Journal of Integer Sequences, Vol. 15 (2012),
Article 12.7.5

On Integers for Which the Sum of Divisors is the Square of the Squarefree Core

Kevin A. Broughan
Department of Mathematics
University of Waikato
Private Bag 3105
Hamilton, New Zealand
kab@waikato.ac.nz

Jean-Marie De Koninck
Départment de mathématiques et de statistique
Université Laval
Québec G1V 0A6
Canada
jmdk@mat.ulaval.ca

Imre Kátai
Department of Computer Algebra
Pázmány Péter sétány I/C
H-1117 Budapest
Hungary
katai@compalg.inf.elte.hu

Florian Luca
Centro de Ciencias Matemáticas
Universidad Nacional Autonoma de México
C. P. 58089
Morelia, Michoacán
México
fluca@matmor.unam.mx

Abstract

We study integers $n > 1$ satisfying the relation $\sigma(n) = \gamma(n)^2$, where $\sigma(n)$ and $\gamma(n)$ are the sum of divisors and the product of distinct primes dividing n, respectively. We show that the only solution n with at most four distinct prime factors is $n = 1782$. We show that there is no solution which is fourth power free. We also show that the number of solutions up to $x > 1$ is at most $x^{1/4+\epsilon}$ for any $\epsilon > 0$ and all $x > x_\epsilon$. Further, call n primitive if no proper unitary divisor d of n satisfies $\sigma(d) \mid \gamma(d)^2$. We show that the number of primitive solutions to the equation up to x is less than x^ϵ for $x > x_\epsilon$.

1 Introduction

At the Western Number Theory conference in 2000, the second author asked for all positive integer solutions n to the equation

$$\sigma(n) = \gamma(n)^2 \tag{1}$$

(denoted "De Koninck's equation"), where $\sigma(n)$ is the sum of all positive divisors of n, and $\gamma(n)$ is the product of the distinct prime divisors of n, the so-called "core" of n. It is easy to check that $n = 1$ and $n = 1782$ are solutions, but, as of the time of writing, no other solutions are known. A computer search for all $n \le 10^{11}$ did not reveal any other solution. The natural conjecture (coined the "De Koninck's conjecture") is that there are no other solutions. It is included in Richard Guy's compendium [1, Section B11].

It is not hard to see, and we prove such facts shortly, that any non-trivial solution n must have at least three prime factors, must be even, and can never be squarefree. The fourth author [2] has a derivation that the number of solutions with a fixed number of prime factors is finite. Indeed, he did this for the broader class of positive solutions n to the equation $\sigma(n) = a\gamma(n)^K$ where $K \ge 2$ and $1 \le a \le L$ with K and L fixed parameters. Other than this, there has been little progress on De Koninck's conjecture.

Here, we show that the above solutions $n = 1,\ 1782$ are the only ones having $\omega(n) \le 4$. As usual, $\omega(n)$ stands for the number of distinct prime factors of n. The method relies on elementary upper bounds for the possible exponents of the primes appearing in the factorization of n and then uses resultants to solve the resulting systems of polynomial equations whose unknowns are the prime factors of n.

We then show that if an integer n is fourth power free (i.e. $p^4 \nmid n$ for all primes p), then n cannot satisfy De Koninck's equation (1). We then count the number of potential solutions n up to x. Pollack and Pomerance [4], call a positive integer n to be *prime–perfect* if n and $\sigma(n)$ share the same set of prime factors. Obviously, any solution n to the De Koninck's equation is also prime–perfect. Pollack and Pomerance show that the set of prime–perfect numbers is infinite and the counting function of prime–perfects $n \le x$ has cardinality at most $x^{1/3+o(1)}$ as $x \to \infty$. By using the results of Pollack and Pomerance, we show that the number of solutions $n \le x$ to De Koninck's equation is at most $x^{1/4+\epsilon}$ for any $\epsilon > 0$ and all $x > x_\epsilon$.

By restricting to so-called "primitive" solutions, using Wirsing's method [5], we obtain an upper bound of $O(x^\epsilon)$ for all $\epsilon > 0$. The notion of primitive that is used is having no proper unitary divisor $d \mid n$ satisfying $\sigma(d) \mid \gamma(d)^2$. In a final section of comments, we make some remarks about the related problem of identifying those integers n such that $\gamma(n)^2 \mid \sigma(n)$.

In summary: the aim of this paper is to present items of evidence for the truth of De Koninck's conjecture, and to indicate the necessary structure of a possible counter example.

Any non-trivial solution other than 1782 must be even, have one prime divisor to power 1 and possibly one prime divisor to a power congruent to 1 modulo 4, with other odd prime divisors being to even powers. At least one prime divisor must appear with an exponent 4 or more. Finally, any counter example must be greater than 10^{11}.

We use the following notations, most of which have been recorded already: $\sigma(n)$ is the sum of divisors, $\gamma(n)$ is the product of the distinct primes dividing n, if p is prime $v_p(n)$ is the highest power of p which divides n, $\omega(n)$ is the number of distinct prime divisors of n, and $\mathcal{K}$ is the set of all solutions to $\sigma(n) = \gamma(n)^2$. The symbols p, q, p_i and q_i with $i = 1, 2, \ldots$ are reserved for odd primes.

2 Structure of solutions

First we derive the shape of the members of $\mathcal{K}$.

Lemma 1. *If $n > 1$ is in $\mathcal{K}$, then*

$$n = 2^e p_1 \prod_{i=2}^{s} p_i^{a_i},$$

where $e \geq 1$ and a_i is even for all $i = 3, \ldots, s$. Furthermore, either a_2 is even in which case $p_1 \equiv 3 \pmod 8$, or $a_2 \equiv 1 \pmod 4$ and $p_1 \equiv p_2 \equiv 1 \pmod 4$.

Proof. Firstly, we note that n must be even: indeed, if $n > 1$ satisfies $\sigma(n) = \gamma(n)^2$ and n is odd, then $\sigma(n)$ must be odd so that the exponent of each prime dividing n must be even, making n a perfect square. But then $n < \sigma(n) = \gamma(n)^2 \leq n$, a contradiction.

Secondly, since n is even, it follows that $2^2 \| \gamma(n)^2$. Write

$$n = 2^e \prod_{i=1}^{s} p_i^{a_i}$$

with distinct odd primes $p_1, \ldots, p_s$ and positive integer exponents $a_1, \ldots, a_s$, where the primes are arranged in such a way that the odd exponents appear at the beginning and the even ones at the end. Using the fact that $\sigma(2^e) = 2^{e+1} - 1$ is odd, we get that $2^2 \| \prod_{i=1}^{s} \sigma(p_i^{a_i})$. Thus, there are at most two indices i such that $\sigma(p_i^{a_i})$ is even, with all the other indices being odd. But if p is odd and $\sigma(p^a)$ is also odd, then a is even. Thus, either only a_1 is odd, or only a_1 and a_2 are odd. Now let us show that there is at least one exponent which is 1. Assuming that this is not so, the above argument shows that $a_1 \geq 3$ and that $a_i \geq 2$ for $i = 2, \ldots, s$. Thus,

$$4p_1^2 \prod_{i=2}^{s} p_i^2 = \gamma(n)^2 = \sigma(n) \geq \sigma(2)\sigma(p_1^3) \prod_{i=2}^{s} \sigma(p_i^2) > 3p_1^3 \prod_{i=2}^{s} p_i^2,$$

leading to $p_1 < 4/3$, which is impossible. Hence, $a_1 = 1$. Finally, if a_2 is even, then $2^2 \| \sigma(p_1)$ showing that $p_1 \equiv 3 \pmod 8$, while if a_2 is odd, then $2 \| \sigma(p_1)$ and $2 \| \sigma(p_2^{a_2})$, conditions which easily lead to the conclusion that $p_1 \equiv p_2 \equiv 1 \pmod 4$ and $a_2 \equiv 1 \pmod 4$. $\square$

3 Solutions with $\omega(n) \le 4$

Theorem 2. *Let $n \in \mathcal{K}$ with $\omega(n) \le 4$. Then $n = 1$ or $n = 1782$.*

Proof. Using Lemma 1, we write $n = 2^\alpha pm$, where $\alpha > 0$ and m is coprime to $2p$.

We first consider the case $p = 3$. If additionally $m = 1$, we then get that $\sigma(n) = 6^2$, and we get no solution. On the other hand, if $m > 1$, then $\sigma(m)$ is a divisor of $\gamma(n)^2/4$ and must therefore be odd. This means that every prime factor of m appears with an even exponent. Say $q^\beta \| m$. Then

$$\sigma(q^\beta) = q^\beta + \cdots + q + 1$$

is coprime to $2q$ and is larger than $3^2 + 3 + 1 > 9$. Thus, there exists a prime factor of m other than 3 or q, call it r, which divides $q^\beta + \cdots + q + 1$, implying that it also divides m and that it appears in the factorization of m with an even exponent. Since $\omega(n) \le 4$, we have $m = q^\beta r^\gamma$. Now

$$q^\beta + \cdots + q + 1 = 3^i r^j \qquad \text{and} \qquad r^\gamma + \cdots + r + 1 = 3^k q^\ell,$$

where $i + k \le 2$ and $j, \ell \in \{1, 2\}$. Thus,

$$(q^\beta + \cdots + q + 1)(r^\gamma + \cdots + r + 1) = 3^{i+k} q^\ell r^j.$$

The left–hand side of this equality is greater than or equal to $3q^\beta r^\gamma$. In the case where $\beta > 2$, we have $\beta \ge 4$, so that $q^4 r^2 \le q^\beta r^\alpha \le 9q^2 r^2$, giving $q \le 3$, which is a contradiction. The same contradiction is obtained if $\gamma > 2$.

Thus, $\beta = \gamma = 2$. If $l = j = 2$, we then get that

$$(q^2 + q + 1)(r^2 + r + 1) = 3^{i+k} q^2 r^2,$$

leading to $\sigma(2^\alpha) \mid 3^{2-i-j}$. The only possibility is $\alpha = 1$ and $i + j = 1$, showing that $i = 0$ or $j = 0$. Since the problem is symmetric, we treat only the case $i = 0$. In that case, we get $q^2 + q + 1 = r^2$, which is equivalent to $(2q + 1)^2 + 3 = (2r)^2$, which has no convenient solution (q, r).

If $j = \ell = 1$, we then get that

$$q^2 r^2 < (q^2 + q + 1)(r^2 + r + 1) < 9qr,$$

implying that $qr < 9$, which is false.

Hence, it remains to consider the case $j = 2$ and $\ell = 1$, and viceversa. Since the problem is symmetric in q and r, we only look at $j = 2$ and $\ell = 1$. In that case, we have

$$q^2 r^2 < (q^2 + q + 1)(r^2 + r + 1) = 3^{i+k} r^2 q,$$

so that $q < 3^{i+k}$. Since $q > 3$, this shows that $i = k = 1$ and $q \in \{5, 7\}$. Therefore, $r^2 + r + 1 = 75,\ 147$, and neither gives a convenient solution n.

From now on, we can assume that $p > 3$, so that $p + 1 = 2^u m_1$, where $u \in \{1, 2\}$ and $m_1 > 1$ is odd. Let q be the largest prime factor of m_1. Clearly, $p + 1 \ge 2q$, so that $q < p$. Moreover, since $\omega(n) \le 4$ we have

$$p < 4q^4 < q^6,$$

so that $q > p^{1/6}$. Let again β be such that $q^\beta \| n$. We can show that $\beta \le 77$. Indeed, assuming that $\beta \ge 78$, we first observe that

$$p^{13} < q^{78} \le q^\beta < \sigma(q^\beta),$$

and write

$$\sigma(q^\beta) = 2^v m_2,$$

where $v \in \{0,1\}$ and m_2 is coprime to $2q$. If m_2 divides p^2, we get that

$$p^{13} < \sigma(q^\beta) \le 2p^2,$$

which is a contradiction. Thus, there exists another prime factor r of n, and $m_2 \le p^2 r^2$. Hence,

$$p^{13} < \sigma(q^\beta) < 2p^2 r^2 < p^3 r^2,$$

implying that $r > p^5$. Let γ be such that $r^\gamma \| n$. Then

$$r + 1 \le \sigma(r^\gamma) \le 2p^2 q^2 < p^5,$$

which is a contradiction. Thus, $\beta \le 77$.

Say r doesn't appear in the factorization of $(p+1)\sigma(q^\beta)$. Then we need to solve the system of equations

$$p + 1 = 2^u q^w \qquad \text{and} \qquad q^\beta + \cdots + q + 1 = 2^v p^z,$$

where $\beta \in \{1, \ldots, 77\},\ u \in \{1,2\},\ 0 \le v \le 2 - u,\ \{w, z\} \subseteq \{1,2\}$, which we can solve with resultants. This gives us a certain number of possibilities for the pair (p,q). If $\omega(n) = 3$, we have $\sigma(n) = 4p^2q^2$, and we find n. If $\omega(n) = 4$, then $\sigma(r^\gamma)$ is a divisor of $2p^2q^2$ and we find certain possibilities for the pair (r,γ). Then we extract n from the relation $\sigma(n) = 4p^2q^2r^2$.

Now say r appears in the factorization of $(p+1)\sigma(q^\beta)$. We then write

$$p + 1 = 2^u q^w r^\delta \qquad \text{and} \qquad \sigma(q^\beta) = 2^v p^z r^\eta, \tag{2}$$

where $u \in \{1,2\},\ w \in \{1,2\},\ 0 \le v \le 2 - u,\ z \in \{0,1,2\},\ \delta + \eta \in \{1,2\}$. If $z = 0$, then since $q > p^{1/6}$, we have that

$$q < \sigma(q^\beta) \le 2r^2 < r^3,$$

so that $r > q^{1/3} > p^{1/18}$. Now $\gamma \le 89$, for if not, then

$$p^5 < r^{90} \le r^\gamma < \sigma(r^\gamma) < 2p^2q^2 < p^5,$$

which is false.

Suppose now that $z > 0$. Then

$$q^w r^\delta < p < 4q^w r^\delta \tag{3}$$

from the first relation of (2), while

$$\frac{q^\beta}{2r^\eta} < p^z < \frac{2q^\beta}{r^\eta} \tag{4}$$

from the second relation of (2). If $z = 1$, we get from (3) and (4) that

$$r^{\delta+\eta} < 2q^{\beta-w} \qquad \text{and} \qquad r^{\delta+\eta} > \frac{q^{\beta-w}}{8}.$$

From the above left inequality and the fact that $\delta + \eta \geq 1$, we read that $\beta - w \geq 1$, and then from the right one that $9r^2 > 8r^{\delta+\eta} > q^{\beta-w} \geq q$, and thus $r^2 \geq 3r > q^{1/2}$, so that $r > q^{1/4} > p^{1/24}$. It now follows easily that $\gamma \leq 119$, for if not, then $\gamma \geq 120$ would give

$$p^5 < r^{120} \leq r^\gamma < \sigma(r^\gamma) \leq 2p^2q^2 < p^5,$$

which is a contradiction. Finally, if $z = 2$, we get from (4) that

$$\frac{q^{\beta/2}}{\sqrt{2}r^{\eta/2}} < p < \frac{\sqrt{2}q^{\beta/2}}{r^{\eta/2}},$$

which combined with (3) yields

$$r^{\delta+\eta/2} < \sqrt{2}q^{\beta/2-w} \qquad \text{and} \qquad r^{\delta+\eta/2} > \frac{q^{\beta/2-w}}{4\sqrt{2}}.$$

From the above left inequality and because $\delta + \eta/2 \geq 1/2$, we read that $\beta/2 > w$, implying that $\beta/2 - w \geq 1/2$. Thus,

$$4\sqrt{2}r^2 \geq 4\sqrt{2}r^{\delta+\eta/2} > q^{\beta/2-w} \geq q^{1/2}$$

and therefore

$$r^8 > 32r^4 \geq (4\sqrt{2}r^{\delta+\eta/2})^2 > q > p^{1/6},$$

showing that $r > p^{1/48}$. This shows that $\gamma \leq 239$, for if $\gamma \geq 240$, then

$$p^5 < r^{240} \leq r^\gamma < \sigma(r^\gamma) < 2p^2q^2 < p^5,$$

which is a contradiction. Thus, we need to solve

$$\begin{aligned} p+1 &= 2^u q^w r^\delta; \\ \sigma(q^\beta) &= 2^v p^z r^\eta; \\ \sigma(r^\gamma) &= 2^\lambda p^s q^t, \end{aligned}$$

where $1 \leq \beta \leq 77$, $1 \leq \gamma \leq 239$, $u \in \{1, 2\}$, $u + v + \lambda \leq 2$, $1 \leq w \leq 2$, $w + t \leq 2$, $\delta + \eta \in \{1, 2\}$, $z \in \{0, 1, 2\}$ and $s \in \{0, 1, 2\}$. This can be solved with resultants and it gives us a certain number of possibilities for the triplet (p, q, r). From $\sigma(n) = 4p^2q^2r^2$, we extract n by solving the equation for α, given p, q and r. Failure to detect an integer value for α means the candidate solution fails. A computer program went through all these steps and confirmed the conclusion of Theorem 2.

□

4 The case of fourth power free n

Theorem 3. *If $n > 1$ is in $\mathcal{K}$, then n is not fourth power free.*

Proof. Let us assume that the result is false, that is, that there exists some $n \in \mathcal{K}$ which is fourth power free. By Lemma 1 we can write

$$n = 2^e p_1 p_2^{a_2} \prod_{i=1}^{k} q_i^2,$$

where $a_2 \in \{0, 1\}$. Let $\mathcal{Q} = \{q_1, \ldots, q_k\}$. The idea is to exploit the fact that there exist at most two elements $q \in \mathcal{Q}$ such that $q \equiv 1 \pmod 3$. If there were three or more such elements, then 3^3 would divide $\prod_{q\in\mathcal{Q}} \sigma(q^2)$ and therefore a divisor of $\gamma(n)^2$, which is a contradiction.

We begin by showing that $k \le 8$. To see this, let

$$\mathcal{R} = \left\{ r \in \mathcal{Q} : \gcd\left(\sigma(r^2), \prod_{q\in\mathcal{Q}} q \right) = 1 \right\}.$$

Then $\prod_{r\in\mathcal{R}} \sigma(r^2)$ divides p_1^2 (if $a_2 = 0$) and $p_1^2 p_2^2$ if $a_2 > 0$. It follows that $\sigma(r^2)$ is either a multiple of p_1 or of p_2 for each $r \in \mathcal{R}$. Since there can be at most two r's for which $\sigma(r^2)$ is a multiple of p_1, and at most two r's for which $\sigma(r^2)$ is a multiple of p_2, we get that $\#\mathcal{R} \le 4$. When $r \in \mathcal{Q} \setminus \mathcal{R}$, we have, since $\sigma(r^2) > 9$, that $\sigma(r^2) = r^2 + r + 1$ is a multiple of some prime $q_{i_r} > 3$ for some $q_{i_r} \in \mathcal{Q}$. Now, since q_{i_r} is a prime divisor of $r^2 + r + 1$ larger than 3, it must satisfy $q_{i_r} \equiv 1 \pmod 3$. Since i_r can take the same value for at most two distinct primes r, and there are at most two distinct values of the index i_r, we get that $k - \#\mathcal{R} \le 4$, which implies that $k \le 8$, as claimed.

Next rewrite the equation $\sigma(n) = \gamma(n)^2$ as

$$\left(\frac{2^{e+1} - 1}{4}\right) \prod_{i=1}^{k} \left(\frac{q_i^2 + q_i + 1}{q_i^2}\right) = \left(\frac{p_1^2}{p_1 + 1}\right)\left(\frac{p_2^{2\delta_2}}{\sigma(p_2^{a_2})}\right), \tag{5}$$

where $\delta_2 = 0$ if $a_2 = 0$ and $\delta_2 = 1$ if $a_2 > 0$. The left–hand side of (5) is at most

$$\left(\frac{2^{e+1} - 1}{4}\right)\left(\prod_{q\le 23} \frac{q^2 + q + 1}{q^2}\right) < 0.73(2^{e+1} - 1). \tag{6}$$

First assume that $a_2 = 0$. Then the right–hand side of (5) is

$$\frac{p_1^2}{p_1 + 1} \ge \frac{9}{4} = 2.25. \tag{7}$$

If $e = 1$, then the left–hand side of inequality (5) is, in light of (6), smaller than $0.73(2^2 - 1) < 2.22$, which contradicts the lower bound provided in (7). Thus, $e \in \{2, 3\}$, and

$$\frac{p_1^2}{p_1 + 1} \le 0.73(2^4 - 1) = 10.95,$$

so that $p_1 \leq 11$. Since $p_1 \equiv 3 \pmod 8$, we get that $p_1 \in \{3, 11\}$. If $p_1 = 11$, then $3 \in \mathcal{Q}$. If $p_1 = 3$, then since $e \in \{2, 3\}$, we get that either 5 or 7 is in $\mathcal{Q}$.

If $3 \in \mathcal{Q}$, then $13 \mid 3^2 + 3 + 1$, $61 \mid 13^2 + 13 + 1$ and $97 \mid 61^2 + 61 + 1$ are all three in $\mathcal{Q}$ and are congruent to 1 modulo 3, a contradiction.

If $5 \in \mathcal{Q}$, then $31 \mid 5^2 + 5 + 1$, $331 \mid 31^2 + 31 + 1$ and $7 \mid 331^2 + 331 + 1$ are all in $\mathcal{Q}$, a contradiction.

If $7 \in \mathcal{Q}$, then $7, 19 \mid 7^2 + 7 + 1$ and $127 \mid 19^2 + 19 + 1$ are all in $\mathcal{Q}$, a contradiction.

Assume next that $a_2 > 0$. Then, by Lemma 1, $p_1 \equiv p_2 \equiv 1 \pmod 4$. Since $e \in \{1, 2, 3\}$, it follows that one of $3, 5, 7$ divides n.

If $3 \mid n$, then $3 \in \mathcal{Q}$.

If $5 \mid n$, and 5 is one of p_1 or p_2, then $3 \mid \sigma(p_1 p_2^{a_2}) \mid n$, while if $5 \in \mathcal{Q}$, then $31 = 5^2 + 5 + 1$ is not congruent to 1 modulo 4 and divides n, implying that it belongs to $\mathcal{Q}$, and thus $3 \mid 31^2 + 31 + 1 \mid n$.

Finally, if $7 \mid n$, then 7 cannot be p_1 or p_2, meaning that 7 is in $\mathcal{Q}$ and therefore that $3 \mid 7^2 + 7 + 1$, which implies that $3 \mid n$.

To sum up, it is always the case that when $a_2 > 0$, necessarily 3 divides n.

Hence, $13 = 3^2 + 3 + 1$ divides n, so that either $13 \in \mathcal{Q}$, or not. If $13 \notin \mathcal{Q}$, then $7 \mid 13 + 1$ is in $\mathcal{Q}$, in which case $19 \mid 7^2 + 7 + 1$ divides n and it is not congruent to 1 modulo 4, implying that $19 \in \mathcal{Q}$ and thus that $127 \mid 19^2 + 19 + 1$ divides n and is not congruent to 1 modulo 4, so that $127 \in \mathcal{Q}$. Hence, all three numbers $7, 19, 127$ are in $\mathcal{Q}$, which again is a contradiction.

If $13 \in \mathcal{Q}$, then $61 \mid 13^2 + 13 + 1$ divides n.

If 61 is one of p_1 or p_2, then $31 \mid \sigma(p_1 p_2^{a_2})$ and $31 \equiv 3 \pmod 4$, so that $31 \in \mathcal{Q}$. Next $331 \mid 31^2 + 31 + 1$ is a divisor of n and it is not congruent to 1 modulo 4, implying that it belongs to $\mathcal{Q}$ and therefore that $13, 31, 331$ are all in $\mathcal{Q}$, a contradiction.

Finally, if $61 \in \mathcal{Q}$, then $97 \mid 61^2 + 61 + 1$ is a divisor of n. If $97 \in \mathcal{Q}$ we get a contradiction since 13 and 61 are already in $\mathcal{Q}$, while if 97 is one of p_1 or p_2, then $7 \mid \sigma(p_1 p_2^{a_2})$ is a divisor of n and therefore necessarily in $\mathcal{Q}$, again a contradiction. $\square$

5 Counting the elements in $\mathcal{K} \cap [1, x]$

Let $\mathcal{K}(x) = \mathcal{K} \cap [1, x]$.

Theorem 4. *The estimate*

$$\#\mathcal{K}(x) \leq x^{1/4 + o(1)}$$

holds as $x \to \infty$.

Proof. By Theorem 1.2 in [4], we have $\#\mathcal{K}(x) = x^{1/3 + o(1)}$ as $x \to \infty$. It remains to improve the exponent 1/3 to 1/4. We recall the following result from [4].

Lemma 5. *If $\sigma(n)/n = N/D$ with $(N, D) = 1$, then given $x \geq 1$ and $d \geq 1$*

$$\#\{n \leq x : D = d\} = x^{o(1)}$$

as $x \to \infty$.

Now let $n \in \mathcal{K}(x)$, assume that $n > 1$ and write it in the form $n = A \cdot B$ with A squarefree, B squarefull and $(A, B) = 1$. By Lemma 1, we have $A \in \{1, p_1, 2p_1, p_1p_2, 2p_1p_2\}$. Then

$$\frac{N}{D} = \frac{\sigma(n)}{n} = \frac{\gamma(n)^2}{n} = \frac{\gamma(A)^2}{A} \cdot \frac{\gamma(B)^2}{B} = \frac{A}{B/\gamma(B)^2}, \tag{8}$$

and $(A, B/\gamma(B)^2) = 1$. Since $\sigma(n) > n$, it follows that $B/\gamma(B)^2 < A$. Thus,

$$B/\gamma(B)^2 < \sqrt{AB/\gamma(B)^2} \le \sqrt{n} \le \sqrt{x}.$$

By Lemma 1 again, we can write $B = \delta C^2 D$, where C is squarefree, D is 4-full, $\delta \in \{1, 2^3\}$, and where δ, C and D are pairwise coprime. Then $B/\gamma(B)^2 = \delta/\gamma(\delta)^2 \times D/\gamma(D)^2$, so therefore $D/\gamma(D)^2 \le B/\gamma(B)^2 < x^{1/2}$. Because D is 4-full it follows that $D/\gamma(D)^2$ is squarefull and so the number of choices for $D/\gamma(D)^2$ is $O(x^{1/4})$. Hence, the number of choices for $B/\gamma(B)^2 \in \{D/\gamma(D)^2, 2D/\gamma(D)^2\}$ is also $O(x^{1/4})$, which together with Lemma 5 and formula (8) implies the desired conclusion. □

A positive integer d is said to be a *unitary divisor* of n if $d \mid n$ and $(d, n/d) = 1$; it is said to be a *proper unitary divisor* of n if it also satisfies $1 < d < n$. We will say that an integer $n \in \mathcal{K}(x)$ is *primitive* if no proper unitary divisor d of n satisfies $\sigma(d) \mid \gamma(d)^2$. Let us denote this subset of $\mathcal{K}(x)$ by $\mathcal{H}(x)$. Elements of $\mathcal{H}(x)$ can be considered as the *primitive solutions* of $\sigma(n) = \gamma(n)^2$. For example, the number $n = 1782 \in \mathcal{H}(x)$ since, although the proper divisor $d = 6$ of n satisfies $\sigma(d) \mid \gamma(d)^2$, it fails to be unitary. Also, it is interesting to observe that the condition $\sigma(d) \mid \gamma(d)^2$ seems to be very restrictive: for instance, the only positive integers $d < 10^8$ satisfying this condition are 6 and 1782; this is already an indication that the set $\mathcal{H}(x)$ is very thin. As a matter of fact, we now prove the following result.

Theorem 6. *Let $\epsilon > 0$ be given. Then, given any $\epsilon > 0$,*

$$\#\mathcal{H}(x) = O(x^{\epsilon}).$$

Proof. Let $n \in \mathcal{H}(x)$ and assume that $x > 0$ is large. Let a be the largest divisor of n such that all prime factors $p \mid a$ satisfy $p \le \log x$. Write $n = a \cdot b$ and write down the standard factorization of b into primes as

$$b = p_1^{\beta_1} \cdots p_k^{\beta_k}, \qquad \text{where} \qquad p_1 < \cdots < p_k.$$

Set $M := \lceil \log x / \log\log x \rceil$. Then, since $b \le n \le x$ and since for each i, we have $\log x < p_i$, we get

$$(\log x)^{\beta_1 + \cdots + \beta_k} < p_1^{\beta_1} \cdots p_k^{\beta_k} = b \le x,$$

implying that

$$\beta_1 + \cdots + \beta_k < \frac{\log x}{\log\log x}, \quad \text{so that} \quad k \le M.$$

Now assume that the positive integer a is given and that there is some positive integer b such that $n = a \cdot b$ is a primitive element of $\mathcal{K}$. We will show how to find b from a using the knowledge of the exponents $\beta_1, \ldots, \beta_k$.

Firstly, if a is already primitive, we then have $b = 1$. So, suppose that a is not primitive.

Since $\sigma(a)\sigma(b) = \gamma(a)^2\gamma(b)^2$, and the two factors on the right hand side are coprime, we must have

$$d := \frac{\sigma(a)}{(\sigma(a), \gamma(a)^2)} \mid \gamma(b)^2.$$

Hence, let p_1 be the least prime dividing the left–hand side of the above relation. Note that the left–hand side is not 1, since otherwise we would have $\sigma(a) \mid \gamma(a)^2$, which is not possible since n is primitive.

Now replace a by $ap_1^{\beta_1}$ and proceed. If at step $i < k$, we have $d = 1$, then the choice of the β_i's for $i = 1, \dots, k$ fails to generate an element of $\mathcal{K}$. We can then move on to the next choice. With success at every step, we generate b from a by finding primes $p_1, \dots, p_k$ such that $a \cdot p_1^{\beta_1} \cdots p_k^{\beta_k} \in \mathcal{K}$.

To complete the proof, we only need to find an upper bound for

$$\#\{\text{choices for } a\} \cdot \#\{\text{choices for } (\beta_1, \dots, \beta_k)\},$$

and this is the same as in Wirsing's proof [5] or [3, Theorem 7.8, pp. 1008-1010] for the case of multiperfect numbers:

$$\begin{aligned}
\#\{\text{choices for } (\beta_1, \dots, \beta_k)\} &\leq \#\{(\beta_1, \dots, \beta_k) : \beta_1 + \cdots + \beta_k \leq M\} \leq 2^M, \\
\#\{\text{choices for } a\} &\leq \#\{n \leq x : p \mid n \Rightarrow p \leq \log x\} \\
&\leq \#\{n \leq x : p \mid n \Rightarrow \log^{\frac{3}{4}} x < p \leq \log x\} \\
&\quad \times \#\{n \leq x : p \mid n \Rightarrow p \leq \log^{\frac{3}{4}} x\} \\
&\leq 2^{4M} \times 2^M = 2^{5M},
\end{aligned}$$

in which case we obtain the upper bound

$$2^{6M} = x^{\frac{6 \log 2}{\log \log x}} = x^{o(1)} \qquad \text{as} \qquad x \to \infty,$$

for the number of primitive $n \in \mathcal{K}(x)$, which completes the proof of this theorem. $\square$

6 Final remarks

Here, we briefly consider another question related to the problem of De Koninck, namely the one which consists in identifying those integers n satisfying $\gamma(n)^2 \mid \sigma(n)$. There is an infinite set of solutions $n = 2^i 3^j$ with $i \equiv 5 \pmod 6$, $j \equiv 1 \pmod 2$. If $n = 2^i 3^j$ satisfies $\gamma(n)^2 \mid \sigma(n)$, then these two congruence conditions are also satisfied.

Indeed, first let $i = 5+6k$, $j = 1+2m$ and $n = 2^i 3^j$. Then $\sigma(2^i) = 2^{6(k+1)} - 1 \equiv 0 \pmod 9$ and $3^{2(m+1)} - 1 \equiv 0 \pmod 8$ so that $3^2 \mid \sigma(2^i)$ and $2^2 \mid \sigma(3^j)$.

Now assume that $n = 2^i 3^j$ and that the integers r and s are such that $2^2 \mid \sigma(3^s)$ and $3^2 \mid \sigma(2^r)$. It is well–known and quite easy to prove by elementary arguments that

$$\begin{aligned}
v_2(\sigma(3^s)) + 1 &= v_2((3+1)(s+1)) \geq 3, \text{ and} \\
v_3(\sigma(2^r)) &= v_3((2+1)(r+1)) \geq 2,
\end{aligned}$$

so by the first of these equations s is odd. By the second equation, we see that $r \equiv 2 \pmod 3$, so that $r \equiv 2 \pmod 6$ or $r \equiv 5 \pmod 6$. If the first of these was true, then r would be even, so that $3 \mid \sigma(2^r)$ would not be possible. Thus, we must have $r \equiv 5 \pmod 6$.

Observe that this infinite set $2^i 3^j$ does not exhaust all of the non-trivial solutions, even those with only two distinct prime factors. For example, $n = p^{q-2}q^{p-2}$ with $p = 2,\ q = 1093$ or $p = 83,\ q = 4871$ are both solutions, since in either case we have

$$p^2 \mid q^{p-1} - 1 \quad \text{and} \qquad q^2 \mid p^{q-1} - 1,$$

and such divisibilities yield $p^2 q^2 \mid \sigma(p^{q-2}q^{p-2})$. Note also that there are many non-trivial solutions with 3 prime factors, for example 17 solutions up to 10^6 and 25 up to 4×10^6. Typical solutions have the form

$$\{2^3 3^3 5^5,\ 2^5 3^5 7^1,\ 2^9 3^4 11^1\}.$$

As a final note, let us mention that, given any arbitrary integer $k \geq 2$, one can easily check that the more general property $\gamma(n)^k | \sigma(n)$ is indeed satisfied by infinitely many positive integers n, namely those of the form

$$n = 2^{2i3^{k-1}-1} 3^{j2^{k-1}-1} \qquad (i \geq 1,\ j \geq 1).$$

7 Acknowledgements

Research of F. L. was supported in part by Grant SEP-CONACyT 79685, of I. K. by a grant from the European Union and the European Social Fund and of J-M.D.K. by a grant from NSERC.

References

[1] R. K. Guy, *Unsolved Problems in Number Theory, Third Edition*, Springer, 2004.

[2] F. Luca, On numbers n for which the prime factors of $\sigma(n)$ are among the prime factors of n, *Result. Math.*, **45** (2004), 79–87.

[3] C. Pomerance and A. Sárközy, Combinatorial Number Theory, in *Handbook of Combinatorics vol I*, R. L. Graham, M. Grötschel and L. Lovàsz, eds., Elsevier Science, 1995.

[4] P. Pollack and C. Pomerance, Prime–perfect numbers, *INTEGERS*, to appear.

[5] E. Wirsing, Bemerkung zu der Arbeit über vollkommene Zahlen, *Math. Ann.* **137** (1959), 316–318.

2010 *Mathematics Subject Classification*: Primary 11A25; Secondary 11A41.
Keywords: sum of divisors, squarefree core of an integer, De Koninck's conjecture.

Received February 23 2012; revised versions received July 26 2012; August 14 2012; September 3 2012. Published in *Journal of Integer Sequences*, September 8 2012.

Return to Journal of Integer Sequences home page.