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Solar Wind Fluctuations and the von Kármán–Howarth Equations: The role of fourth-order correlations

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Abstract. The von Kármán–Howarth (vKH) hierarchy of equations relate the second-order correlations of the turbulent fluctuations to the third-order ones, the third-order to the fourth-order, and so on. We recently demonstrated [1] that for MHD, self-similar solutions to the vKH equations seem to require at least two independent similarity lengthscales (one for each Elsässer energy), so that compared to hydrodynamics a richer set of behaviors seems likely to ensue. Moreover, despite the well-known anisotropy of MHD turbulence with a mean magnetic field ($\mathbf{B}_0$), the equation for the second-order correlation does *not* contain explicit dependence on $\mathbf{B}_0$. We show that there is, however, *implicit* dependence on $\mathbf{B}_0$ via the third-order correlations, which themselves have both explicit $\mathbf{B}_0$-dependence and also their own implicit dependence through fourth-order correlations. Some subtleties and consequences of this implicit-explicit balance are summarized here.

In addition, we present an analysis of simulation results showing that the evolution of turbulence can depend strongly on the initial fourth-order correlations of the system. This leads to considerable variation in the energy dissipation rates. Some associated consequences for MHD turbulence are discussed.

1. BACKGROUND

Multiple lines of evidence suggest that MHD turbulence is an important aspect of solar wind fluctuations. In many cases, available solar wind observations show a direct correspondence with features seen in turbulence simulations. For example, the probability density function (pdf) for the **v**–**b** alignment angle [2], the kurtosis and the scale-dependent kurtosis of the magnetic fluctuations [3], the statistics of magnetic discontinuities [4–7], and properties of magnetic reconnection [8–10] all show such correspondences. Here, **v** and **b** are the fluctuation velocity and magnetic fields.

A common feature of these, and other, examples is the nongaussian nature of the associated statistics. As is well known, a gaussian distribution is completely determined by its second-order moment (the variance σ^2), with all higher-order moments being simple functions of σ. In contrast, for nongaussian distributions the higher order moments are, in general, independent quantities; that is, they are not fully known once the variance is known.

For Navier–Stokes (NS) and MHD turbulence, nongaussian pdfs are inherent. Indeed, although some quantities may have pdfs which are *approximately* gaussian, there can be no truly gaussian turbulence since all odd moments for gaussian fluctuations are necessarily zero. However, the celebrated Kolmogorov third-order law [11–14] requires that (certain) third-order moments are nonzero if a turbulent cascade is active, since they are proportional to the nonzero energy flux, ε. For example, in the case of isotropic NS turbulence the law takes the form

$$\langle (\delta u_\parallel)^3 \rangle = -\frac{4}{5}\varepsilon r, \tag{1}$$

where $\delta u_\parallel(r) = \mathbf{r}\cdot[\mathbf{v}(\mathbf{x}+\mathbf{r})-\mathbf{v}(\mathbf{x})]/r$ is the longitudinal component of the velocity increment at scale $r = |\mathbf{r}|$.

One can then ask, how do higher-order correlations influence the development of MHD turbulence? Here we investigate this issue using two approaches: the von Kármán–Howarth (vKH) equations and numerical simulations with initial conditions chosen to have different values of some fourth-order correlations.

2. MHD VON KÁRMÁN–HOWARTH EQUATIONS

This well-known hierarchy of equations relates the evolution of the second-order correlations to third-order ones, the evolution of third-order correlations to fourth-order ones, and so on out to infinite order [1, 14–16]. We begin by sketching their derivation and then comment on a few important aspects of them.

We employ Elsässer variables, $\mathbf{z}^\pm(\mathbf{x},t) = \mathbf{v}\pm\mathbf{b}$, in Alfvén speed units, to write the incompressible 3D MHD equations—*with* a DC magnetic field $\mathbf{B}_0$, as

$$\partial_t z_i^\pm = -\left(z_k^\mp \mp B_{0k}\right)\partial_k z_i^\pm - \partial_i P + \nu\partial_k\partial_k z_i^\pm. \tag{2}$$

Here P is the total pressure and ν is the kinematic viscosity, assumed equal to the resistivity. Using this equa-

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tion, and its partner evaluated at the displaced position $\mathbf{x}' = \mathbf{x} + \mathbf{r}$, it is straightforward to construct the equation for the evolution of the two-point single-time correlation functions,

$$R^{\pm}(\mathbf{r},t) = \langle \mathbf{z}^{\pm}(\mathbf{x},t) \cdot \mathbf{z}^{\pm}(\mathbf{x}+\mathbf{r},t) \rangle = \langle \mathbf{z}^{\pm} \cdot \mathbf{z}^{\pm\prime} \rangle, \quad (3)$$

where the prime symbol indicates evaluation at $\mathbf{x}'$, the angle brackets denote averaging over the position vector $\mathbf{x}$, and spatial homogeneity is assumed. One obtains [e.g. 1, 13, 14]

$$\partial_t R^{\pm} = -\frac{\partial}{\partial r_k} \left[\hat{Q}_k^{\pm}(\mathbf{r}) - \hat{Q}_k^{\pm}(-\mathbf{r}) \right] + 2\nu \frac{\partial^2 R^{\pm}}{\partial r_k \partial r_k}, \quad (4)$$

where $\hat{Q}_k^{\pm}(\mathbf{r}) = \langle z_k^{\mp\prime} \mathbf{z}^{\pm} \cdot \mathbf{z}^{\pm\prime} \rangle$ is a (two-point) third-order correlation. Equation (4) is often called the MHD von Kármán–Howarth equation.[1] Its Fourier transform with respect to $\mathbf{r}$ yields the equation for the evolution of the Elsässer energy spectra, $E^{\pm}(\mathbf{k},t)$.

Several points are noteworthy. First, clearly the equation is not closed, since third-order correlations appear in it. It is again straightforward to derive evolution equations for these correlations, and they in turn are not closed because of their dependence upon fourth-order correlations. The process leads to an infinite hierarchy of vKH equations.

Second, the DC field—present in the MHD equations (2)—is *not* present in the vKH equation (4). On the face of it, this is perplexing given the known anisotropy of $R^{\pm}(\mathbf{r})$, and $E^{\pm}(\mathbf{k})$, with respect to the direction of $\mathbf{B}_0$. The anisotropy is seen in experiments and simulations, and supported by theory [e.g., 17–24]. So the question is, how can the equation which determines the energy spectrum be independent of $\mathbf{B}_0$ when the spectrum shows well-established dependence on $\mathbf{B}_0$? The resolution is that the dependence is *implicit*, via the third-order correlations. This becomes evident once the vKH equations for the latter are obtained [e.g., 1]. For example, one finds,

$$\begin{aligned} \partial_t \langle z_k^{-\prime} \mathbf{z}^{+} \cdot \mathbf{z}^{+\prime} \rangle \quad = \quad & -2B_{0k} \langle \mathbf{z}^{+} \cdot \mathbf{z}^{+\prime} \partial' z_k^{-\prime} \rangle \\ & + \frac{\partial}{\partial r_m} \langle z_m^{-} z_k^{-\prime} \mathbf{z}^{+} \cdot \mathbf{z}^{+\prime} \rangle \\ & + 8 \text{ other 3rd/4th order corrns}, \end{aligned} \quad (5)$$

which is explicitly dependent upon $\mathbf{B}_0$. It is also implicitly dependent, since the fourth-order correlations are functions of $\mathbf{B}_0$ in similar fashion. The development of anisotropy thus arises in a somewhat subtle way, with the implicit dependence "cascading" in from the infinite-order correlations but the explicit dependence entering at third (and higher) order.

Both these aspects—the closure problem and the absence of $\mathbf{B}_0$ in the equation for the energy spectrum—fold in to the development of phenomenological models for the energy spectrum. One can view such models as closure approximations to the steady-state version of (the Fourier transform of) Eq. (4). For example, in developing their phenomenological models of the energy spectrum, Iroshnikov [25] and Kraichnan [26] (IK) both recognised that the timescale associated with (some) triple correlations would be B_0-dependent [see Eq. (5)], and that this timescale would influence the energy spectrum [27, 28].[2]

3. FOURTH-ORDER CORRELATIONS

We now provide some numerical examples of how the evolution of incompressible MHD turbulence can be affected by the initial values of fourth-order correlations. Of particular interest are the following three (normalized) quantities:

$$\Sigma_{v\omega} = \frac{\langle (\mathbf{v} \cdot \omega)^2 \rangle}{\langle \mathbf{v}^2 \rangle \langle \omega^2 \rangle}, \quad \Sigma_{jb} = \frac{\langle (\mathbf{j} \cdot \mathbf{b})^2 \rangle}{\langle \mathbf{j}^2 \rangle \langle \mathbf{b}^2 \rangle}, \quad \Sigma_{vb} = \frac{\langle (\mathbf{v} \cdot \mathbf{b})^2 \rangle}{\langle \mathbf{v}^2 \rangle \langle \mathbf{b}^2 \rangle}, \quad (6)$$

where $\omega = \nabla \times \mathbf{v}$ and $\mathbf{j} = \nabla \times \mathbf{b}$. There are other such correlations, but these three have the benefit of corresponding to local alignment properties of the three nonlinear terms in the "primitive" form of the MHD equations: $\partial_t \mathbf{v} \sim \mathbf{v} \times \omega + \mathbf{j} \times \mathbf{b}$, and $\partial_t \mathbf{b} \sim \nabla \times (\mathbf{v} \times \mathbf{b})$ [30, 31].

We choose four different initial conditions (ICs). Three of these are based on the Taylor–Green vortex (characterized by a single large-scale wavevector) for $\mathbf{v}$ and relatives of it for $\mathbf{b}$; they are identical to those employed in Lee *et al.* [32], where they are referred to as the "insulating", "conducting", and "alternative" runs. They each have an initial energy of $E = 0.25$, equally split between the kinetic and magnetic components, and the initial cross helicity and magnetic helicity are zero.

Germaine to our interest here, is that the fourth-order correlation Σ_{vb} differs between these ICs, whereas $\Sigma_{v\omega}$ and Σ_{jb} are initially zero in each case. Thus, these Lee *et al.* runs each have a distinct $\mathbf{b}$ and Σ_{vb}, but most other familiar parameters are identical or very similar.

The final run we consider starts with broadband Fourier fluctuations, with $3 \leq k \leq 6$ and gaussian random phases. It can be considered as a more typical turbulent

[1] Since there are an infinite number of vKH equations, it is more accurately called the second-order vKH equation for the traced Elsässer correlation functions.

[2] Although the IK phenomenologies address the change to the energy spectrum associated with the strength of $\mathbf{B}_0$, they do not take proper account of the anisotropy associated with the *direction* of $\mathbf{B}_0$. See, e.g., [20, 28, 29] for discussion on this point.

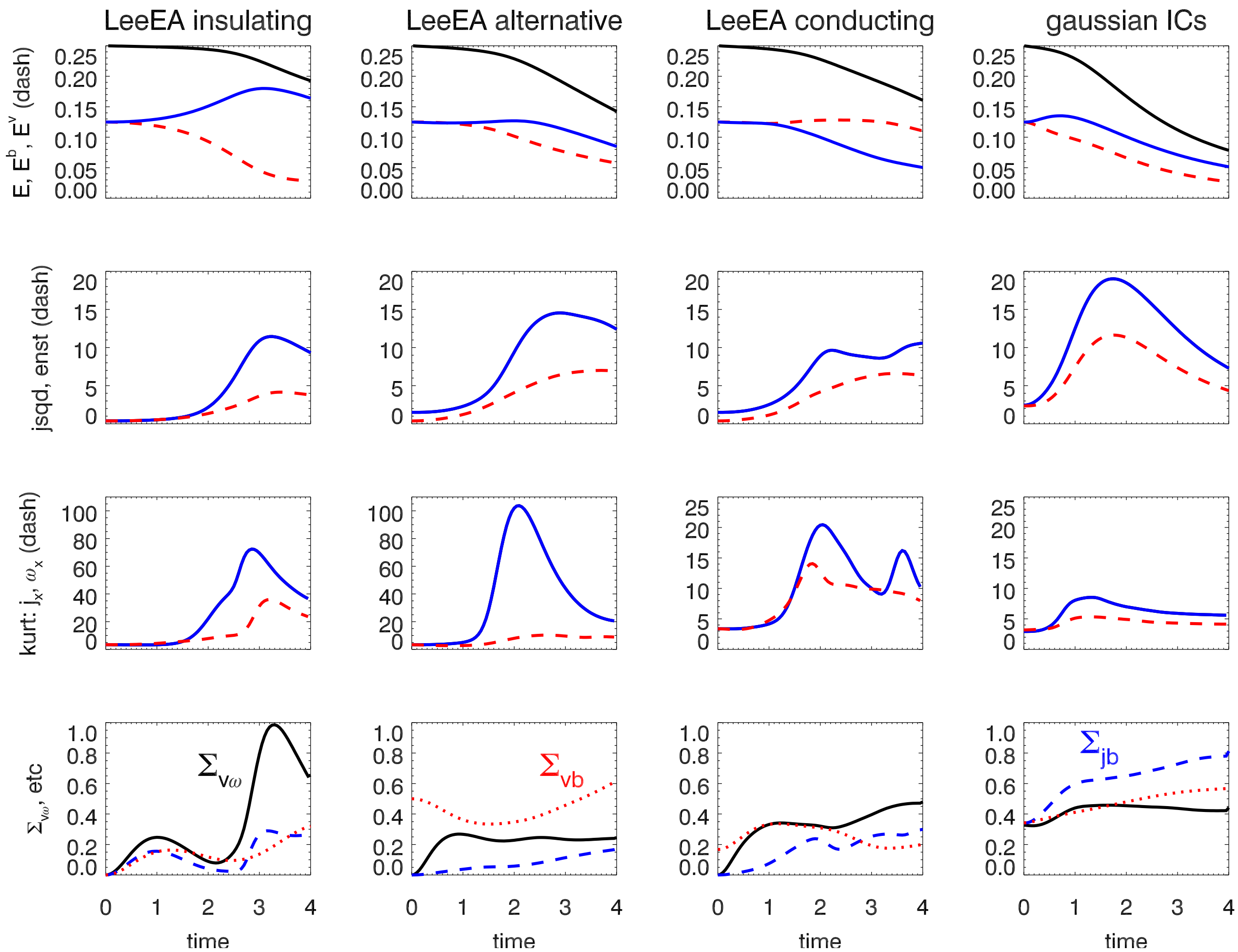

FIGURE 1. (color online) Time histories for energies, etc for four runs with distinct initial conditions (see main text). Each column corresponds to a different run. The third row is the kurtosis of the x components of the electric current and vorticity.

initial state than the Lee *et al.* ICs.

The equations are solved using a Fourier pseudospectral code with RK2 timestepping. The runs are unforced and have no DC field. The resolution reported here is 512^3, with $\nu = 1.1 \times 10^{-3}$ (equal to the resistivity). This value of ν ensures that all the runs are well-resolved, i.e., that the cutoff wavenumber ($k_{\text{wall}} = N/2 = 256$) is at least triple the maximum Kolmogorov dissipation wavenumber, which is important for obtaining accurate higher-order statistics like the kurtosis of $\mathbf{j}$: $\langle|\mathbf{j}|^4\rangle/\langle|\mathbf{j}|^2\rangle^2$ [e.g., 33, 34].

Figure 1 displays time histories for the energies and some second and fourth order quantities for each run. Focusing first on the top three rows of the figure, we see that the 'conducting' run is a little anomalous compared to the others. These others are all qualitatively similar, with the magnetic energy predominant, and the enstrophy $\langle\omega^2\rangle/2$, mean-square current $\langle\mathbf{j}^2\rangle/2$, and kurtoses having a single peak, with the magnetic quantities being larger. Although the energy decays are qualitatively quite similar, there are significant quantitative differences (see also the spectra plots in [32]). In particular the decay rates and the ratio of kinetic to magnetic energies vary markedly across the runs. Another evident feature is the strong nongaussianity of the kurtosis of each component of $\mathbf{j}$ (only that for j_x is shown), with their maxima being well above the gaussian kurtosis value of 3.

Considering now the evolution of the Σs defined in Eq. (6) (Fig. 1, bottom row), there are both similarities and differences between the runs. General statements concerning the evolution are perhaps hard to make; however, there is a tendency for each Σ to grow, although often not monotonically. The timing of the local extrema in the kurtosis of j_x is correlated with changes in Σ_{jb} (e.g., extrema, change in growth rate). Teasing out the details of how higher-order correlations like the Σs impact the flow dynamics may prove to be rather difficult. Nonetheless, these examples suggest that along with well-known parameters like the energy, Alfvén ratio, cross helicity, and magnetic helicity, fourth-order correlations like the

Σs defined by Eq. (6) may also need to be taken into account when trying to ascertain the dynamics and/or long-time states (see [1] for discussion regarding the consequences for universality of MHD turbulence).

For the future, we intend to examine the anomalous nature of the 'conducting' run in more detail. In a freely decaying run it is unusual to see the kinetic energy being dominant. Moreover, in this particular case it is close to constant. What is responsible for this behavior? The relationship between it and the likewise unusual double-peak behavior of $\langle \mathbf{j}^2 \rangle$ and the kurtosis of j_x is also of interest.

4. CONCLUSIONS

We have highlighted how the absence of the DC magnetic field from the second-order vKH equation is really only an apparent absence since the unclosed nature of the equation supports an implicit dependence on $\mathbf{B}_0$, through the third-order correlations. The $\mathbf{B}_0$-dependence of the energy spectrum has been used in physical phenomenologies of the spectrum for many decades, starting with the Iroshnikov and Kraichnan models [25, 26]. However, the formal mathematical arguments justifying it do not appear to be well known. These arise from consideration of the vKH hierarchy and its unclosed nature.

We have also presented several numerical examples revealing the correspondence between different (kinetic and magnetic) energy decay rates and several fourth-order correlations. Much remains to be worked out regarding cause and effect in this regard.

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