Variable Separation for Heat and Schrödinger equations

by

Gregory John Reid

A thesis
submitted in partial fulfilment
of the requirements for the degree
of

Doctor of Philosophy

Mathematics Department,
University of Waikato,
Hamilton, New Zealand.

March 1984

To my Parents

Abstract

All R-separable coordinate systems are classified for the equation

$$\Delta_m \Psi + 2\varepsilon \frac{\partial}{\partial t} \Psi = E\Psi \ , \qquad (*)$$

where $\Delta_m = \sum_{u=1}^{m} \frac{\partial^2}{(\partial y^u)^2}$ and the variables t and y^u are real. An R-separable coordinate system for (*) is a coordinate system $y^u = y^u(x^k)$, $t = t(x^k)$ such that (*) admits a solution of form $\Psi = e^{R(x^j)} \prod_{i=1}^{m+1} \Psi_i(x^i; c_1, \cdots, c_{m+1})$. For $\varepsilon = \frac{i}{2}$ and $\varepsilon = -\frac{1}{2}$, $E = 0$, (*) yields standardised versions of the Schrödinger and Heat equations respectively. Recognition of (*) as a symmetry-reduced version of the Helmholtz equation on $m+2$ dimensional Minkowski space, for which there is a well developed theory of separation, is central to the solution of the classification problem. The solution is built from the case $\varepsilon = 0$ which has been solved by Kalnins and Miller in an article submitted to the Transactions of the American Mathematical Society.

We find the operators and separation equations for (*) and give a detailed treatment of the physically interesting case $m=3$. The operators are always in the enveloping algebra of the Schrödinger algebra. This means that much of the special function theory relating to the separated solutions can be reduced to problems in the representation theory of this algebra.

Various complex extensions of (*) are treated. In particular a class of systems is uncovered leading to nontrivial R-separation for the Hamilton-Jacobi equation corresponding to (*). A formula is derived for expressing the second order Killing tensors describing separation in flat spaces in terms of the enveloping algebra. This formula forms the basis of a program written in the symbolic language MACSYMA capable of producing all the time consuming details of separation.

Acknowledgements

This thesis could not have been written without the early work of Charles Boyer, Ernie Kalnins, and Bill Miller. To all of these people, my thanks. I would like to thank Ernie Kalnins especially for his time, suggestions and encouragement.

I am grateful to Terry Robb for introducing me to MACSYMA, and for the time he spent helping me learn the language. It is a pleasure to thank Bronwyn Beder for her help in preparing this thesis, and also to thank my flatmates and friends for their tolerance. I will remember the staff and students of Mathematics Department as being both friendly and helpful. I readily acknowledge the assistance of the staff of the Computer Center in using the typesetting language VTROFF: the quality of the typeset is due to their efforts.

Table of Contents

Abstract		iii
Acknowledgements		iv
Contents		v
I Introduction		
1.1	Statement of Problem	1
1.2	Background	3
1.3	Thesis Outline	10
II The Real Heat and Schrödinger Equations		
2.1	Introduction	12
2.2	Passage to the Hamilton-Jacobi equation in Minkowski space	14
2.3	Reducing the Hamilton-Jacobi equation	22
2.4	Metric and coordinates	29
2.5	Graphical Representation of coordinates	41
2.6	Operators	46
2.7	R-separable solutions	57
2.8	Applications	64
III The Complex Heat and Schrödinger equations		
3.1	Introduction	76
3.2	Heat-type systems with one nonorthogonal entry	78
3.3	Non Heat-type systems for the Helmholtz equation	92
3.4	R-separation for the Hamilton-Jacobi equation	97
Appendix A	Tables of R-separable systems	102
Appendix B	Symbolic Programming in Separation of Variables	115
Appendix C	The SEPCAL.V program	128
Appendix D	Useful general results	135
Bibliography		138

I

Introduction

1.1 Statement of Problem

Variable separation is one of the major techniques for solving partial differential equations. In this thesis a complete treatment of the variable separation problem is given for the equation

$$\Delta_m \Psi + 2\varepsilon \partial_t \Psi = E\Psi , \qquad (\Delta_m = \sum_{u=1}^{m} \partial_{y^u y^u}) , \tag{*}$$

in the real variables y^u and t. In addition a partial treatment is given when these variables are complex. When $\varepsilon = i/2$ this equation is a standardised version of the "constant potential Schrödinger equation" which will be referred to as the "Schrödinger equation". If $\varepsilon = -1/2$ and E=0 then (*) is a standardised version of the Heat or diffusion equation.

The coordinate system

$$y^u = y^u(\mathbf{x}), \quad u = 1,2, \ldots, m ,$$

$$t = t(\mathbf{x}), \ \mathbf{x} = (x^1, \ldots, x^{m+1}), \tag{1.1.1}$$

is *R-separable* if there are complex analytic functions Ψ, Ψ_j and R, such that (*) admits a solution of form

$$\Psi(\mathbf{x},\mathbf{c}) = e^{R(\mathbf{x})} \prod_{j=1}^{m+1} \Psi_j(x^j, \mathbf{c}) \quad , \tag{1.1.2}$$

where $\mathbf{c} = (c_1, \ldots, c_{m+1})$ are the $m+1$ separation constants. It is the main task of this thesis to classify these systems. At first it might seem that any solution of (*) is R-separable but the independence of R from the constants c_j severely limits the possible R-separable solutions. Pure separation corresponds to $R = 0$ and trivial R-separation to

$$R = \sum_{i=1}^{m+1} R_i(x^i) . \tag{1.1.3}$$

Variable separation for (*) has its origins in the Classical Mechanics of the last century. We now trace the development of the subject.

1.2 Background

The History of variable separation dates back to Liouville (1846) who found a large class of separable systems for the Hamilton-Jacobi equation of Classical Mechanics:

$$H(p_i, x^i) = E, \qquad p_i = \partial W / \partial x^i, \quad i = 1, \ldots, n. \tag{1.2.1}$$

The most powerful method of solving this equation is to find a *complete integral*. This method is treated in many standard texts on Classical Mechanics such as Pars (1965). A *complete integral* is a regular n-parameter family of solutions $W = W(x^j; \lambda_1, \ldots, \lambda_n)$ of (1.2.1) such that $det(\partial^2 W / \partial x^i \partial \lambda_j) \neq 0$. The λ_j are referred to as *constants* or *integrals of the motion* and satisfy

$$\{ H, \lambda_j \}_P = 0 \tag{1.2.2}$$

where $\{ \, , \, \}_P$ denotes the Poisson Bracket.

If

$$W = \sum_{i=1}^{n} W_i(x^i; \lambda_1, \ldots, \lambda_n) \tag{1.2.3}$$

then the coordinates x^i are said to be separable and the λ_i are called the *separation constants*. In this case (1.2.3) is equivalent to a set of n ordinary differential equations, the *separation equations*.

It was Stäckel (1891) who first raised the subject from an art to a science. He considered the motion of a particle in a Riemannian space with orthogonal nondegenerate metric (i.e. $g_{ij} = 0$ for $i \neq j$ and $\det(g_{ij}) = g \neq 0$):

$$ds^2 = g_{ij} dx^i dx^j \tag{1.2.4}$$

Here the Einstein summation convention has been used and the term "Riemannian" is understood to mean "pseudo-Riemannian". Both of these conventions will be assumed unless otherwise stated. Stäckel found that the condition for the corresponding Hamilton-Jacobi equation

$$g^{ij} p_i p_j = E \, , \quad (E \neq 0), \tag{1.2.5}$$

to be separable was that there exists a matrix $(\psi_{ij}) = (\psi_{ij}(x^i))$ such that

$$g^{ii} = \frac{\psi^{i1}}{\psi} \tag{1.2.6}$$

where $\psi = \det(\psi_{ij})$. The matrix (ψ_{ij}) is called a "Stäckel matrix" and its i-th row is a function of x^i alone. The g^{ii} are the $i,1$ cofactors of (ψ_{ij}) or, equivalently the first row of the inverse of (ψ_{ij}). Stäckel determined the constants of the motion:

$$\lambda_i = \sum_{j=1}^{n} \frac{\psi^{ij}}{\psi} p_j^2 \ , \qquad (1.2.7)$$

and showed that they are in *involution*, i.e.,

$$\{\lambda_i \ , \lambda_j\}_P = 0 \qquad (1.2.8)$$

For elementary treatments of Stäckel's results see Pars (1965) and Hagihara (1970).

Levi-Civita (1904) provided some useful results for Hamiltonians in the more general form (1.2.1). He found a set of integrability conditions for solutions of the type (1.2.3) to exist. However the integration of these conditions for Hamiltonians quadratic in the momenta p_i proved difficult. An indirect method was furnished by Dall'Acqua (1912) who managed to derive the separation equations whose form had been conjectured by Burgatti (1911).

With the advent of Quantum mechanics in the 1920's the method of variable separation was extended to the Helmholtz(or time independent Schrödinger) equation by Robertson (1927). This equation when expressed in coordinate invariant form in a Riemannian space V_n with metric (1.2.4) is

$$\Delta_n \Psi = E \Psi \qquad (1.2.9)$$

where $\Delta_n = g^{-1/2}\partial_i(g^{1/2}g^{ij}\partial_j \ )$ is the Laplace-Beltrami operator. The Helmholtz equation is separable if it has a complete solution of form

$$\Psi = \prod_{i=1}^{n} \Psi_i(x^i;c_1, \ldots, c_n) \qquad (1.2.10)$$

where substitution of Ψ into (1.2.9) reduces it to n second order linear ordinary differential equations for the Ψ_i. Robertson (1927) showed that the Helmholtz equation was separable in orthogonal coordinates provided that the corresponding Hamilton-Jacobi equation (1.2.5) was separable and that extra constraints called the Robertson or Schrödinger conditions were satisfied (see (D9)). The passage between the Helmholtz equation (1.2.5) and the Hamilton-Jacobi equation (1.2.9) is analogous to that between Classical and Quantum mechanics. In Kalnins and Miller (1983) additive separation was defined as the fundamental kind of separation for partial differential equations. The conditions of separation for the Helmholtz equation are derived by setting $\psi = e^{W}$, and reducing its product separation to additive separation.

Thus far only the general functional form of the separable orthogonal metrics had been given (the entries in the Stäckel matrix could be arbitrary functions). Few attempts had been made at classifying the separable systems occurring in particular Riemannian spaces. The first effort in this direction was that of Eisenhart (1934). In this classic paper he applied the flat space curvature conditions (see (D11))

$$R_{ijkl} \equiv 0 \ , \quad 1 \leq i,j,k,l \leq 3 \ , \tag{1.2.11}$$

coupled with Stäckel's conditions (1.2.6) to give a complete list of separable orthogonal systems on three dimensional real Euclidean space $\mathbb{R}^3$(see Appendix A, Tables 2 and 5 for this list). Eisenhart (1934) also found some useful general results. He gave a geometric counterpart for the Robertson condition in orthogonal coordinates by showing that there it was equivalent to $R_{ij} = 0$ (see (D12) for the definition of this tensor). In Einstein spaces such as flat and constant curvature space this means that the Hamilton-Jacobi and the Helmholtz equations separate in exactly the same orthogonal coordinate systems. Eisenhart uncovered a useful set of necessary and sufficient conditions for the Stäckel form (1.2.6) (see Lemma 2.4.1), and was able to characterise the constants of the motion that give rise to variable separation. He showed that if a set of constants of the motion were in involution and also possessed n simultaneous normalizable eigenvector fields, then coordinates could be chosen such that the Hamiltonian (1.2.5) was separable (see Eisenhart (1949) for the relevant definitions). This result was not only important in that it gave a coordinate free description of separation but also in other ways which will soon be discussed.

Only sporadic advances were made over the next thirty years. Olevski (1950) classified separable systems on three dimensional spaces of constant curvature and a few results were obtained by Moon and Spencer (1952). Useful summaries of separable coordinate systems are contained in Moon and Spencer (1961) and Morse and Feshbach (1953). Agostinelli (1958a,b) studied classes of solutions to $R_{ij} = 0$ and later extended these results to nonorthogonal cases. Carter (1968) showed that the Hamilton-Jacobi equation for a rotating black hole was not only separable but could be derived and generalised simply using separation of variables theory.

In the meantime the theory of symmetry (or Lie) groups was becoming established as an invaluable tool in Physics. For example in the Hydrogen atom, just from a knowledge of its symmetry group SO(4), it proved possible to exploit the commutation relations and calculate many physical quantities: energy levels, transition probabilities, wave functions etc. It was some time before these ideas flourished in the field of variable separation. Several simple systems had been characterised in terms of second order symmetry operators, but the explicit statement of the relationship between symmetry and variable separation appeared for the first time in Winternitz and Fris (1965). These authors gave group theoretic characterisations of the separable coordinate systems in two dimensional spaces of constant curvature.

Let us discuss some of these concepts in more detail. The *symmetry group* of a partial differential equation is the group of transformations which leaves its form invariant (see Bluman and Cole (1974), Miller (1977)). Equivalently it is the group of transformations which maps solutions of the equation into other solutions of the same equation. Using the methods of Bluman and Cole, the symmetry algebra corresponding to the symmetry group of the Hamilton-Jacobi equation (1.2.5) can be derived by solving the relation

$$\{ H \, , \mu_j \}_P = 0 \tag{1.2.12}$$

for the Killing vectors $\mu_j = a_j^i(\mathbf{x})p_i + b_j(\mathbf{x})$. The μ_j form a Lie algebra under the action of the Poisson Bracket. The symmetries of the corresponding Helmholtz equation are realised as first order partial differential operators via the commutator bracket (see Bluman and Cole (1974))

$$[\Delta_n \, , \tilde{\mu}] = 0 \tag{1.2.13}$$

where $\tilde{\mu}_j = a_j^i(\mathbf{x})\partial_i + b_j(\mathbf{x})$. There is a direct correspondence between the Killing vectors for the Hamilton-Jacobi equation and the first order symmetries of the Helmholtz equation:

$$\mu_j = a_j^i p_i + b_j \;\rightarrow\; \tilde{\mu}_j = a_j^i \partial_i + b_j \tag{1.2.14}$$

For spaces of constant curvature, Delong (1982) has shown that the second order Killing tensors are in the enveloping algebra. A second order Killing tensor is a tensor a^{ij} which satisfies Killing's equations: $a^{ij}_{;k} + a^{jk}_{;i} + a^{ki}_{;j} = 0$. ($;i$ denotes the covariant derivative as it is defined in Eisenhart (1949)). We will sometimes loosely refer to the constant of the motion $a^{ij}p_i p_j$ derived from this tensor as also being a Killing tensor. Delong's statement implies that if the Killing vectors μ_j are a basis for the space of Killing vectors and λ_m is any second order Killing tensor, then there are constants b^{jk} such that

$$\lambda_m = a_m^{ij} p_i p_j = \sum_{j,k} b^{jk} \mu_j \mu_k \tag{1.2.15}$$

It follows from Eisenhart's results that any orthogonal separable system in a space of constant curvature is uniquely characterised by n Killing tensors which are in the enveloping algebra. Using the results of Kalnins and Miller (1983) it can be shown that the separation constants c_i for the Helmholtz equation are eigenvalues of a set of n second order partial differential operators, i.e.

$$L_m \Psi = c_m \Psi \ , \quad 1 \le m \le n \ , \tag{1.2.16}$$

where

$$L_m = A_m^{ij}(\mathbf{x})\partial_{ij} + B_m^{i}(\mathbf{x})\partial_i + C_m(\mathbf{x}) \tag{1.2.17}$$

Notice that (1.2.16) implies that

$$[L_j \ , L_k] = 0 \ , \tag{1.2.18}$$

in analogy to the involution property for Killing tensors (1.2.8). The results of Kalnins and Miller (1983) show that

$$A_m^{ij} = a_m^{ij} \tag{1.2.19}$$

by examining the highest order derivative terms of both the Helmholtz and Hamilton-Jacobi equations. It follows that modulo a first order member of the Lie algebra

$$L_m = b_m^{jk}\{\tilde{\mu}_j \ , \tilde{\mu}_k\} \tag{1.2.20}$$

Here $\{\ ,\ \}$ is the *symmetric bracket* which is defined for two operators A and B by

$$\{A \ , B\} = \frac{AB+BA}{2} \tag{1.2.21}$$

Thus it is possible to compute the operators for the Helmholtz equation directly from the Killing tensors for the Hamilton-Jacobi equation in spaces of constant curvature. This will be a useful result since we will be able to formulate the R-separation problem for (*) as one of pure separation for a Hamilton-Jacobi equation.

Another illustration of the importance of the symmetry group is in the case of *ignorable variables*. A variable is ignorable if it is possible to choose a coordinate system in which that variable does not appear explicitly in the Helmholtz or the Hamilton-Jacobi equations. Greek letters will be used to denote such variables. Nonignorable coordinates will be referred to as ***essential variables***. Ignorables correspond to elements of the Lie algebra and from (1.2.18) systems containing N of them correspond to N-dimensional abelian subalgebras.

Since the symmetry operators corresponding to the ignorables are first order partial differential operators, the eigenfunction condition implies that

$$L_\alpha \Psi = \partial_\alpha \Psi = c_\alpha \Psi \tag{1.2.22}$$

in which case the x^α dependence in Ψ is simply

$$\Psi_\alpha = e^{c_\alpha x^\alpha}, \ (\textit{no sum on } \alpha) \tag{1.2.23}$$

The theory of *ignorable* or *cyclic* coordinates for the Hamilton-Jacobi equation is well known (see Pars (1965)). In that case

$$p_\alpha = \partial W / \partial x^\alpha = \lambda_\alpha \tag{1.2.24}$$

so that the x^α dependence in W is

$$W_\alpha = \lambda_\alpha x^\alpha \tag{1.2.25}$$

The remainder of the operators and Killing tensors are second order as has already been discussed.

One solution to the variable separation problem in spaces of constant curvature is to use the relation (1.2.18) to find possible commuting sets of members from the enveloping algebra and then solve for the coefficients. The coordinate systems determined by these sets can then be tested to see if they are separable. The determination of the N-dimensional abelian subalgebras corresponding to ignorable variables will be a key factor in the resolution of the R-separation problem for (*).

Reviews on separation have been written by Huaux (1976) and Benenti (1980b). A lot of the material in this last article is duplicated in Benenti (1980a) where a rigorous definition of equivalence is given for the Hamilton-Jacobi equation. By Benenti's definition two separable coordinate systems for the Hamilton-Jacobi equation are *equivalent* if their separated complete integrals are the same. He shows that by using this definition the only types of equivalence that can occur are (2.1.3a) and (2.1.3b). These equivalences extend quite naturally to the Helmholtz equation since both preserve the separability of that equation. It is also possible to add the equivalence (2.1.3c), as equivalence under the group removes the distinction between the many different coordinate transformations leading to the same functional form of the Helmholtz and Hamilton-Jacobi equations.

Using his definition of equivalence Benenti was able to use the results of Dall'Acqua and Levi-Civita to give the conditions for the separability of the Hamilton-Jacobi equation (1.2.5) in any coordinate system, orthogonal or nonorthogonal. The general form of the contravariant separable metric is given in (D3). With this work Benenti put nonorthogonal separation on a sound footing. When nonorthogonal separation occurs the class of essential variables we have already discussed splits into two further classes: Stäckel variables x^a for which $g^{aa} \neq 0$, and first order variables x^r for which $g^{rr}=0$.(see Appendix D). The first order variables are of null type and systems containing them can not appear in positive definite spaces). Even before Benenti had rigorously established the conditions for nonorthogonal separation, Kalnins and Miller had embarked upon a programme to classify separable coordinate systems for some of the most common partial differential equations of mathematical physics (see Miller (1977) and the references cited therein). Many of these physically important systems were nonorthogonal. Along with the separable coordinates they provided the operators in the enveloping algebra defining the separation.

Kalnins and Miller (1983) give an intrinsic definition of separation for a much larger class of partial differential equations than those considered by Benenti. Using their definition many of the generalisations of the statements for orthogonal coordinates are easily extended to the nonorthogonal case. Helmholtz separation implies Hamilton-Jacobi separation as in the orthogonal case. The conditions for the converse of this statement, the generalisation of the Robertson conditions, are given in (D9). However these conditions are no longer equivalent to $R_{ij} = 0$. The Killing tensors for the Hamilton-Jacobi equation (see (D5),(D6)) are still second order as are the corresponding symmetry operators for the Helmholtz equation. In spaces of constant curvature the work of Delong implies that these Killing tensors are in the enveloping algebra and so our statements (1.2.12)-(1.2.25) made for orthogonal coordinates are still valid. In such spaces much of the variable separation problem can be reduced to algebra, that is, to determining commuting sets of members of the enveloping algebra.

A recent feature has been the solution of the variable separation problem in various spaces of arbitrary dimension. Kalnins and Miller (1982a,b) have provided inductive constructions of all separable systems on real Euclidean space $\mathbb{R}^n$, the real n-Sphere S_n, and the Hyperboloid H_n. In Kalnins, Miller and Reid (1983) all orthogonal separable systems have been classified on n-dimensional complex Euclidean space $E(n,\mathbb{C})$.

Many of the important special functions of mathematical physics: Bessel, Hypergeometric, Mathieu etc. arise as the separated solution of a partial differential equation. Miller (1977) has shown how the group theoretical characterisations can be used to study these functions and their identities, a possibility first realised by Weisner (1955, 1959a,b). One example: the problem of expanding one set of separable solutions in terms of another reduces to a problem in the representation theory of the Lie symmetry algebra.

1.3 Thesis outline

This thesis takes its place in the programme to classify separable coordinate systems for the important equations of mathematical physics and to characterise these systems with algebraic invariants.

Although (*) is not of Helmholtz type, it is close. It can be taken as a symmetry reduced version of a Helmholtz equation on a higher dimensional space (see § 2.2). Thus the theory which has been discussed in § 1.2 can be brought to bear on (*).

The core of this thesis lies in the resolution of the variable separation problem for the real Heat and Schrödinger equations. This is Chapter 2. An exhaustive treatment is given there of all the details of separation - the separation equations, the R-separation factors and the operators. The early work of Miller (1977) is generalised when it is shown how R-separable systems for the Schrödinger equation are naturally related to the Schrödinger equation with harmonic oscillator and linear potentials. A major deficiency here is that the special function theory is not pursued. This thesis is confined to obtaining the ordinary differential equations representing the separated solutions, and to obtaining the constants of the motion in terms of the enveloping algebra. A rigorous study of these special functions will be made in the future.

Chapter 3 deals with complex extensions of the work in Chapter 2. In the real case covered in Chapter 2, the signature of Minkowski space enables us to reduce the metric to one nonorthogonal element. For the complex case, as the class covered in § 3.4 shows, this is not always possible. If we assume, however, that the metric has only one nonorthogonal element as is done in § 3.2, much of the analysis for the real case can be applied. In this case there is also an exceptional class of systems which is not of *Heat type* i.e. these systems do not lead to R-separation for the Heat equation.

Finally there are some interesting sidelines. A program has been written in the symbolic language MACSYMA capable of producing all the time consuming details of separation in flat spaces. An essential element of the program was the derivation of a helpful general formula, which for flat spaces converts second order Killing tensors into the corresponding members of the enveloping algebra. Given the Stäckel matrix and the coordinate transformations the program could among other details calculate the metric, the separation equations and most importantly, the Killing tensors defining the separation in terms of the enveloping algebra. This program is useful not only in checking, but also in providing results from which general solutions can be guessed. Furthermore it is used to calculate the details of separation for the unusual class of systems given in § 3.4, an almost impossible task by hand.

II

The Real Heat and Schrödinger equations

2.1 Introduction

In this chapter we completely solve the R-separation problem for (*) when the variables y^u and t are real. Both the Heat and Schrödinger equations studied in this chapter have diverse applications. If $\varepsilon=-1/2$ and $E=0$ then (*) is the Heat equation, and Ψ could represent the temperature of a fluid, or the density of one gas diffusing through another. When $\varepsilon=i/2$, (*) is the Schrödinger equation, and Ψ is the wave function with $|\Psi|^2$ being proportional to the probability density function for a free particle. By keeping ε unspecified we are able to treat these two equations simultaneously.

Fortunately the case $\varepsilon=0$ has been solved by Kalnins and Miller (1982a). This case, the enumeration of all separable systems on $\mathbb{R}^m$, forms the building block of our solution. Another valuable reference is Miller (1977) where the R-separable coordinate systems are classified for both the Schrödinger and Heat equations in one and two spatial dimensions (i.e. $m=1$ or 2).

Even though it will not be possible to solve the classification problem using group theory alone it will be a great help to us. Using the techniques of Bluman and Cole (1974), the infinitesimal generators

$$L = a^i(\mathbf{Y})\partial_{Y^i}+b(\mathbf{Y}) \tag{2.1.1}$$

for the Lie algebra of (*)'s symmetry group are found by solving the relation

$$[\ Q,\ L] = M(\mathbf{Y})Q, \tag{2.1.2}$$

where $Q = \Delta_m + 2\varepsilon\partial_t - E$ and $\mathbf{Y} = (y^1, \ldots, y^m, t)$. This is the analogous relation to (1.2.13), the equation used to find the symmetries of the Helmholtz equation. The infinitesimal generators L can also be interpreted as symmetry operators in the sense that they map solutions to solutions: if Φ is a solution of (*) then so too is $L\Phi$.

To simplify this study we will say that two R-separable coordinate systems $\{\mathbf{x}\}$ and $\{\bar{\mathbf{x}}\}$ are *equivalent* if

$$a. \quad \bar{x}^a = f^a(x^a) \ , \tag{2.1.3a}$$

$$b. \quad \bar{x}^\alpha = c^\alpha_\beta\, x^\beta + \sum_a g_a(x^a) \quad , \ \det(c^\alpha_\beta) \neq 0 \ , \tag{2.1.3b}$$

or

c. *they are related by the action of the symmetry group.* (2.1.3c)

The c^α_β above are real constants and the x^a are essential variables. The equivalences (2.1.3a) and (2.1.3b) are those that Benenti (1980a) derives using his definition of equivalence for the Hamilton-Jacobi equation. In a similar manner to the Helmholtz equation these equivalences extend naturally to (*), since they preserve this equations R-separability. Again the equivalence (2.1.3c) can be added because equivalence under the group removes the distinction between many different coordinate transformations leading to the same functional form of (*).

The structure of this chapter is as follows. In §2.2 we transform (*) to the Helmholtz equation in Minkowski space, and exploit the fact that both this equation and the even simpler Hamilton-Jacobi equation separate in the same coordinate systems. §2.2 is also introductory in nature as it outlines much of the basic material needed for our study. In §2.3 we determine the possible sets of commuting Killing vectors (or abelian sub algebras) characterising separable systems for the Hamilton-Jacobi equation. Using these abelian subalgebras a simple form is then found for the metric. As a result the Riemann curvature conditions are solved in §2.4 and the metric completely determined. The technical work of finding the coordinate transformations is also carried out in §2.4. This work is summarised in graphical form in §2.5. In §2.6 we determine the operators for (*) and in §2.7 the R-separable solutions. The applications for the Heat and Schrödinger equations are discussed in §2.8 and the results for $m = 1, 2,$ and 3 are tabulated in Appendix A.

2.2 Passage to the Hamilton-Jacobi equation in Minkowski space

The problem of finding all R-separable systems for (*) is shown to be equivalent to that of finding all separable systems for the Helmholtz equation (2.2.1) defined on Minkowski space with symmetry (2.2.2). The conditions for separability of (2.2.1) are best derived from its classical counterpart, the Hamilton-Jacobi equation (2.2.14). We show that the symmetry algebras of (*) and (2.2.14) are isomorphic to the Schrödinger algebra when $E=0$ and to the Galilean algebra when $E \neq 0$. The central result is the following theorem.

Theorem 2.1

There is a mapping between R-separable coordinate systems of (*) and separable coordinate systems for the Helmholtz equation on $n(=m+2)$ dimensional Minkowski space,

$$\Box_n \overline{\Psi} = E\overline{\Psi}, \quad n=m+2, \tag{2.2.1}$$

whose solutions $\overline{\Psi}$ are eigenfunctions of the symmetry operator

$$\partial_{x^n} = \frac{1}{2}(\partial_{\bar{y}^{n-1}} + \partial_{\bar{y}^n}) \tag{2.2.2}$$

with associated eigenvalue ε. (i.e. $\partial_{x^n}\overline{\Psi} = \varepsilon\overline{\Psi}$).

Here the D'Alembertian operator on Minkowski space with the coordinates $\bar{y}^i$, $i = 1,2,\ldots,n$, is defined by

$$\Box_n \equiv \partial_{\bar{y}^1\bar{y}^1} + \cdots + \partial_{\bar{y}^{n-1}\bar{y}^{n-1}} - \partial_{\bar{y}^n\bar{y}^n}. \tag{2.2.3}$$

Proof

We first construct the mapping from (*) to (2.2.1). Suppose

$$y^j = y^j(x^1, \ldots, x^{n-1}), j = 1, 2, \ldots, n-2,$$

$$t = t(x^1, \ldots, x^{n-1}), \tag{2.2.4}$$

is an R-separable system for (*) then there are functions Ψ, Ψ_j and R such that

$$\Psi = e^R \prod_{j=1}^{n-1} \Psi_j(x^j) \tag{2.2.5}$$

Consider the coordinate system $\bar{x}^i$

$$\bar{y}^j = y^j(\bar{x}^1, \quad , \bar{x}^{n-1}), j = 1, 2, \quad n-2,$$

$$\bar{y}^{n-1} - \bar{y}^n = 2t(\bar{x}^1, \quad , \bar{x}^{n-1}),$$

$$\bar{y}^{n-1} + \bar{y}^n = \bar{x}^n - R/\varepsilon \tag{2.2.6}$$

This is a system in Minkowski space with ignorable $\bar{x}^n$, corresponding to the symmetry operator (2.2.2) whose eigenvalue we shall specify as ε. Let

$$\bar{\Psi} = e^{\varepsilon \bar{x}^n} \prod_{j=1}^{n-1} \Psi_j(\bar{x}^j)$$

$$= e^{[\varepsilon(\bar{y}^{n-1} + \bar{y}^n) + R]} \prod_{j=1}^{n-1} \Psi_j(\bar{x}^j) \tag{2.2.7}$$

As Ψ satisfies (*), $\bar{\Psi}$ is easily shown to satisfy (2.2.1). In other words (2.2.6) is a separable coordinate system for the Helmholtz equation (2.2.1). We now show that the mapping from (2.2.4) to (2.2.6) is onto by constructing its inverse. If $\{\bar{\mathbf{x}}\}$ is a separable system for (2.2.1) with symmetry operator (2.2.2) then

$$\bar{y}^j = y^j(\bar{x}^1, \ldots, \bar{x}^{n-1}), j = 1, 2, \ldots, n-2,$$

$$\bar{y}^{n-1} - \bar{y}^n = 2t(\bar{x}^1, \ldots, \bar{x}^{n-1}),$$

$$\bar{y}^{n-1} + \bar{y}^n = \bar{x}^n + f(\bar{x}^1, \ldots, \bar{x}^{n-1}) \tag{2.2.8}$$

which if we let

$$f = -R/\varepsilon \tag{2.2.9}$$

is the image of (2.2.4), and the theorem is proved. Q.E.D.

In fact (2.2.1) separates in the same coordinate systems for each nonzero value of E. Care must be taken when $E=0$. When this occurs (2.2.1) is the wave equation which in general separates in additional coordinate systems as is shown in Kalnins and Miller (1982c). However if we define

$$\tilde{\Psi} = e^{-Et/2\varepsilon}\Psi \tag{2.2.10}$$

then by substitution:

$\tilde{\Psi}$ *is an R-separable solution* for (*) *with* $E=0$

iff

Ψ *is an R-separable solution* for (*).

The presence of the symmetry (2.2.2) has eliminated these additional systems. This means that a classification for nonzero E will also yield all possible systems when $E=0$. Again care must be taken as the symmetry group of the wave equation is larger: it admits additional conformal symmetries. In essence, some of the systems which are inequivalent for (*) when $E \neq 0$ become equivalent under the action of the extra symmetries when $E=0$. We will return to this point later. All that we need to know at the moment is that everything can be obtained from the nonzero case.

One of the advantages of transforming our problem to the Helmholtz equation (2.2.1) is that to such an equation we can naturally associate a Riemannian geometry. Here (2.2.1) corresponds to the metric

$$ds^2 = (dy^1)^2 + \qquad + (dy^{n-1})^2 - (dy^n)^2 \tag{2.2.11}$$

which is the fundamental metric of a general n-dimensional Minkowski space $E(n,1)$. The classification of the separable coordinate systems can now be framed as the classification of the metrics

$$ds^2 = g_{ij} dx^i dx^j \tag{2.2.12}$$

that can occur in Minkowski space (i.e. a space with fundamental form (2.2.11)) and separate the Helmholtz equation (2.2.1). Such a classification can be accomplished using the results of Riemannian geometry. Since the space is Minkowski these results imply that all the components of the Riemann curvature tensor given in (D11) are identically zero, i.e.

$$R_{ijkl} \equiv 0 \quad , \quad 1 \le i,j,k,l \le n \tag{2.2.13}$$

Since (2.1.1) is a Helmholtz equation the discussion in §1.2 implies that every separable coordinate system for this equation also separates the corresponding Hamilton-Jacobi equation

$$H = P_1^{\,2} + \qquad + P_{n-1}^{\,2} - P_n^{\,2} = E \quad , \quad P_i = \partial W / \partial y^i \quad , \tag{2.2.14}$$

with Minkowski space metric (2.2.11) and symmetry operator (2.2.2). As mentioned in §1.2, not every separable system for the Hamilton-Jacobi necessarily separates the Helmholtz equation. Helmholtz separation only occurs if the generalised Robertson conditions given in (D9) are also satisfied. These conditions depend on the metric components and are especially complicated in metrics with many nonorthogonal components. In our case they can be shown to be always satisfied after the metric has been reduced to the simple form (2.3.1) given in §2.3. The argument is the same as that used for the complex case (see (3.2.3) and (3.2.4)). Thus the Hamilton-Jacobi equation and the Helmholtz equation (2.2.1) separate in exactly the same coordinate systems. This result can also be proved by making the following observation which avoids introducing the technical details of the generalised Robertson condition. The separability of (2.2.1) in all separable systems for the Hamilton-Jacobi equation (2.2.14) is ultimately demonstrated in §2.7 by obtaining the corresponding separation equations for (*). In summary our problem has been reduced to the much easier one of finding all separable systems for the Hamilton-Jacobi equation (2.2.14).

Working with the Hamilton-Jacobi equation is more convenient since its form (1.2.5) is not as complicated as that of the Helmholtz equation (1.2.9) and is more closely related to the metric (2.2.12)(it's just the inverse). Using his definition of *equivalence*, Benenti (1980a) was able give the conditions for separability of the Hamilton-Jacobi equation (D2). He showed that each of his classes contained a *canonical* separable system $\{x^a, x^r, x^\alpha\}$ with contravariant metric (D3). The form of (D3) implies that the first order variables are of null-type, and the signature of Minkowski space which is $n-1$ thus limits the number of these variables (n_2) to 1. A discussion of null variables is given in Eisenhart (1949).

Since our problem will be solved using the Hamilton-Jacobi equation (2.2.14), we first find the Killing vectors corresponding to the first order symmetries of (2.2.1) and (*) when $E \neq 0$. To find these Killing vectors we solve (1.2.12) in the standard coordinates y^i:

$$\{H, \mu\}_P = 0 \tag{2.2.15}$$

where $\mu = a^j(\mathbf{y})P_j$. When $E \neq 0$ the Hamilton-Jacobi equation (2.2.14) admits a Lie algebra of Killing vectors which is the Lie algebra of Minkowski space $e(n,1)$. All the coordinate systems we are considering possess the symmetry (2.2.2) and so (1.2.18) implies that all symmetries of (2.2.1) must commute with (2.2.2). The Killing vector counterpart of this relation, (1.2.8), is

$$\{\hat{\varepsilon}, \mu\}_P = 0 \quad , \hat{\varepsilon} = \frac{1}{2}(P_{n-1} + P_n) \tag{2.2.16}$$

where $\hat{\varepsilon}$ is the Killing vector corresponding to (2.2.2). This extra condition confines us to a subalgebra of $e(n,1)$ - the Galilean algebra g_m of dimension $\frac{1}{2}m(m+1)+2$ and basis

$$\hat{\varepsilon},\ P_u,\ M_{uv} = y^u P_v - y^v P_u,\ B_u = y^u \hat{\varepsilon} - tP_u\ , K_{-2} = P_t . \tag{2.2.17}$$

Here $u,v \in U = \{1, \ldots, n-2\}$. In general the indices u and v will be taken from the set U. Since the above Killing vectors are a basis, any Killing vector has the form

$$\lambda_\alpha = \rho_\alpha^u P_u + m_\alpha^{uv} M_{uv} + \beta_\alpha^u B_u + \kappa_\alpha K_{-2} . \tag{2.2.18}$$

where the ρ_α^u, m_α^{uv}, β_α^u and κ_α are all real constants. The commutation relations for g_m may be derived from Table 2.2.1. The symmetry algebra for the Helmholtz equation (2.2.1) ($E \neq 0$) is again g_m with the identifications

$$\hat{\varepsilon} \to \frac{1}{2}(\partial_{n-1} + \partial_n),\ P_u \to \partial_u,\ M_{uv} \to y^u \partial_v - y^v \partial_u ,$$

$$B_u \to y^u \hat{\varepsilon} - t\,\partial_u,\ K_{-2} \to \partial_t . \tag{2.2.19}$$

By solving (2.1.2) for (*) we find that g_m is also the symmetry algebra for (*), except that the operator $\hat{\varepsilon}$ is replaced by its eigenvalue ε.

If $E=0$ the symmetries of (2.2.1) are of the form $a^j P_j + f$ where $\{H, a^j P_j + f\}_P = M(\Upsilon)H$ and then there are two extra symmetries. When expressed as Killing vectors these are

$$K_2 = -t^2 P_t - t\sum_u y^u P_u + \frac{1}{2}\hat{\varepsilon}\sum_u (y^u)^2 ,$$

$$D = \sum_u y^u P_u + 2tP_t . \tag{2.2.20}$$

•

and as operators for (*)

$$K_2 = -t^2\partial_t - t\sum_u y^u \partial_u - \frac{1}{2}mt + \frac{1}{2}\varepsilon\sum_u (y^u)^2 ,$$

$$D = \sum_u y^u \partial_u + 2t\,\partial_t + \frac{1}{2}m \tag{2.2.21}$$

These satisfy the commutation relations

$$[D,P_u]=-P_u,\ [D,M_{uv}]=0,\ [D,B_u]=B_u,\ [D,K_{\pm 2}]=\pm 2K_{\pm 2},$$

$$[K_2,P_u]=-B_u,\ [K_2,M_{uv}]=0,\ [K_2,M_{uv}]=0,\ [K_2,B_u]=0,\ [K_2,K_{-2}]=D. \quad (2.2.22)$$

This enlarged algebra is the Schrödinger algebra s_m of dimension $\frac{1}{2}m(m+3)+4$.

Geometrically, the P_u and M_{uv} are generators of space translations and rotations under which (*) is clearly invariant. Less obvious are the transformations corresponding to the B_u's. These are the Galilean or velocity boosts and illustrate the fact that (*) retains its form in uniformly moving frames of reference. When $E=0$ we have the additional conformal Killing vectors D and K_2. D is the generator of the dilatation symmetry $\Psi(\mathbf{y},t) \to \Psi(\alpha\mathbf{y},\alpha^2 t)$. The action of K_2 is rather complicated and it does not have a simple geometrical interpretation.

One of the chief applications of these symmetry algebras will be to determine the abelian subalgebras corresponding to sets of ignorable variables. Again the action of the group helps us choose simple representatives for these subalgebras. The group acts on the algebra g_m via its *adjoint action*. In general, for two members of a Lie algebra L_1 and L_2, the adjoint action of L_1 on L_2 is given by

$$e^{aL_1}L_2 e^{-aL_1} = e^{a\, Ad\, L_1}L_2 \quad , \qquad (2.2.23)$$

where $Ad\, L_1(L_2) \equiv [L_1, L_2]$. A proof of this result is given by Hausner and Schwartz (1968). The adjoint actions for the Galilean algebra are summarised in Table 2.2.1.

Table 2.2.1 Adjoint Actions for the Galilean Algebra

$L_1 \backslash L_2$	P_w	M_{wz}	B_w	K_{-2}
P_u		M_{wz} + $a(\delta_{uw}P_z - \delta_{uz}P_w)$	$B_w + \frac{a\delta_{uw}\varepsilon}{2}$	
M_{uv}	P_w + $a(\delta_{vw}P_u - \delta_{uw}P_v)$ $+a^2 \cdots$ †	M_{wz} + $a(\delta_{vw}M_{uz} - \delta_{vz}M_{uw}$ $+\delta_{uz}M_{vw} - \delta_{uw}M_{vz})$ $+a^2 \cdots$ †	B_w + $a(\delta_{vw}B_u - \delta_{uw}B_v)$ $+a^2 \cdots$ †	
B_u	$P_w - \frac{a\delta_{uw}\varepsilon}{2}$	M_{wz} + $a(\delta_{uw}B_z - \delta_{uz}B_w)$		$K_{-2}+aP_u - \frac{a^2\varepsilon}{4}$
K_{-2}			$B_w - aP_w$	

Table 2.2.1: each entry represents $e^{a\,Ad\,L_1}(L_2)$, e.g. $e^{a\,Ad\,P_u}(B_w) = B_w + \frac{a\delta_{uw}\varepsilon}{2}$. If the adjoint has no effect, i.e. $e^{a\,Ad\,L_1}(L_2) = L_2$, then there is no entry. From (2.2.23), the commutation relations are simply the coefficients of a in the table. For example $[M_{uv}, P_w] = \delta_{vw}P_u - \delta_{uw}P_v$ where δ_{jk} is the Kronecker delta.

† -The adjoint action of M_{uv} on the P_w's is $e^{aAdM_{uv}}(\rho_u P_u + \rho_v P_v) = \rho_u' P_u + \rho_v' P_v$ where ρ_u' and ρ_v' are determined by $\begin{bmatrix} \rho_u' \\ \rho_v' \end{bmatrix} = \begin{bmatrix} \cos(a) & \sin(a) \\ -\sin(a) & \cos(a) \end{bmatrix} \begin{bmatrix} \rho_u \\ \rho_v \end{bmatrix}$. It acts on the B_u's in exactly the same manner. $(u, v, w, z \in U = \{1, \ldots, n-2\})$.

The operators P_u, B_u, t generate the $2m+1$ dimensional Weyl algebra w_m and the M_{uv}'s generate the $\frac{1}{2}m(m-1)$ - dimensional orthogonal algebra $o(m)$. If we define the operators

$$A_1 = D, \; A_2 = K_2 + K_{-2}, \; A_3 = K_{-2} - K_2 \tag{2.2.24}$$

these satisfy the commutation relations

$$[A_1, A_2] = -2A_3, \; [A_3, A_1] = 2A_2, \; [A_2, A_3] = 2A_1 \tag{2.2.25}$$

and form a basis for the Lie algebra $sl(2, \mathbb{R})$. It follows that the structure of s_m is

$$s_m = (sl(2, \mathbb{R}) \oplus o(m)) \oplus w_m \tag{2.2.26}$$

where $\oplus$ represents the direct sum and $\oplus$ the indirect sum. Similarly the Galilean algebra has structure

$$g_m = (t_1 \oplus o(m)) \oplus w_m \tag{2.2.27}$$

where t_1 is the one dimensional algebra of time translations. Using standard results from Lie theory, these operators can be exponentiated to obtain the Schrödinger and Galilean groups. These groups act on the space of locally analytic functions of the real variables y^j, t and map solutions of (*) into solutions. Expressions for the actions of these groups appear in Miller (1977).

2.3 Reducing the Hamilton-Jacobi equation

In this section we find simple forms for both the metric and the Killing vectors, summarising these results in the following theorem.

Theorem 2.3.1

All Hamilton-Jacobi separable coordinate systems for (2.2.14) with Killing vector $\hat{\varepsilon}$, are equivalent to a coordinate system associated with the Hamilton-Jacobi equation

$$g^{11}p_1{}^2 + \qquad + g^{(n-2)(n-2)}p_{n-2}{}^2 + g^{nn}p_n{}^2 + 2p_{n-1}p_n = E . \tag{2.3.1}$$

Moreover if one variable is first order, and there are r ignorables $x^{\alpha(1)}, \ldots, x^{\alpha(r)}$ in addition to x^n, the corresponding Killing vectors are

$$\lambda_n = \frac{1}{2}(P_{n-1} + P_n) = \hat{\varepsilon} ,$$

$$\lambda_{\alpha(1)} = M_{12} , \ldots , \lambda_{\alpha(p)} = M_{2p-1,2p} ,$$

$$\lambda_{\alpha(p+1)} = P_{2p+1} , \ldots , \lambda_{\alpha(q)} = P_{p+q} ,$$

$$\lambda_{\alpha(q+1)} = B_{p+q+1} - \rho^{p+q+1}P_{p+q+1}, \ldots , \lambda_{\alpha(r)} = B_{p+r} - \rho^{p+r}P_{p+r}. \tag{2.3.2}$$

If there are no first order variables then all the Killing vectors except for one‡ will be those in (2.3.2).

Proof

Let x^a be a Stäckel variable while x^r is first order. From (2.2.8)

$$g_{ni} = \frac{\partial t}{\partial x^i} \tag{2.3.3}$$

so that

$$\frac{\partial t}{\partial x^a} = 0 , \quad \frac{\partial t}{\partial x^\alpha} = g_{n\alpha}(x^r) , \tag{2.3.4}$$

since inversion of (D3) shows that for Stäckel variables $g_{ai} = \delta^{ai}(g^{aa})^{-1}$. The equations in (2.3.4) can be integrated to give

$$t = h(x^r) + \sum_\alpha g_{n\alpha}(x^r)x^\alpha . \tag{2.3.5}$$

‡ - This Killing vector can only be calculated from the coordinate transformations (2.4.30). It has form $K_{-2} + \beta^u B_u$, but it's knowledge was not needed to produce the reduced form of the contravariant metric (2.3.1). It may be regarded as a special subcase when the first order variable has become ignorable.

Now from (2.3.5) and (2.2.8)

$$\lambda_\alpha = y^u{}_{,\alpha} P_u + g_{n\alpha}(P_{n-1} - P_n) + \tfrac{1}{2} f_{,\alpha}(P_{n-1} + P_n)\ , \alpha \neq n\ , \tag{2.3.6}$$

so (2.2.18) implies

$$g_{n\alpha} = \kappa_\alpha\ . \tag{2.3.7}$$

We know from §2.2 that $n_2 \leq 1$.

If $n_2 = 0$ then

$$t = \kappa_\alpha x^\alpha \to x^{n-1} \tag{2.3.8}$$

under a general linear motion amongst the ignorables.

Consider $n_2 = 1$. If for some β, $\kappa_\beta \neq 0$, we can use a general linear motion and then the freedom (2.1.3b) to transform t to x^{n-2}, *i.e.*,

$$t \to h(x^{n-1}) + x^{n-2} \to x^{n-2}. \tag{2.3.9}$$

Thus from (2.3.7), (g_{ij}) has form

$$\begin{pmatrix}
\bullet & & & & & & & \\
 & & & & & & & \\
 & \bullet & & & & & & \\
 & & \bullet & \bullet & & \bullet & \bullet & 0 \\
 & & \bullet & & & & \bullet & 0 \\
 & & & & & & & \\
 & & \bullet & & & & \bullet & 1 \\
 & & \bullet & \bullet & & \bullet & \bullet & 0 \\
 & & 0 & & 0 & 1 & 0 & 0
\end{pmatrix}$$

(2.3.10a)

The variable x^{n-1} is first order, thus $g^{(n-1)(n-1)} = 0$ and by taking the inverse of (2.3.10a), (g^{ij}) has form

$$\begin{pmatrix}
\cdot & & & & & & & \\
 & & & & & & & \\
 & \cdot & & & & & & \\
 & & \cdot & \cdot & \cdot & 0 & \cdot & \cdot \\
 & & \cdot & & & 0 & \cdot & \\
 & & & & & & & \\
 & & \cdot & & \cdot & & & \cdot \\
 & & 0 & & & 0 & 0 & 1 \\
 & & \cdot & \cdot & \cdot & 0 & 0 & \cdot \\
 & & \cdot & & \cdot & 1 & \cdot & \cdot
\end{pmatrix} \tag{2.3.10b}$$

The vectors

$$\mathbf{X}^{n-1} = (0,0, \cdots ,0,0,1,0)$$

$$\mathbf{X}^{n-2} = (0,0, \cdots ,0,1,0,0) \tag{2.3.11}$$

are orthogonal and null with respect to (g^{ij}), which is impossible in Minkowski space (see Eisenhart (1949)). Thus

$$\kappa_\alpha = 0, \text{ for } each \ \alpha. \tag{2.3.12}$$

It is now possible to use the freedom (2.1.3a) to change variables so that

$$h(x^{n-1}) \to x^{n-1} . \tag{2.3.13}$$

We see that whether x^{n-1} is first order or ignorable the coordinate transformations are

$$y^u = y^u(x^1, \ldots , x^{n-1}) ,$$

$$y^{n-1} - y^n = 2x^{n-1} = 2t ,$$

$$y^{n-1} + y^n = x^n + f(x^1, \ldots , x^{n-1}). \tag{2.3.14}$$

This is progress as the time coordinate t has been identified as just x^{n-1}.

From (2.2.18) and (2.3.14)

$$\lambda_\alpha = \rho_\alpha^u P_u + m_\alpha^{uv} M_{uv} + \beta_\alpha^u B_u \ , \alpha \neq n-1. \tag{2.3.15}$$

Let $x^{n-1} = c$ and $x^n = k$ in (2.3.14) where c and k are both constants. The metric (2.2.11) becomes

$$ds^2 = \sum_{u}^{n-2} (dy^u)^2 \tag{2.3.16}$$

with corresponding Hamilton-Jacobi equation

$$\sum_{u} P_u{}^2 = \bar{E} \tag{2.3.17}$$

From (2.3.14) the coordinate transformations for this system are

$$y^u = y^u(x^1, \ldots, x^{n-2}; x^{n-1}=c) \tag{2.3.18}$$

Technically it is not possible to take both x^{n-1} and x^n to be constants since the metric (2.2.11) then becomes singular. It will be legitimate in our case, however, as the properties we will derive do not depend on the singularity of the metric.

Separable solutions $\Psi=\prod_{u=1}^{n-2}\Psi_u(x^u)\ \Psi_{n-1}(x^{n-1})e^{cx^n}$ will also yield separable solutions of (2.3.17) parameterised by c and k. Thus (2.3.18) is a separable coordinate system for the Hamilton-Jacobi equation (2.3.17). All such systems have been classified by Kalnins and Miller (1982a). They find that there is an equivalent separable system $\{\bar{\mathbf{x}}\}$,

$$\bar{x}^u = T^u(x^v; c), \tag{2.3.19}$$

in which the Killing vectors are

$$\begin{aligned} \bar{\lambda}_{\alpha(1)} &= \bar{M}_{12}, \ \ldots, \bar{\lambda}_{\alpha(p)} = \bar{M}_{2p-1,2p}, \\ \bar{\lambda}_{\alpha(p+1)} &= \bar{P}_{2p+1}, \ \ldots, \bar{\lambda}_{\alpha(r)} = \bar{P}_{p+r} \end{aligned} \tag{2.3.20}$$

Now consider

$$\begin{aligned} \bar{x}^u &= T^u(x^v; c=0) \\ \bar{x}^{n-1} &= x^{n-1} \\ \bar{x}^n &= x^n \end{aligned} \tag{2.3.21}$$

which is an equivalent separable system for (2.3.14). Setting $x^{n-1} = x^n = 0$, a constant, (2.3.15) becomes

$$\bar{\lambda}_{\alpha(s)} = \bar{\rho}_s^u \bar{P}_u + \bar{m}_s^{uv} \bar{M}_{uv}, \alpha(s) \neq n-1, n \tag{2.3.22}$$

Comparing with (2.3.20) the $\bar{\rho}_s^u$'s and $\bar{m}_s^{uv}$'s are determined and with the restriction $x^{n-1} = x^n = constant = 0$, removed

$$\lambda_{\alpha(1)} = M_{12} + \beta_1^u B_u , \qquad , \lambda_{\alpha(p)} = M_{2p-1,2p} + \beta_p^u B_u ,$$

$$\lambda_{\alpha(p+1)} = P_{2p+1} + \beta_{p+1}^u B_u , \quad \ldots , \bar{\lambda}_{\alpha(r)} = P_{p+r} + \beta_r^u B_u , \ \alpha(s) \neq n-1, n. \qquad (2.3.23)$$

We can assume that

$$\beta_s^u = 0 , s = p+1, \ldots , q ,$$

$$\beta_s^v \neq 0 , for\ some\ v , s = q+1, \ldots , r , \qquad (2.3.24)$$

and will show that

$$\beta_s^u = 0 , s = 1, \ldots , p. \qquad (2.3.25)$$

First consider the Killing vector

$$\lambda_{\alpha(1)} = M_{12} + \beta_1^u B_u . \qquad (2.3.26)$$

From Table 2.2.1 its possible to take

$$\beta_1^1 = \beta_1^2 = 0 \qquad (2.3.27)$$

by using the action of B_1 and B_2. Now using rotations independent of x^1 and x^2,

$$\lambda_{\alpha(1)} \to M_{12} + \beta\, B_3 , \qquad (2.3.28)$$

where

$$\beta^2 = \sum_u (\beta_1^u)^2 . \qquad (2.3.29)$$

Again letting $x^{n-1} = c$ and $x^n = k$,

$$\lambda_{\alpha(1)} \to M_{12} - \beta c\ P_3 , \qquad (2.3.30)$$

which must be a Killing vector for a separable system on $\mathbb{R}^{n-2}$. The argument used by Kalnins and Miller (1982a) to derive (2.3.20) shows that this is only possible if $\beta = 0$, *i.e.*, from (2.3.29)

$$\beta_1^u = 0 . \qquad (2.3.31)$$

This result is easily generalised to show that (2.3.25) holds.

We will now show that

$$\beta_s^u = 0 , u \neq s+p , q+1 \leq s \leq r \qquad (2.3.32)$$

Using rotations independent of y^{s+p}

$$\lambda_{\alpha(s)} \rightarrow P_{s+p} + \beta_s^{s+p} B_{s+p} + \beta\, B_v \text{ , some } v \neq s+p \text{ ,} \tag{2.3.33}$$

where

$$\beta = [\sum_{u \neq s+p} (\beta_s^u)^2]^{1/2} \tag{2.3.34}$$

Further using the rotation $M_{(s+p)v}$ we obtain

$$\lambda_{\alpha(s)} \rightarrow \frac{1}{\kappa}[\beta_s^{s+p} P_{s+p} - \beta\, P_v\,] + \kappa\, B_{s+p} \tag{2.3.35}$$

where from (2.3.24)

$$\kappa = [\beta^2 + (\beta_s^{s+p})^2]^{1/2} \tag{2.3.36}$$

is nonzero. Applying the action of K_{-2}

$$\lambda_{\alpha(s)} \rightarrow B_{s+p} - \bar{\beta}\, P_v \tag{2.3.37}$$

where κ has been absorbed in $x^{\alpha(s)}$ and

$$\bar{\beta} = \frac{\beta}{\kappa^2}. \tag{2.3.38}$$

From (2.2.8) and (2.3.37)

$$dx^{\alpha(s)} = \frac{dy^{s+p}}{\frac{1}{2}(y^n - y^{n-1})} = \frac{dy^v}{-\bar{\beta}} = \frac{dy^w}{0} = \frac{dy^{n-1}}{\frac{1}{2}y^{s+p}} = \frac{dy^n}{\frac{1}{2}y^{s+p}}$$

$$, w \neq v, s+p \text{ ,} \tag{2.3.39}$$

which implies

$$y^{s+p} = -x^{n-1}x^{\alpha(s)} + A_{s+p} \text{ ,}$$

$$y^v = -\bar{\beta}x^{\alpha(s)} + A_v \text{ ,}$$

$$y^w = A_w \text{ ,}$$

$$y^{n-1}+y^n = x^n - \frac{1}{2}x^{n-1}(x^{\alpha(s)})^2 + A_{s+p}x^{\alpha(s)} + D \text{ ,}$$

$$y^{n-1}-y^n = 2x^{n-1} \text{ ,} \tag{2.3.40}$$

where $A_1, \ldots, A_{n-2}$ and D do not depend on $x^{\alpha(s)}$ and x^{n-1}. If $\bar{\beta} \neq 0$, and $x^{n-1} = c$, a constant, the work of Kalnins and Miller (1982a) implies that A_{s+p}

and A_v are not functions of the x^u, so they depend on x^{n-1} alone. This is impossible since if it occurred y^{s+p}, y^v and $y^{n-1} - y^n$ would be dependent. Thus $\bar{\beta} = 0$ and together with (2.3.34) and (2.3.38), (2.3.32) is verified.

The results (2.3.25) and (2.3.32) show that the Killing vectors have the form (2.3.2) given in Theorem 2.3.1 and this will be used to prove the other part of the theorem. Using (2.3.2)

$$dx^{\alpha(s)} = \frac{dy^{2s-1}}{-y^{2s}} = \frac{dy^{2s}}{-y^{2s-1}} = \frac{dy^{\kappa(s)}}{0}, \; 1 \le s \le p \;, \kappa(s) \ne 2s-1,\, 2s \;,$$

$$dx^{\alpha(s)} = \frac{dy^{p+s}}{1} = \frac{dy^{\kappa(s)}}{0}, \; p+1 \le s \le q \;, \kappa(s) \ne p+s \;,$$

$$dx^{\alpha(s)} = \frac{dy^{p+s}}{[\rho_{p+s} - \frac{1}{2}(y^{n-1} - y^n)]} = \frac{dy^{n-1}}{\frac{1}{2}y^{p+s}} = \frac{dy^n}{\frac{1}{2}y^{p+s}} = \frac{dy^{\kappa(s)}}{0},$$

$$q+1 \le s \le r \;, \kappa(s) \ne p+s,\, n-1 \,,\, n-2$$

$$dx^n = \frac{dy^u}{0} = \frac{dy^{n-1}}{1/2} = \frac{dy^n}{1/2}, \tag{2.3.41}$$

we obtain the coordinates

$$y^{2s-1} = A_{2s-1}\cos(x^{\alpha(s)} + A_{2s}) \;,$$

$$y^{2s} = A_{2s-1}\sin(x^{\alpha(s)} + A_{2s}) \;, \; 1 \le s \le p \;,$$

$$y^{p+s} = x^{\alpha(s)} + A_{p+s} \quad , p+1 \le s \le q \;,$$

$$y^{p+s} = [\rho_{p+s} - x^{n-1}]x^{\alpha(s)} + A_{p+s} \;, q+1 \le s \le r \;,$$

$$y^s = A_s \quad , r+1 \le s \le n-2 \;,$$

$$y^{n-1}+y^n = x^n + \sum_{s=q+1}^{r} (\frac{1}{2}[\rho_{p+s} - x^{n-1}](x^{\alpha(s)})^2 + A_{p+s}x^{\alpha(s)}) + D \;,$$

$$y^{n-1}-y^n = 2x^{n-1} \tag{2.3.42}$$

Here $A_1, \quad , A_{n-2}, D$ are functions of x^{n-1} and the Stäckel variables alone. Taking x^{n-1} to be constant, the work of Kalnins and Miller (1982a) implies that

$$A_{2s} \;, 1 \le s \le p+1 \;, A_{p+s} \;, p+1 \le s \le r \;, \tag{2.3.43}$$

are not functions of the Stäckel variables which means they are dependent on x^{n-1} alone. Using the transformations,

$$x^{a(s)} + A_{2s} \to x^{a(s)},\ 1 \le s \le p\ ,$$

$$x^{a(s)} + A_{p+s} \to x^{a(s)},\ p+1 \le s \le q\ ,$$

$$x^{a(s)} \to x^{a(s)} - \frac{A_{p+s}}{[\rho_{p+s} - x^{n-1}]},\ q+1 \le s \le r\ , \qquad (2.3.44)$$

it can be assumed that

$$A_{2s} = 0,\ 1 \le s \le p$$

$$A_{p+s} = 0,\ p+1 \le s \le q. \qquad (2.3.45)$$

With the aid of the results of (2.3.42), (2.3.45) and the fact that for Stäckel variables $g_{ai} = \delta^{ai}(g^{aa})^{-1}$, the metric can be written as

$$ds^2 = \sum_u g_{uu}(dx^u)^2 + g_{(n-1)(n-1)}(dx^{n-1})^2 + 2dx^{n-1}dx^n \qquad (2.3.46)$$

Inversion of (2.3.46) gives the Hamilton-Jacobi equation in the form (2.3.1). Q.E.D.

By finding the symmetries and using the restrictive signature of Minkowski space we have reduced the nonorthogonal part of the Hamilton-Jacobi equation to one off-diagonal element.

2.4 Metric and coordinates

In the previous section we established a simple form (2.3.1) for the contravariant metric and this will enable us to use the curvature conditions (2.2.13) to determine this form exactly. We will also find the coordinate transformations.

Theorem 2.4.1

The Hamilton-Jacobi equation can be written

$$\sum_{q \in Q} \frac{1}{\sigma_q} \sum_{b \in B_q} \bar{g}^{bb} p_b{}^2 + 2p_{n-1}p_n + \sum_{q \in Q} \frac{V_q}{\sigma_q} p_n{}^2 = E \qquad (2.4.1)$$

where

$$Q = \{ q_i : q_1 < q_2 < \qquad < q_l \} \tag{2.4.2}$$

is some subset of $U = \{1,\dots,m\}$ and the sets

$$B_{q_i} = \{ q_i, q_i + 1, \quad , q_{i+1} - 1 \}, \ 1 \le i \le l , \tag{2.4.3}$$

form a partition of U. Here σ_q is a function of x^{n-1} alone, and if $b \in B_q$

$$\bar{g}^{bb} = \bar{g}^{bb}(x^c) \ , \ V_q = V_q(x^c) \tag{2.4.4}$$

where $c \in B_q$.

As an illustration of the notation used in Theorem 2.4.1 see (2.5.14). There $U = Q = \{1,2\}$, $B_1 = \{1\}$ and $B_2 = \{2\}$. In (2.5.17) $Q = \{1\}$, $B_1 = \{1,2\}$ and $\sigma_1 = |(x^3)^2 + 1|^{\frac{1}{2}}$. To prove the Theorem we will make use of the following equivalent condition for the Stäckel matrix found by Eisenhart (1934).

Lemma 2.4.1

The nonsingular $Q \times Q$ matrix (ψ_{ij}) is a Stäckel matrix if and only if

$$\partial_{jk} \log(\frac{\psi}{\psi^{i1}}) = \partial_j \log(\frac{\psi}{\psi^{i1}}) \partial_k \log(\frac{\psi}{\psi^{i1}}) - \partial_j \log(\frac{\psi}{\psi^{i1}}) \partial_k \log(\frac{\psi}{\psi^{j1}})$$

$$- \partial_k \log(\frac{\psi}{\psi^{i1}}) \partial_j \log(\frac{\psi}{\psi^{k1}}) \quad ,1 \le j < k \le Q. \tag{2.4.5}$$

We now prove Theorem 2.4.1.

Proof of Theorem 2.4.1

For notational convenience we need only assume $n_2 = 1$ since the functional form for $n_2 = 0$ is just a special case. Consider the matrix

$$(\Phi_{ij}) = \begin{pmatrix} \psi_{ab} & 0 \\ A_{\alpha\beta} & I_{MM} \end{pmatrix} \tag{2.4.6}$$

where

$$A_{\beta a} = -A_a^{\beta\beta}, \ \beta = n_1 + 1 \ , \ . \ . \ . \ , n - 2 \tag{2.4.7}$$

is an $M \times N$ matrix ($N = n_1 + 1$, $M = n - 2 - n_1$) and I_{MM} is the $M \times M$ identity matrix.

This matrix has the properties

$$g^{\alpha\alpha} = \frac{\Phi^{\alpha 1}}{\Phi}, \; g^{\beta\beta} = \frac{\Phi^{\beta 1}}{\Phi}, \; g^{(n-1)n} = 1 = \frac{\Phi^{(n-1)1}}{\Phi}. \tag{2.4.8}$$

These will enable us to make extensive use of Lemma 2.4.1. It follows immediately from (2.3.1) and Lemma 2.4.1 that

$$\partial_{v(n-1)}\log g_{uu} = \partial_v \log g_{uu} \partial_{n-1}\log g_{uu} - \partial_v \log g_{uu} \partial_{n-1}\log g_{vv} \tag{2.4.9}$$

It is easily shown that

$$R_{vuu(n-1)} = \frac{1}{2} g_{uu} \partial_{v(n-1)}\log(g_{uu}) + \frac{g_{uu}}{4}[\, \partial_v log\,(g_{uu})\, \partial_{n-1} log\,(g_{uu})$$

$$- \partial_v log\,(g_{uu})\, \partial_{n-1} log\,(g_{vv})\,] = 0 \tag{2.4.10}$$

Substitution of (2.4.9) into [] in (2.4.10) gives

$$\partial_{v(n-1)}\log(g_{uu}) = 0. \tag{2.4.11}$$

This implies

$$g_{uu} = \sigma_u(x^{n-1})\, \bar{g}_{uu} \tag{2.4.12}$$

where $\bar{g}_{uu}$ does not depend on x^{n-1}. From (2.4.9) and (2.4.11)

$$\partial_v \log(g_{uu})\, \partial_{n-1}\log(\sigma_u / \sigma_v) = 0. \tag{2.4.13}$$

We can define an equivalence relation $\sim$ on U by $u \sim v$ if σ_u is proportional to σ_v, and by rescaling coordinates there is no loss in assuming that $\sigma_u = \sigma_v$ on each equivalence class. By reordering the indices, the sets of the partition can be taken to be the B_q's of Theorem 2.4.1. Furthermore, (2.4.13) implies that the $\bar{g}_{bb}$'s are only functions of those x^c's in their class, *i.e.* they satisfy property (2.4.4) in Theorem 2.4.1. Defining

$$V_q = \sum_{b \in B_q} \bar{g}^{bb} A_b^{nn} \tag{2.4.14}$$

the Hamilton-Jacobi equation takes the form (2.4.1) given in Theorem 2.4.1. Q.E.D.

Theorem 2.4.2

The spaces with metrics

$$d\bar{s}_q^{\,2} = \sum_{b \in B_q} \bar{g}_{bb}\,(dx^b)^2 \tag{2.4.15}$$

are separable, flat and positive definite. Also we have

$$\partial_{x^c x^d} V_q = 0\,,\quad c \neq d\,,\ c\,,d \in B_q\,. \tag{2.4.16}$$

Proof : We first show that the metrics (2.4.15) are differentially flat, separable and positive definite. The flatness conditions $R_{ijkl} = 0$ with $i,j,k,l \in B_q$, are equivalent to $\bar{R}_{ijkl} = 0$, *i.e.*, those for the metrics in (2.4.15). It follows that the spaces associated with these metrics are flat. Since the metrics in (2.4.15) are orthogonal the separability conditions (2.4.5) for $i,j,k \in B_q$ can be applied to show that these metrics are also separable. To show that the metrics in (2.4.15) are positive definite we first compute the eigenvalues of (2.4.1). These are

$$\xi_u = g^{uu}\,, \xi_\pm = \tfrac{1}{2}[g^{nn} \pm \sqrt{(g^{nn})^2 + 4}\,]. \tag{2.4.17}$$

Regardless of the value of g^{nn} the eigenvalues ξ_+ and ξ_- are positive and negative respectively. The remaining eigenvalues ξ_u must be positive since the space is Minkowski with signature n-1. It now follows from (2.4.1) that

$$\sigma_q > 0 \text{ and } \bar{g}_{uu} > 0\,, \tag{2.4.18}$$

redefining σ_q and $\bar{g}_{uu}$ to be their negatives if necessary. This shows that the metrics (2.4.15) are positive definite.

To prove condition (2.4.16) of the Theorem we notice that since V_q is given by (2.4.14), the extension of Lemma 2.4.1 (see equation (2.4.8)) can be used to obtain

$$\partial_{cd}\, V_q + \partial_c\, V_q\, \partial_d \log(g_{cc}) + \partial_d\, V_q\, \partial_c \log(g_{dd}) = 0 \tag{2.4.19}$$

Alternatively this result can be obtained by considering the Hamilton-Jacobi equation (artificially created for our purpose) $\sum_{b \in B_q} \bar{g}^{bb} p_b^2 + \sum_q V_q p_{n-1}^2$. As this equation is separable the results of Lemma 2.4.1 can be applied: equation (2.4.19) is derived from (2.4.5) with $j = c$, $k = d$ and $i = n-1$. Combining this result with the curvature conditions $R_{c(n-1)(n-1)d} = 0$, which are equivalent to

$$-\frac{1}{2}\partial_{cd} V_q + \partial_c V_q \partial_d \log(g_{cc})/4 + \partial_d V_q \partial_c \log(g_{dd})/4 = 0, \qquad (2.4.20)$$

we obtain condition (2.4.16). Q.E.D.

We now go on to determine the exact form of the unknown functions $\bar{g}_{bb}$, σ_q, V_q, and hence obtain the classification we are seeking. First since the spaces defined by the metrics (2.4.15) are positive definite, we can use the results of Kalnins and Miller (1982a), in which they completely classified such spaces; this determines the $\bar{g}_{uu}$'s. By transforming to standard cartesian coordinates on each $\mathbb{R}^{n_q}$, the σ_q and V_q are determined. In transforming back to general separable coordinates, however, the separability is not necessarily preserved, so we find the compatibility conditions to ensure this preservation.

Since the spaces defined by the metrics are flat and positive definite we can choose standard coordinates z^b :

$$z^b = z^b(x^c),\ b,\ c \in B_q. \qquad (2.4.21)$$

and

$$d\bar{s}_q{}^2 = \sum_{b \in B_q} (dz^b)^2. \qquad (2.4.22)$$

Working in terms of these coordinates (2.4.19) implies that

$$\partial_{z^c z^d} V_q = 0 \quad c,\ d \in B_q,\ c \neq d. \qquad (2.4.23)$$

$R_{(n-1)cc(n-1)} = 0$ is equivalent to

$$\frac{1}{2}\sigma_q{}'' - \frac{(\sigma_q{}')^2}{4\sigma_q} - \frac{1}{2}\sum_{t \in Q} \frac{V_{t,cc}}{\sigma_t} = 0 \qquad (2.4.24)$$

where the prime denotes differentiation with respect to x^{n-1}. Together with (2.4.23) this last equation implies that

$$V_q = \sum_{b \in B_q} \left(\frac{\zeta_q}{4}(z^b)^2 + \gamma_b z^b\right) + \delta \qquad (2.4.25)$$

and

$$2\sigma_q \sigma_q{}'' - (\sigma_q{}')^2 = \zeta_q,\quad \sigma_q > 0,\ \zeta_q,\ \delta \in \mathbb{R}. \qquad (2.4.26)$$

Making a transformation of form $x^n \to x^n + g(x^{n-1})$ we can take $\delta = 0$ in (2.4.25). When $\zeta_q \neq 0$ it is possible to translate z^b so that $\gamma_b \to 0$. To solve (2.4.26) differentiate it to obtain $\sigma'''=0$ and substitute the resulting quadratic into the original equation (2.4.26). Application of the coordinate freedoms (2.1.3) leads to the five possibilities $I \to IV\pm$ in Table 2.4.1.

Table 2.4.1 Possibilities for $\sigma(x^{n-1})$

type	σ_q	ζ_q	γ_q
I	1	0	*arbitrary*
II	$(x^{n-1}+v_q)^2$	0	*arbitrary*
III	$(x^{n-1}+v_q)$	-1	0
$IV\pm$	$\lvert(x^{n-1}+v_q)^2 \pm w_q^2\rvert$	$\pm 4w_q^2, w_q \neq 0$	0

The constants ζ_q, γ_b, v_q and w_q are all real.

A knowledge of the lower dimensional cases, plus the form of the coordinate transformations (2.3.42), and a little guess work leads to the transformations for $n=3(m=1)$. They are

$$y^1 = z^1\sigma^{\frac{1}{2}} + \frac{1}{2}\gamma\int\int\sigma^{-3/2},$$

$$y^2 + y^3 = x^3 - \sigma'(z^1)^2/4 - \frac{1}{2}\gamma\, z^1\sigma^{\frac{1}{2}}\int\sigma^{-3/2} - \frac{\gamma^2}{8}\int(\int\sigma^{-3/2})^2,$$

$$y^2 - y^3 = 2x^2, \tag{2.4.27}$$

where $\int f(y) = \int_{y=y_0}^{x^2} f(y)dy$. We now show that this 1-dimensional case forms the building block for all others. Define

$$F_{uq}(z^u, x^{n-1}) = z^u\sigma_q^{\frac{1}{2}} + \frac{1}{2}\gamma_u\int\int\sigma_q^{-3/2}, \tag{2.4.28}$$

and

$$G_{uq}(z^u, x^{n-1}) = \sigma_q{}'(z^u)^2/4 + \frac{1}{2}\gamma_u\, z^u\sigma_q^{\frac{1}{2}}\int\sigma_q^{-3/2} + \frac{\gamma_u^2}{8}\int(\int\sigma_q^{-3/2})^2, \tag{2.4.29}$$

then the coordinate transformations are

$$y^u = F_{uq} \ ,$$

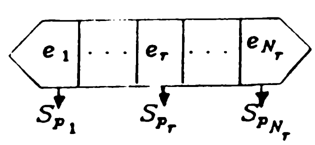
$$y^{n-1} + y^n = x^n - \sum_{q \in Q} \sum_{b \in B_q} G_{bq} \ ,$$

$$y^{n-1} - y^n = 2x^{n-1} \qquad (2.4.30)$$

These results can be checked by computing the metric using (2.4.30) and then taking its inverse to obtain the Hamilton-Jacobi equation (2.4.1). In the above transformations we will say that the γ_u term is *attached* to the z^u coordinate.

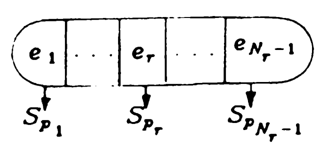
To obtain the coordinate transformations in terms of the x^i we simply substitute the expressions given for the z^b in (2.4.21). Not all the systems thus obtained will be separable. Naturally they will be separable when $\gamma_u = 0$ for all u and $\zeta_q = 0$, or for all values of these parameters when the coordinates are cartesian. Given a certain separable system (2.4.21) the problem is to work out the form of V_q (or equivalently the values of the parameters γ_u and ζ_q) to ensure separation in the variables x^i.

To tackle this problem it is necessary to know just what the possible systems (2.4.21) are. We give a brief summary of the solution provided by Kalnins and Miller (1982a). They show that it is possible to decompose $\mathbb{R}^n$ into a direct sum of subspaces $\mathbb{R}^{n_r}$ in such a way that the separable coordinates on each of these are of either "elliptic" or "parabolic" type with graphical representations



(2.4.32A)

or



(2.4.32B)

respectively. We will say that $\mathbb{R}^n$ *splits* into the subspaces $\mathbb{R}^{n_r}$. Cartesian coordinates on $I\!R^{n_r}$ for case A are given by

$$z^i = ({}_{N_r}w_j)({}_{p_j}s_{q_j}) \; ; 1 \le i \le n_r, \; 1 \le j \le N_r \tag{2.4.33A}$$

and for case B

$$z^1 = ({}_{N_r}w_1),$$

$$z^i = ({}_{N_r}w_j)({}_{p_j}s_{q_j}) \; ; 2 \le i \le n_r, \; 2 \le j \le N_r. \tag{2.4.33B}$$

Here the ${}_{p_j}s_{q_j}$, $1 \le q_j \le p_j + 1$ are coordinates on the p_j-dimensional sphere S_{p_j} and therefore satisfy

$$\sum_{q_j = 1}^{p_j + 1} {}_{p_j}s_{q_j}^2 = 1. \tag{2.4.34}$$

In (2.4.32) the sphere S_{p_j} is said to be *attached* to e_j or equivalently to the coordinate ${}_{N_r}w_j$. When $p_j = 0$, S_{p_j} is the "zero dimensional" or "trivial" sphere. Equivalently there is no sphere attached to ${}_{N_r}w_j$.

For elliptic-type coordinates A

$${}_{N_r}w_j^{\,2} = c_r^2 \frac{\prod_{l=1}^{N_r}(x_r^l - e_j^r)}{\prod_{l \ne j}(e_l^r - e_j^r)} \quad , j = 1, \ldots, N_r, \tag{2.4.35A}$$

where

$$e_1^r < x_r^1 < \cdots < e_{N_r}^r < x_r^{N_r}, \; c_r, e_j^r \in I\!R.$$

For parabolic coordinates B

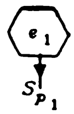
$${}_{N_r}w_1 = \frac{c_r}{2}\left(\sum_{l=1}^{N_r} x_r^l + \sum_{l=1}^{N_r - 1} e_l^r\right)$$

$${}_{N_r}w_j^{\,2} = -c_r^2 \frac{\prod_{l=1}^{N_r}(x_r^l - e_{j-1}^r)}{\prod_{l \ne j-1}(e_l^r - e_{j-1}^r)} \quad , j = 2, \ldots, N_r, \tag{2.4.35B}$$

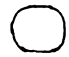
where

$$x_r^1 < e_1^r < \qquad < e_{N_r - 1}^r < x_r^{N_r}, \; c_r, e_j^r \in I\!R$$

If the case $N_r = 1$ is treated in the same way as $N_r > 1$, it is possible to have the elliptic and parabolic systems

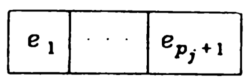


and



corresponding to the metrics $ds^2 = \frac{(dx^1)^2}{x^1 - e_1}$ and $ds^2 = (dx^1)^2$ respectively. Both of these systems are equivalent via the scaling transformation (2.1.3a). This explains why the parabolic case does not appear when $N_r = 1$. This will help clarify some of the exceptional behavoir that we will later encounter for $N_r = 1$.

The classification also depends on the possible separable systems on the spheres S_{p_j}. The structure of these systems is independent of the V_q and σ_q terms. Separable systems on S_{p_j} can be built from irreducible blocks, each having graphical representation

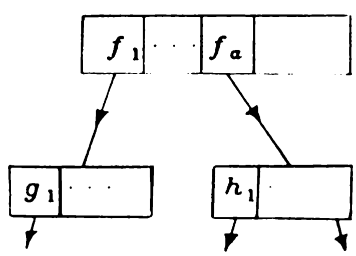


(2.4.36)

The coordinate transformations for this block are

$$ {}_pS_i^2 = c^2 \frac{\prod_{j=1}^{p} (x^j - e_i)}{\prod_{j \neq i} (e_j - e_i)} \quad , 1 \le i \le p+1 \tag{2.4.37} $$

where $e_1 < x^1 < \cdots < x^p < e_{p+1}$. A general graph for S_{p_j} has form



(2.4.38)

The coordinates for (2.4.38) can be built in multiplicative-iterative fashion from those of (2.4.37). This process is best understood by consulting Kalnins and Miller (1982a) and the examples provided in Tables 2 and 5 of Appendix A.

Applying these results to our case, decomposing the space corresponding to $d\bar{s}_q{}^2$, corresponds to partitioning B_q into subsets E_r in just the same way that we partitioned U into the subsets B_q (i.e. the set E_r is labelled by its minimum value r). For example in (2.5.17) $E_1=\{1\}$, $E_2=\{2\}$, and in (2.5.22) $E_1=\{1,2\}$. Using this partition

$$d\bar{s}_q{}^2 = \sum_{r \in \mathcal{R}\cap B_q} d\bar{s}_r{}^2 \tag{2.4.39}$$

where $d\bar{s}_r{}^2$ is the infinitesimal distance

$$d\bar{s}_r{}^2 = \sum_{b \in E_r} \bar{g}_{bb}(d\bar{x}^b)^2 \tag{2.4.40}$$

Here $\mathcal{R}$ is the set of all the (minimum) indices r (e.g. in both (2.5.14) and (2.5.17) $\mathcal{R}=\{1,2\}$). For simplicity on each irreducible block E_r of form (2.4.32A) or (2.4.32B) the coordinates $x^r, x^{r+1}, \ldots, x^{r+n_r-1}$ are relabelled as $x^1, x^2, \ldots, x^{n_r}$. V_r for each of these blocks is defined as in (2.4.14) but with b restricted to E_r. Confining ourselves to one of these blocks E_r we systematically determine V_r.

Suppose that $\zeta_q \neq 0$(i.e. σ_q is of types III or $IV\pm$) on E_r. Here

$$V_r = \sum_{i=1}^{n_r} \frac{\zeta_q}{4}(x^i)^2 \tag{2.4.41}$$

and from (2.4.33) and (2.4.34),

$$V_r = \frac{\zeta_q}{4}\sum_{j=1}^{N_r} {}_{N_r}w_j^2. \tag{2.4.42}$$

Substituting for the ${}_{N_r}w_j$ from (2.4.35) and expanding in partial fractions in terms of the e_j's we find after some time that for the elliptic case A

$$V_r = \frac{\zeta_q c_r{}^2}{4}\{\sum_{i=1}^{N_r} x^i - \sum_{i=1}^{N_r} e_i\}. \tag{2.4.43A}$$

and for the parabolic case B

$$V_r = \frac{\zeta_q c_r^2}{16}\{2\sum_{i=1}^{N_r}(x^i)^2-(\sum_{i=1}^{N_r}x^i)^2+6\sum_{i=1}^{N_r}x^i\sum_{t=1}^{N_r-1}e_t$$

$$-(\sum_{t=1}^{N_r-1}e_t)^2-2\sum_{t=1}^{N_r-1}e_t^2] \tag{2.4.43B}$$

However only the V_r of (2.4.43A) satisfies (2.4.16). This means that if $\zeta_q \neq 0$ only elliptical-type coordinates are separable.

Now consider $\zeta_q=0$ and suppose that at least one of the γ_i's (γ_k say) is nonzero. In this case σ_q is of type I or II and

$$V_r = \sum_1^{n_r}\gamma_i z^i = \sum_{j=1}^{N_r} {}_{N_r}w_j \sum_{q_j=1}^{p_j+1}\gamma^j_{q_j}\, {}_{p_j}s_{q_j} \tag{2.4.44}$$

Let x^l be any of the separable coordinates on the sphere S_{p_1}. From (2.4.16)

$$\partial_{x^1 x^l} V_r(x^1,x^2=e_2,\ldots,x^{N_r}=e_{N_r}) = 0 = \partial_{x^1}({}_{N_r}w_1)\partial_{x^l}(\sum_{q_1=1}^{p_1+1}\gamma^1_{q_1}\, {}_{p_1}s_{q_1}) \tag{2.4.45}$$

since ${}_{N_r}w_k(x^k=e_k)=0$. Now $\partial_{x^1}({}_{N_r}w_1)\neq 0$ so that

$$\partial_{x^l}\left[\sum_{q_1=1}^{p_1+1}\gamma^1_{q_1}\, {}_{p_1}s_{q_1}\right] = 0\,, \tag{2.4.46}$$

i.e., $\sum_{q_1=1}^{p_1+1}\gamma^1_{q_1}\ {}_{p_1}s_{q_1} = A$, a constant. Such a relation can not exist among the ${}_{p_1}s_{q_1}$'s unless $p_1=0$. No sphere can be attached to the coordinate ${}_{N_r}w_1$. This is demonstrated as follows. As the $\gamma^1_{q_1}$'s are real, rotations can be used to pass to an equivalent set of separable coordinates, ${}_{p_1}\tilde{s}_{q_1}$, for the sphere S_{p_1},such that

$$\sum_{q_1=1}^{p_1+1}\gamma^1_{q_1}\, {}_{p_1}s_{q_1} \to [\sum_{q_1}(\gamma^1_{q_1})^2]^{\frac{1}{2}}\, {}_{p_1}\tilde{s}_1 \tag{2.4.47}$$

Since $\sum_{q_1}(\gamma^1_{q_1})^2 \neq 0$ equation (2.4.46) now implies

$$\partial_{x^l}({}_{p_1}\tilde{s}_1) = 0 \quad , \quad \text{for } \mathit{all}\ l \tag{2.4.48}$$

This is only possible if $p_1=0$ i.e. no sphere is attached to the ${}_{N_r}w_1$ coordinate.

Still assuming $\zeta_q=0$ and $\gamma_k\neq 0$ consider the elliptic case.

Let $N_r = 1$. No spheres are attached to ${}_1w_1$ by the argument above and so $z^1 = {}_1w_1 = x^1$. Condition (2.4.16) is satisfied and the system is separable.

Let $N_r \geq 2$. Using the above argument and the symmetry of the elliptic coordinate transformations (2.4.35A), shows that no spheres (beside the trivial ones) can be attached to any of the ${}_{N_r}w_j$ coordinates. Let x^1 and x^2 to be any two of the elliptic-type coordinates in (2.4.35A). Equation (2.4.16) yields

$$\partial_{x^1 x^2} V_r = 0 = \sum_i \frac{\gamma_i \; {}_{N_r}w_i}{4(x^1-e_i)(x^2-e_i)} . \tag{2.4.49}$$

Multiplying this last expression by $(x^2-e_2)^{\frac{1}{2}}$ and setting $x_i = e_i$ for $i \geq 2$, we find that $\gamma_2 = 0$. This result is easily generalised to show that $\gamma_i = 0$, for all i: contradicting our initial assumption that one of the γ_i's was nonzero. Under these conditions, it follows that the elliptic block E_r leads to separability only if $V_r = 0$.

In the parabolic case using the same methods we find that only γ_1 can be nonzero and then

$$\begin{aligned} V_r &= \gamma_1 \; {}_{N_r}w_1 \\ &= \tfrac{1}{2}\gamma_1 c_r (x^1 + \cdots + x^{N_r} + e_1 + \cdots + e_{N_r - 1}). \end{aligned} \tag{2.4.50}$$

i.e. parabolic coordinates are separable in this case.

Finally if $\gamma_1 = 0 = \zeta_q$ (i.e. σ_q is of type I or II), then both elliptic or parabolic type blocks are possible.

A more elegant proof of the above results will be given for the complex case in §3.2. That proof is equally applicable here and does not use the explicit form of the coordinate transformations. We have determined all Euclidean coordinate systems that can combine with a given V_q to form a separable system for (2.4.1). Thus using the results of §2.2 the R-separation problem for (*) has been solved. This procedure is systematised in graphical form in the next section.

2.5 Graphical Representation of Coordinates

We develop a graphical calculus to represent the R-separable coordinate systems for (*), illustrating this procedure in detail for $m = 1,2,3$ -the cases of physical interest.

From (2.2.4) and (2.4.30) the coordinate transformations for (*) are given by

$$y^u = x^u \sigma_q^{\frac{1}{2}} + \frac{1}{2}\gamma_u \int\int \sigma_q^{-3/2} \quad , u \in B_q, \ q \in Q,$$

$$t = x^{n-1}, \qquad (2.5.1)$$

and these can be given the graphical representation

$$\overset{L_1}{\boxed{G_1}} \qquad \overset{L_q}{\boxed{G_q}} \qquad (2.5.2)$$

Here G_q is a separable system on $\mathbb{R}^{n(B_q)}$ and the box indicates that there is just one function σ_q on G_q, while the Latin number $L_q(I, II, III$ or $IV\pm)$ specifies its type. G_q is one of Kalnins and Miller's graphs and will in general have form

$$G_q^1 \qquad G_q^r \qquad (2.5.3)$$

where each G_q^r is one of the elliptic or parabolic types given in (2.4.32A) and (2.4.32B).

At the end of the last section we found the compatibility conditions for a σ_q - G_q^r combination to be separable. To summarise these results we start with a given separable Euclidean system G_q^r and then give the compatible σ functions.

If G_q^r is of elliptic-type then L_q could take any of its values $I \to IV\pm$. However the case $\zeta_q = 0$, $\gamma_u \neq 0$ where x^u is one of the coordinates on G_q^r can only occur if G_q^r has form ⟨e⟩ . If the block G_q^r is of parabolic-type it is only compatible with the σ-types I and II since it was shown in §2.4 that $\zeta_q \neq 0$ did not satisfy (2.4.16). In the allowed cases I and II, the γ_u term *attaches* itself to the coordinate ${}_{N_r}w_1$ of (2.4.35B).

These results are now generalised to a block of form

$$\overset{L_q}{\boxed{G_q}} \qquad (2.5.4)$$

If all the G_q^r 's are of elliptic-type then L_q can take any of it's values, $I \to IV\pm$. The parameters γ_u on G_q can only be nonzero if the blocks G_q^r to which they are attached have form $\langle e \rangle$.

If at least one G_q^r is of parabolic-type, then L_q is restricted to types I or II. The exceptional case $N=1$ can be given our uniform general treatment if we regard it as being eqivalent to the two systems ◯ and $\langle e \rangle$ as discussed in §2.4. A γ_u term can be attached to ◯ (no spheres can be attached to this graph). Spheres can be attached to $\langle e \rangle$ and this is compatible with $\zeta_q = \neq 0$ but not with $\zeta_q = 0$ and $\gamma_u \neq 0$.

The parameters w_q and v_q can be normalised, since by making the separability preserving transformations

$$x^{n-1} \to ax^{n-1} + b \ , \tag{2.5.5}$$

we can take one v_q to be zero and one w_q to be 1. Further normalisations are possible when $E=0$ because of the extra conformal symmetries K_2 and D. For instance, consider the case when one of the L_q's is I in (2.5.2). The coordinates on this block are

$$y^b = z^b + \gamma_b (x^{m+1})^2/4 \quad , b \in B_q \ . \tag{2.5.6}$$

The dilatation D acts on the coordinates as

$$y^b \to cy^b \ , \quad t \to c^2 t \ , \quad c \in \mathbb{R} \ . \tag{2.5.7}$$

If this action is combined with the *equivalence* transformations

$$z^b \to cz^b \ , \quad x^{m+1} \to c^2 x^{m+1} \ , \tag{2.5.8}$$

and $c = \gamma_b^{-1/3}$ then γ_b can be normalised to 1 or 0. If L_q is II on a block, then the same normalisation is possible by similar methods. There are more equivalences possible under the conformal symmetries, but the discussion of these will be postponed until the operators have been determined in §2.6. It is not possible to simultaneously normalise all the γ_b's. There is a different coordinate system for each value of γ_b. Thus in general, there is an infinite number of R-separable coordinate systems for (*) and the numbering of these systems in Appendix A is for tidiness only.

The following procedure emerges for the construction of all R-separable systems for (*).

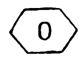
A. Construct the graphs representing all separable systems on $\mathbb{R}^m$.

B. For each of these construct all possible boxings.

C. From the discussion above determine all possible σ's compatible with the boxings.

We now go through this procedure for $m = 1$.

A. There is only one possible separable system on $\mathbb{R}^1$

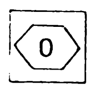


(2.5.9)

corresponding to the choice of coordinate

$$z^1 = x^1. \tag{2.5.10}$$

B. There is only one possible boxing:



(2.5.11)

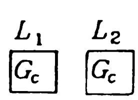
C. The resulting types with their coordinate transformations are listed in Table 1 of Appendix A.

We have displayed possible normalisations of the parameters v_q, w_q in brackets alongside L_q, but have not substituted their values in the coordinate transformations since the unnormalised forms will be needed for the $m = 2$ and $m = 3$ classifications.

For $m = 2$ the separable systems resulting from step A are listed in Table 2 of Appendix A. There are three classes of boxings arising from step B:

(2.5.12a)

where G is the elliptic, parabolic or polar system of Table 2 in Appendix A,



(2.5.12b)

and

$$\overset{L_1}{\boxed{G_c \quad G_c}} \tag{2.5.12c}$$

where G_c is $\langle 0 \rangle$. The only new systems are those of type (2.5.12a) as it will be shown that the remaining systems can be derived from the $m=1$ case. The unsplit class a types are listed in Table 3 of Appendix A. The class b systems are mixtures of the $m=1$ systems

$$\overset{L_1}{\boxed{G_c}} \quad \text{and} \quad \overset{L_2}{\boxed{G_c}} \tag{2.5.13}$$

where $\sigma_1 \neq \sigma_2$, and these are listed in Table 4 of Appendix A. $\mathbb{R}^2$ has *split* into $\mathbb{R} \oplus \mathbb{R}$ and so these coordinate systems are referred to as *splitting types*. For example, the system 5: *II*, *III* $(v_2=0)$ in Table 4 has coordinate transformations

$$y^1 = (x^3+v_1)x^1 + \gamma_1/4(x^3+v_1),$$

$$y^2 = |x^3|^{\frac{1}{2}} x^2,$$

$$t = x^3 \tag{2.5.14}$$

that are easily derived from systems *II* and *III* in Table 1. The remaining systems of class c are simply combinations of the $m=1$ systems

$$\overset{L_1}{\boxed{G_c}} \quad \text{and} \quad \overset{L_1}{\boxed{G_c}} \tag{2.5.15}$$

where $\sigma_1 = \sigma_2$, that is class b with $L_1 = L_2$. They are listed in Table 4 of Appendix A. For example the coordinate transformations for $IV{+}(v_1=0,\, w_1^2=1)$ can be derived from those of the system

$$\overset{IV+(v_1=0,\; w_1^2=1)}{\boxed{\langle 0 \rangle}} \tag{2.5.16}$$

given in Table 1. They are

$$y^1 = |(x^3)^2 + 1|^{\frac{1}{2}} x^1$$

$$y^2 = |(x^3)^2 + 1|^{\frac{1}{2}} x^2$$

$$t = x^3 \tag{2.5.17}$$

Miller (1977) has also classified the R-separable systems for $m=2$ and developed many of their properties but misses splitting types such as (2.5.14) and (2.5.17). In general, his classification only includes those mixing types of class b for which one of the coordinates is $y^1 = x^1$. These omissions are rectified in Kalnins and Miller (1979).

For general m the R-separable systems have form

$$\overset{L_1}{\boxed{G}} \quad , \tag{2.5.18}$$

where G is of parabolic or elliptic-type, or they will be a mixture of systems like (2.5.2). In this case (*) is equivalent to the equations

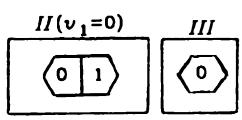
$$(\Delta_q + 2\varepsilon\partial_t)\Psi_q = T_q \Psi_q, \quad \Psi = \prod_{q \in Q} \Psi_q \quad . \tag{2.5.19}$$

where

$$\Psi_q = \Psi_q(x^b), \; b \in B_q . \tag{2.5.20}$$

All such mixtures can be classified from the lower dimensional equations given in (2.5.19).

The classification for $m=3$ for the unsplit types given in Table 5 of Appendix A has not appeared before. These systems can of course be derived from our general procedure but we list them for easy access. The splitting types for $m=3$ can be derived from the tables for $m=1$ and $m=2$ given in Appendix A. For example consider



(2.5.21)

for which the coordinates are

$$y^1 = cx^4\cosh(x^1)\cos(x^2) ,$$

$$y^2 = cx^4\sinh(x^1)\sin(x^2) ,$$

$$y^3 = |x^4 + v_3|^{\frac{1}{2}} x^3 ,$$

$$t = x^4 , \tag{2.5.22}$$

and these can be obtained from the tables for $m=1$ and $m=2$.

2.6 Operators

Each R-separable system for (*) is characterised as a commuting set of second order partial differential operators that are in the enveloping algebra of (*). These operators are derived from those representing separable systems on $\mathbb{R}^m$.

We will first derive the operators for (2.2.1) from the Killing tensors λ_i for the Hamilton-Jacobi equation (2.4.1) associated with (2.2.1). The Killing tensors characterising separable systems for (2.4.1) are of second order, and the work of Delong (1982) implies that they are in the enveloping algebra of $e(n, 1)$. In other words if the Killing vectors μ_j are a basis for $e(n, 1)$ then there are real constants a^{jk} such that

$$\lambda_i = \sum_{j,k} a^{jk}\{\mu_j, \mu_k\}, \quad 1 \le i,j,k \le n . \tag{2.6.1}$$

Recall that $\{\ ,\ \}$ is the Symmetric Bracket which is defined by

$$\{\mu_j, \mu_k\} = \frac{\mu_j\mu_k + \mu_k\mu_j}{2} \tag{2.6.2}$$

From the discussion surrounding (1.2.20) the corresponding operators $\tilde{\lambda}_i$ for (2.2.1) may be obtained by the identification $\mu_j \to \tilde{\mu}_j$ defined in (1.2.14) so that

$$\tilde{\lambda}_i = \sum_{j,k} a^{jk}\{\tilde{\mu}_j, \tilde{\mu}_k\}. \tag{2.6.3}$$

These will also be operators for (*) with the identification $\hat{\varepsilon} \to \varepsilon$ since

$$\tilde{\lambda}_i\bar{\Psi} = l_i\bar{\Psi} \ \ iff \ \ \tilde{\lambda}_i\Psi = l_i\Psi \tag{2.6.4}$$

where $\bar{\Psi}$ and Ψ are defined in (2.2.7).

The Killing tensors can be determined from (D6). They are

$$\lambda_i = \sum_{a=1}^{n-2}\frac{\Phi^{ai}}{\Phi}p_a^2 + 2\frac{\Phi^{(n-1)i}}{\Phi}p_{n-1}p_n + \sum_{a=1}^{n-2}\frac{\Phi^{ai}}{\Phi}A_a^{nn}p_n^2$$

$$i = 1, 2, \dots, n-1, \tag{2.6.5}$$

where (Φ_{ij}) is the Stäckel matrix defined in (2.4.6). We first find this matrix in terms of the Stäckel matrices for the embedded separable Euclidean systems $d\bar{s}_q^2$. These are the $n(B_q)\times n(B_q)$ dimensional matrices, $(\Phi_q)_{ij}$, such that

$$\bar{g}^{bb} = \frac{\Phi_q^{b1}}{\Phi_q}, \quad \Phi_q = \det((\Phi_q)_{ij}). \tag{2.6.6}$$

The matrix (Φ_{ij}) can be taken as

$$\begin{pmatrix} 0 & \boxed{\Phi_1} & \cdot & 0 & & 0 \\ \cdot & & \cdot & \cdot & \cdot & \\ \cdot & \cdot \quad 0 & & \boxed{\Phi_q} & \cdot & \cdot \\ & \cdot \quad \cdot & & & & \\ & & & \cdot & & \cdot \\ 0 & & & & & \\ 1 & \frac{-1}{\sigma_1} \quad 0 & \cdot & \frac{-1}{\sigma_q} \quad 0 & \cdot & \cdot \end{pmatrix} \tag{2.6.7}$$

since this matrix satisfies

$$g^{bb} = \frac{\Phi^{b1}}{\Phi} = \frac{1}{\sigma_q}\frac{\Phi_q^{b1}}{\Phi_q}, \quad \text{for } b \in B_q. \tag{2.6.8}$$

By substituting for (Φ_{ij}) from (2.6.7) into (2.6.5) the Killing tensors are

$$\lambda_v = \sum_{b \in B_q} \frac{\Phi_q^{bv}}{\Phi_q} p_b^2 + U_v p_n^2 ,$$

$$v = 1, 2, \quad \ldots, m,$$

$$\lambda_{m+1} = E. \tag{2.6.9}$$

where

$$U_v = \sum_{b \in B_q} \frac{\Phi_q^{bv}}{\Phi_q} A_b^{nn} . \tag{2.6.10}$$

In (2.6.9) the terms $\sum_{a \in B_q} \frac{\Phi_q^{bv}}{\Phi_q} p_b^2$ can be recognised as having the same form as that for the constants of the motion for the separable system $d\bar{s}_q^2$. This fact will guide us in what follows.

The constant of the motion λ_v may also be written

$$\lambda_v = \mathbf{p}^t \Lambda_v \mathbf{p} \tag{2.6.11}$$

where

$$\mathbf{p} = \begin{bmatrix} p_1 \\ \\ \\ p_n \end{bmatrix}, \text{ and}$$

$$\Lambda_v = \begin{bmatrix} 0 & & & \cdot \\ & & 0 & \\ & \Lambda_v^E & & \\ 0 & & \ddots & \\ \cdot & & & U_v \end{bmatrix} \tag{2.6.12}$$

Here Λ_v^E is a diagonal matrix whose elements as determined by (2.6.9) are $(\Lambda_v^E)_{bc} = \delta^{bc} \frac{\Phi_q^{bv}}{\Phi_q}$. It is what will be called the matrix form of the Euclidean constants of the motion arising on $d\bar{s}_q^2$. We express λ_v in terms of the y^i coordinates by using the transformation matrix J:

$$J_{ab} = \partial y^b / \partial x^a. \tag{2.6.13}$$

In the y^i coordinates

$$\lambda_v = \mathbf{P}^t J^t \Lambda_v J \mathbf{P} \tag{2.6.14}$$

where

$$\mathbf{P} = \begin{bmatrix} P_1 \\ \\ \cdot \\ P_n \end{bmatrix}, \quad P_i = \partial_{y^i} W \tag{2.6.15}$$

and t denotes matrix transposition. Using (2.6.13) J is given by

$$\begin{pmatrix} \sigma_1^{\frac{1}{2}} J_1 & & & \mathbf{c}_1 & \mathbf{c}_1 \\ & & 0 & & \\ & \sigma_q^{\frac{1}{2}} J_q & & \mathbf{c}_q & \mathbf{c}_q \\ & 0 & & & \cdot \\ & & & & \cdot \\ B_1 & B_q & & X & Y \\ 0 & 0 & & 1/2 & 1/2 \end{pmatrix} \tag{2.6.16}$$

where from (2.4.30)

$$(\mathbf{c}_q)_a = -\frac{1}{2}\partial_{x^a} \sum_{b \in B_q} G_{bq} \text{ and } (J_q)_{ab} = \partial z^b / \partial x^a \tag{2.6.17}$$

As our final result is independent of the quantities B_q, X and Y we need not calculate them. Finally

$$\boxed{\begin{aligned} \lambda_v &= \sigma_q \mathbf{P}_q^t \tilde{\Lambda}_v^E \mathbf{P}_q \\ &- \mathbf{P}_q^t \varepsilon \left[\sigma_q{}' \sigma_q^{\frac{1}{2}} \tilde{\Lambda}_v^E \mathbf{z}_q + \sigma_q^{\frac{1}{2}} \left[\int \sigma_q^{-3/2} \right] \tilde{\Lambda}_v^E \gamma_q \right] + F_v \varepsilon^2 . \end{aligned}} \tag{2.6.18}$$

In this equation

$$F_v = 4\mathbf{c}_q^t \tilde{\Lambda}_v^E \mathbf{c}_q + U_v ,$$

$$\tilde{\Lambda}_v^E = J^t \Lambda_v^E J \tag{2.6.19a}$$

and γ_q, $\mathbf{P}_q$ and $\mathbf{z}_q$ are $n(B_q) \times 1$ vectors:

$$(\gamma_q)_b = \gamma_b \ , \ (\mathbf{P}_q)_b = P_b \ , \ (\mathbf{z}_q)_b = z^b \ , \ b \in B_q \tag{2.6.19b}$$

From (2.6.19a) $\tilde{\Lambda}_v^E$ is just Λ_v^E in the z^b coordinates. Equation (2.6.18) implies that the constants of the motion for $v \in B_q$ do not depend on the structure of (*) on any of the other blocks. This is natural since (*) is equivalent to the set of lower dimensional equations given in (2.5.19). The problem of finding the operators for a general system has now reduced to the determination of the constants of the motion on one of the blocks

$$\begin{array}{c} L_q \\ \boxed{G_q} \end{array} \qquad (2.6.20)$$

Kalnins and Miller (1982a) give specific formulae for the Euclidean constants of the motion $\tilde{\lambda}_v^E$. On each irreducible block they find that

$$P_b^2,\ M_{ab}^2 \text{ and } \{M_{qb}, P_b\},\ a,\ b \in B_q, \qquad (2.6.21)$$

are a basis so that

$$\tilde{\lambda}_v^E = \sum_b A^b P_b^2 + \sum_b B^b \{M_{qb}, P_b\} + \sum_{a<b} C^{ab} M_{ab}^2 \qquad (2.6.22)$$

$$A^b,\ B^b,\ C^{ab} \in \mathbb{R}.$$

The reader is referred to Kalnins and Miller (1982a) for the determination of the constants in (2.6.22). Since (2.6.18) provides us with a relation of form

$$\lambda_v = \lambda_v(\tilde{\lambda}_v^E, \sigma_q) \qquad (2.6.23)$$

all that has to be done for each of the σ's for $I \to IV\pm$, is to determine the images of the Euclidean constants of the motion in (2.6.21). In each case we will find an expression λ' in the enveloping algebra such that λ is determined to within a term in ε^2 i.e.

$$\lambda = \lambda' + F'\varepsilon^2. \qquad (2.6.24)$$

(In this discussion prime is not the derivative). Since λ is a constant of the motion and λ' is in the enveloping algebra

$$\{E, \lambda\}_P = 0 = \{E, \lambda' + F'\varepsilon^2\}_P = \{E, F'\varepsilon\}_P \qquad (2.6.25)$$

Solving this relation with $E = P_1^2 + \qquad + P_{n-1}^2 - P_n^2$ we find that F' is a constant, but since ε is already on the orbit

$$\lambda = \lambda' \qquad (2.6.26)$$

in (2.6.24).

The images of the Euclidean operators are displayed in Table 2.6.1, however one example will be done in detail to show how they are derived. Consider the term $\{M_{qb}, P_b\}$ which appears in constants of the motion in parabolic-type coordinates. The matrix form $\tilde{\Lambda}^E$ for this term is

$$\begin{bmatrix} & & -z^b/2 & \\ & & 0 & \\ -z^b/2 & 0 & z^q & 0 \\ & & \vdots & \\ & & 0 & \end{bmatrix} \tag{2.6.27}$$

as it is easily checked that $\mathbf{P}^t \tilde{\Lambda}^E \mathbf{P} = \{M_{qb}, P_b\}$. From (2.4.30) and (2.4.50)

$$y^q = z^q + \gamma_q (x^{n-1})^2/4 \ ,$$

$$y^b = z^b \qquad b \neq q$$

The calculation for the constant of the motion corresponding to $\{M_{qb}, P_b\}$ goes as follows:

$$\begin{aligned} \lambda(\{M_{qb}, P_b\}, \sigma=1) &= \{z^q P_{y^b} - z^b P_{y^q} \ , P_b\} - \mathbf{P}_q^t x^{n-1} \tilde{\Lambda}^E \boldsymbol{\gamma}_q \ \varepsilon + F\varepsilon^2 \\ &= \{M_{qb}, P_b\} - \frac{1}{4}\gamma_q (x^{n-1})^2 P_b^2 + \frac{1}{2}\gamma_q x^{n-1} P_b \varepsilon + F'\varepsilon^2 \\ &= \{M_{qb}, P_b\} - \gamma_q B_b^2/4 \end{aligned} \tag{2.6.28}$$

since $F'=0$ from the discussion above. The corresponding operator for (2.2.1) and (*) is obtained via (2.6.3) but it may be unambiguously written as (2.6.28) by removing the tildes. Occasionally P_b has been used to represent both the Euclidean constant of the motion $\partial W/\partial z^b$ and $\partial W/\partial y^b$, a constant of the motion for the Hamilton-Jacobi equation (2.4.1). Sometimes, as in (2.6.28), we have written P_{y^b} to emphasise the difference.

Table 2.6.1 Images of Euclidean Operators

$\tilde{\Lambda}^E$	L_q			
	I	II	III	$IV\pm$
P_b^2	$P_b^2+\gamma_b\varepsilon B_b$	$(v_q P_b-B_b)^2$ $+\gamma_b\varepsilon P_b$	$v_q P_b^2$ $-\{P_b,B_b\}$	$(v_q P_b-B_b)^2$ $\pm w_q^2 P_b^2$
M_{ab}^2	M_{ab}^2	M_{ab}^2	M_{ab}^2	M_{ab}^2
$\{M_{qb},P_b\}$	$\{M_{qb},P_b\}$ $-\gamma_q B_b^2/4$	$\{M_{qb},v_q P_b-B_b\}$ $-\gamma_q P_b^2/4$	does not occur	does not occur

The results in Table 2.6.1 can be used to obtain the operators for any m but we will go through these results in detail for $m=1$ and $m=2$.

When $m=1$ the Euclidean operator is P_1^2, and its images can be read from the row containing P_b^2 in Table 2.6.1 obtaining the results listed in Table 1 of Appendix A.

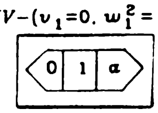
When $m=2$ the operators resulting from the unmixed systems are listed in Table 3 of Appendix A. The operators for the mixing systems are simply those of the component one dimensional systems. For example the two Euclidean operators for (2.5.14) are P_1^2 and P_2^2 and by using Table 2.6.1 their images are

$$(v_1P_1-B_1)^2+\gamma_1\varepsilon P_1,\ \text{and}\ -\{P_1,B_1\}. \tag{2.6.29}$$

Alternatively the same result is easily obtained from Table 1 of Appendix A.

In the three dimensional case the operators have not been listed in Appendix A, but the following example will show how they are obtained. Consider system 4 of Table 6 in Appendix A

$IV-(v_1=0,\ w_1^2=1)$



(2.6.30)

The operators for ellipsoidal coordinates are easily derived from the work of Kalnins and Miller (1982a) but we also record them in Table 5. They are

$$P_1^2 + P_2^2 + P_3^2,$$

$$\mathbf{J}\cdot\mathbf{J} + c^2[(1+a)P_1^2 + aP_2^2 + P_3^2], \tag{2.6.31}$$

$$J_2^2 + aJ_3^2 + c^2aP_1^2.$$

Employing Table 2.6.1

$$J_b^2 \to J_b^2,\ P_b^2 \to B_b^2 - P_b^2,\ b = 1, 2, 3 \tag{2.6.32}$$

so that the operators for (2.6.30) are

$$\sum_b (B_b^2 - P_b^2),$$

$$\mathbf{J}\cdot\mathbf{J} + c^2[(1+a)(B_1^2 - P_1^2) + a(B_2^2 - P_2^2) + B_3^2 - P_3^2], \tag{2.6.33}$$

$$J_2^2 + aJ_3^2 + c^2a(B_1^2 - P_1^2).$$

The operators for the $m=3$ mixing types are also computed in the same fashion. For instance the operators for the mixing type (2.5.21) can be derived from the tables for the one and two dimensional cases. They are

$$B_1^2 + B_2^2,\ M_{12}^2 + c^2B_1^2 \text{ and } v_3P_3^2 - \{P_3, B_3\}. \tag{2.6.34}$$

All R-separable systems for (*) have been characterised as commuting sets of second order partial differential operators which are members of the enveloping algebra. An immediate application for this characterisation is now given.

All separable systems on $\mathbb{R}^m$ possess the Casimir operator $\sum_{u=1}^{m} P_u^2$. If $E=0$ and our systems have form

$$\overset{L_1}{\boxed{G}} \tag{2.6.35}$$

then the operators corresponding to this invariant for each of the σ types are:

$$\begin{array}{ll} I & \sum_1^m P_b^2 + \gamma_1 \varepsilon B_1 \\ II & \sum_1^m B_b^2 + \gamma_1 \varepsilon P_1 \\ III & \sum_1^m \{P_b, B_b\} \\ IV\pm & \sum_1^m B_b^2 \pm \sum_1^m P_b^2 \end{array} \qquad (2.6.36)$$

By examining the forms of the operators in (2.2.19)→(2.2.20) the following substitutions hold on the solution space of (*) when $E=0$.

$$\sum_1^m P_b^2 \to -2\varepsilon K_{-2},\ \sum_1^m \{P_b, B_b\} \to \varepsilon D \text{ and } \sum_1^m B_b^2 \to 2\varepsilon K_2 \ . \qquad (2.6.37)$$

With these substitutions, the new expressions for the operators in (2.6.36) are summarised in Table 2.6.2.

L_1 Operator Table 2.6.2	
I	$\varepsilon[\gamma_1 B_1 - 2K_{-2}]$
II	$\varepsilon[2K_2 + \gamma_1 P_1]$
III	εD
$IV\pm$	$\varepsilon[K_2 \mp K_{-2}]$

Each of the operators in Table 2.6.2 is first order and can be diagonalised to reduce (*) by a dimension. This feature is discussed by Miller (1977) for the one and two dimensional heat equations. His claim, however, that this is a feature of all systems of those dimensions does not hold. For example the operators (2.6.29) of the splitting type (2.5.14) can never be made first order. This subject will be discussed in greater depth in §2.8.

The knowledge of the operators will now be exploited to find the extra equivalences for $E=0$ that were mentioned in §2.5. For a system like (2.6.35) it is possible, as was explained in §2.5, to normalise γ_1 to 0 or 1. Thus there are the possibilities for L_1:

1 a. $I(\gamma_1=0)$

b. $I(\gamma_1=1)$

2 a. $II(v_1=0,\gamma_1=0)$

b. $II(v_1=0,\gamma_1=1)$

3 $III(v_1=0)$

4 a. $IV-(v_1=0,w_1^2=-1)$

b. $IV+(v_1=0,w_1^2=1)$ (2.6.38)

The extra equivalences occur under the action of the operator $A_3=K_{-2}-K_2$. The adjoint action of this operator is given in Table 2.6.3.

Table 2.6.3 Adjoint Action of A_3

λ	$e^{sAdA_3}\lambda$
P_a	$cos(s)P_a+sin(s)B_a$
M_{ab}	M_{ab}
B_a	$cos(s)B_a-sin(s)P_a$
A_1	$A_1cos(2s)-A_2sin(2s)$
A_2	$A_2cos(2s)-A_1sin(2s)$

The adjoint action of A_3 on any element $\{L_1,L_2\}$ of the enveloping algebra is easily shown to be

$$e^{s\,Ad\,A_3}(\{L_1,L_2\}) = \{e^{s\,Ad\,A_3}(L_1),e^{s\,Ad\,A_3}(L_2)\} \quad (2.6.39)$$

by using (2.2.23). Consider the action of A_3 when $s=-\pi/4$ and for convenience let $J=e^{\frac{-\pi}{4}A_3}$. Then from (2.2.23)

$$e^{\frac{-\pi}{4}Ad\,A_3}(L) = J\;L\;J^{-1} \quad (2.6.40)$$

The Euclidean operators for (*) are

$$\tilde{\lambda}_v^E = \sum_b A_v^b P_b^2 + \sum_b B_v^b \{M_{qb}, P_b\} + \sum_{a<b} C_v^{ab} M_{ab}^2 \tag{2.6.41}$$

where $v = 1, \quad , m+1$. For case 3 in (2.6.38) the corresponding operators as obtained from Table 2.6.1 are

$$\lambda_v = -\sum_b A_v^b \{P_b, B_b\} + \sum_{a<b} C_v^{ab} M_{ab}^2 \tag{2.6.42}$$

If the action of J is applied to these operators using Table 2.6.3 we obtain

$$\lambda_v{}' = -\sum_b A_v^b \{\frac{P_b}{\sqrt{2}} - \frac{B_b}{\sqrt{2}}, \frac{B_b}{\sqrt{2}} + \frac{P_b}{\sqrt{2}}\} + \sum_{a<b} C_v^{ab} M_{ab}^2$$

$$= \sum_b A_v^b (B_b^2 - P_b^2)/2 + \sum_{a<b} C_v^{ab} M_{ab}^2 \tag{2.6.43}$$

since $P_b \to \frac{P_b}{\sqrt{2}} - \frac{B_b}{\sqrt{2}}$, $B_b \to \frac{B_b}{\sqrt{2}} + \frac{P_b}{\sqrt{2}}$ and $M_{ab} \to M_{ab}$ under the action of J. The expression obtained in (2.6.43) is precisely the one that would have been been obtained from the same Euclidean operators for system 4a in (2.6.38). In similar fashion it can be shown that the systems 1a and 1b are equivalent to the systems 2a and 2b respectively under the action of

$$J^2 = e^{-\frac{\pi}{2}A_3} \tag{2.6.44}$$

We have demonstrated that the systems in (2.6.38) collapse to the systems 1a, 1b, 3 and 4b. The action of J on solutions of (*) is

$$J\Phi(\mathbf{y},t) = \left[\frac{\sqrt{2}}{(1+t)}\right]^{m/2} \exp\left[\frac{\varepsilon \mathbf{y}\cdot\mathbf{y}}{2(1+t)}\right] \Phi\left(\frac{\sqrt{2}\mathbf{y}}{t+1}, \frac{t-1}{t+1}\right)$$

$$J^2\Phi(\mathbf{y},t) = t^{-m/2} \exp\left[\frac{\varepsilon \mathbf{y}\cdot\mathbf{y}}{2t}\right] \Phi\left(\frac{\mathbf{y}}{t}, \frac{-1}{t}\right)$$

$$J^4\Phi(\mathbf{y},t) = -\Phi(-\mathbf{y},t)$$

$$J^8\Phi(\mathbf{y},t) = \Phi(\mathbf{y},t) \tag{2.6.45}$$

The expression for J^2 is particularly notable: it is the *Appell* transform, and its importance for the theory of the Heat equation is discussed in Widder (1975). The above work is a generalisation of that of Miller (1977) for the one and two dimensional heat equations.

In conclusion, it is possible to exploit the conformal symmetries for unsplit systems when $E=0$ to show that some systems that look different are actually equivalent under the action of the Schrödinger Group.

2.7 R-separable solutions

We now investigate the R-separable solutions of (*). To accomplish this it is necessary to determine both the R-separation factors and the *separation equations*. The separation equations for (*) are the ordinary differential equations determining the Ψ_j functions in (2.2.5). From (2.2.7) these are the same as the corresponding functions for the Helmholtz equation (2.2.1). Therefore this equation is used to find the separation equations for (*). The form of (2.2.1) in the separable coordinates x^i can be found by using this equations local coordinate description given in (1.2.9). It is

$$\sum_{q\in Q}\frac{1}{\sigma_q}\sum_{b\in B_q}[\bar{g}_q^{-1/2}\partial_b(\bar{g}^{bb}\bar{g}_q^{1/2}\partial_b\Psi) + \bar{g}^{bb}A_b^{nn}\partial_{nn}\Psi]$$

$$+ 2\partial_{(n-1)n}\Psi + \frac{1}{2}\sum_{q\in Q}n(B_q)(\log\sigma_q)'\partial_n\Psi = E\Psi \qquad \textbf{(2.7.1)}$$

where

$$\bar{g}_q = \det(\bar{g}_{bc}),\ b,c \in B_q. \qquad (2.7.2)$$

and A_b^{nn} is defined in (2.4.14). When $\Psi = \prod_{i=1}^{n}\Psi_i(x^i)$ is substituted into (2.7.1) and the resulting equation is divided by Ψ we obtain the separation equations:

$$\Psi_n{}' = \varepsilon\Psi_n, \qquad (2.7.3)$$

and

$$2\varepsilon\Psi'_{n-1} + \sum_{q\in Q}\frac{s_q}{\sigma_q}\Psi_{n-1} + \frac{\varepsilon}{2}\sum_{q\in Q}n(B_q)(\log\sigma_q)'\Psi_{n-1} = E\Psi_{n-1}. \qquad (2.7.4)$$

Here s_q is the eigenvalue of the operator S_q whose action on Ψ is

$$S_q \Psi = \sum_{b \in B_q} [\bar{g}_q^{-1/2} \partial_b (\bar{g}^{bb} \bar{g}_q^{1/2} \partial_b \Psi) + (\varepsilon)^2 \bar{g}^{bb} A_b^{nn} \Psi] = s_q \Psi \tag{2.7.5}$$

The remaining separation equations are derived from (2.7.5) by noticing it is the Helmholtz equation on $d\bar{s}_q^2$ with an extra term

$$\sum_{b \in B_q} (\varepsilon)^2 \bar{g}^{bb} A_b^{nn} \Psi. \tag{2.7.6}$$

By a simple modification of Kalnins and Miller's results for $\mathbb{R}^{n(B_q)}$ we obtain these equations.

The one dimensional elliptic case is an exception to the general construction of the separation equations that we are about to give. The separation equation for this case is

$$\frac{d^2 \Psi_1}{(dx^1)^2} + \{\varepsilon^2 [\zeta_1 (x^1)^2 / 4 + \gamma_1 x^1] - s_1\} \Psi_1 = 0 . \tag{2.7.7}$$

Returning to our general treatment we can confine ourselves to a block E_r of form (2.4.32A) or (2.4.32B). For simplicity on each of these irreducible blocks the coordinates $x^r, x^{r+1}, \ldots, x^{r+n_r-1}$ are relabelled as $x^1, x^2, \ldots, x^{n_r}$ just as in our discussion of the coordinate transformations at the end of §2.4. The additional term contributed to the separation equations by (2.7.6) for the elliptic-type A coordinates x^a is

$$Z_a = -\frac{(\varepsilon)^2 c_r^4 \zeta_q}{16} (x^a)^{N_r - 1} (x^a - \sum_{b=1}^{N_r} e_b)$$

$$, a = 1, 2, \ldots, N_r . \tag{2.7.8A}$$

For the parabolic-type B coordinates x^a this term is

$$Z_a = \frac{(\varepsilon)^2 c_r^3 \gamma_1}{8} (x^a)^{N_r - 1} (x^a + \sum_{b=1}^{N_r - 1} e_b)$$

$$, a = 1, 2, \ldots, N_r . \tag{2.7.8B}$$

The separation equations in both of these cases are

$$(P_a / Q_a)^{\frac{1}{2}} \frac{d}{dx^a}[(P_a Q_a)^{\frac{1}{2}} \frac{d}{dx^a}\Psi_a] + \tag{2.7.9}$$

$$\{\sum_{b=1}^{N_r} \frac{\prod_{c \neq b}(e_b - e_c)}{(x^a - e_b)} t_b + \sum_{b=1}^{N_r} l_b (x^a)^{N_r - b} + Z_a\}\Psi_a = 0.$$

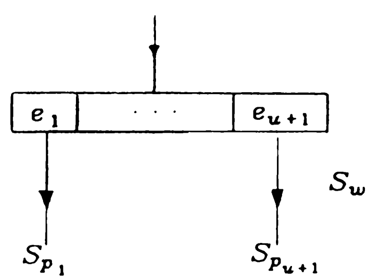
In this last equation

$$P_a = \prod_{b=1}^{N_r}(x^a - e_b), \quad Q_a = \prod_{b=1}^{N_r}(x^a - e_b)^{p_b} \tag{2.7.10}$$

$$t_b = \begin{cases} 0 & , \ p_b = 0 \\ j_b(j_b + p_b - 1) & , \ p_b \neq 0 \end{cases}$$

where p_b is the dimension of the sphere S_{p_b} attached to e_b (see (2.4.32A) and (2.4.32B)). If there is no sphere attached then $p_b = 0$. From (2.7.5) there is no loss in assuming $l_1 = -s_1$. The constant t_b is the eigenvalue of the Helmholtz equation on the sphere S_{p_b}. The form of t_b given in (2.7.10) appears in Kalnins and Miller (1982a) and is derived from the spectral theory of the Sphere.

We now consider the separation equations for the remaining variables i.e. those on the attached spheres S_{p_b}. As was mentioned in §2.4 the separable systems for such spaces have been classified by Kalnins and Miller (1982a). If we trace down one of the tree graphs they use to represent these systems it will have form (see (2.4.38))



(2.7.11)

The separation equations for the variables $x^1, \ldots, x^u$ on S_w are

$$(P_a / Q_a)^{\frac{1}{2}} \frac{d}{dx^a} \left[(P_a Q_a)^{\frac{1}{2}} \frac{d}{dx^a} \Psi_a \right] + \tag{2.7.12}$$

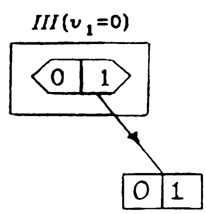
$$\left\{ \sum_{b=1}^{u+1} \frac{\prod_{c \neq b} (e_b - e_c)}{(x^a - e_b)} t_b + \sum_{b=1}^{u} l_b (x^a)^{u-b} \right\} \Psi_a = 0 , \quad a = 1, 2, \ldots, u .$$

with the same definitions for P_a, Q_a and t_b except that the number of elliptic or parabolic coordinates N_r is replaced by u. We can of course take $-l_1$ as the value of the Helmholtz equation on the sphere S_w and via the spectral theory to be $-\mu(\mu+w-1)$ (see Talman (1968)). These separation equations are unaffected by the terms V_q and σ_q, and are just the same as they would be on Euclidean space.

The separation equation for x^{n-1}, (2.7.4), can be directly integrated:

$$\Psi_{n-1} = \{ \prod_{q \in Q} \sigma_q^{-n(B_q)/4} \} \, exp \left(\frac{1}{2\varepsilon} [E x^{n-1} - \sum_{q \in Q} s_q \int \frac{dx^{n-1}}{\sigma_q}] \right). \tag{2.7.13}$$

As an example consider system 9 in Table 6 of Appendix A:



(2.7.14)

The separation equations for this system are

$$(x^a)^{\frac{1}{2}} \frac{d}{dx^a} \left[x^a (x^a - 1) \frac{d}{dx^a} \Psi_a \right] +$$

$$\left\{ \frac{j_2^2}{(x^a - 1)} + -s_1 x^a + l_2 + \frac{(\varepsilon)^2 c_r^4}{16} x^a (x^a - 1) \right\} \Psi_a = 0, \; a = 1, 2, \tag{2.7.15}$$

and

$$[x^3(x^3-1)]^{\frac{1}{2}} \frac{d}{dx^3} ([x^3(x^3-1)]^{\frac{1}{2}} \frac{d}{dx^3} \Psi_3) - j_2^2 \Psi_3 = 0. \tag{2.7.16}$$

Here s_1, l_2 and j_2^2 are the separation constants. These coordinates are in standard form but they could easily be transformed to the more familiar expressions in terms of cos, sinh etc. that appear in Appendix A. Indeed letting $x^3 = \sin^2\vartheta$, (2.7.16) becomes

$$\frac{1}{4}\frac{d^2\Psi_\vartheta}{d\vartheta^2} + j_2^2\Psi_\vartheta = 0$$

We also have from (2.7.13)

$$\Psi_4 = |x^4|^{\frac{-3}{4} - \frac{s_1}{2\varepsilon}} e^{Ex^4/2\varepsilon} \qquad (2.7.17)$$

In order to fully determine the R-separable solutions we now find the R-separation factor R. From (2.2.9) and (2.4.30)

$$R = -\varepsilon f = \varepsilon \sum_{q \in Q} \sum_{b \in B_q} G_{bq} \qquad (2.7.18)$$

If we define

$$R_r = \varepsilon \sum_{b \in E_r} G_{bq} \qquad (2.7.19)$$

then

$$R = \sum_{q \in Q} \sum_{r \in \mathcal{R}} R_r \ . \qquad (2.7.20)$$

and as a result the essential structure of the R-separation factor is solely dependent on the structure of each of the irreducible blocks E_r. Employing (2.4.50) it can be assumed that $\gamma_{r+1} = \gamma_{r+2} = \cdots = 0$, with γ_r being nonzero and zero in the parabolic and elliptic cases respectively. By using (2.7.18) we obtain

$$R_r = \varepsilon \left\{ \frac{\sigma_q'}{4} \sum_{b \in E_r} (z^b)^2 + \frac{1}{2}\gamma_r z^r \sigma_q^{\frac{1}{2}} \int \sigma_q^{-3/2} \right\}$$

$$+ \frac{\varepsilon\, \gamma_r^2}{8} \int \left(\int \sigma_q^{-3/2} \right)^2 \qquad (2.7.21)$$

The last term of R_r is a function of x^{n-1} alone and so only contributes to trivial

R-separation. It is absorbed in Ψ_{n-1} by redefining this function:

$$\Psi_{n-1} \to \{\prod_{q\in Q} \sigma_q^{-n(B_q)/4}\}$$

$$\times \exp\left[\frac{Ex^{n-1}}{2\varepsilon} + \sum_{q\in Q}\sum_{b\in B_q} [\frac{\varepsilon\gamma_b^2}{8}\int(\int\sigma_q^{-3/2})^2 - \frac{s_q}{2\varepsilon}\int\frac{dx^{n-1}}{\sigma_q}] \right] \quad \textbf{(2.7.22)}$$

and then

$$R_r = \varepsilon\{\frac{\sigma_q{}'}{4}\sum_{b\in E_r}(z^b)^2 + \frac{1}{2}\gamma_r z^r \sigma_q^{\frac{1}{2}}\int\sigma_q^{-3/2}\}. \quad (2.7.23)$$

Using the results of (2.4.39) we have

$$\sum_{b\in E_r}(z^b)^2 = \begin{cases} c_r^2\sum_1^{N_r}(x^i - e_i), \\ \text{if } E_r \textit{ is of elliptic-type} \\ \frac{c_r^2}{4}[2\sum_1^{N_r}(x^i)^2 - (\sum_1^{N_r}x^i)^2 + 6\sum_1^{N_r}x^i\sum_1^{N_r-1}e_t - (\sum_1^{N_r-1}e_t)^2 - 2\sum_1^{N_r-1}e_t^2], \\ \text{if } E_r \textit{ is of parabolic-type} \end{cases} \quad (2.7.24)$$

R is now determined for each of the types I through $IV\pm$. For I, when $\sigma_q = 1$ on E_r,

$$R_r = \frac{1}{2}\varepsilon\gamma_r z^r x^{n-1} \quad (2.7.25)$$

If the coordinates on E_r are elliptic-type then $R_r = 0$ except in the cartesian case when

$$R_r = \frac{1}{2}\varepsilon\gamma_r x^r x^{n-1}. \quad (2.7.26)$$

If the coordinates are of parabolic-type then

$$R_r = \varepsilon\gamma_r c_r x^{n-1}[\sum_1^{N_r-1}x^i + \sum_1^{N_r}e_t]/4 \quad (2.7.27)$$

In this last case Ψ_{n-1} may be redefined so that

$$R_r = \varepsilon\gamma_r c_r x^{n-1}\sum_1^{N_r-1}x^i/4 \quad (2.7.28)$$

In both cases nontrivial R-separation occurs only if some γ_r is nonzero.

In case *II* when $\sigma_q = (x^{n-1} + v_q)^2$

$$R_r = \frac{\varepsilon}{2}\{(x^{n-1}+v_q)\sum(z^b)^2 - \gamma_r z^r / 4(x^{n-1}+v_q)\} \tag{2.7.29}$$

If the coordinates are elliptic-type then by scaling

$$R_r \rightarrow \frac{\varepsilon}{2}c_r^2 x^{n-1}\sum_1^{N_r} x^i \tag{2.7.30}$$

except for the cartesian case where

$$R_r \rightarrow \frac{\varepsilon}{2}\{(x^{n-1}+v_q)(x^r)^2 - \gamma_r x^r / 4(x^{n-1}+v_q)\} \tag{2.7.31}$$

When the coordinates are of parabolic-type the result is messy and is obtained by substituting the expression for $\sum(z^b)^2$ given in (2.7.24) into (2.7.29).

For types *III* and *IV*±, only elliptic type coordinates are possible on E_r and

$$R_r = \frac{\varepsilon}{4}\sigma_q \sum_{b \in E_r}(z^b)^2 \tag{2.7.32}$$

In case III when $\sigma_q = |x^{n-1} + v_q|$, only trivial R-separation occurs and $R \rightarrow 0$, by absorbing functions of x^i into the separated solutions Ψ_i.

In *IV*±, by similar redefinitions of the Ψ_i

$$R_r^{\pm} \rightarrow \tau_{\pm}\frac{\varepsilon c_r^2}{2}x^{n-1}\sum x^i \tag{2.7.33}$$

where $\tau_{\pm} = sign((x^{n-1}+v_q)^2 \pm w_q^2)$. In each case the R-factor is independent of the variables on the spheres S_{p_b}. The R-separation factors corresponding to the unmixed types m = 1, 2, 3 are given in Appendix A. The R-separation factors for the two dimensional mixing types may be obtained simply from those for the one dimension. Similar comments apply for m = 3.

In one and two dimensions the separation equations can be solved to give known special functions. Miller (1977) lists these results and investigates some of their properties (e.g.bases and overlaps for the Schrödinger equation). In higher dimensions however, the separation equations lead to new special functions about which little is known. Development of the properties of these functions is a problem for future research.

2.8 Applications

In (*) ε could take any complex value, but we are more interested in the Schrödinger equation ($\varepsilon = \frac{1}{2}i$) and the Heat or Diffusion equation ($\varepsilon = -\frac{1}{2}$, $E=0$). Some interesting relations between (*), the harmonic oscillator and linear potential Schrödinger equations will be discussed. Finally there are some brief comments on how the results in this chapter could be applied to moving boundary value problems.

Consider the Schrödinger equation with a potential $V(\mathbf{y},t)$:

$$\Delta_m \Psi + i\partial_t \Psi + V\Psi = 0 . \tag{2.8.1}$$

Although only systems for which $V=0$ have been classified, these results can be extended to many cases in which this restriction has been removed. Just as for (*), (2.8.1) may be considered as a symmetry reduced version of

$$\Box_{m+2}\Psi + V\Psi = 0 , \tag{2.8.2}$$

where the operator $\hat{\varepsilon} = \frac{1}{2}(P_{m+1} + P_{m+2})$ has been diagonalised. Reasoning as previously, all R-separable systems for (2.8.1) may be classified by finding all separable systems for (2.8.2) with $\hat{\varepsilon}$ diagonal. Once again the well developed theory of separation for equations like (2.8.2) helps us out. In Benenti (1980b) it was shown that (2.8.2) is separable if and only if

1. Equation (2.8.2) with $V=0$ is separable.
2. V has Stäckel form; that is,in analogy with (2.4.14)

$$V = \sum_{i=1}^{n_1+n_2} \frac{\psi^{i1}}{\psi} v_i(x^i) \tag{2.8.3}$$

where each v_i is an arbitrary function of x^i alone and $(\psi_{ij}(x^i))$ is the Stäckel matrix (see (D4)). In other words the R-separable systems for (2.8.1) are the same as those for (*) but with the added constraint that V must take Stäckel form. For example the general type of potential compatible with the spherical type II coordinates of Table 6 in Appendix A is

$$V = \frac{1}{x^4}\left[v_1(x^1) + \frac{1}{(x^1)^2}\left[v_2(x^2) + \frac{v_3(x^3)}{\sin^2(x^2)}\right]\right] \tag{2.8.4}$$

There are cases where more can be said about the potential. Boyer (1974) classified all time-independent potentials for (2.8.1) which admitted symmetry groups. To find the symmetry algebras of these groups he solved the relation (2.1.2):

$$[\tilde{Q}, L] = M\tilde{Q} \tag{2.8.5}$$

where $\tilde{Q} = \Delta_m + i\partial_t + V$. As all such systems are invariant under time translations Boyer only considered those admitting at least one other symmetry calling them *nontrivial*. He found that all these algebras were isomorphic to some subalgebra of the Schrödinger algebra s_m. In effect adding a potential to the free particle equation has restricted us to a subalgebra of that equation's symmetry algebra s_m. Boyer gave a complete list of potentials and their corresponding algebras for the three dimensional case. In general these potentials contained arbitrary functions. For instance the form of potential admitting the algebra $o(3)\oplus t_1$ (rotations and time translations) was $V(r)$. Exact forms were obtained for those potentials invariant under the Schrödinger algebra of maximal dimension $\frac{1}{2}m(m+3)+4$. Boyer found the following possibilities:

1. $V = 0$ - Free Particle
2. $V = y^1$ - Linear Potential
3. $V = \frac{\pm 1}{4}\mathbf{y}\cdot\mathbf{y}$ - Attractive(-) and Repulsive(+) Harmonic Oscillators (2.8.6)

The equivalence of these 3 cases is at first surprising, since one would expect the harmonic oscillator to break translational symmetry and the linear potential to break rotational symmetry. This will become clearer after we have made a transformation from the Schrödinger to the Heisenberg pictures. The above equivalences were first demonstrated by Neiderer (1973) together with the explicit forms of their associated isometric mappings.

The generators of (2.2.19) and (2.2.20)(when $E=0$) will be used to represent the Schrödinger algebra for the above 3 cases. It must be remembered that for the different potentials it is only the commutation relations that are preserved and not the explicit forms of the operators.

Boyer (1974) showed that these operators could be interpreted as a Lie algebra of skew symmetric operators on the Hilbert space $L_2(\mathbb{R}^m)$ of complex-valued Lesbesgue square integrable functions with respect to the scalar product

$$(\Psi_1, \Psi_2) = \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \Psi_1^*(\mathbf{y},t)\Psi_2(\mathbf{y},t)\,dy^1 \cdots dy^m \tag{2.8.7}$$

where the * denotes the complex conjugate. We will first follow his argument for the free particle case. To do this set $t=0$ everywhere and make the replacement

$$K_{-2} = \partial_t \to i\,\Delta_m = \bar{K}_{-2} \tag{2.8.8}$$

that holds on the solution space of free particle equation. Consider $f \in L_2(\mathbb{R}^m)$. Let

$$\Psi(\mathbf{y},t) = e^{t\bar{K}_{-2}} f(\mathbf{y}) \tag{2.8.9}$$

then substituting Ψ in (*)

$$\bar{K}_{-2}\Psi = \partial_t \Psi \tag{2.8.10}$$

i.e. Ψ also satisfies the free particle equation. This shows how to obtain the solution Ψ at a time t given a knowledge of Ψ at $t=0$: we are passing from the Schrödinger to the Heisenberg pictures. Kato (1966) gives the following useful integral operator form of $e^{-t\bar{K}_{-2}}$:

$$\Psi(\mathbf{y},t) = e^{-t\bar{K}_{-2}}\,\Psi(\mathbf{y},0) \tag{2.8.11}$$

$$= l.i.m.\,(4\pi i t)^{-m/2}\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty} exp[-|\mathbf{x}-\mathbf{y}|^2/4it]\Psi(\mathbf{x},0)dx^1\cdots dx^m$$

Here l.i.m. refers to the ***limit in the mean*** as it is defined in Kato. The operator $e^{t\bar{K}_{-2}}$ is unitary so the inner product (2.8.7) is independent of t of time translations. Similarly setting $t=0$ the other generators become

$$M_{ab} \to \bar{M}_{ab} = y^u\partial_v - y^v\partial_u\ ,$$

$$P_b \to \bar{P}_b = \partial_b\ ,$$

$$B_b \to \bar{B}_b = iy^b\ ,$$

$$K_{-2} \to \bar{K}_{-2} = i\Delta_m\ ,$$

$$D \to \bar{D} = \sum y^b\partial_b + m/2\ ,$$

$$K_2 \to \bar{K}_2 = i\sum(y^b)^2\ . \tag{2.8.12}$$

These operators satisfy the same commutation relations as the unbarred operators. It can be verified by using the adjoint action and the commutation relations that if μ is any of the unbarred operators then

$$e^{t\bar{K}_{-2}}\,\bar{\mu}\,e^{-t\bar{K}_{-2}} = \mu \tag{2.8.13}$$

This shows how the barred and unbarred operators are related.

Now consider

$$\Phi(\mathbf{y},t) = e^{t\bar{A}_3} f(\mathbf{y}) \tag{2.8.14}$$

where $f(\mathbf{y}) \in L_2(\mathbb{R}^m)$ and $\bar{A}_3 = \bar{K}_{-2} - \bar{K}_2$. From (2.8.12)

$$\partial_t \Phi = \bar{A}_3 \Phi = i\Delta_m \Phi - i\,\mathbf{y}\cdot\mathbf{y}\,\Phi \tag{2.8.15}$$

- Φ satisfies the equation for the attractive harmonic oscillator. It follows that the operator

$$A(t) = e^{t\bar{A}_3} e^{-t\bar{K}_{-2}} \tag{2.8.16}$$

maps solutions of the free particle equation to the harmonic oscillator equation i.e. if Ψ is a solution of the free particle equation then

$$\Phi = A(t)\Psi \tag{2.8.17}$$

is a solution of the attractive harmonic oscillator equation. In a similar vein each of the linear potential and repulsive oscillator equations are related by time dependent operators to the free particle equation. Using the results of Table 2.6.2 the equations in (2.8.6) may be written

1. $V = 0$ $\quad \partial_t \Psi = \bar{K}_{-2}\Psi$
2. $V = y^1$ $\quad \partial_t \Psi = (\bar{K}_{-2} - \bar{B}_1)\Psi$
3. $V = \frac{\pm 1}{4}\mathbf{y}\cdot\mathbf{y}$ $\quad \partial_t \Psi = (\bar{K}_{-2} \pm \bar{K}_2)\Psi$ $\qquad$ (2.8.18)

These results are now used to interpret R-separation for the free particle equation. Given any unsplit R-separable system for this equation, the Casimir operator always yields one of the operators in Table 2.6.2. If this operator was A_3 then there are others $O_2, O_3, \ldots, O_m$ such that the free particle equation is equivalent to

$$A_3\Psi = \lambda_1\Psi,\ O_2\Psi = \lambda_2\Psi,\ \ldots,\ O_m\Psi = \lambda_m\Psi\ , \qquad \lambda_i \in \mathbb{R} \tag{2.8.19}$$

If we pass to the equivalent system

$$\bar{A}_3\Phi = \partial_t \Phi \qquad (2.8.20)$$

using $A(t)$ the operators for this new system will be

$$\bar{A}_3\Phi = \lambda_1\Phi,\ \ \bar{O}_2\Phi = \lambda_2\Phi,\ \ldots,\ \bar{O}_m\Phi = \lambda_m\Phi \qquad (2.8.21)$$

Consequently, the Hamiltonian $\bar{A}_3$ has been diagonalised, the variable t is ignorable, and as a result (2.8.20) is purely separable. We have shown that every R-separable system for the free particle equation possessing the operator A_3 is equivalent to a purely separable system for the attractive harmonic oscillator equation (2.8.20). The other cases can be treated in a similar manner. In each case unsplit R-separable systems of the free particle equation correspond to purely separable systems of the linear potential, attractive or repulsive oscillator equations. Miller (1977) gives this treatment for $m=2$ and also provides the explicit form of the mapping $A(t)\Psi=\Phi$. It has been shown that all the above equations are equivalent under the action of the Schrödinger group with this equivalence having a direct bearing on their separability properties. All three equations should be studied as a unit.

It is interesting to note that the V of (2.4.14) is analogous to the potentials of the four types of Hamiltonians. Further, it is easily shown using the results of §2.4, that given a general R-separable system (2.5.1) of (*) the coordinate system

$$y^i = z^i\,,$$
$$t = x^{m+1} \qquad (2.8.22)$$

provides a purely separable system for

$$\Delta_m\Psi + i\,\partial_t\Psi + \sum_{q\in Q} V_q\Psi = E\,\Psi \qquad (2.8.23)$$

It must be noted that solutions of (*) can be mapped to solutions of (2.8.23) via the Schrödinger group only when $E=0$, and the R-separable system is of unsplit type.

It is possible to obtain potential terms for (2.8.1) by diagonalising various operators for higher dimensional versions of (*). For example consider

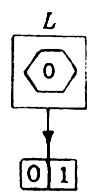
$$\Delta_m\Psi - \sum_{w=1}^{N} \frac{a_w^2}{(y^w)^2}\Psi + i\,\partial_t\Psi = E\Psi\,. \qquad (2.8.24)$$

Taking $a_w = 0$ for each w shows that the R-separable systems for (2.8.24) form a subclass of those for (*). These systems may be obtained from

$$\Delta_{m+N}\Psi + i\,\partial_t\Psi = E\Psi \tag{2.8.25}$$

by diagonalising the symmetry operators $M_{12}, M_{34}, \dots, M_{(2N-1)2N}$. The graphs for (2.8.25) can be identified in that the N circles $\boxed{0|1}$ corresponding to the above operators must be attached to their lowest extremities. Another way to achieve this classification is to consider all the graphs for the R-separable systems of (*) and choose those for which the circles $\boxed{0|1}$ can be attached to form an R-separable system for (2.8.25).

The classification for $m=1$ appears in Miller (1977). It is obtained from (2.8.25) with $N=1$ by diagonalising M_{12}. The only possible graphs for (2.8.25) are 7 → 10 of Table 3 in Appendix A, i.e.,



(2.8.26)

where L: $I \to IV\pm$. The coordinates for (2.8.24) are

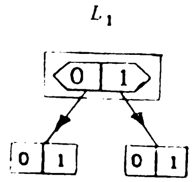
$$y^1 = \sigma_L^{\frac{1}{2}} x^1 . \tag{2.8.27}$$

Alternatively the only way the circle $\boxed{0|1}$ can be attached to the $m=1$ systems to form $m=2$ systems is in the manner above. The R-separation factors and separation equations are easily obtained from those for (2.8.26). Miller (1977) develops many of the spectral properties and special function identities associated with this classification.

All the possibilities for $m=2$ have form

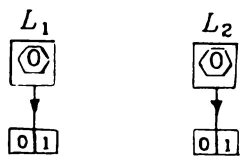
$$\Delta_2\Psi - \left(\frac{a_1^2}{(y^1)^2} + \frac{a_2^2}{(y^2)^2}\right)\Psi + i\,\partial_t\Psi = E\Psi \tag{2.8.28}$$

If both a_1 and a_2 are nonzero, the only ways two circles can be attached separately to the graphs in Table 2 of Appendix A are



(2.8.29a)

and



(2.8.29b)

where $L_1, L_2 \colon I \to IV\pm$. The systems of (2.8.29b) are split types and can be derived from the results for $m=1$. The coordinates for (2.8.29a) are

$$y^1 = \sigma_L^{\frac{1}{2}} \cosh(x^1)\cos(x^2)\cos(x^3) \ ,$$

$$y^2 = \sigma_L^{\frac{1}{2}} \cosh(x^1)\cos(x^2)\sin(x^3) \ ,$$

$$y^3 = \sigma_L^{\frac{1}{2}} \sinh(x^1)\sin(x^2)\cos(x^4) \ ,$$

$$y^4 = \sigma_L^{\frac{1}{2}} \sinh(x^1)\sin(x^2)\sin(x^4) \ ,$$

$$t = x^5 \ , \qquad L \colon I \to IV\pm \ . \tag{2.8.30}$$

The R-separation factors and the separation equations are just those for 4-dimensional system with graph (2.8.29a). Using our general methods we reproduced the results of Boyer (1976). In the same article Boyer explored some of the special function theory related to this case.

The other operators resulting from the orbit analysis of §2.3 may also be diagonalised. By diagonalising P_u the free particle equation in m dimensions is transformed into the constant potential Schrödinger equation in $m-1$ dimensions. This may seem trivial but it shows that (*) with $E=0$ is the primitive equation of our study. Diagonalising $B_w - v_w P_w$ yields the time dependent potential term

$$\frac{a_w^2}{(x^{m+1}+v_w)^2} \ \Psi \tag{2.8.31}$$

In general the equations

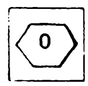
$$\Delta_m \Psi + \left\{ \sum_{w=1}^{A} \frac{-a_w^2}{(y^w)^2} + \sum_{w=2A+1}^{N} \frac{a_w^2}{(x^{m+1}+v_w)^2} \right\} \Psi + i\partial_t \Psi = E\Psi \qquad (2.8.32)$$

can be derived from (2.8.2) with $V=0$ by diagonalising the operators

$$M_{12}, \quad , M_{(2A-1)2A}, B_{2A+1}-v_{2A+1}P_{2A+1}, \quad , B_N - v_N P_N \qquad (2.8.33)$$

The corresponding graphs are obtained by attaching the circles corresponding to the M_{ij}'s to the lowest parts of the graph and then adding the systems

$II(v_w; \gamma_w = 0)$



(2.8.34)

These results can be generalised by replacing the circles mentioned above by p-dimensional spheres. The eigenvalues $-a_w^2$ can then be replaced by the more general forms $-\mu_w(\mu_w + p - 1)$.

Many of the properties that held for the Schrödinger equation also hold for the Heat equation. A potential term is analogous to a heating term or a distribution of sources for the Heat equation. For R-separability this term must take Stäckel form, just as it did for the Schrödinger equation. It is not possible however, to introduce a Hilbert space structure directly as for the Schrödinger equation. Rosenbloom and Widder (1959) do manage to obtain some expansion theorems by using a time independent form similar to (2.8.7). The main difficulty is that the resulting operators are not all unitary.

We now focus our attention on the coordinate surfaces $x^i = constant = x_0^i$. For R-separable coordinate systems (2.8.1) implies that

$$z^u(x^k) = (y^u - \tfrac{1}{2}\gamma_u \int\int \sigma_q^{-3/2})\sigma_q^{-\frac{1}{2}},$$

$$x^{m+1} = t \qquad (2.8.35)$$

If $\sigma_q = 1$ and $\gamma_q = 0$, for all q, then these surfaces are independent of time, and represent separable coordinate systems for

$$\Delta_m \Psi = E\,\Psi \qquad (2.8.36)$$

From the formulae given for the coordinates in (2.4.35) it can be shown that these surfaces are all confocal quadrics:

$$\sum_{u=1}^{m} \beta_u \frac{(y^u - \alpha_u)}{(x^a - \eta_u)} = c^2 ; \quad a = 1, \qquad , m \tag{2.8.37}$$

In three dimensions these yield such familiar quadrics as ellipsoids, hyperboloids, paraboloids, planes etc. If time dependence is allowed, then these surfaces are again the confocal quadrics of (2.8.37) except that

$$y^u \to [y^u - \tfrac{1}{2}\gamma_u \int\int \sigma_q^{-3/2}]\sigma_q^{-\frac{1}{2}} \,|_{x^{n-1}=t_0} \tag{2.8.38}$$

These surfaces are now parameterised by time - they move. To illustrate, consider the spherical systems 17 → 20 which appear in Table 6 of Appendix A. The coordinates for these systems are:

$$y^1 = \sigma_L^{\frac{1}{2}} x^1 \cos(x^2) \,,$$

$$y^2 = \sigma_L^{\frac{1}{2}} x^1 \sin(x^2)\cos(x^3) \,,$$

$$y^3 = \sigma_L^{\frac{1}{2}} x^1 \sin(x^2)\sin(x^3) \,,$$

$$t = x^4 \qquad , L : I \to IV\pm \tag{2.8.39}$$

The coordinate surfaces are:

$$(x^1)^2 = constant = [(y^1)^2+(y^2)^2+(y^3)^2]/\,\sigma_L(t)$$

$$\tan^2(x^2) = constant = [(y^2)^2+(y^3)^2]/\,(y^1)^2$$

$$\tan(x^3) = constant = y^3/\,y^2 \tag{2.8.40}$$

The surfaces representing the angles x^2 and x^3 are stationary. Indeed for any R-separable system it can be shown that the coordinate surfaces associated with the attached spheres are motionless.

When $L=I$ the coordinate surface $x^1 = x_0^1$ are simply the stationary spheres of radius x_0^1. In case II, when $\sigma = t^2$, these spheres are expanding with speed x_0^1. When $\sigma = t$ the spheres x_0^1 are expanding in such a way that their surface area is increasing uniformly with time. In $IV\pm$ the radius at time t is

$$r(t) = (t^2 \pm 1)^{\frac{1}{2}} x_0^1 \,. \tag{2.8.41}$$

These results could be used to solve moving boundary value problems where the initial data is specified on a boundary moving in one of the ways above. Bluman and Cole (1974) provide several examples of these moving boundaries for the

Heat equation in one spatial dimension. Their similarity solutions are those we obtain when $m=1$. In general the geometry of these boundaries is that of confocal quadrics and there are only the five dependencies on time $I \to IV\pm$. Despite these restrictions it seems possible to solve a reasonable class of physical problems by applying perturbation analysis to the boundaries obtained by these methods.

The classification of all R-separable systems has been achieved using direct methods. We were helped by the restrictive signature of Minkowski space, which enabled us to reduce the metric to one off-diagonal element so that the curvature conditions could be successfully applied. In Chapter 3 this work is extended to provide results for the case of one first order and one ignorable coordinate in complex Euclidean space. There the metric has only one nonorthogonal entry so that many of the same methods can be used. These methods are hard to generalise when working with spaces where the signature is less definite such as $E(n,2)$ or with complex Euclidean space $E(\mathbb{C},n)$ where the signature does not help at all. The many nonorthogonal terms in the metrics of such spaces mean that the curvature conditions are often impossible to solve. However progress was made recently when Kalnins et al. (1983) extended some forgotten work of Böcher (1894) to classify large classes of separable coordinate systems on $E(n,\mathbb{C})$ and on the n-dimensional complex sphere. It was found that if there are only Stäckel and ignorable variables (i.e. none are first order) then all separable systems are *limits* of elliptic coordinates. In this way even nonorthogonal systems may be obtained. Roughly these *limits* refer to limits taken amongst the parameters associated with separable systems. For our purposes they are best understood by examining the examples that we will provide. These limits enable the easy computation of the operators and coordinate transformations. The remaining gap in the classification is for first order variables. In this study we have obtained many systems with one first order variable for the Helmholtz equation. It will be shown how all these systems can be derived from the type IV(harmonic oscillator) coordinates by using limits and in so doing we offer a clue towards the solution of the general first order problem.

Putting $[(x^2+v)+w^2] = (x^2-\alpha_1)(x^2-\alpha_2)$ the associated Minkowski metric for system 4 in Table 1 of Appendix A is

$$ds^2 = (x^2-\alpha_1)(x^2-\alpha_2)(dx^1)^2 - \frac{(\alpha_1-\alpha_2)^2(x^1)^2(dx^2)^2}{4(x^2-\alpha_1)(x^2-\alpha_2)} + 2dx^2dx^3 \qquad (2.8.42)$$

If we make the transformations $\alpha_2 \to -1/\delta^2$, $x^1 \to \delta x^1$ and take the limit of (2.8.42) as $\delta \to 0$ we obtain the metric for system 3 in Table 1 of Appendix A. To obtain the associated metric of system 2:

$$ds^2 = (x^2-\alpha_1)^2(dx^1)^2 - \frac{\gamma x^1}{(x^2-\alpha_1)^2}(dx^2)^2 + 2dx^2dx^3, \qquad (2.8.43)$$

the substitutions

$$x^1 \to (\delta x^1 + \frac{2\gamma}{\delta}),\ \alpha_1 \to \frac{\alpha_1}{\delta},\ \alpha_2 \to \frac{\alpha_1}{\delta} - \delta,\ x^2 \to \frac{x^2}{\delta}, \qquad \textbf{(2.8.44)}$$

$$x^3 \to \delta x^3 + \gamma^2/2\delta(x^2-\alpha_1).$$

are made in (2.8.42) and the limit $\delta \to 0$ is taken. The metric of system 1 is obtained from (2.8.43) by making the transformations

$$\alpha_1 \to \frac{1}{\delta},\ x^1 \to \delta x^1,\ \gamma \to \frac{\gamma}{\delta^3} \qquad (2.8.45)$$

and letting δ go to zero.

This method also works for the operators: using limits they can all be derived from the operator for system IV. When expressed in terms of α_1 and α_2 this operator is

$$\{B_1-\alpha_1P_1, B_1-\alpha_2P_1\} \qquad (2.8.46)$$

If this operator is multiplied by δ and the substitution $\alpha_2 \to -1/\delta$ is made, then in the limit $\delta \to 0$ we obtain $\{P_1, B_1-\alpha_1P_1\}$ - the operator for system 3. Under the adjoint action of K_{-2} the operator (2.8.46) is equivalent to

$$\{B_1, B_1-\eta P_1\} \qquad (2.8.47)$$

If the adjoint actions e^{2aAdP_1} and e^{bAdB_1} are applied this becomes

$$\{B_1+a\varepsilon, B_1+a\varepsilon-\eta(P_1+b\varepsilon)\}. \qquad (2.8.48)$$

If we put $a=-\gamma/\eta$, $b=-2\gamma/\eta^2$ and let $\eta \to 0$ then (2.8.48) is

$$B_1^2+\gamma\varepsilon P_1 \qquad (2.8.49)$$

- the operator for system 2 of Table 1 in Appendix A has been obtained as a limit. To get the last system, the actions of $e^{-bAdK_{-2}}$ and e^{2aAdB_1} are applied to (2.8.49) and it is then

$$(B_1 + a\varepsilon + bP_1)^2 + \gamma\varepsilon P_1 \tag{2.8.50}$$

Multiplying this operator by δ^2 and subtracting the term in ε^2 (since ε is already one of the operators) we have

$$\delta^2[B_1^2 + b^2P_1^2 + 2a\varepsilon B_1 + 2b\{B_1, P_1\} + (2ab+\gamma)\varepsilon P_1] \tag{2.8.51}$$

Making the transformations

$$a \to \gamma/2\delta^2,\ b \to 1/\delta,\ \gamma \to -\gamma/\delta^3 \tag{2.8.52}$$

and letting δ go to zero we obtain

$$P_1^2 + \varepsilon\gamma B_1 \tag{2.8.53}$$

which is the operator for system 1. This process can be generalised to obtain arbitrary m-dimensional unsplit systems from type IV elliptic coordinates. It seems difficult, however, to obtain many of the split systems from the generalisation of the metric (2.8.42) because of the number of extra parameters (the v_q's etc.) the splitting types possess. In that case the splitting types can be derived from a system like (2.5.2) where each L_q is IV and each G_q is an $n(B_q)$-dimensional elliptic system. This is more complicated than in the case of $\mathbb{R}^n$, where only one parent metric (that of general elliptic coordinates) was needed. At the very least, it seems difficult to obtain these first order type systems from orthogonal systems.

III

The Complex Heat and Schrödinger equations

3.1 Introduction

In this chapter we consider the complex generalisation of (*)

$$\Delta_m \Psi + 2\varepsilon \frac{\partial}{\partial t} \Psi = E\Psi \ , \qquad (3.1.1)$$

,i.e., y^u, t and E are now complex. Since this is the complex case the coordinates may be scaled by any complex number. Thus ε can be normalised to $\frac{i}{2}$ or $\frac{-1}{2}$: the complex Heat and Schrödinger equations are equivalent. It is easy to check that Theorem 2.3.1 still holds: all the R-separable systems for (3.1.1) can be classified by classifying those for the complex version of (2.2.1). As this is the complex case, for tidiness sake, we can let $\bar{y}^n \to i\bar{y}^n$. Equation (2.1.1) becomes

$$\Delta_m \bar{\Psi} = (\partial_{y^1 y^1} + \cdots + \partial_{y^n y^n}) \bar{\Psi} = E\, \bar{\Psi} \qquad (3.1.2)$$

and the symmetry operator (2.2.2) is now

$$\partial_{x^n} = \tfrac{1}{2}(\partial_{y^{n-1}} - i\partial_{y^n}) \qquad (3.1.3)$$

The symmetry algebra of (3.1.2) is the complexification of the symmetry algebra for (*). When $E=0$ it is the complex Schrödinger algebra

$$S_{m\mathbb{C}} = (sl(2,\mathbb{C}) \oplus o(m,\mathbb{C})) \oplus w_{m\mathbb{C}} \ , \qquad (3.1.4)$$

with the same commutation relations and adjoint actions as in the real case (see Table 2.2.1). When $E \neq 0$ the symmetry algebra is the complexification of the Galilean algebra (2.2.27).

In summary the R-separation problem for (3.1.1) becomes the separation problem for the Helmholtz equation defined on complex Euclidean space $E(n,\mathbb{C})$. Indeed the problem of classifying the R-separable systems for (3.1.1) is equivalent to that of classifying systems for the Helmholtz equation. These two equations are embedded within each other. This can be demonstrated by noting that when $\varepsilon=0$, (3.1.1) is just the Helmholtz equation in m dimensions. Thus we will confine ourselves to classifying separable metrics for (3.1.2), with the understanding that the details for (3.1.1) can be easily derived from the results of the last chapter.

Studying complex equations like (3.1.1) is a two-edged activity. On the one hand it enables one to tidily treat the resulting real forms (examples of which will be given). On the other hand solving the complex problem is always more difficult. In the previous chapter, the signature of Minkowski space enabled us to reduce the metric to one nonorthogonal entry. Here the signature does not help at all. In general the separable metrics can contain many nonorthogonal terms (see § 3.4 for an example).

The classification of separable systems on $E(n,\mathbb{C})$ is an unsolved problem. The orthogonal systems have been classified in Kalnins, Miller and Reid (1983). Some progress has been made with applying the methods of that article to nonorthogonal systems consisting of only Stäckel and ignorable variables. The first order problem remains a mystery. In this chapter we offer some clues by finding solutions to the case where the metric has one nonorthogonal entry and 1 first order variable. The Hamilton-Jacobi equation is then

$$g^{11}p_1^2 + \cdots + g^{(n-2)(n-2)}p_{n-2}^2 + 2g^{(n-1)n}p_{n-1}p_n + g^{nn}p_n^2 = E. \qquad (3.1.5)$$

When $g^{n(n-1)}=1$ many of the same methods used in the real case can be applied. This case is the subject of §3.2. All such systems will be shown to possess the symmetry operator (3.1.3). Thus by the arguments of §2.2 they are of heat type: they all provide R-separable systems for the complex equation (3.1.1). By assuming that $g^{n(n-1)}\neq 1$ in §3.3, systems are classified that are not of heat type. These systems do not R-separate (3.1.1) via the mapping given in §2.2. However they can be embedded in a higher dimensional Schrödinger equation by our previous discussion and do give insights into the general first order problem.

In §3.4 we cover a class of systems with two first order coordinates. This class is the first documented occurrence of nontrivial R-separation for the Hamilton-Jacobi equation.

3.2 Heat-type systems with one nonorthogonal entry

In this section the contravariant metric is assumed to have one nonorthogonal entry $g^{n(n-1)}=1$, and one first order variable x^{n-1}. It will be shown that these systems must be of heat-type.

Theorems 2.4.1 and 2.4.2 are readily proved with no changes except that signature dependent statements are removed (e.g. the positive definite condition in Theorem 2.4.2).

In summary the Hamilton-Jacobi equation can be written

$$\sum_{q\in Q}\frac{1}{\sigma_q}\sum_{b\in B_q}\bar{g}^{bb}p_b^2 + 2p_{n-1}p_n + \sum_{q\in Q}\frac{V_q}{\sigma_q}p_n^2 = E \tag{3.2.1}$$

where in the now complex standardised coordinates $z^b = z^b(x^c)$,

$$V_q = \sum_{b\in B_q}\left[\frac{\zeta_q(z^b)^2}{4} + \gamma_b z^b\right] \tag{3.2.2}$$

The separation equations for the Helmholtz equation corresponding to (3.2.1) will not be derived here. Thus it will not be possible to verify Helmholtz separability directly as was done in the real case. Instead the general Schrödinger conditions (D9) are used. From the form of (3.2.1), (D9a) is easily verified. Condition (D9b) is

$$g^{-1/2}\partial_{n-1}g^{1/2} = \sum_{a=1}^{n-2}g^{aa}E_a^n(x^a) + E_{n-1}^n(x^{n-1}) \tag{3.2.3}$$

Here $g = \prod_{a=1}^{n-2}\sigma_a\bar{g}_{aa} = \prod_{q\in Q}\sigma_q^{n(B_q)}\prod_{a=1}^{n-2}\bar{g}_{aa}$. Equation (3.2.3) is satisfied with

$E_a^n = 0$ and

$$E_{n-1}^n = \frac{1}{2}\sum_{q\in Q}n(B_q)\frac{d}{dx^{n-1}}\log(\sigma_q) \ , \tag{3.2.4}$$

Thus any separable coordinate system with Hamilton-Jacobi equation (3.2.1) is also Helmholtz separable and the form of the Helmholtz separation equations is given in (D10). Incidentally this is the same result as in the real case (see (2.7.1) and (2.7.4)).

The coordinate transformations are

$$y^u = F_{uq}\left(z^u(x^v), x^{n-1}\right).$$

$$y^{n-1}+i\,y^n = x^n - \sum_{q\in Q}\sum_{b\in B_q} G_{bq}$$

$$y^{n-1}-i\,y^n = 2x^{n-1} \;, \tag{3.2.5}$$

with the details being the same as in (2.4.30). (Remember $y^n \to iy^n$). In particular these transformations imply that $\partial_{x^n} = \frac{1}{2}(P_{n-1}-iP_n)$,i.e., all the systems are of heat-type.

To obtain the coordinate transformations in terms of the x^i, the expressions $z^b = z^b(x^c)$ are substituted into (3.2.5). Not all such systems will be separable. As in the real case, the problem is to work out the form of the "potential" V_q leading to separation in the variables x^i. The key will be the compatibility condition (2.4.16): $\partial_{ij} V_q = 0$.

As a first step the complex orthogonal separable systems $z^b = z^b(x^c)$ are determined. These have been classified in Kalnins, Miller and Reid (1983). A brief summary of that solution is now given.

The solution is similar to that for $\mathbb{R}^n$, in which separable systems are represented by tree graphs. Again $\mathbb{C}^n$ can be decomposed into a direct sum of subspaces $\mathbb{C}^{n_r}$. To simplify matters we will confine ourselves to one of these irreducible blocks and unambiguously remove the subscripts q and r.

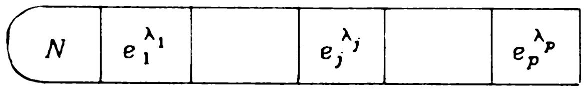
Consider as a prototype the elliptic-type A system with graph (2.4.32A) and no spheres attached. This is also an orthogonal separable system for $\mathbb{C}^{n_r}$ where the variables x^a and z^b are now complex-valued. The metric in these coordinates is

$$ds^2 = \sum_{i=1}^{N} \frac{\prod_{j\neq i}(x^i - x^j)}{\prod_{j=1}^{N}(x^i - e_j)} (dx^i)^2 \;, \qquad (N = N_r). \tag{3.2.6}$$

Kalnins, Miller and Reid (1983) found that the case where the roots can occur with various multiplicities also provides orthogonal separable systems on $E(n,\mathbb{C})$. That is, the restriction that the roots e_j are different is dropped. These metrics are

$$ds^2 = \sum_{i=1}^{N} \frac{\prod_{j\neq i}(x^i - x^j)}{\prod_{j=1}^{p}(x^i - e_j)^{\lambda_j}} (dx^i)^2 \;. \tag{3.2.7}$$

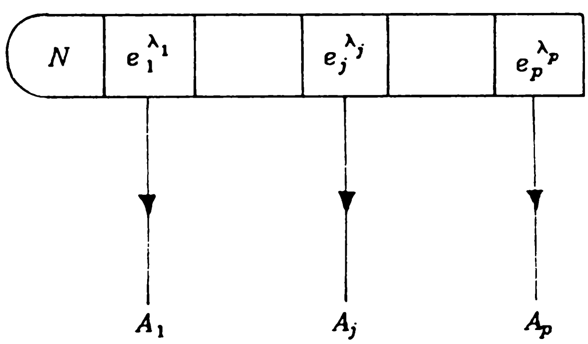
where λ_j is the multiplicity of the root e_j. The corresponding graph is



(3.2.8)

where $\lambda_1 + \ldots + \lambda_p = N$. Naturally the same generalisation applies to parabolic coordinates but now with $\lambda_1 + \cdots + \lambda_p = N-1$. Parabolic coordinates are obtained from elliptic coordinates by a limiting process whereby one of the roots goes to infinity at a specified rate. Indeed the number of the roots may take any value between 0 and N.

To complete the classification it is shown that only complex spheres $S_{p\mathbb{C}}$ can be attached to a root of multiplicity one and only complex Euclidean spaces can be attached to a root of multiplicity greater than one. Thus a general graph has form



(3.2.9)

where $0 \le \lambda_1 + \cdots + \lambda_p \le N$ and

$$A_j = \begin{cases} S_{p_j\mathbb{C}} \ ; & \text{if } \lambda_j = 1 \\ E_{p_j\mathbb{C}} \ ; & \text{if } \lambda_j > 1 \end{cases} \tag{3.2.10}$$

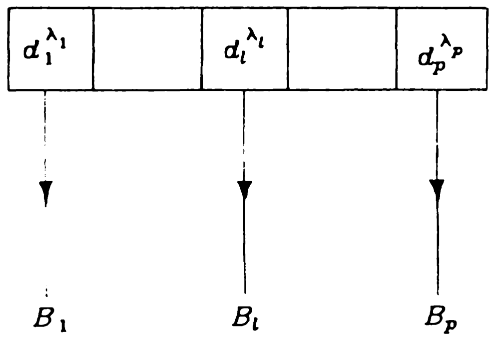
Of course there is always the trivial option of no space being attached to a root.

The metric corresponding to (3.2.9) is

$$ds^2 = \sum_{i=1}^{N} \frac{\prod_{j \neq i}(x^i - x^j)}{\prod_{j=1}^{p}(x^i - e_j)^{\lambda_j}} (dx^i)^2 + \sum_{j=1}^{p} \frac{\prod_{i=1}^{N}(x^i - e_j)}{\prod_{i \neq j}(e_i - e_j)} ds_j^2 \tag{3.2.11}$$

Here ds_j^2 is the infinitesimal distance for the space A_j. The letters h, i, j will be used to denote variables from the leading block while k and l will be used for variables on the attached spaces A_j.

The spheres S_{p_j} are constructed in much the same way. They have graphical representation



(3.2.12)

where $\lambda_1 + \cdots + \lambda_p = n_j + 1$ and

$$B_l = \begin{cases} S_{v_l \mathbb{C}} \; ; & \text{if } \lambda_l = 1 \\ E_{v_l \mathbb{C}} \; ; & \text{if } \lambda_l > 1 \end{cases} \tag{3.2.13}$$

$(x^1, \ldots, x^{n_j}$ are the coordinates on the leading block). The metric ds_j^2 for S_{p_j} is (3.2.11) with $\sum \lambda_j = n_j + 1$.

Next the potential term $V(=V_q)$ is determined. Equation (2.4.16) implies

$$\begin{aligned} V &= v_1(x^1) + \cdots + v_m(x_m) \\ &= \sum_{u=1}^{m} g^{uu} A_u^{nn}(x^u) \;,\; (m = n-2). \end{aligned} \tag{3.2.14}$$

Suppose that x^{k_j} is a coordinate on one of the attached spaces A_j. If x^1 is one of the coordinates on the leading block then from (3.2.11) and (3.2.14)

$$\partial_{x^1 x^{k_j}} V = 0 \tag{3.2.15}$$

$$= \partial_{x^1}\left(\frac{\prod_{k \neq j}(e_k - e_j)}{(x^1 - e_j)\prod_{i=2}^{N}(x^i - e_j)} \right) \partial_{x^{k_j}}\left[\sum_{l_j} \tilde{g}^{l_j l_j} A_{l_j}^{nn}(x^{l_j}) \right]$$

where $ds_j^2 = \sum_{l_j} \tilde{g}_{l_j l_j}(dx^{l_j})^2$. Equation (3.2.15) implies $\partial_{x^{k_j}} V = 0$ i.e. from (3.2.14)

$$V = \sum_{i=1}^{N} v_i(x^i) \tag{3.2.16}$$

Thus V is independent of the coordinates on the attached spaces A_j just as it was in the real case. In the following theorem V is completely determined.

Theorem 3.2.1

$$V = A\sum_{i=1}^{N} x^i + B \quad , \quad (A,\ B\ \textit{constants}\ ), \quad \textit{where} \tag{3.2.17}$$

$$A = \begin{cases} \zeta, & \text{if } \sum_{i=1}^{p} \lambda_i = N, \\ \gamma, & \text{if } \sum_{i=1}^{p} \lambda_i < N \\ (\gamma \textit{ is arbitrary} \text{ and } \zeta=0 \text{ for } \textit{this last case}\ ) \end{cases} \tag{3.2.18}$$

This theorem is a generalisation of the work at the end of §2.4.

Proof of Theorem 3.2.1: To prove this theorem we take a different route than in the real case. The explicit form of the coordinate transformations is not used. Instead the compatibility problem is solved using the remaining n curvature conditions $R_{a(n-1)(n-1)a} = 0$. They are equivalent to

$$-\frac{1}{2}\zeta g_{aa} + u_a{}' = \frac{1}{2}\frac{g_{aa,a}}{g_{aa}}u_a - \frac{1}{2}\sum_{b\neq a}\frac{g_{aa,b}}{g_{bb}}u_b \tag{3.2.19}$$

where

$$u_a(x^a) = \frac{d}{dx^a}\, v_a(x^a), \quad \text{and} \quad u_a{}' = \frac{du_a}{dx^a} \tag{3.2.20}$$

Remember that we are considering an irreducible block E_r attached to the function σ where $2\sigma\sigma'' - (\sigma')^2 = \zeta$ (the label q has been dropped).

Of course for the variables x^k on the attached spaces $u_k = 0$. From (3.2.11)

$$-\frac{1}{2}\zeta g_{ii}+u_i{}' = \frac{1}{2}\left[\sum_{j\neq i}\frac{1}{x^i-x^j} - \sum_{j=1}^{p}\frac{\lambda_j}{x^i-e_j}\right]u_i$$

$$+\frac{1}{2}\sum_{j\neq i}\frac{g_{ii}}{(x^i-x^j)g_{jj}}u_j \qquad (3.2.21)$$

If this expression is multiplied by (x^i-x^h) and then we set $x^h=x^i$ (permissible since none of the denominators vanish) it becomes

$$\frac{1}{2}u_i(x^i)+\frac{1}{2}\frac{g_{ii}}{g_{hh}}\Big|_{x^h=x^i}u_h(x^i) = 0 \qquad (3.2.22)$$

Since $\frac{g_{ii}}{g_{hh}}\Big|_{x^h=x^i} = -1$,

$$u_i(x^i) = u_h(x^i) \qquad (3.2.23)$$

i.e. all the u_i's have the same functional form ($u_i=u$ say). We now add the expressions for $R_{i(n-1)(n-1)i}$ and $R_{h(n-1)(n-1)h}$, and making use of (3.2.23), take the limit $x^h\to x^i$ of the result. The only things to be careful of are vanishing denominators. The calculation goes as follows:

$$-\frac{1}{2}\zeta\lim_{x^h\to x^i}(g_{ii}+g_{hh})+\lim_{x^h\to x^i}(u'(x^i)+u'(x^h))$$

$$=\frac{1}{2}\lim_{x^h\to x^i}\frac{u(x^i)-u(x^h)}{x^i-x^h}-\frac{1}{2}\lim_{x^h\to x^i}\frac{\frac{g_{hh}}{g_{ii}}u(x^i)-\frac{g_{ii}}{g_{hh}}u(x^h)}{x^i-x^h}$$

$$+\sum_{j\neq i,h}\lim_{x^h\to x^i}\left\{\frac{1}{2}\frac{g_{ii}}{(x^i-x^j)g_{jj}}u(x^j)+\frac{1}{2}\frac{g_{hh}}{(x^h-x^j)g_{jj}}u(x^j)\right\}.$$

Simplifying this equation gives:

$$2u' = \frac{1}{2}\lim_{x^h\to x^i}\frac{u(x^i)-u(x^h)}{x^i-x^h}-\frac{1}{2}\lim_{x^h\to x^i}\frac{\frac{g_{hh}}{g_{ii}}u(x^i)-\frac{g_{ii}}{g_{hh}}u(x^h)}{x^i-x^h}$$

$$=\frac{1}{2}u'+\frac{1}{2}u' = u' \qquad (3.2.24)$$

using l'Hôpital's rule. Thus $u' = 0$ and

$$V = A\sum_{1}^{N}x^i + B \qquad (3.2.25)$$

fulfilling condition (3.2.17) of Theorem 3.2.1.

Equation (3.2.21) is now

$$-\frac{1}{2}\zeta g_{ii} = \frac{1}{2}A\left[\sum_{j\neq i}\frac{1}{x^i-x^j} - \sum_{j=1}^{p}\frac{\lambda_j}{x^i-e_j} - \sum_{j\neq i}\frac{g_{ii}}{(x^i-x^j)g_{jj}}\right] \qquad (3.2.26a)$$

where

$$g_{kk} = \frac{\prod_{j\neq k}(x^k-x^j)}{\prod_{j=1}^{p}(x^k-e_j)^{\lambda_j}}$$

In the case where $\lambda_j=1$ for $1\le j\le N$, tedious use of partial fractions shows that (3.2.26a) implies $A=\zeta$ (the same result as in the real elliptic case). Equation (3.2.26a) becomes

$$-\frac{1}{2}g_{ii} = \frac{1}{2}\left[\sum_{j\neq i}\frac{1}{x^i-x^j} - \sum_{j=1}^{N}\frac{1}{x^i-e_j} - \sum_{j\neq i}\frac{g_{ii}}{(x^i-x^j)g_{jj}}\right] \qquad (3.2.26b)$$

Since (3.2.26b) is continuous in the parameters e_j we can let $e_2\to e_1$ in that equation. It then becomes (3.2.26a) with $\lambda_1=2$, $\lambda_j=1$, $j\neq 1$, and implies that $A=\zeta$: solving the case where $\lambda_1=2$. This argument is easily generalised to show that if $\sum_j\lambda_j=N$ then (3.2.26a) implies that $A=\zeta$.

Now let $e_N\to\infty$ in (3.2.26b). This equation becomes

$$0 = \left[\sum_{j\neq i}\frac{1}{x^i-x^j} - \sum_{j=1}^{N-1}\frac{1}{x^i-e_j} - \sum_{j\neq i}\frac{\bar{g}_{ii}}{(x^i-x^j)\bar{g}_{jj}}\right] \qquad (3.2.26c)$$

where

$$\bar{g}_{ii} = \frac{\prod_{j\neq i}(x^i-x^j)}{\prod_{j=1}^{N-1}(x^i-e_j)}$$

The $\bar{g}_{ii}$'s are the metric coefficients for parabolic coordinates. Comparison of (3.2.26c) with (3.2.26a) shows that for these coordinates $\zeta=0$ and A is arbitrary, a result already obtained in the real case. The case where the $\sum_j\lambda_j=N-1$ can be treated in the same manner as that for elliptic coordinates: by letting $e_j\to e_k$ etc. The results are identical: $\zeta=0$ and A arbitrary($A=\gamma$ say).

If $\sum_j \lambda_j \leq N-2$, a rerun of the argument for $\sum_j \lambda_j = N-1$ gives the same results: $\zeta=0$ and $A=\gamma$ where γ is arbitrary. The remaining conditions (3.2.18) of Theorem 3.2.1 have been satisfied. Q.E.D.

When $\sum_j \lambda_j = N$ Theorem 3.2.1 implies that $V=\zeta\sum x^j + B = \sum_{i=1}^{m} \zeta(z^i)^2/4$: the coordinate transformations are completely determined.

If $\sum \lambda_j \leq N-1$ Theorem 3.2.1 implies that $\zeta=0$ but the γ_i terms are still unknown. The expression of potentials of the form V in (3.2.17), in terms of the cartesian coordinates z^i, for orthogonal systems with graphs of form (3.2.8) has been given in Boyer et al. (1983). There it was shown that

$$\sum_1^N x^i - \sum_{j=1}^{p} \lambda_j e_j = \begin{cases} \sum_1^m (z^i)^2/4 \ , & \text{if } \lambda = \sum_{i=1}^{p} \lambda_j = N \quad (3.2.27a) \\ {}^{*}z^1 \ , & \text{if } \lambda = \sum_{i=1}^{p} \lambda_j = N-1 \quad (3.2.27b) \\ {}^{**}z^1 - iz^2 \ , & \text{if } \lambda = \sum_{i=1}^{p} \lambda_j \leq N-2 \quad (3.2.27c) \end{cases}$$

* - z^1 can be determined from Boyer et al. (1983). There the z^i are derived by limiting processes from the z^i's for elliptic coordinates.

** - Once again the particular z^i meant here can be derived by limiting processes using the work of Boyer et al. (1983). One way of detecting z^1 and z^2 is to notice that they are the only coordinates such that $z^1 - iz^2$ has the sum form given in (3.2.27b).

In the two and three dimensional cases the results of (3.2.27) hold even when there are spaces A_j attached (i.e. when the systems have graphs of form (3.2.9)). Indeed we conjecture that (3.2.27) is still valid for any orthogonal separable system on $E(n,\mathbb{C})$. One reason, though not a proof, is that V is independent of the coordinates on the attached spaces A_j.

Equation (3.2.27) can be used to give an alternative derivation of the results for $\lambda = N$ given in Theorem 3.2.1. Using (3.2.27a)

$$V = A\sum_1^N x^i + B$$

$$= A\left(\sum_1^N x^i - \sum_1^p \lambda_j e_j\right) \qquad (3.2.28)$$

since there is no loss in assuming that $B=-A\sum_1^p \lambda_j e_j$. ($B$ can be easily altered by the transformations $x^n \to x^n + Cx^{n-1}$). Using (3.2.27a) $V = A\sum_1^m (z^i)^2/4$ and from (3.2.2)

$$A = \zeta \tag{3.2.29}$$

the result already obtained in Theorem 3.2.1.

If $\lambda = N-1$ (and no spaces A_j are attached) then (3.2.27b) implies

$$V = A\left[\sum_1^N x^i - \sum_1^p \lambda_j e_j\right]$$

$$= Az^1 \tag{3.2.30}$$

so that

$$A = \gamma_1 , \quad \gamma_j = 0 , \quad j \neq 1 \tag{3.2.31}$$

If $\lambda \leq N-2$ (and no spaces A_j are attached) then

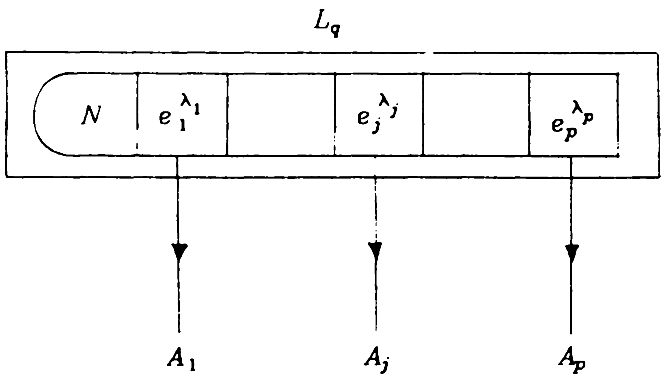
$$V = A(z^1 - i\, z^2)$$

$$= \gamma_1 z^1 + \gamma_2 z^2 \tag{3.2.32}$$

where $A=\gamma_1$, $\gamma_2 = -i\gamma_1$ and $\gamma_j \neq 0$ for $j \neq 1,2$.

The conditions of Theorem 3.2.1 readily yield the results for the real case. If the leading block is of elliptic-type then $\lambda_j = 1$ for $j=1,\ldots,N$. Thus from (3.2.18) and (3.2.27a) we obtain the conditions for the elliptic case (see (2.4.43A) and (2.7.24)). The discrepancy in the sign of $\sum\lambda_j$ between the V of (3.2.27) and that for the real parabolic case (2.4.50) is caused by the definition that was taken for the cartesian coordinates (2.4.35B). For consistency here we could have chosen $-\sum_1^{N_r-1} e_j$ in the definition of ${}_{N_r}w_1$ instead of $\sum_1^{N_r-1} e_j$ without any change in the infinitesimal distance.

In parallel with §2.5 a graphical calculus can be devised to represent these results. The rules are much the same. Analogous to the elliptic-type A coordinates is the graph

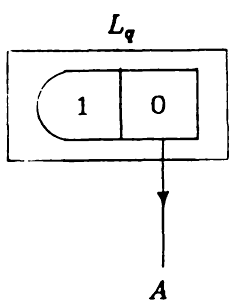


(3.2.33)

where $\sum_{j=1} \lambda_j = N$. Here L_q can take any of the values $I \to IV$, and $\gamma_i = 0$.

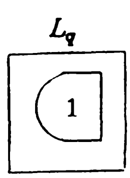
In analogy to the real parabolic case, all the other graphs have form (3.2.33) with $\sum_{j=1} \lambda_j < N$. Here L_q can only be I or II and $\zeta = 0$ always.

As in our real treatment the case $N = 1$ is an exception. When this happens there are two possibilities. The first parallels the elliptic one dimensional case and has graph



(3.2.34A)

where L_q can be I, II, III or IV. Paralleling the parabolic case is the system with graph



(3.2.34B)

where L_q could be I or II. As there are no roots no spaces can be attached to this graph. Since (1 | 0) and (1) are equivalent by the scaling transformation

(2.1.3a), both linear and quadratic potentials are possible for $N=1$.

To illustrate the above procedure the classification is carried out for $m=1$ and 2.

If $m=1$ the R-separable systems are those listed in Table 1 of Appendix A, but with understanding that the variables x^i, y^i and t are now complex.

If $m=2$ then there are six orthogonal separable systems on $E(2,\mathbb{C})$. Four of these are the complexifications of the systems in Table 2 of Appendix A. There are a further two truly complex systems and these are listed in Table 7 of the same Appendix. System A of that table corresponds to the degenerate form of elliptic coordinates obtained when the roots have become equal. Since $\lambda=\lambda_1+\lambda_2=2$, all the cases L_q: $I \to IV$ are possible. These are listed in Table 8 of Appendix A. The real forms corresponding to this case are obtained by making the replacements $z^1 \to u^1$, $z^2 \to i\,u^2$ (u^1 and u^2 real). The R-separable systems under these replacements are for

$$(\partial_{u^1u^1} - \partial_{u^2u^2})\Psi + 2\varepsilon\partial_t\Psi = E\Psi \tag{3.2.35}$$

This is an example of what was referred to as a real form in §3.1. Using this technique systems derived from metrics with varying signatures can be classified. (However not all complex systems have corresponding real forms).

System B of Table 7 is the degenerate form of elliptic coordinates obtained when both roots have been taken to infinity. Since $\lambda=0$, only the σ-types I and II are possible. The γ_i terms attach themselves to the z^1-iz^2 coordinate since this is x^1+x^2.

The mixing types are the complexifications of those in Table 4 of Appendix A.

Operators: The general formula (2.6.18) is equally applicable in the complex case. As in the real case the essential problem is to work out the the images of members of the enveloping algebra characterising orthogonal systems.

In general any member of the enveloping algebra will have the form (B6). The operators for orthogonal complex systems can be derived from Kalnins, Miller and Reid (1983).

In the following table the results of Table 2.6.1 are generalised to a larger class of members of the enveloping algebra needed in the complex case.

Table 3.2.1 Images of Euclidean Operators

L_q	**Images of the Euclidean Operator $P_a P_b$**
I	$P_a P_b + \frac{1}{2}\varepsilon(\gamma_a B_b + \gamma_b B_a)$
II	$\{v_q P_a - B_a, v_q P_b - B_b\} + \frac{1}{2}\varepsilon(\gamma_a P_b + \gamma_b P_a)$
III	$v_q P_a P_b - \frac{1}{2}[\{P_a, B_b\} + \{P_b, B_a\}]$
IV	$\{v_q P_a - B_a, v_q P_b - B_b\} + w_q^2 P_a P_b$
	Images of the Euclidean Operator $\{M_{ab}, M_{cd}\}$ (if $\gamma_a = \gamma_b = \gamma_c = \gamma_d = 0$)
L_q	$\{M_{ab}, M_{cd}\}$ $(L_q : I \rightarrow IV)$
	Images of the Euclidean Operator $\{M_{qb}, P_b\}$ (if $\gamma_b = 0$)
I	$\{M_{qb}, P_b\} - \gamma_q B_b^2 / 4$
II	$\{M_{qb}, v_q P_b - B_b\} - \gamma_q P_b^2 / 4$
III,IV	Do not occur
	Images of the Euclidean Operator $\{M_{ab} - iM_{ac}, P_b - iP_c\}$ (if $\gamma_a \neq 0$ and $\gamma_b = \gamma_c = 0$)
I	$\{M_{ab} - iM_{ac}, P_b - iP_c\} - \gamma_a (B_b - iB_c)^2 / 4$
II	$\{M_{ab} - iM_{ac}, v_q(P_b - iP_c) - (B_b - iB_c)\} - \gamma_a (P_b - iP_c)^2 / 4$
III,IV	Do not occur
	Images of the Euclidean Operator $\{M_{ab}, P_a - iP_b\}$ (if $\gamma_b = -i\gamma_a$)
I	$\{M_{ab}, P_a - iP_b\} - i\gamma_a (B_a - iB_b)^2 / 4$
II	$\{M_{ab}, v_q(P_a - iP_b) - (B_a - iB_b)\} - i\gamma_a (P_a - iP_b)^2 / 4$
III,IV	Do not occur

When $a=b$ the results of the real case in Table 2.6.1 are reproduced. As in §2.6 notation has been abused. The Euclidean operator P_a in Table 3.2.1 is ∂_{z^a} while its image in the above table is $P_a=\partial_{y^a}$.

As in §2.6 we provide the details of one of the calculations needed to produce Table 3.2.1. Suppose we are dealing with Euclidean constants of the motion containing the terms

$$M_{ab}\,M_{cd} \tag{3.2.36}$$

Suppose that $\gamma_a=\gamma_b=\gamma_c=\gamma_d=0$. Then

$$y^u=\sigma^{\frac{1}{2}}z^u \quad , \; u=a,b,c,d.$$

Equation (2.6.18), the formula for computing the image λ_I, gives

$$\lambda_I \;=\; \sigma \mathbf{P}^t\, \tilde{\Lambda}^E\, \mathbf{P} - \mathbf{P}^t\, \varepsilon\, \sigma' \sigma^{\frac{1}{2}}\, \tilde{\Lambda}^E\, \mathbf{z} \tag{3.2.37}$$

where $\mathbf{z}=(z^1,z^2,\cdots)^t$. By going through the possibilities for a,b,c,d we find that

$$\tilde{\Lambda}^E\,\mathbf{z} \;=\; 0 \quad (0 \textit{ is the zero vector}). \tag{3.2.38}$$

For example consider a,b,c,d to be all distinct. There is no loss in assuming $a=1$, $b=2$, $c=3$, $d=4$.

The matrix $\tilde{\Lambda}^E$ is

$$\begin{pmatrix} 0 & 0 & \frac{1}{2}z^2z^4 & -\frac{1}{2}z^2z^3 & 0 & \cdots \\ 0 & 0 & -\frac{1}{2}z^1z^4 & \frac{1}{2}z^1z^3 & & \\ \frac{1}{2}z^2z^4 & -\frac{1}{2}z^1z^4 & 0 & 0 & & \\ -\frac{1}{2}z^2z^3 & \frac{1}{2}z^1z^3 & 0 & 0 & & \\ 0 & 0 & 0 & 0 & & \\ & & & & \ddots & \end{pmatrix}$$

It is easily verified that

$$M_{12}M_{34} \;=\; \mathbf{P}^t\tilde{\Lambda}^E\,\mathbf{P} \tag{3.2.39}$$

Direct calculation shows that (3.2.38) is satisfied. From (3.2.37)

$$\lambda_I = \sigma \mathbf{P}^t \tilde{\Lambda}^E \mathbf{P} = M_{ab} M_{cd} \tag{3.2.40}$$

Thus the row containing M_{ab}^2 in Table 2.6.1 is generalised in the complex case to

$$\{M_{ab}^E \, , \, M_{cd}^E\} \to \{M_{ab} \, , \, M_{cd}\} \tag{3.2.41}$$

under the condition that $\gamma_a = \gamma_b = \gamma_c = \gamma_d = 0$, regardless of whether L_q is I, II, III, or IV. The E superscript has been used in (3.2.41) to emphasise that the operators on the left hand side of that equation are the Euclidean operators while those on the right are their images. The work of Kalnins et al. (1983) implies that all the operators on the attached spaces are sums of elements of form (3.2.36). The images of these operators are determined by (3.2.41).

Consider the case when $m=3$. The defining operators for the corresponding orthogonal $E(3,\mathbb{C})$ systems can be derived from the work of Kalnins et al. (1983). Alternatively they are listed in Kalnins and Miller (1976).

When $\sum \lambda_j = N = 3$ all the operators are of the form $\sum_{a,b,c,d} A^{abcd} M_{ab} M_{cd} + C^{ab} P_a P_b$ $(1 \le a,b,c,d \le 3)$. The images of these operators can be obtained from Table 3.2.1.

When $\sum \lambda_j = N-1 = 2$ then we need to compute the images of the two further operators $\{M_{ab} - iM_{ac}, P_b - iP_c\}$ and $\{M_{qb}, P_b\}$.

If $\sum \lambda_j \le N-2$ the only new Euclidean operator whose image needs to be found is $\{M_{ab}, P_a - iP_b\}$.

In three dimensions at least, the above observations show that Table 3.2.1 is sufficient to work out all the operators. We conjecture that Table 3.2.1 is sufficient to work out all the operators in m dimensions.

Example: consider system 6 of Table 8. The complex orthogonal system from which it is derived is system B of Table 7 in Appendix A. The defining operators for that orthogonal system are also supplied in the same table. They are

$$P_1^2 + P_2^2 \quad , \quad -i\{M_{12}, P_1 - iP_2\} + \frac{1}{4}(P_1 + iP_2)^2 \tag{3.2.42}$$

To find the images of these operators we compute the following:

$$\{M_{12}, P_1 - iP_2\} \to \{M_{12}, -(B_1 - iB_2)\} - i\gamma_1 (P_1 - iP_2)^2/4 \ ,$$

$$P_b^2 \to B_b^2 + \varepsilon \gamma_b P_b \quad , \quad b = 1,2,$$

$$P_1 P_2 \to \{B_1, B_2\} + \frac{1}{2}\varepsilon \gamma_1 (P_2 - i\, P_1) \tag{3.2.43}$$

These results allow us to find the two operators for system 6 which are given in Table 8.

3.3 Non Heat-type systems for the Helmholtz equation

We find a general class of contravariant metrics with one nonorthogonal entry $g^{(n-1)n}$. When $g^{(n-1)n}=1$ the class yields the systems of §3.2. But when $g^{(n-1)n}\neq 1$ the resulting metrics are not of heat-type. Only a sketch of this latter case is provided: the operators and coordinate transformations are not given.

We give as a generalisation of (2.4.1) the class of Hamilton-Jacobi equations

$$E = \sum_{c\in C} g^{cc}p_c^2 + \frac{1}{\prod_{a\in A} x^a}\times \tag{3.3.1}$$

$$\times\left\{\sum_{q\in Q}\frac{1}{\sigma_q}\sum_{b\in B_q}\bar{g}^{bb}p_b^2 + 2p_{n-1}p_n + \left[\sum_{q\in Q}\frac{V_q}{\sigma_q} - \tau^2\prod_{a\in A}x^a\sum_{a\in A}\frac{g^{aa}}{(x^a)^4}\right]p_n^2\right\}$$

Here the V_q, σ_q have the same significance they had in Theorem 2.4.1 and τ is an arbitrary constant. Also $A=\{1,2,\ldots,n_A\}$ is a subset of $C=\{1,2,\ldots,n_A,\ldots,n_C\}$. The sets Q and B_q are the same as those defined in (2.4.2) and (2.4.3). Note that now $q_1=n_C+1$.

Using the methods of §2.4 the orthogonal metric ds_C^2 corresponding to $\sum_{c\in C} g^{cc}p_c^2$ can be shown to be the metric of a flat space. That space *splits* into a sum of flat disjoint subspaces each with metric of form (3.2.11). One of these, is taken to be

$$ds_A^2 = \sum_{a\in A}\frac{\prod_{a^*\neq a}(x^a-x^{a^*})}{\prod_{j=1}^{p}(x^a-e_j)^{\lambda_j}}(dx^a)^2 + \sum_{j=1}^{p}\frac{\prod_{a\in A}(x^a-e_j)}{\prod_{k\neq j}(e_k-e_k)}ds_j^2 \tag{3.3.2}$$

where $1\le a,a^*\le n_A$. We also suppose that the root e_1 has multiplicity at least two. There is no loss in taking $e_1=0$

This hefty notation is now illustrated with a few examples. If $n_A=0$, (3.3.1) collapses to the form of (2.4.1). The most simple example occurs when $n_A=m=2$ i.e. on $E(4,\mathbb{C})$. The Hamilton-Jacobi equation is then

$$\frac{(x^1)^2p_1^2-(x^2)^2p_2^2}{(x^1-x^2)} + \frac{2}{x^1x^2}p_3p_4 - \frac{\tau^2}{(x^1-x^2)}\left[\frac{1}{(x^1)^2} - \frac{1}{(x^2)^2}\right]p_4^2 = E \quad (3.3.3)$$

with corresponding metric

$$ds^2 = (x^1-x^2)[(dx^1/x^1)^2-(dx^2/x^2)^2] + 2x^1x^2dx^3dx^4-\tau^2(x^1+x^2)(dx^3)^2 \quad (3.3.4)$$

This system appeared in Kalnins and Miller (1979) together with its coordinate transformations and defining operators. As they noted at the time it was the single nonorthogonal system they found which was not of heat-type. In general any system with $g^{(n-1)n}\neq 1$ and one first order variable is not of heat type. This can be explained as follows. If a system is of heat-type then it must have the symmetry operator $\frac{1}{2}(P_{n-1}-iP_n)$. It follows that its coordinate transformations are of form (2.2.8). A simple modification of the argument at the beginning of §2.3 (see (2.3.3)-(2.3.10)) shows that $g^{(n-1)n}$ can be scaled to 1.

To explain the technical notation used in (3.3.1) we give the next simplest example. This occurs when $n_A=2$ and $n=5$. The Hamilton-Jacobi equation is then

$$E = \frac{(x^1)^2p_1^2-(x^2)^2p_2^2}{(x^1-x^2)} + \quad (3.3.5)$$

$$+ \frac{1}{x^1x^2}\left\{\frac{p_3^2}{\sigma_3(x^4)} + 2p_4p_5 + \left[\frac{\tau^2(x^1+x^2)}{x^1x^2} + \frac{V_3}{\sigma_3}\right]p_5^2\right\}$$

with metric

$$ds^2 = (x^1-x^2)[(dx^1/x^1)^2-(dx^2/x^2)^2] + x^1x^2\sigma_3(dx^3)^2 + 2x^1x^2dx^4dx^5$$

$$- [\tau^2(x^1+x^2) + x^1x^2\frac{V_3}{\sigma_3}](dx^4)^2 \quad (3.3.6)$$

The curvature condition $R_{3443}=0$ is equivalent to

$$2\sigma_3''\sigma_3 - (\sigma_3')^2 = \zeta_3 + \tau^2\sigma_3^2 \quad (3.3.7)$$

where

$$\tfrac{1}{2}\zeta_3 = V_{3,33} \quad (3.3.8)$$

Differentiating (3.3.7) gives

$$\sigma_3''' - \tau^2 \sigma_3' = 0 \tag{3.3.9}$$

Solving this equation for σ' and substituting back into (3.3.7) gives

$$\sigma_3 = \begin{cases} 1 \ , \quad \text{or} & (3.3.10a) \\ U_3 e^{\tau x^4} + W_3 e^{-\tau x^4} + \sqrt{4 U_3 W_3 - \frac{\zeta_3}{\tau^2}} & (3.3.10b) \end{cases}$$

The constants U_3 and W_3 are arbitrary. From (3.3.8)

$$V_3 = \frac{\zeta_3 (x^3)^2}{4} + \gamma_3 x^3 + \delta \tag{3.3.11}$$

Notice that if $\sigma = 1$ then (3.3.7) implies $\zeta_3 = -\tau^2$. We have completely determined the unknown functions in the metric (3.3.6). This system is not of heat-type because the component $g^{45} = 1/x^1 x^2$ in (3.3.5) can not be scaled to 1. It is an example of a system with one first order variable not of the type in §3.2.

We will now determine the V_q and the σ_q for the general class of Hamilton-Jacobi equations in (3.3.1). Since the root $e_1 = 0$ has multiplicity at least 2 then from (3.3.2)

$$g^{aa} = \frac{\sum_{i=0}^{n_A - 2} r_i (x^a)^i (x^a)^2}{\prod_{a^* \neq a} (x^a - x^{a^*})} \tag{3.3.12}$$

The r_i's are constants which can be determined from (3.3.2) ($\sum_{i=0}^{n_A-2} r_i (x^a)^{i+2}$ $= \prod_{j=1}^{p} (x^a - e_j)^{\lambda_j}$). The following identities are verified by using partial fractions:

$$\sum_{a \in A} \frac{(x^a)^v}{\prod_{a^* \neq a} (x^a - x^{a^*})} = \begin{cases} 0 & , \ v = 0, \ldots, n_A - 2 \\ \dfrac{(-1)^{n_A}}{\prod_{a \in A} x^a} & , \ v = -1 \\ \dfrac{(-1)^{n_A}}{\prod_{a \in A} x^a} \sum_{a \in A} \dfrac{1}{x^a} & , \ v = -2 \end{cases} \tag{3.3.13}$$

Using these identities

$$\sum g^{aa}\frac{1}{(x^a)^4} = \frac{(-1)^{n_A}}{\prod_{a\in A} x^a}\left[r_0\sum_{a\in A}\frac{1}{x^a} + r_1\right] \tag{3.3.14}$$

The metric corresponding to (3.3.1) can now be written

$$ds^2 = \sum_{c\in C} g_{cc}(dx^c)^2 + \prod_{a\in A} x^a \Big(\sum_{q\in Q}\frac{1}{\sigma_q}\sum_{b\in B_q}\bar{g}_{bb}(dx^b)^2 +$$

$$+ 2dx^{n-1}dx^n + [r_0\tau^2(-1)^{n_A}\sum_{a\in A}\frac{1}{x^a} - \sum_{q\in Q}\frac{V_q}{\sigma_q}](dx^{n-1})^2 \Big)$$

(3.3.15)

The r_1 term has been removed by the transformation $x^n \to x^n - \frac{1}{2}(-1)^{n_A} r_1 x^{n-1}$. From (3.3.12)

$$r_0 = \prod_{a\in A} x^a(-e_a)^{\lambda_a} \tag{3.3.16}$$

The conditions $R_{b(n-1)(n-1)c} = 0$, $b \neq c$ imply,

$$V_q = \sum_{b\in B_q} v_b(x^b) \quad . \tag{3.3.17}$$

as in §3.2. The curvature conditions $R_{b(n-1)(n-1)b} = 0$ are equivalent to

$$2\sigma_q''\sigma_q - (\sigma_q')^2 = \zeta_q + \tau^2 r_0^2 \sigma_q^2 \tag{3.3.18}$$

The spaces with metrics $ds^2 = \sum_{b\in B_q}\bar{g}_{cc}(dx^c)^2$ are flat (using the methods of §2.4). In terms of the standardised coordinates z^b on each of these spaces

$$\frac{\zeta_q}{2} = V_{q,bb} \tag{3.3.19}$$

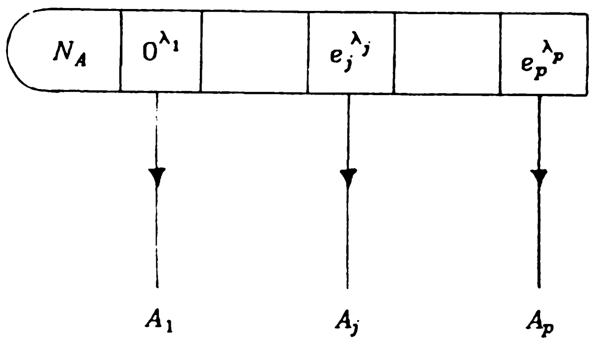
These equations have the solution as in (3.3.7) and (3.3.8). Thus

$$\sigma_q = \begin{cases} 1\ , \quad \text{or} & (3.3.20a) \\ U_q e^{\tau x^{n-1}} + W_q e^{-\tau x^{n-1}} + \sqrt{4U_q W_q - \frac{\zeta_q}{\tau^2}} & (3.3.20b) \end{cases}$$

The constants U_q and W_q are arbitrary. From (3.3.19)

$$V_q = \sum_{b\in B_q}[\ \zeta_q(z^b)^2/4 + \gamma_b z^b\ ] \tag{3.3.21}$$

We will show how the above systems can be represented with graphs similar to those in §3.2. Following our discussion in §3.2 the metric $ds_A{}^2 = \sum_A g_{aa}(dx^a)^2$ has graph



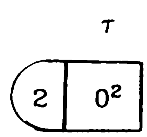
(3.2.22)

where $\lambda_1 \geq 2$. The separable systems $ds_q^2 = \sum_{b \in B_q} \bar{g}_{bb}(dx^b)^2$ are flat and can also be represented by the graphs given in §3.2. The compatibility problem for the potential terms V_q has the same solution as in §3.2. (the determining equations for V_q, (3.3.17) and (3.3.21), are the same as (3.2.12) and (3.2.2)).

The labelling of the σ-functions depends on the value of τ in (3.3.18). If $\tau=0$ then σ can be any of the types I, II, III or IV. Incidentally, here the metric collapses to that given in §3.2. If $\tau \neq 0$ then σ is of form (3.3.19a) (i.e. type I) or of form (3.3.19b). This latter form will be called type V. A σ function of type V is exactly specified by U_q, W_q and ζ_q. The potential V_q attached to this block is specified by all these parameters in addition to the γ_b's. Thus the label given to such a block is

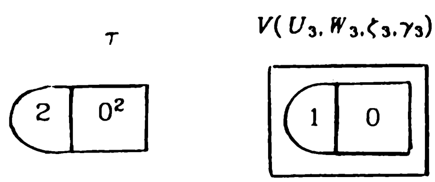
$$L_q = V(U_q, W_q, \zeta_q, \gamma_b) \tag{3.2.23}$$

As an example the system (3.3.3) has graph



(3.2.24)

(τ is attached to the leading block). The system (3.3.5) has graph



(3.2.25)

The Schrödinger conditions for the class covered in this section can also be verified ,i.e., all these systems are also Helmholtz separable. The operators and the coordinate transformations have not been calculated but the same technique used in §3.2 should help in finding these details, i.e., standardisation of the coordinates in the manner of §2.4 (see (2.4.27)-(2.4.30)) to find the general form of the coordinate transformations. Also one might be able to construct a general formula like (2.6.18) for calculating the operators from the Euclidean ones on the space $E(n-2,\mathbb{C})$.

The techniques used to derive the general class (3.3.1) are direct ones requiring arduous calculations involving the Stäckel and curvature conditions. The effectiveness of these methods is heavily dependent on the metric having only one nonorthogonal entry. It seems difficult to generalise these techniques to solve the general first order problem.

3.4 R-separation for the Hamilton-Jacobi equation

Initially the aim of this thesis was to classify all heat-type separable systems on $E(\mathbb{C},5)$. This aim was abandoned in favour of the complete results obtained for the case of one first order variable. Only partial results were obtained beyond this case. Nonetheless this partial classification yielded the particularly interesting class which is the subject of this section. This class gives rise to the first instance, to our knowledge, of nontrivial R-separation for the Hamilton-Jacobi equation

$$H = g^{ij} p_i p_j = E \quad , \qquad E \neq 0 . \tag{3.4.1}$$

This class is an extension of that which was the subject of an article by Kalnins and Reid (1982).

By "R-separation" we mean that there is a nonzero function $R(\mathbf{x})$ such that (3.4.1) admits a complete solution of the form

$$W = R(\mathbf{x}) + \sum_{i=1}^{n} W_i(x^i; E, \lambda_2, \quad \ldots, \lambda_n) \tag{3.4.2}$$

As in the corresponding definition for the Helmholtz equation R need not depend on the separation constants λ_i. The R-separation is said to be trivial if $\partial^2 R / \partial x^i \partial x^j = 0$, for all $i \neq j$.

Before the work of Kalnins and Miller (1980) R-separation was only thought to exist for the Laplace equation. These investigators have found nonorthogonal systems for the Helmholtz equation which are nontrivially R-separable.

The class of metrics we consider is the one which is the subject of the MACSYMA example given in Appendix B. It is the class with coordinate transformations (B38), Hamilton-Jacobi equation (B41), metric (B43) and is uniquely defined by the Killing tensors (B47).

E. Kalnins observed that if the transformation

$$x^5 \to \tilde{x}^5 + b\log(x^2 - x^1) \tag{3.4.3}$$

is made in (B38) then R-separation results for the Hamilton-Jacobi equation in the new coordinates $x^1, x^2, x^3, x^4, \tilde{x}^5$. Since $z^5 = \tilde{x}^5$, this new Hamiltonian is (B41) with $b = 0$, i.e.

$$\tilde{H} = P\, p_3^2 + \frac{2}{(x^1 - x^2)}[p_2(p_4 + x^2 p_3) + p_1(p_4 + x^1 p_3)] + p_5^2 \tag{3.4.4}$$

where $P = -Q(x^1+x^2) + R(x^1+x^2)/(x^2-x^1) + S/(x^2-x^1)$. In detail suppose

$$W = W_1(x^1) + W_2(x^2) + \lambda_3 x^3 + \lambda_4 x^4 + \lambda_5 x^5 \tag{3.4.5}$$

is a separable solution for (B41). Substituting for x^5 from (3.4.3) into (3.4.5)

$$\tilde{W} = \alpha\log(x^2 - x^1) + W_1(x^1) + W_2(x^2) + \lambda_3 x^3 + \lambda_4 x^4 + \lambda_5 \tilde{x}^5 \;,$$

$$(\alpha = \lambda_5 b)\,, \tag{3.4.5}$$

is an R-separable solution of (3.4.4). The parameter α is arbitrary. If $\alpha = 0$ the R-separation is trivial. When $\alpha \neq 0$ the R-separation is nontrivial since $\partial_{x^1 x^2} R \neq 0$ for the R-separation factor $R = \alpha\log(x^1 - x^2)$. The separation equations for $\tilde{W}$ are derived from (B44) with $\lambda_5 b$ replaced by α. They are

$$2(\lambda_4 + x^1\lambda_3)p_1 = -2\alpha\lambda_3 + \lambda_3^2(S + Rx^1 + Q(x^1)^2) - x^1\lambda_5^2 + \lambda_2 + \lambda_1 x^1 \;,$$

$$2(\lambda_4 + x^2\lambda_3)p_2 = \lambda_3^2(Rx^2 - Q(x^2)^2) + x^2\lambda_5^2 - \lambda_2 - \lambda_1 x^2 \;,$$

$$p_\alpha = \lambda_\alpha \;, \qquad \alpha = 3,4,5 \,. \tag{3.4.7}$$

There is no loss in taking $\lambda_1 = E$

The Killing tensors describing this R-separation are derived from those in (B47), by replacing $\lambda_5 b$ with α. They are

$$\lambda_2 = -\alpha(P_3 - iP_4) + K + Q\lambda_Q + R\lambda_R + S\lambda_S \ ,$$

$$\lambda_3 = P_1 - iP_2 \ ,$$

$$\lambda_4 = -\frac{1}{2}(M_{13} - iM_{23}) + \frac{i}{2}(M_{14} - iM_{24}) \ ,$$

$$\lambda_5 = P_5 \ , \tag{3.4.8}$$

where

$$K = M_{13}(P_1 - iP_2) - iM_{34}(P_3 - iP_4) + P_2(M_{14} - iM_{24}) + iM_{12}P_3 \ ,$$

$$\lambda_Q = -\frac{1}{4}[(M_{13} - iM_{23}) - i(M_{14} - iM_{24})]^2 \ ,$$

$$\lambda_R = \frac{1}{2}(P_3 - iP_4)[(M_{13} - iM_{23}) - i(M_{14} - iM_{24})] \ ,$$

$$\lambda_S = -\frac{1}{2}(P_1 - iP_2)(P_3 - iP_4) \tag{3.4.9}$$

Some additional explanation is needed here. The freedom to choose an equivalent set of Killing tensors $\sum_j C_{ij}\lambda_j$ for (B47) has been used. For example in the above set λ_3 is the negative of λ_3 in (B47). Simplification with the aid of the identities (B15) and (B28) has also taken place.

In addition the result has implications for the corresponding Helmholtz equation. If $\Psi = \Psi_1(x^1)\Psi(x^2)\ e^{\sum_3^5 \lambda_\alpha x^\alpha}$ is a separable solution of the Helmholtz equation (B47) then under the transformation (3.4.3)

$$\Psi = (x^2 - x^1)^\alpha \Psi_1(x^1)\Psi(x^2) e^{\lambda_3 x^3 + \lambda_4 x^4 + \lambda_5 x^5} \ , \ (\alpha = \lambda_5 b) \ , \tag{3.4.10}$$

will be a solution of the transformed Helmholtz equation (which is (B42) with $b = 0$). Again we have nontrivial R-separation with a family of R-separation factors $M_\alpha(\mathbf{x}) = (x^2 - x^1)^\alpha$ depending on the parameter α. The operators characterising this separation are as usual obtained from the correspondence (1.2.20), i.e.,

$$L_2 = -\alpha(P_3 - iP_4) + \tilde{K} + Q\tilde{\lambda}_Q + R\tilde{\lambda}_R + S\tilde{\lambda}_S \ ,$$

$$L_3 = P_1 - iP_2 \ ,$$

$$L_4 = -\frac{1}{2}(M_{13}-iM_{23})+\frac{i}{2}(M_{14}-iM_{24}) \ ,$$

$$L_5 = P_5 \ , \tag{3.4.11}$$

where now $P_i=\partial/\partial z^i$, $M_{ij}=z^i\partial_{z^j}-z^j\partial_{z^i}$ and

$\tilde{\lambda}_R = \frac{1}{2}\{P_3-iP_4, (M_{13}-iM_{23})-i(M_{14}-iM_{24})\}$ etc.

The operator L_2 is "non-self-adjoint" for $\alpha\neq 0$. (The adjoint A^* of an operator A is defined by

$$\langle A^*f \ , h\rangle = \langle f \ , Ah\rangle \tag{3.4.12}$$

A second order operator S is said to be self adjoint if $S^*=S$). In the case of pure separation, $\alpha=0$ and the operator L_2 is self-adjoint. If we are to have a satisfactory theory of variable separation we must be able to account for this phenomenon and classify it when it occurs.

This section has been of mainly theoretical interest, and is a generalisation of a class of four dimensional systems which appeared in an article by Kalnins and Reid (1982). That class is obtained from ours by diagonalising P_5 and taking $Q=R=S=0$. The essential elements are the same: nontrivial R-separation depending on a free parameter and Helmholtz separation characterised by a non-self-adjoint operator. We have also managed to fill a gap in the classification of Kalnins and Miller (1979) by providing the operators for the four dimensional system obtained by diagonalising P_5 when Q, R and S are nonzero.

Appendices

Appendices

Appendix A Tables of R-separable systems

The main purpose of this appendix is to summarise in tabular form results concerning R-separation for $m = 1, 2$ and 3.

In all the tables $\sigma_{II} \to \sigma_{IV\pm}$ are the functions of Table 2.4.1, that is

$$\sigma_{II} = (x^{m+1}+v)^2 ,$$

$$\sigma_{III} = |x^{m+1}+v| ,$$

and

$$\sigma_{IV\pm} = |(x^{m+1}+v)^2 \pm w^2| , \quad x^{m+1} = t \tag{A1}$$

As we have already mentioned

$$\tau = sign((x^{m+1}+v)^2 \pm w^2). \tag{A2}$$

In the tables for the unsplit systems we have unambiguously removed the subscripts from the parameters so that $v_1 \to v$, $w_1 \to w$ and $\gamma_1 \to \gamma$.

The general forms for elliptic and parabolic coordinates in (2.4.35) can be transformed to give their more familiar appearance in low dimensions. For example we make the transformations

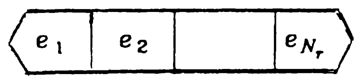
$$\sqrt{x^1} \to \cosh(x^1), \ \sqrt{x^2} \to \cos(x^2) \tag{A3}$$

to give elliptic coordinates their 'usual' appearance in Table 2. It must be remembered however that in higher dimensions this is not always possible. For cases like the ellipsoidal coordinates of Table 5 the general forms of (2.4.35) must be used. In Table 5, $J_1 = M_{32}$, $J_2 = M_{13}$ and $J_3 = M_{21}$.

In the tables some of the parameters e_r have been normalised. In general by making the transformations

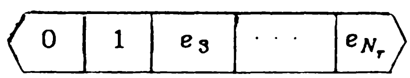
$$x^i \to ax^i + b\,, \quad e_i \to ae_i + b \tag{A4}$$

an elliptic block



(A5)

can become



(A6)

Analogous remarks apply to parabolic blocks and to those representing separable systems on the spheres S_{p_b}. As a final remark, the split types for $m = 3$ may be obtained from the results for $m = 1$ and $m = 2$(i.e. from Tables 1 and 3). For a brief discussion see (2.5.19) → (2.5.22).

Table 1 R-separable Coordinates and Operators for

$$(\partial^2_{y^1} + 2\varepsilon\partial_t)\Psi = E\Psi$$

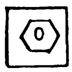
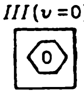
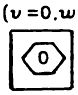
	Graph	Coordinates $\{y^1, t\}$ $t = x^2$ for *all systems*	Operator R-separation factor R
1.	I 	$y^1 = x^1 + \gamma(x^2)^2/4$	$P_1^2 + \gamma\varepsilon B_1$ $R = \varepsilon\gamma x^1 x^2/2$
2.	$II(v=0)$ 	$y^1 = \sigma_{II}^{1/2} x^1 + \gamma/4(x^2+v)$	$(vP_1 - B_1)^2 + \gamma\varepsilon P_1$ $R = \frac{1}{2}\varepsilon[(x^2+v)(x^1)^2$ $-\gamma x^1/4(x^2+v)]$
3.	$III(v=0)$ 	$y^1 = \sigma_{III}^{1/2} x^1$	$vP_1^2 - \{P_1, B_1\}$ $R = 0$
4.	$IV\pm(v=0, w^2=1)$ 	$y^1_\pm = \sigma_{IV\pm}^{1/2} x^1$	$(vP_1 - B_1)^2 \pm w^2 P_1^2$ $R = \tau_\pm \varepsilon (x^1)^2 x^2/2$

Table 2 Separable Coordinates and Operators for

$$p_1^2 + p_2^2 = E$$

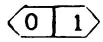
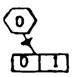
Name & Graph	Coordinates	Operator
Elliptic G_e ≡ 	$z_e^1 = c\cosh(x^1)\cos(x^2)$ $z_e^2 = c\sinh(x^1)\sin(x^2)$	$M_{12}^2 + c^2 P_1^2$
Parabolic G_p ≡ 	$z_p^1 = \frac{1}{2}[(x^1)^2 - (x^2)^2]$ $z_p^2 = x^1 x^2$	$\{M_{12}, P_2\}$
Polar G_r ≡ 	$z_r^1 = x^1\cos(x^2)$ $z_r^2 = x^1\sin(x^2)$	M_{12}^2
Cartesian 	$z^1 = x^1$ $z^2 = x^2$	P_1^2

Table 3 Unsplit R-separable Coordinates and Operators for

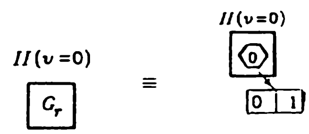
$$(\Delta_2 + 2\varepsilon\partial_t)\Psi = E\Psi$$

	Graph	Coordinates $\{y^u, t\}$ $t = x^3$ for *all systems*	Operators R-separation factor R
1.	I G_e	$y^u = z_e^u$	$P_1^2 + P_2^2$ $M_{12}^2 + c^2 P_1^2$ $R = 0$
2.	II ($v=0$) G_e	$y^u = \sigma_{II}^{1/2} z_e^u$	$(vP_1 - B_1)^2 + (vP_2 - B_2)^2$ $M_{12}^2 + c^2(vP_1 - B_1)^2$ $R = \varepsilon c^2 x^3(cosh^2(x^1) + cos^2(x^2))/2$
3.	III ($v=0$) G_e	$y^u = \sigma_{III}^{1/2} z_e^u$	$v(P_1^2 + P_2^2) - \{P_1, B_1\} - \{P_2, B_2\}$ $M_{12}^2 + c^2(vP_1^2 - \{P_1, B_1\})$ $R = 0$
4.	IV± ($v=0, w^2=1$) G_e	$y_\pm^u = \sigma_{IV\pm}^{1/2} z_e^u$	$(vP_1 - B_1)^2 + (vP_2 - B_2)^2 \pm w^2(P_1^2 + P_2^2)$ $M_{12}^2 + c^2[(vP_1 - B_1)^2 \pm w^2 P_1^2]$ $R = \varepsilon\tau_\pm c^2 x^3(\cosh^2(x^1) + \cos^2(x^2))/2$
5.	I G_p	$y^1 = z_p^1 + \gamma(x^3)^2/4$ $y^2 = z_p^2$	$P_1^2 + P_2^2 + \gamma\varepsilon B_1$ $\{M_{12}, P_2\} - \gamma B_2^2/4$ $R = \varepsilon\gamma x^3((x^1)^2 - (x^2)^2)/4$
6.	II ($v=0$) G_p	$y^1 = \sigma_{II}^{1/2} z_p^1 + \gamma/4(x^3+v)$ $y^2 = \sigma_{II}^{1/2} z_p^2$	$(vP_1 - B_1)^2 + (vP_2 - B_2)^2 + \gamma\varepsilon P_1$ $\{M_{12}, vP_2 - B_2\} - \gamma P_2^2/4$ $R = \frac{\varepsilon}{8}[(x^3+v)((x^1)^2+(x^2)^2)^2 - \frac{\gamma((x^1)^2-(x^2)^2)}{2(x^3+v)}]$
7.	I G_r	$y^u = z_r^u$	$P_1^2 + P_2^2$ M_{12}^2 $R = 0$

Table 3 (continued)

	Graph	Coordinates $\{y^u, t\}$ $t = x^3$ for *all systems*	Operators R-separation factor R
8.	$II(v=0)$ G_r	$y^u = \sigma_{II}{}^{1/2} z_r^u$	$(vP_1 - B_1)^2 + (vP_2 - B_2)^2$ M_{12}^2 $R = \varepsilon x^3 (x^1)^2 / 2$
9.	$III(v=0)$ G_r	$y^u = \sigma_{III}{}^{1/2} z_r^u$	$v(P_1^2 + P_2^2) - \{P_1, B_1\} - \{P_2, B_2\}$ M_{12}^2 $R = 0$
10.	$IV\pm(v=0, w^2=1)$ G_r	$y_\pm^u = \sigma_{IV\pm}{}^{1/2} z_r^u$	$(vP_1 - B_1)^2 + (vP_2 - B_2)^2$ $\pm w^2 (P_1^2 + P_2^2)$ M_{12}^2 $R = \varepsilon \tau_\pm x^3 (x^1)^2 / 2$

Table 3: The results of Table 2 have been used to simplify the presentation of this table. For example in system 7



and $y^u = \sigma_{II}{}^{1/2} z_r^u$ is

$$y^1 = |x^3 + v| x^1 \cos(x^2)$$
$$y^2 = |x^3 + v| x^1 \sin(x^2).$$

(Recall that $t = x^3$ for these systems).

Table 4 Split R-separable Coordinates for

$$(\Delta_2 + 2\varepsilon\partial_t)\Psi = E\Psi$$

Class b Splitting Types			
1. $I \quad II(v_2=0)$	2. $I \quad III(v_2=0)$	3. $I \quad IV\pm(v_2=0, w_2^2=1)$	
4. $II(v_1=0) \quad II(v_2\neq 0)$	5. $II \quad III(v_2=0)$	6. $II \quad IV\pm(v_2=0, w_2^2=1)$	
7. $III(v_1=0) \quad III(v_2\neq 0)$	8. $III \quad IV\pm(v_2=0, w_2^2=1)$		
9. $IV\pm \quad IV\pm$	10. $IV\pm \quad IV\mp$		
Class c Splitting Types			
1. I	2. $II(v_1=0)$	3. $III(v_1=0)$	4. $IV\pm(v_1=0, w_1^2=1)$

Table 4: An entry $L_1 \; L_2$ in this table corresponds to the split coordinate system

$$y^1 = \sigma_{L_1}^{1/2} x^1 + \frac{1}{2}\gamma_1 \int\int \sigma^{-3/2}$$

$$y^2 = \sigma_{L_2}^{1/2} x^2 + \frac{1}{2}\gamma_2 \int\int \sigma^{-3/2}$$

$$t = x^3 \;, \qquad L_i : I \rightarrow IV\pm$$

An entry L_1 in part c of Table 4 has coordinate transformations as above but with $L_1 = L_2$.

Table 5 Unsplit Separable Coordinates and Operators for $p_1^2 + p_2^2 + p_3^2 = E$

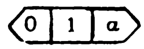
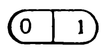
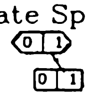
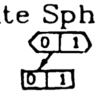
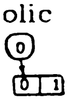
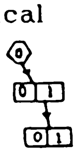
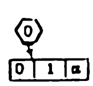
Name & Graph	Coordinates	Operators
Ellipsoidal G_{eo} = 	$z_{eo}^1 = c\left[\frac{x^1x^2x^3}{a}\right]^{\frac{1}{2}}$ $z_{eo}^2 = c\left[\frac{(x^1-1)(x^2-1)(x^3-1)}{(1-a)}\right]^{\frac{1}{2}}$ $z_{eo}^3 = c\left[\frac{(x^1-a)(x^2-a)(x^3-a)}{a(a-1)}\right]^{\frac{1}{2}}$ $0<x^1<1<x^2<a<x^3$	$\mathbf{J}\cdot\mathbf{J}+c^2[(1+a)P_1^2+aP_2^2+P_3^2]$ $J_2^2+aJ_3^2+c^2aP_1^2$
Paraboloidal G_{po} = 	$z_{po}^1 = \frac{1}{2}c\,(x^1+x^2+x^3-1)$ $z_{po}^2 = c\,[-x^1x^2x^3]^{\frac{1}{2}}$ $z_{po}^3 = c\,[(x^1-1)(x^2-1)(x^3-1)]^{\frac{1}{2}}$ $x^1<0<x^2<1<x^3$	$\{J_2,P_3\}-\{J_3,P_2\}+c\,(P_1^2+P_3^2)$ $J_1^2+c\,\{J_3,P_2\}$
Prolate Spheroidal G_{ps} = 	$z_{ps}^1 = c\cosh(x^1)\cos(x^2)$ $z_{ps}^2 = c\sinh(x^1)\sin(x^2)\cos(x^3)$ $z_{ps}^3 = c\sinh(x^1)\sin(x^2)\sin(x^3)$	$\mathbf{J}\cdot\mathbf{J}-c^2(P_2^2+P_3^2)$ J_1^2
Oblate Spheroidal G_{os} = 	$z_{os}^1 = c\cosh(x^1)\cos(x^2)\cos(x^3)$ $z_{os}^2 = c\cosh(x^1)\cos(x^2)\sin(x^3)$ $z_{os}^3 = c\sinh(x^1)\sin(x^2)$	$\mathbf{J}\cdot\mathbf{J}+c^2(P_2^2+P_3^2)$ J_1^2
Parabolic G_{pa} = 	$z_{pa}^1 = \frac{1}{2}[(x^1)^2-(x^2)^2]$ $z_{pa}^2 = x^1x^2\cos(x^3)$ $z_{pa}^2 = x^1x^2\sin(x^3)$	$\{J_2,P_3\}-\{J_3,P_2\}$ J_1^2
Spherical G_{sp} = 	$z_{sp}^1 = x^1\cos(x^2)$ $z_{sp}^2 = x^1\sin(x^2)\cos(x^3)$ $z_{sp}^3 = x^1\sin(x^2)\sin(x^3)$	$\mathbf{J}\cdot\mathbf{J}$ J_1^2
Conical G_{co} = 	$z_{co}^1 = x^1\left[\frac{x^2x^3}{a}\right]^{\frac{1}{2}}$ $z_{co}^2 = x^1\left[\frac{(x^2-1)(x^3-1)}{1-a}\right]^{\frac{1}{2}}$ $z_{co}^3 = x^1\left[\frac{(x^2-a)(x^3-a)}{a(a-1)}\right]^{\frac{1}{2}}$ $x^1>0,\ 0<x^2<1<x^3<a$	$\mathbf{J}\cdot\mathbf{J}$ $J_2^2+aJ_3^2$

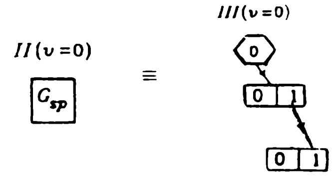
Table 6 Unsplit R-separable Coordinates and R-factors for
$(\Delta_3 + 2\varepsilon\partial_t)\Psi = E\Psi$

	Graph	Coordinates $\{y^u, t\}$ $t = x^4$ *for all systems*	R-separation factor R
1.	I $\boxed{G_{eo}}$	$y^u = z^u_{eo}$	0
2.	$II(\nu=0)$ $\boxed{G_{eo}}$	$y^u = \sigma_{II}{}^{½} z^u_{eo}$	$\frac{1}{2}\varepsilon c^2 x^4(x^1+x^2+x^3)$
3.	$III(\nu=0)$ $\boxed{G_{eo}}$	$y^u = \sigma_{III}{}^{½} z^u_{eo}$	0
4.	$IV\pm(\nu=0, w^2=1)$ $\boxed{G_{eo}}$	$y^u_\pm = \sigma_{IV\pm}{}^{½} z^u_{eo}$	$\frac{1}{2}\varepsilon\tau_\pm c^2 x^4(x^1+x^2+x^3)$
5.	I $\boxed{G_{po}}$	$y^1 = z^1_{po} + \gamma(x^4)^2/4$ $y^2 = z^2_{po}$ $y^3 = z^3_{po}$	$\varepsilon c\gamma x^4(x^1+x^2+x^3)/4$
6.	$II(\nu=0)$ $\boxed{G_{po}}$	$y^1 = \sigma_{II}{}^{½} z^1_{po} + \gamma/4x^4$ $y^2 = \sigma_{II}{}^{½} z^2_{po}$ $y^3 = \sigma_{II}{}^{½} z^3_{po}$	$\frac{1}{2}\varepsilon\{c^2x^4[2\sum_1^3(x^i)^2-(\sum_1^3 x^i)^2+6\sum_1^3 x^i]$ $-\gamma c\sum_1^3 x^i/8x^4\}$
7.	I $\boxed{G_{ps}}$	$y^u = z^u_{ps}$	0
8.	$II(\nu=0)$ $\boxed{G_{ps}}$	$y^u = \sigma_{II}{}^{½} z^u_{ps}$	$\varepsilon c^2x^4(\cosh^2(x^1)+\cos^2(x^2))/2$
9.	$III(\nu=0)$ $\boxed{G_{ps}}$	$y^u = \sigma_{III}{}^{½} z^u_{ps}$	0
10.	$IV\pm(\nu=0, w^2=1)$ $\boxed{G_{ps}}$	$y^u_\pm = \sigma_{IV\pm}{}^{½} z^u_{ps}$	$\varepsilon\tau_\pm c^2x^4(\cosh^2(x^1)+\cos^2(x^2))/2$
11.	I $\boxed{G_{os}}$	$y^u = z^u_{os}$	0
12.	$II(\nu=0)$ $\boxed{G_{os}}$	$y^u = \sigma_{II}{}^{½} z^u_{os}$	$\varepsilon c^2x^4(\cosh^2(x^1)+\cos^2(x^2))/2$

Table 6 (continued)

	Graph	Coordinates $\{y^u, t\}$ $t = x^4$ for *all systems*	R-separation factor R
13.	$III(\nu=0)$ G_{os}	$y^u = \sigma_{III}{}^{1/2} z^u_{os}$	0
14.	$IV\pm(\nu=0, w^2=1)$ G_{os}	$y^u_\pm = \sigma_{IV\pm}{}^{1/2} z^u_{os}$	$\varepsilon\tau_\pm c^2 x^4(\cosh^2(x^1)+\cos^2(x^2))/2$
15.	I G_{pa}	$y^1 = z^1_{pa} + \gamma(x^4)^2/4$ $y^2 = z^2_{pa}$ $y^3 = z^3_{pa}$	$\varepsilon\gamma x^4[(x^1)^2-(x^2)^2]/4$
16.	$II(\nu=0)$ G_{pa}	$y^1 = \sigma_{II}{}^{1/2} z^1_{pa} + \gamma/4x^4$ $y^2 = \sigma_{II}{}^{1/2} z^2_{pa}$ $y^3 = \sigma_{II}{}^{1/2} z^3_{pa}$	$\frac{\varepsilon}{8}\{x^4((x^1)^2+(x^2)^2)^2 - \frac{\gamma((x^1)^2-((x^2)^2)}{2x^4}\}$
17.	I G_{sp}	$y^u = z^u_{sp}$	0
18.	$II(\nu=0)$ G_{sp}	$y^u = \sigma_{II}{}^{1/2} z^u_{sp}$	$\frac{1}{2}\varepsilon x^4(x^1)^2$
19.	$III(\nu=0)$ G_{sp}	$y^u = \sigma_{III}{}^{1/2} z^u_{sp}$	0
20.	$IV\pm(\nu=0, w^2=1)$ G_{sp}	$y^u_\pm = \sigma_{IV\pm}{}^{1/2} z^u_{sp}$	$\frac{1}{2}\varepsilon\tau_\pm x^4(x^1)^2$
21.	I G_{co}	$y^u = z^u_{co}$	0
22.	$II(\nu=0)$ G_{co}	$y^u = \sigma_{II}{}^{1/2} z^u_{co}$	$\frac{1}{2}\varepsilon x^4(x^1)^2$
23.	$III(\nu=0)$ G_{co}	$y^u = \sigma_{III}{}^{1/2} z^u_{co}$	0
24.	$IV\pm(\nu=0, w^2=1)$ G_{co}	$y^u_\pm = \sigma_{IV\pm}{}^{1/2} z^u_{co}$	$\frac{1}{2}\varepsilon\tau_\pm x^4(x^1)^2$

Table 6: In analogy to the case $m=2$ we have used the results of Table 5 to simplify the presentation of Table 6. For instance in system 19



and $y^u = \sigma_{III}^{1/2} z_{sp}^u$ is

$$y^1 = |x^4+v|^{\frac{1}{2}} x^1 \cos(x^2)$$

$$y^2 = |x^4+v|^{\frac{1}{2}} x^1 \sin(x^2)\cos(x^3)$$

$$y^3 = |x^4+v|^{\frac{1}{2}} x^1 \sin(x^2)\sin(x^3) .$$

Table 7 Complex Orthogonal Separable systems for $E(2,\mathbb{C})$

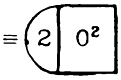
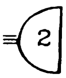
Graph	Metric & Operator	Coordinates
A. $G_A \equiv$ 	$ds^2 = c^2 \frac{(x^1-x^2)}{4}[(dx^1/x^1)^2 - (dx^2/x^2)^2]$	$\overset{1}{z}_A - i\,\overset{2}{z}_A = c^2\sqrt{x^1x^2}$
	$M_{12}^2 + c^2(P_1 - i\,P_2)^2$	$\overset{1}{z}_A + i\,\overset{2}{z}_A = \sqrt{x^1/x^2} + \sqrt{x^2/x^1}$
B. $G_B \equiv$ 	$ds^2 = (x^1 - x^2)[(dx^1)^2 - (dx^2)^2]$	$\overset{1}{z}_B - i\,\overset{2}{z}_B = x^1 + x^2$
	$-i\{M_{12}, P_1 - i\,P_2\} + (P_1 + iP_2)^2/4$	$\overset{1}{z}_B + i\,\overset{2}{z}_B = \frac{1}{2}(x^1 - x^2)^2$

Table 7: The other four complex separable systems for $E(2,\mathbb{C})$ are those listed in Table 2 with it now being understood that z^i and x^i are complex.

Table 8 Unsplit R-separable Coordinates and Operators for the complex equation $(\Delta_2 + 2\varepsilon\partial_t)\Psi = E\Psi$

	Graph	Coordinates $\{y^u, t\}$ $t = x^3$ for *all systems*	Operators
1.	I G_A	$y^u = z^u_A$	$P_1^2 + P_2^2$ $M_{12}^2 + c^2(P_1 - i P_2)^2$
2.	$II(\nu=0)$ G_A	$y^u = \sigma_{II}{}^{1/2} z^u_A$	$B_1^2 + B_2^2$ $M_{12}^2 + c^2(B_1 - i B_2)^2$
3.	$III(\nu=0)$ G_A	$y^u = \sigma_{III}{}^{1/2} z^u_A$	$-\{P_1, B_1\} - \{P_2, B_2\}$ $M_{12}^2 - c^2\{P_1 - i P_2, B_1 - i B_2\}$
4.	$IV(\nu=0, w^2=1)$ G_A	$y^u_\pm = \sigma_{IV\pm}{}^{1/2} z^u_A$	$B_1^2 + B_2^2 + P_1^2 + P_2^2$ $M_{12}^2 + c^2[(B_1 - iB_2)^2 + (P_1 - iP_2)^2]$
5.	I G_B	$y^1 = z^1_B + \gamma(x^3)^2/4$ $y^2 = z^2_B - i\gamma(x^3)^2/4$	$P_1^2 + P_2^2 + \gamma\varepsilon(B_1 - iB_2)$ $-i\{M_{12}, P_1 - iP_2\} + (P_1 + iP_2)^2/4 + \frac{\gamma\varepsilon}{2}(B_1 + iB_2) - \frac{\gamma}{4}(B_1 - iB_2)^2$
6.	$II(\nu=0)$ G_B	$y^1 = \sigma_{II}{}^{1/2} z^1_B + \gamma/4x^3$ $y^2 = \sigma_{II}{}^{1/2} z^2_B - i\gamma/4x^3$	$B_1^2 + B_2^2 + \gamma\varepsilon(P_1 - iP_2)$ $-i[\{M_{12}, -(B_1 - iB_2)\} + -\frac{i\gamma}{4}(P_1 - iP_2)^2] + +\frac{1}{4}[(B_1 + iB_2)^2 + 2\gamma\varepsilon(P_1 + iP_2)]$

Table 8: The other complex systems in two dimensions are the complexifications of those in Table 3, and Table 4. Also see those tables for an explanation of the notation used here.

Appendix B Symbolic Programming in Separation of Variables

Symbolic programming refers to the capability of performing analytic calculations such as integration, differentiation etc. on computers. There has been much progress in recent years. Gerdt et al. (1980) have written a comprehensive review comparing various symbolic languages such as MACSYMA and REDUCE, and detailing some of the applications in mathematics and physics. An excellent example provided in that review is that of Delaunay who spent 20 years writing a monumental work on Celestial mechanics. He derived an analytic theory for the moon correct to small quantities of the seventh order. Deprit, Henrard and Rom (1970), in just one year, reproduced Delaunay's results using a symbolic language. Remarkably, only one error was found in Delaunay's multivolume work which contained more than 40,000 equations! We are lucky to have one of the most versatile symbolic languages, MACSYMA (see Mathlab (1983)), available at Waikato University on its VAX computer. It was T. Robb, a graduate student at Waikato, who first pointed out that many of the arduous calculations required in variable separation could be performed with ease using MACSYMA.

In this Appendix we give an outline of a program capable of producing all the time consuming details of separation on flat manifolds. In the process of writing this program, an explicit formula was derived for expressing the constants of the motion $\lambda_m = a_m^{ij} p_i p_j$ in terms of the enveloping algebra. Finally we give an application for this program. We find all the details of separation for the interesting class of systems in §3.4, an almost impossible task were it attempted by hand. The program itself, SEPCAL.V, is given in Appendix C together with a rough explanation of the MACSYMA commands it uses.

Equation (D6) shows that the constants of the motion may be computed given the Stäckel matrix , the $B_r^{r\alpha}$, and the $A_b^{\alpha\beta}$. We will go one step further here and show how these Killing tensors can be expressed in terms of the enveloping algebra. The initial problem is that these tensors, λ_m, are expressed in terms of the separable coordinates $\{x^i\}$,i.e.,

$$\lambda_m = a_m^{ij}(\mathbf{x}) \frac{\partial W}{\partial x^i} \frac{\partial W}{\partial x^j} \tag{B1}$$

To gain a coordinate free description they are expressed in terms of the cartesian coordinates z^i. In analogy to (2.6.14) and (2.6.15)

$$\lambda_m = a_m^{ij} p_i p_j = \tilde{a}_m^{ij} \frac{\partial W}{\partial z^i} \frac{\partial W}{\partial z^j} \tag{B2}$$

where

$$\tilde{a}_m = J^t a_m J \tag{B3}$$

(see line 178 of the program SEPCAL.V given in Appendix C).

Here $(J)_{ik} = (\partial z^k / \partial x^i)$ is the transformation matrix

$$(\tilde{a}_m)_{ij} = \tilde{a}_m^{ij} \quad \text{and} \quad (a_m)_{ij} = a_m^{ij} \tag{B4}$$

Thus

$$\lambda_m = \mathbf{P}^t \tilde{a}_m \mathbf{P} \tag{B5}$$

where $\mathbf{P} = (P_1, \ldots, P_n)^t$ and $P_i = \partial_{z^i} W$. As has been mentioned in the Introduction, for flat spaces these Killing tensors can be expressed in terms of the enveloping algebra - i.e. there are constants T^{ijkl}, T^{ijk}, T^{jk} such that

$$\lambda_m = T^{ijkl} M_{ij} M_{kl} + T^{ijk} M_{ij} P_k + T^{ij} P_i P_j \tag{B6}$$

(Recall that the Einstein summation convention is assumed). The problem is to determine these constants. Since $M_{ij} = z^i P_j - z^j P_i$ ($P_i = \dfrac{\partial W}{\partial z^i}$), (B6) implies that

$$\tilde{a}_m^{ij} = A_{kl}^{ij} z^k z^l + B_k^{ij} z^k + C^{ij} \tag{B7}$$

In the following argument the relationship between the constants A_{kl}^{ij}, B_k^{ij}, C^{ij} and the constants in (B6) is determined. The terms

$$A_{kl}^{ij} z^k z^l p_i p_j \tag{B8}$$

correspond to the

$$a^{rstu} M_{rs} M_{tu} \tag{B9}$$

part of the enveloping algebra. Expanding

$$A_{kl}^{ij} z^k z^l p_i p_j = (a^{kilj} - a^{kilj} - a^{ijkl} + a^{ikjl}) z^k z^l p_i p_j$$

$$=> 2A_{kl}^{ij} z^k z^l = [a^{kilj} - a^{iklj} - a^{kijl} + a^{ikjl}$$

$$+ \quad a^{kjli} - a^{kjil} - a^{jkli} + a^{jkil}] z^k z^l$$

$$
\begin{aligned}
\Rightarrow 4A^{ij}_{kl} &= [\, a^{kilj} - a^{kijl} - a^{iklj} + a^{ikjl} \\
&\quad + a^{kjli} - a^{kjil} - a^{jkli} + a^{jkil} \,] \\
D^{jkil} &\qquad\qquad\qquad\qquad\qquad\qquad\qquad D^{ikjl} \\
&\quad + [\, a^{likj} - a^{lijk} - a^{ilkj} + a^{iljk} \\
&\quad + a^{ljki} - a^{ljik} - a^{jlki} + a^{jlik} \,] \\
&= D^{jkil} + D^{ikjl}
\end{aligned}
\tag{B10}
$$

where

$$
\begin{aligned}
D^{jkil} &= a^{kilj} - a^{kijl} - a^{iklj} + a^{ikjl} \\
&\quad + a^{ljki} - a^{ljik} - a^{jlki} + a^{jlik}
\end{aligned}
\tag{B11}
$$

Here D^{jkil} is the overall coefficient of $M_{jk} M_{il}$ except when $k=l$ and $i=j$. In that case it is twice the coefficient of M_{ij}^2.

The D^{ijkl} possess the Riemann-type symmetries

$$
\begin{aligned}
D^{ikjl} &= -D^{iklj} \\
D^{ikjl} &= -D^{kijl} \\
D^{ikjl} &= -D^{jlik}
\end{aligned}
\tag{B12}
$$

Since D^{ijkl} is the overall coefficient, the order $i<j$, $k<l$ and $i<k$ can be imposed on the indices.

If none of the indices are equal, then

$$
A^{ij}_{kl} z^k z^l p_i p_j = \sum_{r=1}^{n-3} \sum_{s=r+1}^{n-2} \sum_{t=s+1}^{n-1} \sum_{u=t+1}^{n} T^{rstu} \tag{B13}
$$

where

$$
T^{rstu} = D^{rstu} M_{rs} M_{tu} + D^{rtsu} M_{rt} M_{su} + D^{rust} M_{ru} M_{st} \tag{B14}
$$

and without loss of generality $r<s<t<u$.
Making use of the Riemann-type identity :

$$
M_{ru} M_{st} + M_{rs} M_{tu} + M_{rt} M_{us} = 0 \;, \tag{B15}
$$

and substituting for $M_{rs} M_{tu}$ in (B14)

$$A^{ij}_{kl} z^k z^l p_i p_j = \sum_{r=1}^{n-3} \sum_{s=r+1}^{n-2} \sum_{t=s+1}^{n-1} \sum_{u=t+1}^{n} [(D^{rstu} - D^{rust}) M_{rs} M_{tu}$$

$$+ (D^{rtsu} + D^{rust}) M_{rt} M_{su}] \tag{B16}$$

This last equation, when combined with (B10) and (B12) implies

$$A^{ij}_{kl} z^k z^l p_i p_j = 4 \sum_{r=1}^{n-3} \sum_{s=r+1}^{n-2} \sum_{t=s+1}^{n-1} \sum_{u=t+1}^{n} [A^{rt}_{su} M_{rs} M_{tu} + A^{rs}_{tu} M_{rt} M_{su}] \tag{B17}$$

If two indices are the same then

$$A^{ij}_{kl} z^k z^l p_i p_j = \sum_{r=1}^{n-2} \sum_{s=r+1}^{n-1} \sum_{u=s+1}^{n} T^{rsu} \tag{B18}$$

where

$$T^{rsu} = D^{rsru} M_{rs} M_{ru} + D^{rssu} M_{rs} M_{su} + D^{rusu} M_{ru} M_{su}$$

$$(\text{from(B10)}) = 2A^{rr}_{su} M_{rs} M_{ru} + 4A^{rs}_{su} M_{rs} M_{su} + 2A^{rs}_{uu} M_{ru} M_{su} \tag{B19}$$

The case where more than two of the indices are the same can only occur as two pairs, and then

$$A^{ij}_{kl} z^k z^l p_i p_j = \sum_{r=1}^{n-1} \sum_{s=r+1}^{n} T^{rs} \tag{B20}$$

where

$$T^{rs} = \frac{1}{2} D^{rsrs} M^2_{rs}$$

$$= A^{rr}_{ss} M^2_{rs} \tag{B21}$$

Now consider,

$$B^{ij}_k z^k p_i p_j \tag{B22}$$

which in terms of the enveloping algebra must be

$$a^{rst} M_{rs} P_t \tag{B23}$$

Expanding,

$$B^{ij}_k z^k p_i p_j = (a^{kij} - a^{ikj}) z^k p_i p_j$$

$$=> \quad 2B_k^{ij} z^k = (a^{kij} - a^{ikj} + a^{kji} - a^{jki}) z^k \qquad \text{(B24)}$$

Taking $\frac{\partial}{\partial z^l}$ of (B24)

$$2B_l^{ij} = a^{lij} - a^{ilj} + a^{lji} - a^{jli} = a^{lij} + a^{ilj} \qquad \text{(B25)}$$

where a^{rst} is the overall coefficient of $M_{rs} P_t$ and we can assume $r < s$.

Consider terms with all indices different (i.e. $r < s < t$), then

$$B_k^{ij} z^k p_i p_j = \sum_{r=1}^{n-2} \sum_{s=r+1}^{n-1} \sum_{t=s+1}^{n} T^{rst} \qquad \text{(B26)}$$

where

$$T^{rst} = a^{rst} M_{rs} P_t + a^{str} M_{st} P_r + a^{rts} M_{rt} P_s \qquad \text{(B27)}$$

Substituting the identity

$$M_{st} P_r = M_{rt} P_s - M_{rs} P_t \qquad \text{(B28)}$$

and using (B25)

$$T^{rst} = -2B_s^{rt} M_{rs} P_t - 2B_t^{rs} M_{rt} P_s \qquad \text{(B29)}$$

If two of the indices are the same then

$$B_k^{ij} z^k p_i p_j = \sum_{r=1}^{n-1} \sum_{s=r+1}^{n} T^{rs} \qquad \text{(B30)}$$

where

$$T^{rs} = a^{rsr} M_{rs} P_r + a^{rss} M_{rs} P_s \qquad \text{(B31)}$$

Using (B25) this is

$$T^{rs} = 2B_r^{rs} M_{rs} P_r - 2B_s^{rs} M_{rs} P_s \qquad \text{(B32)}$$

The form of the $C_{rs} P_r P_s (r \neq s)$ and $C_{rr} P_r^2$ terms is easily computed.

Collecting these results:

$$\lambda_m = 4\sum_{r=1}^{n-3}\sum_{s=r+1}^{n-2}\sum_{t=s+1}^{n-1}\sum_{u=t+1}^{n}[A_{su}^{rt}M_{rs}M_{tu} + A_{tu}^{rs}M_{rt}M_{su}]$$
$$+\sum_{r=1}^{n-2}\sum_{s=r+1}^{n-1}\sum_{u=s+1}^{n}[2A_{su}^{rr}M_{rs}M_{ru}$$
$$+4A_{su}^{rs}M_{rs}M_{su}$$
$$+2A_{uu}^{rs}M_{ru}M_{su}$$
$$-2B_{u}^{rs}M_{ru}P_s - 2B_{s}^{ru}M_{rs}P_u]$$
$$+\sum_{r=1}^{n-1}\sum_{s=r+1}^{n}[A_{ss}^{rr}M_{rs}^2$$
$$+2B_{r}^{rs}M_{rs}P_r - 2B_{s}^{rs}M_{rs}P_s$$
$$+2C_{rs}P_rP_s]$$
$$+\sum_{r=1}^{n}C_{rr}P_r^2 \qquad \text{(B33)}$$

Example: We express the second order Killing tensor

$$\lambda = z^1z^3P_2P_4 - z^1z^4P_2P_3 - z^2z^3P_1P_4 + z^2z^4P_1P_3$$

in terms of the enveloping algebra. Equation (B7) implies that

$$A_{13}^{24} = A_{24}^{13} = \frac{1}{4} \quad , A_{14}^{23} = A_{23}^{14} = -\frac{1}{4}$$

The formula (B33) then implies

$$\lambda = 4A_{24}^{13}M_{12}M_{34} = M_{12}M_{34} \quad ,$$

a result which is easily checked.

The formula (B33) is incorporated in lines 280 to 293 of the SEPCAL.V program. There $A\sim[r,t,s,u] \equiv A_{su}^{rt}$ and $B[r,s,t] \equiv B_t^{rs}$.

The dimension $D(n)$ of the enveloping algebra is easily calculated from (B33):

$$D(n) = 2\,{}^nC_4 + 5\,{}^nC_3 + 4\,{}^nC_2 + {}^nC_1$$
$$= \frac{n(n+1)^2(n+2)}{12} \qquad \text{(B34)}$$

The dimension rapidly grows with n. When $n=5$, $D(5)=105$. The dimension formula has already been given by other investigators (e.g. see Delong (1982)).

The Killing vector part of the argument works in a similar and even simpler fashion:

$$\lambda_a = \sum_{i<j} c^{ij} M_{ij} + \sum_i d^i P_i = \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} c^{ij}(z^i P_j - z^j P_i) + \sum_{i=1}^{n} d^i P_i \qquad \text{(B35)}$$

This calculation is carried out in lines 316-331 of the SEPCAL.V program. There $c\sim[i,j] \equiv c^{ij}$ and $d\sim[i] \equiv d^i$.

The only remaining difficulty is that $\tilde{a}_m^{ij}$ is a function of the x^i 's, and is not yet expressed in the z^i 's as in (B7). The difficulty is resolved by obtaining the coefficients A_{kl}^{ij}, B_k^{ij}, and C^{ij} by differentiation:

$$A_{kl}^{ij} = \frac{\partial^2 \tilde{a}_m^{ij}}{\partial z^k \partial z^l}$$

Upon application of the Chain rule

$$A_{kl}^{ij} = \frac{\partial x^p}{\partial z^k} \frac{\partial}{\partial x^p} \left[\frac{\partial x^q}{\partial z^l} \frac{\partial}{\partial x^q} \tilde{a}_m^{ij} \right]$$

$$= (J^{-1})_{kp} \frac{\partial}{\partial x^p} \left[(J^{-1})_{lq} \frac{\partial}{\partial x^q} \tilde{a}_m^{ij} \right] \qquad \text{(B36)}$$

(lines 196-198 of the program SEPCAL.V).

Only knowledge of the inverse of the transformation matrix J is needed.

To obtain the B_u^{ij} and the C_k^{ij} is a simple matter:

$$B_u^{ij} = \frac{\partial}{\partial z^u} (\tilde{a}_m^{ij} - A_{kl}^{ij} z^k z^l)$$

$$C^{ij} = \tilde{a}_m^{ij} - A_{kl}^{ij} z^k z^l - B_k^{ij} z^k \qquad \text{(B37)}$$

(line 239 of SEPCAL.V).

The result, (B33), is the heart of the program and its most useful function; computing the constants of the motion in terms of the enveloping algebra. For completeness, the program "SEPCAL.V" has been included in Appendix C. A list of a few symbols and their meanings is also given there. The description is not complete, and is only meant to be a guide to the program's general nature.

The separation equations, the Helmholtz equation, the Schrödinger conditions, and the inverse of the Stäckel matrix have explicit formulae (see the example in Appendix D), and therefore have been easily incorporated in the program SEPCAL.V.

Early in the program (lines 82-90) the product of the contravariant and covariant metric term is calculated. If the product is not the identity (which it should be) an error message is printed. This is an invaluable check. Once past this stage the results should be totally reliable.

The program can also perform calculations in spaces which are not flat. At present it is being used to check the results for separation obtained by Kalnins and Miller in certain spaces of low dimension.

Time consumption becomes a problem in more than five dimensions. Various things have been done about this: all the symmetries in the A^{ij}_{kl}, and B^{ij}_{k} have been exploited (see lines 220-272 of SEPCAL.V). In addition, since the A^{ij}_{kl} etc. are constants, evaluation of these at $\mathbf{x} = \mathbf{x}_0$ often speeds the simplification process. This facility is available if it is needed. Nevertheless, it does perform all the details for 2-dimensional elliptical coordinates in 1 minute and 50 seconds. That's got to be faster than doing it by hand!

In the following example the program is used to compute the details of separation for the class of metrics that is the subject of §3.4. All the systems in that class possess 2 null coordinates and 3 ignorable coordinates. We first present the input file.

Example of the SEPCAL.V program

Input file

```
/* The following commands put the output file in VTROFF notation,
   make the program distinguish between upper and lower case letters,
   and add "~" to the programs alphabet */
(typeset  true, bothcases: true, declare("~",alphabetic))$

/* We enter the dimension of the space n (=5) and the number of first
   order variables n2 (=2) */
(n 5,n1 0 ,n2:2)$

/* Next the values of the B[u,r] and C[u,v,r] necessary for
   the definition of the Hamilton-Jacobi equation are given */
(B[3,1]  x[1], B[4,1]  1, B[5,1]: 0)$
(B[3,2] -x[2], B[4,2] -1, B[5,2]: 0)$
/* All the components of C[i,j,k] are set to zero */
 C[i,j,k] = 0$
/* The nonzero components of C[i,j,k] are now entered */
(C[3,5,1]  b, C[5,5,1]  x[1], C[5,5,2]  x[2])$
 C[3,3,1]  -Q*x[1]^2 - x[1]*R - S$
 C[3,3,2]  -Q*x[2]^2 + x[2]*R$

Schrodinger  true$
/* The above command causes the program to compute the Schrodinger
   conditions given in (D9) */

/* The Stackel matrix is entered */
 PHI matrix([ x[1], x[2] ],
            [  1  ,  1   ])$
/*                                             i   i  j
   We enter the coordinate transformations z = z (x ) implicitly
   defined by the linear equations y[i] */
 y[1]  z[1]-%i*z[2] =  x[2]-x[1]$
 y[2]  z[1]+%i*z[2] = (x[1]+x[2])*x[4] - 2*x[3]
            + Q*(x[2]^2-x[1]^2) - 1/2*R*(x[1]+x[2])*(1+2*log(x[1]-x[2]))
            - S*log(x[1]-x[2])$
 y[3]  z[3]+%i*z[4] = (x[1]-x[2])*x[4]
            -1/2*Q*(x[1]-x[2])^2 + 1/2*R*(x[1]-x[2])*(1-2*log(x[1]-x[2]))$
 y[4]  z[3]-%i*z[4] =  x[1]+x[2]$
 y[5]         z[5] =  x[5] - b*log(x[2]-x[1])$
/* The following solves for the z[i] and puts them in a form acceptable to
   the program */
 linsolve([y[1],y[2],y[3],y[4],y[5]],[z[1],z[2],z[3],z[4],z[5]])
 ,globalsolve: true$

/* The program SEPCAL.V is loaded */
 load("SEPCAL.V")$
 hj(n,n1,n2,z,A,B,C,PHI)$
 for i thru 5 do display(y[i]);
 kill(z, M)$
/* The following line puts the Killing tensor L[2] calculated by SEPCAL.V into
   a nicer form by picking out the coefficients of the constants Q,R,S and b */
 H[2]  ratcoeff(L[2],Q)*Q + ratcoeff(L[2],R)*R + ratcoeff(L[2],S)*S
     + ratcoeff(L[2],b)*b + ev(L[2], Q=0,R=0,S=0,b=0);
```

The output file was printed with the aid of the computer typesetting language VTROFF but for a little doctoring to tidy the presentation and give equation numbers. For example on a Visual Display Unit one would see

```
      [               x      ]
      [     1          2     ]
      [ - -------  -------   ]
      [   x  - x   x  - x    ]
      [    2    1   2    1   ]
F  =  [                      ]
      [               x      ]
      [     1          1     ]
      [   -------  - ------- ]
      [   x  - x     x  - x  ]
      [    2    1     2    1 ]
```

for the matrix F instead of the VTROFF version in (B40): MACSYMA output is quite readable on a ordinary terminal screen. Normally MACSYMA does not use superscripts so here the contravariant quantities z^i are subscripted.

Output file

The coordinate transformations are:

for i thru 5 do display(y[i]);

$$z_1 - i\,z_2 = x_2 - x_1$$

$$z_1 + i\,z_2 = -\log(x_1 - x_2)S - \frac{(x_2 + x_1)(2\log(x_1 - x_2) + 1)R}{2}$$

$$+ (x_2^2 - x_1^2)\,Q + (x_2 + x_1)x_4 - 2x_3$$

$$z_3 + i\,z_4 = \frac{(x_1 - x_2)(1 - 2\log(x_1 - x_2))R}{2} - \frac{(x_1 - x_2)^2 Q}{2} + (x_1 - x_2)x_4$$

$$z_3 - i\,z_4 = x_2 + x_1$$

$$z_5 = x_5 - b\log(x_2 - x_1) \qquad \text{(B38)}$$

The Stackel matrix is

$$\Phi = \begin{bmatrix} x_1 & x_2 \\ 1 & 1 \end{bmatrix} \qquad \text{(B39)}$$

The inverse of the Stackel matrix is

$$F = \begin{vmatrix} -\frac{1}{x_2 - x_1} & \frac{x_2}{x_2 - x_1} \\ \frac{1}{x_2 - x_1} & -\frac{x_1}{x_2 - x_1} \end{vmatrix} \tag{B40}$$

The Hamilton-Jacobi equation is

$$-\frac{2p_3p_5b}{x_2-x_1} + \frac{p_3^2(S + (x_2+x_1)R + (x_1^2 - x_2^2)Q)}{x_2-x_1} + p_5^2$$
$$-\frac{2x_2p_2p_3}{x_2-x_1} - \frac{2p_2p_4}{x_2-x_1} - \frac{2x_1p_1p_3}{x_2-x_1} - \frac{2p_1p_4}{x_2-x_1} = E \tag{B41}$$

The Helmholtz equation is

$$-\frac{2\frac{\partial^2\Psi}{\partial x_3\,\partial x_5}b}{x_2-x_1} + \frac{\frac{\partial^2\Psi}{\partial x_3^2}(S + (x_2+x_1)R + (x_1^2 - x_2^2)Q)}{x_2-x_1} + \frac{\partial^2\Psi}{\partial x_5^2} - \frac{2\frac{\partial\Psi}{\partial x_3}}{x_2-x_1}$$
$$-\frac{2\frac{\partial^2\Psi}{\partial x_2\,\partial x_4}}{x_2-x_1} - \frac{2x_2\frac{\partial^2\Psi}{\partial x_2\,\partial x_3}}{x_2-x_1} - \frac{2\frac{\partial^2\Psi}{\partial x_1\,\partial x_4}}{x_2-x_1} - \frac{2x_1\frac{\partial^2\Psi}{\partial x_1\,\partial x_3}}{x_2-x_1} = E\Psi$$

(B42)

The metric is

$$ds^2 = \frac{(dx_2-dx_1)^2(b^2 + (x_1-x_2)S + (x_1^2 - x_2^2)R + (x_2^3 - x_1x_2^2 - x_1^2x_2 + x_1^3)Q)}{(x_2-x_1)^2}$$
$$-\frac{2\,dx_2\,dx_5\,b}{x_2-x_1} + \frac{2\,dx_1\,dx_5\,b}{x_2-x_1} + dx_5^2 - 2x_2\,dx_1\,dx_4 + 2x_1\,dx_2\,dx_4 - 2\,dx_2\,dx_3 + 2\,dx_1\,dx_3$$

(B43)

The Hamilton-Jacobi separation equations are

$$p_1(2\lambda_4 + 2x_1\lambda_3) = -2\lambda_3\lambda_5 b - \lambda_3^2(-S - x_1R - x_1^2Q) - x_1\lambda_5^2 + \lambda_2 + x_1\lambda_1$$
$$p_2(-2\lambda_4 - 2x_2\lambda_3) = -\lambda_3^2(x_2R - x_2^2Q) - x_2\lambda_5^2 + \lambda_2 + \lambda_1x_2$$
$$p_3 = \lambda_3$$
$$p_4 = \lambda_4$$
$$p_5 = \lambda_5 \tag{B44}$$

The Schrodinger conditions are

$$\frac{2}{x_2-x_1} = \frac{E_{3.2}-E_{3.1}}{x_2-x_1}$$

$$0 = \frac{E_{4.2}-E_{4.1}}{x_2-x_1}$$

$$0 = \frac{E_{5.2}-E_{5.1}}{x_2-x_1} \tag{B45}$$

The Helmholtz separation equations are

$$0 = (2l_4 + 2x_1 l_3)\frac{d}{dx_1}\Psi_1 +$$

$$(2l_3 l_5 b - l_3^2 S - x_1 l_3^2 R - x_1^2 l_3^2 Q + l_5 E_{5.1} + x_1 l_5^2 + l_4 E_{4.1} + l_3 E_{3.1} - l_2 - x_1 l_1)\Psi_1$$

$$0 = (-2l_4 - 2x_2 l_3)\frac{d}{dx_2}\Psi_2 +$$

$$(x_2 l_3^2 R - x_2^2 l_3^2 Q + l_5 E_{5.2} + x_2 l_5^2 + l_4 E_{4.2} + l_3 E_{3.2} - l_2 - l_1 x_2)\Psi_2$$

$$\frac{d}{dx_3}\Psi_3 = l_3\Psi_3$$

$$\frac{d}{dx_4}\Psi_4 = l_4\Psi_4$$

$$\frac{d}{dx_5}\Psi_5 = l_5\Psi_5 \tag{B46}$$

The constants of the motion are

```
H[2]: ratcoeff(L[2],Q)*Q + ratcoeff(L[2],R)*R + ratcoeff(L[2],S)*S
   + ratcoeff(L[2],b)*b + ev(L[2], Q=0,R=0,S=0,b=0);
```

$$H_2(=\lambda_2) = (i P_4 - P_3 + i P_2 - P_1) P_5 b$$

$$+ \frac{1}{2} S \left[(P_2 + i P_1) P_4 + (i P_2 - P_1) P_3 + P_2^2 + 2i P_1 P_2 - P_1^2 \right]$$

$$+ \frac{1}{2} R \left[(i M_{24} - M_{23} - M_{14} - i M_{13}) P_4 \right.$$

$$\left. + (- M_{24} - i M_{23} - i M_{14} + M_{13}) P_3 \right]$$

$$- \frac{1}{4} Q \left[2 M_{12} M_{34} + M_{24}^2 + (2i M_{23} + 2i M_{14} - 4 M_{13}) M_{24} \right.$$

$$\left. - M_{23}^2 - 2i M_{13} M_{23} - M_{14}^2 - 2i M_{13} M_{14} + M_{13}^2 \right]$$

$$- M_{34} P_4 - i P_3 M_{34} + i M_{12} P_3 - i P_2 M_{24} + M_{14} P_2 - i M_{13} P_2 + P_1 M_{13}$$

$$\lambda_3 = i P_2 - P_1$$

$$\lambda_4 = \frac{M_{24}}{2} + \frac{i M_{23}}{2} + \frac{i M_{14}}{2} - \frac{M_{13}}{2}$$

$$\lambda_5 = P_5 \qquad \text{(B47)}$$

The Schrödinger conditions (B45) are easily solved to give:

$$E_{3,2} = 2 \text{ and } E_{3,1} = E_{4,2} = E_{4,1} = E_{5,2} = E_{5,1} = 0 \ , \qquad \text{(B48)}$$

thus determining the unknown functions in the Helmholtz separation equations (B46).

Appendix C The SEPCAL.V program

In this appendix we include the MACSYMA program SEPCAL.V, together with a list explaining some of the symbols used in the program. These explanations are sketchy, and not intended as a substitute for the MACSYMA reference manual (see Mathlab(1983)). If more information is required the reader should write to the author of this thesis.

The symbols in the following list are entered in roughly the order they occur throughout first the input file, then the output file, and finally the SEPCAL.V program.

List of Symbols

/* */	comment delimiters
typeset: true	puts the file in VTROFF notation
$ and ;	command line terminators: if $ the result is not displayed, if ; then it is
bothcases: true	the program distinguishes between upper and lower case letters
declare("~",alphabetic)	the symbol "~" is added to the program alphabet, i.e. now possible to define a~[i,j] etc.
n	the dimension of the space
n1	n_1, the number of Stäckel variables
n2	n_1+n_2, where n_2 is the number of first-order variables
A[u,v,a]	$A_a^{uv}(x^a)$, (the indices u and v are used to denote ignorables i.e. $u,v \longleftrightarrow \alpha,\beta$)
C[u,v,r]	$A_r^{uv}(x^r)$
B[u,r]	$B_r^{ur}(x^r)$
PHI, F	the Stäckel matrix Φ and its inverse $F(=\Phi^{-1})$
z[i]	standard cartesian coordinates z^i
x[i]	separable coordinates x^i
linsolve([y[i]],[z[i]])	solves the linear equations y[i] for the variables z[i]
load("SEPCAL.V")	the input program SEPCAL.V is loaded
hj(n,n1,n2,z,A,B,C,PHI)	essentially the input program is a function hj of all the variables n, n1, etc.
ttyoff: true	enables the program to be loaded without its lengthy contents being displayed

and :=	value assignment and function assignment respectively. e.g. x:5 is $x = 5$; f(x):x^2 is $f(x) = x^2$
LIST, lsimp	if the switch lsimp is set to true, i.e. lsimp: true , then LIST may be set to [x[1]=a1,...,x[n]=an] to help speed the simplification of expressions which we know to be constants
g[i,j]	the components of g^{ij}
J,Ji	the transformation matrix and its inverse respectively
HJ,HE	the Hamilton-Jacobi and Helmholtz equations respectively
se[i]	the Hamilton-Jacobi separation equations
SE[i]	the Helmholtz separation equations
E[u,a]	$E_a^u(x^a)$, the unknown functions needing to be determined by the Schrödinger conditions for the complete definition of the Helmholtz separation equations. (see (D9))
Schrodinger	if Schrodinger = true then the program displays the Schrodinger conditions for the unknown functions $E_a^a(x^a)$ (see (D9))
ratsimpx(f),moresimp	if moresimp ≠ true (by default), then ratsimpx = ratsimp which performs rational simplifications of functions. if moresimp = true then some more powerful simplification function than ratsimp can be called e.g. ratsimpx = trigsimp
allcons	by default allcons = true and all the Killing tensors are computed. If however allcons: false and nix = ni and nfx: nf then only those Killing tensors L[j]: $ni \leq j \leq nf$ are calculated
kill(x)	kills the variable x and all its associated properties; useful if x is wanted for something else or if computer space needs to be saved
sum(k[i],i,a,b)	$\sum_{i=a}^{b} k_i$
diff(f,x,k)	$\frac{\partial^k f}{\partial x^k}$

The SEPCAL.V Program

(The line numbers are not part of the program and are only for reference)

```
ttyoff: true$
/* We enter the entire program as a block function hj().
   The contents are not displayed in the output files
   because of the command ttyoff: true$ */
hj(n,n1,n2,z,A,B,C,PHI):= block([],
gcprint:false,
print("The coordinate transformations are:"),
for m thru n do display(z[m]),
print("The Stackel matrix is"),
display(PHI),

if lsimp#true then LIST:[],
/*The following 2 lines are designed to print out an error message if the
list chosen to speed the simplification process does not have the required
n elements*/
if lsimp=true and length(LIST)#n then
(print("**ERROR** your list does not have n elements"),return(ttt) ),

/*This section computes F, the inverse of the Stackel matrix PHI*/
print("The inverse of the Stackel matrix is"),
F: block([?varlist:?varlist],genmatrix(PHIX,n2,n2)^^-1) ,
F: ratsimp(ev(F,PHIX=PHI)),
display(F),

/*First we introduce the symmetries A[u,v,a]=A[v,u,a],C[u,v,a]=C[v,u,a]*/
for a thru n1   do
 for u from n2+1 thru n do
  for v from u     thru n do
   (A[v,u,a]: A[u,v,a]),

for r from n1+1 thru n2 do
 for u from n2+1 thru n  do
  for v from u     thru n  do
   (C[v,u,r]: C[u,v,r]),

/* Here we enter the contravariant metric whose components are g[i,j]*/
for i thru n do
 for j thru n do
  g[i,j]: 0 ,

for a thru n1 do
 g[a,a]: F[a,1] ,

for r from n1+1 thru n2 do
 for u from n2+1 thru n do
  g[r,u]: g[u,r]: F[r,1]*B[u,r] ,

for u from n2+1 thru n do
 for v from u     thru n do
  g[u,v]: g[v,u]: ratsimp( sum( A[u,v,a]*F[a,1],a,1,n1 )
                         + sum( C[u,v,r]*F[r,1],r,n1+1,n2 )) ,

/*HJ is just the Hamilton-Jacobi equation*/
HJ: sum(sum(g[i,j]*p[i]*p[j],i,1,n),j,1,n)=E,
print("The Hamilton-Jacobi equation is"),
print(HJ),

/*Next the transformation matrix J is computed, and subsequently the
sqg, the square root of the determinant of the metric by using sqg=det(J)*/
J: genmatrix(JJ,n,n),
sqg: block([?varlist:?varlist],determinant(J)),
JJ[i,k]:= ratsimp(diff( z[k],x[i] ))   ,
J: ev(J),
sqg: ratsimp(ev(sqg)),

print("The Helmholtz equation is"),
for i thru n do depends(PSI,x[i]),
HE: sum(sum(g[i,j]*diff(diff(PSI,x[i]),x[j]),i,1,n),j,1,n)
   +sum( ratsimp(sum(diff(sqg*g[i,j],x[i]),i,1,n)/sqg) * diff(PSI,x[j]),j,1,n)
   = E*PSI,
```

```
print(HE),

/* The matrix gsup representing the contravariant metric g[i,j] is given
   and gsub the matrix representing the metric is computed using J  */
gsup:genmatrix(g,n,n),
gsub: ratsimp(J.transpose(J)),

metric: ds^2 = sum(sum(gsub[i,j]*dx[i]*dx[j],i,1,n),j,1,n),
print("The metric is"),
print(metric),

/* Here we check that gsub.gsup = I the identity matrix. If they don't
   an error message is printed */
II:trigsimp(gsub.gsup),
I:ident(n),
if II#I then (
print("**ERROR** the metric and the HJ-equation displayed are not compatible.
The product of their two matrices is:"),
print(II),
return(uuu)  ),

/* Evaluation of the Hamilton-Jacobi separation equations */
print("The Hamilton-Jacobi separation equations are"),
for a thru n1 do
  se[a]: p[a]^2 = - sum(sum(A[u,v,a]*'L[u]*'L[v],u,n2+1,n),v,n2+1,n)
                  + sum(PHI[b,a]*'L[b],b,1,n2),

for r from n1+1 thru n2 do
  se[r]: sum(2*B[u,r]*'L[u],u,n2+1,n)*p[r] =
                  - sum(sum(C[u,v,r]*'L[u]*'L[v],u,n2+1,n),v,n2+1,n)
                  + sum(PHI[b,r]*'L[b],b,1,n2),

for u from n2+1 thru n do
  se[u]: p[u] = 'L[u],

for m thru n do print(se[m]),

/* This section is designed to give the Schrodinger conditions determining
the unknown functions E[u,s] in the Helmholtz separation equations      */
if Schrodinger=true then
   ( print("the Schrodinger conditions are:"),
       for a thru n1 do
        ( SC[a]: diff(log(sqg*g[a,a]), x[a]) = f~[a] , display(SC[a]) ),
       for u from n2+1 thru n do
        ( SC[r]: sum( 1/sqg*diff(sqg*g[r,u], x[r]), r,n1+1,n2)
                      = sum(F[t,1]*E[u,t], t,1,n2) , display(SC[r]) )
   ),

/* Evaluation of the Helmholtz separation equations */
print("The Helmholtz separation equations are"),
for a thru n1 do
  SE[a]: diff(diff(PSI[a],x[a]),x[a])
       + ratsimp( (diff(sqg*g[a,a],x[a])/(sqg*g[a,a])) ) * diff(PSI[a],x[a])
       + ratsimp( sum(sum(A[u,v,a]*'l[u]*'l[v],u,n2+1,n),v,n2+1,n)
                 +sum(E[u,a]*'l[u],u,n2+1,n)
                 -sum(PHI[b,a]*'l[b],b,1,n2) ) * PSI[a]
       = 0,

for r from n1+1 thru n2 do
  SE[r]: sum(2*B[u,r]*'l[u],u,n2+1,n) * diff(PSI[r],x[r])
       + ratsimp( sum(sum(C[u,v,r]*'l[u]*'l[v],u,n2+1,n),v,n2+1,n)
                 +sum(E[u,r]*'l[u],u,n2+1,n)
                 -sum(PHI[b,r]*'l[b],b,1,n2) ) * PSI[r]
       = 0,

for u from n2+1 thru n do
  SE[u]: diff(PSI[u],x[u]) = 'l[u]*PSI[u],

for m thru n do print(SE[m]),
```

```
/* The inverse Ji of the transformation matrix J is computed */
Ji: block([?varlist:?varlist],genmatrix(JX,n,n)^^-1) ,
Ji: ratsimp(ev(Ji,JX=J)),

/* THE NEXT SECTION IS THE HEART OF THE PROGRAM - WHERE THE CONSTANTS
   OF THE MOTION ARE COMPUTED */
/* If allcons=false this enables the user to only compute those
   constants of the motion between the the input values of ni and nf.
   This is useful if the program is taking too long to run */
if allcons=false then (nix: ni ,nfx: nf),
if allcons#false then (nix: 2  ,nfx: n2),

print("The constants of the motion are"),
for m from nix thru nfx do
 (
/* First the constants are computed in the x[i],p[i] coordinates */
for i thru n do
 for j thru n do
   a[i,j,m]: 0 ,

for b thru n1 do
  a[b,b,m]: F[b,m] ,

for r from n1+1 thru n2 do
 for u from n2+1 thru n do
   a[r,u,m]: a[u,r,m]: F[r,m]*B[u,r] ,

for u from n2+1 thru n do
 for v from u    thru n do
   a[u,v,m]: a[u,v,m]: ratsimp( sum( A[u,v,a]*F[a,m],a,1,n1 )
                              + sum( C[u,v,r]*F[r,m],r,n1+1,n2 ) ),

/* We now get down to the business of finding the A[r,s,t,u]'s
  etc. needed to compute the constants of the motion in terms of the
  enveloping algebra. First evaluate a~ = transpose(J).am.J */
  aa[i,j]:= a[i,j,m],
  am: genmatrix(aa,n,n),
  a~: ratsimp( transpose(J).am.J ),
  kill(aa,am),

/* The following are the necessary differentations etc. to find
   the constants for the coefficients in the enveloping algebra */
Da~[r,s,t]:= 0,
fDa~[r,s,t]:= ratsimp( sum(Ji[t,k]*diff(a~[r,s],x[k]), k,1,n) ),

for r from 1   thru n-2 do
 for s from r+1 thru n-1 do
  for t from s+1 thru n   do
   (Da~[r,s,t]: fDa~[r,s,t],Da~[r,t,s]: fDa~[r,t,s]),

for r from 1 thru n-1 do
 for s from 1 thru n   do
  (Da~[r,s,s]:fDa~[r,s,s],Da~[r,r,s]:fDa~[r,r,s],Da~[r,s,r]:-1/2*Da~[r,r,s]),

 A~[r,s,t,u]:= 0,
fA~[r,s,t,u]:= ratsimp(sum(ev(
      Ji[u,k]*ev(ff[k]), ff[k]: ev(diff(Da~[r,s,t],x[k]),diff)
                                  ,LIST), k,1,n)/2),

/* LIST refers to the possibilty of substituting the values
   LIST = [x[1]=value1,...,x[n]=valuen], to speed the calculation
   of the constants fA~[r,s,t,u] */

for r from 1   thru n-3 do
 for s from r+1 thru n-2 do
  for t from s+1 thru n-1 do
   for u from t+1 thru n   do
    (A~[r,t,s,u]: fA~[r,t,s,u],A~[r,s,t,u]: fA~[r,s,t,u]),
```

```
for r from 1   thru n-2 do
 for s from r+1 thru n-1 do
  for u from s+1 thru n   do
   (A~[r,s,s,u]  fA~[r,s,s,u],A~[r,r,s,u]  fA~[r,r,s,u],
                                A~[r,s,u,u]. fA~[r,s,u,u] ),

for r from 1   thru n-1 do
 for s from r+1 thru n   do
  (A~[r,r,s,s]. fA~[r,r,s,s]),

/* In this section we exploit the symmetries of the constants A[r,s,t,u]
   etc. in order to remove unnecessary computations and thus save time */

AA~[i,j,k,l]:= block([Q],
Q: [min(i,j),max(i,j),min(k,l),max(k,l)],
if Q[1] # min(i,j,k,l) then
Q: [Q[3],Q[4],Q[1],Q[2]],
if Q[1] = min(i,j,k,l) and Q[1] = Q[3] and Q[2] > Q[4] then
Q: [Q[3],Q[4],Q[1],Q[2]],
if Q[4] = max(i,j,k,l) then
        if Q[1] = Q[3] or Q[2] = Q[4] then
        AA~[i,j,k,l]: -1/2*A~[Q[1],Q[3],Q[2],Q[4]]
        else
        AA~[i,j,k,l]:      A~[Q[1],Q[2],Q[3],Q[4]]
else AA~[i,j,k,l]: -A~[Q[1],Q[3],Q[4],Q[2]]-A~[Q[1],Q[4],Q[3],Q[2]],
AA~[i,j,k,l]
                    ),

fA~z[r,s,t]:= ratsimp( sum(AA~[r,s,t,l]*z[l], l,1,n) ),
 fB~[r,s,t]:= ratsimp( ev( Da~[r,s,t]-2*fA~z[r,s,t] , LIST) ),

for r from 1   thru n-2 do
 for s from r+1 thru n-1 do
  for u from s+1 thru n   do
   ( A~z[r,s,u]: fA~z[r,s,u],A~z[r,u,s]: fA~z[r,u,s],
      B~[r,s,u]:  fB~[r,s,u], B~[r,u,s]: fB~[r,u,s] ),

for r from 1   thru n-1 do
 for s from r+1 thru n   do
 ( A~z[r,s,r]: fA~z[r,s,r],A~z[r,s,s]: fA~z[r,s,s],A~z[r,r,s]: fA~z[r,r,s],
    B~[r,s,r]:  fB~[r,s,r], B~[r,s,s]:  fB~[r,s,s],B~[r,r,s]:  fB~[r,r,s] ),

AA~z[i,j,k]:= block([Q],
Q: [min(i,j),max(i,j)],
        if k < Q[1] then
        AA~z[i,j,k]: -A~z[k,Q[1],Q[2]]-A~z[k,Q[2],Q[1]]
        else
        AA~z[i,j,k]:  A~z[Q[1],Q[2],k],
        if i=j and i=k then
        AA~z[i,j,k]: 0,
AA~z[i,j,k]
                   ),

BB~[i,j,k]:= block([Q],
Q: [min(i,j),max(i,j)],
        if k < Q[1] then
        BB~[i,j,k]: -B~[k,Q[1],Q[2]]-B~[k,Q[2],Q[1]]
        else
        BB~[i,j,k]:  B~[Q[1],Q[2],k],
        if i=j and i=k then
        BB~[i,j,k]: 0,
BB~[i,j,k]
                   ),
for r thru n do
 for s from r thru n do
  ( C~[r,s]  ratsimp(
    ev(a~[r,s]-sum( (AA~z[r,s,l]+BB~[r,s,l])*z[l] ,l,1,n ) , LIST,infeval)
                    )
  ),
```

```
          if n>3 then
           L[m] 4*sum(sum(sum(sum(A~[r,t,s,u]*M[r,s]*M[t,u]
                                  +A~[r,s,t,u]*M[r,t]*M[s,u]
                                     ,u,t+1,n),t,s+1,n-1),s,r+1,n-2),r,1,n-3)
                      +sum(sum(sum(2*A~[r,r,s,u]*M[r,s]*M[r,u]
                                  +4*A~[r,s,s,u]*M[r,s]*M[s,u]
                                  +2*A~[r,s,u,u]*M[r,u]*M[s,u]
                            -2*B~[r,s,u]*M[r,u]*P[s]-2*B~[r,u,s]*M[r,s]*P[u]
                                     ,u,s+1,n),s,r+1,n-1),r,1,n-2)
                           +sum(sum( A~[r,r,s,s]*M[r,s]^2
                            +2*B~[r,s,r]*M[r,s]*P[r]-2*B~[r,s,s]*M[r,s]*P[s]
                                    +2*C~[r,s]*P[r]*P[s]
                                     ,s,r+1,n),r,1,n-1)
                              +sum(C~[r,r]*P[r]^2,r,1,n)
          else if n=3 then
           L[m]:        sum(sum(sum(2*A~[r,r,s,u]*M[r,s]*M[r,u]
                                  +4*A~[r,s,s,u]*M[r,s]*M[s,u]
                                  +2*A~[r,s,u,u]*M[r,u]*M[s,u]
                            -2*B~[r,s,u]*M[r,u]*P[s]-2*B~[r,u,s]*M[r,s]*P[u]
                                     ,u,s+1,n),s,r+1,n-1),r,1,n-2)
                           +sum(sum( A~[r,r,s,s]*M[r,s]^2
                            +2*B~[r,s,r]*M[r,s]*P[r]-2*B~[r,s,s]*M[r,s]*P[s]
                                    +2*C~[r,s]*P[r]*P[s]
                                     ,s,r+1,n),r,1,n-1)
                              +sum(C~[r,r]*P[r]^2,r,1,n)
          else if n=2 then
           L[m]:           sum(sum( A~[r,r,s,s]*M[r,s]^2
                            +2*B~[r,s,r]*M[r,s]*P[r]-2*B~[r,s,s]*M[r,s]*P[s]
                                    +2*C~[r,s]*P[r]*P[s]
                                     ,s,r+1,n),r,1,n-1)
                              +sum(C~[r,r]*P[r]^2,r,1,n),

  display(L[m]),
  kill(A~,B~,C~,AA~z,fA~z,fB~,BB~,AA~,A~z,AA~,fA~,fDa~,Da~)
 ),

/*The constants c~[i,j] and d~[i] for the KILLING VECTORS now calculated*/
for m from n2+1 thru n do
  (
    for j from 1 thru n do
      for i from 1 thru j do
        c~[j,i]:- (c~[i,j]:
ratsimp(sum( ev( Ji[j,k]*ev(gg[k]), LIST, gg[k]: ev(diff(JJ[m,i],x[k]),diff))
                                   ,k,1,n))
  ),

    for i thru n do
      d~[i]: ratsimp( ev(JJ[m,i]-sum(c~[i,j]*z[j],j,1,n) , LIST) ),

    L[m]: sum(d~[i]*P[i],i,1,n)-sum(sum(c~[i,j]*M[i,j],i,1,j-1),j,2,n),
    display(L[m]),
    kill(c~,d~)
  ),

/*If the equation happens to be of heat type we use the following
section to simplify the constants of the motion*/
if heat=true then
        (kill(B,B~),
              print("Expressed in terms of B[j] = (M[j,n-1] - iM[j,n])/2,
              epsilon = (P[n-1]-iP[n])/2, the constants of the motion are:"),
        for j thru n-2 do
         (M[j,n-1]: B[j] + B~[j],  M[j,n]: %i*(B[j] - B~[j])),
        P[n-1]: epsilon + K[-2], P[n]: %i*(epsilon - K[-2]),
        for m from 2 thru n do
          ( L[m]: ratsimp(ev(L[m])), display(L[m]) )

        )

)$
ttyoff:false $
```

Appendix D General Results

In this appendix we collect some general results concerning separation on a Riemannian manifold with metric

$$ds^2 = g_{ij}\, dx^i dx^j \tag{D1}$$

Using his definition of equivalence Benenti (1980a,b) was able to give the conditions for separability of the Hamilton-Jacobi equation

$$H = g^{ij} p_i p_j = E \tag{D2}$$

He showed that each of his classes contained a *canonical* separable system $\{x^a, x^r, x^\alpha\}$ with contravariant metric

$$(g^{ij}) = \begin{pmatrix} \delta^{ab} g^{aa} & 0 & 0 \\ 0 & 0 & g^{r\alpha} \\ 0 & g^{r\alpha} & g^{\alpha\beta} \end{pmatrix} \begin{matrix} n_1 \\ n_2 \\ n_3 \end{matrix} \tag{D3}$$

Here $n_3 \geq n_2$ and the integer indices a, r, α vary in the ranges $1 \leq a, b \leq n_1$; $n_1 + 1 \leq r \leq n_1 + n_2$; $n_1 + n_2 + 1 \leq \alpha, \beta \leq n_1 + n_2 + n_3 = n$. The nonzero components of the contravariant metric (2.19) are

$$g^{aa} = \frac{\psi^{a1}}{\psi}, \quad g^{r\alpha} = B_r^{r\alpha}(x^r)\, \frac{\psi^{r1}}{\psi}, \quad g^{\alpha\beta} = \sum_b A_b^{\alpha\beta}(x^b)\, \frac{\psi^{b1}}{\psi}, \tag{D4}$$

where ψ and ψ^{i1} are the determinant and $i1$ cofactors of the $(n_1+n_2) \times (n_1+n_2)$ Stäckel matrix $(\psi_{ij}(x^i))$. Furthermore, $\partial_\alpha g^{ij} = 0$ for each *ignorable* variable x^α. The variables x^a and x^r are referred to as *Stäckel* and *first order* variables respectively. Together these variables form the class of essential variables that was mentioned in the Introduction.

The constants of the motion are given in Kalnins and Miller (1981). They are

$$\begin{aligned} a_m^{ij} p_1 p_j &= \lambda_m, \quad 1 \leq m \leq n_1 + n_2 \\ P_\alpha &= \lambda_\alpha, \quad n_1 + n_2 + 1 \leq \alpha \leq n \end{aligned} \tag{D5}$$

where

$$(a_m^{ij}) = \begin{pmatrix} \delta^{ab} a_m^{bb} & 0 & 0 \\ 0 & 0 & a_m^{r\alpha} \\ 0 & a_m^{r\alpha} & a_m^{\alpha\beta} \end{pmatrix} \tag{D6}$$

The nonzero components of this tensor are

$$a_m^{bb} = \frac{\psi^{bm}}{\psi}, \quad a_m^{r\alpha} = B_r^{r\alpha}(x^r)\,\frac{\psi^{rm}}{\psi}, \quad a_m^{\alpha\beta} = \sum_b A_b^{\alpha\beta}(x^b)\frac{\psi^{bm}}{\psi}, \tag{D7}$$

(It is straightforward, though tedious, to verify $\{\lambda_i, \lambda_j\}_P = 0$). The Helmholtz equation

$$g^{-\frac{1}{2}}\partial_i(g^{\frac{1}{2}} g^{ij}\partial_j \Psi) = E\Psi \tag{D8}$$

is separable provided that the corresponding Hamilton-Jacobi equation (D2) is separable, and the additional (Schrödinger) conditions are satisfied. That is there are functions $f_a(x^a)$, $E_a^\alpha(x^a)$ and $E_r^\alpha(x^r)$ such that

$$\partial_a \log(g^{\frac{1}{2}} g^{aa}) = f_a(x^a)\ , \tag{D9a}$$

$$\sum_{r=n_1+1}^{n_1+n_2} g^{-\frac{1}{2}}\partial_r(g^{\frac{1}{2}} g^{r\alpha}) = \sum_{a=1}^{n_1} \frac{\psi^{a1}}{\psi} E_a^\alpha(x^a) + \sum_{r=n_1+1}^{n_1+n_2} \frac{\psi^{r1}}{\psi} E_r^\alpha(x^r)\ . \tag{D9b}$$

$$\alpha = n_1+n_2+1, \ldots, n$$

These conditions can be derived from the work of Kalnins and Miller (1983).
If the Helmholtz equation is separable then the Helmholtz separation equations are:

$$\Psi_a'' + f_a\Psi_a' + \left\{\sum_\alpha E_a^\alpha l_\alpha + \sum_{\alpha,\beta} A_a^{\alpha\beta} l_\alpha l_\beta - \sum_{t=1}^{n_1+n_2} \psi_{at}(x^a) l_t\right\}\Psi_a = 0$$

$$, a = 1, \ldots, n_1, \tag{D10a}$$

$$2\sum_\alpha B^{r\alpha} l_\alpha \Psi_r' + \left\{\sum_\alpha E_r^\alpha l_\alpha + \sum_{\alpha,\beta} A_r^{\alpha,\beta} l_\alpha l_\beta - \sum_{t=1}^{n_1+n_2} \psi_{rt}(x^r) l_t\right\}\Psi_r = 0$$

$$, r = n_1+1, \ldots, n_1+n_2 \tag{D10b}$$

$$\psi_\alpha' = l_\alpha \psi_\alpha \qquad , \alpha = n_1+n_2+1, \ldots, n \tag{D10c}$$

Here the l_i are the eigenvalues of the operators characterising the separation,

$\Psi = \prod_{i=1}^{n} \Psi_i(x^i)$ is the separable solution, and $\Psi_i{}' = \dfrac{d\Psi_i}{dx^i}$.

The components of the Riemann curvature tensor are given by

$$R_{hijk} = \frac{1}{2}(g_{ij,hk} + g_{hk,ij} - g_{hj,ik} - g_{ik,hj})$$

$$+ g^{lm}([ij,l][hk,m] - [hj,l][ik,m]) \qquad \text{(D11)}$$

where

$$[a\,b\,,c\,] = \frac{1}{2}(g_{ac,b} + g_{bc,a} - g_{ab,c})$$

The Ricci tensor is defined by

$$R_{ij} = g^{hk} R_{hijk} \qquad \text{(D12)}$$

Bibliography

Agostinelli C

(1958a), Atti Acc. Sc. Torino **92**, 411-443.

(1958b), Boll. U.M.I. **13** (3), 11-23.

Benenti S and Francaviglia M

(1980a), *Separability Structures in Riemannian manifolds*, Proceedings of the Conference on "Differential Geometrical Methods in Mathematical Physics", held in Salamanca 1979, Lecture Notes in Mathematics **836**, (Springer-Verlag, Berlin).

(1980b), *The theory of separability of the Hamilton-Jacobi equation and its applications to General Relativity*, in General Relativity and Gravitation, **1**, edited by Held A (Plenum, New York).

Bluman G and Cole J (1974), *Similarity Methods for Differential Equations*, Applied Mathematical Sciences **13** (Springer, New York).

Böcher M (1894), *Die Reihenentwickelungen der Potentialtheorie* (Leipzig).

Boyer C P

(1974), *The maximal kinematical invariance group for an arbitrary potential*, Helv. Phys. Acta **47**, 589-605.

(1976), *Lie theory and separation of variables for the equation* $iU_t + \Delta_2 U - (\alpha / x_1^2 + \beta / x_2^2) U = 0$, SIAM J. Math. Anal. **7**, 230-263.

Boyer C P, Kalnins E G and Miller W Jr. (1983), *Stäckel-equivalent integrable Hamiltonian systems*, University of Waikato Mathematics Research Report No. 120.

Burgatti P (1911), Rend. Acc. Naz. Lincei **20**, 108-111.

Carter B (1968), *Hamilton-Jacobi and Schrödinger Separable Solutions of Einstein's Equations*, Comm. math. phys. **10**, 280-310; Math. Review **39**, #1198.

Dall'Acqua A (1912), Rend. Circ. Mat. Palermo **33** , 341-351.

Delong R P (1982), *Killing Tensors and the Hamilton-Jacobi equation*, Doctoral Thesis (School of Mathematics, University of Minnesota).

Eisenhart L P

(1934), *Separable systems of Stäckel*, Ann. Math. **35** , 284-305.

(1949), *Riemannian Geometry*, (Princeton University Press, Princeton).

Gerdt V P, Tarasov O V and Shirkov D V (1980), *Analytic calculations on digital computers for applications in physics and mathematics*, Sov. Phys. Usp. **23** (1), 59-77. (Amer. Inst. of Physics).

Hagihara Y (1970), Celestial Mechanics, Vol 1, (MIT Press, Cambridge, Massachusetts, London).

Hausner M and Schwartz J (1968), *Lie Groups and Lie Algebras*, (Gordon & Breach, New York).

Huaux A (1976), Ann. Mat. Pura Appl. **108** , 251-282.

Kalnins E G and Miller W Jr.

(1976), *Lie theory and separation of variables. 9. Orthogonal R-separable coordinate systems for the wave equation* $\Psi_{tt} - \Delta_2\Psi = 0$, J. Math. Phys. **17** , 331-355.

(1977), *Lie theory and the wave equation in space-time. 3. Semi subgroup coordinates*, J. Math. Phys. **18** , 271.

(1979), *Non-orthogonal separable coordinate systems for the flat 4-space Helmholtz equation*, J. Phys. A: Math. Gen **12** , 1129-1147.

(1980), *Some remarkable R-separable coordinate systems for the Helmholtz equation*, Letters in Math. Phys. **4** , 469.

(1981), *Killing Tensors and nonorthogonal variable separation for Hamilton-Jacobi equations*, SIAM J. Math. Anal. **12** , 617-629.

(1982a). *Separation of variables on n-dimensional Riemannian manifolds 1 The n-sphere and Euclidean n-space*. University of Minnesota Mathematics Report 81-158 (to be published).

(1982b). *Separation of variables on n-dimensional Riemannian manifolds 2. The n-dimensional hyperboloid H_n*, University of Minnesota Mathematics Report 81-160 (to be published).

(1982c). *Separation of variables on n-dimensional Riemannian manifolds 3. Conformally Euclidean spaces C_n*, University of Minnesota Mathematics Report 81-171 (to be published).

(1983). *Intrinsic Characterisation of Variable Separation for the Partial Differential Equations of Mechanics*, in Proc. of the IUTAM-ISIMM Symposium on "Modern Developments in Analytical Mechanics", edited by Benenti S, Francaviglia M, and Lichnerowicz A (Academia delle Scienze di Torino, Torino).

Kalnins E G, Miller W Jr., and Reid G J (1983), *Separation of variables for complex Riemannian spaces of constant curvature 1. Orthogonal separable coordinates for $S_{n\mathbb{C}}$ and $E_{n\mathbb{C}}$*, University of Waikato Mathematics Research Report No. 116 (accepted by Proceedings of the Royal Society).

Kalnins E G and Reid G J (1982), *R-Separation for the Hamilton-Jacobi Equation*, Letters in Mathematical Physics, **6** , 97-100.

Kato T (1966), *Perturbation Theory for Linear Operators*, page 493 (Springer, New York).

Levi-Civita T (1904), *Sulla integrazione della equazione di Hamilton-Jacobi per separazione di variabili*, Math. Ann. **59** , 383-397. For some of the same results also see Hagihara (1970), 78-80.

Liouville J (1846), *Sur quelques cas particuliers où les équations du mouvement d'un point matériel peuvent s'integrer*, J. de Math. 11 , 345-378.

Mathlab Group (1983), *MACSYMA Reference Manual*, Version 10 (MIT, Cambridge, Massachusetts).

Miller W Jr. (1977) *Symmetry and Separation of Variables*, (Reading, M: Addison-Wesley).

Moon P and Spencer D

(1952), *Separability Conditions for the Laplace and Helmholtz Equations*, J. Franklin Inst. **253** , 585.

(1961), *Field Theory Handbook* (Springer, Berlin)

Morse P and Feshbach H (1953), *Methods of Theoretical Physics*, Part I (McGraw-Hill, New York).

Neiderer U (1973), *The maximal kinematical group of the harmonic oscillator*, Helv. Phys. Acta. **46** , 191-200.

Olevski P (1950), *The separation of variables in the equation* $\Delta_3 u + \lambda u = 0$ *for spaces of constant curvature in two and three dimensions*, Mat. Sb. **27** (69), 379-426.

Pars L A (1965), A Treatise on Analytical Dynamics (Heineman, London).

Robertson H P (1927), *Bemerkung über separierbare Systeme in der Wellenmechanik*, Math. Annalen **98** , 749-752.

Rosenbloom P and Widder D V (1959), *Expansions in terms of heat polynomials and associated functions*, Trans. Amer. Math. Soc. **92** , 220-266.

Stäckel P (1891), *Uber die Intergration der Hamilton-Jacobischen Differentialgleichung mittelst separation der variabelng* (Habiliationsschrift, Halle).

Talman J D

(1968), *Special Functions (A Group-theoretic Approach)*, (Benjamin, New York).

Weisner L

(1955), *Group-theoretic origin of certain generating functions* Pacific J. Math. **5** , 1033-1039.

(1959a) *Generating functions for Bessel functions*, Canad. J. Math. **11**, 148-155.

(1959b) *Generating functions for Hermite functions*, Canad. J. Math. **11**, 141-147.

Winternitz P and Fris I (1965), *Invariant expansions of relativistic amplitudes and subgroups of the proper Lorentz group*, Soviet Physics JNP **1**, 636-643.

Widder D V (1975), *The Heat Equation*, (Academic Press, New York).