) \cong \mathrm{Hom}\big(V_f \otimes_{\mathbb{Z}_p} \mathbb{Z}_p(k-1), K_f\big).$$

Throughout we shall assume that $f^\rho = f$, *therefore* $\overline{a_n(f)} = a_n(f)$ *for every* n. In particular, if one fixes a $G_{\mathbb{Q}}$-stable lattice in $T \subset V_f$ then clearly $T \subset V_{f^\rho} = V_f$, and one obtains a self-duality

$$T \cong \mathrm{Hom}_{\mathbb{Z}_p}\big(T(k-1), \mathcal{O}_{K_f}\big) \tag{8.3}$$

of free rank two $\mathcal{O}_{K_f}$-modules, with a continuous action of $G_{\mathbb{Q}} = \mathrm{Gal}\left(\overline{\mathbb{Q}}/\mathbb{Q}\right)$.

In the following discussion, we will choose T to be the Galois-stable lattice generated by the image of $H^1\big(Y(N_f)(\mathbb{C}), \mathrm{Sym}^{k-2}_{\mathbb{Z}_p}(\underline{H}^1_p)\big)$ inside the dual space V_f^*. To be more precise, if $\pi : \mathbb{E}^{\mathrm{univ}} \twoheadrightarrow Y(N_f)$ denotes the projection from the universal elliptic curve $\mathbb{E}^{\mathrm{univ}}$ with a $\Gamma(N_f)$-structure to the modular curve, then one writes

$$\underline{H}^1_p = R^1\pi_*\mathbb{Z}_p(1) = \left(R^1\pi_*\mathbb{Z}_p\right)^{\mathrm{dual}}$$

for the smooth étale sheaf of rank two on $Y(N_f)[1/p]$, which we can then localise at the maximal ideal of the Hecke algebra corresponding to the newform $f = f^\rho$ (for more details of this construction, we refer the reader to [Kat04] and [Del08, §2.4]).

Let us now suppose that the odd prime p satisfies both $p \nmid a_p(f)$ and $p^2 \nmid N_f$, and we use $\alpha_p(f)$ to denote the p-adic unit root of $X^2 - a_p(f)X + \eta(p)p^{k-1}$. Because of the self-duality assumption, $\alpha_p(f)$ is likewise the unit root of the dual polynomial $X^2 - a_p(f^\rho)X + \overline{\eta}(p)p^{k-1}$ as the newforms f^ρ and f coincide.

Theorem 8.2. *If one has $f^\rho = f$, there exist norm-compatible zeta-elements*

$$\underline{\mathbf{z}}^{\mathrm{Kato}}(f) = \left\{\mathbf{z}^{\mathrm{Kato}}_{p^m}(f)\right\}_{m\geq 1} \in \mathbb{Q} \otimes \varprojlim_{m} H^1_{\text{ét}}\big(\mathbb{Z}[\mu_{p^m}, 1/p], T(k-1)\big)$$

such that $\mathrm{LOG}_{\mathbb{Q}_p, V_f^*} \circ \mathrm{loc}_p\left(\underline{\mathbf{z}}(f) \otimes \mathfrak{d}\right)$ *is (up to a scalar) the p-adic L-function $L_p(f)$ in [MTT86], where $\mathfrak{d}$ generates the eigenspace $\mathbf{D}_{\mathrm{cris}}(V_f)^{\varphi = \alpha_p(f)}$. If $\mathrm{im}(\overline{\rho}_f)$ contains a conjugate of $\mathrm{SL}_2(\mathbb{F}_p)$ then the element $\underline{\mathbf{z}}^{\mathrm{Kato}}(f)$ is p-integral.*

If $\overline{\rho}_f : G_{\mathbb{Q}} \to \mathrm{GL}_2(\mathbb{F}_{p^e})$ has large enough image then the stated result is already known from [Kat04]. However if $\mathrm{im}(\overline{\rho}_f)$ is not big enough, it was previously only shown that there are zeta-elements mapping to $L_{p,(S)}(f)$ where S consists of the primes $\ell \mid N_f$ with $\ell \neq p$; one may then view the theorem above as confirming that *primitive* zeta-elements have no poles along the cyclotomic direction even if $\overline{\rho}_f$ has too small of an image (see the closely related work of C.-H. Kim [Kim23] who discusses other applications).

Proof. We begin by choosing a pair of integers $c, d \in \mathbb{Z}_{\geq 2}$ coprime to $6 \cdot N_f$. For each $j \in \{0, \ldots, k-2\}$, the construction of Kato [Kat04, §12] yields zeta-elements

$$ {}_{c,d}\underline{\mathbf{z}}(j) = \left\{ {}_{c,d}\mathbf{z}_{p^m}\big(f,k,j,\mathrm{supp}(pN_f)\big) \right\}_m \in \varprojlim_m H^1_{\text{ét}}\big(\mathbb{Z}[\mu_{p^m},1/p],T^*(-j)\big) $$

such that ${}_{c,d}\underline{\mathbf{z}}(j) = {}_{c,d}\underline{\mathbf{z}}(0) \otimes (\zeta_{p^m})_m^{\otimes -j}$, and for $S = \mathrm{supp}(N_f) \setminus \{p\}$:

$$ \mathrm{LOG}_{\mathbb{Q}_p,V_f^*(-j)}\left({}_{c,d}\underline{\mathbf{z}}(j) \otimes \mathfrak{d} \right) = \big(c^2 - c^{k+1-j}\cdot\sigma_c\big)\big(d^2 - d^{j+1}\cdot\sigma_d\big)\cdot \mathrm{Tw}_j\left(L_{p,(S)}(f)\right) $$

where σ_c (resp. σ_d) maps to c (resp. d) under κ_{cyc}.

If one considers the Iwasawa module, $\mathscr{Z}^{\text{prim}}$ say, generated by the ${}_{c,d}\underline{\mathbf{z}}(j)$ as c and d both vary then one can remove these junk factors at the cost of p-integrality—these modified zeta-elements $\underline{\mathbf{z}}(j)$ à priori lie inside $\mathbb{Q}\otimes\varprojlim_m H^1_{\text{ét}}\big(\mathbb{Z}[\mu_{p^m},1/p],T^*(-j)\big)$. Moreover, if $\mathrm{im}(\overline{\rho}_f)$ does indeed contain a conjugate of $\mathrm{SL}_2(\mathbb{F}_{p^e})$ then there is no loss of p-integrality when constructing $\underline{\mathbf{z}}(j)$, as is outlined further in [Kat04; Del08; Kim23]. In terms of their special values, at a critical twist $j \in \{0,\dots,k-2\}$ one finds

$$ \mathrm{LOG}_{\mathbb{Q}_p,V_f^*(-j)}\left(\underline{\mathbf{z}}(j)\otimes\mathfrak{d}\right) = \mathrm{Tw}_j\left(L_{p,(S)}(f)\right). \tag{8.4} $$

Bearing in mind the decomposition of $L_p(f)$ arising from Propositions 7.2 and 7.4, one next constructs a dual version of the zeta-elements $\underline{\mathbf{z}}(j)$.

For simplicity, assume that the image of $\overline{\rho}_f$ *contains a conjugate of* $\mathrm{SL}_2(\mathbb{F}_p)$. Recall from [Kat04] or [Kim23, Lemma 2.2] that for each choice of branch character ϕ, the $\mathcal{O}_{K_f}[\![\gamma-1]\!]$-module $\varprojlim_m H^1_{\text{ét}}\big(\mathbb{Z}[\mu_{p^m},1/p],T^*(-j)\big)^{(\phi^{-1})}$ will be free of rank one; let $\mathfrak{e}_{\phi^{-1}}(j)$ be a generator (which is well-defined up to an element of $\mathcal{O}_{K_f}[\![\gamma-1]\!]^\times$). We again pick integers $c',d' \in \mathbb{Z}_{\geq 2}$ coprime to $6\cdot N_f$, and can then directly write

$$ {}_{c',d'}\underline{\mathbf{z}}(j)^{(\phi^{-1})} = {}_{c',d'}\mathscr{F}_{\overline{\phi}}(\gamma-1)\cdot\mathfrak{e}_{\phi^{-1}}(j) \text{ for some } {}_{c',d'}\mathscr{F}_{\overline{\phi}}(\gamma-1) \in \mathcal{O}_{K_f}[\![\gamma-1]\!]\,. $$

We define a *dual Kato zeta-element* inside $\varprojlim_m H^1_{\text{ét}}\big(\mathbb{Z}[\mu_{p^m},1/p],T^*(-j)\big)^{(\phi)}$ as

$$ {}_{c',d'}\underline{\mathbf{z}}^\star_{\overline{\phi}}(j) := \mathrm{Tw}_{2-k}\left({}_{c',d'}\mathscr{F}_{\overline{\phi}}(\gamma^{-1}-1)\right)\cdot\mathfrak{e}_{\phi}(j)\,. $$

Note that if we multiply $\mathfrak{e}_{\phi^{-1}}(j)$ by an element of $\mathcal{O}_{K_f}[\![\gamma-1]\!]^\times$ then this changes ${}_{c',d'}\underline{\mathbf{z}}^\star_{\overline{\phi}}(j)$ by an element of $\mathcal{O}_{K_f}[\![\gamma-1]\!]^\times$ as well. To normalise this element correctly, we first observe that $\mathrm{LOG}_{\mathbb{Q}_p,V_f^*(-j)}\left({}_{c',d'}\underline{\mathbf{z}}^{\star}_{\phi}(j)\otimes\mathfrak{d}\right)$ belongs to

$$\left(c'^2-c'^{3+j}\cdot\overline{\phi}(\sigma_{c'})\sigma_{c'}^{-1}\right)\left(d'^2-d'^{k-j-1}\cdot\overline{\phi}(\sigma_{d'})\sigma_{d'}^{-1}\right)\cdot\mathrm{Tw}_{k-2-j}\left(L_{p,(S)}^{\phi^{-1}}(f)\right)^{\bullet}\cdot\mathcal{O}_{K_f}[\![\gamma-1]\!]^{\times}$$

where we have applied the functoriality $\mathrm{LOG}_{\mathbb{Q}_p,V_f^*(-j)}\left(-^{\bullet}\right)=\mathrm{LOG}_{\mathbb{Q}_p,V_f^*(-j)}(-)^{\bullet}$. It follows that at every branch ϕ, there is a compatible choice of $\mathfrak{e}_{\phi}(j)$ and $\mathfrak{e}_{\phi^{-1}}(j)$ so that ${}_{c',d'}\underline{\mathbf{z}}^{\star}_{\phi}(j)$ is sent under the homomorphism $\mathrm{LOG}_{\mathbb{Q}_p,V_f^*(-j)}\left(-\otimes\mathfrak{d}\right)$ to the power series element

$$\left(c'^2-c'^{3+j}\cdot\overline{\phi}(\sigma_{c'})\sigma_{c'}^{-1}\right)\left(d'^2-d'^{k-j-1}\cdot\overline{\phi}(\sigma_{d'})\sigma_{d'}^{-1}\right)\cdot\mathrm{Tw}_{k-2-j}\left(L_{p,(S)}^{\phi^{-1}}(f)\right)^{\bullet}.$$

As a result, setting ${}_{c',d'}\underline{\mathbf{z}}^{\star}(j):=\sum_{\phi}{}_{c',d'}\underline{\mathbf{z}}^{\star}_{\phi}(j)$ one deduces that

$$\mathrm{LOG}\left({}_{c',d'}\underline{\mathbf{z}}^{\star}(j)\right)=\left(c'^2-c'^{3+j}\cdot\sigma_{c'}^{-1}\right)\left(d'^2-d'^{k-j-1}\cdot\sigma_{d'}^{-1}\right)\cdot\mathrm{Tw}_{k-2-j}\left(L_{p,(S)}(f)\right)^{\bullet}.$$

Applying the same procedure which allowed the removal of the extraneous factors $\left(c^2-c^{k+1-j}\cdot\sigma_c\right)\left(d^2-d^{j+1}\cdot\sigma_d\right)$ from ${}_{c,d}\underline{\mathbf{z}}(j)$, we obtain dual (primitive) elements inside $\mathbb{Q}\otimes\varprojlim_m H^1_{\text{ét}}\left(\mathbb{Z}[\mu_{p^m},1/p],T^*(-j)\right)$, namely

$$\underline{\mathbf{z}}^{\star}(j):=\left(\left(c'^2-c'^{3+j}\cdot\sigma_{c'}^{-1}\right)\left(d'^2-d'^{k-j-1}\cdot\sigma_{d'}^{-1}\right)\right)^{-1}\cdot{}_{c',d'}\underline{\mathbf{z}}^{\star}(j).$$

These dual elements will be sent to the S-depleted dual of the p-adic L-function under Perrin-Riou's big logarithm map, i.e.

$$\mathrm{LOG}_{\mathbb{Q}_p,V_f^*(-j)}\left(\underline{\mathbf{z}}^{\star}(j)\otimes\mathfrak{d}\right)=\mathrm{Tw}_{k-2-j}\left(L_{p,(S)}(f)\right)^{\bullet}. \tag{8.5}$$

Since $\mathrm{im}(\overline{\rho}_f)$ contains a conjugate of $\mathrm{SL}_2(\mathbb{F}_{p^e})$, each $\underline{\mathbf{z}}^{\star}(j)$ will also be p-integral.

Fix a branch $\phi:\Delta\to\overline{\mathbb{Q}_p}^{\times}$. By Proposition 7.4, Hypothesis (S^{dep}) holds for $M(f)$ and $S=\Sigma^{\mathrm{bad}}(f)$ and hence Proposition 7.2 gives us a factorisation

$$L_p^{\phi}(f)=\mathcal{W}^{(\pm),\phi}\big(M(f)\big)\cdot\mathcal{Y}_1^{\phi}\big(M(f)\big)\times\mathrm{Tw}_{k-2}\left(L_{p,(S)}^{\phi^{-1}}(f)\right)^{\bullet}+\mathcal{Y}_2^{\phi}\big(M(f)\big)\times L_{p,(S)}^{\phi}(f) \tag{8.6}$$

for some $\mathcal{Y}_1^{\phi},\mathcal{Y}_2^{\phi}\in\mathcal{O}_{K_f}[\![\gamma-1]\!]$ and $\mathcal{W}^{(\pm),\phi}\in\mathcal{O}_{K_f}[\![\gamma-1]\!]^{\times}$, where $\phi(-1)=\pm1$. One may therefore define each ϕ-component of the primitive zeta-elements by

$$\underline{\mathbf{z}}^{\mathrm{Kato}}(f,j)^{(\phi)}:=\mathrm{Tw}_j\left(\mathcal{W}^{(\pm),\phi}\right)\cdot\mathrm{Tw}_j\left(\mathcal{Y}_1^{\phi}\right)\cdot\underline{\mathbf{z}}^{\star}(j)^{(\phi)}+\mathrm{Tw}_j\left(\mathcal{Y}_2^{\phi}\right)\cdot\underline{\mathbf{z}}(j)^{(\phi)}$$

and $\mathrm{LOG}_{\mathbb{Q}_p,V_f^*}\circ\mathrm{loc}_p\left(\underline{\mathbf{z}}^{\mathrm{Kato}}(f,0)^{(\phi)}\right)$ coincides with the right-hand side of (8.6). Here we have employed Equations (8.4) and (8.5) to compute the images of $\underline{\mathbf{z}}(j)^{(\phi)}$ and $\underline{\mathbf{z}}^{\star}(j)^{(\phi)}$, respectively. Thus, at any critical twist $j\in\{0,\dots,k-2\}$:

$$\mathrm{LOG}_{\mathbb{Q}_p,V_f^*(-j)}\circ\mathrm{loc}_p\left(\underline{\mathbf{z}}^{\mathrm{Kato}}(f,j)^{(\phi)}\otimes\mathfrak{d}\right)=\mathrm{Tw}_j\left(L_p^{\phi}(f)\right).$$

Finally, if one now puts $\underline{\mathbf{z}}^{\mathrm{Kato}}(f):=\sum_{\phi}\underline{\mathbf{z}}^{\mathrm{Kato}}(f,0)^{(\phi)}$ then the proof is finished. N.B. If $\mathrm{im}(\overline{\rho}_f)$ contains no conjugate of $\mathrm{SL}_2(\mathbb{F}_p)$, the above argument works fine, except one has to tensor $\varprojlim_m H^1_{\text{ét}}\left(\mathbb{Z}[\mu_{p^m},1/p],T^*(-j)\right)$ by $\mathbb{Q}$ throughout.

■

8.2.2 Beilinson-Flach zeta-elements

Let $f_1=\sum_{n=1}^{\infty}a_n(f_1)\cdot q^n$ and $f_2=\sum_{n=1}^{\infty}a_n(f_2)\cdot q^n$ be primitive forms of type (k_1,N_{f_1},η_1) and (k_2,N_{f_2},η_2) respectively, wherein $k_1>k_2>0$. We denote by K the finite extension of $\mathbb{Q}_p$ which contains the image of $a_n(f_1)$ and $a_n(f_2)$ over ι_p for each $n\in\mathbb{N}$. We suppose throughout that $f_1^{\rho}=f_1$ and $f_2^{\rho}=f_2$, and we set $N:=\mathrm{lcm}(N_{f_1},N_{f_2})$. For each choice of $i\in\{1,2\}$, we shall write T_{f_i} for the $G_{\mathbb{Q}}$-lattice generated by the image of $H^1\left(Y(N)(\mathbb{C}),\mathrm{Sym}^{k-2}_{\mathbb{Z}_p}(\underline{H}^1_p)\right)$ inside the contragredient representation $V_{f_i}^*$. Let us further suppose that the fixed prime number $p\neq 2$ satisfies the indivisibilities $p\nmid a_p(f_1)$ and $p^2\nmid N_{f_1}$.

Theorem 8.3. *There exist primitive norm-compatible Beilinson-Flach elements*

$$\underline{\mathcal{BF}}^{\mathrm{prim}}_{(j)}=\left\{\mathcal{BF}^{\mathrm{prim}}_{(j),p^m}\right\}_{m\geq 1}\in\mathbb{Q}\otimes\varprojlim_m H^1_{\text{ét}}\Big(\mathbb{Z}\big[\mu_{p^m},1/pN\big],T_{f_1}\otimes T_{f_2}(-j)\Big)$$

such that for $j\in\{0,\dots,k_1-k_2-1\}$, *their image under Perrin-Riou's map is*

$$\mathrm{LOG}_{\mathbb{Q}_p,V_{f_1}^*\otimes V_{f_2}^*(-j)}\circ\mathrm{loc}_p\left(\underline{\mathcal{BF}}^{\mathrm{prim}}_{(j)}\otimes\left(\mathfrak{d}'_{f_1}\otimes\mathfrak{d}''f_2\right)\right)=\mathfrak{h}_{f_1}\cdot\mathcal{N}x_p^j\big(L_p(f_1\otimes f_2)\big).$$

Here $\mathfrak{d}'_{f_1}$ *generates*[1] *the rank one module* $M_{\mathrm{dR}}(f_1)/\,\mathrm{Fil}^1$ *as it appears in [KLZ17, Proposition 10.1.1(2)] and* $\mathfrak{d}''_{f_2}\in\mathrm{Fil}^1 M_{\mathrm{dR}}(f_2)$ *is the differential associated to* f_2 *as in 10.1.1(1) of op. cit.*

---

[1] In fact $\mathfrak{d}'_{f_1}$ is given by "η_{f_1}" in the notation of [KLZ17, Proposition 10.1.1], while $\mathfrak{d}''_{f_2}$ refers to "ω_{f_2}".

Note that the relation between imprimitive Beilinson-Flach elements and the imprimitive p-adic L-function in *op. cit.* is initially obtained outside of the critical region of interpolation, and makes use of Hida's theory for p-ordinary families [Hid88]. However the theorem above involves the inside of the critical strip, where special values of L-functions are directly connected to Bloch–Kato Selmer groups.

Proof. Let us keep the same notations/definitions employed in Section 7.3.2. In particular, one recalls from the proof of Corollary 7.6 that $S = \Sigma_{\mathrm{(V)}}^{\mathrm{bad}} \cup \Sigma_{\mathrm{(VI)}}^{\mathrm{bad}}$ and that there is a decomposition

$$\begin{aligned} L_p^{\phi}(\tilde{f}_1 \otimes \tilde{f}_2) &= \mathcal{Y}_2^{\phi}(M) \times L_{p,(S)}^{\phi}(\tilde{f}_1, \tilde{f}_2) \\ &\quad + \mathcal{W}^{(\varepsilon_0),\phi}(M) \cdot \mathcal{Y}_1^{\phi}(M) \times \mathrm{Tw}_{k_1-k_2-1}\, L_{p,(S)}^{\phi^{-1}}\big(\tilde{f}_1^{\rho}, \tilde{f}_2^{\rho}\big)^{\bullet} \end{aligned}$$

with $\mathcal{Y}_1^{\phi}(M), \mathcal{Y}_2^{\phi}(M) \in \mathcal{O}_K[\![\gamma - 1]\!]$. Moreover $L_p^{\phi}(f_1 \otimes f_2) = \frac{\Omega(\tilde{f}_1)}{\Omega(f_1)} \times L_p^{\phi}\big(\tilde{f}_1 \otimes \tilde{f}_2\big)$, therefore one obtains the identity

$$\begin{aligned} \mathfrak{h}_{f_1} \cdot L_p^{\phi}(f_1 \otimes f_2) = \frac{\mathfrak{h}_{f_1}}{\mathfrak{h}_{\tilde{f}_1}} \cdot \frac{\Omega(\tilde{f}_1)}{\Omega(f_1)} \times \Big(&\mathcal{Y}_2^{\phi}(M) \times \mathfrak{h}_{\tilde{f}_1} \cdot L_{p,S}^{\phi}\big(\tilde{f}_1, \tilde{f}_2\big) \\ &+ \mathcal{W}^{(\varepsilon_0),\phi}(M) \cdot \mathcal{Y}_1^{\phi}(M) \times \mathfrak{h}_{\tilde{f}_1} \cdot \mathrm{Tw}_{k_1-k_2-1}\, L_{p,(S)}^{\phi^{-1}}\big(\tilde{f}_1^{\rho}, \tilde{f}_2^{\rho}\big)^{\bullet}\Big). \end{aligned} \tag{8.7}$$

Applying [KLZ17, Theorem 10.2.2] to the pair of twisted newforms $\tilde{f}_1 := f_1 \otimes \varepsilon$ and $\tilde{f}_2 := f_2 \otimes \varepsilon^{-1}$ where as before $\varepsilon = \prod_{\mathfrak{v} \in \Sigma_{\mathrm{(II)}}^{\mathrm{bad}} \cup \Sigma_{\mathrm{(III)}}^{\mathrm{bad}}} \varepsilon_{\mathfrak{v}}$, it directly follows that

$$\begin{aligned} \mathrm{LOG}_{\mathbb{Q}_p, V_{\tilde{f}_1}^* \otimes V_{\tilde{f}_2}^*(-j)} \circ \mathrm{loc}_p \left({}_c\mathcal{BF}_{(j)}^{\tilde{f}_1, \tilde{f}_2} \otimes \big(\mathfrak{d}'_{\tilde{f}_1} \otimes \mathfrak{d}''_{\tilde{f}_2}\big)\right) &= \upsilon_{N_{\tilde{f}_1}}(\tilde{f}_1)^{-1}(-1)^{j+1} \\ \times \Big(c^2 - c^{-(k_1+k_2-4-2j)} \cdot \overline{\eta_1 \varepsilon^2}(c) \cdot \overline{\eta_2 \varepsilon^{-2}}(c)\Big) &\times \mathfrak{h}_{\tilde{f}_1} \cdot \mathcal{N}x_p^j\big(L_p(\tilde{f}_1, \tilde{f}_2)\big) \end{aligned}$$

where ${}_c\mathcal{BF}_{(j)}^{\tilde{f}_1, \tilde{f}_2} \in \varprojlim_m H^1_{\text{ét}}\Big(\mathbb{Z}\big[\mu_{p^m}, 1/pN\big], T_{\tilde{f}_1} \otimes T_{\tilde{f}_2}(-j)\Big)$, $c > 1$ is coprime to $6pN$, and $\upsilon_{N_{\tilde{f}_1}}(\tilde{f}_1)$ is the pseudo-eigenvalue of the Atkin-Lehner operator acting on $\tilde{f}_1$.

Remark 18.

(i) From the definition of $\tilde{f}_1$ and $\tilde{f}_2$, it is clear that $V_{\tilde{f}_1}^* \otimes V_{\tilde{f}_2}^* \cong V_{f_1}^* \otimes V_{f_2}^*$, in which case ${}_c\mathcal{BF}_{(j)}^{\tilde{f}_1, \tilde{f}_2}$ takes coefficients within the same sheaf as $\underline{\mathcal{BF}}_{(j)}^{\mathrm{prim}}$.

Furthermore, assuming Hypothesis (BI) as in [KLZ17], one can remove the junk factor involving c to obtain a p-integral element $\mathcal{BF}_{(j)}^{\tilde{f}_1,\tilde{f}_2}$ satisfying

$$ {}_c\mathcal{BF}_{(j)}^{\tilde{f}_1,\tilde{f}_2} = \Big(c^2 - c^{-(k_1+k_2-4-2j)} \cdot \overline{\eta_1\varepsilon^2}(c) \cdot \overline{\eta_2\varepsilon^{-2}}(c)\Big) \cdot \mathcal{BF}_{(j)}^{\tilde{f}_1,\tilde{f}_2}\,. $$

If Hypothesis (BI) does not hold, then one has to first tensor by $\mathbb{Q}$ to obtain $\mathcal{BF}_{(j)}^{\tilde{f}_1,\tilde{f}_2}$.

(ii) The element $\mathfrak{d}'_{\tilde{f}_1}\otimes\mathfrak{d}''_{\tilde{f}_2}$ coincides with $\mathfrak{d}'_{f_1}\otimes\mathfrak{d}''_{f_2}$ since $V^*_{\tilde{f}_1}\otimes V^*_{\tilde{f}_2} \cong V^*_{f_1}\otimes V^*_{f_2}$ and if ξ generates $\mathrm{Hom}_{\mathcal{O}_K}\big(\mathcal{F}^{-+}(T_{f_1}\otimes T_{f_2}), \mathcal{O}_K\big)$ (with the notation $\mathcal{F}^{-+}$ as in §11 of *op. cit.*) then the quotient $\xi \times \big(G(\eta_1^{-1})G(\eta_2^{-1})\cdot\mathfrak{d}'_{\tilde{f}_1}\otimes\mathfrak{d}''_{\tilde{f}_2}\big)^{-1}$ must generate the congruence ideal for $\tilde{f}_1$. This explains why the congruence number $\mathfrak{h}_{\tilde{f}_1}$ shows up in the formula.

(iii) Under Perrin-Riou's logarithm map, $\mathcal{BF}_{(j)}^{\tilde{f}_1,\tilde{f}_2}$ is sent to the imprimitive function $L_p\big(\tilde{f}_1,\tilde{f}_2\big)$. If we form the depletion at the set $S = \Sigma^{\mathrm{bad}}_{(\mathrm{V})} \cup \Sigma^{\mathrm{bad}}_{(\mathrm{VI})}$, then one has a factorisation

$$ L^{\phi}_{p,(S)}\big(\tilde{f}_1,\tilde{f}_2\big) = \prod_{\mathfrak{v}\in\Sigma^{\mathrm{bad}}_{(\mathrm{V})}} \mathscr{L}^{\phi}_{\mathfrak{v}}\big(\tilde{f}_1,\tilde{f}_2;\gamma\big) \;\times \prod_{\mathfrak{v}\in\Sigma^{\mathrm{bad}}_{(\mathrm{VI})}} \mathscr{L}^{\phi}_{\mathfrak{v}}\big(\tilde{f}_1,\tilde{f}_2;\gamma\big) \times L^{\phi}_{p}\big(\tilde{f}_1,\tilde{f}_2\big) $$

wherein the imprimitive local Euler factors $\mathscr{L}^{\phi}_{\mathfrak{v}}\big(\tilde{f}_1,\tilde{f}_2;\gamma\big)$ interpolate the corresponding $\iota_p\big(L_{\mathfrak{v}}(\tilde{f}_1,\tilde{f}_2,\phi\cdot\psi,k_2+j)\big)$ at every finite order character $\psi : \mathrm{Gal}\left(F^{\mathrm{cyc}}_{p,\infty}/F\right)_0 \to \overline{\mathbb{Q}}_p^{\times}$ which is trivial on Δ.

Combining (i) and (ii), we reinterpret the p-adic regulator formula as

$$ \mathrm{LOG}\circ\mathrm{loc}_p\Big(\mathcal{BF}_{(j)}^{\tilde{f}_1,\tilde{f}_2} \otimes \big(\mathfrak{d}'_{\tilde{f}_1}\otimes\mathfrak{d}''_{\tilde{f}_2}\big)\Big) = \frac{(-1)^{j+1}}{\upsilon_{N_{\tilde{f}_1}}(\tilde{f}_1)} \times \mathfrak{h}_{\tilde{f}_1}\cdot\mathcal{N}x_p^j\big(L_p(\tilde{f}_1,\tilde{f}_2)\big)\,. $$

Bearing (iii) in mind, one next defines S-depleted Beilinson-Flach elements as

$$ \Big(\mathcal{BF}_{(j),(S)}^{\tilde{f}_1,\tilde{f}_2}\Big)^{\phi} := \prod_{\mathfrak{v}\in\Sigma^{\mathrm{bad}}_{(\mathrm{V})}} \mathrm{Tw}_j\big(\mathscr{L}^{\phi}_{\mathfrak{v}}(\tilde{f}_1,\tilde{f}_2;\gamma)\big) \times \prod_{\mathfrak{v}\in\Sigma^{\mathrm{bad}}_{(\mathrm{VI})}} \mathrm{Tw}_j\big(\mathscr{L}^{\phi}_{\mathfrak{v}}(\tilde{f}_1,\tilde{f}_2;\gamma)\big) \times \Big(\mathcal{BF}_{(j)}^{\tilde{f}_1,\tilde{f}_2}\Big)^{\phi} $$

and it follows immediately that at each branch $\phi : \Delta \to \overline{\mathbb{Q}}_p$,

$$ \mathrm{LOG}\circ\mathrm{loc}_p\Big(\Big(\mathcal{BF}_{(j),(S)}^{\tilde{f}_1,\tilde{f}_2}\Big)^{\phi} \otimes \big(\mathfrak{d}'_{\tilde{f}_1}\otimes\mathfrak{d}''_{\tilde{f}_2}\big)\Big) = \frac{(-1)^{j+1}}{\upsilon_{N_{\tilde{f}_1}}(\tilde{f}_1)} \times \mathfrak{h}_{\tilde{f}_1}\cdot\mathcal{N}x_p^j\big(L^{\phi}_{p,(S)}(\tilde{f}_1,\tilde{f}_2)\big)\,. \tag{8.8} $$

In the decomposition (8.7), half of the expression is already interpolated by $\big(\mathcal{BF}^{\tilde{f}_1,\tilde{f}_2}_{(j),(S)}\big)^{\phi}$ and we must produce a 'dual element' interpolating the other half. Now it was shown in [KLZ17, Theorem 11.4.3] that the $\mathcal{O}_K[\![\gamma-1]\!]$-module

$$\varprojlim_m \widetilde{H}^1_{\text{ét}}\big(\mathbb{Z}\big[\mu_{p^m}, 1/pN\big], T_{\tilde{f}_1} \otimes T_{\tilde{f}_2}(-j); \Delta^{(\emptyset)}\big)^{(\phi)}$$

is torsion-free of rank one at the ϕ-branches where $L_p^{\phi}\big(\tilde{f}_1, \tilde{f}_2\big)$ is not identically zero. In their terminology, $\widetilde{H}^1_{\text{ét}}\big(-; \Delta^{(\emptyset)}\big)$ is the subset of cocycles that are unramified at primes $\ell \neq p$ dividing N, and the Beilinson-Flach elements belong to this subspace. Fix a generator $\mathfrak{e}_{\phi^{-1}}(j)$ of $\varprojlim_m \widetilde{H}^1_{\text{ét}}\big(-; \Delta^{(\emptyset)}\big)^{(\phi^{-1})}$ and choose $c' > 1$ such that $c' \nmid 6pN$. Hence one obtains an isomorphism $\varprojlim_m \widetilde{H}^1_{\text{ét}}\big(\mathbb{Z}\big[\mu_{p^m}, 1/pN\big], T_{\tilde{f}_1^{\rho}} \otimes T_{\tilde{f}_2^{\rho}}(-j); \Delta^{(\emptyset)}\big)^{(\phi^{-1})} \cong \mathcal{O}_K[\![\gamma-1]\!] \cdot \mathfrak{e}_{\phi^{-1}}(j)$, and one may express

$$\Big({}_{c'}\mathcal{BF}^{\tilde{f}_1^{\rho},\tilde{f}_2^{\rho}}_{(j)}\Big)^{\phi^{-1}} = {}_{c'}\mathscr{G}_{\overline{\phi}}(\gamma-1) \cdot \mathfrak{e}_{\phi^{-1}}(j) \text{ for some } {}_{c'}\mathscr{G}_{\overline{\phi}}(\gamma-1) \in \mathcal{O}_K[\![\gamma-1]\!]\,.$$

Let us therefore define the *dual Beilinson-Flach element* by taking

$$\Big({}_{c'}\mathcal{BF}^{\tilde{f}_1^{\rho},\tilde{f}_2^{\rho},\star}_{(j)}\Big)^{\phi^{-1}} := \mathrm{Tw}_{k_2+1-k_1}\big({}_{c'}\mathscr{G}_{\overline{\phi}}(\gamma^{-1}-1)\big) \cdot \mathfrak{e}_{\phi}(j)\,.$$

Because $\mathfrak{e}_{\phi^{-1}}(j)$ is only well-defined up to multiplication by an element of $\mathcal{O}_K[\![\gamma-1]\!]^{\times}$, we select $\mathfrak{e}_{\phi}(j)$ and $\mathfrak{e}_{\phi^{-1}}(j)$ compatibly in order to normalise ${}_{c'}\mathcal{BF}^{\tilde{f}_1^{\rho},\tilde{f}_2^{\rho},\star}$ so that

$$\begin{aligned}
&\mathrm{LOG}_{\mathbb{Q}_p, V^*_{\tilde{f}_1^{\rho}} \otimes V^*_{\tilde{f}_2^{\rho}}(-j)} \circ \mathrm{loc}_p \left(\Big({}_{c'}\mathcal{BF}^{\tilde{f}_1^{\rho},\tilde{f}_2^{\rho},\star}_{(j)}\Big)^{\phi^{-1}} \otimes \big(\mathfrak{d}'_{\tilde{f}_1^{\rho}} \otimes \mathfrak{d}''_{\tilde{f}_2^{\rho}}\big)\right) \\
&\quad = \upsilon^{-1}_{N_{\tilde{f}_1^{\rho}}}\big(\tilde{f}_1^{\rho}\big) \cdot (-1)^{j+1+k_2-k_1} \times \Big(c'^2 - c'^{k_1-3k_2+2-2j} \cdot \eta_1\varepsilon^2(c') \cdot \eta_2\varepsilon^{-2}(c')\Big) \\
&\qquad \times \mathfrak{h}_{\tilde{f}_1} \cdot \mathcal{N}x_p^{k_1-k_2-1-j}\Big(L_p^{\phi^{-1}}\big(\tilde{f}_1^{\rho}, \tilde{f}_2^{\rho}\big)\Big)^{\bullet}
\end{aligned}$$

where we have once again applied the functoriality $\mathrm{LOG}_{\mathbb{Q}_p}\big(-^{\bullet}\big) = \mathrm{LOG}_{\mathbb{Q}_p}(-)^{\bullet}$. Fortunately, one can remove the dependence on the positive integer c' by setting

$$\Big(\mathcal{BF}^{\tilde{f}_1^{\rho},\tilde{f}_2^{\rho},\star}_{(j)}\Big)^{\phi^{-1}} := \big(c'^2 - c'^{k_1-3k_2+2-2j} \cdot \eta_1\varepsilon^2(c') \cdot \eta_2\varepsilon^{-2}(c')\big)^{-1} \cdot \Big({}_{c'}\mathcal{BF}^{\tilde{f}_1^{\rho},\tilde{f}_2^{\rho},\star}_{(j)}\Big)^{\phi^{-1}}$$

at the cost of tensoring by $\mathbb{Q}$ when the Hypothesis (BI) of *op. cit.* does not hold. Moreover, one may S-deplete the dual elements constructed above by defining

$$\left(\mathcal{BF}^{\tilde{f}_1^\rho,\tilde{f}_2^\rho,\star}_{(j),(S)}\right)^{\phi^{-1}} := \prod_{\mathfrak{v}\in S} \mathrm{Tw}_{k_1-k_2-1-j}\left(\mathscr{L}_{\mathfrak{v}}^{\phi^{-1}}(\tilde{f}_1^\rho,\tilde{f}_2^\rho;\gamma^{-1})\right)\cdot\left(\mathcal{BF}^{\tilde{f}_1^\rho,\tilde{f}_2^\rho,\star}_{(j)}\right)^{\phi^{-1}}$$

in which case,

$$\begin{aligned}&\mathrm{LOG}_{\mathbb{Q}_p,V^*_{\tilde{f}_1^\rho}\otimes V^*_{\tilde{f}_2^\rho}(-j)}\circ\mathrm{loc}_p\left(\left(\mathcal{BF}^{\tilde{f}_1^\rho,\tilde{f}_2^\rho,\star}_{(j),(S)}\right)^{\phi^{-1}}\otimes\left(\mathfrak{d}'_{\tilde{f}_1^\rho}\otimes\mathfrak{d}''_{\tilde{f}_2^\rho}\right)\right)\\ &=\frac{(-1)^{j+1+k_2-k_1}}{\upsilon_{N_{\tilde{f}_1^\rho}}(\tilde{f}_1^\rho)}\times\mathfrak{h}_{\tilde{f}_1}\cdot\mathcal{N}x_p^{k_1-k_2-1-j}\left(L_{p,S}^{\phi^{-1}}(\tilde{f}_1^\rho,\tilde{f}_2^\rho)\right)^{\bullet}.\end{aligned}\tag{8.9}$$

The following identities are consequences of assuming $f_1^\rho = f_1$ and $f_2^\rho = f_2$, together with the definition of twisted newforms $\tilde{f}_1 := f_1\otimes\varepsilon$ and $\tilde{f}_2 := f_2\otimes\varepsilon^{-1}$:

- $\mathfrak{d}'_{f_1}\otimes\mathfrak{d}''_{f_2} = \mathfrak{d}'_{\tilde{f}_1}\otimes\mathfrak{d}''_{\tilde{f}_2} = \mathfrak{d}'_{\tilde{f}_1^\rho}\otimes\mathfrak{d}''_{\tilde{f}_2^\rho}$
- $V^*_{f_1}\otimes V^*_{f_2}\cong V^*_{\tilde{f}_1}\otimes V^*_{\tilde{f}_2}\cong V^*_{\tilde{f}_1^\rho}\otimes V^*_{\tilde{f}_2^\rho}$
- $T_{f_1}\otimes T_{f_2}\cong T_{\tilde{f}_1}\otimes T_{\tilde{f}_2}\cong T_{\tilde{f}_1^\rho}\otimes T_{\tilde{f}_2^\rho}$.

According to [Del24, Lemma 3.2], the quotient of the numbers $\frac{\mathfrak{h}_{f_1}}{\Omega(f_1)}$ and $\frac{\mathfrak{h}_{\tilde{f}_1}}{\Omega(\tilde{f}_1)}$ is a p-adic unit[2] which we denote by $\mathfrak{u}_{f_1}\in\mathcal{O}_K^\times$ under the map $\iota_p\circ\iota_\infty^{-1}$. It follows that one may rewrite Equation (8.7) in terms of zeta-elements via

$$\begin{aligned}\mathfrak{h}_{f_1}\cdot\mathcal{N}x_p^j\left(L_p^\phi(f_1\otimes f_2)\right) &= \mathfrak{u}_{f_1}\times\Big(\mathcal{Y}_2^\phi\big(M(j)\big)\times\mathfrak{h}_{\tilde{f}_1}\cdot\mathcal{N}x_p^j\left(L^\phi_{p,(S)}(\tilde{f}_1,\tilde{f}_2)\right)\\ &\quad+\mathcal{W}^{(\varepsilon_0),\phi}\big(M(j)\big)\cdot\mathcal{Y}_1^\phi\big(M(j)\big)\times\mathfrak{h}_{\tilde{f}_1}\cdot\mathcal{N}x_p^{k_1-k_2-1-j}\left(L^{\phi^{-1}}_{p,(S)}(\tilde{f}_1^\rho,\tilde{f}_2^\rho)\right)^{\bullet}\Big)\\ &=\mathfrak{u}_{f_1}\times\Bigg(\mathcal{Y}_2^\phi\big(M(j)\big)\frac{\upsilon_{N_{\tilde{f}_1}}(\tilde{f}_1)}{(-1)^{j+1}}\\ &\qquad\cdot\mathrm{LOG}_{\mathbb{Q}_p,V^*_{f_1}\otimes V^*_{f_2}(-j)}\circ\mathrm{loc}_p\left(\left(\mathcal{BF}^{\tilde{f}_1,\tilde{f}_2}_{(j),(S)}\right)^{\phi}\otimes\left(\mathfrak{d}'_{f_1}\otimes\mathfrak{d}''_{f_2}\right)\right)\\ &\quad+\mathcal{W}^{(\varepsilon_0),\phi}\big(M(j)\big)\cdot\mathcal{Y}_1^\phi\big(M(j)\big)\times\frac{\upsilon_{N_{\tilde{f}_1^\rho}}(\tilde{f}_1^\rho)}{(-1)^{j+1+k_2-k_1}}\\ &\qquad\cdot\mathrm{LOG}_{\mathbb{Q}_p,V^*_{f_1}\otimes V^*_{f_2}(-j)}\circ\mathrm{loc}_p\left(\left(\mathcal{BF}^{\tilde{f}_1^\rho,\tilde{f}_2^\rho,\star}_{(j),(S)}\right)^{\phi^{-1}}\otimes\left(\mathfrak{d}'_{f_1}\otimes\mathfrak{d}''_{f_2}\right)\right)\Bigg)\end{aligned}$$

where we applied both Equations (8.8) and (8.9) to obtain the second line. Finally, if we define $\underline{\mathcal{BF}}^{\mathrm{prim}}_{(j)}$ to be the p-adic zeta-element with ϕ-branch as

$$\left(\underline{\mathcal{BF}}^{\mathrm{prim}}_{(j)}\right)^{\phi} := \mathfrak{u}_{f_1}\times\left(\mathcal{Y}_2^\phi\big(M(j)\big)\times\frac{\upsilon_{N_{\tilde{f}_1}}(\tilde{f}_1)}{(-1)^{j+1}}\cdot\left(\mathcal{BF}^{\tilde{f}_1,\tilde{f}_2}_{(j),(S)}\right)^{\phi}\right.$$

---

[2] In fact, the statement of [Del24, Lemma 3.2] relies on an Ihara's Lemma type condition which has subsequently been proven in [Mal25].

$$+\,\mathcal{W}^{(\varepsilon_0),\phi}\big(M(j)\big)\cdot\mathcal{Y}_1^{\phi}\big(M(j)\big)\times\frac{\upsilon_{N_{\tilde{f}_1^{\rho}}}(\tilde{f}_1^{\rho})}{(-1)^{j+1+k_2-k_1}}\cdot\Big(\mathcal{BF}_{(j),(S)}^{\tilde{f}_1^{\rho},\tilde{f}_2^{\rho},\star}\Big)^{\phi^{-1}}\Bigg)$$

then $\underline{\mathcal{BF}}_{(j)}^{\mathrm{prim}}=\sum_{\phi}\big(\underline{\mathcal{BF}}_{(j)}^{\mathrm{prim}}\big)^{\phi}$ must satisfy the interpolation property given in the statement of the theorem, and the proof is finished. ■

Remark 19. If one further assumes that Hypothesis (BI) of *op. cit.* is true, then the elements $\underline{\mathcal{BF}}_{(j)}^{\mathrm{prim}}$ constructed above are p-integral, in other words $\big(\underline{\mathcal{BF}}_{(j)}^{\mathrm{prim}}\big)^{\phi}\in\varprojlim_m H^1_{\text{ét}}\big(\mathbb{Z}[\mu_{p^m},1/pN],T_{f_1}\otimes T_{f_2}(-j)\big)^{(\phi)}$ at each ϕ-branch and any $j\in\{0,\ldots,k_1-k_2-1\}$.

8.2.3 The symmetric square case

Let f be the exact same newform that we considered in Section 8.2.1. There are already two Euler systems available for the symmetric square of f, namely the Loeffler–Zerbes version from [LZ19], as well as the elements constructed by Skinner–Vincentelli in [SV24]. Unfortunately, applying our replenishment techniques directly to these elements proves troublesome and this section outlines exactly why these obstructions arise.

We discuss the Loeffler–Zerbes approach first, which a formula of Dasgupta in (c.f. [Das16, Theorem 1])

$$L_p(f\otimes f\otimes\psi,s)=L_p\big(\operatorname{Sym}^2(f)\otimes\psi,s\big)\times L_p(\psi\eta,s-k+1)\,.$$

Let $p>5$ be a prime satisfying $p\nmid a_p(f)$ and $p\nmid N_f$. To avoid any exceptional zero scenarios (see [LZ19, §3]), we will also fix a Dirichlet character ψ such that $\psi\eta(p)\neq 1$. The image of the Beilinson-Flach elements takes values in one of the summands

$$T_f^{\otimes 2}=T_f\otimes_{\mathcal{O}_{K_f}}T_f\cong\operatorname{Sym}^2(T_f)\oplus\bigwedge\nolimits^2 T_f$$

depending on the sign of ψ; hence, the imprimitive element ${}_c\mathcal{BF}_{(j)}^{f,f}$ belongs to

$$\begin{cases}\varprojlim_m H^1_{\text{ét}}\Big(\mathbb{Z}\big[\mu_{\mathfrak{c}(\psi)\cdot p^m},1/pM\big]^+,\ \operatorname{Sym}^2\big(T_f\big)\big(\psi\kappa_{\mathrm{cyc}}^{-j}\big)\Big) & \text{if } \psi(-1)=(-1)^{j+1}\\ \varprojlim_m H^1_{\text{ét}}\Big(\mathbb{Z}\big[\mu_{\mathfrak{c}(\psi)\cdot p^m},1/pM\big]^+,\ \bigwedge^2 T_f\big(\psi\kappa_{\mathrm{cyc}}^{-j}\big)\Big) & \text{if } \psi(-1)=(-1)^{j}\,.\end{cases}$$

For reasons of parity, the dual imprimitive element $\big({}_{c'}\mathcal{BF}^{f^\rho,f^\rho,\bigstar}_{(j)}\big)^{\phi^{-1}}$ takes coefficients in (a Tate twist of) the *opposite* summand to $\big({}_{c}\mathcal{BF}^{f,f}_{(j)}\big)^{\phi}$, thus there is no way of combining them together using the decomposition in Section 7.2 as they belong to incompatible cohomology groups. Indeed the most one can say is that the primitive zeta-element lies in $\varprojlim_m H^1_{\text{ét}}\big(\mathbb{Z}\big[\mu_{p^m}, 1/pM\big]^+, T_f \otimes T_f\big)$, which is not enough to proceed.

The Euler system constructed by Skinner and Vincentelli seems to be a better fit. Let $\mathrm{Ad}^0(V_f)$ denote the adjoint representation where the superscript '0' means the submodule of endomorphisms whose trace is 0. For each positive square-free integer M coprime to p, the authors [SV24, Theorem 5.11.4] have constructed cocycle systems

$$\underline{\mathbf{C}}(\mathbf{F}) = \big\{\mathbf{C}_{Mp^m}(\mathbf{F})\big\}_{m\geq 1} \in \mathbb{Q}\otimes \varprojlim_{m} H^1\Big(\mathbb{Q}(\mu_{p^m M})^+, \mathrm{Ad}^0\left(T_{\mathbf{F}}\right)(1)\Big)$$

attached to a Hida family $\mathbf{F}$ which passes through the p-stabilisation of the form f. Recall that there is an isomorphism, $\mathrm{Sym}^2(V_f) \cong \mathrm{Ad}^0(V_f) \otimes \kappa_{\mathrm{cyc}}\eta^{-1}$ so it is suitable for us to work with the above Euler system (in fact, there is no need to twist by a non-trivial Dirichlet character ψ in this case). If $M = 1$ and for every even weight specialization f, the image under Perrin-Riou's regulator map yields

$$\mathrm{LOG}_{\mathbb{Q}_p} \circ \mathrm{loc}_p^-\,\big(\underline{\mathbf{C}}(f)(j)\big) = \Xi_f{\cdot}\mathfrak{h}_f{\cdot}\mathcal{N}x_p^j\big(L_p(\mathrm{Sym}^2(f), s)\big) \times \prod_{\mathfrak{v}\in\Sigma^{\mathrm{bad}}_{(\mathrm{IV})}} \mathscr{L}_{\mathfrak{v}}\big(\,\mathrm{Sym}^2(f);\gamma\big)$$

where Ξ_f interpolates the product $U_f \cdot M_f \cdot \chi^2(Q_{\mathrm{rps}})$ at odd characters χ (for more details see §8.3 of *op. cit.*) and $L_p\big(\,\mathrm{Sym}^2(f,s)\big)$ is a re-normalisation of Liu's p-adic L-function constructed in [Liu20].

We should point out that by meticulously choosing the volume sections where the Eisenstein series are defined, most of the missing Euler factors were restored by the authors in [SV24] aside from the exceptional supersingular primes, which are denoted by $\Sigma^{\mathrm{bad}}_{(\mathrm{IV})}$ in Section 7.3.3. This is another advantage of using Liu's approach. The obstruction here is that the p-adic interpolation properties are only known for the right-half of the critical twists, that is at

$s \in \{k, \ldots, 2k-2\}$. The remaining critical values $s \in \{1, \ldots, k-1\}$ exhibit a contrasting algebraicity behavior for the complex L-function, and there is no explicit reciprocity law available at these twists (see [SV24, Theorem 7.6.15]).

Remark 20. It has been established in [SV24, §10] that the global cohomology group $\varprojlim_m H^1_{\text{ét}}\big(\mathbb{Z}\big[\mu_{p^m}, 1/pN\big]^+, \mathrm{Ad}^0(T_f)(1)\big)^{(\phi^{-1})}$ is a rank one torsion-free $\mathcal{O}_K[\![\gamma-1]\!]$-module *provided Greenberg's $\mathcal{L}$-invariant attached to* $\mathrm{Sym}^2(f)$ *is non-zero* (which is known at all but finitely many classical specialisations of $\mathbf{F}$ from Hida). This 'free of rank one condition' is necessary to produce the dual zeta-element but without the reciprocity law, one cannot then relate it to the dual p-adic L-function.

Bibliography

[AL21] Suman Ahmed and Meng Fai Lim. "On the signed Selmer groups of congruent elliptic curves with semistable reduction at all primes above p". In: *Acta Arith.* 197.4 (2021), pp. 353–377.

[AV75] Yvette Amice and Jacques Vélu. "Distributions p-adiques associées aux séries de Hecke". In: *Journées arithmétiques de Bordeaux*. Astérisque 24-25. Société mathématique de France, 1975, pp. 119–131.

[AS86] Ash Avner and Glenn Stevens. "Modular forms in characteristic ℓ and special values of their L-functions". In: *Duke Math. J.* 53.3 (1986), pp. 849–868.

[Ben97] Dennis Benois. "p-adic periods and explicit reciprocity laws". In: *Journal Reine Angew. Math.* 493 (1997), pp. 115–151.

[Ber+14] Massimo Bertolini et al. "p-adic L-functions and Euler systems: a tale in two trilogies". In: *Automorphic Forms and Galois Representations*. Vol. 414. London Math. Society Lecture Note Series. Cambridge Univeristy Press, 2014, pp. 52–101.

[BK90] Spencer Bloch and Kazuya Kato. "Tamagawa numbers of motives and L-functions". In: *The Grothendieck Festschrift, 1*. Progress in Math. Boston, MA: Birkhäuser Boston, 1990, pp. 333–400.

[BCP] Weib Bosma, John Cannon, and Catherine Playoust. *The MAGMA Computer Algebra System*. Version 2.27-8, the University of Sydney.

[Bru67] Armand Brumer. "On the units of algebraic number fields". In: *Mathematika* 14.2 (1967), pp. 121–124.

[Bum97] Daniel Bump. *Automorphic Forms and Representations.* Cambridge: Cambridge University Press, 1997.

[BL21] Kazim Büyükboduk and Antonio Lei. "Iwasawa theory of elliptic modular forms over imaginary quadratic fields at non-ordinary primes". In: *Int. Math. Res. Not. IMRN* 2021.14 (2021), pp. 10654–10730.

[CKL17] Francesc Castella, Chan-Ho Kim, and Matteo Longo. "Variation of anticyclotomic Iwasawa invariants in Hida families". In: *Algebra Number Theory* 11.10 (2017), pp. 2339–2368.

[CK17] Suh Hyun Choi and Byoung Du Kim. "Congruences of two-variable p-adic L-functions of congruent modular forms of different weights". In: *Ramanujan J.* 43.1 (2017), pp. 163–195.

[Coa91] John Coates. "Motivic p-adic L-functions". In: *L-functions and Arithmetic.* London Mathematical Society Lecture Note Series. Cambridge University Press, 1991, pp. 141–172.

[CPR89] John Coates and Bernadette Perrin-Riou. "On p-adic L-functions attached to motives over $\mathbb{Q}$". In: *Adv. Stud. Pure Math.* 17 (1989), pp. 23–54.

[CS87] John Coates and Claus-Günther Schmidt. "Iwasawa theory for the symmetric square of an elliptic curve". In: *Journal Reine Angew. Math.* 375/376 (1987), pp. 104–156.

[Col79] Robert Coleman. "Division values in local fields". In: *Invent. Math.* 53 (1979), pp. 91–116.

[Col98] Pierre Colmez. "Théorie d'Iwasawa des représentations de de Rham d'un corps local". In: *Ann. of Math.* 148 (1998), pp. 458–571.

[CS18] Andrew Corbett and Abhishek Saha. "On the order of vanishing of newforms at cusps". In: *Math. Res. Lett.* 25.6 (2018), pp. 1771–1804.

[DD97] Andrzej Dabrowski and Daniel Delbourgo. "S-adic L-functions attached to the symmetric square of a newform". In: *Proc. Lond. Math. Soc.* 74.3 (1997), pp. 559–611.

[Das16] Samit Dasgupta. "Factorization of p-adic Rankin L-series". In: *Invent. Math.* 205 (2016), pp. 221–268.

[Del08] Daniel Delbourgo. *Elliptic curves and big Galois representations.* Vol. 356. London Math Society Lecture Note Series. Cambridge University Press, 2008, pp. 1–292.

[Del16] Daniel Delbourgo. "Exceptional zeroes of p-adic L-functions over non-abelian field extensions". In: *Glasg. Math. J.* 58.2 (2016), pp. 385–432.

[Del20] Daniel Delbourgo. "Variation of the analytic λ-invariant over a solvable extension". In: *Proc. Lond. Math. Soc.* 120 (2020), pp. 918–960.

[Del21] Daniel Delbourgo. "Variation of the algebraic λ-invariant over a solvable extension". In: *Math. Proc. Camb. Phil. Soc.* 170 (2021), pp. 499–521.

[Del24] Daniel Delbourgo. "On the Iwasawa μ-invariant and λ-invariant associated to tensor products of newforms". In: *Ann. Inst. Fourier* 74 (2024), pp. 451–502.

[DL20] Daniel Delbourgo and Antonio Lei. "Heegner cycles and congruences between anticyclotomic p-adic L-functions over CM-extensions". In: *New York Journal of Math.* 26 (2020), pp. 496–525.

[Del69] Pierre Deligne. "Formes modulaires et reprêsentations l-adiques". In: *Séminaire Bourbaki* 179 (1969), Exp. No. 355, 139–172.

[Del73] Pierre Deligne. “Les constantes des equations fonctionnelles des fonctions L”. In: *Modular Functions of One Variable II.* Berlin, Heidelberg: Springer Berlin Heidelberg, 1973, pp. 501–597.

[Del79] Pierre Deligne. “Valeurs de fonctions L et periodes integrales”. In: *Proc. Symp. Pure Math.* 33.2 (1979), pp. 313–346.

[Dia91] Fred Diamond. “Congruence primes for cusp forms of weight $k \geq 2$”. In: *Courbes modulaires et courbes de Shimura, Orsay, 1987-1988.* Astérisque 196/197. Société mathématique de France, 1991, pp. 205–213.

[DFG04] Fred Diamond, Matthias Flach, and Li Guo. “The Tamagawa number conjecture of adjoint motives of modular forms”. In: *Ann. Sci. Éc. Norm. Supér.* 37.5 (2004), pp. 663–727.

[Dim05] Mladen Dimitrov. “Galois representations modulo p and cohomology of Hilbert modular varieties”. In: *Ann. Sci. Éc. Norm. Supér.* 38.4 (2005), pp. 505–551.

[Dim09] Mladen Dimitrov. “On Ihara’s lemma for Hilbert modular varieties”. In: *Comp. Math.* 145.5 (2009), pp. 1114–1146.

[Dis17] Daniel Disegni. “The p-adic Gross-Zagier formula on Shimura curves”. In: *Compos. Math.* 153 (2017), pp. 1987–2074.

[DD07] Tim Dokchitser and Vladimir Dokchitser. “Computations in non-commutative Iwasawa theory”. In: *Proc. Lond. Math. Soc.* 94.1 (2007), pp. 211–272.

[Edi92] Bas Edixhoven. “The weight in Serre’s conjectures on modular forms”. In: *Invent. Math.* 109.3 (1992), pp. 563–594.

[EPW06] Matthew Emerton, Robert Pollack, and Tom Weston. “Variation of Iwasawa invariants in Hida families”. In: *Invent. Math.* 163 (2006), pp. 523–580.

[FJ95] Gerd Faltings and Bruce Jordan. "Crystalline cohomology and $\mathrm{GL}(2,\mathbb{Q})$". In: *Israel Journal of Math.* 90 (1995), pp. 1–66.

[FM87] Jean Marc Fontaine and William Messing. "p-adic periods and p-adic étale cohomology". In: *Contemporary Math.* 67 (1987), pp. 179–207.

[Gar90] Paul Garrett. *Holomorphic Hilbert Modular Forms.* Belmont, California: Brooks/Cole Publishing Company, 1990.

[Gee88] Gerard van der Geer. *Hilbert Modular Surfaces.* Springer-Verlag Berlin Heidelberg, 1988.

[GJ78] Stephen Gelbart and Hervé Jacquet. "A relation between automorphic representations of GL(2) and GL(3)". In: *Ann. Sci. Éc. Norm. Supér.* 11 (1978), pp. 471–542.

[Gou90] Fernando Gouvêa. "Deforming Galois representations: controlling the conductor". In: *J. Number Theory* 34.1 (1990), pp. 95–113.

[Gre89] Ralph Greenberg. "Iwasawa theory for p-adic representations". In: *Adv. Stud. Pure Math.* 17 (1989), pp. 97–137.

[Gre94] Ralph Greenberg. "Iwasawa theory and p-adic deformation of motives". In: *Motives, 2.* Vol. 55. Proceedings of Symposia in Pure Mathematics 2. American Mathematical Society, 1994, pp. 193–224.

[Gre97] Ralph Greenberg. "Iwasawa theory for elliptic curves". In: *Arithmetic Theory of Elliptic Curves, Cetraro 1997.* Vol. 1716. Lecture Notes in Math. 1997, pp. 51–144.

[GS93] Ralph Greenberg and Glenn Stevens. "p-adic L-functions and p-adic periods of modular forms". In: *Invent. Math.* 111 (1993), pp. 407–448.

[GV00] Ralph Greenberg and Vinayak Vatsal. "Iwasawa invariants of elliptic curves". In: *Invent. Math.* 142.2 (2000), pp. 17–63.

[Gro90] Benedict Gross. “A tameness criterion for Galois representations associated to modular forms (mod p)”. In: *Duke Math. Journal* 61.2 (1990), pp. 445–517.

[Gue95] Pavel Guerzhoy. “Jacobi-Eisenstein series and p-adic interpolation of symmetric squares of cusp forms”. In: *Ann. I. Fourier* 45 (1995), pp. 605–625.

[Ham23] Parham Hamidi. “Residual supersingular Iwasawa theory over quadratic imaginary fields”. In: *Bull. Korean Math. Soc.* 60.4 (2023), pp. 1035–1059.

[HL14] Robert Harron and Antonio Lei. “Iwasawa theory for symmetric powers of CM modular forms at non-ordinary primes”. In: *J. Théor. Nombres Bordeaux* 26.3 (2014), pp. 673–708.

[HL19] Jeffrey Hatley and Antonio Lei. “Arithmetic properties of signed Selmer groups at non-ordinary primes”. In: *Ann. Inst. Fourier (Grenoble)* 69.3 (2019), pp. 1259–1294.

[HLV22] Jeffrey Hatley, Antonio Lei, and Stefano Vigni. “Λ-submodules of finite index of anticyclotomic plus and minus Selmer groups of elliptic curves”. In: *Manuscripta Math.* 167.3-4 (2022), pp. 589–612.

[Hid88] Haruzo Hida. “On p-adic Hecke algebras for GL_2 over totally real fields”. In: *Ann. of Math.* 128.2 (1988), pp. 295–384.

[Hid91] Haruzo Hida. “On p-adic L-functions of $\mathrm{GL}(2)\times\mathrm{GL}(2)$ over totally real fields”. In: *Ann. I. Fourier* 41.2 (1991), pp. 311–391.

[Hid93] Haruzo Hida. *Elementary Theory of L-functions and Eisenstein Series.* London Mathematical Society Student Texts. Cambridge University Press, 1993.

[Hid94] Haruzo Hida. “On the critical values of L-functions of $\mathrm{GL}(2)$ and $\mathrm{GL}(2) \times \mathrm{GL}(2)$”. In: *Duke Math. J.* 74.2 (1994), pp. 431–529.

[Hid06] Haruzo Hida. *Hilbert Modular Forms and Iwasawa Theory.* Oxford Mathematical Monographs. Oxford University Press, 2006.

[HY24] Ming-Lun Hsieh and Shunsuke Yamana. "Four-variable p-adic triple product L-functions and the trivial zero conjecture". In: *Math. Ann.* 390.2 (2024), pp. 889–976.

[Hum80] James Humphreys. *Arithmetic Groups.* Springer Berlin, Heidelberg, 1980.

[Iwa59] Kenkichi Iwasawa. "On Γ-extensions of algebraic number fields". In: *Bull. Amer. Math. Soc.* 65 (1959), pp. 183–226.

[JL70] Hervé Jacquet and Robert Langlands. "Automorphic forms on GL(2)". In: *Lecture Notes in Mathematics.* Vol. 114. Heidelberg: Springer-Berlin, 1970, pp. 1–548.

[Jan88] Uwe Jannsen. "Continuous étale cohomology". In: *Math. Ann.* 280.3 (1988), pp. 207–245.

[KS24] Takenori Kataoka and Takamichi Sano. "On Euler systems for motives and Heegner points". In: *J. Assoc. Math. Res.* 2.2 (2024), pp. 154–208.

[Kat04] Kazuya Kato. "p-adic Hodge theory and values of zeta functions of modular forms". In: *Cohomologie p-adiques et applications arithmétiques (III).* Astérisque 295. Société mathématique de France, 2004, pp. 117–290.

[KKT96] Kazuya Kato, Masato Kurihara, and Takeshi Tsuji. *Local Iwasawa theory of Perrin-Riou and syntomic complexes.* unpublished preprint. 1996.

[KW09] Chandrashekhar Khare and Jean-Pierre Wintenberger. "Serre's modularity conjecture (I)". In: *Invent. Math.* 178 (2009), pp. 485–504.

[Kim09] Byoung Du Kim. "The Iwasawa invariants of the plus/minus Selmer groups". In: *Asian J. Math.* 13.2 (2009), pp. 181–190.

[Kim21] Byoung Du Kim. "Congruences of algebraic p-adic L-functions and the Main Conjecture of Iwasawa Theory". In: *J. Number Theory* 226 (2021), pp. 168–212.

[Kim23] Chan-Ho Kim. *Refined applications of Kato's Euler systems for modular forms.* 2023. URL: `https://arxiv.org/abs/2203.12157`.

[KLP19] Chan-Ho Kim, Jaehoon Lee, and Gautier Ponsinet. *On the Iwasawa invariants of Kato's zeta elements for modular forms.* Preprint at `https://arxiv.org/pdf/1909.01764`. 2019.

[KLZ17] Guido Kings, David Loeffler, and Sarah Zerbes. "Rankin-Eisenstein classes and explicit reciprocity laws". In: *Camb. J. Math.* 5 (2017), pp. 1–122.

[Kob03] Shin-ichi Kobayashi. "Iwasawa theory for elliptic curves at supersingular primes". In: *Invent. Math.* 152 (2003), pp. 1–36.

[Lei11] Antonio Lei. "Iwasawa theory for modular forms at supersingular primes". In: *Compos. Math.* 147.3 (2011), pp. 803–838.

[LLZ10] Antonio Lei, David Loeffler, and Sarah Zerbes. "Wach modules and Iwasawa theory for modular forms". In: *Asian J. Math.* 14.4 (2010), pp. 475–528.

[LLZ11] Antonio Lei, David Loeffler, and Sarah Zerbes. "Coleman maps and p-adic regulator". In: *Algebra Number Theory* 5.8 (2011), pp. 1095–1131.

[LLZ14] Antonio Lei, David Loeffler, and Sarah Zerbes. "Euler systems for Rankin-Selberg convolutions of modular forms". In: *Ann. of Math.* 180 (2014), pp. 653–771.

[Liu20] Zheng Liu. "p-adic L-functions for ordinary families on symplectic groups". In: *J. Inst. Math. Jussieu* 19.4 (2020), pp. 1287–1347.

[Liv89] Ron Livné. “On the conductors of mod l Galois representations coming from modular forms”. In: *J. Number Theory* 31.2 (1989), pp. 133–141.

[LMFDB] The LMFDB Collaboration. *The L-functions and modular forms database.* `https://www.lmfdb.org`. [Online; accessed 13 January 2026]. 2026.

[LZ19] David Loeffler and Sarah Zerbes. “Iwasawa theory for the symmetric square of a modular form”. In: *Journal Reine Angew. Math.* 752 (2019), pp. 179–210.

[Mal25] Simone Maletto. *An integral two variable p-adic L-function for the χ-twisted symmetric square representation associated to an ordinary modular form.* PhD. thesis. University of British Columbia. 2025.

[Man76] Yu. Manin. “Non-Archimedean integration and Jacquet-Langlands p-adic L-functions”. In: *Russ. Math. Surv.* 31.1 (1976), pp. 5–57.

[MS21] Jeffrey Manning and Jack Shoton. “Ihara’s Lemma for Shimura curves over totally real fields via patching”. In: *Math. Ann.* 379 (2021), pp. 187–234.

[MSD74] Barry Mazur and Peter Swinnerton-Dyer. “Arithmetic of Weil curves”. In: *Invent. Math.* 25 (1974), pp. 1–61.

[MTT86] Barry Mazur, John Tate, and Jeremy Teitelbaum. “On p-adic analogues of the conjectures of Birch and Swinnerton-Dyer”. In: *Invent. Math.* 84 (1986), pp. 1–48.

[MW84] Barry Mazur and Andrew Wiles. “Class fields of abelian extensions of $\mathbb{Q}$”. In: *Invent. Math.* 76 (1984), pp. 179–330.

[Miy71] Toshitsune Miyake. “On automorphic forms on GL_2 and Hecke operators”. In: *Ann. of Math.* 94.1 (1971), pp. 174–189.

[Neu99] Jürgen Neukirch. *Algebraic Number Theory.* Springer-Verlag Berlin, Heidelberg, 1999.

[Och00] Tadashi Ochiai. “Control theorem for Bloch-Kato’s Selmer groups of p-adic representations”. In: *J. Number Theory* 82.1 (2000), pp. 69–90.

[Pan89] A. A. Panchishkin. “Convolutions of Hilbert modular forms and their non-Archimedean analogues”. In: *Math. USSR-Sb.* 64.2 (1989), pp. 571–584.

[Pan94] Alexei A. Panchishkin. “Motives over totally real fields and p-adic L-functions”. In: *Ann. I. Fourier* 44.4 (1994), pp. 989–1023.

[PR94] Bernadette Perrin-Riou. “Théorie d’Iwasawa des représentations p-adiques sur un corps local”. In: *Invent. Math.* 115 (1994), pp. 81–149.

[PSR87] I. Piatetski-Shapiro and Stephen Rallis. “Rankin triple L functions”. In: *Compos. Math.* 64 (1987), pp. 31–115.

[Pol03] Robert Pollack. “On the p-adic L-function of a modular form at a supersingular prime”. In: *Duke Math. Journal* 118.3 (2003), pp. 523–558.

[PR04] Robert Pollack and Karl Rubin. “The main conjecture for CM elliptic curves at supersingular primes”. In: *Ann. of Math.* 159 (2004), pp. 447–464.

[PW11] Robert Pollack and Tom Weston. “On anticyclotomic μ-invariants of modular forms”. In: *Compos. Math.* 147.5 (2011), pp. 1353–1381.

[Pon20] Gautier Ponsinet. “On the structure of signed Selmer groups”. In: *Math. Zeit* 294.3-4 (2020), pp. 1635–1658.

[Ran39] Robert A. Rankin. “Contributions to the theory of Ramanujan’s function $\tau(n)$ and similar arithmetic functions: II. The order of the Fourier coefficients of integral modular forms". In: *Math. Proc. Camb. Philos. Soc.* 35.3 (1939), pp. 357–372.

[RS23] Anwesh Ray and R. Sujatha. "Euler characteristics and their congruences for multisigned Selmer groups". In: *Canad. J. Math.* 75.1 (2023), pp. 298–321.

[RSV23] Anwesh Ray, R. Sujatha, and Vinayak Vatsal. "Iwasawa invariants for symmetric square representations". In: *Res. Math. Sci.* 10.27 (2023), pp. 1–43.

[Ros16] Giovanni Rosso. "Derivative at $s = 1$ of the p-adic L-function of the symmetric square of a Hilbert modular form". In: *Israel Journal of Math.* 215 (2016), pp. 255–315.

[Rub91] Karl Rubin. "The "main conjectures" of Iwasawa theory for imaginary quadratic fields". In: *Invent. Math.* 103.1 (1991), pp. 25–68.

[Rub00] Karl Rubin. *Euler Systems.* Annals of Mathematics Studies. Princeton University Press, 2000.

[Sch88] Claus-Günther Schmidt. "p-adic measures attached to automorphic representations of GL(3)". In: *Invent. Math.* 92 (1988), pp. 597–631.

[Sch90] Anthony Scholl. "Motives for modular forms". In: *Invent. Math.* 100 (1990), pp. 419–430.

[Sel40] Atle Selberg. "Bemerkungen über eine Dirichletsche Reihe, die mit der Theorie der Modulformen nahe verbunden ist". In: *Arch. Math. Naturvid.* 43 (1940), pp. 47–50.

[Ser60] Jean-Pierre Serre. "Classes des corps cyclotomiques". In: *Séminaire Bourbaki : années 1958/59 - 1959/60, exposés 169-204.* Séminaire Bourbaki 5. talk:174. Société mathématique de France, 1960, pp. 83–93.

[Ser72] Jean-Pierre Serre. “Propriétés galoisiennes des points d’ordre fini des courbes elliptiques”. In: *Invent. Math.* 15 (1972), pp. 259–331.

[ST68] Jean-Pierre Serre and John Tate. “Good reduction of abelian varieties”. In: *Ann. of Math.* 88.3 (1968), pp. 492–517.

[SW93] Thomas Shemanske and Lynne Walling. “Twists of Hilbert modular forms”. In: *Trans. Amer. Math. Soc.* 338.1 (1993), pp. 375–403.

[Shi61] Goro Shimura. “On the zeta-functions of the algebraic curves uniformized by certain automorphic functions”. In: *J. Math. Soc. Japan* 13.3 (1961), pp. 275–331.

[Shi62] Goro Shimura. “On Dirichlet series and abelian varieties attached to automorphic forms”. In: *Ann. of Math.* 76.2 (1962), pp. 237–294.

[Shi78] Goro Shimura. “The special values of the zeta functions associated with Hilbert modular forms”. In: *Duke Math. J.* 45.3 (1978), pp. 637–679.

[Sil94] Joseph Silverman. *Advanced Topics in the Arithmetic of Elliptic Curves.* Vol. 151. Graduate Texts in Mathematics. Springer Science+Business Media New York, 1994.

[SU14] Christopher Skinner and Eric Urban. “The Iwasawa Main Conjectures for (GL_2)”. In: *Invent. Math.* 195 (2014), pp. 1–277.

[SV24] Christopher Skinner and Marco Vincentelli. *An Euler system for the adjoint of a modular form.* 2024. URL: `https://drive.google.com/file/d/1L3r5RujWxsytTwUa93xmBGUaVI-9Oquc/view`.

[Spr12] Florian Ito Sprung. “Iwasawa theory for elliptic curves at supersingular primes: A pair of main conjectures”. In: *J. Number Theory* 132.7 (2012), pp. 1483–1506.

[Spr17] Florian Ito Sprung. “On pairs of p-adic L-functions for weight two modular forms”. In: *Algebra Number Theory* 11.4 (2017), pp. 885–928.

[Tag02] Yuichiro Taguchi. “Induction formula for the Artin conductors of mod l Galois representations”. In: *Proc. Amer. Math. Soc.* 130.10 (2002), pp. 2865–2869.

[Tat67] John Tate. “Fourier analysis in number fields and Hecke’s zeta function”. In: *Algebraic Number Theory.* Ed. by A. Fröhlich and J.W.S Cassels. Washington, D.C.: Thompson Book Company Inc., 1967, pp. 305–347.

[Tat79] John Tate. “Number Theoretic Background”. In: *Automorphic forms, representations and L-functions, Part 2.* Ed. by A. Borel and W. Casselman. Vol. 33. Providence, R.I.: Proc. Sympos. Pure Math., XXXIII, Amer. Math. Soc., 1979, pp. 3–26.

[Tay89] Richard Taylor. “On Galois representations associated to Hilbert modular forms”. In: *Invent. Math.* 98 (1989), pp. 265–280.

[Vat98] Vinayak Vatsal. “Canonical periods and congruence formulae”. In: *Duke Math. J.* 98.2 (1998), pp. 397–419.

[Viš76] M. M. Višik. “Non-Archimedean measures connected with Dirichlet series”. In: *Math USSR-Sb.* 28.2 (1976), pp. 216–228.

[Wil90] Andrew Wiles. “The Iwasawa conjecture for totally real fields”. In: *Ann. of Math.* 131.3 (1990), pp. 493–540.

[Wil95] Andrew Wiles. “Modular elliptic curves and Fermat’s last theorem”. In: *Ann. of Math.* 141.3 (1995), pp. 443–551.