TOPICS IN NUMBER THEORY

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Abstract

The thesis is concerned mainly with topics in the theory of Riemann's zeta function, but it also includes some contributions to prime number theory and the study of the Möbius function. New conditions are stated for the validity of the quasi-Riemann hypothesis $RH(\sigma_0)$, that $\zeta(s) \neq 0$ for $\sigma > \sigma_0$. The orders and oscillatory behaviour of a variety of summatory functions are considered in this context. Particular study is made of the sums defined by

$$A_{-1}(x) = \sum_{n \leq x} \frac{\lambda(n)}{n}, \quad A_k(x) = \sum_{n \leq x} A_{k-1}(n)$$

and

$$B_{-1}(x) = \sum_{n \leq x} \frac{\mu(n)}{n}, \quad B_k(x) = \sum_{n \leq x} B_{k-1}(n)$$

$(k = 0, 1, 2 \ldots)$, incomplete sums of the form

$$\sum_{n \leq x^\delta} \lambda(n) n^\kappa \left[\frac{x}{n}\right] \quad \text{and} \quad \sum_{n \leq x^\delta} \frac{\lambda(n)}{e^{n/x} - 1},$$

and summatory functions associated with the coefficients of the Dirichlet series representations for

$$\{(1 - \frac{2}{2^s})\zeta(s)\}^{1/k} \ (k = 1, 2, 3, \ldots) \quad \text{and} \quad \frac{1}{\zeta(s)} \prod_p (1 - \frac{1}{[p^k]^s})^{-1} \ (k > 1).$$

On the same topic results are proven about connections between $RH(\sigma_0)$ and the distribution of the set H_N of Farey numbers of order N. Some general theorems concerning the sums

$$\sum_{p_k \in H_N} (p_k^r - \frac{1}{r+1}), \quad (r = 1, 2, 3, \ldots),$$

are established which allow known results in the cases $r = 2, 3$ to be extended to $r = 4, 5, 6$. Other analytic studies include a series representation for Riemann's Ξ function, and a theorem improving earlier results concerning the number of zeros of $f(\lambda + it)$ with $0 < t < T$, and fixed λ between 0 and 1, where $f(s) = \pi^{-s/2}\,\Gamma(s/2)\zeta(s)$.

The remaining topics are more arithmetic in nature. An attempt is made to show by elementary methods that the order of the Tchebychef difference $\psi(x) - x$ is not greater than that of the Möbius sum $M(x)$, and although only partially successful, the attempt improves on previously published elementary results. Some theorems are proved which relate the order of $J(x)$, the maximum number of consecutive integers each of which is divisible by at least one prime $\leqslant x$, to the problems of the least prime in an arithmetic progression and the order of the difference between consecutive primes. Two sections contain a direct attack by elementary methods on the problem of getting estimates of $A_0(T)$ and $B_0(T)$ which would be equivalent to $RH(\sigma_0)$ for some $\sigma_0 < 1$, and the problem is reduced, roughly speaking, to finding sufficiently large N such that the solutions $x_1, x_2, \ldots, x_T$ of the system

$$\sum_{T \geqslant n \geqslant k} [\tfrac{n}{k}]x_n = [\tfrac{N}{k}] - [\tfrac{N}{T+1}] \quad (k = 1, 2, \ldots, T)$$

are predominantly of one sign.

The closing sections in the thesis deal with Möbius functions of ordered semigroups of a certain type, and lead to a conjecture that the Möbius functions in a way characterize the ordered multiplicative structure.

Some evidence supporting the conjecture is given in the case of the ordinary Möbius function μ.

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P.B. BRAUN.

Introduction

This thesis is concerned mainly with topics in the theory of Riemann's zeta function, but it also includes some contributions to prime number theory and the study of the Möbius function.

The first eight sections deal with a variety of necessary and/or sufficient conditions for the quasi-Riemann hypothesis $RH(\sigma_0)$ that $\zeta(s) \neq 0$ for $Re s > \sigma_0$. It is investigated how $RH(\sigma_0)$ is related to the behaviour of certain summatory functions involving the Möbius function μ, or the Liouville function λ, or one of a class of functions of which λ and μ are special cases. There are also results relating $RH(\sigma_0)$ to the distribution of Farey numbers.

In the next four sections some isolated topics are studied: two concerned directly with the zeta function, and two concerned with the distribution of prime numbers.

The problem of $RH(\sigma_0)$ is taken up again in sections 13 and 14, where a technique is developed that may shed some light on the behaviour of two of the summatory functions considered earlier on in relation to $RH(\sigma_0)$.

The final two sections are concerned with Möbius functions defined on ordered semi-groups. A conjecture about the Möbius function is formulated, and some supporting evidence for this conjecture is given.

The following summary previews in more detail the content of this thesis.

Section 1 is introductory and uses standard methods to establish connections between $RH(\sigma_0)$ and the orders and average orders of the summatory functions

$$S_\kappa(x) = \sum_{n\leqslant x} \lambda(n)n^\kappa \quad \text{and} \quad M_\kappa(x) = \sum_{n\leqslant x} \mu(n)n^\kappa.$$

Higher averages, defined by

$$A_{-1}(x) = S_{-1}(x), \quad A_k(x) = \sum_{n\leqslant x} A_{k-1}(n) \quad (k = 0, 1, 2, \ldots)$$

$$B_{-1}(x) = M_{-1}(x), \quad B_k(x) = \sum_{n\leqslant x} B_{k-1}(n) \quad (k = 0, 1, 2, \ldots)$$

are considered in sections 2 and 3, and it is shown how their order and their oscillatory behaviour relate to $RH(\sigma_0)$. This generalizes and improves known results concerning $B_0(x)$ and $A_0(x)$. It is noted that further generalization is possible by using, instead of λ and μ, a class of functions of which λ and μ are special cases. This theme is taken up again in section 7.

Section 4 deals with incomplete sums, such as

$$\sum_{n\leqslant x^\delta} \lambda(n)n^\kappa [\tfrac{x}{n}] .$$

Further conditions for $RH(\sigma_0)$ are obtained, and attention is drawn to some unanswered questions in this context

Section 5 begins with a brief survey of earlier results linking $RH(\sigma_0)$ with the distribution of the set H_N of Farey numbers of order N. Some general theorems are proven concerning the sums

$$\sum_{p_k \in H_N} (p_k^r - \frac{1}{r+1}) \qquad (r = 1, 2, 3, \ldots).$$

These allow known results in the cases $r = 2, 3$ to be extended to $r = 4, 5, 6$.

In section 6 general results are established concerning the Dirichlet expansion of any real power of a Dirichlet series. These are used to show that $RH(\frac{1}{2})$ is equivalent to a sequence of conditions involving the sum function of the Dirichlet coefficients of $\{(1 - \frac{2}{2^s})\zeta(s)\}^{1/k}$ $(k = 1, 2, 3, \ldots)$. A consequence of Lindelöf's hypothesis is also noted.

In section 7 a class of functions $\lambda^{(k)}$ is introduced by

$$\frac{1}{\zeta(s)} \prod_p (1 - \frac{1}{[p^k]^s})^{-1} = \sum_{n=1}^{\infty} \frac{\lambda^{(k)}(n)}{n^s}$$

where k ranges over all real numbers > 1. Results, similar to those for λ and μ in earlier sections, are then obtainable for $k \geqslant 2$. The case $1 < k < 2$ is also studied, and connections with $RH(\frac{1}{k})$ are demonstrated.

Links between $RH(\sigma_0)$ and the order of

$$\sum_{n \leq x} \frac{\lambda(n)}{e^{n/x} - 1}$$

are examined in section 8. A general result for Dirichlet series is obtained first. This does not seem to have been stated before, and it is used as one of the principal tools in the discussion.

Section 9 gives a series representation, believed to be new, for Riemann's Ξ function. This may be of some interest in itself, but no significant application of this representation has been found.

Section 10 contains a theorem improving earlier results concerning the number of zeros of the real and imaginary parts of $f(\lambda + it)$ with $0 < t < T$ and fixed λ between 0 and 1, where $f(s) = \pi^{-s/2}\Gamma(s/2)\zeta(s)$.

An attempt is made in section 11 to show by elementary arguments that the order of the Tchebychef difference $\psi(x) - x$ is not greater than that of the Möbius sum $M_0(x)$. The attempt is only partially successful, but yields an improvement on previously published elementary results.

Let $J(x)$ be the maximum number of consecutive integers each of which is divisible by at least one prime $\leq x$. In Section 12, it is shown how knowledge of the order of $J(x)$ leads to information about the least prime in an arithmetic progression and about the difference between consecutive primes. (However, it appears difficult to obtain sufficiently good estimation of $J(x)$.)

Sections 13 and 14 contain a direct attack, by elementary methods, on the problem of getting estimates of $A_0(x)$ and $B_0(x)$ which would, according to the results stated in section 1, be equivalent to $RH(\sigma_0)$ for some $\sigma_0 < 1$. Preliminary results concerning weighted sums involving λ and μ are obtained in section 13. The ideas developed there are applied in section 14 to obtain estimates of $A_0(T)$ and $B_0(T)$ whose quality, roughly speaking, depends on finding sufficiently large N such that the solutions $x_1, x_2, \ldots, x_T$ of the system

$$\sum_{T \geq n \geq k} [\tfrac{n}{k}] x_n = [\tfrac{N}{k}] - [\tfrac{N}{T+1}] \quad (k = 1, 2, \ldots, T)$$

are predominantly of one sign.

Section 15 begins with some observations about the Möbius function of ordered semigroups of a certain type. This leads to the conjecture that the Möbius function in a way characterizes the ordered multiplicative structure. Some evidence supporting this, as far as the ordinary μ is concerned, is given in section 15 and, more explicitly, in section 16.

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Section 1.

Some statements equivalent to the quasi-Riemann hypothesis.

As usual in Number Theory, let s be a complex variable, $\sigma = \mathrm{Re}\,s$, $t = \mathrm{Im}\,s$. Let ζ be Riemann's zeta function, and, for $1 > \sigma_0 \geq \frac{1}{2}$, let $RH(\sigma_0)$ be the statement

$$\zeta(s) \neq 0 \quad \text{for} \quad \sigma > \sigma_0.$$

We refer to this statement as the 'quasi-Riemann hypothesis'. With our notation, $RH(\frac{1}{2})$ will then signify the Riemann hypothesis proper.

In this and later sections we have occasion to use the following result for expressing a Dirichlet series as an integral. The proof of this result is a simple application of a well-known technique but is included here for the sake of completeness.

Proposition 1.

Let $a : \mathbb{N} \to \mathbb{C}$ satisfy

$$A(x) = \sum_{n \leq x} a(n) = O(x^{\Delta})$$

as $x \to \infty$. Then for $\sigma > \Delta$,

$$\sum_{n=1}^{\infty} \frac{a(n)}{n^s} = s\int_1^{\infty} \frac{A(x)}{x^{s+1}}\,dx .$$

Proof:

For $\sigma > \Delta$,

$$\sum_{n=1}^{\infty} \frac{a(n)}{n^s} = \lim_{N\to\infty} \sum_{n=1}^{N} \frac{a(n)}{n^s}$$

$$= \lim_{N\to\infty} \sum_{n=1}^{N} \frac{A(n) - A(n-1)}{n^s} \quad (A(0) = 0)$$

$$= \lim_{N\to\infty} \left\{ \sum_{n=1}^{N-1} A(n) \left[\frac{1}{n^s} - \frac{1}{(n+1)^s} \right] + \frac{A(N)}{N^s} \right\}$$

$$= \lim_{N\to\infty} \left\{ \sum_{n=1}^{N-1} s \int_n^{n+1} \frac{A(x)}{x^{s+1}}\, dx + \frac{A(N)}{N^s} \right\}$$

$$= s \int_1^{\infty} \frac{A(x)}{x^{s+1}}\, dx,$$

and the function defined by the integral is analytic for $\sigma > \Delta$.

For real κ let

$$S_\kappa(x) = \sum_{n\leq x} \lambda(n) n^\kappa, \quad M_\kappa(x) = \sum_{n\leq x} \mu(n) n^\kappa \quad ,$$

$$h_\kappa(x) = \sum_{n\leq x} \lambda(n) n^{\kappa-1}, \quad g_\kappa(x) = \sum_{n\leq x} \mu(n) n^{\kappa-1},$$

$$H_\kappa(x) = \sum_{n\leq x} h_\kappa(n), \quad G_\kappa(x) = \sum_{n\leq x} g_\kappa(n) \quad ,$$

where λ is Liouville's function, and μ is the Möbius function.

Proposition 2.

Let either $\kappa = -1$ or $\kappa > -\sigma_0$. Then the following statements are equivalent:

(i) $RH(\sigma_0)$,

(ii) $\forall\ \varepsilon > 0,\ S_\kappa(x) = O(x^{\sigma_0+\kappa+\varepsilon})$ as $x \to \infty$,

(iii) $\forall\ \varepsilon > 0,\ H_{\kappa+1}(x) = O(x^{\sigma_0+1+\kappa+\varepsilon})$ as $x \to \infty$,

(iv) $\forall\ \varepsilon > 0,\ M_\kappa(x) = O(x^{\sigma_0+\kappa+\varepsilon})$ as $x \to \infty$,

(v) $\forall\ \varepsilon > 0,\ G_{\kappa+1}(x) = O(x^{\sigma_0+1+\kappa+\varepsilon})$ as $x \to \infty$.

Proof:

We show that (i) $\Longleftrightarrow$ (ii) $\Rightarrow$ (iii) $\Rightarrow$ (i). The proof that (i) $\Longleftrightarrow$ (iv) $\Rightarrow$ (v) $\Rightarrow$ (i) is similar.

To show that (i) $\Rightarrow$ (ii) suppose that $RH(\sigma_0)$ is true and consider first the case $\kappa = -1$. The method in Titchmarch [1], pages 282-283, can be modified to argue that $\zeta(s) = O(t^\varepsilon)$, $\frac{1}{\zeta(s)} = O(t^\varepsilon)$ as $t \to \infty$, for every $\sigma > \sigma_0$, and every $\varepsilon > 0$. Now let

$$f(s) = \zeta(2s)/\zeta(s).$$

Then for every $\sigma > \sigma_0$ and any $\varepsilon > 0$, $f(s) = O(t^\varepsilon)$ as $t \to \infty$, and by Titchmarch [1], page 6,

$$\sum_{n=1}^{\infty} \frac{\lambda(n)}{n^s} = f(s) \quad \text{for } \sigma > 1.$$

Also it is clear that $f(1) = 0$. Using a procedure similar to that in Titchmarch [1], page 315 we thus get

$$S_{-1}(x) = \sum_{n\leq x} \frac{\lambda(n)}{n}$$

$$= \frac{1}{2\pi i}\int_{2-iT}^{2+iT} f(w+1)\ \frac{x^w}{w}\, dw + O(\frac{x^2}{T})$$

$$= \frac{1}{2\pi i}\left[\int_{2-iT}^{\sigma_0-1+\delta-iT} + \int_{\sigma_0-1+\delta-iT}^{\sigma_0-1+\delta+iT} + \int_{\sigma_0-1+\delta+iT}^{2+iT}\right] f(w+1)\frac{x^w}{w}\,dw +$$

$$+ O\left(\frac{x^2}{T}\right),$$

$$= O(T^{-1+\varepsilon}\, x^2) + O(T^{\varepsilon}\, x^{\sigma_0-1+\delta})$$

as $x \to \infty$, provided $\varepsilon > 0$, and $0 < \delta < 1 - \sigma_0$. Hence, choosing $T = x^3$, for every $\varepsilon > 0$,

$$S_{-1}(x) = O(x^{\sigma_0-1+\varepsilon}) \quad \text{as} \quad x \to \infty,$$

i.e. (i) $\Rightarrow$ (ii) when $\kappa = -1$.

That (i) $\Rightarrow$ (ii) when $\kappa > -\sigma_0$ can now be deduced as follows. If $\kappa > -\sigma_0$ and $\varepsilon > 0$, then

$$S_\kappa(x) = \sum_{n\leq x} (S_{-1}(n) - S_{-1}(n-1))n^{\kappa+1}$$

$$= \sum_{n\leq x} S_{-1}(n)\,(n^{\kappa+1} - (n+1)^{\kappa+1}) +$$

$$+ S_{-1}(x)\,[x+1]^{\kappa+1}$$

$$= O\left(\sum_{n\leq x} n^{\kappa+\sigma_0-1+\varepsilon}\right) + O(x^{\kappa+\sigma_0+\varepsilon}).$$

$$= O(x^{\kappa+\sigma_0+\varepsilon})$$

as $x \to \infty$, for every $\varepsilon > 0$.

To show that (ii) $\Rightarrow$ (i) suppose that for every $\varepsilon > 0$,

$$S_\kappa(x) = O(x^{\kappa+\sigma_0+\varepsilon}) \quad \text{as} \quad x \to \infty.$$

Then, by partial summation,

$\sum_{n=1}^{\infty} \frac{\lambda(n)n^{\kappa}}{n^s}$ converges and represents an analytic function for

$\sigma > \sigma_0 + \kappa$.

Then from

$$\sum_{n=1}^{\infty} \frac{\lambda(n)n^{\kappa}}{n^s} = \frac{\zeta(2s-2\kappa)}{\zeta(s-\kappa)},$$

we see that $\zeta(s)$ is non-zero for $\sigma > \sigma_0$.

To show that (ii) $\Rightarrow$ (iii) suppose that $\kappa = -1$ or $\kappa > -\sigma_0$, and that

$$\forall\ \varepsilon > 0,\quad S_{\kappa}(x) = O(x^{\sigma_0+\kappa+\varepsilon}) \quad \text{as} \quad x \to \infty.$$

Then, via (i), also

$$\forall\ \varepsilon > 0,\quad S_{\kappa+1}(x) = O(x^{\sigma_0+\kappa+1+\varepsilon}) \quad \text{as} \quad x \to \infty.$$

But
$$S_{\kappa+1}(x) = \sum_{n \leq x} (S_{\kappa}(n) - S_{\kappa}(n-1))n$$

$$= - \sum_{n \leq x} S_{\kappa}(n) + S_{\kappa}(x)[x+1],$$

so that

(1)
$$H_{\kappa+1}(x) = \sum_{n \leq x} h_{\kappa+1}(n)$$

$$= \sum_{n \leq x} S_{\kappa}(n)$$

$$= [x+1]\ S_{\kappa}(x) - S_{\kappa+1}(x)$$

$$= O(x^{\sigma_0+\kappa+1+\varepsilon})$$

as $x \to \infty$, for every $\varepsilon > 0$.

To show that (iii) $\Rightarrow$ (i), note first that the estimate,

$$S_\kappa(x) = O(x^{\kappa+1}) \quad \text{as} \quad x \to \infty,$$

is trivial for $\kappa > -1$, and follows for $\kappa = -1$ from

$$S_{-1}(x) = \frac{1}{x} \sum_{n \le x} \lambda(n) \left[\frac{x}{n}\right] + \frac{1}{x} \sum_{n \le x} \lambda(n) \left\{\frac{x}{n}\right\}$$

$$= \frac{1}{x} [\sqrt{x}] + O(1)$$

as $x \to \infty$.

Consequently, using proposition 1, and Titchmarch [1], page 6, we have

$$\frac{\zeta(2s-2\kappa)}{\zeta(s-\kappa)} = \sum_{n=1}^{\infty} \frac{\lambda(n) n^\kappa}{n^s}$$

(2) $$= s \int_1^\infty \frac{S_\kappa(x)}{x^{s+1}} \, dx$$

(3) $$= s \int_1^\infty \frac{x \, S_\kappa(x)}{x^{s+2}} \, dx$$

for $\sigma > \kappa + 1$, $\kappa \geq -1$.

Also, replacing s by $s + 1$, and κ by $\kappa + 1$ in (2), for $\sigma > \kappa + 1$, $\kappa \geq -2$

(4) $$\frac{\zeta(2s-2\kappa)}{\zeta(s-\kappa)} = (s+1) \int_1^\infty \frac{S_{\kappa+1}(x)}{x^{s+2}} \, dx \, .$$

Hence from (3) and (4), for $\sigma > \kappa + 1$, $\kappa \geq -1$,

(5) $$\frac{1}{s(s+1)} \frac{\zeta(2s-2\kappa)}{\zeta(s-\kappa)} = \int_1^\infty \frac{x \, S_\kappa(x) - S_{\kappa+1}(x)}{x^{s+2}} \, dx \, .$$

From (1) we easily see

$$H_{\kappa+1}(x) = x \, S_\kappa(x) - S_{\kappa+1}(x) +$$

$$+ O(x^{\kappa+1})$$

as $x \to \infty$, and so from (5) for $\sigma > \kappa + 1$, $\kappa \geq -1$,

(6) $$\frac{1}{s(s+1)} \frac{\zeta(2s-2\kappa)}{\zeta(s-\kappa)} = \int_1^\infty \frac{H_{\kappa+1}(x)}{x^{s+2}}\, dx + E_\kappa(s),$$

where $E_\kappa(s)$ is analytic for $\sigma > \kappa$.

Finally if (iii) holds, i.e. if

$$\forall\ \varepsilon > 0,\ H_{\kappa+1}(x) = O(x^{\sigma_0+\kappa+1+\varepsilon}), \quad \text{as} \quad x \to \infty,$$

then the RHS of (6) is analytic for $\sigma > \sigma_0 + \kappa + \varepsilon$, and hence $\zeta(s)$ must be non-zero for $\sigma > \sigma_0$.

Corollary:

Let $\zeta(s)$ have zeros on $\sigma = \sigma_1 > 0$.

Let either $\kappa = -1$ or $\kappa > -\sigma_1$.

Then

(i) $\forall\ \varepsilon > 0,\ H_{\kappa+1}(x) = \Omega(x^{\kappa+1+\sigma_1-\varepsilon})$ as $x \to \infty$,

(ii) $\forall\ \varepsilon > 0,\ G_{\kappa+1}(x) = \Omega(x^{\kappa+1+\sigma_1-\varepsilon})$ as $x \to \infty$,

(iii) $\forall\ \varepsilon > 0,\ S_\kappa(x) = \Omega(x^{\kappa+\sigma_1-\varepsilon})$ as $x \to \infty$,

(iv) $\forall\ \varepsilon > 0,\ M_\kappa(x) = \Omega(x^{\kappa+\sigma_1-\varepsilon})$ as $x \to \infty$.

Proof of (i):

Suppose the statement

$$\forall\ \varepsilon > 0,\ H_{\kappa+1}(x) = \Omega(x^{\kappa+1+\sigma_1-\varepsilon}) \quad \text{as} \quad x \to \infty,$$

is false. Then there exists $\varepsilon^* > 0$ such that

$$H_{\kappa+1}(x) = O(x^{\kappa+1+\sigma_1-\varepsilon^*}) \quad \text{as} \quad x \to \infty,$$

and hence from the previous proposition

$$\zeta(s) \text{ is zero free for } \sigma > \sigma_1 - \varepsilon^*,$$

which contradicts the initial assumption. (ii), (iii), and (iv) follow similarly.

Note 1. Since $\zeta(s)$ does have zeros on $\sigma = \frac{1}{2}$ the statements of the corollary, with σ_1 replaced by $\frac{1}{2}$, are all true.

Note 2. The most familiar functions appearing in the literature are

$$S(x) = S_0(x) = \sum_{n\le x} \lambda(n), \quad M(x) = M_0(x) = \sum_{n\le x} \mu(n),$$

$$h(x) = h_0(x) = \sum_{n\le x} \frac{\lambda(n)}{n}, \quad g(x) = g_0(x) = \sum_{n\le x} \frac{\mu(n)}{n},$$

$$H(x) = H_0(x) = \sum_{n\le x} h_0(n), \quad G(x) = G_0(x) = \sum_{n\le x} g_0(n).$$

We now prove an extension of the previous proposition in a specialised case.

Let $S^*(x) = \sum_{n\le x} \lambda(n)\{\frac{x}{n}\}$.

Proposition 3.

Let $1 > \sigma_0 \ge \frac{1}{2}$. The following statements are equivalent:

(i) $\forall\ \varepsilon > 0,\ H(x) = O(x^{\sigma_0+\varepsilon})$ as $x \to \infty$,

(ii) $\forall\ \varepsilon > 0,\ h(x) = O(x^{\sigma_0-1+\varepsilon})$ as $x \to \infty$,

(iii) $\forall\ \varepsilon > 0,\ S(x) = O(x^{\sigma_0+\varepsilon})$ as $x \to \infty$,

(iv) $\forall\ \varepsilon > 0,\ S^*(x) = O(x^{\sigma_0+\varepsilon})$ as $x \to \infty$,

(v) $\forall\ \varepsilon > 0,\ S(x) - S^*(x) = O(x^{\sigma_0+\varepsilon})$ as $x \to \infty$,

(vi) $RH(\sigma_0)$.

Proof:

We have (i) $\Longleftrightarrow$ (ii) $\Longleftrightarrow$ (iii) $\Longleftrightarrow$ (vi) from proposition 2.

From (1),

$$xh(x) = H(x) + S(x) + O(1) \quad \text{as} \quad x \to \infty.$$

Also,

$$(7) \qquad xh(x) - S^*(x) = \sum_{n \leq x} \lambda(n) \left[\frac{x}{n}\right]$$

$$= [\sqrt{x}] ,$$

and hence from these two equations

$$(8) \qquad H(x) = S^*(x) - S(x) + O(x^{\frac{1}{2}}) \quad \text{as} \quad x \to \infty.$$

From (7), (ii) $\Longleftrightarrow$ (iv), and from (8), (i) $\Longleftrightarrow$ (v), thus completing the proof.

Note 3. In the previous proposition (ii) $\Longrightarrow$ (i) holds for every pair of functions k, K such that

$$K(x) = \sum_{n \leq x} k(n), \text{ and in this}$$

sense (i) is weaker than (ii), and in the next section we develop this theme further.

Note 4. A corresponding result to proposition 3 holds for the functions

$$G(x),\ g(x),\ M(x),\ M^*(x).$$

Note 5. Turan's conjecture that $h(x) > 0$, for $x > 1$, has been upset by numerical investigation (Haselgrove, C.B. [1]) but we note in the next section that the argument of Lehmer and Selberg [1], that $G(x)$ changes sign infinitely often as $x \to \infty$, does not apply to $H(x)$ if RH($\frac{1}{2}$) is true.

Section 2.

Further statements equivalent to $RH(\sigma_0)$.

The notion of 'weakness' we mention in note (3), section 1, manifests itself in higher averages.

Let $$A_{-1}(x) = \sum_{n\le x} \frac{\lambda(n)}{n},$$

and for any integer $k \geq 0$ let

$$A_k(x) = \sum_{n\le x} A_{k-1}(n).$$

In this notation,

$$h(x) = A_{-1}(x),$$

and

$$H(x) = A_0(x).$$

In this section we prove:

Proposition 1.

For any fixed integer $r \geq -1$, the following statements are equivalent:

(i) $RH(\sigma_0)$,

(ii) For every $\varepsilon > 0$, $A_r(x) = O(x^{\sigma_0+r+\varepsilon})$ as $x \to \infty$.

Before proceeding to the proof we establish some helpful lemmas:

Lemma 1.

For every integer $r \geq -1$,

$$A_r(x) = \frac{1}{(r+1)!} x^{r+1} \sum_{n\le x} \frac{\lambda(n)}{n} (1 - \frac{n}{x})^{r+1} + O(x^r) \quad \text{as} \quad x \to \infty.$$

Proof:

For $r = -1$, the truth of the above statement is seen from the definition of $A_{-1}(x)$.

Also,

$$A_0(x) = \sum_{k \leq x} A_{-1}(k)$$

$$= \sum_{k \leq x} \sum_{n \leq k} \frac{\lambda(n)}{n}$$

$$= \sum_{n \leq x} \frac{\lambda(n)}{n} \sum_{n \leq k \leq [x]} 1$$

$$= \sum_{n \leq x} \frac{\lambda(n)}{n} ([x] - n + 1)$$

$$= \sum_{n \leq x} \frac{\lambda(n)}{n} (x-n) + O(1)$$

$$= \frac{1}{1!} x^1 \sum_{n \leq x} \frac{\lambda(n)}{n} (1-\frac{n}{x})^1 + O(1), \text{ as } x \to \infty,$$

and we see the proposition is true for $r = 0$.

Now suppose the proposition is true for $r = R \geq 0$.

Then

$$A_{R+1}(x) = \sum_{k \leq x} A_R(k)$$

$$= \sum_{k \leq x} \frac{1}{(R+1)!} k^{R+1} \sum_{n \leq k} \frac{\lambda(n)}{n} (1-\frac{n}{k})^{R+1} + $$

$$+ O\left(\sum_{k \leq x} k^R\right)$$

$$= \frac{1}{(R+1)!} \sum_{k \leq x} \sum_{n \leq k} \frac{\lambda(n)}{n} (k-n)^{R+1} + O(x^{R+1})$$

$$= \frac{1}{(R+1)!} \sum_{n\le x} \frac{\lambda(n)}{n} \sum_{n\le k\le [x]} (k-n)^{R+1} + O(x^{R+1})$$

(1) $$= \frac{1}{(R+1)!} \sum_{n\le x} \frac{\lambda(n)}{n} \sum_{0\le k\le [x]-n} k^{R+1} + O(x^{R+1}) \quad \text{as} \quad x \to \infty.$$

Now

(2) $$\sum_{k=1}^{b} k^{R+1} = \frac{1}{R+2} b^{R+2} + \sum_{i=1}^{R+1} C_{R+1,i}\, b^{i}$$

where the coefficients $C_{R+1,i}$ are independent of b. Consequently, from (1) and (2),

(3) $$A_{R+1}(x) = \frac{1}{(R+2)!} \sum_{n\le x} \frac{\lambda(n)}{n} ([x]-n)^{R+2} + \\ + \frac{1}{(R+1)!} \sum_{n\le x} \frac{\lambda(n)}{n} \sum_{i=1}^{R+1} C_{R+1,i} ([x]-n)^{i} + \\ + O(x^{R+1}).$$

But

$$\sum_{n\le x} \frac{\lambda(n)}{n} \sum_{i=1}^{R+1} C_{R+1,i} ([x]-n)^{i}$$

$$= \sum_{n\le x} \frac{\lambda(n)}{n} \sum_{i=1}^{R+1} C_{R+1,i} \sum_{t=0}^{i} \binom{i}{t} [x]^{i-t} (-n)^{t}$$

$$= \sum_{i=1}^{R+1} C_{R+1,i} \sum_{t=0}^{i} \binom{i}{t} [x]^{i-t} (-1)^{t} \sum_{n\le x} \lambda(n) n^{t-1}$$

$$= O\left(\sum_{i=1}^{R+1} \sum_{t=0}^{i} x^{i} \right)$$

$$= O(x^{R+1})$$

as $x \to \infty$, since, as noted in section 1, $\sum_{n\le x} \lambda(n) n^{t-1} = O(x^{t})$, as $x \to \infty$, for each integer $t \ge 0$.

Thus it follows from (3) that

$$(4) \qquad A_{R+1}(x) = \frac{1}{(R+2)!} \sum_{n\leq x} \frac{\lambda(n)}{n} ([x]-n)^{R+2} + O(x^{R+1})$$

as $x \to \infty$.

Finally,

$$\sum_{n\leq x} \frac{\lambda(n)}{n} ([x]-n)^{R+2}$$

$$= \sum_{n\leq x} \frac{\lambda(n)}{n} ((x-n)-\{x\})^{R+2}$$

$$= \sum_{n\leq x} \frac{\lambda(n)}{n} (x-n)^{R+2} + \sum_{n\leq x} \frac{\lambda(n)}{n} \sum_{1\leq t\leq R+2} (x-n)^{R+2-t}(-1)^t\{x\}^t \binom{R+2}{t}$$

$$= \sum_{n\leq x} \frac{\lambda(n)}{n} (x-n)^{R+2} + \sum_{n\leq x} \frac{\lambda(n)}{n} \sum_{1\leq t\leq R+2} \binom{R+2}{t} \sum_{0\leq s\leq R+2-t} x^{R+2-t-s}(-1)^s n^s (-1)^t\{x\}^t \binom{R+2-t}{s}$$

$$= \sum_{n\leq x} \frac{\lambda(n)}{n} (x-n)^{R+2} + \sum_{1\leq t\leq R+2} \binom{R+2}{t} \sum_{0\leq s\leq R+2-t} (-1)^{s+t} \{x\}^t x^{R+2-t-s} \binom{R+2-t}{s} \sum_{n\leq x} \lambda(n) n^{s-1}$$

But, as noted in section 1, for $s \geq 0$

$$\sum_{n\leq x} \lambda(n) n^{s-1} = O(x^s)$$

as $x \to \infty$. Hence

$$(5) \qquad \sum_{n\leq x} \frac{\lambda(n)}{n} ([x]-n)^{R+2} - \sum_{n\leq x} \frac{\lambda(n)}{n} (x-n)^{R+2}$$

$$= O\left\{ \sum_{1\leq t\leq R+2} \sum_{0\leq s\leq R+2-t} x^{R+2-t} \right\} = O(x^{R+1})$$

as $x \to \infty$. The lemma now follows from (4) and (5), and the principle of induction.

Recalling the notation

$$S_\kappa(x) = \sum_{n \le x} \lambda(n) n^\kappa$$

we next have

Lemma 2.

For every integer $r \geq -1$,

$$A_r(x) = \frac{1}{(r+1)!} \sum_{\kappa=0}^{r+1} \binom{r+1}{\kappa} (-1)^\kappa \, x^{r+1-\kappa} \, S_{\kappa-1}(x) +$$

$$+ \, O(x^r) \quad \text{as} \quad x \to \infty.$$

Proof:

From lemma (1)

$$A_r(x) = \frac{1}{(r+1)!} \sum_{n \le x} \frac{\lambda(n)}{n} \sum_{\kappa=0}^{r+1} \binom{r+1}{\kappa} x^{r+1-\kappa} (-1)^\kappa \, n^\kappa +$$

$$+ \, O(x^r)$$

$$= \frac{1}{(r+1)!} \sum_{\kappa=0}^{r+1} \binom{r+1}{\kappa} x^{r+1-\kappa} (-1)^\kappa \sum_{n \le x} \lambda(n) n^{\kappa-1} +$$

$$+ \, O(x^r)$$

as $x \to \infty$.

Thus

(6) $$A_r(x) = \frac{1}{(r+1)!} \sum_{\kappa=0}^{r+1} \binom{r+1}{\kappa} (-1)^\kappa \, x^{r+1-\kappa} \, S_{\kappa-1}(x) +$$

$$+ \, O(x^r) \quad \text{as} \quad x \to \infty.$$

Lemma 3.

For $\sigma > r + 1$, and every integer $r \geq -1$,

$$\int_1^\infty \frac{A_r(x)}{x^{s+1}}\,dx = \frac{1}{s(s-1)\ldots(s-r-1)}\;\frac{\zeta(2s-2r)}{\zeta(s-r)} + P_r(s)$$

where $P_r(s)$ is analytic for $\sigma > r$.

Proof:

We have noted in (2), Section 1, that

$$\frac{\zeta(2s-2\kappa)}{\zeta(s-\kappa)} = s\int_1^\infty \frac{S_\kappa(x)}{x^{s+1}}\,dx$$

for $\kappa \geq -1$, and $\sigma > \kappa + 1$.

Writing $s-r+\kappa$ for s in this formula we have

$$\frac{\zeta(2s-2r)}{\zeta(s-r)} = (s-r+\kappa)\int_1^\infty \frac{S_\kappa(x)}{x^{s-r+\kappa+1}}\,dx$$

for $\sigma > r + 1$, with $\kappa \geq -1$.

Hence

$$\int_1^\infty \frac{x^{r-\kappa+1}S_{\kappa-1}(x)}{x^{s+1}}\,dx = \frac{1}{(s-r+\kappa-1)}\;\frac{\zeta(2s-2r)}{\zeta(s-r)} \tag{7}$$

for $\sigma > r + 1$ with $\kappa \geq 0$.

Consequently, from lemma 2,

$$\int_1^\infty \frac{A_r(x)}{x^{s+1}}\,dx$$

$$= \frac{1}{(r+1)!}\sum_{\kappa=0}^{r+1}\binom{r+1}{\kappa}(-1)^\kappa \int_1^\infty \frac{x^{r+1-\kappa}\,S_{\kappa-1}(x)\,dx}{x^{s+1}} + P_r(s),$$

where $P_r(s)$ is analytic for $\sigma > r$.

Then from (7) we have for $\sigma > r+1$,

$$(8) \qquad \int_1^\infty \frac{A_r(x)}{x^{s+1}}\,dx$$

$$= \frac{1}{(r+1)!} \sum_{\kappa=0}^{r+1} \binom{r+1}{\kappa} (-1)^\kappa \frac{1}{(s-r+\kappa-1)} \frac{\zeta(2s-2r)}{\zeta(s-r)}$$

$$+ P_r(s) .$$

Using the 'cover up' rule for partial fractions we easily see that

$$\frac{1}{(r+1)!} \sum_{\kappa=0}^{r+1} \binom{r+1}{\kappa} (-1)^\kappa \frac{1}{(s-r+\kappa-1)}$$

$$= \frac{1}{s(s-1)\dots(s-r-1)} ,$$

and hence from (8),

$$(9) \qquad \int_1^\infty \frac{A_r(x)}{x^{s+1}} = \frac{1}{s(s-1)\dots(s-r-1)} \frac{\zeta(2s-2r)}{\zeta(s-r)} +$$

$$+ P_r(s)$$

for $\sigma > r+1$, where $P_r(s)$ is analytic for $\sigma > r$.

<u>Proof of proposition 1:</u>

For integer $r \geq -1$ let T_r be the statement:

For every $\varepsilon > 0$, $A_r(x) = O(x^{\sigma_0+r+\varepsilon})$ as $x \to \infty$.

From proposition 3, section 1, we have

$$\mathrm{RH}(\sigma_0) \iff T_{-1} \iff T_0 .$$

Clearly, $T_r \Rightarrow T_{r+1}$ for all $r \geq 0$. It thus suffices to show

$T_r \Rightarrow RH(\sigma_0)$ for any fixed $r \geq -1$, and this follows readily from (9).

Note 1. With $B_{-1}(x) = \sum_{n \leq x} \frac{\mu(n)}{n}$

and

$$B_k(x) = \sum_{n \leq x} B_{k-1}(n)$$

for integer $k \geq 0$, the method of proof of the preceding proposition leads to an analogue of lemma (3). Namely,

for $\sigma > r + 1$, and integer $r \geq -1$,

$$\int_1^\infty \frac{B_r(x)}{x^{s+1}}\, dx = \frac{1}{s(s-1)\ldots(s-r-1)\zeta(s-r)} + Q_r(s), \tag{10}$$

where $Q_r(s)$ is analytic for $\sigma > r$. We consequently have

Proposition 2.

For any fixed integer $r \geq -1$, the following statements are equivalent

(i) $RH(\sigma_0)$,

(ii) For every $\varepsilon > 0$, $B_r(x) = O(x^{\sigma_0+r+\varepsilon})$ as $x \to \infty$.

Proof: c.f. Proposition 1.

Note 2. Although we are concentrating mainly on the Möbius function and the Liouville function the preceding propositions apply to the class of functions $\{\tau^{(k)}\}$ defined for $k = 2, 3, \ldots$ by

$$\sum_{n=1}^{\infty} \frac{\tau^{(k)}(n)}{n^s} \zeta(s) = \zeta(ks), \quad (\sigma > 1),$$

where we have $\tau^{(2)} \equiv \lambda$, and, in a sense, $\tau^{(\infty)} \equiv \mu$.

Section 3.

Some results on the oscillatory behaviour of certain summatory functions involving μ and λ.

Let $A_r(x)$ and $B_r(x)$ be defined as in section 2. Let $\bar{\sigma}$ satisfy $\frac{1}{2} \leqslant \bar{\sigma} < 1$ and be such that $\zeta(s) = 0$ has a solution with $\sigma \geqslant \bar{\sigma}$. From propositions 1 and 2, section 2, it follows that

$$\forall \quad \varepsilon > 0, \quad A_r(x) = \Omega(x^{r+\bar{\sigma}-\varepsilon}),$$

and

$$\forall \quad \varepsilon > 0, \quad B_r(x) = \Omega(x^{r+\bar{\sigma}-\varepsilon}),$$

as $x \to \infty$. Actually, we can say more than this.

Proposition 1.

Let r be an integer, $r \geqslant -1$, and let K be any real number. Then for every $\varepsilon > 0$,

$$B_r(x) - K\, x^{r+\bar{\sigma}-\varepsilon}$$

changes sign infinitely often as $x \to \infty$.

i.e. $\qquad \forall \quad \varepsilon > 0, \quad B_r(x) = \Omega\pm(x^{r+\bar{\sigma}-\varepsilon}) \quad \text{as} \quad x \to \infty.$

Proof:

For $\sigma > r + 1$, $r \geqslant 0$, let the Dirichlet series $L_r(s)$ be defined by

$$L_r(s) = \sum_{n=1}^{\infty} \frac{B_{r-1}(n) - Kn^{r-1+\bar{\sigma}-\varepsilon}}{n^s}, \quad \text{where} \quad 0 < \varepsilon < \bar{\sigma}.$$

From proposition 1, section 1,

$$L_r(s) = \sum_{n=1}^{\infty} \frac{B_{r-1}(n)}{n^s} - K\,\zeta(s-r+1-\bar{\sigma}+\varepsilon)$$

$$= s\int_1^{\infty} \frac{B_r(x)}{x^{s+1}}\,dx - K\,\zeta(s-r+1-\bar{\sigma}+\varepsilon)$$

Hence, from 10, section 2,

$$(1) \qquad L_r(s) = \frac{s}{s(s-1)\;....\;(s-r-1)\,\zeta(s-r)} - K\zeta(s-r+1-\bar{\sigma}+\varepsilon) +$$

$$+\; Q_r(s),$$

where $Q_r(s)$ is regular for $\sigma > r$.

Suppose that the coefficients of the series for $L_r(s)$ are eventually of one sign. Then by a classical theorem of Landau, the series has a singularity at the real point on the line of convergence of the series. But the first term in (1), $\dfrac{1}{(s-1)(s-2)\ldots(s-r-1)\,\zeta(s-r)}$, has singularities at $s = 1;\ 2,\ \ldots,\ r$ and $s = r + \rho$, where ρ is a zero of $\zeta(s)$. Since $\zeta(s)$ has no real zeros with $s \geq 0$ the first term has no real singularities with $\sigma > r$. The second term in (1), $K\,\zeta(s-r+1-\bar{\sigma}+\varepsilon)$, has no singularities at all for $\sigma > r + \bar{\sigma} - \varepsilon$. Hence $L_r(s)$ has no real singularity for $\sigma > r + \bar{\sigma} - \varepsilon$, and the abscissa of convergence of the Dirichlet series for $L_r(s)$ must be less than or equal to $r + \bar{\sigma} - \varepsilon$. Hence $L_r(s)$ is analytic for $\sigma > r + \bar{\sigma} - \varepsilon$, and thus, from (1), $\zeta(s)$ must be non-zero for $\sigma > \bar{\sigma} - \varepsilon$, which contradicts the definition of $\bar{\sigma}$. If follows that the coefficients of the Dirichlet series for $L_r(s)$ cannot be ultimately of one sign, and this completes the proof.

<u>Corollary 1</u>:

Let r be an integer, $r \geq -1$, and let K be any real number.

Then $\forall \quad \varepsilon > 0, \; B_r(x) - K\, x^{r+\frac{1}{2}-\varepsilon}$

changes sign infinitely often as $x \to \infty$.

Proof:

This follows since $\bar{\sigma} \geq \frac{1}{2}$.

As a corollary to the method of proof of proposition 1 we also have

Corollary 2.

Let r be an integer, $r \geq -1$, and let K be any real number. Let $1 \geq \sigma_0 \geq \frac{1}{2}$. If $B_r(x) - K\, x^{r+\sigma_0}$ is eventually of one sign as $x \to \infty$, then $\mathrm{RH}(\sigma_0)$ is true.

Proof:

Let $B_r(x) - K\, x^{r+\sigma_0}$ be eventually of one sign as $x \to \infty$. Then with σ_0 playing the role of $\bar{\sigma}$ in the equations leading up to (1) we find

$$L_r(s) = \frac{1}{(s-1)\ldots.(s-r-1)\,\zeta(s-r)} - K\,\zeta(s-r+1-\sigma_0) +$$

$$+ \; Q_r(s),$$

where $Q_r(s)$ is regular for $\sigma > r$. As in proposition 1 we then have $L_r(s)$ analytic for $\sigma > \sigma_0 + r$ and consequently $\zeta(s) \neq 0$ for $\sigma > \sigma_0$.

Note 1. Analagous results to proposition 1 hold for the corresponding summatory functions associated with $\tau^{(k)}$ for $k = 3, 4, \ldots,$ where we recall

$$\sum_{n=1}^{\infty} \frac{\tau^{(k)}(n)}{n^s}\,\zeta(s) = \zeta(ks), \quad (\sigma > 1).$$

However, for $k = 2$, $\tau^{(2)} \equiv \lambda$, and the equation corresponding to (10) is

$$L_r(s) = \frac{1}{(s-1)(s-2)\dots(s-r-1)} \frac{\zeta(2s-2r)}{\zeta(s-r)} +$$

$$- K\zeta(s-r+1-\bar{\sigma}+\varepsilon) + P_r(s).$$

The pole of $\zeta(2s-2r)$ at $s = r + \frac{1}{2}$ prevents the argument in proposition 1 following here in the case $\bar{\sigma} = \frac{1}{2}$. But for $\bar{\sigma} > \frac{1}{2}$ the corresponding result holds.

i.e.

Proposition 2.

Let $\bar{\sigma}$ satisfy $\frac{1}{2} < \bar{\sigma} < 1$ and be such that $\zeta(s) = 0$ has a solution with $\sigma \geqslant \bar{\sigma}$. Let r be an integer, $r \geqslant -1$. Then for every $\varepsilon > 0$,

$$A_r(x) = \Omega_{\pm}(x^{r+\bar{\sigma}-\varepsilon})$$

as $x \to \infty$.

Proof:

Similar to that of proposition 1. A result corresponding to corollary 1 cannot be stated here for the $A_r(x)$, since proposition 2 assumes $\bar{\sigma} > \frac{1}{2}$, and if RH($\frac{1}{2}$) is true it is conceivable that the $A_r(x)$ are eventually of one sign as $x \to \infty$ for some $r \geqslant R > -1$.

However, we do have an analogue of corollary 2, for the $A_r(x)$. Namely,

Corollary 3.

Let r be an integer, $r \geqslant -1$, and let K be any real number. Let $1 \geqslant \sigma_0 \geqslant \frac{1}{2}$. If $A_r(x) - Kx^{r+\sigma_0}$ is eventually of one sign as $x \to \infty$ then RH(σ_0) is true.

Proof:

Similar to that of corollary 2.

Note 2. These results improve and generalise the result of Lehmer and Selberg [1], that $B_0(x) - K$ changes sign infinitely often as $x \to \infty$, and generalise the well known result that if $\frac{H(x)}{x^{\frac{1}{2}}}$ is either bounded above or below then RH($\frac{1}{2}$) is true.

Section 4.

Incomplete sums involving the Möbius and Liouville functions.

Let $\theta = \inf\{\sigma_0 : RH(\sigma_0) \text{ is true}\}$

$= \inf\{\sigma_0 : \zeta(s) \neq 0 \text{ for } \sigma > \sigma_0\}.$

Let $\alpha(\kappa) = \inf\{\xi : S_\kappa(x) = O(x^\xi) \text{ as } x \to \infty\}.$

From proposition 2, section 1, and its corollary, we have

$$\alpha(\kappa) = \theta + \kappa \quad \text{if} \quad \kappa = -1 \quad \text{or} \quad \kappa > -\theta.$$

Let

$$T_{\kappa,\delta}(x) = \sum_{n \leq x^\delta} \lambda(n) n^\kappa \left[\frac{x}{n}\right],$$

and

$$\beta(\delta,\kappa) = \inf\{\xi : T_{\kappa,\delta}(x) = O(x^\xi) \quad \text{as} \quad x \to \infty\},$$

where $0 \leq \delta \leq 1$, and either $\kappa = 0$ or $\kappa > 1 - \theta$.

We would expect $T_{\kappa,\delta}(x)$ to behave rather like $xS_{\kappa-1}(x^\delta)$ when δ is not too close to 1, using $\frac{x}{n}$ as an approximation for $[\frac{x}{n}]$. However, when δ is close to 1 we would expect the comparison to be more delicate since, for example,

$$T_{0,1}(x) = \sum_{n \leq x} \lambda(n) \left[\frac{x}{n}\right] = [\sqrt{x}],$$

whilst

$$xS_{-1}(x) = \Omega(x^{\theta-\varepsilon}) \quad \text{as} \quad x \to \infty \quad \text{if} \quad \varepsilon > 0.$$

Proposition 1.

Suppose that $\kappa = 0$ or $\kappa > 1 - \theta$. Then

$$\beta(\delta,\kappa) = 1 + \delta(\theta+\kappa-1)$$

for $0 \leq \delta < \frac{1}{2-\theta}$.

Proof:

For convenience we put $\beta(\delta,\kappa) = \beta$. Then

(1) $$x\, S_{\kappa-1}(x^{\delta}) = T_{\kappa,\delta}(x) + \sum_{n\leq x^{\delta}} \lambda(n)n^{\kappa}\{\frac{x}{n}\} .$$

Hence, for every $\varepsilon > 0$,

(2) $$x\, S_{\kappa-1}(x^{\delta}) = O(x^{\beta+\varepsilon}) + O(x^{\delta(1+\kappa)}), \quad \text{as} \quad x \to \infty.$$

Also, from proposition 2, corollary, section 1, for every $\varepsilon > 0$,

(3) $$x\, S_{\kappa-1}(x^{\delta}) = \Omega(x^{1+\delta(\theta+\kappa-1-\varepsilon)}) \quad \text{as} \quad x \to \infty.$$

From (2) and (3), for sufficiently small positive ε,

$$\beta < \delta(1+\kappa) \Rightarrow \beta + \varepsilon \leq \delta(1+\kappa)$$

$$\Rightarrow \delta(\kappa+1) > 1 + \delta(\theta+\kappa-1-\varepsilon)$$

$$\Rightarrow \delta > \frac{1}{2-\theta+\varepsilon} .$$

Consequently, for sufficiently small positive ε,

$$\delta < \frac{1}{2-\theta} \Rightarrow \delta \leq \frac{1}{2-\theta+\varepsilon} .$$

$$\Rightarrow \beta + \varepsilon > \delta(1+\kappa) .$$

Hence, from (2) and (3)

$$\delta < \frac{1}{2-\theta} \Rightarrow \beta + \varepsilon > 1 + \delta(\theta+\kappa-1-\varepsilon) \ .$$

Then, letting $\varepsilon \to 0$,

(4) $$\delta < \frac{1}{2-\theta} \Rightarrow \beta \geqslant 1 + \delta(\theta+\kappa-1).$$

Also, using proposition 2, section 1, for all $\varepsilon > 0$,

(5) $$T_{\kappa,\delta}(x) = x\, S_{\kappa-1}(x^{\delta}) - \sum_{n\leqslant x^{\delta}} \lambda(n) n^{\kappa}\{\tfrac{x}{n}\}$$

$$= O(x^{1+\delta(\theta+\kappa-1+\varepsilon)}) + O(x^{\delta(\kappa+1)})$$

as $x \to \infty$, and from the definition of β

(6) $$T_{\kappa,\delta}(x) = \Omega(x^{\beta-\varepsilon})$$

as $x \to \infty$.

Since for sufficiently small positive ε,

$$\delta < \frac{1}{2-\theta} \Rightarrow \delta \leqslant \frac{1}{2-\theta+\varepsilon}$$

$$\Rightarrow \delta(\kappa+1) \leqslant 1 + \delta(\theta+\kappa-1-\varepsilon),$$

we have from (5) and (6) that

$$1 + \delta(\theta+\kappa-1+\varepsilon) > \beta - \varepsilon$$

if $\delta < \frac{1}{2-\theta}$.

Letting $\varepsilon \to 0$,

(7) $$\beta \leqslant 1 + \delta(\theta+\kappa-1) \quad \text{if} \quad 0 \leqslant \delta < \frac{1}{2-\theta} \ .$$

Comparing (4) and (7) we obtain the desired result.

For the omitted δ-interval we have

Proposition 2.

Suppose that either $\kappa = 0$ or $\kappa > 1 - \theta$. Then

$$\beta(\delta,\kappa) \leq 1 + \delta(\theta+\kappa-1)$$

for $\frac{1}{2-\theta} \leq \delta \leq 1$.

Proof:

We treat the cases $\kappa = 0$ and $\kappa > 1 - \theta$ separately.

Firstly for $\kappa = 0$, we have

$$T_{0,\delta}(x) = \sum_{n \leq x^\delta} \lambda(n)\left[\frac{x}{n}\right]$$

$$= [\sqrt{x}] - \sum_{x^\delta < n \leq x} \lambda(n)\left[\frac{x}{n}\right]$$

$$= [\sqrt{x}] - \sum_{x^\delta < n \leq x} (S(n) - S(n-1))\left[\frac{x}{n}\right] .$$

Hence

$$T_{0,\delta}(x) = [\sqrt{x}] - \sum_{[x^\delta] \leq n \leq x} S(n)\left(\left[\frac{x}{n}\right] - \left[\frac{x}{n+1}\right]\right) + \tag{8}$$

$$+ S(x^\delta)\left[\frac{x}{[x^\delta]}\right] .$$

Since $S(x) = O(x^{\theta+\varepsilon})$ as $x \to \infty$,

$$S(x^\delta) = O(x^{\delta(\theta+\varepsilon)}) \quad \text{as} \quad x \to \infty. \tag{9}$$

Also, from Gelfond and Linnik [1] we have with $n > \sqrt{x}$,

$$\left[\frac{x}{n}\right] - \left[\frac{x}{n+1}\right] = \begin{cases} 1, & n = \left[\frac{x}{k}\right] , \\ 0, & n \neq \left[\frac{x}{k}\right] , \end{cases} \tag{10}$$

where $k < \sqrt{x}$ is an integer.

Applying (10) to (8), noting that $\delta \geqslant {}^2/_3$ here, it follows that

$$T_{0,\delta}(x) = [\sqrt{x}] - \sum_{k \leqslant \frac{x}{[x^\delta]}} S(\tfrac{x}{k}) + S(x^\delta)\left[\frac{x}{[x^\delta]}\right].$$

Hence, from (9),

$$(11) \qquad T_{0,\delta}(x) = [\sqrt{x}] + O(x^{\theta\delta-\delta+1+\varepsilon'}).$$

Finally since $\theta \geqslant \frac{1}{2}$, $\delta \leqslant 1$ we see $\theta\delta - \delta + 1 \geqslant \frac{1}{2}$, and so from (11) we have

$$\beta(\delta, 0) \leqslant \theta\delta - \delta + 1.$$

We now consider the case in which $\kappa > 1 - \theta$. For convenience put $N = [x^\delta]$. Then for every $\varepsilon > 0$,

$$T_{\kappa,\delta}(x) = \sum_{n \leqslant N} (S_\kappa(n) - S_\kappa(n-1))[\tfrac{x}{n}]$$

$$= \sum_{n \leqslant N} S_\kappa(n)\left([\tfrac{x}{n}] - [\tfrac{x}{n+1}]\right) + S_\kappa(N)[\tfrac{x}{N+1}]$$

$$= O\left(\sum_{n \leqslant N} n^{\theta+\kappa+\varepsilon}\left([\tfrac{x}{n}] - [\tfrac{x}{n+1}]\right) + N^{\theta+\kappa+\varepsilon}[\tfrac{x}{N+1}]\right)$$

$$= O\left(\sum_{n \leqslant N} (n^{\theta+\kappa+\varepsilon} - (n-1)^{\theta+\kappa+\varepsilon})[\tfrac{x}{n}]\right)$$

$$= O\left(x \sum_{n \leqslant N} n^{\theta+\kappa+\varepsilon-2}\right)$$

$$= O(x\, N^{\theta+\kappa+\varepsilon-1})$$

$$= O(x^{1+\delta(\theta+\kappa+\varepsilon-1)})$$

as $x \to \infty$, since $\theta + \kappa > 1$.

Hence, letting $\varepsilon \to 0$,

$$\beta(\delta,\kappa) \leqslant 1 + \delta(\theta+\kappa-1) \; .$$

Proposition 3.

Suppose that $\kappa > 1 - \theta$. Then

$$\beta(\delta,\kappa) = 1 + \delta(\theta+\kappa-1)$$

at $\delta = 1$. i.e. $\beta(1,\kappa) = \theta + \kappa$.

Proof:

From proposition 2 we have

$$\beta(1,\kappa) \leqslant \theta + \kappa \, ,$$

and it thus suffices to show that $\beta(1,\kappa) \geqslant \theta + \kappa$. For convenience we set

$$A(x) = \sum_{n \leqslant x} \lambda(n) n^{\kappa} [\frac{x}{n}] \quad .$$

Then

$$S_{\kappa}(x) = \sum_{n \leqslant x} \lambda(n) n^{\kappa}$$

$$= \sum_{n \leqslant x} \lambda(n) n^{\kappa} \sum_{r \leqslant \frac{x}{n}} \mu(r) [\frac{x}{rn}]$$

$$= \sum_{r \leqslant x} \mu(r) \sum_{n \leqslant \frac{x}{r}} \lambda(n) n^{\kappa} [\frac{x}{rn}] \quad .$$

Thus

$$(12) \qquad S_{\kappa}(x) = \sum_{r \leqslant x} \mu(r) A(\frac{x}{r}) \quad .$$

Now if $\beta(1,\kappa) \leq \sigma_0 + \kappa$ with $\sigma_0 + \kappa > 1$, then from the definition of $\beta(1,\kappa)$ and (12), for every $\varepsilon > 0$,

$$S_\kappa(x) = O\left\{\sum_{r \leq x} \left(\frac{x}{r}\right)^{\sigma_0+\kappa+\varepsilon}\right\}$$

$$= O(x^{\sigma_0+\kappa+\varepsilon}) \qquad \text{as } x \to \infty.$$

Hence from our opening remarks,

$$\sigma_0 + \kappa + \varepsilon \geq \theta + \kappa.$$

Letting $\varepsilon \to 0$, $\sigma_0 \geq \theta$.

Thus $\qquad \beta(1,\kappa) \leq \sigma_0 + \kappa \Rightarrow \sigma_0 \geq \theta$

$$\Rightarrow \beta(1,\kappa) \geq \theta + \kappa$$

and the proposition follows.

Note 1. If $\kappa = 0$ then $\beta(\delta,\kappa) = \frac{1}{2}$ for $\delta = 1$.

i.e. $\beta(1,0) = \frac{1}{2} \leq \theta$.

The problem of a lower bound for $\beta(\delta,\kappa)$ when $\frac{1}{2-\theta} < \delta < 1$ is still under investigation.

Note 2. In this section the only estimate we have used for $\sum_{n \leq x^\delta} \lambda(n) n^\kappa \{\frac{x}{n}\}$ is the trivial estimate

(13) $$\sum_{n \leq x^\delta} \lambda(n) n^\kappa \{\frac{x}{n}\} = O(x^{\delta(1+\kappa)})$$

as $x \to \infty$, although it follows from proposition 2 and (1) that for every $\varepsilon > 0$,

(14) $$\sum_{n \leqslant x^{\delta}} \lambda(n) n^{\kappa} \{\frac{x}{n}\} = O(x^{1+\delta(\theta+\kappa-1)+\varepsilon})$$

as $x \to \infty$, provided $1 \geqslant \delta \geqslant \frac{1}{2-\theta}$, when either $\kappa = 0$ or $\kappa > 1 - \theta$.

We see (14) is a significant improvement on (13) but the problem of improving on (13) in the range $0 \leqslant \delta < \frac{1}{2-\theta}$ remains open. In light comparison with

$$\sum_{n \leqslant x^{\delta}} \lambda(n) n^{\kappa} = O(x^{\delta(\theta+\kappa+\varepsilon)})$$

as $x \to \infty$, (for all $\varepsilon > 0$), we may suspect

$$\sum_{n \leqslant x^{\delta}} \lambda(n) n^{\kappa} \{\frac{x}{n}\} = O(x^{\delta(\theta+\kappa+\varepsilon)})$$

as $x \to \infty$ (for all $\varepsilon > 0$) for the whole range $0 \leqslant \delta \leqslant 1$.

<u>Note 3</u>. The methods and results in this section apply, and are valid, with the Möbius function replacing the Liouville function.

We conclude this section with some remarks on the sum

$$\sum_{n \leqslant x^{\delta}} \lambda(n) n [\frac{x}{n}]^2 \quad .$$

From

$$\sum_{n^2 \leqslant x} n^2 = \sum_{n \leqslant x} n (\sum_{g | n} \lambda(g))$$

$$= \tfrac{1}{2} \sum_{n \leqslant x} \lambda(n) n \ [\frac{x}{n}] \left([\frac{x}{n}] + 1 \right)$$

we have

(15) $$\sum_{n \leqslant x} \lambda(n) n [\frac{x}{n}]^2 + \sum_{n \leqslant x} \lambda(n) n [\frac{x}{n}] = O(x^{3/2}).$$

The possibility that each term

$$\sum_{n\leq x} \lambda(n)n[\frac{x}{n}]^2, \qquad \sum_{n\leq x} \lambda(n)n\ [\frac{x}{n}]$$

has order greater than $^3/_2$ and that cancellation occurs to produce the $O(x^{3/2})$ term on the RHS of (15) is open, particularly in view of the corresponding μ-equation

$$\sum_{n\leq x} \mu(n)n[\frac{x}{n}]^2 + \sum_{n\leq x} \mu(n)n[\frac{x}{n}] = 1$$

where, from the analogue of proposition 3 we necessarily have

$$\sum_{n\leq x} \mu(n)n[\frac{x}{n}] = \Omega(x^{3/2-\varepsilon}) \quad \text{as} \quad x \to \infty.$$

However, we note from proposition 3 that

$$\sum_{n\leq x} \lambda(n)n[\frac{x}{n}] = O(x^{3/2+\varepsilon}) \quad \text{as} \quad x \to \infty,$$

(for every $\varepsilon > 0$), implies RH($\frac{1}{2}$), and there is consequently some point in examining the orders of

$$\sum_{n\leq x^\delta} \lambda(n)n[\frac{x}{n}], \quad \text{and} \quad \sum_{n\leq x^\delta} \lambda(n)n[\frac{x}{n}]^2 .$$

For $0 \leq \delta \leq 1$ we have

$$\beta(\delta,1) = \inf\{\, \xi : \sum_{n\leq x^\delta} \lambda(n)n[\frac{x}{n}] = O(x^\xi) \quad \text{as} \quad x \to \infty\} ,$$

and we let

$$\gamma(\delta,1) = \inf\{\, \xi : \sum_{n\leq x^\delta} \lambda(n)n[\frac{x}{n}]^2 = O(x^\xi) \quad \text{as} \quad x \to \infty\} .$$

Proposition 4.

For $0 \leq \delta < \frac{1}{2-\theta}$,

$$\gamma(\delta,1) - \beta(\delta,1) = 1 - \delta.$$

Proof:

The methods of proposition (1) apply here so the detail is not repeated.

Section 5.

Farey Series and $RH(\sigma_0)$.

Connections between statements involving properties of Farey series and $RH(\sigma_0)$ were discovered by Franel and Landau, [1], and certain extensions of their results were developed by Kopriva, [1]. Some of their arguments were non-elementary in nature but Zulauf [4] provided elementary arguments for all the preceding results. In this section we survey these results and develop a lemma which has application in Zulauf [4], and include a further result in the theory.

For complex $s = \sigma + it$ let

$$Z(s,x) = \sum_{m \leq x} \frac{1}{m^s}$$

and

$$Y(s,x) = \sum_{n \leq x} \mu(n)\, Z(s,\frac{x}{n}) \quad .$$

From the method of proof in Titchmarch, [1], page 67, we see that for $\sigma > 1$,

$$Z(s,x) = \zeta(s) - \frac{x^{1-s}}{s-1} + O(x^{-\sigma})$$

as $x \to \infty$. Thus for $\sigma > 1$, $Z(s,x)$ increases and tends to a limit as $x \to \infty$, and we thus expect the behaviour of $Y(s,x)$ as $x \to \infty$ to be similar to that of $M(x)$. In fact, this turns out to be the case for $\sigma > 1 - \theta$ as the following proposition shows.

Proposition 1.

Suppose that $\sigma > 1 - \sigma_0$, Then $RH(\sigma_0)$ is true if and only if for every $\varepsilon > 0$, $Y(s,x) = O(x^{\sigma_0+\varepsilon})$ as $x \to \infty$.

Proof:

Firstly, we note that

$$Y(s,x) = \sum_{n\leq x} \mu(n) \sum_{m\leq x/n} \frac{1}{m^s}$$

$$= \sum_{m\leq x} \frac{1}{m^s} \sum_{n\leq x/m} \mu(n) ,$$

and so

(1) $$Y(s,x) = \sum_{m\leq x} \frac{1}{m^s} M(\frac{x}{m}) .$$

Thus, if for $\varepsilon > 0$, we have

$$M(x) = 0(x^{\sigma_0+\varepsilon}) \text{ as } x \to \infty, \text{ then}$$

(2) $$Y(s,x) = 0\left(\sum_{m\leq x} \frac{1}{m^\sigma} (\frac{x}{m})^{\sigma_0+\varepsilon}\right) \text{ as } x \to \infty$$

$$= 0(x^{\sigma_0+\varepsilon}) \text{ as } x \to \infty.$$

Now from (1) we have

$$x^s\, Y(s,x) = \sum_{m\leq x} (\frac{x}{m})^s M(\frac{x}{m}) ,$$

and hence, using the Möbius inversion formula

(3) $$x^s\, M(x) = \sum_{m\leq x} \mu(m) (\frac{x}{m})^s\, Y(s,\frac{x}{m}) .$$

It then follows from (3) that if for every $\varepsilon > 0$, $Y(s,x) = 0(x^{\sigma_0+\varepsilon})$ as $x \to \infty$, then

(4) $$M(x) = 0(x^{-\sigma} \sum_{m\leq x} (\frac{x}{m})^{\sigma+\sigma_0+\varepsilon})$$

$$= 0(x^{\sigma_0+\varepsilon}) \text{ as } x \to \infty.$$

The proposition now follows from proposition 2, section 1, and the fact that (2) and (4) imply each other.

The sums $Y(s,x)$ have certain connections with Farey series which we develop from the following lemma.

$$\text{Let } K_N = \{ \tfrac{a}{b} : (a,b) = 1, \ 0 < b \leq N, \ a, b \in \mathbb{N} \}.$$

i.e. let K_N be the set of all positive rational numbers whose denominators in reduced form do not exceed N.

Let $p_1, p_2, p_3, \ldots$ be the elements of K_N, enumerated in ascending order of magnitude, and let

$$K = \Phi(N) = \sum_{n \leq N} \varphi(n), \quad \text{where } \varphi \text{ is}$$

Euler's function. We note that $p_K = 1$ and with $p_0 = 0$, $p_0, p_1, \ldots, p_K$ are the elements of the Farey series of order N.

Let $H_N = \{p_1, p_2, \ldots, p_K\}$.

<u>Lemma 1.</u>

Let f be any function defined on the positive rational numbers, and let x be any positive number.

Then

$$\sum_{\substack{p_k \in K_N \\ p_k \leq x}} f(p_k) = \sum_{1 \leq b \leq N} M\left(\frac{N}{b}\right) \sum_{a \leq bx} f\left(\frac{a}{b}\right) .$$

<u>Proof:</u>

$$\sum_{1 \leq b \leq N} M\left(\frac{N}{b}\right) \sum_{a \leq bx} f\left(\frac{a}{b}\right) = \sum_{1 \leq b \leq N} \sum_{\substack{a \leq bx \\ (a,b)=1}} \sum_{d \leq N/b} f\left(\frac{ad}{bd}\right) M\left(\frac{N}{bd}\right)$$

$$= \sum_{1 \leq b \leq N} \sum_{\substack{a \leq bx \\ (a,b)=1}} f\left(\frac{a}{b}\right) \sum_{d \leq N/b} M\left(\frac{N}{bd}\right)$$

$$= \sum_{1 \le b \le N} \sum_{\substack{a \le bx \\ (a,b)=1}} f(\tfrac{a}{b})$$

$$= \sum_{\substack{p_k \in K_N \\ p_k \le x}} f(p_k) \ .$$

Note 1. This lemma is well known in the special case $x = 1$. i.e. the case in which the sum on the LHS extends over all positive Farey fractions of order N. (see for example, Edwards, [1], p.264).

Note 2. Using the lemma with the choice $f(t) = t^{-s}$ we get

(5)
$$\sum_{\substack{p_k \in K_N \\ p_k \le x}} \frac{1}{p_k^{\ s}} = \sum_{1 \le b \le N} M(\tfrac{N}{b}) \sum_{a \le bx} (\tfrac{b}{a})^s$$

$$= \sum_{1 \le b \le N} M(\tfrac{N}{b}) b^s \ Z(s,bx).$$

But from (1),

$$Y(-s,N) = \sum_{1 \le b \le N} M(\tfrac{N}{b}) b^s,$$

and hence

$$\sum_{\substack{p_k \in K_N \\ p_k \le x}} \frac{1}{p_k^{\ s}} = Y(-s,N) \ Z(s,bx).$$

For $\sigma > 1$, letting $x \to \infty$, we thus have

(6)
$$\zeta_N^*(s) = \sum_{p_k \in K_N} \frac{1}{p_k^{\ s}} = Y(-s,N) \ \zeta(s).$$

Also, with $x = 1$,

(7) $$\sum_{p_k \in H_N} \frac{1}{p_k^s} = \sum_{1 \leq b \leq N} b^s \, M(\tfrac{N}{b}) \, Z(s,b) \quad .$$

With r a non-negative integer let

$$Z(-r,b) = \sum_{n \leq b} n^r$$

$$= \sum_{1 \leq j \leq r+1} C_{r,j} b^j$$

for each natural number b, where for fixed r the coefficients $C_{r,j}$ are independent of b.

Proposition 2.

For every integer $r \geq 2$,

$$\sum_{p_k \in H_N} (p_k^r - \frac{1}{r+1}) = \frac{1}{2} + \sum_{j=1}^{r-1} C_{r,j} \; Y(r-j,N).$$

Proof:

From (7)

$$\sum_{p_k \in H_N} p_k^r = \sum_{1 \leq b \leq N} b^{-r} \, M(\tfrac{N}{b}) \; Z(-r,b)$$

$$= \sum_{1 \leq b \leq N} b^{-r} \, M(\tfrac{N}{b}) \sum_{1 \leq j \leq r+1} C_{r,j} \, b^j$$

$$= \sum_{1 \leq j \leq r+1} C_{r,j} \sum_{1 \leq b \leq N} M(\tfrac{N}{b}) b^{j-r}$$

Hence, from (1),

(8) $$\sum_{p_k \in H_N} p_k^r = \sum_{1 \leq j \leq r+1} C_{r,j} \; Y(r-j,N) \; .$$

With $r = 0$,

$$\Phi(N) = \sum_{p_k \in H_N} 1 = C_{0,1}\, Y(-1,N),$$

and so

(9) $$\sum_{p_k \in H_N} 1 = Y(-1,N)$$

Also, since $C_{r,r+1} = \frac{1}{r+1}$, for all $r \geq 0$ we have from (8),

(10) $$\sum_{p_k \in H_N} (p_k^r - \frac{1}{r+1}) = \sum_{1 \leq j \leq r} C_{r,j}\, Y(r-j,N).$$

The proposition follows from (10) on noting from (1) that

(11) $$Y(0,N) = \sum_{n \leq N} M(\frac{N}{n}) = 1,$$

and that $C_{r,r} = \frac{1}{2}$ for $r \geq 1$.

In particular, for $r = 2,3$ we have

(12) $$\sum_{p_k \in H_N} (p_k^2 - \frac{1}{3}) = \frac{1}{2} + \frac{1}{6}\, Y(1,N),$$

and

(13) $$\sum_{p_k \in H_N} (p_k - \frac{1}{4}) = \frac{1}{2} + \frac{1}{4}\, Y(1,N).$$

The known equivalence (Kopriva [1], Zulauf [1]) of the statements

(i) $\mathrm{RH}(\sigma_0)$,

(ii) $\forall\ \varepsilon > 0$, $\sum_{p_k \in H_N} (p_k^2 - \frac{1}{3}) = O(N^{\sigma_0+\varepsilon})$ as $N \to \infty$,

(iii) $\forall\ \varepsilon > 0$, $\sum_{p_k \in H_N} (p_k^3 - \frac{1}{4}) = O(N^{\sigma_0+\varepsilon})$ as $N \to \infty$, can now be deduced from (12) and (13) using proposition 1.

Note 3. The original discovery of a connection between $RH(\sigma_0)$ and the distribution of Farey numbers by Franel and Landau [1] was in the form of the equivalence of the three statements

(iv) $\forall \ \varepsilon > 0, \quad M(N) = O(N^{\sigma_0+\varepsilon})$ as $x \to \infty$,

(v) $\forall \ \varepsilon > 0, \quad \sum_{p_k \in H_N} (p_k - \frac{k}{\Phi(N)})^2 = O(N^{2\sigma_0-2+\varepsilon})$ as $N \to \infty$,

(vi) $\forall \ \varepsilon > 0, \quad \sum_{p_k \in H_N} \left| (p_k - \frac{k}{\Phi(N)}) \right| = O(N^{\sigma_0+\varepsilon})$ as $N \to \infty$.

An elementary proof that (iv), (v) and (iv) are equivalent can be obtained by repeated use of lemma 1. (See Zulauf [4]). Statements (ii) and (iii) are weaker than (v) and (vi) in the sense that (v) $\Rightarrow$ (vi) $\Rightarrow$ (ii) $\wedge$ (iii) using only the fact that H_N has $\Phi(N)$ elements.

Note 4. From (8) in the case $r = 1$, and (9) and (11), we only get

$$\sum_{p_k \in H_N} p_k = C_{11}\ Y(0,N) + C_{12}\ Y(-1,N)$$
$$= \frac{1}{2} + \frac{1}{2} \sum_{p_k \in H_N} 1 \quad .$$

i.e.

$$\sum_{p_k \in H_N} (p_k - \frac{1}{2}) = \frac{1}{2} \ ,$$

which is an essentially trivial relationship. However, with the aid of lemma 1 it can be shown (Zulauf [4]) that

$$\sum_{\substack{p_k \in H_N \\ p_k \leq \frac{1}{2}}} \left(p_k - \frac{1}{4}\right) = \frac{1}{4} - \frac{1}{8} Y(1,N) + \frac{1}{16} Y\left(1,\frac{N}{2}\right).$$

From (10) and proposition (1) we easily see that $RH(\sigma_0)$ implies for every $\varepsilon > 0$ that

$$\sum_{p_k \in H_N} \left(p_k^r - \frac{1}{r+1}\right) = O(N^{\sigma_0+\varepsilon})$$

as $N \to \infty$, for any fixed integer $r \geq 2$. We have noted that the converse implications are valid in the cases $r = 2, 3$. We are not aware of a proof that the converse implication is valid for each fixed $r \geq 2$, and we now prove a result which moves in this direction.

<u>Proposition 3.</u>

Let r be a fixed integer, $r \geq 3$. Suppose that for every $\varepsilon > 0$,

$$\sum_{p_k \in H_N} \left(p_k^r - \frac{1}{r+1}\right) = O(N^{\sigma_0+\varepsilon})$$

as $N \to \infty$. Then either $RH(\sigma_0)$ is true or every zero of $\zeta(s)$ in $\sigma > \sigma_0$ is a zero of $f_r(s) = \sum_{j=1}^{r-1} C_{r,j}\, \zeta(s+r-j)$

<u>Proof.</u>

For $\sigma > 1$, $b > 0$,

$$\frac{\zeta(s+b)}{\zeta(s)} = \sum_{m=1}^{\infty} \frac{\mu(m)}{m^s} \sum_{n=1}^{\infty} \frac{\frac{1}{n^b}}{n^s}$$

$$= \sum_{k=1}^{\infty} \frac{c(k)}{k^s} ,$$

where

$$c(k) = \sum_{n|k} \mu\left(\frac{k}{n}\right) \frac{1}{n^b} ,$$

and therefore, by (1)

$$\sum_{k\le x} c(k) = \sum_{k\le x} \sum_{n|k} \mu\left(\frac{k}{n}\right) \frac{1}{n^b}$$

$$= \sum_{n\le x} \frac{1}{n^b} \sum_{m\le x/n} \mu(m)$$

$$= \sum_{n\le x} \frac{1}{n^b} M\left(\frac{x}{n}\right)$$

$$= Y(b,x) \ .$$

Hence, from proposition 1, section 1, we have for $\sigma > 1$, $b > 0$,

$$(14) \qquad \frac{\zeta(s+b)}{\zeta(s)} = s\int_1^\infty \frac{Y(b,x)}{x^{s+1}}\, dx \ .$$

Now suppose for every $\varepsilon > 0$,

$$\sum_{p_k \in H_N} \left(p_k^r - \frac{1}{r+1}\right) = O(N^{\sigma_0+\varepsilon})$$

as $N \to \infty$, for fixed $r > 2$.

Then from proposition 2, for every $\varepsilon > 0$,

$$\sum_{j=1}^{r-1} C_{r,j}\, Y(r-j,N) = O(N^{\sigma_0+\varepsilon})$$

as $N \to \infty$. But then

$$s\int_1^\infty \sum_{j=1}^{r-1} C_{r,j}\, Y(r-j,x)\, x^{-s-1}\, dx$$

defines a function which is analytic for $\sigma > \sigma_0$.

However, for $\sigma > 1$,

$$s\int_1^\infty \sum_{j=1}^{r-1} C_{r,j}\, Y(r-j,x)\, x^{-s-1}\, dx$$

$$= s \sum_{j=1}^{r-1} C_{r,j} \int_1^{\infty} Y(r-j,x)\, x^{-s-1} dx$$

$$= \sum_{j=1}^{r-1} C_{r,j} \frac{\zeta(s+r-j)}{\zeta(s)}$$

$$= \frac{1}{\zeta(s)} \sum_{j=1}^{r-1} C_{r,j}\, \zeta(s+r-j)$$

from (14), and the proposition now follows.

Using this proposition we can now provide the following additional result

Proposition 4.

Let $1 \geq \sigma_0 \geq \frac{1}{2}$.

The following statements are equivalent

(i) For every $\varepsilon > 0$,

$$\sum_{p_k \in H_N} (p_k^4 - \frac{1}{5}) = O(N^{\sigma_0+\varepsilon}) \quad \text{as} \quad N \to \infty,$$

(ii) For every $\varepsilon > 0$,

$$\sum_{p_k \in H_N} (p_k^5 - \frac{1}{6}) = O(N^{\sigma_0+\varepsilon}) \quad \text{as} \quad N \to \infty,$$

(iii) For every $\varepsilon > 0$,

$$\sum_{p_k \in H_N} (p_k^6 - \frac{1}{7}) = O(N^{\sigma_0+\varepsilon}) \quad \text{as} \quad N \to \infty,$$

(iv) $RH(\sigma_0)$.

Proof:

In view of the preceding discussion and the last proposition it suffices to show that certain linear combinations of $\zeta(s+1), \ldots, \zeta(s+5)$ are non-zero for $\sigma \geq \sigma_0 \geq \frac{1}{2}$.

Since

$$\sum_{b=1}^{N} b^4 = \frac{1}{5}N^5 + \frac{1}{2}N^4 + \frac{1}{3}N^3 - \frac{1}{30}N ,$$

$$\sum_{b=1}^{N} b^5 = \frac{1}{6}N^6 + \frac{1}{2}N^5 + \frac{5}{12}N^4 - \frac{1}{12}N^2 ,$$

and

$$\sum_{b=1}^{N} b^6 = \frac{1}{7}N^7 + \frac{1}{2}N^6 + \frac{1}{2}N^5 - \frac{1}{6}N^3 + \frac{1}{42}N$$

the actual linear combinations of $\zeta(s+1), \ldots, \zeta(s+5)$ corresponding to cases (i), (ii) and (iii) are

(15) $$f_4(s) = -\frac{1}{30}\zeta(s+3) + \frac{1}{3}\zeta(s+1),$$

(16) $$f_5(s) = -\frac{1}{12}\zeta(s+3) + \frac{5}{12}\zeta(s+1),$$

and

(17) $$f_6(s) = \frac{1}{42}\zeta(s+5) - \frac{1}{6}\zeta(s+3) + \frac{1}{2}\zeta(s+1)$$

respectively. We now prove that $f_4(s)$, $f_5(s)$, and $f_6(s)$ are non-zero for $\sigma \geq \frac{1}{2}$, treating (15) and (16) together and (17), which seems to require more care, separately.

Firstly, we note that for $\sigma \geq \frac{1}{2}$,

(18) $$\left|\frac{\zeta(s+3)}{\zeta(s+1)}\right| = \left|\sum_{n=1}^{\infty}\frac{1}{n^{s+3}}\right| \cdot \left|\sum_{n=1}^{\infty}\frac{\mu(n)}{n^{s+1}}\right|$$

$$\leq \sum_{n=1}^{\infty}\frac{1}{n^{7/2}} \cdot \sum_{n=1}^{\infty}\frac{1}{n^{3/2}}$$

$$\leq \zeta(7/2) \cdot \zeta(3/2) \quad .$$

From Jahnke, Emde and Lösch [1], p. 41, we have

(19) $$\zeta(^3/_2) \leqq 2.613 \quad ,$$

(20) $$\zeta(^7/_2) \leqq 1.128 \quad ,$$

(21) $$\zeta(^{11}/_2) \leqq 1.0253 \ .$$

Hence from (18), (19), and (20) with $\sigma > \frac{1}{2}$,

$$\left|\frac{\zeta(s+3)}{\zeta(s+1)}\right| \leqslant 2.948 \ .$$

From (15), $f_4(s) = 0$ with $\sigma \geqslant \frac{1}{2}$ implies

$$\left|\frac{\zeta(s+3)}{\zeta(s+1)}\right| = 10 > 2.948 \ .$$

From (16), $f_5(s) = 0$ with $\sigma \geqslant \frac{1}{2}$ implies

$$\left|\frac{\zeta(s+3)}{\zeta(s+1)}\right| = 5 > 2.948.$$

Hence $f_4(s)$, $f_5(s)$ are non-zero for $\sigma \geqslant \frac{1}{2}$.

To complete the corresponding result for $f_6(s)$ we note for $\sigma \geqslant \frac{1}{2}$.

$$\left| \frac{1}{6}\frac{\zeta(s+3)}{\zeta(s+1)} - \frac{1}{42}\frac{\zeta(s+5)}{\zeta(s+1)} \right|$$

$$= \left|\frac{1}{6}\left(1 + \sum_{n=2}^{\infty}\frac{1}{n^{s+3}}\right) - \frac{1}{42}\left(1 + \sum_{n=2}^{\infty}\frac{1}{n^{s+5}}\right)\right| \left|\sum_{n=1}^{\infty}\frac{\mu(n)}{n^{s+1}}\right|$$

$$\leqslant \left|\frac{1}{7} + \frac{1}{6}\sum_{n=2}^{\infty}\frac{1}{n^{s+3}} - \frac{1}{42}\sum_{n=2}^{\infty}\frac{1}{n^{s+5}}\right| \sum_{n=1}^{\infty}\frac{1}{n^{3/2}}$$

$$\leqslant \left(\frac{1}{7} + \frac{1}{6}\sum_{n=2}^{\infty}\frac{1}{n^{7/2}} + \frac{1}{42}\sum_{n=2}^{\infty}\frac{1}{n^{11/2}}\right)\zeta(^3/_2)$$

$$\leqslant \left(\frac{1}{7} + \frac{1}{6}(\zeta(^7/_2) - 1) + \frac{1}{42}\left(\zeta\left(\frac{11}{2}\right) - 1\right)\right)\zeta(^3/_2)$$

$$\leqslant (.1429 + .0214 + .0007) \times 2.613$$

$$\leqslant .432 \ , \quad \text{from (19), (20), and (21).}$$

Since $f_6(s) = 0$ implies

$$\left|\frac{1}{6}\frac{\zeta(s+3)}{\zeta(s+1)} - \frac{1}{42}\frac{\zeta(s+5)}{\zeta(s+1)}\right| = \frac{1}{2},$$

we have $f_6(s) \neq 0$ for $\sigma > \frac{1}{2}$.

Section 6.

Dirichlet series connected with $\zeta(s)$.

Let $N(\sigma,T)$ be the number of zeros $\beta + i\gamma$ of $\zeta(s)$ such that $\beta > \sigma$, $0 < t \leqslant T$. Backlund [1] proved that the Lindelöf hypothesis, [for every $\varepsilon > 0$, $\zeta(\sigma + it) = O(t^{\varepsilon})$ as $t \to \infty$, for $\sigma \geqslant \frac{1}{2}$], is equivalent to the statement $N(\sigma,T+1) - N(\sigma,T) = O(\log T)$ as $T \to \infty$, for every $\sigma > \frac{1}{2}$. Littlewood [1] proved, that RH($\frac{1}{2}$) implies the Lindelöf hypothesis.

In this section we derive two statements: one equivalent to RH($\frac{1}{2}$) and one which is implied by the Lindelöf hypothesis, and note a curious implication which follows from the statements. Firstly we determine some general properties of Dirichlet series.

Let $c : \mathbb{C} \to \mathbb{N}$ satisfy $c(1) = 1$ and be such that the Dirichlet series

$$L(s) = \sum_{n=1}^{\infty} \frac{c(n)}{n^s}$$

has a finite abscissa of convergence σ_0. Let σ_3 be the abscissa of absolute convergence of $L(s)$. We know $\sigma_0 \leqslant \sigma_3 \leqslant \sigma_0 + 1$.

For $\sigma > \sigma_3$, and with $\sigma > \sigma' > \sigma_3$ we note

$$\begin{aligned} \left|\sum_{n=2}^{\infty} \frac{c(n)}{n^s}\right| &\leqslant \sum_{n=2}^{\infty} \frac{|c(n)|}{n^{\sigma}} \\ &= \sum_{n=2}^{\infty} \frac{|c(n)|}{n^{\sigma'}} \cdot \frac{1}{n^{\sigma-\sigma'}} \\ &\leqslant \frac{1}{2^{\sigma-\sigma'}} \sum_{n=2}^{\infty} \frac{|c(n)|}{n^{\sigma'}} \\ &< 1 \end{aligned}$$

for sufficiently large σ.

Hence finite numbers σ_1, σ_2 are defined by

$$\sigma_1 = \inf\{\sigma' : \left|\sum_{n=2}^{\infty} \frac{c(n)}{n^\sigma}\right| < 1 \quad \text{for} \quad \sigma > \sigma'\}$$

and

$$\sigma_2 = \inf\{\sigma' : \sum_{n=2}^{\infty} \frac{|c(n)|}{n^\sigma} < 1 \quad \text{for} \quad \sigma > \sigma\}$$

and we also have $\sigma_2 \geqslant \sigma_1$.

Proposition 1.

Let $\alpha \in \mathbb{R}$. Then $L(s)^\alpha$ can be represented by a Dirichlet series which is absolutely convergent for $\sigma > \sigma_2$.

Proof:

Let $M(s) = \sum_{n=2}^{\infty} \frac{c(n)}{n^s}$ and $\hat{M}(\sigma) = \sum_{n=2}^{\infty} \frac{|c(n)|}{n^\sigma}$.

Then for $\sigma > \sigma_2$

$$|M(s)| \leqslant \hat{M}(\sigma) < 1,$$

and hence

(1) $$L(s)^\alpha = (1 + M(s))^\alpha = \sum_{r=0}^{\infty} \binom{\alpha}{r} M(s)^\alpha$$

where the sum is in fact absolutely convergent for $\sigma > \sigma_1$.

Also for $\sigma > \sigma_2$,

(2) $$(1 + \hat{M}(\sigma))^\alpha = \sum_{r=0}^{\infty} \binom{\alpha}{r} \hat{M}(\sigma)^r$$

where the sum is absolutely convergent for $\sigma > \sigma_2$.

But for $\sigma > \sigma_3$ (and so for $\sigma > \sigma_2$)

(3) $$M(s)^r = \sum_{n=1}^{\infty} \frac{b^{(r)}(n)}{n^s}$$

where

$$b^{(r)}(n) = \sum_{\substack{m_1 m_2 \ldots m_r = n \\ m_1, m_2 \ldots, m_r \geq 2}} c(m_1)c(m_2) \ldots c(m_r),$$

and

(4) $$\hat{M}(\sigma)^r = \sum_{n=1}^{\infty} \frac{\hat{b}^{(r)}(n)}{n^\sigma}$$

where

$$\hat{b}^{(r)}(n) = \sum_{\substack{m_1 m_2 \ldots m_r = n \\ m_1, m_2, \ldots, m_r \geq 2}} |c(m_1)c(m_2) \ldots c(m_r)|,$$

and so

(5) $$\left| \frac{b^{(r)}(n)}{n^s} \right| \leq \frac{\hat{b}^{(r)}(n)}{n^\sigma} \quad .$$

Using (4) in (2) we see the double sum

$$\sum_{r=0}^{\infty} \binom{\alpha}{r} \sum_{n=1}^{\infty} \frac{\hat{b}^{(r)}(n)}{n^s} \text{ converges absolutely for } \sigma > \sigma_2.$$

Then using (3) in (1) and noting (5) it follows that

$$\sum_{r=0}^{\infty} \binom{\alpha}{r} \sum_{n=1}^{\infty} \frac{b^{(r)}(n)}{n^\sigma} \text{ converges absolutely for } \sigma > c_2.$$

We may consequently invert the order of summation in this sum, and so, from (3) and (1), for $\sigma > \sigma_2$,

(6) $$L(s)^\alpha = \sum_{n=1}^{\infty} \left\{ \sum_{r=0}^{\infty} \binom{\alpha}{r} b^{(r)}(n) \right\} \frac{1}{n^s} \quad .$$

This relationship must of course hold for $\sigma > \sigma_4$, where σ_4 is the abscissa of convergence of the Dirichlet series on the RHS, and $\sigma_0 \leq \sigma_4 \leq \sigma_3$.

Corollary:

Let
$$\delta(x) = \left[\frac{\log(x+1)}{\log 2} + 1\right],$$

$$L(s)^{\alpha} = \sum_{n=1}^{\infty} \frac{c^{(\alpha)}(n)}{n^s}, \quad C_{\alpha}(x) = \sum_{n \leq x} c^{(\alpha)}(n),$$

and
$$B_r(x) = \sum_{n \leq x} b^{(r)}(n), \quad \text{for} \quad r = 0,1,2, \ldots \quad .$$

Then
$$C_{\alpha}(x) = \sum_{r=0}^{\delta(x)} \binom{\alpha}{r} B_r(x) \; .$$

Proof:

From (3) we have
$$b^{(r)}(n) = 0 \quad \text{for} \quad 1 \leq n < 2^r,$$

and hence
$$b^{(r)}(n) = 0 \quad \text{for} \quad r \geq \delta(x), \quad \text{with} \quad n \leq x.$$

Then from (6),
$$C_{\alpha}(x) = \sum_{n \leq x} \sum_{r=0}^{\infty} \binom{\alpha}{r} b^{(r)}(n)$$
$$= \sum_{n \leq x} \sum_{r=0}^{\delta(x)} \binom{\alpha}{r} b^{(r)}(n)$$
$$= \sum_{r=0}^{\delta(x)} \binom{\alpha}{r} \sum_{n \leq x} b^{(r)}(n)$$

$$= \sum_{r=0}^{\delta(x)} \binom{\alpha}{r} B_r(x).$$

We now examine the particular case

$$L(s) = (1 - \frac{2}{2^s})\zeta(s)$$

and all terms defined such as $M(s)$, σ_2, $B_r(x)$, etc. refer to this particular choice.

Proposition 2.

If $RH(\frac{1}{2})$ is true then for all positive integers k,

(7) $$\forall \ \varepsilon > 0, \ \sum_{r=0}^{\delta(x)} \binom{1/k}{r} B_r(x) = O(x^{\frac{1}{2}+\varepsilon}) \quad \text{as} \quad x \to \infty.$$

Conversely, if (7) holds for infinitely many positive integers k then $RH(\frac{1}{2})$ is true.

(The O-terms in (7) may depend on k).

Proof:

Firstly, suppose $RH(\frac{1}{2})$ is true. Then for $k = 1, 2, \ldots$ each function f_k defined by

$$f_k(s) = ((1 - \frac{2}{2^s})\zeta(s))^{1/k}$$

satisfies

(a) Defined and analytic for $\sigma > \frac{1}{2}$,

(b) $\forall \ \varepsilon > 0$, $f_k(s) = O(t^{\varepsilon})$ as $t \to \infty$, for $\sigma > \frac{1}{2}$,

and according to proposition 1,

(c) $$f_k(s) = \sum_{n=1}^{\infty} \frac{c^{(1/k)}(n)}{n^s} \quad \text{for} \quad \sigma > \sigma_2 .$$

It then follows by a known theorem on Dirichlet series (see for example Titchmarch [2], p.302) that each series

$$\sum_{n=1}^{\infty} \frac{c^{(1/k)}(n)}{n^s} \quad \text{is}$$

convergent for $\sigma > \frac{1}{2}$, and consequently from another well known result that for every $\varepsilon > 0$,

$$\sum_{n\leq x} c^{(1/k)}(n) = O(x^{\frac{1}{2}+\varepsilon}) \quad \text{as} \quad x \to \infty,$$

for $k = 1,2,\ldots$. The implication now follows on noting from the corollary to proposition 1 that

$$\sum_{n\leq x} c^{(1/k)}(n) = \sum_{r=0}^{\gamma(x)} \binom{1/k}{r} B_r(x).$$

Conversely, suppose that for every $\varepsilon > 0$,

$$\sum_{n\leq x} c^{(1/k)}(n) = O(x^{\frac{1}{2}+\varepsilon}) \quad \text{as} \quad x \to \infty,$$

for infinitely many positive integers k.

Then each series $\displaystyle\sum_{n=1}^{\infty} \frac{c^{(1/k)}(n)}{n^s} = f_k(s)$

converges for $\sigma > \frac{1}{2}$, and defines a function which is analytic for $\sigma > \frac{1}{2}$.

Then from

$$f_k(s)^k = (((1 - \frac{2}{2^s})\zeta(s))^{1/k})^k$$

$$= (1 - \frac{2}{2^s})\zeta(s)$$

for $\sigma > \frac{1}{2}$, we see that a zero of $\zeta(s)$ in $\sigma > \frac{1}{2}$ would have infinite multiplicity.

Hence $\zeta(s) \neq 0$ for $\sigma > \frac{1}{2}$.

In contrast to the last proposition we next have

Proposition 3.

The Lindelöf hypothesis implies

$$\forall \quad \varepsilon > 0, \quad B_r(x) = O(x^{\frac{1}{2}+\varepsilon}) \quad \text{as} \quad x \to \infty,$$

for $r = 1,2, \ldots$.

(The O-constants may depend on r).

Proof:

Assuming the Lindelöf hypothesis, for each positive integer r

$$M(s)^r = ((1 - \frac{2}{2^s})\zeta(s) - 1)^r$$

satisfies

(a) Defined and regular for all s,

(b) $\forall \quad \varepsilon > 0, \quad M(s)^r = O(t^{\varepsilon})$ as $t \to \infty$, for $\sigma > \frac{1}{2}$,

and in the notation of the proof of proposition 1,

(c) $M(s)^r = \sum_{n=1}^{\infty} \frac{b^{(r)}(n)}{n^s}$ for $\sigma > \sigma_2$.

Hence, by much the same argument used in the proof of proposition 2, each series

$$\sum_{n=1}^{\infty} \frac{b^{(r)}(n)}{n^s} \quad \text{is convergent for} \quad \sigma > \tfrac{1}{2}.$$

Thus, for every $\varepsilon > 0$, and each integer $r \geq 1$,

$$B_r(x) = \sum_{n \leq x} b^{(r)}(n)$$
$$= O(x^{\frac{1}{2}+\varepsilon})$$

as $x \to \infty$.

Note 1. From proposition 2 and 3 and our opening remarks we thus have

$$\forall \quad \varepsilon > 0, \quad \sum_{r=0}^{\delta(x)} \binom{1/k}{r} B_r(x) = O(x^{\frac{1}{2}+\varepsilon}) \quad \text{as} \quad x \to \infty,$$

for $k = 1, 2 \ldots,$ implies

$$\forall \quad \varepsilon > 0, \; B_r(x) = O(x^{\frac{1}{2}+\varepsilon}) \quad \text{as} \quad x \to \infty \quad \text{for} \quad r = 1,2,\ldots$$

Note 2. We have noted that the O-terms may depend on more than one parameter, however the possibility that the converse implication can be argued is still under investigation.

Section 7.

More Dirichlet series related to $\zeta(s)$.

We have noted that the results of sections 1 and 2 apply to the corresponding summatory functions for the functions $\tau^{(k)}$ defined by

$$\sum_{n=1}^{\infty} \frac{\tau^{(k)}(n)}{n^s} \cdot \zeta(s) = \zeta(ks), \quad (\sigma > 1),$$

$k = 2,3,\ldots,$ and the restriction of k to integral values is necessary for the $\tau^{(k)}$ to be well defined.

This restriction can be avoided by defining functions $\lambda^{(k)}$ for real $k \geq 1$ by

$$\sum_{n=1}^{\infty} \frac{\lambda^{(k)}(n)}{n^s} \cdot \zeta(s) = \prod_p \left(1 - \frac{1}{[p^k]^s}\right)^{-1}$$

for $\sigma > 1$. We see $\lambda^{(k)} = \tau^{(k)}$ for $k = 2,3, \ldots$.

In the range $k \geq 2$ the earlier results we mentioned will apply to the wider class $\lambda^{(k)}$. In the range $2 > k > 1$ the logical relationships between the orders of the summatory functions and RH($\frac{1}{2}$) are not so definite. In this section we prove a 'sample' result concerned with the range $2 > k > 1$, and a complete result for the orders of the sums

$$\sum_{n \leq x} \lambda^{(k)}(n)$$

for the range $k \geq 1$.

We begin with what may be viewed as a weaking of Turan's now disproved conjecture that $h(x) \geq 0$ for $x > 1$.

(Actually, from corollary 2, section 3 the statement $h(x) + Ax^{-\frac{1}{2}} \geq 0$ for $x \geq 1$ implies RH($\frac{1}{2}$)).

Let $h^{(k)}(x) = \sum_{n \leq x} \frac{\lambda^{(k)}(n)}{n}$, $S^{(k)}(x) = \sum \lambda^{(k)}(n)$, and

$H^{(k)}(x) = \sum_{n \leq x} h^{(k)}(n)$.

Proposition 1.

Suppose for some fixed k satisfying $2 > k > 1$, and $A > 0$, we have

$$h^{(k)}(x) + Ax^{-1+1/k} \geq 0 \quad \text{for} \quad x \geq 1.$$

Then RH$(1/k)$ is true.

Proof:

For convenience we let

$$f_k(s) = \prod_p \left(1 - \frac{1}{[p^k]^s}\right)^{-1}, \quad (\sigma > 1/k).$$

For $\sigma > 1/k$, $f_k(s)$ can be expressed as a Dirichlet series and we have

$$f_k(s) = \sum_{n=1}^{\infty} \frac{a^{(k)}(n)}{n^s}, \quad \text{where}$$

(1) $$a^{(k)}(n) = \sum_{g|n} \lambda^{(k)}(g),$$

and from the product expansion for $f_k(s)$ we also have $a^{(k)}(n) \geq 0$ for $n \geq 1$.

Since the Dirichlet series for $f_k(s)$ converges for $\sigma > 1/k$ it follows by a well known theorem that

(2) $$A^{(k)}(x) = \sum_{n \leq x} a^{(k)}(n)$$

$$= O(x^{(1/k)+\varepsilon})$$

as $x \to \infty$, for every $\varepsilon > 0$.

Applying the Möbius inversion formula to (1),

$$\lambda^{(k)}(n) = \sum_{g|n} \mu(\frac{n}{g})a^{(k)}(g),$$

and hence

(3)
$$S^{(k)}(x) = \sum_{n\leq x} \lambda^{(k)}(n)$$

$$= \sum_{n\leq x} \sum_{g|n} \mu(\frac{n}{g})a^{(k)}(g)$$

$$= \sum_{n\leq x} a^{(k)}(n)M(\frac{x}{n}).$$

Since $a^{(k)}(n) \geq 0$ for $n \geq 1$, and $|M(x)| \leq x$ we thus have

(4)
$$|S^{(k)}(x)| \leq x \sum_{n\leq x} \frac{a^{(k)}(n)}{n} .$$

But

$$\sum_{n\leq x} \frac{a^{(k)}(n)}{n} = \sum_{n\leq x} \frac{A^{(k)}(n) - A^{(k)}(n-1)}{n}$$

$$= \sum_{n\leq x} \frac{A^{(k)}(n)}{n(n+1)} + \frac{A^{(k)}(x)}{[x+1]}$$

$$= O\left(\sum_{n\leq x} \frac{n^{(1/k)+\varepsilon}}{n(n+1)}\right) + O\left(\frac{x^{(1/k)+\varepsilon}}{x}\right)$$

$$= O(1)$$

as $x \to \infty$, and so from (4), choosing ε so that $0 < \varepsilon < 1 - 1/k$,

(5)
$$S^{(k)}(x) = O(x) \quad \text{as} \quad x \to \infty.$$

Using (5), we can now see that

(6)
$$h^{(k)}(x) = \sum_{n\leq x} \frac{\lambda^{(k)}(n)}{n}$$

$$= \sum_{n \leq x} \frac{S^{(k)}(n) - S^{(k)}(n-1)}{n}$$

$$= \sum_{n \leq x} \frac{S^{(k)}(n)}{n(n+1)} + \frac{S^{(k)}(x)}{[x+1]}$$

$$= O(\log x)$$

as $x \to \infty$.

Hence, using (6),

$$(7) \qquad H^{(k)}(x) = \sum_{n \leq x} h^{(k)}(n)$$

$$= \sum_{m \leq x} \sum_{n \leq m} \frac{\lambda^{(k)}(n)}{n}$$

$$= \sum_{n \leq x} \frac{\lambda^{(k)}(n)}{n} ([x] - n + 1)$$

$$= x\, h^{(k)}(x) - S^{(k)}(x) + O(\log x)$$

as $x \to \infty$.

From proposition 1, section 1, for $\sigma > 0$,

$$(8) \qquad \frac{f_k(s+1)}{\zeta(s+1)} = \sum_{n=1}^{\infty} \frac{\frac{\lambda^{(k)}(n)}{n}}{n^s}$$

$$= s \int_1^{\infty} \frac{h^{(k)}(x)}{x^{s+1}}\, dx \quad ,$$

and for $\sigma > 1$,

$$(9) \qquad \frac{f_k(s)}{\zeta(s)} = \sum_{n=1} \frac{\lambda^{(k)}(n)}{n^s}$$

$$= s \int_1^{\infty} \frac{S^{(k)}(x)}{x^{s+1}}\, dx \ .$$

Writing $s-1$ for s in (8), and using (9), we thus see that for $\sigma > 1$,

$$\text{(10)} \qquad \int_1^\infty \frac{x\,h^{(k)}(x) - S^{(k)}(x)}{x^{s+1}}\,dx = \frac{1}{(s-1)s}\,\frac{f_k(s)}{\zeta(s)} \quad .$$

From (7) and (10) we thus see that for $\sigma > 1$,

$$\text{(11)} \qquad \int_1^\infty \frac{H^{(k)}(x)}{x^{s+1}}\,dx = \frac{1}{(s-1)s}\,\frac{f_k(s)}{\zeta(s)} + C(s),$$

where $C(s)$ is analytic for $\sigma > 0$.

We note that (11) is essentially a special case of lemma 3 section 2, which could have been applied here, once (6) was obtained.

Now suppose $h^{(k)}(x) + An^{-1+1/k} \geqslant 0$ for $x \geqslant 1$. Then the Dirichlet series

$$\sum_{n=1}^\infty \frac{h^{(k)}(n) + An^{-1+1/k}}{n^s} \quad \text{has a singularity}$$

at the real point on its line of convergence.

However, from proposition 1, section 1, and (11) for $\sigma > 1$,

$$\text{(12)} \qquad \sum_{n=1}^\infty \frac{h^{(k)}(n) + An^{-1+1/k}}{n^s} = s\int_1^\infty \frac{H^{(k)}(x)}{x^{s+1}}\,dx + A\zeta(s+1-1/k)$$

$$= \frac{1}{(s-1)}\,\frac{f_k(s)}{\zeta(s)} + A\zeta(s+1-1/k)$$

$$+ C(s),$$

where $C(s)$ is regular for $\sigma > 0$. Since $f_k(s)$ has no real zeros for $\sigma > 1/k$, and $\zeta(s+1-1/k)$ is analytic for $\sigma > 1/k$ we see from the RHS of (12) that the series on the LHS of (12) converges and is analytic for $\sigma > 1/k$. Then from the RHS of (12) it follows that $\zeta(s) \neq 0$ for $\sigma > 1/k$.

Recalling the definition

$$\theta = \inf\{\sigma_0 : RH(\sigma_0) \text{ is true}\}$$

we next prove -

Proposition 2.

Let k be real, $k \geqslant 1$. Then

$$\forall \quad \varepsilon > 0, \quad \sum_{n \leqslant x} \lambda^{(k)}(n) = O(x^{\varepsilon + \text{Max}\{\theta, 1/k\}})$$

as $x \to \infty$.

Proof:

We have already noted in section 1 that Littlewood's argument reproduced in Titchmarch [1] p. 282-283 can be modified to prove

$$\forall \quad \varepsilon > 0, \quad 1/\zeta(s) = O(t^{\varepsilon}), \quad \text{as} \quad t \to \infty,$$

for each $\sigma > \theta$. Also, since

$$\sum_{n=1}^{\infty} \frac{a^{(k)}(n)}{n^s} \quad (= f_k(s)) \quad \text{is absolutely convergent for}$$

$\sigma > 1/k$, we have $f_k(s) = O(1)$ as $t \to \infty$ for each $\sigma > 1/k$.

Now for $\sigma > 1$, we have

$$\sum_{n=1}^{\infty} \frac{\lambda^{(k)}(n)}{n^s} = \frac{f_k(s)}{\zeta(s)}. \tag{13}$$

Since $f_k(s)/\zeta(s)$ is analytic for $\sigma > \text{Max}\{1/k, \theta\}$, and

$$\forall \quad \varepsilon > 0 \;; \quad f_k(s)/\zeta(s) = O(t^{\varepsilon}) \quad \text{as} \quad t \to \infty,$$

for each $\sigma > \text{Max}\{1/k, \theta\}$, it follows from (13) and Titchmarch [2], p.302

that

$$\sum_{n=1}^{\infty} \frac{\lambda^{(k)}(n)}{n^s} \text{ is convergent for } \sigma > \operatorname{Max}\{1/k, \theta\}.$$

The proposition now follows:

Proposition 3.

Let k be real, $k > 1$. Then

$$\forall \ \varepsilon > 0, \quad \sum_{n \leq x} \lambda^{(k)}(n) = \Omega(x^{-\varepsilon + \operatorname{Max}\{1/k,\theta\}}) \quad .$$

Proof:

Firstly, suppose $k > \frac{1}{\theta}$ and there exists $\varepsilon^* > 0$ satisfying $k(\theta - \varepsilon^*) > 1$ such that

$$\sum_{n \leq x} \lambda^{(k)}(n) = O(x^{\theta - \varepsilon^*})$$

as $x \to \infty$. Then the equation

$$\sum_{n=1}^{\infty} \frac{\lambda^{(k)}(n)}{n^s} \cdot \zeta(s) = \prod_p \left(1 - \frac{1}{[p^k]^s}\right)^{-1}$$

is valid in the form written for $\sigma > \theta - \varepsilon^*$, which then shows that the zeros of $\zeta(s)$ in $\operatorname{Re} s \in (\theta - \varepsilon^*, \theta]$ are zeros of

$\prod_p \left(1 - \frac{1}{[p^k]^s}\right)^{-1}$. The proposition for $k > \frac{1}{\theta}$ follows from this contradiction. We now suppose $\frac{1}{\theta} \geq k > 1$. If for some $\varepsilon^* > 0$ we have

$$\sum_{n \leq x} \lambda^{(k)}(n) = O(x^{(1/k) - \varepsilon^*}) \quad \text{as } x \to \infty, \quad \text{then for}$$

$\sigma > 1/k - \varepsilon^*$ we have

$$\sum_{n=1}^{\infty} \frac{\lambda^{(k)}(n)}{n^s} \zeta(s) = f_k(s).$$

This contradicts the fact that $f_k(s)$ has a simple pole at $s = 1/k$, and the proposition now follows.

In conclusion we note from propositions 2 and 3 that if RH(½) is true the distinctive behaviour change in the $\lambda^{(k)}$ is at $k = 2$, and the corresponding function is Liouville's function.

Section 8.

A representation for Dirichlet series.

Let $f(s) = \sum_{n=1}^{\infty} \frac{a(n)}{n^s}$ be any Dirichlet series satisfying

$|a(n)| \leq C$ for $n \geq 1$, where C is a constant independent of n. We note that the series defining f is absolutely convergent for $\sigma > 1$.

For each integer $r \geq -1$ let coefficients A_r be defined by

$$\frac{1}{e^z - 1} = \sum_{r=-1}^{\infty} A_r z^r, \quad \text{for} \quad 0 < |z| < 2\pi.$$

Then $A_r = \frac{1}{(r+1)!} B_{r+1}$, where the B_r are the Bernoulli numbers.

Proposition 1.

For $\sigma > 1$, and any fixed δ satisfying $0 < \delta \leq 1$,

$$\int_1^{\infty} \left\{ \sum_{n \leq x^{\delta}} \frac{a(n)}{(e^{n/x} - 1)} \right\} x^{-s-1} dx = \sum_{r=-1}^{\infty} \frac{A_r}{s+r} f\left(\frac{s+r-r\delta}{\delta}\right) .$$

Proof:

Let $F_r(x) = \sum_{n \leq x} a(n) n^r$. Then from proposition 1, section 1,

$$f(s-r) = s \int_1^{\infty} \frac{F_r(x)}{x^{s+1}} dx$$

for $\sigma > r + 1$, with $r \geq -1$, and hence

(1) $$f(s) = (s+r) \int_1^{\infty} \frac{F_r(x)}{x^{s+r+1}} dx$$

for $\sigma > 1$, with $r \geq -1$.

Now for $\sigma > 1$, $0 < \delta \leq 1$, we easily see that

(2)
$$\int_1^\infty \left[\sum_{n \leq x^\delta} \frac{a(n)}{(e^{n/x}-1)} \right] x^{-s-1}\, dx$$

$$= \int_1^\infty \sum_{n \leq x^\delta} a(n) \sum_{r=-1}^\infty A_r \left(\frac{n}{x}\right)^r x^{-s-1}\, dx$$

$$= \sum_{r=-1}^\infty A_r \int_1^\infty \sum_{n \leq x^\delta} a(n) n^r x^{-s-r-1}\, dx$$

$$= \sum_{r=-1}^\infty A_r \int_1^\infty \frac{F_r(x^\delta)}{x^{r+s+1}}\, dx .$$

It follows from (1) that

(3)
$$\int_1^\infty \frac{F_r(x^\delta)}{x^{r+s+1}}\, dx = \frac{1}{(s+r)} f\left(\frac{s+r-r\delta}{\delta}\right) ,$$

and hence from (2) and (3) we obtain the required result.

<u>Corollary</u>:

For $\sigma > 1$,

$$\int_1^\infty \left[\sum_{n \leq x} \frac{a(n)}{(e^{n/x}-1)} \right] x^{-s-1} dx = f(s) \sum_{r=-1}^\infty \frac{A_r}{s+r} .$$

<u>Proof</u>:

This follows choosing $\delta = 1$.

<u>Note 1</u>. The restriction $0 < \delta \leq 1$ in proposition 1 is necessary since the Laurent series for $\frac{1}{e^z-1}$ is convergent only for $0 < |z| < 2\pi$, and consequently the interchange of operations following (2) is only valid for $0 < \left|\frac{n}{x}\right| < 2\pi$.

Note 2. The function defined by $\sum_{r=-1}^{\infty} \frac{A_r}{s+r}$ is analytic in the entire plane except for simple poles at $s = 1,0,-1,-3,-5, \ldots$.

For $\sigma > 1$, we easily see that

$$q(s) = \sum_{r=-1}^{\infty} \frac{A_r}{s+r}$$

$$= \int_0^1 \frac{x^{s-1}}{e^x-1}\, dx$$

$$= \int_1^{\infty} \frac{x^{-s-1}}{e^{1/x}-1}\, dx \, .$$

We note from Titchmarch [1], p.18, that

$$\Gamma(s)\zeta(s) = \int_0^{\infty} \frac{x^{s-1}}{e^x-1}\, dx,$$

for $\sigma > 1$, and hence

$$q(s) = \Gamma(s)\zeta(s) - \int_1^{\infty} \frac{x^{s-1}}{(e^x-1)}\, dx$$

$$= \Gamma(s)\zeta(s) + p(s) \, ,$$

where $p(s)$ is analytic in the whole plane.

We next examine some aspects of the case $a(n) = \lambda(n)$ for $n \geqslant 1$, where λ is Liouville's function.

Proposition 2.

$$RH(\sigma_0) \Rightarrow \forall\ \varepsilon > 0, \quad \sum_{n \leq x} \frac{\lambda(n)}{e^{n/x}-1} = O(x^{\sigma_0+\varepsilon}) \quad \text{as} \quad x \to \infty.$$

Proof:

Let $w(x) = \frac{1}{x} - \frac{1}{e^x-1}$.

Then $w'(x) = \frac{1}{(e^x-1)^2}\{e^x - (\frac{e^x-1}{x})^2\}$, and since $e^{x/2} \leq \frac{e^x-1}{x}$ for $x \geq 0$ it follows that $w'(x) \leq 0$ for $x \geq 0$. We thus see that $w(x)$ decreases from $\frac{1}{2}$ to 0 as x increases from 0 to ∞.

Now

$$x\,h(x) - \sum_{n\leq x} \frac{\lambda(n)}{e^{n/x}-1} = \sum_{n\leq x} \lambda(n)\left\{\frac{x}{n} - \frac{1}{e^{n/x}-1}\right\}$$

$$= \sum_{n\leq x} \lambda(n)w(\frac{n}{x})$$

$$= \sum_{n\leq x} (S(n) - S(n-1))w(\frac{n}{x})$$

$$= \sum_{n\leq x} S(n)\,(w(\frac{n}{x}) - w(\frac{n+1}{x})) +$$

$$+ S([x])w(\frac{[x]+1}{x}) \quad .$$

From proposition 3, section 1, and our opening remarks it thus follows that $RH(\sigma_0)$ implies for $\varepsilon > 0$,

$$\left|x\,h(x) - \sum_{n\leq x} \frac{\lambda(n)}{e^{n/x}-1}\right| \leq C_\varepsilon\left[\sum_{n\leq x} n^{\sigma_0+\varepsilon}\left\{w(\frac{n}{x}) - w(\frac{n+1}{x})\right\} + \right.$$

$$\left. + [x]^{\sigma_0+\varepsilon}\, w(\frac{[x]+1}{x})\right]$$

$$= C_\varepsilon \sum_{n\leq x} \left\{n^{\sigma_0+\varepsilon} - (n-1)^{\sigma_0+\varepsilon}\right\}w(\frac{n}{x})$$

$$= O\left(\sum_{n\leq x} n^{\sigma_0-1+\varepsilon}\right)$$

$$= O(x^{\sigma_0+\varepsilon})$$

as $x \to \infty$.

But from proposition 3, section 1, $RH(\sigma_0)$ implies $\forall\ \varepsilon > 0$, $x\ h(x) = O(x^{\sigma_0+\varepsilon})$ as $x \to \infty$, and the proposition now follows.

We can also prove a weak converse of proposition 2.

<u>Proposition 3.</u>

Let $1 > \sigma_0 \geq \tfrac{1}{2}$, and for every $\varepsilon > 0$, let

$$\sum_{n\leq x} \frac{\lambda(n)}{e^{n/x}-1} = O(x^{\sigma_0+\varepsilon}) \quad \text{as} \quad x \to \infty.$$

Then every zero of $\zeta(s)$ in $\sigma > \sigma_0$ is a zero of $q(s)$.

<u>Proof:</u>

From the corollary to proposition 1, with $\sigma > 1$, we have

$$\int_1^\infty \left(\sum_{n\leq x} \frac{\lambda(n)}{e^{n/x}-1}\right) x^{-s-1}dx = q(s)\ \frac{\zeta(2s)}{\zeta(s)}$$

The integral on the LHS defines a function which is analytic for $\sigma > \sigma_0$, and the proposition follows.

In contrast to this last result we have

<u>Proposition 4.</u>

For any fixed $\delta > 1$,

$$\sum_{n\leq x^\delta} \frac{\lambda(n)}{e^{n\pi/x}-1} = \tfrac{1}{2}\, x^{\frac{1}{2}} - \tfrac{1}{2} + O\left(x^{\frac{1}{2}}e^{-\pi x}\right) + O\left(xe^{-\pi x^{\delta-1}}\right)$$

as $x \to \infty$.

Proof:

Let $\omega(x) = \sum_{n=1}^{\infty} e^{-n^2\pi x}$ for real $x > 0$.

Then

$$\omega(x) = \sum_{n=1}^{\infty} \left(\sum_{g|n} \lambda(g)\right) e^{-n\pi x}$$

$$= \sum_{n=1}^{\infty} \frac{\lambda(n)}{e^{n\pi x}-1} .$$

From Davenport [1], page 64 we have

$$\omega(\tfrac{1}{x}) = \tfrac{1}{2} x^{\frac{1}{2}} - \tfrac{1}{2} + x^{\frac{1}{2}} \omega(x).$$

Hence

$$\text{(1)} \qquad \sum_{n \leq x^\delta} \frac{\lambda(n)}{e^{n\pi/x}-1} = \tfrac{1}{2} x^{\frac{1}{2}} - \tfrac{1}{2} + x^{\frac{1}{2}} \omega(x) + - \sum_{n > x^\delta} \frac{\lambda(n)}{(e^{n\pi/x}-1)} .$$

But $\omega(x) = O(e^{-\pi x})$ as $x \to \infty$, and

$$\left|\sum_{n > x^\delta} \frac{\lambda(n)}{e^{n\pi/x}-1}\right| \leq \sum_{n > x^\delta} \frac{1}{e^{n\pi/x}-1}$$

$$= O\left\{x e^{-\pi x^{\delta-1}}\right\}$$

as $x \to \infty$, and the proposition now follows using these estimates in (1).

Section 9.

A series representation for Riemann's Ξ function.

In standard notation

$$\xi(s) := \tfrac{1}{2} s(s-1)\pi^{-s/2}\Gamma(\frac{s}{2})\zeta(s),$$

and

$$\Xi(t) = \xi(\frac{1}{2} + it),$$

where ζ is Riemann's zeta-function.

A well-known integral representation for $\Xi(t)$ is

$$\Xi(t) = 2\int_0^\infty \phi(u)\cos(ut)du,$$

where

$$\phi(u) = 2e^{-u/2}\frac{d}{du}\{e^{3u}\psi'(e^{2u})\}$$

for $|\mathrm{Im}\ u| < \frac{\pi}{4}$, and

$$\psi(x) = \sum_{n=1}^{\infty} e^{-n^2\pi x}$$

for $\mathrm{Re}\ x > 0$. In this section we obtain a series representation for $\Xi(t)$.

Let

$$\vartheta(t) = \sum_{k=-\infty}^{\infty} \phi(\frac{2k\pi}{t}) .$$

Proposition 1.

For $t > 0$,

$$\Xi(t) = \sum_{n=1}^{\infty} \mu(n)\{\frac{\pi\vartheta(nt)}{nt} - \frac{\Xi(0)}{2}\} .$$

<u>Proof</u>.

Spira [1], remarks that

$$\phi(u) = \phi(-u),$$

as a consequence of the functional equation

$$\psi(x) + \frac{1}{2} = x^{-1/2}\{\psi(\frac{1}{x}) + \frac{1}{2}\} .$$

We may therefore write

$$\Xi(t) = \int_{-\infty}^{\infty} \phi(u)\cos(ut)du.$$

Writing nt for t, where n is any integer,

$$\Xi(nt) = \int_{-\infty}^{\infty} \phi(u)\cos(unt)du.$$

Hence, for $t > 0$,

(1) $$\Xi(nt) = \frac{1}{t}\int_{-\infty}^{\infty} \phi(\frac{u}{t})\cos(nu)du.$$

It is easily seen that

$$\psi'(x) = O(e^{-\pi x}), \quad \text{and} \quad \psi''(x) = O(e^{-\pi x}) ,$$

as $x \to \infty$, and it follows readily from this that

(2) $$\phi(u) = O(e^{9u/2}e^{-\pi e^{2u}})$$

as $u \to \infty$. Also, since $\phi(-u) = \phi(u)$, we have

(3) $$\phi(-u) = O(e^{9u/2}e^{-\pi e^{2u}})$$

as $u \to \infty$.

With estimates (2) and (3) we can justify transforming (1) as follows:

$$\Xi(nt) = \frac{1}{t}\sum_{k=-\infty}^{\infty}\int_{2k\pi}^{2(k+1)\pi}\phi(\tfrac{u}{t})\cos(nu)du$$

$$= \frac{1}{t}\sum_{k=-\infty}^{\infty}\int_{0}^{2\pi}\phi(\frac{u+2k\pi}{t})\cos(nu)du$$

$$= \frac{1}{t}\int_{0}^{2\pi}\{\sum_{k=-\infty}^{\infty}\phi(\frac{u+2k\pi}{t})\}\cos(nu)du \quad .$$

Hence

(4) $$\Xi(nt) = \frac{1}{t}\int_{0}^{2\pi}\Phi(u,t)\cos(nu)du,$$

where

$$\Phi(u,t) = \sum_{k=-\infty}^{\infty}\phi(\frac{u+2k\pi}{t}) \quad .$$

$\Phi(u,t)$ is a regular even function of u, for $|\mathrm{Im}\frac{u}{t}| < \frac{\pi}{4}$, and it is periodic in u with period 2π. By (4) we therefore have the Fourier expansion

$$\Phi(u,t) = \frac{t}{2\pi}\Xi(0) + \frac{t}{\pi}\sum_{n=1}^{\infty}\Xi(nt)\cos(nu).$$

Setting $u = 0$, and noting that

$$\vartheta(t) = \Phi(0,t)$$

we get

(5) $$\sum_{n=1}^{\infty}\Xi(nt) = \frac{\pi}{t}\vartheta(t) - \frac{\Xi(0)}{2} .$$

We see in Titchmarch [1], pages 214-215, that

$$\Xi(t) = O(t^{A}e^{-\pi t/4}) ,$$

as $t \to \infty$. Hence it is easily seen that

$$\sum_{n=1}^{\infty} \sum_{k=1}^{\infty} \mu(n)\Xi(knt)$$

converges absolutely, and consequently that

$$(6) \qquad \sum_{n=1}^{\infty} \sum_{k=1}^{\infty} \mu(n)\Xi(knt) = \sum_{m=1}^{\infty} \{\sum_{n|m} \mu(n)\}\,\Xi(mt)$$

$$= \Xi(t) \quad .$$

Thus from (5) and (6)

$$\Xi(t) = \sum_{n=1}^{\infty} \mu(n)\left\{\frac{\pi\vartheta(nt)}{nt} - \frac{\Xi(0)}{2}\right\}.$$

<u>Note:</u> This result appears in Braun [4] .

Section 10.

A problem connected with the zeros of Riemann's zeta function

Let

$$f(s) = \pi^{-s/2}\Gamma(\tfrac{s}{2})\zeta(s),$$

where ζ is Riemann's zeta function. Suppose that λ is a fixed real number such that $\frac{1}{2} \le \lambda < 1$, and let

$$R(t) = \operatorname{Re} f(\lambda+it), \quad I(t) = \operatorname{Im} f(\lambda+it).$$

Finally let $N_R(\lambda,T)$ and $N_I(\lambda,T)$ be the number of zeros of $R(t)$ and $I(t)$, respectively, in the interval $0 < t < T$, multiple zeros being counted according to their multiplicity.

Berlowitz [1] proved that $N_R(\lambda,T)$ and $N_I(\lambda,T)$ are unbounded as $T \to \infty$. Berndt [2] improved on this result by showing that there exists a positive constant A such that, for all sufficiently large T,

$$N_R(\lambda,T) > AT, \quad N_I(\lambda,T) > AT.$$

The following proposition is a further improvement.

Proposition 1.

For every λ such that $\frac{1}{2} \le \lambda < 1$, there exists a positive constant A such that, for all sufficiently large T,

$$N_R(\lambda,T) \geq \frac{1}{2\pi} T\log T - AT, \quad N_I(\lambda,T) \geq \frac{1}{2\pi} T\log T - AT.$$

Proof:

We shall prove the theorem for $N_R(\lambda,T)$; the proof for $N_I(\lambda,T)$ is the same, except for obvious changes of notation. Let $\frac{1}{2} < \lambda < 1$, $T > 0$. Without loss of generality we may assume that T is not the imaginary part of a zero of $\zeta(s)$.

Let L be the straight line segment joining the points λ and $\lambda + iT$ modified, if need be, by small semicircular indentations so that any zeros of $\zeta(s)$ with real part λ lie to the left of L. It is obvious that the number of distinct zeros of $\operatorname{Re} f(s)$ on L is not less than $[\frac{1}{\pi} \Delta_L \arg f(s)]$. If $\zeta(s)$, and hence $f(s)$, has a zero of order n at $\lambda + it_0$ then $R(t)$ has a zero of the same or higher order at t_0; on the other hand, if γ be the semicircular detour by which L avoids $\lambda + it_0$, then $\operatorname{Re} f(s)$ has at most $n + 1$ distinct zeros on γ, the radius of γ being assumed to have been chosen sufficiently small. Hence it follows easily that

(1) $$N_R(\lambda,T) \geqslant [\frac{1}{\pi} \Delta_L \arg f(s)] - N_0(\lambda,T),$$

where $N_0(\lambda,T)$ is the number of zeros of $\zeta(s)$ on the straight line segment from λ to $\lambda + iT$. It is well known that $N_0(\lambda,T) = O(T)$, and it is easily shown (c.f. the Lemma below) that

(2) $$\Delta_L \arg f(s) = \tfrac{1}{2}\, T\log T + O(T) .$$

By (1) and (2) the theorem is thus established.

Lemma

If L is defined as in the proof of the theorem above, then (2) holds.

Proof:

Let L' be the contour which is symmetric to L about the line of real part $\frac{1}{2}$; and let C be the simple closed contour which goes along L from λ to $\lambda + iT$ then straight from $\lambda + iT$ to $1 - \lambda + iT$, then along L' from $1 - \lambda + iT$ to $1 - \lambda$, and finally straight from $1 - \lambda$ back to λ.

It is well known that the number of zeros of $f(s)$ (or, equivalently, of $\zeta(s)$) inside C is $\frac{1}{2\pi} T\log T + O(T)$. Hence

$$(3) \qquad \Delta_C \arg f(s) = T\log T + O(T).$$

But $f(s)$ is real on the real axis, and $f(1-s) = f(s)$, and $f(\bar{s}) = \overline{f(s)}$ for all s. Hence

$$(4) \qquad \Delta_C \arg f(s) = 2\Delta_L \arg f(s) + 2\Delta_K \arg f(s),$$

where K is the straight line segment from $\lambda + iT$ to $\frac{1}{2} + iT$.

But

$$\Delta_K \arg(\pi^{-s/2}) = \Delta_K \operatorname{Im} \log(\pi^{-s/2}) = 0,$$

$$\Delta_K \arg \Gamma(\tfrac{s}{2}) = \Delta_K \operatorname{Im} \log \Gamma(\tfrac{s}{2}) = O(1),$$

as may be seen by Stirling's formula, and

$$\Delta_K \arg \zeta(s) = - \int_{\frac{1}{2}+iT}^{\lambda+iT} \operatorname{Im} \left\{\frac{\zeta'(s)}{\zeta(s)}\right\} ds = O(\log T),$$

as may be shown by the method given in Davenport [1], pages 103-104.

Hence

$$(5) \qquad \Delta_K \arg f(s) = O(\log T).$$

The desired result (2) now follows from (3), (4) and (5).

Note 1. By using improved estimates for $N_0(\lambda,T)$ and for the number of zeros of $\zeta(s)$ inside C (see Titchmarch [1], theorems 9.4 and 9.19), it can easily be seen that the term AT in proposition 1 may be replaced by

$$\frac{1}{2\pi}(1 + \log(2\pi))T + o(T).$$

Note 2. Proposition 1 appeared in Braun and Zulauf [3]. It was also obtained, independently, by Levinson [1].

Note 3. It has subsequently been noted by Montgomery [1], that proposition 1 can, by a simple additional argument, be sharpened to

$$N_R(\lambda,T) = \frac{1}{2\pi} T\log T + O(T), \quad N_I(\lambda,T) = \frac{1}{2\pi} T\log T + O(T).$$

Section 11

An elementary connection between the orders of $g(x)$ and $\psi(x) - x$.

The extent to which arguments in number theory can be made elementary is of ongoing interest, and in this section we examine a problem in which elementary methods appear to be far less efficient than non-elementary techniques.

Let ψ be Tchebychef's function defined by

$$\psi(x) = \sum_{n \leq x} \Lambda(n)$$

$$= \sum_{n \leq x} \sum_{d|n} \mu(d) \log \frac{n}{d} .$$

The following statements can be proved equivalent using non-elementary methods,

With $0 < \lambda < \frac{1}{2}$,

$T_1 \equiv$ for every $\varepsilon > 0$, $g(x) = O(x^{-\lambda+\varepsilon})$ as $x \to \infty$,

$\Leftrightarrow$

$T_2 \equiv$ for every $\varepsilon > 0$, $\psi(x) - x = O(x^{1-\lambda+\varepsilon})$ as $x \to \infty$,

a path of proof being

$$T_1 \Rightarrow RH(1-\lambda) \Rightarrow T_2 \Rightarrow RH(1-\lambda) \Rightarrow T_1 .$$

We are unaware of any previous record of an elementary proof that $T_2 \Rightarrow T_1$.

Here we offer an elementary proof of a result which, at least for small values of λ, comes close to the desired goal.

Proposition 1.

Let $0 < \lambda < \frac{1}{2}$. If $g(x) = O(x^{-\lambda})$ as $x \to \infty$, then $\psi(x) - x = O(x^{1/(1+\lambda)})$ as $x \to \infty$.

Proof:

For each natural number n let $p(n)$ be defined by

$$\sum_{1 \leq r \leq n} \frac{1}{r} = \log n + \gamma + p(n).$$

It is well known that $p(n) = O(\frac{1}{n})$ as $n \to \infty$.

Firstly we note

$$\sum_{n \leq x} \sum_{d|n} \mu(d) \sum_{1 \leq r \leq n/d} \frac{1}{r}$$

$$= \sum_{n \leq x} \sum_{d|n} \mu(d) (\log \frac{n}{d} + \gamma + p(\frac{n}{d}))$$

$$= \psi(x) + \gamma + \sum_{n \leq x} \sum_{d|n} \mu(d) p(\frac{n}{d}),$$

and since

$$\sum_{n \leq x} \sum_{d|n} \mu(d) p(\frac{n}{d}) = \sum_{n \leq x} p(n) M(\frac{x}{n}),$$

we have

(1) $$\sum_{n \leq x} \sum_{d|n} \mu(d) \sum_{1 \leq r \leq n/d} \frac{1}{r} = \psi(x) + \gamma + \sum_{n \leq x} p(n) M(\frac{x}{n}) .$$

Now if $g(x) = O(x^{-\lambda})$ as $x \to \infty$, we have

(2) $$M(x) = \sum_{n\leq x} (g(n) - g(n-1)n$$

$$= - \sum_{n\leq x} g(n) + ([x]+1)g(x)$$

$$= O(x^{1-\lambda}) \quad \text{as} \quad x \to \infty.$$

Then examining the last term in (1),

$$\sum_{n\leq x} p(n)M(\frac{x}{n}) = O\left(\sum_{n\leq x} \frac{1}{n}(\frac{x}{n})^{1-\lambda}\right)$$

$$= O(x^{1-\lambda}) \quad \text{as} \quad x \to \infty,$$

and hence from (1),

(3) $$\sum_{n\leq x} \sum_{d|n} \mu(d) \sum_{1\leq r\leq n/d} \frac{1}{r} = \psi(x) + O(x^{1-\lambda}) \quad \text{as} \quad x \to \infty.$$

On the other-hand,

$$\sum_{n\leq x} \sum_{d|n} \mu(d) \sum_{1\leq r\leq n/d} \frac{1}{r} = \sum_{1\leq r\leq x} \frac{1}{r} \sum_{\substack{d|n \\ d\leq n/r \\ n\leq x}} \mu(d)$$

$$= \sum_{1\leq r\leq x} \frac{1}{r} \sum_{d\leq x/r} \mu(d)([\frac{x}{d}] - r+1)$$

$$= \sum_{1\leq r\leq x} \frac{1}{r} \sum_{d\leq x/r} \mu(d)[\frac{x}{d}] +$$

$$- \sum_{r\leq x} M(\frac{x}{r}) + \sum_{r\leq x} \frac{1}{r} M(\frac{x}{r}) .$$

Since $\sum_{r\leq x} M(\frac{x}{r}) = 1,$ and

$$\sum_{r\leq x} \frac{1}{r} M\left(\frac{x}{r}\right) = O\left(\sum_{r\leq x} \frac{1}{r} \left(\frac{x}{r}\right)^{1-\lambda}\right)$$

$$= O(x^{1-\lambda}) \quad \text{as} \quad x \to \infty,$$

it follows that

$$\text{(4)} \qquad \sum_{n\leq x} \sum_{d|n} \mu(d) \sum_{1\leq r\leq n/d} \frac{1}{r} = \sum_{1\leq r\leq x} \frac{1}{r} \sum_{d\leq x/r} \mu(d) \left[\frac{x}{d}\right] + $$

$$+ O(x^{1-\lambda}) \quad \text{as} \quad x \to \infty.$$

From (3) and (4) we thus have

$$\text{(5)} \qquad \psi(x) = \sum_{1\leq r\leq x} \frac{1}{r} \sum_{d\leq x/r} \mu(d) \left[\frac{x}{d}\right] + O(x^{1-\lambda}) \quad \text{as} \quad x \to \infty.$$

With

$$\left[\frac{x}{d}\right] = \frac{x}{d} - \left\{\frac{x}{d}\right\}$$

in (5), and noting that

$$\sum_{1\leq r\leq x} \frac{1}{r} \sum_{d\leq x/r} \frac{\mu(d)}{d} = 1$$

we obtain

$$\text{(6)} \qquad \psi(x) = x - \sum_{1\leq r\leq x} \frac{1}{r} \sum_{d\leq x/r} \mu(d) \left\{\frac{x}{d}\right\} +$$

$$+ O(x^{1-\lambda}) \quad \text{as} \quad x \to \infty.$$

We next obtain a satisfactory estimate for the double sum in (6).

Let T be any fixed natural number satisfying $2 \leq T \leq [\sqrt{x}] - 1$.

Then

(7) $$\sum_{1\le r\le x}\frac{1}{r}\sum_{d\le x/r}\mu(d)\{\frac{x}{d}\} = \sum_{1\le r\le T}\frac{1}{r}\sum_{d\le x/r}\mu(d)\{\frac{x}{d}\} + $$

$$+ \sum_{T<r\le x}\frac{1}{r}\sum_{d\le x/r}\mu(d)\{\frac{x}{d}\}$$

$$= \sum_{1\le r\le T}\frac{1}{r}\sum_{d\le x/r}\mu(d)\{\frac{x}{d}\} + O(\frac{x}{T})$$

as $x \to \infty$, where the O-term is uniform with respect to T.

Now since,

$$\sum_{d\le x}\mu(d)\{\frac{x}{d}\} = x\,g(x) - 1$$
$$= O(x^{1-\lambda})$$

as $x \to \infty$, we have

(8) $$\sum_{1\le r\le T}\frac{1}{r}\sum_{d\le x/r}\mu(d)\{\frac{x}{d}\} = \sum_{1\le r\le T}\frac{1}{r}\sum_{d\le x}\mu(d)\{\frac{x}{d}\} +$$

$$- \sum_{2\le r\le T}\frac{1}{r}\sum_{x/r<d\le x}\mu(d)\{\frac{x}{d}\}$$

$$= - \sum_{2\le r\le T}\frac{1}{r}\sum_{x/r<d\le x}\mu(d)\{\frac{x}{d}\} +$$

$$+ O(x^{1-\lambda}\log T)$$

uniformly, as $x \to \infty$.

Thus from (7) and (8),

(9) $$\sum_{1\le r\le x}\frac{1}{r}\sum_{d\le x/r}\mu(d)\{\frac{x}{d}\} = - \sum_{2\le r\le T}\frac{1}{r}\sum_{x/r<d\le x}\mu(d)\{\frac{x}{d}\} +$$

$$+ O(\frac{x}{T}) + O(x^{1-\lambda}\log T) \quad \text{as} \quad x \to \infty.$$

For convenience we let

$$J_k = ([\frac{x}{k+1}],\ [\frac{x}{k}]]$$

for positive integer $k \leq x$.

Then (9) can be written in the form

$$(10) \qquad \sum_{1\leq r\leq x} \frac{1}{r} \sum_{d\leq x/r} \mu(d)\{\frac{x}{d}\} = - \sum_{2\leq r\leq T} \frac{1}{r} \sum_{k\leq r-1} \sum_{d\in J_k} \mu(d)\{\frac{x}{d}\} + $$

$$+ O(\frac{x}{T}) + O(x^{1-\lambda} \log T)$$

as $x \to \infty$.

For $d \in J_k$, we have $[\frac{x}{d}] = k$ and so $\{\frac{x}{d}\} = \frac{x}{d} - k$.

From (10) we thus have

$$(11) \qquad \sum_{1\leq r\leq x} \frac{1}{r} \sum_{d\leq x/r} \mu(d)\{\frac{x}{r}\} = - \sum_{2\leq r\leq T} \frac{1}{r} \sum_{k\leq r-1} \sum_{d\in J_k} \mu(d)(\frac{x}{d} - k)$$

$$= - \sum_{2\leq r\leq T} \frac{x}{r}(g(x) - g(\frac{x}{r})) +$$

$$+ \sum_{2\leq r\leq T} \frac{1}{r} \sum_{k\leq r-1} k(M(\frac{x}{k}) - M(\frac{x}{k+1})) +$$

$$+ O(\frac{x}{T}) + O(x^{1-\lambda} \log T), \quad \text{as} \quad x \to \infty.$$

But

$$\sum_{2\leq r\leq T} \frac{x}{r} (g(x) - g(\frac{x}{r})) = O\left(\sum_{2\leq r\leq T} \frac{x}{r} (\frac{x}{r})^{-\lambda}\right)$$

$$= O(x^{1-\lambda}T^{\lambda})$$

as $x \to \infty$, and

$$\sum_{2\leq r\leq T} \frac{1}{r} \sum_{1\leq k\leq r-1} k\left(M\left(\frac{x}{k}\right) - M\left(\frac{x}{k+1}\right)\right)$$

$$= \sum_{2\leq r\leq T} \frac{1}{r}\left[\sum_{k\leq r} M\left(\frac{x}{k}\right) - r\, M\left(\frac{x}{r}\right)\right]$$

$$= O\left[\sum_{r\leq T} \frac{1}{r} \sum_{k\leq r} \left(\frac{x}{k}\right)^{1-\lambda}\right] + O\left[\sum_{r\leq T} \left(\frac{x}{r}\right)^{1-\lambda}\right]$$

$$= O(x^{1-\lambda}T^{\lambda}), \quad \text{as} \quad x \to \infty.$$

Hence, from (11) we obtain

(12) $$\sum_{1\leq r\leq x} \frac{1}{r} \sum_{d\leq x/r} \mu(d)\left\{\frac{x}{d}\right\} = O(x^{1-\lambda}T^{\lambda}) + O\left(\frac{x}{T}\right) \quad \text{as} \quad x \to \infty.$$

Then from (6) and (12),

$$\psi(x) = x + O(x^{1-\lambda}T^{\lambda}) + O\left(\frac{x}{T}\right)$$

as $x \to \infty$, and the proposition now follows choosing $T = [x^{\lambda/(\lambda+1)}]$.

Note 1. A somewhat more economical proof of proposition 1 appeared in Braun and Zulauf, [5].

Note 2. In Gelfond and Linnik, [1], p.54-55, there is an elementary argument that $M(x) = o(x)$ as $x \to \infty$, implies $\psi(x) - x = o(x)$ as $x \to \infty$.

If $M(x) = O(x^{1-\lambda})$ we obtain, again by elementary methods, the following improvement.

Proposition 2.

Let $0 < \lambda < \frac{1}{2}$. If $M(x) = O(x^{1-\lambda})$ as $x \to \infty$, then $\psi(x) - x = O(x^{1/(1+\lambda)})$ as $x \to \infty$.

Proof:

We see from Braun and Zulauf [5] that $M(x) = O(x^{1-\lambda}) \Rightarrow g(x) = O(x^{-\lambda})$ as $x \to \infty$, can be argued by elementary methods. We thus have an elementary proof of proposition 2.

Section 12.

Consecutive integers covered by a given set of primes.

We say that an integer n is covered by a given set, P, of primes if and only if $p|n$ for at least one $p \in P$.

Let $P_r = \{p_1, p_2, \ldots, p_r\}$ be the set of the first r primes, and let

$$Q_r = \{p_1^{\alpha_1} p_2^{\alpha_2} \ldots p_r^{\alpha_r} : \alpha_1 \geq 0, \ldots, \alpha_r \geq 0, \alpha_1 \in \mathbb{Z}\}.$$

i.e. let Q_r be the semigroup generated by P_r. We may regard the equation

$$1 = \sum_{n \leq N} \mu(n) [\frac{N}{n}]$$

as a special case of

$$\phi_r(N) = \sum_{n \in Q_r} \mu(n) [\frac{N}{n}]$$

$=$ The number of positive integers $\leq N$ not covered by P_r.

Let

$$J(x) = \text{Max}\{[N] : \exists \ K \geq 1 : \phi_{\pi(x)}(N+K) = \phi_{\pi(x)}(K)\} .$$

Then $J(x)$ is the maximum number of consecutive numbers each of which is covered by $P_{\pi(x)}$. In the following we show how certain order estimates for $J(x)$ relate to the problems of

(i) the least prime in an arithmetic progression, and

(ii) the distance, (in the asymptotic sense) between consecutive primes.

The problem of the actual order of $J(x)$ though remains unsolved.

Throughout this section let $(D,l) = 1$ with $0 < l < D$, $D \geqslant 2$.

Proposition 1.

Let M and B be integers with $B > 0$. Let P be any given set of primes. Suppose $Dm + l$ is covered by P for all $m \in [M+1, M+B]$. Then there exists a positive integer K such that n is covered by P for all $n \in [K+1, K+B]$.

Proof:

For each $m \in [M+1, M+B]$ choose $p_m \in P$ such that $Dm + l \equiv 0 \pmod{p_m}$. The p_m may not necessarily be distinct, but since $(D,l) = 1$ we note that if $Dm + l \equiv 0 \pmod{p_m}$ and $Dm' + l \equiv 0 \pmod{p_m}$ then $m \equiv m' \pmod{p_m}$. Thus by the Chinese remainder theorem we can choose a positive integer K such that

$$K \equiv -m \pmod{p_m} \quad \text{for} \quad m = 1, 2, \ldots, B.$$

Then

$$K + m \equiv 0 \pmod{p_m} \quad \text{for} \quad m = 1, 2, \ldots, B, \text{ and}$$

proposition 1 follows.

Proposition 2.

Let $0 < \alpha \leqslant 1$. Suppose that $J(x) \leqslant Ax^{2-\alpha}$ for all $x \geqslant 2$. Then the least prime of the form $Dm + l$ does not exceed $(A+1)^{2/\alpha} D^{2/\alpha}$.

Proof:

Suppose that for $1 \leqslant m \leqslant \left[\frac{(A+1)^{2/\alpha} D^{2/\alpha} - l}{D}\right]$ each $Dm + l$ is composite. Since if a number n is composite it has a prime divisor $\leqslant \sqrt{n}$ it follows that the set of primes $\leqslant (A+1)^{1/\alpha} D^{1/\alpha}$ cover the numbers $Dm + l$ for

$$1 \leqslant m \leqslant \left[\frac{(A+1)^{2/\alpha} D^{2/\alpha} - l}{D}\right].$$

Then from proposition 1 it follows that

$$J((A+1)^{1/\alpha} D^{1/\alpha}) \geqslant \left[\frac{(A+1)^{2/\alpha} D^{2/\alpha} - l}{D}\right].$$

Hence from our initial assumption, $J(x) \leqslant Ax^{2-\alpha}$, we have

$$A(AD+D)^{(2/\alpha)-1} \geqslant \left[\frac{(AD+D)^{2/\alpha} - l}{D}\right] > \frac{(AD+D)^{2/\alpha} - 2D}{D}$$

Whence

$$2 > (AD+D)^{(2/\alpha)-1} \left\{\frac{AD+D}{D} - A\right\}$$

$$= (AD+D)^{(2/\alpha)-1} > D^{(2/\alpha)-1} \geqslant D,$$

since $0 < \alpha \leqslant 1$ and, obviously, $A > 0$. But $D \geqslant 2$ by assumption. Hence there must be a prime $Dm + l$ with

$$1 \leqslant m \leqslant \left[\frac{(A+1)^{2/\alpha} D^{2/\alpha} - l}{D}\right].$$

Proposition 3.

Let $\beta \geqslant 1$, $A > 0$. Suppose that $J(x) \leqslant Ax^{2\beta/(\beta+1)}$ for all $x \geqslant 2$. Then for every positive integer $N \geqslant 1$ there is at least one prime of the form $Dm + l$ with $N \leqslant m \leqslant [(A+1)^{\beta} D^{\beta} N]$.

Proof:

For convenience we let

$$B = [(A+1)^{\beta} D^{\beta} N].$$

As in the previous proposition if $Dm + l$ is composite for $m = N$, $N + 1, \ldots, B - 1$, then each such number is covered by a prime less than or equal to $(D(B-1) + l)^{\frac{1}{2}}$.

Since $(D(B-1) + l)^{\frac{1}{2}} < (DB)^{\frac{1}{2}} < N^{\frac{1}{2}}(AD+D)^{(\beta+1)/2}$

$$(1) \qquad J(N^{\frac{1}{2}}(AD+D)^{(\beta+1)/2}) \geq B - N .$$

However, from our initial assumption

$$\begin{aligned} J(N^{\frac{1}{2}}(AD+D)^{(\beta+1)/2}) &\leq AN^{\beta/(\beta+1)}(AD+D)^{\beta} \\ &< AN(AD+D)^{\beta} \\ &= N(A+1)^{\beta+1} D^{\beta} - N(AD+D)^{\beta} \\ &< N(A+1)^{\beta+1} D^{\beta} - ND \\ &\leq N(A+1)^{\beta+1} D^{\beta} - 2N, \end{aligned}$$

and hence,

$$(2) \qquad J(N^{\frac{1}{2}}(AD+D)^{(\beta+1)/2}) \leq B - 2N .$$

From (1) and (2) it thus follows that at least one of $Dm + l$ with $m = N, N + 1, \ldots, B - 1$ is prime.

Corollary:

If $J(x) = O(x^{\gamma})$ as $x \to \infty$, for some $\gamma < 2$, then there are infinitely many primes of the form $Dm + l$.

Proof:

In this case we may write $\gamma = 2\beta/(\beta+1)$ for some $\beta \geq 1$, and then apply the proposition for $N = N_i$, for an integer sequence $\{N_i\}$ satisfying

$$N_{i+1} > (A+1)^{\beta} D^{\beta} N_i$$

for $i \geq 1$.

Proposition 4.

Let $\frac{1}{2} > \gamma \geq 0$, $A > 0$. Suppose that $J(x) \leq Ax^{1+2\gamma}$ $\forall x \geq 2$. Then

(a) For all $N \geq N_0$ there is a prime in $[[N - AN^{\frac{1}{2}+\gamma}], N]$,

(b) $p_{n+1} - p_n = O(p_n^{\frac{1}{2}+\gamma})$ as $n \to \infty$, where p_n is the nth prime.

Proof:

(a) We assume that $N - AN^{\frac{1}{2}+\gamma} \geq 1$ for $N \geq N_0$.

If every number in $[[N - AN^{\frac{1}{2}+\gamma}], N]$ is composite then every number in this interval is covered by the set of primes $\leq \sqrt{N}$.

Consequently, from the definition of $J(x)$,

$$A(N^{\frac{1}{2}})^{1+2\gamma} \geq N - [N - AN^{\frac{1}{2}+\gamma}] + 1,$$

and hence

$$AN^{\frac{1}{2}+\gamma} \geq AN^{\frac{1}{2}+\gamma} + 1$$

thus establishing (a).

(b) With $N = p_{n+1}$ in (a) we see

$$p_{n+1} - p_n < Ap_{n+1}^{\frac{1}{2}+\gamma} + 1$$

$$= O(p_n^{\frac{1}{2}+\gamma}) \quad \text{as} \quad n \to \infty .$$

Note 1. These results are somewhat related to a more specialised result proved in Formenko [1].

Note 2. Jutila [1] proves that Linnik's constant L satisfies $L \leq 80$. The same result would follow from proposition 2 if it could be shown that $\alpha \geq \frac{1}{40}$.

Note 3. Huxley [1] proves that

$$p_{n+1} - p_n < p_n^{\delta}$$

for sufficiently large n, whenever $\delta > \frac{7}{12}$. A similar result would follow from proposition 4 if it could be shown that $\gamma \leq \frac{1}{12}$.

Thus Jutila's result is implied by $\forall\ x \geq 1,\ J(x) \leq Ax^{79/40}$,
and Huxley's result is implied by $\forall\ x \geq 1,\ J(x) \leq Ax^{7/6}$.

Section 13.

Weighted sums involving λ and μ.

We recall that one standard definition for the Möbius function

$\mu : \mathbb{N} \to \{-1,0,1\}$ is

$$\sum_{g|n} \mu(g) = \begin{cases} 1 & \text{if } n = 1, \\ 0 & \text{if } n > 1, \end{cases}$$

and that the corresponding definition for the Liouville function $\lambda : \mathbb{N} \to \{-1,1\}$ is

$$\sum_{g|n} \lambda(g) = \begin{cases} 1 & \text{if } n = \square, \\ 0 & \text{if } n \neq \square. \end{cases}$$

As in earlier sections we let

$$M(x) = \sum_{n\leq x} \mu(n), \qquad S(x) = \sum_{n\leq x} \lambda(n),$$

$$g(x) = \sum_{n\leq x} \frac{\mu(n)}{n}, \qquad h(x) = \sum_{n\leq x} \frac{\lambda(n)}{n},$$

$$G(x) = \sum_{n\leq x} g(n), \qquad H(x) = \sum_{n\leq x} h(n).$$

In this section we examine certain sums of the form

$$\sum_{n\leq N} \mu(n)\, P(N,n) = F(N) \tag{1}$$

where $P(N,n) \geq P(N, n+1) \geq 1$ for $1 \leq n \leq N-1$, and investigate some circumstances in which a knowledge of the order of $P(N,1)$ and $F(N)$ could lead to new information about the order of $M(N)$.

The form (1) occurs naturally in weighted summations.

For example:

$$\sum_{n\leq N} 1 \sum_{g|n} \mu(g) = \sum_{n\leq N} \mu(n) [\frac{N}{n}],$$

and hence, from the definition of μ,

$$\sum_{n\leq N} \mu(n) [\frac{N}{n}] = 1.$$

The corresponding equation for the Liouville function is

$$\sum_{n\leq N} \lambda(n) [\frac{N}{n}] = [\sqrt{N}] .$$

Generally, we have

$$\sum_{n\leq N} \mu(n)P(N,n) = \sum_{n\leq N} (M(n) - M(n-1))P(N,n)$$

$$= \sum_{n\leq N-1} M(n)(P(N,n) - P(N,n+1)) +$$

$$+ M(N)P(N,N),$$

and hence

(2) $$\sum_{n\leq N-1} M(n)(P(N,n) - P(N,n+1)) + M(N)P(N,N) = F(N).$$

Also, trivially,

(3) $$P(N,1) = \sum_{1\leq n\leq N-1} (P(N,n) - P(N,n+1)) + P(N,N)$$

With the conditions

(i) $P(N,n) \geq P(N,n+1) \geq 1,\quad 1 \leq n \leq N-1,$

(ii) $P(N,n)$ integer valued for $1 \leq n \leq N$, we see from (2) and (3) that if the orders of $P(N,1)$ and $F(N)$ can be kept low, (i.e. less than one), the possibility of obtaining non-trivial estimates for $M(N)$ then arises.

We consider some examples of the form (1), which do however involve functions whose orders are not known.

Example 1

Let $P(N,n) = [\sqrt{\frac{N}{n}}]$ for $1 \leq n \leq N$. We see that conditions (i) and (ii) are satisfied with this choice. Actually, (1) becomes the familiar equation

$$\sum_{n\leq N} \mu(n)[\sqrt{\frac{N}{n}}] = S(N) \quad .$$

Thus if $F(N) = S(N) = O(N^{\alpha})$ as $N \to \infty$, we know from proposition 2, section 1, that the order of $M(N)$ is the same as the order of $S(N)$. This can be argued directly though, by a simple elementary method.

Indeed,

$$F(N) = \sum_{n\leq N} \mu(n)[\sqrt{\frac{N}{n}}]$$

$$= \sum_{n\leq N} \mu(n) \sum_{k\leq\sqrt{\frac{N}{n}}} 1$$

$$= \sum_{k^2\leq N} \sum_{n\leq N/k^2} \mu(n)$$

$$= \sum_{k^2\leq N} M(\frac{N}{k^2}) \quad ,$$

and hence

$$M(N) = \sum_{r\leq\sqrt{N}} \mu(r) \sum_{k^2\leq N/r^2} M(\frac{N}{r^2k^2})$$

$$= O\left(\sum_{r\leq\sqrt{N}} (\frac{N}{r^2})^{\alpha}\right)$$

$$= O(N^{\alpha}) \quad \text{as} \quad N \to \infty,$$

provided $\alpha > \frac{1}{2}$, and $F(N) = O(N^{\alpha})$ as $N \to \infty$.

Example 2.

$$\text{Let}\quad P(N,n) = \begin{cases} [\frac{N}{T+1}] , & 1 \leqslant n \leqslant T , \\ [\frac{N}{n}] , & T+1 \leqslant n \leqslant N , \end{cases}$$

where $T = [N^{\beta}]$, for fixed β satisfying $0 < \beta < 1$.

Again we see that conditions (i) and (ii) are met, and that the equation corresponding to (1) is

$$[\frac{N}{T+1}] \, S(T) + \sum_{n>T} \mu(n) [\frac{N}{n}] = F(N).$$

We now show that if $\beta \leqslant \frac{1}{2-\alpha}$ where α satisfies $\frac{1}{2} < \alpha < 1$, and $F(N) = O(NT^{\alpha-1})$ as $N \to \infty$, then for every $\varepsilon > 0$, $M(T) = O(T^{\alpha+\varepsilon})$ as $T \to \infty$.

To see this, we firstly note that

$$S(T) = \sum_{n \leqslant T} (h(n) - h(n-1))n$$

$$= - \sum_{n \leqslant T} h(n) + (T+1)H(T).$$

i.e.

$$(4) \qquad S(T) = -H(T) + (T+1)H(T) \quad .$$

Hence

$$\sum_{n \leqslant N} \mu(n)P(N,n) = [\frac{N}{T+1}] \sum_{k \leqslant T} \lambda(k) + \sum_{k>T} \lambda(k)[\frac{N}{k}]$$

$$= \frac{N}{T+1} S(T) + [\sqrt{N}] - \sum_{k \leqslant T} \lambda(k)[\frac{N}{k}] + O(T)$$

$$= \frac{N}{T+1} S(T) - N\,h(T) + [\sqrt{N}] + O(T).$$

$$= -\frac{N}{T+1} H(T) + [\sqrt{N}] + O(T),$$

as $T \to \infty$, using (4).

Thus if $F(N) = O(NT^{\alpha-1})$ as $N \to \infty$, it follows that $H(T) = O(T^{\alpha})$ as $T \to \infty$.

Consequently, from proposition 2, section 1, for every $\varepsilon > 0$, $M(T) = O(T^{\alpha+\varepsilon})$, as $T \to \infty$. We are unaware of a direct elementary argument which can be used to obtain this estimate for $M(T)$ from the given conditions.

The literature does not appear to contain an example of the form

$$\sum_{n \leq N} \mu(n) P(N,n) = F(N),$$

satisfying simultaneously the three conditions:

(i) $P(N,n) \geq P(N,n+1) \geq 1$, for $1 \leq n \leq N-1$,

(ii) $P(N,1) = O(N^{\alpha})$, as $N \to \infty$ with $0 < \alpha < 1$,

(iii) $F(N) = O(N^{\alpha})$, as $N \to \infty$, with $0 < \alpha < 1$.

In examples 1 and 2, conditions (i) and (ii) are met but the order of the corresponding $F(N)$ is not known. By contrast, if we relax condition (i) to 'asymptotically monotonic' we can provide an example meeting conditions (ii) and (iii) with $\alpha = \frac{1}{2}$.

Example 3.

Let

$$r(x) = [x] - \sum_{n \geq 1} \left[\frac{x}{n(n+1)}\right].$$

Then

$$\sum_{n \leq N} \mu(n) r\left(\frac{N}{n}\right) = 1 - \sum_{k(k+1) \leq N} 1$$

$$= O(N^{\frac{1}{2}}).$$

as $N \to \infty$.

Here, we do not necessarily have

$$r(\frac{N}{n}) \geq r(\frac{N}{n+1}) \quad \text{for} \quad 1 \leq n \leq N-1,$$

but we next prove that

$$r(x) = -\zeta(\tfrac{1}{2})x^{\frac{1}{2}} + O(x^{1/3}),$$

as $x \to \infty$, and this gives

$$r(\frac{N}{n}) \geq r(\frac{N}{n+1}), \quad \text{for} \quad 1 \leq n \leq AN^{1/7}.$$

A cursory comparison of examples (1) and (3) suggests the possibility $S(N) = O(N^{\frac{1}{2}})$, and in section 14 we note a similar type of comparison which can be made in formulae which involve $H(N)$.

Let l be defined on $\mathbb{N}$ so that

(i) $\quad l(1) = 0, \quad l(n) \geq 0 \quad \text{if} \quad n \geq 2,$

(ii) $\quad \sum_{n \leq x} l(n) = x^{\frac{1}{2}} + O(1) \quad \text{as} \quad x \to \infty,$

(iii) $\quad \sum_{n \leq x} \frac{l(n)}{n} = 1 + o(1) \quad \text{as} \quad x \to \infty.$

Let

$$r_l(x) = [x] - \sum_{n \leq x} l(n)[\frac{x}{n}].$$

<u>Proposition 1.</u>

$$r_l(x) = -\zeta(\tfrac{1}{2})x^{\frac{1}{2}} + O(x^{1/3}),$$

as $x \to \infty$.

Proof:

Firstly, we prove that

$$(5) \qquad \sum_{n \leq x} \frac{l(n)}{n} = 1 - \frac{1}{x^{\frac{1}{2}}} + O\left(\frac{1}{x}\right) \quad ,$$

as $x \to \infty$.

Indeed, from (iii) we have

$$(6) \qquad \sum_{n \leq x} \frac{l(n)}{n} + \lim_{y \to \infty} \sum_{x<n\leq y} \frac{l(n)}{n} = 1.$$

Letting $p(x) = \sum_{n \leq x} l(n)$, we then note that

$$\sum_{x<n\leq y} \frac{l(n)}{n} = \sum_{x<n\leq y} \frac{p(n) - p(n-1)}{n}$$

$$= \sum_{[x]\leq n \leq [y]} \frac{p(n)}{n(n+1)} - \frac{p(x)}{[x]} + \frac{p(y)}{[y+1]}$$

Hence, from condition (ii),

$$\sum_{x<n\leq y} \frac{l(n)}{n} = \sum_{[x]\leq n\leq y} \frac{n^{\frac{1}{2}}}{n(n+1)} - \frac{1}{x^{\frac{1}{2}}} + \frac{1}{y^{\frac{1}{2}}} + O\left(\frac{1}{x}\right) + O\left(\frac{1}{y}\right)$$

as $y \to \infty$, with $x < y$.

Consequently,

$$\lim_{y\to\infty} \sum_{x<n\leq y} \frac{l(n)}{n} = \sum_{n=[x]}^{\infty} \frac{n^{\frac{1}{2}}}{n(n+1)} - \frac{1}{x^{\frac{1}{2}}} + O\left(\frac{1}{x}\right)$$

$$= \frac{1}{x^{\frac{1}{2}}} + O\left(\frac{1}{x}\right),$$

and (5) now follows from (6).

Further,

$$(7) \qquad \sum_{n\leq x} l(n)\ [\frac{x}{n}] = \sum_{m\leq x} l(m) \sum_{n\leq x/m} 1$$

$$= \sum_{n\leq x} \sum_{m\leq x/n} l(m)$$

$$= \sum_{n\leq x} p(\frac{x}{n}) \quad .$$

Also, for any fixed integer k satisfying $1 \leq k \leq x$ we have

$$(8) \qquad \sum_{k\leq n\leq x} p(\frac{x}{n}) = \sum_{k\leq m\leq x} \sum_{n\leq x/m} l(n)$$

$$= \sum_{n\leq x/k} l(n)\,([\frac{x}{n}] - k + 1)$$

$$= \sum_{n\leq x/k} l(n)\,[\frac{x}{n}] - (k-1)p(\frac{x}{k}).$$

Hence, from (7) and (8),

$$(9) \qquad \sum_{n\leq x} l(n)\,[\frac{x}{n}] = \sum_{n\leq k-1} p(\frac{x}{n}) + \sum_{n\leq x/k} l(n)\,[\frac{x}{n}] +$$

$$- (k-1)p(\frac{x}{k}).$$

We now obtain estimates for the terms on the RHS of (9).

Firstly,

$$\sum_{n\leq k-1} p(\frac{x}{n}) = \sum_{n\leq k-1} (\frac{x}{n})^{\frac{1}{2}} + O(k)$$

$$= x^{\frac{1}{2}}(2k^{\frac{1}{2}} + \zeta(\tfrac{1}{2}) + O(1/k^{\frac{1}{2}})) + O(k).$$

i.e.

$$(10) \qquad \sum_{n\leq k-1} p\left(\frac{x}{n}\right) = 2(xk)^{\frac{1}{2}} + \zeta(\tfrac{1}{2})x^{\frac{1}{2}} + O\left(\left(\frac{x}{k}\right)^{\frac{1}{2}}\right) + O(k).$$

Also,

$$(11) \qquad \sum_{n\leq x/k} l(n)\left[\frac{x}{n}\right] = x \sum_{n\leq x/k} \frac{l(n)}{n} + O\left(\sum_{n\leq x/k} l(n)\right)$$

$$= x - (kx)^{\frac{1}{2}} + O(k) + O\left(\left(\frac{x}{k}\right)^{\frac{1}{2}}\right)$$

using (5).

Finally,

$$(12) \qquad (k-1)p\left(\frac{x}{k}\right) = (xk)^{\frac{1}{2}} + O\left(\left(\frac{x}{k}\right)^{\frac{1}{2}}\right) + O(k).$$

Using (10), (11) and (12) in (9) we thus have

$$\sum_{n\leq x} l(n)\left[\frac{x}{n}\right] = x + \zeta(\tfrac{1}{2})x^{\frac{1}{2}} + O\left(\left(\frac{x}{k}\right)^{\frac{1}{2}}\right) + O(k),$$

and the choice $k = [x^{1/3}]$ leads to the result.

<u>Note 1.</u> The method of proof here has been applied to general theorems on special divisor problems (Zulauf and Braun [1]).

<u>Note 2.</u> Putting $l(n) = \begin{cases} 1 & \text{if } n = m(m+1), \\ 0 & \text{otherwise} \end{cases}$

we get $r_l(x) = r(x)$ and conditions (i), (ii), (iii) are easily seen to be satisfied.

In the next section we have occasion to use definite upper and lower bounds for $r(x)/x^{\frac{1}{2}}$.

Proposition 2.

For $x \geqslant 1$,

$$[\sqrt{x}] \leqslant r(x) \leqslant 2[\sqrt{x}].$$

Proof:

With $l(n)$ as defined in note 2, we have

$$r(x) = [x] - \sum_{n \leqslant x} l(n) \left[\frac{x}{n}\right].$$

Setting

$$p = p(x) = \sum_{n \leqslant x} l(n)$$

we have

(13) $$p(p+1) \leqslant [x] < (p+1)(p+2).$$

Also,

(14) $$\sum_{n \leqslant x} \frac{l(n)}{n} = \sum_{m(m+1) \leqslant x} \left(\frac{1}{m} - \frac{1}{m+1}\right)$$

$$= 1 - \frac{1}{p+1}.$$

Hence, from (14),

$$r(x) \geqslant [x] - \sum_{n \leqslant x} l(n)\frac{x}{n}$$

$$\geq [x] - x + \frac{x}{p+1},$$

$$> \frac{x}{p+1} - 1.$$

Since $r(x)$ is an integer, we thus have

(15) $$r(x) \geq \left[\frac{x}{p+1}\right].$$

Also, using (14),

$$r(x) \leq x - \sum_{n \leq x} l(n)\left(\frac{x}{n} - 1\right)$$

$$= x - x + \frac{x}{p+1} + p$$

$$= \frac{x}{p+1} + p,$$

and consequently, since $r(x)$ and p are integers we have

(16) $$r(x) \leq \left[\frac{x}{p+1}\right] + p.$$

If $p(p+1) \leq x < (p+1)^2$ then

$$[\sqrt{x}] = p = \left[\frac{x}{p+1}\right],$$

and from (15) and (16) it follows that $[\sqrt{x}] \leq r(x) \leq 2[\sqrt{x}]$.

If $(p+1)^2 \leq x < (p+1)(p+2)$ then

$$[\sqrt{x}] = p+1 = \left[\frac{x}{p+1}\right],$$

and from (15) and (16) we obtain

$$[\sqrt{x}] \leq r(x) \leq 2[\sqrt{x}] - 1.$$

Section 14.

Some aspects of $H(x)$ and $G(x)$.

For any integers N and T with $1 \leq T \leq N$ let $x_i(N,T)$ $(i = 1,2, \ldots, T)$ be the unique solution of the system of equations

$$\sum_{k \leq n \leq T} x_n [\tfrac{n}{k}] = [\tfrac{N}{k}] - [\tfrac{N}{T+1}] , \quad (k = 1,2,..,T).$$

Proposition 1.

(1) $$H(T) = \frac{T+1}{N} \sum_{n \leq T} x_n(N,T)[\sqrt{n}] + O(\frac{T^2}{N}) ,$$

and

(2) $$G(T) = \frac{T+1}{N} \sum_{n \leq T} x_n(N,T) \quad + \quad O(\frac{T^2}{N})$$

as $T \to \infty$, (and consequently $N \to \infty$), where the constants implicit in the O – terms are independent of N and T.

Proof:

For convenience we write $x_n = x_n(N,T)$.

Then

$$\sum_{n \leq T} x_n [\sqrt{n}] = \sum_{n \leq T} x_n \sum_{k \leq n} \lambda(k) [\tfrac{n}{k}]$$

$$= \sum_{k \leq T} \lambda(k) \sum_{k \leq n \leq T} x_n [\tfrac{n}{k}]$$

$$= \sum_{k \leq T} \lambda(k) \;([\tfrac{N}{k}] - [\tfrac{N}{T+1}])$$

$$= N\, h(T) - \frac{N}{T+1} \;S(T) +$$

$$- \sum_{k \leq T} \lambda(k)\{\frac{N}{k}\} + \{\frac{N}{T+1}\} S(T)$$

$$= \frac{N}{T+1} H(T) + O(T), \quad \text{as} \quad T \to \infty, \quad \text{using (4), section 13.}$$

Hence the result for $H(T)$, with a similar method leading to (2).

We now examine the equation which corresponds to (1) with $[\sqrt{n}]$ replaced by $r_l(n)$ as defined in section (13), recalling that

$$r_l(n) = - \zeta(\tfrac{1}{2})[\sqrt{n}] + O(n^{1/3}).$$

<u>Proposition 2.</u>

For all integer N,T satisfying $N \geq T \geq 1$,

$$\frac{T+1}{N} \sum_{n \leq T} x_n(N,T) r_l(n) = 2T^{1/2} + O(\frac{T}{N}^{3/2}) + O(1)$$

as $T \to \infty$, (and consequently $N \to \infty$), where the constants implicit in the O-notation are independent of N and T but may depend on the choice of the function l.

<u>Proof:</u>

For convenience we write $x_n = x_n(N,T)$. Using the definition of x_n and $r_l(n)$,

$$\sum_{n \leq T} x_n r_l(n) = \sum_{n \leq T} x_n (n - \sum_{k \leq n} l(k)[\frac{n}{k}])$$

$$= \sum_{n \leq T} x_n n - \sum_{k \leq T} l(k) \sum_{k \leq n \leq T} x_n[\frac{n}{k}]$$

$$= N - [\frac{N}{T+1}] - \sum_{k \leq T} l(k) ([\frac{N}{k}] - [\frac{N}{T+1}])$$

$$= N - \sum_{k \leq T} l(k)[\frac{N}{k}] + [\frac{N}{T+1}] \left(\sum_{k \leq T} l(k) - 1\right)$$

$$= \frac{2N}{T^{1/2}} + O(\frac{N}{T}) + O(T^{1/2}), \text{ as } T \to \infty,$$

using (5), section 13, and properties (i) and (ii) of l, as defined in section 13. The proposition now follows.

Corollary:

$$H(T) = \frac{-2}{\zeta(\frac{1}{2})} T^{1/2} + O(1) + O(\frac{T^{3/2}}{N}) +$$

$$+ \frac{T+1}{N} \sum_{n \leq T} x_n(N,T) R_l(n)$$

as $T \to \infty$, where

$$R_l(x) = \frac{1}{\zeta(\frac{1}{2})} r_l(x) + [\sqrt{x}] = O(x^{1/3})$$

as $x \to \infty$.

Proof:

By propositions 1 and 2,

$$H(T) = \frac{T+1}{N} \sum_{n \leq T} x_n [\sqrt{n}]$$

$$= \frac{T+1}{N} \sum_{n \leq T} x_n (- \frac{1}{\zeta(\frac{1}{2})} r_l(n) + R_l(n))$$

$$= \frac{-2}{\zeta(\frac{1}{2})} T^{1/2} + O(\frac{T^{3/2}}{N}) + O(1) +$$

$$+ \frac{T+1}{N} \sum_{n \leq T} x_n R_l(n).$$

Note 1. Selberg [1] notes that if K, k are the upper and lower limits of $H(x)/x^{\frac{1}{2}}$, respectively, (with these constants possibly infinite) then

$$K \geq -\frac{2}{\zeta(\frac{1}{2})} \geq k, \quad \text{but}$$

Ingham [1] states that a method of Schmidt and Landau can be used to obtain the stronger inequalities,

$$K \geq \frac{-2}{\zeta(\frac{1}{2})} + c > \frac{-2}{\zeta(\frac{1}{2})} - c \geq k,$$

where c is a positive constant (finite or infinite) dependent on the complex zeros of $\zeta(s)$.

Before comparing proposition 1 and 2 further we obtain formal expressions for the x_i, and look at some systems of equations which have a superficial resemblance to the original system.

Proposition 3.

For $1 \leq k \leq T$,

$$x_k = \sum_{n \leq [\frac{T}{k}]} \mu(n) \left[\frac{N}{nk}\right] - \sum_{n \leq [\frac{T}{k+1}]} \mu(n) \left[\frac{N}{n(k+1)}\right] + $$
$$- \left(M\left(\frac{T}{k}\right) - M\left(\frac{T}{k+1}\right)\right)\left[\frac{N}{T+1}\right].$$

Proof:

Recalling that

$$\sum_{n \leq T} x_n \left[\frac{n}{h}\right] = \left[\frac{N}{h}\right] - \left[\frac{N}{T+1}\right],$$

for $1 \leq h \leq T$, we have

$$\sum_{r\leq[\frac{T}{k}]} \mu(r) \sum_{n\leq T} x_n[\frac{n}{rk}] = \sum_{r\leq[\frac{T}{k}]} \mu(r) \; [\frac{N}{rk}] + $$

$$- M(\frac{T}{k})[\frac{N}{T+1}] \quad .$$

On the other-hand,

$$\sum_{r\leq[\frac{T}{k}]} \mu(r) \sum_{n\leq T} x_n[\frac{n}{rk}] = \sum_{n\leq T} x_n \sum_{r\leq[\frac{T}{k}]} \mu(r) \; [\frac{n}{rk}]$$

$$= \sum_{k\leq n\leq T} x_n \; .$$

Consequently,

$$\sum_{k\leq n\leq T} x_n = \sum_{r\leq[\frac{T}{k}]} \mu(r) \; [\frac{N}{rk}] - M(\frac{T}{k})[\frac{N}{T+1}] \; ,$$

from which the proposition follows.

We now contrast the original system of equation with other related systems.

Example 1:

Let $y_i \equiv y_i(N,T)$, $(1 \leq i \leq T)$, be the solutions of

$$\sum_{k\leq n\leq T} y_n(\frac{n}{k} - \frac{1}{2} + \frac{1}{2k}) = \frac{N}{k} - \frac{N}{T+1}$$

for $1 \leq k \leq T$. Then $y_n = \dfrac{2N}{(T+1)^2}$, $1 \leq n \leq T$.

This result follows by some simple algebraic manipulation. We note that the average difference between $[\frac{n}{k}]$ and $\frac{n}{k}$ for $n = 1, 2, \ldots, k$ is $\frac{1-k}{2k}$.

Example 2.

Let $y_i = y_i(N,T)$, $i = 1, 2, \ldots, T$, be the solution of

$$\sum_{k\leq n\leq T} y_n\left[\frac{n}{k}\right] = \frac{N}{k} - \frac{N}{T+1} \quad (1 \leq k \leq T).$$

Then

$$\sum_{1\leq n\leq T} y_n = \frac{N}{T+1}\, G(T).$$

Indeed,

$$\sum_{1\leq n\leq T} y_n = \sum_{n\leq T} y_n \sum_{k\leq n} \mu(k)\left[\frac{n}{k}\right]$$

$$= \sum_{k\leq T} \mu(k) \sum_{k\leq n\leq T} y_n\left[\frac{n}{k}\right]$$

$$= \sum_{k\leq T} \mu(k)\left(\frac{N}{k} - \frac{N}{T+1}\right)$$

$$= \frac{N}{T+1}\, G(T).$$

The last step following after an argument similar to that in example 2, section 13.

Now we note that $y_T > 0$, and from Lehmer and Selberg [1] that $G(T)$ changes sign infinitely often. Consequently, we might expect considerable sign difference in the y_i when $G(T) < 0$. We now examine how the sign distribution in the $x_i = x_i(N,T)$ defined by

$$\sum_{k\leq n\leq T} x_n\left[\frac{n}{k}\right] = \left[\frac{N}{k}\right] - \left[\frac{N}{T+1}\right], \quad (1 \leq k \leq T),$$

is related to the order of $H(T)$. For each positive integer T let $E(T)$ be defined as follows.

Let A be the set of all positive integers N such that the solution of the system of equations

$$\sum_{k\leq n\leq T} x_n\left[\frac{n}{k}\right] = \left[\frac{N}{k}\right] - \left[\frac{N}{T+1}\right], \quad (1 \leq k \leq T),$$

satisfies $x_n \geq 0$ for $1 \leq n \leq T$.

$$\text{Let } E(T) = \begin{cases} \text{Max}\{N : N \in A\}, & |A| < \infty, \\ \text{Min}\{N : N \in A \wedge N \geq T^2\}, & |A| = \infty. \end{cases}$$

We note that $T \in A$, (since for $N = T$ the solution is $x_1 = x_2 = \ldots = x_{T-1} = 0$, $x_T = 1$), so that $A \neq \phi$, and $E(T)$ is well defined.

Proposition 4.

$$H(T) = O(T^{1/2}) + O\left(\frac{T^2}{E(T)}\right) \quad \text{as} \quad T \to \infty.$$

Proof:

From proposition 1 and 2 with, for example, the choice $r(x) = [x] - \sum_{k\geq 1}\left[\frac{x}{k(k+1)}\right]$, we have

$$\text{(3)} \qquad H(T) + O(1) + O\left(\frac{T^2}{N}\right) = \frac{T+1}{N}\sum_{n\leq T} x_n(N,T)[\sqrt{n}],$$

and

$$\text{(4)} \qquad 2T^{1/2} + O\left(\frac{T^{3/2}}{N}\right) + O(1) = \frac{T+1}{N}\sum_{n\leq T} x_n(N,T)r(n).$$

From proposition 2, section 13, we have

$$\text{(5)} \qquad [\sqrt{n}] \leq r(n) \leq 2[\sqrt{n}]$$

for $n \geq 1$. If we now choose $N = E(T)$ then the $x_n(N,T)$ are all non-negative, and we get from (3), (4) and (5),

$$|H(T)| = \frac{T+1}{E(T)} \sum_{n \leq T} x_n [\sqrt{n}] + 0(1) + 0(\frac{T^2}{E(T)})$$

$$\leq \frac{T+1}{E(T)} \sum_{n \leq T} x_n \, r(n) + 0(1) + 0(\frac{T^2}{E(T)})$$

$$= 2T^{1/2} + 0(1) + 0(\frac{T^2}{E(T)})$$

$$= 0(T^{1/2}) + 0(\frac{T^2}{E(T)})$$

as $T \to \infty$, which is the desired result.

Note 1. Since $T \in A$, as noted earlier, we have $E(T) \geq T$. It can be shown that $E(T) \leq 5T^2$ if $T \geq 60$.

Note 2. From proposition 3, section 1, $RH(\sigma_0)$ is true if and only if for all $\varepsilon > 0$, $H(T) = 0(T^{\sigma_0+\varepsilon})$ as $T \to \infty$. Consequently $RH(\sigma_0)$ is true if for every $\varepsilon > 0$, there is a A_ε such that $E(T) > A_\varepsilon T^{2-\sigma_0-\varepsilon}$, for all $T \geq 1$. We next see how the condition restricting the size of $E(T)$ can be considerably relaxed yet can still lead to a result similar to

$$H(T) = 0(T^{1/2}) + 0(\frac{T^2}{E(T)}) \quad \text{as } T \to \infty.$$

Again, with reference to the system

$$\sum_{k \leq n \leq T} x_n(N,T) \, [\frac{n}{k}] = [\frac{N}{k}] - [\frac{N}{T+1}], \quad (1 \leq k \leq T),$$

let

$$P^+ \equiv P^+(N,T) = \sum_{x_n>0} x_n\ [\sqrt{n}],$$

$$P^- \equiv P^-(N,T) = -\sum_{x_n<0} x_n [\sqrt{n}],$$

$$Q^+ \equiv Q^+(N,T) = \sum_{x_n>0} x_n\ r(n),$$

$$Q^- \equiv Q^-(N,T) = -\sum_{x_n<0} x_n\ r(n),$$

with the understanding that

$$P^- = Q^- = 0 \text{ if there does not exist } x_n < 0.$$

Let δ be any fixed real number such that $0 < \delta < 1$. We put $\delta_0 = \frac{\delta}{1-\delta}$ and let

$$B = \{N : P^+ \geq (2+\delta_0)P^- \vee P^- \geq (2+\delta_0)P^+\}.$$

Finally, let

$$E_1(T) = \begin{cases} \text{Max } \{N : N \in B\} & \text{if } |B| < \infty, \\ \text{Min } \{N : N \in B \wedge N \geq E(T) \wedge N \geq T^2\} & \text{if } |B| = \infty. \end{cases}$$

Referring to the definition of A and $E(T)$ preceding proposition 4 we note that $A \subseteq B$ and $E(T) \leq E_1(T)$.

Proposition 5.

$$H(T) = O(T^{1/2}) + O(\frac{T^2}{E_1(T)})$$

as $T \to \infty$.

Proof:

We note that $2 + \delta_0 = \frac{2-\delta}{1-\delta} > 2$.

Let $N = E_1(T)$. Then either

$$(1-\delta)P^+ \geq (2-\delta)P^- \text{ and thus, by (5),}$$

$$\begin{aligned} Q^+ - Q^- &\geq P^+ - 2P^- \\ &\geq \delta(P^+ - P^-) \\ &\geq \delta(1+\delta_0)P^- \\ &\geq 0, \end{aligned}$$

or

$$(1-\delta)P^- \geq (2-\delta)P^+ \text{ and thus, by (5),}$$

$$\begin{aligned} Q^- - Q^+ &\geq P^- - 2P^+ \\ &\geq \delta(P^- - P^+) \\ &\geq \delta(1+\delta_0)P^+ \\ &\geq 0 \quad . \end{aligned}$$

In both cases,

$$|Q^+ - Q^-| \geq \delta|P^+ - P^-|.$$

Hence, from, (3) and (4),

$$|H(T)| = \frac{T+1}{N}|P^+ - P^-| + O(1) + O(\frac{T^2}{E_1(T)}),$$

$$\leq \frac{T+1}{\delta N}|Q^+ - Q^-| + O(1) + O(\frac{T^2}{E_1(T)}),$$

$$\leq \frac{1}{\delta}\left\{2T^{1/2} + O(1) + O\left[\frac{T^{3/2}}{E_1(T)}\right]\right\} + O(1) + O(\frac{T^2}{E_1(T)}) \quad ,$$

$$= O(T^{1/2}) + O(\frac{T^2}{E_1(T)})$$

as $T \to \infty$ which is the desired result.

We next detail a consequence of $RH(\sigma_0)$ being false.

Proposition 6.

If $1 > \sigma_0 \geq \frac{1}{2}$ and $RH(\sigma_0)$ is false, then there exists an $\varepsilon > 0$, and an infinite increasing natural number sequence $\{T_i\}$, such that

$$0 < \frac{Q^+(N,T_i)}{Q^-(N,T_i)} - 1 < T_i^{\frac{1}{2}-\sigma_0-2\varepsilon}$$

for all $N \geq T_i^{2-\sigma_0-\varepsilon}$, and all $i \geq 1$.

Proof:

Suppose $\sigma_0 \geq \frac{1}{2}$ and $RH(\sigma_0)$ is false. Then, by proposition 3, section 1,

$$H(T) = O(T^{\sigma_0+2\varepsilon}) \quad \text{as} \quad T \to \infty,$$

is false for some $\varepsilon > 0$, and consequently by proposition 5, there is an infinite increasing sequence of natural numbers $\{T_i\}$ satisfying

(6) $$E_1(T_i) < T_i^{2-\sigma_0-2\varepsilon} \quad \text{and} \quad |H(T_i)| > 9T_i^{\sigma_0+2\varepsilon}$$

for some fixed $\varepsilon > 0$ satisfying $\sigma_0 + \varepsilon < 1$. For this sequence $\{T_i\}$ we let $P_i^+ = P(N,T_i)$ etc., and let B_i be the set used in the definition of $E_1(T_i)$.

We note that B_i must be finite since otherwise we would have $E_1(T_i) \geq T_i^2$.

Hence if $N > E\,(T_i)$ then $N \notin B_i$, and consequently

$$P_i^+ < (2+\delta_0)P_i^- \quad \text{and} \quad P_i^- < (2+\delta_0)P_i^+.$$

It then follows that $P_i^+ > 0$, $P_i^- > 0$, and

$$\frac{1}{2+\delta_0} < \frac{P_i^+}{P_i^-} < 2+\delta_0.$$

Hence

$$\frac{1}{2+\delta_0} - 1 < \frac{P_i^+}{P_i^-} - 1 < 1+\delta_0.$$

i.e.

$$\frac{-(1+\delta_0)}{2+\delta_0} < \frac{P_i^+}{P_i^-} - 1 < 1+\delta_0,$$

and so

(7) $$\left|\frac{P_i^+}{P_i^-} - 1\right| < 1+\delta_0$$

$$< 2.$$

But from proposition 1 and 2 we have

(8) $$\frac{T_i+1}{N}\,(Q_i^+ - Q_i^-) = 2T_i^{1/2} + O\left(\frac{T_i^{3/2}}{N}\right) + O(1) \quad \text{as } i \to \infty, \text{ and}$$

(9) $$H(T_i) = \frac{T_i+1}{N}\,(P_i^+ - P_i^-) + O\left(\frac{T_i^2}{N}\right) + O(1) \quad \text{as } i \to \infty.$$

Then from (8) with $N \geq T_i^{2-\sigma_0-\varepsilon} > E_1(T_i)$ we have

(10) $$\frac{T_i+1}{N}\,(Q_i^+ - Q_i^-) = 2T_i^{\frac{1}{2}} + O(T_i^{\sigma_0-\frac{1}{2}+\varepsilon}).$$

Since $\sigma_0 + \varepsilon < 1$ we have $\frac{1}{2} > \sigma_0 - \frac{1}{2} + \varepsilon$, and then from (10),

$$Q_i^+ - Q_i^- > 0 \quad \text{for } i > i_0 \text{ say.}$$

From (9) and (10) with $i > i_0$, and $N \geq T_i^{2-\sigma_0-\varepsilon} > E\,(T_i)$, it follows that

$$(11) \qquad H(T_i) = \frac{P_i^+ - P_i^-}{Q_i^+ - Q_i^-}\left\{2T_i^{\frac{1}{2}} + O(T_i^{\sigma_0-\frac{1}{2}+\varepsilon})\right\} + O(T_i^{\sigma_0+\varepsilon})$$

as $i \to \infty$.

Since $|H(T_i)| > 9T_i^{\sigma_0+2\varepsilon}$, it follows from (11) that with $i > i_1$ (say) and $N \geq T_i^{2-\sigma_0-\varepsilon}$ we have

$$\left|\frac{P_i^+ - P_i^-}{Q_i^+ - Q_i^-}\right| > 4T_i^{\sigma_0-\frac{1}{2}+2\varepsilon}.$$

Finally, using (7),

$$0 < Q_i^+ - Q_i^- < \frac{1}{4}\left|P_i^+ - P_i^-\right| T_i^{\frac{1}{2}-\sigma_0-2\varepsilon},$$

and hence

$$0 < \frac{Q_i^+}{Q_i^-} - 1 < \frac{1}{4}\frac{P_i^-}{Q_i^-}\left|\frac{P_i^+}{P_i^-} - 1\right| T_i^{\frac{1}{2}-\sigma_0-2\varepsilon}$$

$$\leq \frac{1}{4}\, 2\cdot 2\, T_i^{\frac{1}{2}-\sigma_0-2\varepsilon}$$

if $N \geq T_i^{2-\sigma_0-\varepsilon}$ and $i \geq i_1$.

The proposition then follows by relabelling the sequence $\{T_i\}_{i>i_1}$.

Section 15.

On a possible characterization of the multiplicative structure of the natural numbers by the Möbius function.

In this section we examine an interpretation of the equations

$$\sum_{1 \leq n \leq N} \mu(n) \left[\frac{N}{n}\right] = 1, \quad N \geq 1, \quad N \in \mathbb{N},$$

which leads to a conjecture about μ. Throughout, (G) will be any commutative semigroup,

$$G = \{\lambda_1, \lambda_2, \lambda_3, \ldots \},$$

for which we have a transitive relation, $<$, and the following properties:

(i) $\lambda_1 < \lambda_2 < \lambda_3 < \ldots$,

(ii) λ_1 is an identity,

(iii) $\lambda_p < \lambda_q \Rightarrow \lambda_n \lambda_p < \lambda_n \lambda_p$,

for $\lambda_p, \lambda_q, \lambda_n \in G$. Relations $>, \geq, \leq,$ are defined in the obvious way. We define $[\ ]$ by

$$\left[\frac{\lambda_N}{\lambda_n}\right] = \begin{cases} 0 & \text{if } n > N, \\ k & \text{if } n \leq N \text{ and } \lambda_n \lambda_k \leq \lambda_N < \lambda_n \lambda_{k+1} . \end{cases}$$

We then define the Möbius function of G, $\hat{\mu} : G \to \mathbb{Z}$, by the equations

$$\sum_{1 \leq n \leq N} \hat{\mu}(\lambda_n) \left[\frac{\lambda_N}{\lambda_n}\right] = 1, \quad N \geq 1, \quad N \in \mathbb{N},$$

but note that an alternative, equivalent definition is

$$\sum_{\lambda_g | \lambda_n} \hat{\mu}(\lambda_g) = \begin{cases} 1, & n = 1, \\ 0 & n > 1, \end{cases}$$

for all $n \in \mathbb{N}$.

Further, if we assume uniqueness of factorisation in G, then we also have an equivalent definition:

$$\hat{\mu}(\lambda_n) = \begin{cases} 1 & \text{if } n = 1, \\ (-1)^r & \text{if } \lambda_n \text{ is the product of } r \text{ distinct generators,} \\ 0 & \text{otherwise (i.e. if } \lambda_n \text{ is divisible by a square generator)} \end{cases}$$

The principal direction of inquiry in this context was addressed to the problem of classifying semigroups for which

$$\forall \; n \in N : \hat{\mu}(\lambda_n) = \mu(n),$$

and we begin with some simple observations:

<u>Proposition 1.</u>

$$\forall \; \lambda_N, \lambda_n \in G: \; [\frac{\lambda_N}{\lambda_n}] = [\frac{N}{n}]$$

if and only if

$$\forall \; \lambda_m, \lambda_n \in G: \; \lambda_n \lambda_m = \lambda_{nm}.$$

<u>Proof:</u>

Suppose that

$$\forall \; \lambda_N, \lambda_n \in G: \; [\frac{\lambda_N}{\lambda_n}] = [\frac{N}{n}] \quad ,$$

and let m and n be positive integers, $m \geq 2$.

Then $$[\frac{\lambda_{nm}}{\lambda_n}] = m,$$

and so

$$\lambda_n \lambda_m \leqslant \lambda_{nm}.$$

On the other hand, obviously,

$$[\frac{\lambda_{nm-1}}{\lambda_n}] = m - 1,$$

and hence

$$\lambda_n \lambda_m > \lambda_{nm-1} .$$

Consequently

$$\lambda_n \lambda_m = \lambda_{nm}$$

for all $n \geqslant 1$ and all $m \geqslant 2$, and the same result holds trivially for all $n \geqslant 1$ if $m = 1$.

Conversely, suppose that

$$\forall \quad \lambda_n, \lambda_m \in G, \quad \lambda_n \lambda_m = \lambda_{nm}.$$

Let N and n be positive integers, $N \geqslant n$. Then, putting $N = rn + s$, $r \geqslant 1$, $0 \leqslant s < n$,

$$\lambda_r \lambda_n = \lambda_{rn} \leqslant \lambda_N < \lambda_{(r+1)n} = \lambda_{r+1} \lambda_n,$$

and hence

$$[\frac{\lambda_N}{\lambda_n}] = [\frac{N}{n}] \quad .$$

<u>Proposition 2.</u>

If

$$\forall \quad \lambda_n, \lambda_m \in G: \quad \lambda_n \lambda_m = \lambda_{nm}$$

then

$$\forall \quad n \in \mathbb{N} : \hat{\mu}(\lambda_n) = \mu(n).$$

Proof:

In this case we have

$$[\frac{\lambda_N}{\lambda_n}] = [\frac{N}{n}] \quad .$$

Hence

$$\sum_{1 \leqslant n \leqslant N} \hat{\mu}(\lambda_n) [\frac{N}{n}] = 1, \quad n \geqslant 1, \quad N \geqslant 1.$$

But the equations

$$\sum_{1 \leqslant n \leqslant N} \mu(n) [\frac{N}{n}] = 1, \quad n \geqslant 1, \quad N \geqslant 1,$$

uniquely define the Möbius function.

Hence

$$\forall \quad n \in \mathbb{N} \ : \hat{\mu}(\lambda_n) = \mu(n).$$

Proposition 3.

If $(G) \subset (\mathbb{R}_+^*)$ is a semigroup of numbers with the usual ordering, and multiplication, and $\forall \ \lambda_n, \lambda_m \in G, \ \lambda_n \lambda_m = \lambda_{nm}$ then there exists $\alpha \in \mathbb{R}_+^*$ such that $\forall \ n \in \mathbb{N}, \ \lambda_n = n^\alpha$.

Proof:

Let p be any prime. We define $\nu_k(p) = \nu_k$ by

(1) $$\lambda_p^{\nu(k)} \geqslant \lambda_2^k > \lambda_p^{\nu(k)-1} \quad .$$

Consequently

$$\frac{\nu(k)}{k} \geqslant \frac{\log \lambda_2}{\log \lambda_p} > \frac{\nu(k)-1}{k},$$

and so

(2) $$\lim_{k\to\infty} \frac{\nu(k)}{k} = \frac{\log \lambda_2}{\log \lambda_p} .$$

But from (1), and our initial assumptions, we also have

$$\lambda_p^{\nu(k)} \geqslant \lambda_2^{k} \geqslant \lambda_p^{\nu(k)-1},$$

and hence

$$p^{\nu(k)} \geqslant 2^k \geqslant p^{\nu(k)-1} .$$

Consequently

(3) $$\lim_{k\to\infty} \frac{\nu(k)}{k} = \frac{\log 2}{\log p} .$$

Hence, putting $\alpha = \dfrac{\log \lambda_2}{\log 2}$, we see from (2) and (3) that $\log \lambda_p = \alpha \log p$, i.e. $\lambda_p = p^{\alpha}$, for all primes p. The proposition now follows.

Consequent investigation into the condition $\hat{\mu}(\lambda_n) = \mu(n)$ led to the

<u>CONJECTURE 1</u>. If $\hat{\mu}(\lambda_n) = \mu(n)$ for all $n \geqslant 1$ then

$\lambda_{nm} = \lambda_n \lambda_m$, for all $n \geqslant 1$, $m \geqslant 1$.

In support of this conjecture we firstly summarise the results of a numerical investigation, Braun [2], where uniqueness of factorization in G was not assumed, and in the next section, give a detailed proof of the conjecture for $nm \leqslant 26$, where we do assume uniqueness of factorization in G.

Consequences of the assumption

$$\sum_{1 \leq n \leq N} \mu(n) [\frac{\lambda_N}{\lambda_n}] = 1, \quad N \geq 1,$$

were investigated in a step-wise manner starting from $N = 1$.

For any fixed $N > 1$ write

$$[\frac{\lambda_N}{\lambda_n}] = [\frac{\lambda_{N-1}}{\lambda_n}] + \delta(n),$$

for $1 \leq n \leq N$.

We then have $\delta(n) \in \{0,1\}$, $1 \leq n \leq N$, and

(4) $$\sum_{1 \leq n \leq N} \mu(n) \delta(n) = 0 .$$

One consistency condition which could be checked with respect to the accumulating data was:

if $$[\frac{\lambda_N}{\lambda_p}] = [\frac{\lambda_{N-1}}{\lambda_p}] + 1$$

$$= k,$$

then

(5) $$[\frac{\lambda_N}{\lambda_k}] = p.$$

Starting with $p = N$, and decreasing, we could use this condition to show that certain $\delta(n)$ were zero. Also $\delta(1) = \delta(N) = 1$. The remaining undecided cases led to a modified form of (4)

(6) $$\sum_{n \in A_N} \mu(n) \delta(n) = 0.$$

Further use of (5) and (6) then determined the feasible factorizations for λ_N, and where more than one choice was available, the factorizations were conditional on some definite preceding structure of factorization. The only possible case up to $N = 7$ is

$$\lambda_1,\ \lambda_2 \text{ prime},\ \lambda_3 \text{ prime},\ \lambda_4 = \lambda_2^2,$$
$$\lambda_5 \text{ prime},\ \lambda_6 = \lambda_2\lambda_3,\ \lambda_7 \text{ prime}.$$

At $N = 8$, two cases arise which cannot be excluded using the preceding data. Namely either $\lambda_8 = \lambda_2^3$ or $\lambda_8 = \lambda_3^2$, and at $N = 9$, the corresponding choices were either $\lambda_9 = \lambda_3^2$ or $\lambda_9 = \lambda_2^3$. Although uniqueness of factorization was not assumed the first case of different factorizations occured at $N = 22$, where the possibility $\lambda_{22} = \lambda_3^3 = \lambda_5^2$ arose, subject to $\lambda_3^2 = \lambda_8$, but then contradiction arose at $N = 24$ with no consistent factorization for λ_{24}.

With increasing N, more logical possibilities for factorizations occur, and the development of all cases was carried along in a lattice structure. For example,

$$\lambda_2\lambda_7 = \lambda_{14} \text{ or } \lambda_{15}$$
$$\lambda_3\lambda_5 = \lambda_{15} \text{ or } \lambda_{14}$$
$$\lambda_3\lambda_7 = \lambda_{21} \text{ or } \lambda_{22}$$
$$\lambda_2\lambda_{11} = \lambda_{22} \text{ or } \lambda_{21}\ .$$

All the cases involving $\lambda_3^2 = \lambda_8$, $\lambda_2^3 = \lambda_8$ lead to contradiction, in the sense above, that eventually an N value was arrived at for which λ_N had no consistent factorization.

At $N = 28$, the possibility

$$\lambda_3^2 = \lambda_8,\ \ \lambda_2^3 = \lambda_9 \text{ was excluded, as was the case}$$
$$\lambda_2\lambda_7 = \lambda_{15},\ \lambda_3\,\lambda_5 = \lambda_{14}.$$

The case $\lambda_{22} = \lambda_3\lambda_7$, $\lambda_{21} = \lambda_2\lambda_{11}$ was excluded at $N = 46$, and for $nm < 23$ we had necessarily $\lambda_n \lambda_m = \lambda_{nm}$, and we refer to Braun [2], for this verification.

The assumption of uniqueness of factorization requires only the consideration of factorizations of λ_N consistent with the third definition we listed for $\hat{\mu}(\lambda_N)$, and this provides a basis for a much quicker method of investigation and verification.

Proposition 4.

Assuming unique factorization in (G): If $\hat{\mu}(n) = \mu(n)$ for $1 \leqslant n \leqslant 240$ then

$$\lambda_m \lambda_n = \lambda_{mn} \quad \text{for } 1 \leqslant nm \leqslant 74.$$

Proof:

A proof is contained in Zulauf [2]. A proof for the ranges $1 \leqslant n \leqslant 68$, $1 \leqslant nm \leqslant 26$ is given in the next section.

It is clear that the step-by-step procedures described here and in the next section cannot be carried on indefinitely. They have also failed to reveal any general pattern suitable for mathematical induction, or to give any clue as to how a general proof of conjecture 1 might be constructed. It seems that more sophisticated tools will have to be used, and with this in mind we explored the possiblity of introducing a 'logarithm' function L on G with real non-negative values, satisfying

(i) $L(\lambda_m\lambda_n) = L(\lambda_m) + L(\lambda_n)$

(ii) $\lambda_m < \lambda_n \Longleftrightarrow L(\lambda_m) < L(\lambda_n)$ for all $\lambda_m, \lambda_n \in G$.

A 'logarithm' of sorts does indeed exist if uniqueness of factorization is assumed in (G), provided that condition (ii) is weakened to

(iia) $\lambda_m < \lambda_n \Rightarrow L(\lambda_m) \leq L(\lambda_n) \Rightarrow \lambda_m \leq \lambda_n$ for all $\lambda_m, \lambda_n \in G$.

We call L a logarithm of G if it satisfies (i) and (iia), and a strong logarithm on G if L satisfies (ii) as well.

Proposition 5.

Assuming unique factorization in (G) there exists a logarithm L (satisfying conditions (i) and (iia)) on G. This logarithm is unique except for a positive factor of proportionality, and can be made unique by assigning a prescribed positive value to $L(\lambda_2)$.

Proof:

See Zulauf and Braun [6].

Putting $|\lambda_n| = \exp L(\lambda_n)$, and calling this the norm of λ_n, it can now be seen that the elements of G are generalized integers in the sense of Beuring [1]. Generalized integers have been studied extensively (c.f. Le Veque [1]), but we have not found any evidence that conjecture 1 has been studied, let alone solved, before. We note that the following conjecture implies conjecture 1.

CONJECTURE 2.

Distinct orderings (satisfying the multiplicative condition) in semi-groups (G) with (countably) infinitely many generators give rise to distinct Möbius functions.

However, if we allow only finitely many generators for G then the equivalent of conjecture 2 is false.

Indeed, with G_1 and G_2, both generated by $\lambda_{p_1}, \lambda_{p_2}$, for G_1 we could have

$$\lambda_1 < \lambda_{p_1} \lesssim \lambda_{p_2} < \lambda_{p_1}{}^2 < \lambda_{p_1} \lambda_{p_2} < \lambda_{p_1}{}^3 < \lambda_{p_2}{}^2 < \ldots\ldots ,$$

and for G_2,

$$\lambda_1 < \lambda_{p_1} < \lambda_{p_2} < \lambda_{p_1}{}^2 < \lambda_{p_1}{}^2 \lambda_{p_2} < \lambda_{p_2}{}^2 < \lambda_{p_1}{}^3 < \ldots\ldots,$$

yet both G_1 and G_2 would have the same Möbius function.

We now look at natural extensions of conjectures 1 and 2 which we derive from the arithmetic nature of μ.

With $n = p_{i_1}{}^{\alpha_1} p_{i_2}{}^{\alpha_2} \ldots p_{i_r}{}^{\alpha_r}$, where the p_{i_j} are distinct prime numbers, we let

$$\nu(n) = \sum_{i=1}^{r} \alpha_i .$$

Then a form of definition for μ is

$$\mu(n) = \begin{cases} 1 & \text{if } n = 1, \\ 0 & \text{if } \exists \ t > 1 : t^2 | n, \\ (-1)^1 & \text{if } \nexists \ t > 1 : t^2 | n, \text{ and } \nu(n) \equiv 1 (\text{mod} 2), \\ (-1)^2 & \text{if } \nexists \ t > 1 : t^2 | n, \text{ and } \nu(n) \equiv 2 (\text{mod} 2), \end{cases}$$

and the fact that μ is weakly-multiplicative is easily verified.

A family of functions which μ belongs to can be defined as follows: Let ξ be an m^{th} primitive root of unity, and let μ_ξ be given by

$$\mu_\xi(n) = \begin{cases} 1 & \text{if } n = 1, \\ 0 & \text{if } \exists\ t > 1 : t^m | n, \\ \xi^r & \text{if } \nexists\ t > 1 : t^m \nmid n,\ \nu(n) \equiv r (\text{mod} m), \\ & \qquad 1 \leqslant r \leqslant m. \end{cases}$$

We see that μ_ξ is weakly multiplicative, and $\mu = \mu_{-1}$. If ξ is an m^{th} primitive root of unity we call μ_ξ an m-primitive Möbius function. There would not seem to be a clear reason why the status of μ, in conjecture 1, should be different from μ_ξ, where μ_ξ is any m-primitive Möbius function, assuming uniqueness of factorization in G, Thus defining $\hat{\mu}_\xi(\lambda_n)$ for all $\lambda_n \in G$ analogously to the way μ_ξ is defined we have

<u>CONJECTURE 1(ξ).</u>

If $\hat{\mu}_\xi(\lambda_n) = \mu_\xi(n)$ for all $n \geqslant 1$, where μ_ξ is an m-primitive Möbius function, then

$$\lambda_n \lambda_m = \lambda_{nm} \qquad \text{for all } n \geqslant 1,\ m \geqslant 1.$$

Section 16.

A small step towards proof of a conjecture concerning the Möbius function.

Let (G) be a semigroup with an ordering relation satisfying the multiplicative conditions in section 15. We now assume uniqueness of factorization in (G) or, what comes to the same, that (G) is a free semi-group. As in section 15, let λ_n be the n^{th} element, and let π_i be the i^{th} generator, according to the ordering of G. Let the Möbius function $\hat{\mu}$ be defined on G as in section 15. Let p_i be the i^{th} prime number. If $\Pi p_i^{\alpha_i}$ is the canonical factorization of the positive integer n then conjecture 1 of section 14 may be stated thus: If

(1) $$\hat{\mu}(\lambda_n) = \mu(n) \quad \text{for all} \quad n \geq 1, \quad \text{then}$$

(2) $$n = \Pi p_i^{\alpha_i} \Rightarrow \lambda_n = \Pi\ \pi_i^{\alpha_i} \text{ for all } \quad n \geq 1.$$

As mentioned in section 15 we now prove (2) for $1 \leq n \leq 26$ assuming (1) for $1 \leq n \leq 68$. Although this represents only small progress towards conjecture 1, the proof is actually quite lengthy, and we divide it into several stages.

Stage 1: The possible factorization of λ_n for $1 \leq n \leq 15$.

Proceeding stepwise $n = 1,2,3, \ldots$ we determine all factorizations of λ_n that are consistent with (1) and with the data obtained in previous steps. We denote by $h(n,i)$ the smallest multiple of π_i which is not less than λ_n. By checking whether $\hat{\mu}(h(n,i)) = \mu(n)$ we see whether $h(n,i)$ is a possible candidate for λ_n, and we do this for all $\pi_i < \lambda_n$. For convenience we write n for λ_n when there is no danger of confusion. It will be clear from the context whether n

denotes an element of G or an integer. We adopt the convention that $n \cdot m$ stands for $\lambda_n \lambda_m$.

Table 1.

Possible Factorization

λ_n	$\hat{\mu}(\lambda_n)$	Stage 1	Stage 2	Stage 3
1	1	unit		
2	-1	$2 = \pi_1$		
3	-1	$3 = \pi_2$		
4	0	2^2		
5	-1	$5 = \pi_3$		
6	1	$2 \cdot 3$		
7	-1	$7 = \pi_4$		
8	0	$2^3 \vee 3^2$		2^3
9	0	$3^2 \vee 2^3$		3^2
10	1	$2 \cdot 5$		
11	-1	$11 = \pi_5$		
12	0	$2^2 \cdot 3$		
13	-1	$13 = \pi_6$		
14	1	$2 \cdot 7 \vee 3 \cdot 5$	$2 \cdot 7$	
15	1	$3 \cdot 5 \vee 2 \cdot 7$	$3 \cdot 5$	

It is clear that 1 must be the unit and that 2 is the smallest generator, i.e. $2 = \pi_1$. Now we consider λ_n for $3 \leqslant n \leqslant 15$.

(n = 3)

λ	$\hat{\mu}(\lambda)$
$h(3,1) = \pi_1 \cdot 2 = \pi_1^2$	0

Since $\hat{\mu}(3) = -1$, we must have $3 = \pi_2$.

(n = 4)

λ	$\hat{\mu}(\lambda)$
$h(4,1) = \pi_1 \cdot 2 = \pi_1^2$	0
$h(4,2) = \pi_2 \cdot 2 = \pi_1\pi_2$	1

Since $\hat{\mu}(4) = 0$ we have $4 = \pi_1^2 = 2^2$.

(n = 5)

λ	$\hat{\mu}(\lambda)$
$h(5,1) = \pi_1 \cdot 3 = \pi_1\pi_2$	1
$h(5,2) = \pi_2 \cdot 2 = \pi_1\pi_2$	1

Since $\hat{\mu}(5) = -1$ we have $5 = \pi_3$.

(n = 6)

λ	$\hat{\mu}(\lambda)$
$h(6,1) = \pi_1 \cdot 3 = \pi_1\pi_2$	1
$h(6,2) = \pi_2 \cdot 2 = \pi_1\pi_2$	1
$h(6,3) = \pi_3 \cdot 2 = \pi_1\pi_3$	1

Since $\hat{\mu}(6) = 1$, and $\pi_1 \pi_2 < \pi_1 \pi_3$ we have $6 = \pi_1 \pi_2 = 2 \cdot 3$.

(n = 7)

λ	$\hat{\mu}(\lambda)$
$h(7,1) = \pi_1 \cdot 4 = \pi_1^3$	0
$h(7,2) = \pi_2 \cdot 3 = \pi_2^2$	0
$h(7,3) = \pi_3 \cdot 2 = \pi_1\pi_3$	1

Since $\hat{\mu}(7) = -1$ we have $7 = \pi_4$.

(n = 8)

λ	$\hat{\mu}(\lambda)$
$h(8,1) = \pi_1 \cdot 4 = \pi_1{}^3$	0
$h(8,2) = \pi_2 \cdot 3 = \pi_2{}^2$	0
$h(8,3) = \pi_3 \cdot 2 = \pi_1 \pi_3$	1
$h(8,4) = \pi_4 \cdot 2 = \pi_1 \pi_4$	1

Since $\hat{\mu}(8) = 0$ we have $8 = \pi_1{}^3 = 2^3 \vee 8 = \pi_2{}^2 = 3^2$.

Two cases now arise for 9.

(n = 9) If $8 = \pi_1{}^3 = 2^3$,

λ	$\hat{\mu}(\lambda)$
$h(9,1) = \pi_1 \cdot 5 = \pi_1 \pi_3$	1
$h(9,2) = \pi_2 \cdot 3 = \pi_2{}^2$	0
$h(9,3) = \pi_3 \cdot 2 = \pi_1 \pi_3$	1
$h(9,4) = \pi_4 \cdot 2 = \pi_1 \pi_4$	1

and, since $\hat{\mu}(9) = 0$, we have $9 = \pi_2{}^2$

If $8 = \pi_2{}^2 = 3^2$,

λ	$\hat{\mu}(\lambda)$
$h(9,1) = \pi_1 \cdot 4 = \pi_1{}^3$	0
$h(9,2) = \pi_2 \cdot 4 = \pi_1{}^2 \pi_2$	0
$h(9,3) = \pi_3 \cdot 2 = \pi_1 \pi_3$	1
$h(9,4) = \pi_4 \cdot 2 = \pi_1 \pi_4$	1

and, since $\hat{\mu}(9) = 0$ and $\pi_1{}^3 < \pi_1{}^2 \pi_2$, we have $9 = \pi_1{}^3 = 2^3$.

(n = 10)

λ	$\hat{\mu}(\lambda)$
$h(10,1) = \pi_1 \cdot 5 = \pi_1 \pi_3$	1
$h(10,2) = \pi_2 \cdot 4 = \pi_1{}^2 \pi_2$	0
$h(10,3) = \pi_3 \cdot 2 = \pi_1 \pi_3$	1
$h(10,4) = \pi_4 \cdot 2 = \pi_1 \pi_4$	1

Since $\hat{\mu}(10) = 1$ and $\pi_1 \ \pi_3 < \pi_1 \ \pi_4$ we have $10 = \pi_1 \cdot \pi_3 = 2 \cdot 5$.

(n = 11)

λ	$\hat{\mu}(\lambda)$
$h(11,1) = \pi_1 \cdot 6 = \pi_1{}^2\pi_2$	0
$h(11,2) = \pi_2 \cdot 4 = \pi_1{}^2\pi_2$	1
$h(11,3) = \pi_3 \cdot 3 = \pi_2\pi_3$	1
$h(11,4) = \pi_4 \cdot 2 = \pi_1\pi_4$	1

Since $\hat{\mu}(11) = -1$ we have $11 = \pi_5$.

(n = 12)

λ	$\hat{\mu}(\lambda)$
$h(12,1) = \pi_1 \cdot 6 = \pi_1{}^2\pi_2$	0
$h(12,2) = \pi_2 \cdot 4 = \pi_1{}^2\pi_2$	0
$h(12,3) = \pi_3 \cdot 3 = \pi_2\pi_3$	1
$h(12,4) = \pi_4 \cdot 2 = \pi_1\pi_4$	1
$h(12,5) = \pi_5 \cdot 2 = \pi_1\pi_5$	1

Since $\hat{\mu}(12) = 0$ we have $\mu(12) = \pi_1{}^2\ \pi_2 = 2^2 \cdot 3$.

(n = 13)

λ	$\hat{\mu}(\lambda)$
$h(13,1) = \pi_1 \cdot 7 = \pi_1\pi_4$	1
$h(13,2) = \pi_2 \cdot 5 = \pi_2\pi_3$	1
$h(13,3) = \pi_3 \cdot 3 = \pi_2\pi_3$	1
$h(13,4) = \pi_4 \cdot 2 = \pi_1\pi_4$	1
$h(13,5) = \pi_5 \cdot 2 = \pi_1\pi_5$	1

Since $\hat{\mu}(13) = -1$ we have $13 = \pi_6$.

$(n = 14)$

λ	$\hat{\mu}(\lambda)$
$h(14,1) = \pi_1 \cdot 7 = \pi_1\pi_4$	1
$h(14,2) = \pi_2 \cdot 5 = \pi_2\pi_3$	1
$h(14,3) = \pi_3 \cdot 3 = \pi_2\pi_3$	1
$h(14,4) = \pi_4 \cdot 2 = \pi_1\pi_4$	1
$h(14,5) = \pi_5 \cdot 2 = \pi_1\pi_5$	1
$h(14,6) = \pi_6 \cdot 2 = \pi_1\pi_6$	1

Since $\hat{\mu}(14) = 1$, and $\pi_1\ \pi_4 < \pi_1\ \pi_5 < \pi_1\ \pi_6$ we have $14 = \pi_1\ \pi_4 = 2 \cdot 7 \vee 14 = \pi_2\ \pi_3 = 3 \cdot 5$. Two cases now arise for 15.

If $14 = 3 \cdot 5 = \pi_2\pi_3$

$(n = 15)$

λ	$\hat{\mu}(\lambda)$
$h(15,1) = \pi_1 \cdot 7 = \pi_1\pi_4$	1
$h(15,2) = \pi_2 \cdot 6 = \pi_1\pi_2^2$	0
$h(15,3) = \pi_3 \cdot 5 = \pi_1^2\pi_3$	0
$h(15,4) = \pi_4 \cdot 2 = \pi_1\pi_4$	1
$h(15,5) = \pi_5 \cdot 2 = \pi_1\pi_5$	1
$h(15,6) = \pi_6 \cdot 2 = \pi_1\pi_6$	1

and, since $\hat{\mu}(15) = 1$ and $\pi_1\ \pi_4 < \pi_1\ \pi_5 < \pi_1\ \pi_6$ we have $15 = \pi_1\ \pi_4 = 2 \cdot 7$.

If $14 = 2 \cdot 7 = \pi_1\ \pi_4$

λ	$\hat{\mu}(\lambda)$
$h(15,1) = \pi_1 \cdot 8 = \pi_1^4 \vee \pi_1\pi_2^2$	0
$h(15,2) = \pi_2 \cdot 5 = \pi_2\pi_3$	1
$h(15,3) = \pi_3 \cdot 3 = \pi_2\pi_3$	1
$h(15,4) = \pi_4 \cdot 3 = \pi_2\pi_4$	1
$h(15,5) = \pi_5 \cdot 2 = \pi_1\pi_5$	1
$h(15,6) = \pi_6 \cdot 2 = \pi_1\pi_6$	1

and, since $\hat{\mu}(15) = 1$, and $\pi_2\ \pi_3 < \pi_3\ \pi_4$, and $\pi_2\ \pi_3 < \pi_1^2\ \pi_3 < \pi_1\ \pi_5 < \pi_1\ \pi_6$, we have $15 = \pi_2\ \pi_3 = 3 \cdot 5$.

The preceding approach could perhaps be a basis for a computer-aided investigation into this problem, but we now detail a method suggested by A. Zulauf which leads to quicker 'human' resolution of some of the doubtful cases.

Let

$$P(\lambda) = \sum_{\substack{\lambda_n \leq \lambda \\ \hat{\mu}(\lambda_n)=1}} 1 \quad ,$$

$$z(\lambda) = \sum_{\substack{\lambda_n \leq \lambda \\ \hat{\mu}(\lambda_n)=0}} 1 \quad ,$$

$$p_j(\lambda) = \begin{cases} 0 & \text{if } \pi_i \, \pi_{i+1} > \lambda, \\ j-i & \text{if } \pi_i \, \pi_j \leq \lambda < \pi_i \pi_{j+1}, \ (j \geq i+1), \end{cases}$$

and

$$z(\rho,\lambda) = \begin{cases} 0 & \text{if } \rho^2 > \lambda, \\ i & \text{if } \lambda_i \rho^2 \leq \lambda < \lambda_{i+1} \rho^2, \quad i \geq 1. \end{cases}$$

It is easily seen that

$$P(\lambda) = 1 + \sum_{\pi_i \pi_{i+1} \leq \lambda} p_i(\lambda), \text{ if } \lambda < \pi_1 \, \pi_2 \, \pi_3 \, \pi_4 \quad ,$$

and

$$z(\lambda) = -\sum_{1 < \rho^2 \leq \lambda} \hat{\mu}(\rho) \; z(\rho,\lambda).$$

<u>Stage 2</u>. A verification that $\lambda_{14} = \pi_1 \, \pi_4$ and $\lambda_{15} = \pi_2 \, \pi_3$.

Table 2

λ_n	$\hat{\mu}(\lambda_n)$	$z(\lambda_n)$	$P(\lambda_n)$	Possible Factorizations: Stage 3	Stage 4	Stage 5	Stage 7
16	0	5	5	2^4			
17	-1	5	5	$17=\pi_7$			
18	0	6	5	$2 \cdot 3^2$			
19	-1	6	5	$19=\pi_8$			
20	0	7	5	$2^2 \cdot 5$			
21	1	7	6		$2\cdot 11 \vee 3\cdot 7$	$3\cdot 7$	
22	1	7	7		$3\cdot 7 \vee 2\cdot 11$	$2\cdot 11$	
23	-1	7	7		$23=\pi_9$		
24	0	8	7		$2^3\cdot 3 \vee 5^2$		$2^3\cdot 3$
25	0	9	7		$5^2 \vee 2^3\cdot 3$		5^2
26	1	9	8		$2\cdot 13$		
27	0	10	8		$3^3 \vee 2^2\cdot 7$		
28	0	11	8		$2^2\cdot 7 \vee 3^3$		
29	-1	11	8		$\pi_{10} \vee 2\cdot 3\cdot 5$		
30	-1	11	8		$2\cdot 3\cdot 5 \vee \pi_{10} \vee \pi_{11}$		
31	-1	11	8		$\pi_{11} \vee 2\cdot 3\cdot 5$		
32	0	12	8				
33	1	12	9				
34	1	12	10				
35	1	12	11				
36	0	13	11	$2^2\cdot 3^2$			
37	-1	13	11				

Firstly

$$p_1(2\cdot3\cdot5) = \text{Max}\{n-1 : 2\pi_n \leqslant 2\cdot3\cdot5 < 2\pi_{n+1}\},$$
$$= \text{Max}\{n-1 : \pi_n \leqslant 3\cdot5 < \pi_{n+1}\} \quad ,$$
$$p_2(2\cdot3\cdot5) = \text{Max}\{n-2 : \pi_n \leqslant 2\cdot5 < \pi_{n+1}\},$$
$$p_3(2\cdot3\cdot5) = 0, \quad \text{since} \quad 5\cdot7 > 2\cdot3\cdot5.$$

From table 1 we see

$$\pi_6 \leqslant 3\cdot5 < \pi_7 \quad ,$$

and

$$\pi_4 \leqslant 10 < \pi_5 \quad .$$

Hence $\qquad p_1(2\cdot3\cdot5) = 5, \quad \text{and} \quad p_2(2\cdot3\cdot5) = 2.$

Consequently

$$P(2\cdot3\cdot5) = 1 + p_1(2\cdot3\cdot5) + p_2(2\cdot3\cdot5) = 8.$$

Then from Table 2, noting that $\hat{\mu}(2\cdot3\cdot5) = -1$, we must have

(3) $\qquad 2\cdot3\cdot5 \in \{29, 30, 31\},$

and hence

(4) $\qquad z(2\cdot3\cdot5) = 11.$

Now from Table 1

$$z(2, 2\cdot3\cdot5) = \text{Max}\{n : 2^2\cdot n \leqslant 2\cdot3\cdot5\}$$
$$= \text{Max}\{n : 2\cdot n \leqslant 3\cdot5\}$$
$$= \begin{cases} 6 & \text{if} \quad 3\cdot5 = 14, \\ 7 & \text{if} \quad 3\cdot5 = 15, \end{cases}$$

$$z(3,\ 2\cdot 3\cdot 5) = \text{Max}\ \{n:\ 3^2\cdot n \leqslant 2\cdot 3\cdot 5\}$$

$$= \text{Max}\ \{n:\ 3\cdot n \leqslant 10\}$$

$$= 3\ ,$$

$$z(5,\ 2\cdot 3\cdot 5) = \text{Max}\ \{n:\ 5^2\cdot n \leqslant 2\cdot 3\cdot 5\}$$

$$= \text{Max}\ \{n:\ 5\cdot n \leqslant 6\}$$

$$= 1.$$

Also $\quad z(6,\ 2\cdot 3\cdot 5) = 0$ since

$$6^2 = (2\cdot 3)\cdot 6 > (2\cdot 3)\cdot 5.$$

Hence

$$z(2\cdot 3\cdot 5) = z(2,\ 2\cdot 3\cdot 5) + z(3,\ 2\cdot 3\cdot 5) + z(5,\ 2\cdot 3\cdot 5)$$

$$= \begin{cases} 10 & \text{if } 3\cdot 5 = 14, \\ 11 & \text{if } 3\cdot 5 = 15\ . \end{cases}$$

Comparing (4) and (5), and consulting Table 1, we thus have

$$14 = 2\cdot 7$$

$$15 = 3\cdot 5\ .$$

<u>Stage 3</u>: Verification that $\lambda_8 = \pi_1{}^3$ and $\lambda_9 = \pi_2{}^2$, and factorization of λ_n for $1 \leqslant n \leqslant 20$.

Since $2\cdot 3\cdot 5 \geqslant 29$ by (3), all $\lambda < 29$ with $\mu(\lambda) = -1$ must be generators. Hence $17 = \pi_7$, $19 = \pi_8$, $23 = \pi_9$. Also, with $15 < \lambda \leqslant 20$ there are three solutions of $\mu(\lambda) = 0$, and candidates which are multiples of 2, 3, 5, ... are

$$2\cdot8,\quad 2\cdot9,\quad 2\cdot10,\quad 2\cdot12,\quad \ldots,$$
$$3\cdot6,\quad 3\cdot8,\quad 3\cdot9,\quad \ldots,$$
$$5\cdot4,\quad 5\cdot5,\quad \ldots.,$$
$$7\cdot4,\quad \ldots,$$
$$11\cdot4,\quad \ldots$$

respectively.

From table 2 since $\mu(3\cdot7) = 1$ we have $3\cdot7 \gtrsim 21$, and so $3\cdot8 > 21$, and $4\cdot7 > 21$.

Noting that $$3\cdot6 = 2\cdot8 \vee 2\cdot9,$$

and $$5\cdot4 = 2\cdot10$$

the three λ satisfying $\hat{\mu}(\lambda) = 0$, $15 < \lambda \leqslant 20$, must be $2\cdot8$, $2\cdot9$ and $2\cdot10$.

Hence
$$16 = 2\cdot8\ ,$$
$$18 = 2\cdot9\ ,$$
$$20 = 2\cdot10 = 2^2\cdot5\ .$$

Next we note from tables 1 and 2 that

(6) $$p_1(2^2\cdot3^2) = \text{Max}\ \{n-1:\ 2\cdot\pi_n \leqslant 2^2\cdot3^2\}$$
$$= \text{Max}\ \{n-1:\ 2\cdot\pi_n \leqslant 2\cdot3^2\ \}$$
$$= \begin{cases} 5 & \text{if}\quad 2\cdot3^2 = 16, \\ 6 & \text{if}\quad 2\cdot3^2 = 18\ , \end{cases}$$

(7) $$p_2(2^2\cdot3^2) = \text{Max}\ \{n-2:\ 3\cdot\pi_n \leqslant 2^2\cdot3^2\}$$
$$= \text{Max}\ \{n-2:\ \pi_n \leqslant 2^2\cdot3\}$$
$$= 3\ ,$$

(8) $$p_3(2^2 \cdot 3^2) = \mathrm{Max}\,\{n-3:\ 5\cdot\pi_n \leqslant 2^2\cdot 3^2\}$$
$$= \begin{cases} 1, & \text{if } 5\cdot 7 < 2^2\cdot 3^2, \\ 0, & \text{if } 5\cdot 7 > 2^2\cdot 3^2, \end{cases}$$

(Note: $5\cdot\pi_5 > 5\cdot 2\cdot 5 = 2\cdot 5^2 > 2\cdot 18 \geqslant 2^2\cdot 3^2$),

$$p_4(2^2\cdot 3^2) = 0 \quad \text{since} \quad 7\cdot 11 > 5\cdot 11 > 2^2\cdot 3^2 .$$

Since $$p(2^2\cdot 3^2) = 1 + p_1(2^2\cdot 3^2) + p_2(2^2\cdot 3^2) + p_3(2^2\cdot 3^2)$$
we have

$$p(2^2\cdot 3^2) \in \{9,10,11\} ,$$

and hence from table 2

$$2^2\cdot 3^2 \in \{33, 34, 35, 36, 37\} .$$

Checking $\hat{\mu}$-values we must have

(9) $$2^2\cdot 3^2 = 36,$$

and hence $p(2^2\cdot 3^2) = 11$.

From (6), (7), and (8) we thus have

(10) $$2\cdot 3^2 = 18 \quad \text{and} \quad 5\cdot 7 < 2^2\cdot 3^2.$$

But $18 = 2\cdot 9$, and hence from Table 1

$$3^2 = 9,\ 2^3 = 8.$$

The factorization of λ_n has thus been determined for $1 \leqslant n \leqslant 20$.

Stage 4: Possible factorization of λ_n for $20 < n \leqslant 31$.

There are four solutions of $\hat{\mu}(\lambda) = 0$ with $20 < \lambda < 2\cdot3\cdot5 \leqslant 31$, and we note

$$5^2 \leqslant 2\cdot3\cdot5 \,,$$

$$3^3 \leqslant 2\cdot3\cdot5 \,,$$

$$3\cdot2^3 \leqslant 2\cdot3\cdot5 \,,$$

$$2^2\cdot7 \leqslant 2\cdot3\cdot5 \,.$$

Since

$$3\cdot5 < 2\cdot8 \leqslant 2^4,$$

and

$$2^2\cdot5 < 3\cdot7,$$

we have

$$2^2\cdot3\cdot5^2 < 2^4\cdot3\cdot7,$$

and hence

(11) $$5^2 < 2^2\cdot7 \,.$$

Since

$$5\cdot7 < 2^2\cdot3^2$$

by (10), and

$$4\cdot5 < 3\cdot7,$$

we have

$$4\cdot5^2\cdot7 < 4\cdot3^3\cdot7,$$

and hence

$$5^2 < 3^3 \,.$$

Also

$$2^3\cdot3 < 3^3 \,,$$

and

$$2^3\cdot3 < 2^2\cdot7 \,.$$

Hence

$$\{2^3 \cdot 3,\ 5^2\} = \{24,\ 25\},$$

and

$$\{3^3,\ 2^2 \cdot 7\} = \{27,\ 28\}\ .$$

Also, since $3 \cdot 8,\ 2 \cdot 14 < 31$ we must have $3 \cdot 7,\ 2 \cdot 11,\ 2 \cdot 13 < 31$, and since $12 \cdot 7 < 8 \cdot 13$ we have $3 \cdot 7 < 2 \cdot 13$. Then from table 2

$$21 = 3 \cdot 7 \ \vee\ 2 \cdot 11,$$

$$22 = 2 \cdot 11 \vee 3 \cdot 7\ ,$$

$$26 = 2 \cdot 13\ \ .$$

Further, since $2 \cdot 3 \cdot 7 > 2^2 \cdot 3^2 = 36$,

$$\{29,\ 30,\ 31\} = \{\pi_{10},\ \pi_{11}\ ,\ 2 \cdot 3 \cdot 5\},$$

thus completing the stage 4 entries in table 2.

<u>Stage 5:</u> Verification that $\lambda_{21} = \pi_2\ \pi_4$ and $\lambda_{22} = \pi_1\ \pi_5$, and factorization of λ_n for $1 \leqslant n \leqslant 23$.

<u>Table 3</u>

<u>Possible Factorization.</u>

λ_n	$\hat{\mu}(\lambda_n)$	$z(\lambda_n)$	$P(\lambda_n)$	Stages 5 and 6
29	-1	11	8	$\pi_{10} \vee 2\cdot3\cdot5$
30	-1	11	8	$2\cdot3\cdot5 \vee \pi_{10} \vee \pi_{11}$
31	-1	11	8	$\pi_{11} \vee 2\cdot3\cdot5$
32	0	12	8	2^5
33	1	12	9	$3\cdot11 \vee 2\cdot17 \vee 5\cdot7$
34	1	12	10	$2\cdot17 \vee 5\cdot7 \vee 3\cdot11$
35	1	12	11	$5\cdot7 \vee 3\cdot11 \vee 2\cdot17$
36	0	13	11	$2^2\cdot3^2$
37	-1	13	11	π_{12}
38	1	13	12	$2\cdot19 \vee 3\cdot13$
39	1	13	13	$3\cdot13 \vee 2\cdot19$
40	0	14	13	$2^5\cdot5$
41	-1	14	13	$2\cdot3\cdot7 \vee \pi_{13}$
42	-1	14	13	$\pi_{13} \vee \pi_{14} \vee 2\cdot3\cdot7$
43	-1	14	13	$\pi_{14} \vee 2\cdot3\cdot7$
44	0	15	13	
45	0	16	13	
46	1	16	14	

Firstly we note from tables 1 and 2 that

$$p_1(2\cdot3\cdot7) = \text{Max }\{n-1 : 2\cdot\pi_n \leqslant 2\cdot3\cdot7\}$$
$$= 7\ ,$$
$$p_2(2\cdot3\cdot7) = \text{Max }\{n-2 : 3\cdot\pi_n \leqslant 2\cdot3\cdot7\}$$
$$= 4\ ,$$

and since $5\cdot7 < 2\cdot3\cdot7$,

and $5\cdot11 > 5\cdot2\cdot5 > 2\cdot3\cdot7$,

$$p_3(2\cdot3\cdot7) = \text{Max }\{n-3 : 5\cdot\pi_n \leqslant 2\cdot3\cdot7\}$$
$$= 1.$$

Also $p_4(2\cdot3\cdot7) = 0$, since $7\cdot11 > 5\cdot11 > 2\cdot3\cdot7$.

Hence

$$P(2\cdot3\cdot7) = 13.$$

Noting that $\hat{\mu}(2\cdot3\cdot7) = -1$ it follows from table 3 that

$$2\cdot3\cdot7 \in \{41,\ 42,\ 43\},$$

and hence

(12) $$z(2\cdot3\cdot7) = 14.$$

However

$$z(2,\ 2\cdot3\cdot7) = \text{Max }\{n\text{: } 2^2n \leqslant 2\cdot3\cdot7\}$$
$$= \text{Max }\{n\text{: } 2n \leqslant 3\cdot7\}$$
$$= \begin{cases} 10, & \text{if } 3\cdot7 < 2\cdot11, \\ 11, & \text{if } 2\cdot11 < 3\cdot7, \end{cases}$$

$$z(3,\ 2\cdot3\cdot7) = \text{Max }\{n\text{: } 3^2n \leqslant 2\cdot3\cdot7\}$$
$$= 4,$$

$$z(5,\ 2\cdot3\cdot7) = \text{Max}\ \{n\colon\ 5^2 n \leq 2\cdot3\cdot7\}$$
$$= 1,$$
$$z(6,\ 2\cdot3\cdot7) = \text{Max}\ \{n\colon\ 6^2 n \leq 2\cdot3\cdot7\}$$
$$= 1,$$

and

$$z(7,\ 2\cdot3\cdot7) = 0, \text{ since } 7^2 > 2\cdot3\cdot7.$$

Hence

$$(13) \qquad z(2\cdot3\cdot7) = -\sum_{r=1}^{6} \mu(r) z(r,\ 2\cdot3\cdot7)$$
$$= \begin{cases} 14 & \text{if } 3\cdot7 < 2\cdot11, \\ 15 & \text{if } 3\cdot7 > 2\cdot11\ . \end{cases}$$

Comparing (12) and (13) we thus have $21 = 3\cdot7$ and $22 = 2\cdot11$, and (2) has been proven for $1 \leq n \leq 23$.

Stage 6: Possible factorizations of λ_n for $31 < n \leq 43$.

The next smallest non-generator λ after $2\cdot3\cdot7$ with $\hat{\mu}(\lambda) = -1$ is either $2\cdot3\cdot11$ or $2\cdot5\cdot7$. Since, as noted before (10),

$$2\cdot3\cdot7 \in \{41,\ 42,\ 43\}$$
$$2\cdot3\cdot7 < 5\cdot11 < \begin{cases} 2\cdot3\cdot11 \\ 2\cdot5\cdot7 \end{cases}$$
$$2\cdot3\cdot7 < 6\cdot10 < \begin{cases} 2\cdot3\cdot11 \\ 2\cdot5\cdot7 \end{cases}$$

it follows that all λ in the range $32 \leq \lambda \leq 43$, except $2\cdot3\cdot7$, with $\hat{\mu}(\lambda) = -1$ are generators.

Thus

$$37 = \pi_{12} \text{ and } \{41,\ 42,\ 43\} = \{\pi_{13},\ \pi_{14},\ 2\cdot3\cdot7\}\ .$$

In $32 \leqslant \lambda \leqslant 43$ only $\lambda = 32$, $\lambda = 40$ satisfy $\hat{\mu}(\lambda) = 0$.

Noting that

$$2^5 < 2^3 5 ,$$

and

$$2^5 < 2^2 \cdot 3^2 < 2 \cdot 3 \cdot 7 ,$$

and

$$2^3 5 < 2 \cdot 3 \cdot 7$$

we necessarily have

$$32 = 2^5, \quad 40 = 2^3 \cdot 5 .$$

All remaining λ in the range under consideration have $\hat{\mu}(\lambda) = 1$, and from tables 1 and 2 possible multiples of 2, 3, 5, 7, .. to fit the positions are

$$2 \cdot 17, \quad 2 \cdot 19, \; 2 \cdot 23, \quad \ldots$$

$$3 \cdot 11, \quad 3 \cdot 13, \; 3 \cdot 17, \quad \ldots$$

$$5 \cdot 7,$$

respectively.

But $\quad 2 \cdot 17 < 2^2 \cdot 3^2 = 36 ,$

$$3 \cdot 11 < 2^2 \cdot 3^2 = 36 ,$$

$$2 \cdot 19 > 2^2 \cdot 3^2 ,$$

$$3 \cdot 13 > 2^2 \cdot 3^2 ,$$

and $\quad 2 \cdot 23 > 2 \cdot 21 = 2 \cdot 3 \cdot 7 .$

Hence

$$\{33, 34, 35\} = \{2 \cdot 17, 3 \cdot 11, 5 \cdot 7\},$$

and

$$\{38, 39\} = \{2 \cdot 19, 3 \cdot 13\} .$$

Thus completing the stage 6 entries of table 3.

<u>Stage 7:</u> Verification that $\lambda_{24} = \pi_1{}^3 \cdot \pi_2$ and $\lambda_{25} = \pi_3{}^2$, and factorization of λ_n for $1 \leqslant n \leqslant 26$.

Table 4

λ_n	$\hat{\mu}(\lambda_n)$	$z(\lambda_n)$	$P(\lambda_n)$
41	-1	14	13
42	-1	14	13
43	-1	14	13
44	0	15	13
45	0	16	13
46	1	16	14
47	-1	16	14
48	0	17	14
49	0	18	14
50	0	19	14
51	1	19	15
52	0	20	15
53	-1	20	15
54	0	21	15
55	1	21	16
56	0	22	16
57	1	22	17
58	1	22	18
59	-1	22	18
60	0	23	18
61	-1	23	18
62	1	23	19
63	0	24	19
64	0	25	19
65	1	25	20
66	-1	25	20
67	-1	25	20
68	0	26	20

$$z(2,\ 2^6) = \text{Max}\ \{n\colon\ 2^2\cdot n \leqslant 2^6\}$$
$$= 16,$$

Since $3^2\cdot 6 < 2^6,\quad 3^2\cdot 8 > 2^6,$

$$z(3,\ 2^6) = \text{Max}\ \{n\colon\ 3^2 n \leqslant 2^6\}$$

$$6,\quad \text{if}\quad 3^2\cdot 7 > 2^6$$
$$7,\quad \text{if}\quad 3^2\cdot 7 \leqslant 2^6\ .$$

Since $2\cdot 5^2 < 2^6,$

and

$$3\cdot 5^2 > 2\cdot 7\cdot 5 > 2\cdot 2^5 = 2^6,$$

$$z(5,\ 2^6) = \text{Max}\ \{n\colon\ 5^2 n \leqslant 2^6\}$$
$$= 2.$$

Since $6^2 \leqslant 2^6,$

and

$$6^2\cdot 2 > 2^6,$$

$$z(6,\ 2^6) = \text{Max}\ \{n\colon\ 6^2 n \leqslant 2^6\}$$
$$= 1.$$

Since $7^2 < (2^3)^2 < 2^6,$

and

$$2\cdot 7^2 > 2\cdot 6^2 > 2^6,$$

$$z(7,\ 2^6) = \text{Max}\ \{n\colon\ 7^2 n \leqslant 2^6\}$$
$$= 1.$$

Since $10^2 > 10\cdot 8 > 8\cdot 8 = 2^6,$

$$z(10,\ 2^6) = 0\ .$$

Consequently

$$z(2^6) = -\sum_{r=1}^{7} \mu(r) z(r,\ 2^6)$$

$$= \begin{cases} 24 & \text{if} \quad 2^6 < 3^2 \cdot 7 , \\ 25 & \text{if} \quad 2^6 > 3^2 \cdot 7 . \end{cases}$$

Noting that $\hat{\mu}(2^6) = 0$ we see from table 3 that

$$2^6 \in \{63, 64\},$$

and hence

(14) $$P(2^6) = 19.$$

However, from table (3),

$$p_1(2^6) = \text{Max } \{n-1 : 2\pi_n \leq 2^6\}$$
$$= 10.$$

Since

$$3 \cdot 19 < 3 \cdot 2^2 \cdot 5 < 2^6,$$

and

$$3 \cdot 23 > 3 \cdot 2 \cdot 11 > 2^6 ,$$

$$p_2(2^6) = \text{Max } \{n-2 : 3\pi_n \leq 2^6\}$$
$$= 6.$$

Since

$$5 \cdot 11 < 5 \cdot 2^2 \cdot 3 < 2^6,$$

and

$$5 \cdot 17 > 2^2 \cdot 2^4 = 2^6 ,$$

$$p_3(2^6) = \text{Max } \{n-3 : 5\pi_n \leq 2^6\}$$

$$= \begin{cases} 2 & \text{if} \quad 2^6 < 5 \cdot 13 , \\ 3 & \text{if} \quad 5 \cdot 13 < 2^6. \end{cases}$$

Since

$$7 \cdot 11 > 7 \cdot 2 \cdot 5 > 2^6,$$

$$p_4(2^6) = 0.$$

Hence

$$P(2^6) = 1 + p_1(2^6) + p_2(2^6) + p_3(2^6)$$

(15) $$= \begin{cases} 18, & \text{if } 2^6 < 5 \cdot 13 \\ 19, & \text{if } 5 \cdot 13 < 2^6 \end{cases}$$

From (14) and (15) we thus have

$$2^6 < 5 \cdot 13.$$

Also, from table 3,

$$3 \cdot 13 < 2^3 \cdot 5.$$

Hence,

$$2^6 \cdot 3 \cdot 13 < 2^3 \cdot 5^2 \cdot 13,$$

and so

$$2^3 \cdot 3 < 5^2.$$

Thus, from table 2,

$$24 = 2^3 \cdot 3,$$

$$25 = 5^2,$$

and thus proposition 4 of section 14 is established.

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Errata

1. "Titchmarch" should be spelled "Titchmarsh" throughout.

2. p.40, line 8: For "(iv), (v) and (iv)" read "(iv, (v) and (vi)."

3. p.47, line 13: For "$c : \mathbb{C} \to \mathbb{N}$" read "$c : \mathbb{N} \to \mathbb{C}$."

4. p.55, line 4 from bottom: For "weaking" read "weakening".

5. p.73, line 8 from bottom: For "Berndt [2]" read "Berndt [1]".

6. p.115, line 13: For $\lambda_n\lambda_p < \lambda_n\lambda_p$ read $\lambda_n\lambda_p < \lambda_n\lambda_q$.

7. p.118, line 2 from bottom: For ν_k read $\nu(k)$.

8. p.126, line 11 and p.148 last line: For "section 14" read "section 15".

9. p.149, Edwards [1]: For "functions" read "function".

10. p.150, LeVeque [1]: For 335-367 read 355-367.