Almost weak Asplund spaces

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Abstract

Continuous convex functions have long been known to be generically differentiable on Euclidean spaces. However, in 1968 Asplund decided to investigate and classify those Banach spaces that possess this Euclidean space property. Specifically, Asplund investigated those Banach spaces on which every continuous convex function is Gâteaux (Fréchet) differentiable at the points of a residual subset of their domain. Such spaces are now known as weak Asplund (Asplund) spaces. While the study of Asplund spaces has flourished into a beautifully detailed story admitting several characterisations, the corresponding theory for weak Asplund spaces has been rather thin. In particular, there are no known characterisations for this class of spaces.

In one attempt to provide such a characterisation, the class of Gâteaux differentiability spaces, that is, those spaces on which every continuous convex function is densely Gâteaux differentiable, was introduced. While it follows from the definition that every weak Asplund space is a Gâteaux differentiability space, it has been a long standing open problem, first asked by Larman and Phelps in 1979, as to whether every Gâteaux differentiability space is a weak Asplund space.

The main goal of this thesis is to construct a Gâteaux differentiability space that is not weak Asplund, thus answering the question of Larman and Phelps from 1979. We achieve this counter-example by considering Banach spaces of the form $(C(K), \|\cdot\|_\infty)$, where K is a compact Hausdorff space.

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Contents

Chapter 1

Introduction

The differentiability properties of continuous convex functions defined on Banach spaces have been studied for many years and numerous results exist in the literature on the existence of points of Gâteaux and Fréchet differentiability of continuous convex functions defined on nice spaces. One avenue of this research is the classification of Banach spaces according to the differentiability properties of continuous convex functions defined on them. For example, the classification of those Banach spaces that satisfy the Euclidean space property that every continuous convex function defined on them is Gâteaux differentiable everywhere, except at the points of a first category set of their domain. Such spaces are called *weak Asplund spaces.* The "main problem" in the study of weak Asplund spaces is to provide an intrinsic characterisation for this class of spaces. In one such attempt, the following class of Banach spaces was considered. A Banach space X is called a *Gâteaux differentiability space* if every continuous convex function defined on it is Gâteaux differentiable at the points of a dense subset of its domain. While it follows from the definition that every weak Asplund space is a Gâteaux differentiability space, the status of the reverse implication is not so clear, and is in fact, the main point of this thesis. The significance of the reverse implication stems from the fact that Gâteaux differentiability spaces admit an intrinsic characterisation in terms of a geometric property of their dual spaces, (Phelps, 1993). Hence, if it could be shown that the class of Gâteaux differentiability spaces coincides with the class of weak Asplund spaces, then there would be a solution to the "main problem". Unfortunately, in Chapter 4 we show that these two classes of spaces are in fact distinct.

We start our quest for a counter-example that will distinguish the weak Asplund spaces from the Gâteaux differentiability spaces in Chapter 1 by reviewing the history of weak Asplund spaces. In doing so we shall introduce several classes of Banach spaces and explore their relationship to the class of weak Asplund spaces.

We begin Chapter 2 with some basic properties of usco mappings including the characterisation of an usco mapping being a "minimal usco". In the second section of Chapter 2 we study "weakly Stegall" spaces that will help us distinguish the weak Asplund spaces from the Gâteaux differentiability spaces. We also show that weakly Stegall spaces admit an internal characterisation, unlike the known situation for Stegall spaces, by considering a two-player topological game. Following this, we give some sufficiency conditions for a topological space to be a weakly Stegall space. In the third section of this chapter, we study the Banach space counterpart of weakly Stegall spaces and in the final section, we review the subdifferential mapping and then use it to establish the connection between weakly Stegall spaces and the class of "almost weak Asplund" spaces. Finally, in this chapter, we establish the relationship between weak Asplund spaces and two of its subclasses, namely, $\tilde{\mathcal{F}}$ and class($\tilde{\mathcal{S}}$).

In chapter 3, we define the "Kalenda compacta" and give a characterisation of when they belong to the classes of fragmentable, Stegall and weakly Stegall spaces. Following this, we examine conditions under which the Kalenda compacta are weakly Stegall but not Stegall. Then we provide a counter-example to show that weakly Stegall spaces are not closed under products. Our goal in the second section of this chapter is to obtain a detailed characterisation of the dual of the $C(K_A)$ spaces for compact Hausdorff spaces K_A belonging to the family of Kalenda compacta. Using this representation theorem we characterise when Banach spaces of the form $C(K_A)$ belong to $\tilde{\mathcal{F}}$ and class($\tilde{\mathcal{S}}$). We then use this characterisation of the $C(K_A)$ spaces to give examples of Banach spaces that are (i) in class($\tilde{\mathcal{S}}$) but not in $\tilde{\mathcal{F}}$, (ii) weak

Asplund but not in $\tilde{\mathcal{F}}$ and (iii) weak Asplund but not in class($\tilde{\mathcal{S}}$). In the final section of this chapter we provide an example of a K_A space such that it is weakly Stegall but $(C(K_A)^*, \text{weak}^*)$ is not weakly Stegall.

In Chapter 4, we fulfill the aim of this thesis by showing that there is a Gâteaux differentiability space that is not weak Asplund. In order to attain our goal, in the first section of this chapter we introduce a new class of topological spaces called "nearly Stegall" spaces and study their permanence properties. This will be followed in the next section by showing that there is an almost weak Asplund space that is not weak Asplund.

In Chapter 5, we present some applications of our earlier results on weakly Stegall spaces and almost weak Asplund spaces. This will be followed in Appendix A by a discussion of some of the main open problems in the study of weak Asplund spaces. Then in Appendix B, we present a Rainwater Seminar by Isaac Namioka and finally in Appendix C, we conclude this thesis with basic notations and a glossary of definitions.

1.1 Historical overview

Let us start by giving the following definitions which are central to the work covered in the rest of this thesis. As far as possible we have left all the technicalities of proofs to the upcoming chapters. We say that a non-empty subset A of a normed linear space X is *convex* if for each pair of points $x, y \in A$ the line segment joining the points x and y is also contained in A. That is,

$$\lambda x + (1-\lambda)y \in A \quad \text{for all } x, y \in A \text{ and } \lambda \in [0,1].$$

A real-valued function $\varphi : A \to \mathbb{R}$ defined on a non-empty open convex subset A of a normed linear space X is said to be *convex* if,

$$\varphi(\lambda x + (1-\lambda)y) \leq \lambda\varphi(x) + (1-\lambda)\varphi(y) \quad \text{for all } x, y \in A \text{ and } \lambda \in [0,1].$$

Throughout the remainder of this thesis we confine our attention to real normed linear spaces.

We now examine two distinct notions of differentiability that are associated with continuous convex functions defined on non-empty open convex subsets of normed linear spaces. Let A be a non-empty open convex subset of a normed linear space X. We say that $\varphi : A \to \mathbb{R}$ is *Gâteaux differentiable* at a point $x \in A$ if there exists a continuous linear functional $x^* \in X^*$ such that

$$x^*(y) := \lim_{\lambda \to 0} \frac{\varphi(x + \lambda y) - \varphi(x)}{\lambda} \text{ exists for all } y \in X.$$

In this case, the linear functional x^* is called the *Gâteaux derivative of* φ *at* x. If the limit above is approached uniformly with respect to all $y \in B_X$-the closed unit ball of X, then φ is said to be *Fréchet differentiable* at the point $x \in A$ and x^* is called the *Fréchet derivative of* φ *at* x.

The first result concerning the differentiability of convex functions is the classical result that every continuous convex function defined on $\mathbb{R}$ is differentiable everywhere except for (at most) a countable set, (Phelps, 1993, p.9). This result generalises to $\mathbb{R}^n$ (either by using Fubini's theorem or the Kuratowski-Ulam theorem) except that one must replace "countable set" by either "measure zero set" or "first category set", (Phelps, 1993, p.11). Indeed even the continuous convex function $(x, y) \mapsto |x|$ is non-differentiable on an uncountable set. The question of how to extend these results to infinite dimensional spaces has several difficulties. For instance, what does one mean by "except for a small set" in infinite dimensions? The two natural candidates are null sets (in a measure theoretic sense) and first category sets (in a Baire categorical sense). The first notion leads to difficulties as there is no natural notion of Haar-measure on an infinite dimensional Banach space (as they are not locally compact). However, the other notion of "smallness" readily lends itself to infinite dimensions. Indeed, using this approach Mazur showed in 1933 that every continuous convex function defined on a separable Banach space is Gâteaux differentiable everywhere except for (at most) a first category subset of its domain. This begged the question as to whether every Banach space satisfies the property that every continuous convex function defined on it is Gâteaux differentiable everywhere except at the points of a first category subset. However, it is not hard to find counter-examples to this question. For example, in (Larman and Phelps, 1979) the authors provide an example of a continuous seminorm that is nowhere Gâteaux differentiable.

Example 1.1 *Define* $p : l^{\infty}(\mathbb{N}) \to \mathbb{R}$ *by,*

$$p(x) := \limsup_{n\to\infty} |x_n|, \text{ where } x = (x_n : n \in \mathbb{N}).$$

Then p *is continuous but nowhere Gâteaux differentiable.*

Mazur's result was revisited three decades later in (Lindenstrauss, 1963) where it was shown that for separable reflexive Banach spaces the conclusions of Mazur's theorem may be strengthened to: Fréchet differentiable everywhere except for (at most) a first category set.

The next major achievement, which has forever changed the way people study this subject is due to Asplund. His idea was to classify Banach spaces according to the differentiability properties of the continuous convex functions defined on them. He called the Banach spaces on which every continuous convex function defined on them is Gâteaux differentiable everywhere except at the points of a first category set, *weak differentiability spaces* while he called the Banach spaces on which every continuous convex function defined on them is Fréchet differentiable everywhere except on a first category set, *strong differentiability spaces*, (Asplund, 1968). These spaces have subsequently become known as *weak Asplund spaces*, (Larman and Phelps, 1979) and *Asplund spaces*, (Namioka and Phelps, 1975) respectively.

The significance of Asplund's approach is that it connects the differentiability of continuous convex functions to the geometrical and topological structure of Banach spaces. Since Fréchet differentiability implies Gâteaux differentiability, we trivially see that every Asplund space is a weak Asplund space. But the converse is not true. For example, l_1 is not an Asplund space as its norm is nowhere Fréchet differentiable, (Phelps, 1960) but it follows from Mazur's theorem that l_1 is weak Asplund, since it is separable.

In Asplund's ground breaking paper from 1968, he also showed that every Banach space that can be renormed to have an equivalent rotund (equivalent locally uniformly rotund) dual norm is a weak differentiability (strong differentiability) space. Recall that a norm on a Banach space X is said to be *rotund* provided there are no line segments in the unit sphere of X, or equivalently, for any $x, y \in S_X$-the unit sphere of X if, $x \neq y$ then $\|x+y\| < 2$. We say that a norm on a Banach space X is *locally uniformly rotund* if $\|x_n - x\| \to 0$ whenever $2\|x\|^2 + 2\|x_n\|^2 - \|x + x_n\|^2 \to 0$ with $x_n, x \in X$ for all $n \in \mathbb{N}$.

The result of Asplund on weak differentiability spaces is a true improvement upon Mazur's result since every separable Banach space can be renormed to have an equivalent rotund dual norm, (Deville et al., 1993) but not every space with rotund dual is separable. For example, every Hilbert space belongs to this category. His result on strong differentiability spaces is also a true improvement upon Lindenstrauss' result since every separable reflexive Banach space can be renormed to have an equivalent locally uniformly rotund dual norm, (Deville et al., 1993) but not every space with locally uniformly rotund dual is separable and reflexive. For example, non-separable Hilbert spaces belong to this category.

Since the time of Asplund a considerable volume of literature has been written on Asplund spaces. In fact one of the major achievements of functional analysis in the late 1970's was when Stegall showed that a Banach space is an Asplund space if, and only if, its dual has the Radon-Nikodým property, (Stegall, 1978). This was the culmination of the results of (Namioka and Phelps, 1975) where the authors provided an elegant characterisation of Asplund spaces in terms of a geometric property of their dual spaces and also of the results of Stegall in (Stegall, 1975) where he proved the striking result that a Banach space X is an Asplund space if, and only if, every separable closed subspace of X has a separable dual. Subsequent to this many other characterisations of Asplund spaces have been discovered (see for example (Bourgin, 1983), (Diestel and J. J. Uhl, 1977), (Giles, 1982) and (Phelps, 1993)).

By contrast, our knowledge of weak Asplund spaces is rather thin. Even the

obvious question as to whether a closed subspace of a weak Asplund space is itself a weak Asplund space is open and, as yet, there do not exist any characterisation for this class of spaces. Thus the study of weak Asplund spaces has been reduced to pursuing two paths. One path is to try to find classes of Banach spaces that are potentially equivalent to weak Asplund spaces (Path I) and the other is to search for sufficient conditions for a Banach space to be a weak Asplund space (Path II).
Path I: In the hope of characterising the class of weak Asplund spaces several classes of Banach spaces and topological spaces have been defined. In one such attempt the authors in (Larman and Phelps, 1979) considered the following class of Banach spaces. A Banach space X is called a *Gâteaux differentiability space* (or GDS for short) if every continuous convex function defined on it is Gâteaux differentiable at the points of a dense, but not necessarily a G_δ subset of X. While it follows from the definition that every weak Asplund space is a Gâteaux differentiability space, the status of the reverse implication has only been recently established, (Moors and Somasundaram, 2004). The importance of Gâteaux differentiability spaces (apart from their obvious similarity to weak Asplund spaces) comes from the fact that they admit a dual characterisation analogous to that of Asplund spaces. The proof of this characterisation was done in two parts. The first in (Larman and Phelps, 1979) and the second by Fabian in 1988 (unpublished).

Theorem 1.2 (Phelps, 1993, Proposition 6.5) *A Banach space X is a Gâteaux differentiability space if, and only if, every non-empty weak* compact convex subset of X^* is the weak* closed convex hull of its weak* exposed points.*

Recall that a point x^* in a weak* compact convex subset C of X^* is *weak* exposed* if there exists an element $x \in X$ such that $x^*(x) > y^*(x)$ for all $y^* \in C\backslash\{x^*\}$. That is, the weak* continuous linear functional $\hat{x} : X^* \to \mathbb{R}$ defined by, $\hat{x}(y^*) := y^*(x)$ attains its maximum value on C at the single point x^*.

In another attempt to characterise the class of weak Asplund spaces, Stegall introduced the following class of topological spaces, which are defined in terms of minimal uscos. A set-valued mapping $\varphi : X \to 2^Y$ acting between topological spaces

X and Y is called an *usco* mapping if for each $x \in X$, $\varphi(x)$ is a non-empty compact subset of Y and for each open set W in Y, $\{x \in X : \varphi(x) \subseteq W\}$ is open in X. An usco mapping $\varphi : X \to 2^Y$ is called a *minimal usco* if its graph does not contain, as a proper subset, the graph of any other usco defined on X. (Recall that by the graph of φ we mean, $\mathrm{Gr}(\varphi) := \{(x, y) \in X \times Y : y \in \varphi(x)\}$.) A topological space X is said to belong to *Stegall spaces* if for every Baire space B and minimal usco $\varphi : B \to 2^X$, φ is single-valued at the points of a residual subset, that is, a set which contains, as a subset, the countable intersection of dense open sets, of B (see (Stegall, 1983) for the original definition of Stegall spaces). Correspondingly, we say that a Banach space X belongs to *class*$(\tilde{\mathcal{S}})$ if (X^*, weak^*) belongs to Stegall spaces.

The class of Stegall spaces has been shown to possess good permanence properties, unlike the class of weak Asplund spaces (Fabian, 1997). These spaces are closed under taking subspaces, countable products, countable union of closed sets and perfect images. It has been shown in (Stegall, 1983) that every member of class$(\tilde{\mathcal{S}})$ is weak Asplund (see also Theorem 2.30). To date, the class$(\tilde{\mathcal{S}})$ spaces form the largest known well-behaved subclass of the class of weak Asplund spaces.

A further class of topological spaces that has played a significant role in the study of weak Asplund spaces is the class of "fragmentable" spaces. The notion of fragmentability is essentially the same as the notion of dentability and the latter first appeared in (Namioka and Phelps, 1975). The formal definition of "fragmentability" was given in (Jayne and Rogers, 1985) in regard to selection theorems. However, this class of spaces also has implications for the study of weak Asplund spaces. A topological space X is said to be *fragmentable* if there exists a metric d on X such that for every $\varepsilon > 0$ and every non-empty set A of X there exists a non-empty subset B of A that is relatively open in A and d-diam$(B) < \varepsilon$. Correspondingly, we say that a Banach space X belongs to $\tilde{\mathcal{F}}$ if (X^*, weak^*) is fragmentable. The connection between fragmentable and Stegall spaces was first established in (Namioka, 1983) where it was shown that every fragmentable space is a Stegall space. In particular, $\tilde{\mathcal{F}} \subseteq$ class$(\tilde{\mathcal{S}})$. This result was first published in (Ribarska, 1987) (see also

Theorem 2.31).

Path II: There exist several sufficiency conditions for a Banach space to be weak Asplund. It has been shown in (Amir and Lindenstrauss, 1968) that weakly compactly generated spaces, that is, Banach spaces which contain a weakly compact set whose linear span is dense in them, can be equivalently renormed to have a rotund dual and hence are weak Asplund. Later Stegall showed that Asplund generated spaces, that is, spaces which contain dense continuous linear images of Asplund spaces, are weak Asplund, (Stegall, 1981) and this result was improved in (Christensen and Kenderov, 1984) to show that subspaces of Asplund generated spaces are weak Asplund.

It has been shown in (Debs, 1985) that if X is a weakly $\mathcal{K}$-analytic space, that is, there exists an usco mapping from $\mathbb{N}^{\mathbb{N}}$ onto (X, weak), then X belongs to class($\tilde{\mathcal{S}}$). Thus weakly $\mathcal{K}$-analytic spaces are weak Asplund. This improves the previously mentioned result of Amir and Lindenstrauss since weakly compactly generated spaces are weakly $\mathcal{K}$-analytic.

A Banach space X is called *Vašák* or *weakly countably determined* if there exists an usco mapping from a subset of $\mathbb{N}^{\mathbb{N}}$ onto (X, weak). It has been shown in (Kenderov, 1987) that Vašák spaces (or weakly countably determined spaces) are weak Asplund. This result was improved in (Ribarska, 1987) where it was shown that Vašák spaces belong to $\tilde{\mathcal{F}}$, thus strengthening the result of (Debs, 1985) mentioned above since weakly $\mathcal{K}$-analytic spaces are weakly countably determined.

After Asplund showed that every Banach space that can be equivalently renormed to have a rotund dual norm is weak Asplund, there was a considerable amount of interest in showing that every Banach space which admits an equivalent Gâteaux smooth norm, that is, a norm which is Gâteaux differentiable at every non-zero point, is weak Asplund. The fact that this would provide a true generalisation of Asplund's result follows from the fact that there are Banach spaces (e.g., $C[0, \omega_1]$) with equivalent smooth norms that cannot be equivalently renormed to have a rotund dual norm, (Talagrand, 1986). (Of course it is well-known that if the dual norm is rotund then the original norm is smooth, (Giles, 1982, p.107).) The first partial solution to this was obtained in (Borwein and Preiss, 1987) when they showed using variational techniques that every smooth Banach space is a Gâteaux differentiability space. Finally in 1990, twenty two years years after Asplund's result, it was eventually shown by Preiss, Phelps and Namioka that every smooth Banach space is weak Asplund.

Theorem 1.3 (Preiss et al., 1990) *If a Banach space X admits an equivalent smooth norm then X belongs to class*$(\tilde{\mathcal{S}})$*. In particular, X is a weak Asplund space.*

The question as to whether every weak Asplund space is a smooth Banach space was answered negatively in (Haydon, 1990). The techniques of Preiss, Phelps and Namioka were used in (Ribarska, 1992) to improve Theorem 1.3 to show that every Banach space with a Gâteaux smooth norm belongs to $\tilde{\mathcal{F}}$. Subsequently, this result has been further improved to show that every Banach space that admits a smooth Lipschitz bump function belongs to $\tilde{\mathcal{F}}$, (Fosgerau, 1992). A similar result may also be found in (Kortezov, 1998).

In summary, the relationship between the classes: weak Asplund spaces, Gâteaux differentiability spaces, $\tilde{\mathcal{F}}$ and class$(\tilde{\mathcal{S}})$ is as follows:

$$\tilde{\mathcal{F}} \subseteq \text{class}(\tilde{\mathcal{S}}) \subseteq \text{weak Asplund spaces} \subseteq \text{Gâteaux differentiability spaces.}$$

The question as to whether any of the set inclusions above can be reversed has seen some recent progress. It has been shown that under additional set theoretic assumptions that there are indeed examples of class$(\tilde{\mathcal{S}})$ that do not belong to $\tilde{\mathcal{F}}$, (Kenderov et al., 2001b) and examples of weak Asplund spaces that do not belong to class$(\tilde{\mathcal{S}})$, (Kalenda, 2002). In Chapter 4, we provide a counter-example (in ZFC) to show that the last set inclusion given above cannot be reversed, (Moors and Somasundaram, 2004).

Chapter 2

Weakly Stegall spaces

In the study of weak Asplund spaces the class of "weakly Stegall" spaces introduced in (Kalenda, 1997) has proven to be very useful. We say that a topological space X is *weakly Stegall* if for every complete metric space M and minimal usco $\varphi : M \to 2^X$, φ is single-valued at some point of M. The study of this class of topological spaces is very natural, especially in the setting of Banach spaces. Since for a Banach space X if (X^*, weak^*) is weakly Stegall then X is a Gâteaux differentiability space (see Theorem 2.29) but not necessarily a weak Asplund space.

The study of weakly Stegall spaces relies upon the study of usco mappings. Therefore we begin this chapter by investigating the basic properties of usco mappings that were defined in Chapter 1.

The aim of the second section of this chapter is to study weakly Stegall spaces. In doing so we shall introduce a topological game that is related to weakly Stegall spaces and use this game to characterise those spaces that are weakly Stegall. We then explore some permanence properties of this class of spaces and provide some sufficient conditions for a topological space to be a weakly Stegall space.

In the third section of this chapter, we study the Banach space counterpart of weakly Stegall spaces. Then, in the final section we introduce the class of "almost weak Asplund" spaces which lies somewhere between the class of weak Asplund spaces and the class of Gâteaux differentiability spaces. We then use the subdifferential mapping associated with a continuous convex function to give the relationship between weakly Stegall spaces and almost weak Asplund spaces.

2.1 Usco and minimal usco mappings

A set-valued mapping $\varphi : X \to 2^Y$ acting from a topological space X into subsets of a topological space (Y, τ) is τ-*upper semicontinuous* if for every open set W in Y, $\{x \in X : \varphi(x) \subseteq W\}$ is open in X. A set-valued mapping $\varphi : X \to 2^Y$ acting from a topological space X into subsets of a topological space Y is called a τ-*usco* if φ is τ-upper semicontinuous and for each $x \in X$, $\varphi(x)$ is a non-empty τ-compact subset of Y. When there is no ambiguity concerning the topology, we shall simply call a τ-usco an usco.

Among the usco mappings we are particularly interested in the subclass of "minimal uscos". An usco mapping $\varphi : X \to 2^Y$ acting from a topological space X into subsets of a topological space Y is called a *minimal usco* if its graph does not contain, as a proper subset, the graph of any other usco defined on X. It is clear from this definition that all single-valued uscos are minimal. However, most interesting minimal uscos are not everywhere single-valued. Two simple examples that illustrate this are the following.

$$\varphi(x) := \begin{cases} \sin\left(\frac{1}{x}\right) & \text{for } x \neq 0 \\ [-1, 1] & \text{for } x = 0 \end{cases}$$

or more generally,

$$\varphi(x) := \begin{cases} \sin\left(\frac{1}{\mathrm{d}(x,C)}\right) & \text{for } x \in [0,1] \backslash C \\ [-1, 1] & \text{for } x \in C \end{cases}$$

where C is any compact, nowhere dense subset of $\mathbb{R}$.

Remark 2.1 *Thus we may note that for a minimal usco on* $\mathbb{R}$, $\{x \in \mathbb{R} : \varphi(x)$ *is not a singleton*$\}$ *may have positive measure.*

Throughout the rest of this thesis we shall be interested in the topological behaviour of minimal uscos. So we shall gather some known facts about minimal uscos which will be needed in the upcoming chapters. We begin with the following well-known characterisation of the minimality of usco mappings, which was first given implicitly in (Christensen, 1982).

Lemma 2.2 (Christensen, 1982) *Let $\varphi : X \to 2^Y$ be an usco mapping acting from a topological space X into subsets of a Hausdorff topological space Y. Then φ is a minimal usco if, and only if, for any open set V in X and closed set K in Y where $\varphi(V) \nsubseteq K$ there exists a non-empty open set $V' \subseteq V$ such that $\varphi(V') \cap K = \emptyset$.*

Proof. (Moors, 1992) Suppose that φ is a minimal usco such that $\varphi(V) \nsubseteq K$ and for each non-empty open subset $V' \subseteq V$ we have that $\varphi(V') \cap K \neq \emptyset$. Then by the upper semicontinuity of φ we must have that $\varphi(x) \cap K \neq \emptyset$ for each $x \in V$. Now let us define a set-valued mapping $\psi : X \to 2^Y$ by,

$$\psi(x) := \begin{cases} \varphi(x) \cap K & \text{for } x \in V \\ \varphi(x) & \text{otherwise.} \end{cases}$$

This is an usco whose graph is contained in the graph of φ. Therefore, by the minimality of φ, we must have that $\psi = \varphi$ and so $\varphi(V) \subseteq K$. This is a contradiction as $\varphi(V) \nsubseteq K$. Hence there must be a non-empty open subset V' of V such that $\varphi(V') \cap K = \emptyset$.

To prove the converse let us suppose that φ satisfies the condition given in the lemma but is not a minimal usco. This means that there is an usco mapping $\psi : X \to 2^Y$ whose graph is strictly contained in the graph of φ. So for some $x \in X$, $\varphi(x) \neq \psi(x)$. Consider $y \in \varphi(x) \backslash \psi(x)$. Since $\psi(x)$ is compact and Y is a Hausdorff space, there exist disjoint open subsets U and W such that $\psi(x) \subseteq U$ and $y \in W$. Now since ψ is upper semicontinuous at x there exists an open neighbourhood V of x such that $\psi(V) \subseteq U$. On the other hand $\varphi(V) \nsubseteq Y \backslash W$. Therefore by the hypothesis $\varphi(V') \subseteq W$ for some non-empty open subset $V' \subseteq V$. However, for any $x \in V'$ we have that $\emptyset \neq \psi(x) = \psi(x) \cap U \subseteq \varphi(x) \cap U = \emptyset$. This is a contradiction and this proves that φ is a minimal usco. □

It can easily be seen that Lemma 2.2 is equivalent to Lemma 2.3.

Lemma 2.3 *Let $\varphi : X \to 2^Y$ be an usco mapping acting from a topological space X into subsets of a Hausdorff topological space Y. Then φ is a minimal usco if, and only if, for each pair of open subsets U of X and W of Y with $\varphi(U) \cap W \neq \emptyset$ there exists a non-empty open subset $V \subseteq U$ such that $\varphi(V) \subseteq W$.*

2.1.1 Properties of usco mappings

We shall now discuss some basic properties of minimal usco mappings. We begin by showing that the minimality of an usco mapping is preserved under composition with a continuous function.

Lemma 2.4 (Namioka, 1983) *Let φ be a minimal usco acting from a topological space X into subsets of a Hausdorff topological space Y and let f be a continuous mapping from Y into a Hausdorff topological space Z. Then the mapping $f \circ \varphi : X \to 2^Z$ defined by, $(f \circ \varphi)(x) := f(\varphi(x))$ is a minimal usco on X.*

Proof. (Borwein and Moors, 1995) It is clear that $f \circ \varphi$ is an usco on X so it remains to show that it is a minimal usco on X. Let us proceed via Lemma 2.2. Consider an open set U in X and a closed set K in Z such that $(f \circ \varphi)(U) \not\subseteq K$. We need to show that $(f \circ \varphi)(V) \cap K = \emptyset$ for some non-empty open subset $V \subseteq U$. Since f is continuous on Y, $f^{-1}(K)$ is a closed subset of Y. Since φ is a minimal usco and $\varphi(U) \not\subseteq f^{-1}(K)$ there exists a non-empty open subset $V \subseteq U$ such that $\varphi(V) \cap f^{-1}(K) = \emptyset$. Hence $(f \circ \varphi)(V) \cap K = \emptyset$ as required. $\square$

Lemma 2.5 (Moors and Giles, 1997, Theorem 1.2, part (iii)) *Let φ be a minimal usco acting from a topological space X into subsets of a Hausdorff topological group $(Y, \cdot)$. Given a continuous mapping T from X into Y, the set-valued mapping $T \cdot \varphi : X \to 2^Y$ defined by, $(T \cdot \varphi)(x) := T(x) \cdot \varphi(x)$ is a minimal usco on X.*

Proof. The mapping $T \cdot \varphi$ is the composition of the continuous mapping $\Pi : Y \times Y \to Y$ defined by, $\Pi(x, y) := x \cdot y$ with the minimal usco mapping $t \mapsto (T(t), \varphi(t))$ from X into $Y \times Y$, and so $T \cdot \varphi$ is a minimal usco on X by Lemma 2.4. $\square$

The fact that every usco contains a minimal usco is shown in the next lemma. We note that Lemma 2.6 was mentioned in (Christensen, 1982) but not explicitly proven there.

Lemma 2.6 (Christensen, 1982) *Let* $\varphi : X \to 2^Y$ *be an usco mapping acting from a topological space* X *into subsets of a Hausdorff topological space* Y*. Then there exists a minimal usco* $\psi : X \to 2^Y$ *such that* $\psi(x) \subseteq \varphi(x)$ *for all* $x \in X$ *(i.e., every usco contains a minimal usco).*

Proof. (Borwein and Moors, 1995, Proposition 1.2) Let $\mathcal{U}$ denote the family of all usco mappings defined on X whose graphs are contained in the graph of φ. Obviously $\mathcal{U} \neq \emptyset$ as the mapping φ is contained in $\mathcal{U}$. We may now partially order $\mathcal{U}$ as follows. If ψ_1 and ψ_2 are members of $\mathcal{U}$, then we write $\psi_1 \leq \psi_2$ if $\psi_1(x) \subseteq \psi_2(x)$ for each $x \in X$. Next we apply Zorn's lemma to show that $(\mathcal{U}, \leq)$ possesses a minimal element. To this end, let $\{\psi_\gamma : \gamma \in \Gamma\}$ be a totally ordered subset of $\mathcal{U}$ and let $\varphi_M : X \to 2^Y$ be defined by, $\varphi_M(x) := \bigcap\{\psi_\gamma(x) : \gamma \in \Gamma\}$. Since each $\psi_\gamma(x)$ is non-empty and compact, $\varphi_M(x)$ too is non-empty and compact. Let W be an open subset of Y and consider $U := \{x \in X : \varphi_M(x) \subseteq W\}$. We need to show that U is open in X. We may, without loss of generality, assume that $U \neq \emptyset$ and consider $x_0 \in U$.

By the finite intersection property, there exists some $\gamma_0 \in \Gamma$ such that $\psi_{\gamma_0}(x_0) \subseteq W$. Therefore there exists an open neighbourhood U_0 of x_0 such that $\psi_{\gamma_0}(U_0) \subseteq W$, which means that $\varphi_M(U_0) \subseteq W$. Therefore $x_0 \in U_0 \subseteq U$ and so U is open in X. From this, it follows that $\varphi_M \in \mathcal{U}$ and $\varphi_M \leq \psi_\gamma$ for each $\gamma \in \Gamma$. Therefore, by Zorn's lemma, $(\mathcal{U}, \leq)$ possesses a minimal element. It is now easy to see that this element is in fact a minimal usco. □

We see that for an usco mapping φ acting from a topological space X into subsets of a Hausdorff topological space Y, the graph of φ is a closed subset of $X \times Y$, when $X \times Y$ is endowed with the product topology. We shall now show that to some extent the converse of this is also true.

Lemma 2.7 (Christensen, 1982) *Let φ be an usco mapping acting from a topological space X into subsets of a topological space Y and let Ω be a set-valued mapping acting from X into non-empty subsets of Y. If* $\mathrm{Gr}(\Omega)$ *is a closed subset of $X \times Y$ and* $\mathrm{Gr}(\Omega) \subseteq \mathrm{Gr}(\varphi)$*, then Ω is an usco mapping on X.*

Proof. (Borwein and Moors, 1995, Proposition 1.3) Since the graph of Ω is closed in $X \times Y$, $\Omega(x)$ is a closed subset of Y for each $x \in X$, and since $\Omega(x) \subseteq \varphi(x)$ for each $x \in X$, $\Omega(x)$ is a compact subset of Y for each $x \in X$. Let W be an open subset of Y. Consider $U := \{x \in X : \Omega(x) \subseteq W\}$. We will show that U is an open subset of X. Firstly, observe that without loss of generality, we may assume that $U \neq \emptyset$. To this end, suppose that $x_0 \in U$. We shall consider two cases.

(i) If $\varphi(x_0) \subseteq W$ then there exists an open neighbourhood U_0 of x_0 such that $\Omega(U_0) \subseteq \varphi(U_0) \subseteq W$, and so $x_0 \in U_0 \subseteq U$.

(ii) On the other hand, suppose that $K := \varphi(x_0) \backslash W \neq \emptyset$. Then for each $y \in K$, choose open sets $U_y \subseteq X$ and $V_y \subseteq Y$ such that $(x_0, y) \in U_y \times V_y$ and $(U_y \times V_y) \cap \mathrm{Gr}(\Omega) = \emptyset$. Since K is compact and $K \subseteq \bigcup\{V_y : y \in K\}$ there exists a finite subcover $\{V_{y_j} : 1 \leq j \leq n\}$ of $\{V_y : y \in K\}$ such that $K \subseteq \bigcup\{V_{y_j} : 1 \leq j \leq n\}$. Let $N_1 := \bigcap\{U_{y_j} : 1 \leq j \leq n\}$ and observe that for each $x \in N_1$, $\Omega(x) \cap \bigcup\{V_{y_j} : 1 \leq j \leq n\} = \emptyset$. Now, $\varphi(x_0) \subseteq W \cup \bigcup\{V_{y_j} : 1 \leq j \leq n\}$, so there exists an open neighbourhood N_2 of x_0 such that $\Omega(N_2) \subseteq \varphi(N_2) \subseteq \bigcup\{V_{y_j} : 1 \leq j \leq n\} \cup W$. Therefore, $\Omega(x) \subseteq W$ for each $x \in U_0 := N_1 \cap N_2$. Hence in both cases, there exists an open neighbourhood U_0 of x_0 such that $x_0 \in U_0 \subseteq U$. This shows that U is open and the proof is complete. □

We now show that the minimality of an usco mapping is preserved when the mapping is restricted to open or dense subsets of the domain space.

Lemma 2.8 (Kalenda, 1999, Lemma 2) *Let $\varphi : X \to 2^Y$ be a minimal usco acting from a topological space X into subsets of a topological space Y. If $A \subseteq X$ is* (i) *open or* (ii) *dense, then the mapping* φ *restricted to* A*, denoted by* $\varphi|_A$*, is a minimal usco.*

Proof. In both cases, $\varphi|_A$ is clearly an usco mapping. So it remains to show that $\varphi|_A$ is minimal.

(i) Let A be a non-empty open subset of X. To show that $\varphi|_A$ is a minimal usco, let W be an open subset of Y and U be an open subset of A such that $\varphi(U) \cap W \neq \emptyset$. Since $A \subseteq X$ is open, U is open in X. Then it follows from the characterisation of the minimality of φ given in Lemma 2.3 that there is a non-empty open set $V \subseteq U$ such that $\varphi|_A(V) = \varphi(V) \subseteq W$.

(ii) Suppose that A is dense in X, U is an open subset of A and W is an open subset of Y. Let U' be an open subset of X such that $U := U' \cap A \neq \emptyset$. Then $\varphi(U') \cap W \neq \emptyset$ and it follows from the characterisation of the minimality of φ given in Lemma 2.3 that there is a non-empty open set $V' \subseteq U'$ with $\varphi|_A(V' \cap A) \subseteq \varphi(V') \subseteq W$. But since A is dense in X, $V' \cap A$ is non-empty. This completes the proof. □

In the next section we shall begin the study of weakly Stegall spaces that will help us distinguish the Gâteaux differentiability spaces from the weak Asplund spaces.

2.2 Properties of weakly Stegall spaces

In (Fabian, 1997, Theorem 3.2.6) the following equivalent formulation of Stegall spaces was established. A topological space X is said to be a *Stegall space* if for every Baire space B and minimal usco $\varphi : B \to 2^X$, φ is single-valued at some point of B. Thus we see that every Stegall space is weakly Stegall as every complete metric space is a Baire space. Although the definition of being weakly Stegall is very similar to that of being Stegall, weakly Stegall spaces admit an internal characterisation unlike the (known) situation for Stegall spaces. To obtain this characterisation we need to consider the following two-player topological game.

2.2.1 Game characterisation of weakly Stegall spaces

Let $\mathcal{U}$ be any open cover of a topological space (X, τ). On X we consider the $\mathcal{G}(\mathcal{U})$*-game* played between two players Σ and Ω. Player Σ goes first (always!) and chooses a non-empty subset A_1 of X. Player Ω must then respond by choosing a non-empty relatively open subset B_1 of A_1 of the form: $B_1 := B_1^* \cap A_1$ with $B_1^* \in \mathcal{U}$. Following this, player Σ must select another non-empty subset $A_2 \subseteq B_1 \subseteq A_1$ and in turn player Ω must again select a non-empty relatively open subset B_2 of A_2 of the form: $B_2 := B_2^* \cap A_2$ with $B_2^* \in \mathcal{U}$. Continuing this procedure indefinitely the players Σ and Ω produce a sequence $\{(A_n, B_n) : n \in \mathbb{N}\}$ of pairs of non-empty subsets called a *play* of the $\mathcal{G}(\mathcal{U})$-game. We shall declare that player Ω *wins* a play $\{(A_n, B_n) : n \in \mathbb{N}\}$ if, $\bigcap_{n=1}^{\infty} B_n$ is at most one point. Otherwise player Σ is said to have won the play. By a *strategy* t for player Σ (*strategy* s for player Ω) we mean a *'rule'* that specifies each move of player Σ (player Ω) in every possible situation. More precisely, a strategy $t := (t_n : n \in \mathbb{N})$ for Σ is a sequence of set-valued mappings such that $\emptyset \neq t_{n+1}(B_1, B_2, \dots, B_n) \subseteq B_n$ for all $n \in \mathbb{N}$. Similarly, a strategy $s := (s_n : n \in \mathbb{N})$ for Ω is a sequence of set-valued mappings such that (i) $\emptyset \neq s_n(A_1, A_2, \dots, A_n) \subseteq A_n$ for all $n \in \mathbb{N}$ and (ii) $s_n(A_1, A_2, \dots, A_n) := (U_n^* \cap A_n$ with $U_n^* \in \mathcal{U})$. The domain of each mapping t_n is precisely the set of all finite sequences $\{B_1, B_2, \dots, B_{n-1}\}$ of length $n-1$ with each B_j being a non-empty relatively open subset of $t_j(B_1, B_2, \dots, B_{j-1})$ of the form: $B_j := B_j^* \cap t_j(B_1, B_2, \dots, B_{j-1})$ with $B_j^* \in \mathcal{U}$. (Note: the sequence of length 0 has been denoted by $\emptyset$.) Likewise, the domain of each mapping s_n is precisely the set of all finite sequences $\{A_1, A_2, \dots, A_n\}$ of length n with each A_j being a non-empty subset of $s_j(A_1, A_2, \dots, A_{j-1})$. Such a finite sequence $\{B_1, B_2, \dots, B_{n-1}\}$ (infinite sequence $\{B_n : n \in \mathbb{N}\}$) is called a *partial t-play* (*t-play*). Similarly a finite sequence $\{A_1, A_2, \dots, A_{n-1}\}$ (infinite sequence $\{A_n : n \in \mathbb{N}\}$) is called a *partial s-play* (*s-play*). A strategy $t := (t_n : n \in \mathbb{N})$ for the player Σ is called a *winning strategy* if each t-play is won by Σ and a strategy $s := (s_n : n \in \mathbb{N})$ for the player Ω is called a *winning strategy* if each s-play is won by Ω. We will say that the $\mathcal{G}(\mathcal{U})$-game on X

is Σ-*unfavourable* if the player Σ does *not* have a winning strategy in this game.

The following lemma incorporates the notion of a "minimal mapping" between topological spaces. A set-valued mapping $\varphi : X \to 2^Y$ acting between topological spaces X and Y is said to be *minimal* if for each pair of open subsets U of X and W of Y such that $\varphi(U) \cap W \neq \emptyset$ there exists a non-empty open subset $V \subseteq U$ such that $\varphi(V) \subseteq W$. This definition is modelled on the characterising property of the minimality of usco mappings, i.e., an usco mapping $\varphi : X \to 2^Y$ acting from a topological space X into subsets of a topological space Y is a minimal usco if, and only if, φ is a "minimal" mapping (see Lemma 2.3).

Let $\mathcal{U}$ be any open cover of a topological space X and let t be any strategy for the player Σ in the $\mathcal{G}(\mathcal{U})$-game played on X. We shall denote by P the space of all t-plays endowed with the Baire metric d, that is, if $p := \{B_n : n \in \mathbb{N}\}$ and $p' := \{B'_n : n \in \mathbb{N}\}$ then $d(p,p') = 0$ if $p = p'$ and otherwise, $d(p,p') = 1/n$, where $n := \min\{i \in \mathbb{N} : B_i \neq B'_i\}$. It is straight forward to verify that (P,d) is a complete metric space, (Kenderov et al., 2001a).

Lemma 2.9 (Kenderov et al., 2001a, Corollary 3) *Let X be a topological space, $\mathcal{U}$ be any open cover of X, t be a strategy for the player Σ in the $\mathcal{G}(\mathcal{U})$-game and let P denote the space of all t-plays, endowed with the Baire metric. Then the set-valued mapping $F : P \to 2^X$ defined by,*

$$F(p) := \bigcap_{n=1}^{\infty} B_n, \quad \textit{where}\ p := \{B_n : n \in \mathbb{N}\}\ \textit{is a t-play}$$

is a minimal mapping.

Our future investigations also rely upon the following theorem of Alexandroff.

Theorem 2.10 (Kelley, 1975, p.208) *A G_δ set in a complete metric space is homeomorphic to a complete metric space (i.e., if G is a G_δ subset of a complete metric space then G is completely metrizable).*

Theorem 2.11 (Kalenda, 1997) *For a completely regular topological space (X,τ), the following properties are equivalent:*

(i) *Every minimal non-empty valued mapping* $\varphi : M \to 2^X$ *acting from a complete metric space* M *into* X *is single-valued at some point of* M.

(ii) *Every minimal non-empty valued mapping* $\varphi : M \to 2^X$ *acting from a complete metric space* M *into* X *is single-valued at the points of a dense subset of* M.

(iii) *Every minimal non-empty valued mapping* $\varphi : M \to 2^X$ *acting from a complete metric space* M *into* X *is single-valued at the points of an everywhere second category subset of* M.

Proof. We see that the implications (iii) $\Rightarrow$ (ii) $\Rightarrow$ (i) are obvious. We will show that (i) $\Rightarrow$ (iii). Let M be a complete metric space and $\varphi : M \to 2^X$ be a minimal non-empty valued mapping such that $U := \{m \in M : \varphi(m)$ is single-valued$\}$ is not an everywhere second category subset of M. Then there is a non-empty open subset V of M such that $U \cap V$ is first category in V. Therefore there exists a dense G_δ subset G of V such that $G \cap U = \emptyset$. Now by applying Lemma 2.8 twice, we get that $\varphi|_G$ is a minimal mapping. As G is a G_δ subset, it follows from Theorem 2.10 that it is completely metrizable. However, this contradicts (i) since $\varphi|_G$ is nowhere single-valued on G. $\square$

Remark 2.12 *All the properties given in Theorem 2.11 also hold when the mapping is an usco and the domain spaces belong to a class of Baire spaces that is closed with respect to taking open subspaces and dense* G_δ *subsets (see Chapter 3). In particular, when the domain space is a complete metric space (see Theorem 2.10) or an* α*-favourable space (see Theorem 5.6).*

Theorem 2.13 (Moors and Somasundaram, 2003, Theorem 1) *For a completely regular topological space* (X, τ) *with weight* $\kappa \geq \aleph_0$*, the following properties are equivalent:*

(i) *The* $\mathcal{G}(\tau)$*-game on* X *is* Σ*-unfavourable.*

(ii) *Every minimal non-empty valued mapping* $\varphi : M \to 2^X$ *acting from a complete metric space* M *of density at most* κ *into* X *is single-valued at some point of* M*.*

(iii) *Every minimal non-empty valued mapping* $\varphi : M \to 2^X$ *acting from a complete metric space* M *into* X *is single-valued at some point of* M*.*

(iv) *Every minimal non-empty valued mapping* $\varphi : M \to 2^X$ *acting from a complete metric space* M *into* X *is single-valued at the points of a dense subset of* M*.*

(v) *Every minimal non-empty valued mapping* $\varphi : M \to 2^X$ *acting from a complete metric space* M *into* X *is single-valued at the points of an everywhere second category subset of* M*.*

Proof. The implications (v) $\Rightarrow$ (iv) $\Rightarrow$ (iii) $\Rightarrow$ (ii) are obvious and the implications (iii) $\Rightarrow$ (iv) $\Rightarrow$ (v) follow from Theorem 2.11. So it is sufficient to show that (ii) $\Rightarrow$ (i) and (i) $\Rightarrow$ (iii). We will first show that (ii) $\Rightarrow$ (i). Let $\mathcal{B}$ be a topological base for (X, τ) with cardinality κ and let $t := (t_n : n \in \mathbb{N})$ be a strategy for the player Σ in the $\mathcal{G}(\tau)$-game played on X. We need to show that t is not a winning strategy, that is, we need to construct a t-play, $\{B_n : n \in \mathbb{N}\}$ in which Ω wins. To accomplish this we consider a new game on X. Namely, we consider the $\mathcal{G}(\mathcal{B}^*)$-game on X, where $\mathcal{B}^* := \mathcal{B} \cup \{X\}$. For this game we inductively define a strategy $t' := (t'_n : n \in \mathbb{N})$ for the player Σ in terms of the strategy t.

First we define $t'_1(\emptyset) := t_1(\emptyset)$ and then under the assumption that for each $1 \leq j \leq n$, t_j has been defined in such a way that each partial t'-play of length at most n is a partial t-play, we define $t'_{n+1}(B_1, B_2, \ldots, B_n) := t_{n+1}(B_1, B_2, \ldots, B_n)$. This completes the definition of $t' := (t'_n : n \in \mathbb{N})$. With this definition we see that every t'-play is a t-play. In fact, t' is essentially the restriction of the strategy t to the $\mathcal{G}(\mathcal{B}^*)$-game. Next we let P denote the space of all t'-plays endowed with the Baire metric d. We claim however that the density of (P, d) is at most κ. To see this, we first observe that there are at most κ partial t'-plays and that for every partial t'-play $\{B_j : 1 \leq j \leq n\}$ we may fix a unique extension to a full t'-play, namely, that defined $B_k := t_k(B_1, B_2, \ldots, B_{k-1})$ (i.e., $B_k^* := X$) for all $k > n$. One may now check that the set of all these extensions is dense in (P, d). Therefore, the density of (P, d) is at most κ.

Next we define a set-valued mapping $F : P \to 2^X$ acting from the complete metric space P into X by,

$$F(p) := \bigcap_{n=1}^{\infty} B_n, \quad \text{where } p := \{B_n : n \in \mathbb{N}\}.$$

If $F(p) = \emptyset$ for some t'-play p, then Ω wins the play and so t is not a winning strategy for the player Σ in the $\mathcal{G}(\tau)$-game played on X and this completes the proof. Hence we shall assume that F has non-empty values. It now follows from Lemma 2.9 that F is a minimal mapping. Now from (ii) we see that $F(p)$ is a singleton for some $p \in P$. Thus p is a t-play in which Ω wins. Therefore t is not a winning strategy for the player Σ in the $\mathcal{G}(\tau)$-game played on X.

Now we will show that (i) $\Rightarrow$ (iii). Let M be a complete metric space and $\varphi : M \to 2^X$ be a minimal non-empty valued mapping. We will show that φ is single-valued at some point of M. To do this we will construct a strategy $t := (t_n : n \in \mathbb{N})$ for the player Σ and use the fact that Σ does not win some t-play. For the sake of notation, we define $W_0 := M$ and $B_0 := X$.

Step 1: Define $t_1(\emptyset) := \varphi(W_1)$ where W_1 is any non-empty open subset of W_0 with $\text{diam}(W_1) < 1$ and $\varphi(W_1) \subseteq B_0$.

Now suppose that the non-empty open sets W_j and t_j have been defined for each t-sequence $(B_1, B_2, \ldots, B_{j-1})$ of length $(j-1)$ (for $1 \le j \le n$) so that

(i) $W_j \subseteq \overline{W}_j \subseteq W_{j-1}$ with $\text{diam}(W_j) < 1/j$ and $\varphi(W_j) \subseteq B_{j-1}$;

(ii) $t_j(B_1, B_2, \ldots, B_{j-1}) := \varphi(W_j)$.

Step $(n+1)$**:** For each t-sequence $(B_1, B_2, \ldots, B_n)$ of length n, by the minimality of φ we can choose a non-empty open set W_{n+1} so that

(i) $W_{n+1} \subseteq \overline{W}_{n+1} \subseteq W_n$ with $\text{diam}(W_{n+1}) < 1/(n+1)$ and $\varphi(W_{n+1}) \subseteq B_n$ and then define

(ii) $t_{n+1}(B_1, B_2, \dots, B_n) := \varphi(W_{n+1})$.

This completes the definition of the strategy $t := (t_n : n \in \mathbb{N})$. Thus for any t-play we have the following:

$$\emptyset \neq \varphi(\bigcap_{n=1}^{\infty} W_n) \subseteq \bigcap_{n=1}^{\infty} \varphi(W_n) = \bigcap_{n=1}^{\infty} B_n.$$

Since the $\mathcal{G}(\tau)$-game on X is Σ-unfavourable there is some t-play $\{B_n : n \in \mathbb{N}\}$ that is won by Ω. This means that $\bigcap_{n=1}^{\infty} B_n$ is either empty or consists of exactly one point. Since from above $\bigcap_{n=1}^{\infty} B_n \neq \emptyset$, it must be the case that the non-empty set $\bigcap_{n=1}^{\infty} B_n$ has exactly one point. This shows that φ is single-valued at the point of $\bigcap_{n=1}^{\infty} W_n \subseteq M$ and this completes the proof. □

Theorem 2.13 enables us to establish the relationship between weakly Stegall spaces and fragmentable spaces. In (Kenderov and Moors, 1996, Theorem 1.1) the authors show that a space (X, τ) is fragmentable if, and only if, the player Ω has a winning strategy in the $\mathcal{G}(X)$-game played on X. This contrasts with the situation for weakly Stegall spaces which are characterised by the lack of a winning strategy for the player Σ. Hence the distinction between being fragmentable and being weakly Stegall is equivalent to the distinction between Ω having a winning strategy and Σ not having a winning strategy.

2.2.2 Game determined spaces

We consider on a topological space X another game known as the *Determination game* and we shall denote it by $DG(X)$. The game $DG(X)$ is a variation of the $\mathcal{G}(\mathcal{U})$-game. The same players Σ and Ω are involved in the $DG(X)$ game and the rules are the same as in the $\mathcal{G}(\mathcal{U})$-game. The only difference is in the definition of winning. The player Ω is said to have *won* a play $p = \{(A_n, B_n) : n \in \mathbb{N}\}$, if the set $K(p) := \bigcap_{n=1}^{\infty} \overline{A_n}$ is either empty or is a compact set in X such that for every open set U containing $K(p)$ there exists some $n \in \mathbb{N}$ with $A_n \subseteq U$. Otherwise the player Σ is said to have won the play. A topological space X is said to be *game* *determined* if the player Ω has a winning strategy in the $DG(X)$ game. Examples of game determined spaces include: p-spaces, Moore spaces, (Kenderov et al., 2001a) and spaces with countable separation (see Chapter 5 for more details).

Before we proceed further we need the following results from (Kenderov et al., 2001a). Let us consider a non-empty valued mapping $F : Z \to X$ acting from a topological space Z into a completely regular space X. Suppose bX is a compactification of X. The closure of the graph of F in $Z \times bX$ is a graph of some usco mapping $\tilde{F} : Z \to bX$. It is easily seen that if F is minimal, then $\tilde{F}$ is minimal as well. Moreover, the graph of $\tilde{F}$ does not contain, as a proper subset, the graph of any other usco with the same domain. Hence we see that $\tilde{F}$ is a minimal usco mapping.

Theorem 2.14 (Kenderov et al., 2001a, Theorem 6) *Let $F : Z \to X$ be a minimal non-empty valued mapping acting from a Baire space Z into a game determined space X. Suppose bX is a compactification of X and $\tilde{F} : Z \to bX$ is the set-valued mapping whose graph coincides with the closure in $Z \times bX$ of the graph of F. Then the set $C := \{z \in Z : \tilde{F}(z) \subseteq X\}$ contains a dense G_δ subset of Z.*

Theorem 2.15 *Let (X, τ) be a game determined topological space. Then X is weakly Stegall if, and only if, every minimal non-empty valued mapping acting from a complete metric space M into X is single-valued at some point of M.*

Proof. The backward implication easily follows from the fact that every minimal usco is a minimal non-empty valued mapping (see Remark 2.12). We shall now consider the converse.

Let $\varphi : M \to 2^X$ be a minimal non-empty valued mapping acting from a complete metric space M into a weakly Stegall space X. Then by Theorem 2.14 there is a dense G_δ subset G of M and a minimal usco $\tilde{\varphi} : G \to 2^X$ such that $\varphi(m) \subseteq \tilde{\varphi}(m)$ for all $m \in G$. Now since G is a G_δ subset of a complete metric space it follows from Theorem 2.10 that G is completely metrizable and since X is weakly Stegall, we have that $\tilde{\varphi}$ is single-valued at some point of G and so φ is single-valued at some point of M; which completes the proof. □

2.2.3 Permanence properties of weakly Stegall spaces

Weakly Stegall spaces enjoy several permanence properties similar to that of Stegall spaces. To describe and prove these properties we will need the following definitions and proposition.

For a topological space X we shall denote by $\mathcal{D}(X)$ the smallest σ-algebra of subsets on X that is stable under the Souslin operation and contains all the open subsets of X. Let us recall the definition of Souslin operation. We say that a family of sets $\mathcal{A}$ is *closed under the Souslin operation* if all the sets of the form

$$\bigcup\left\{\bigcap\{A(\sigma|_n) : n \geq 1\} : \sigma \in \mathbb{N}^{\mathbb{N}}\right\} \text{ with } A(\sigma|_n) \in \mathcal{A}$$

are contained in $\mathcal{A}$. Here we use the notation $\sigma|_n$ to denote $(\sigma_1, \sigma_2, \ldots, \sigma_n)$ when $(\sigma_1, \sigma_2, \ldots, \sigma_n, \ldots) \in \mathbb{N}^{\mathbb{N}}$.

A subset A of a topological space X is called a *Baire property set* if it is a member of the smallest σ-algebra of subsets of X, that contains the Borel subsets of X and the first category sets. The following result is a consequence of Proposition 5.1 and Proposition 5.3 of (Stegall, 1991).

Proposition 2.16 (Moors and Somasundaram, 2002a, Proposition 1) *Let $\varphi : B \to 2^X$ be a minimal usco acting from a Baire space B into a topological space X. If $K \in \mathcal{D}(X)$ and $\varphi^{-1}(K) := \{b \in B : \varphi(b) \cap K \neq \emptyset\}$ is second category then there exist a non-empty open subset U of B and a dense G_δ subset G of U such that $\varphi(G) \subseteq K$.*

Proof. From (Stegall, 1991, Proposition 5.3) it follows that $\varphi^{-1}(K)$ is a Baire property set. The theorem of Szpilrajn-Marczewski says that each Baire property set S has the following representation: $S = (U \setminus F) \cup (F \setminus U)$ where U is an open set and F is a first category set. Therefore we can write $\varphi^{-1}(K)$ as $(U \setminus F) \cup (F \setminus U)$

where U is an open subset of B and F is a first category set. Since $\varphi^{-1}(K)$ is second category, U must be non-empty. Now since F is a first category set, we observe here that $R := U \setminus F$ is a residual set in U. The proof then follows from (Stegall, 1991, Proposition 5.1) where it was shown that there exists a dense G_δ subset G' of B such that for each $b \in G'$ either $\varphi(b) \subseteq K$ or $\varphi(b) \cap K = \emptyset$. Then on $G := G' \cap R$ which is residual in U, $\varphi(b) \subseteq K$ for each $b \in G$. □

A continuous mapping $f : X \to Y$ is called a *perfect mapping* if it is onto, maps closed sets to closed sets and $f^{-1}(y)$ is compact for each $y \in Y$. In (Kalenda, 1999), the author proves part (iii) of the following theorem for the case when each X_n is a closed set in X. But below we generalise this result as in part (iii) of (Moors and Somasundaram, 2002a, Theorem 11).

Theorem 2.17 (Kalenda, 1999, Proposition W2) *Let (X, τ) and (Y, τ') be topological spaces.*

(i) *Let $f : X \to Y$ be a perfect mapping onto Y. If X is a weakly Stegall space then so is Y.*

(ii) *Let $f : X \to Y$ be a continuous one-to-one mapping. If Y is a weakly Stegall space then so is X.*

(iii) *Let $\{X_n : n \in \mathbb{N}\}$ be a cover of X. If each $X_n \in \mathcal{D}(X)$ is a weakly Stegall space then X is a weakly Stegall space.*

(iv) *If X belongs to weakly Stegall spaces and Y belongs to Stegall spaces then the product $X \times Y$ belongs to weakly Stegall spaces.*

Proof.

(i) Let M be a complete metric space and let $\varphi : M \to 2^Y$ be a minimal usco. Since $f : X \to Y$ is a perfect mapping, $y \mapsto f^{-1}(y)$ is an usco mapping. Then $f^{-1} \circ \varphi : M \to 2^X$ is an usco as the composition of an usco with another usco is still an usco. Let ψ be a minimal usco of $f^{-1} \circ \varphi$ whose graph is contained in the graph of $f^{-1} \circ \varphi$. Hence $(f \circ \psi)(m) \subseteq \varphi(m)$ for all $m \in M$. By the minimality of φ we must have that $(f \circ \psi)(m) = \varphi(m)$ for all $m \in M$. Since X is a weakly Stegall space, there is some point $m \in M$ such that $\psi(m)$ is single-valued. Therefore $\varphi(m) = f(\psi(m))$ is also single-valued. Thus the minimal usco $\varphi : M \to 2^Y$ is single-valued at some point of M. Therefore we conclude that Y is a weakly Stegall space.

(ii) Let M be a complete metric space and let $\varphi : M \to 2^X$ be a minimal usco. Since $f : X \to Y$ is continuous, by Lemma 2.4 we have that $f \circ \varphi : M \to 2^Y$ is a minimal usco. Since Y is a weakly Stegall space $(f \circ \varphi)(m) = f(\varphi(m))$ is single-valued at some point $m \in M$. Now since f is one-to-one, $\varphi(m)$ is single-valued and this shows that X is a weakly Stegall space.

(iii) Let M be a complete metric space and let $\varphi : M \to 2^X$ be a minimal usco. For each $n \in \mathbb{N}$, let $M_n := \varphi^{-1}(X_n) := \{m \in M : \varphi(m) \cap X_n \neq \emptyset\}$. Then $M = \bigcup_{n=1}^{\infty} M_n$ and so there is some $n_0 \in \mathbb{N}$ such that M_{n_0} is second category. Therefore by Proposition 2.16, there exist a non-empty open subset U of M and a dense G_δ subset $G \subseteq U$ such that $\varphi(G) \subseteq X_{n_0}$. By Theorem 2.10, G is completely metrizable and by Lemma 2.3 $\varphi|_G$ is a minimal usco. Therefore as X_{n_0} is a weakly Stegall space there is some point $m \in G$ such that $\varphi|_G(m) = \varphi(m)$ is a singleton.

(iv) Let M be a complete metric space and let $\varphi : M \to 2^{X \times Y}$ be a minimal usco. Denote by Π_X the canonical projection of $X \times Y$ onto X. Then by Lemma 2.4, $\Pi_X \circ \varphi : M \to 2^X$ is a minimal usco. Since X is a weakly Stegall space, it follows from Theorem 2.11 that $\Pi_X \circ \varphi : M \to 2^X$ is single-valued at the points of an everywhere second category subset $A \subseteq M$. Denote by Π_Y the canonical projection of $X \times Y$ onto Y. Then by Lemma 2.4, $\Pi_Y \circ \varphi : M \to 2^Y$ is a minimal usco. Since Y is a Stegall space, $\Pi_Y \circ \varphi : M \to 2^Y$ is single-valued at the points of a residual subset B of M. Hence the minimal usco $\varphi : M \to 2^{X \times Y}$ is single-valued at the points of $A \cap B$, which is a non-empty set. Therefore, $X \times Y$ is a weakly Stegall space and the proof is complete. □

In general the product of weakly Stegall spaces is not a weakly Stegall space unlike the case for the classes of fragmentable and Stegall spaces. Further details of this, including the proof, are delayed until Chapter 3. Next we give some sufficiency conditions for a topological space to be a weakly Stegall space.

Lemma 2.18 (Moors and Somasundaram, 2002a, Lemma 4) *Let* $\varphi : X \to 2^Y$ *be a minimal usco acting from a topological space* X *into a topological space* Y*. If* $\{C_k : 1 \leq k \leq n\}$ *is a family of closed subsets of* Y *and* U *is a non-empty open subset of* X *such that* $\varphi(U) \subseteq \bigcup_{k=1}^{n} C_k$ *then there exist a* $k_0 \in \{1, 2, \ldots, n\}$ *and a non-empty open subset* V *of* U *such that* $\varphi(V) \subseteq C_{k_0}$*.*

Proof. For each $k \in \{1, 2, \ldots, n\}$, let $U_k := \{t \in U : \varphi(t) \cap C_k \neq \emptyset\}$, then $\{U_k : 1 \leq k \leq n\}$ is a closed cover of U. Hence for some $k_0 \in \{1, 2, \ldots, n\}$, $V := \text{int}(U_{k_0}) \neq \emptyset$. It now follows from Lemma 2.3 that $\varphi(V) \subseteq C_{k_0}$. □

A subset K of a topological space T is said to be *relatively pseudo-compact in* T if every continuous real-valued mapping defined on T is bounded on K. The following lemma is a slight generalisation of Stone's well-known "lattice formulation" of the Stone-Weierstrass theorem (see (Kelley, 1975, p.244) or (Stone, 1948)).

Lemma 2.19 (Troallic, 2000, Lemma 3.3) *Let* K *be a relatively pseudo-compact subset of a topological space* T *and let* τ_p *denote the topology (on* $C(T)$*) of point-wise convergence on* T*. If* L *is a countable sublattice of* $C(T)$ *and* $f \in \overline{L}^{\tau_p}$ *then for each* $\varepsilon > 0$ *there exist an* $l_\varepsilon \in L$ *and an open set* U_ε *containing* K *such that* $|f(k) - l_\varepsilon(k)| < \varepsilon$ *for all* $k \in U_\varepsilon$*.*

A completely regular space K is said to be *pseudo-countably determined* if there exists a second countable space P and a set-valued mapping ψ from P onto K such that (i) for each $p \in P$, $\psi(p)$ is relatively pseudo-compact in K and (ii) for each $p \in P$ and open set W in K with $\psi(p) \subseteq W$ there exists a neighbourhood U of p such that $\psi(U) \subseteq \overline{W}$. We shall call a set-valued mapping ψ that satisfies conditions (i) and (ii) a *pseudo-usco*.

Theorem 2.20 (Moors and Somasundaram, 2002a, Theorem 12) *Let K be a pseudo-countably determined topological space. Then $C_p(K)$ is weakly Stegall.*

Proof. Let $\varphi : M \to 2^{C(K)}$ be a minimal usco acting from a complete metric space M into $C(K)$. We will show that there is a point $t_\infty \in M$ such that $\varphi(t_\infty)$ is a singleton. Since K is a pseudo-countably determined topological space there exists a second countable space P (with countable base $\{U_n : n \in \mathbb{N}\}$) and a pseudo-usco mapping $\psi : P \to 2^K$ from P onto K, i.e., $K = \psi(P)$. For each $n \in \mathbb{N}$ we shall define $p_n : C(K) \to \mathbb{R}^+ \cup \{\infty\}$ by,

$$p_n(f) := \sup\{|f(k)| : k \in \psi(U_n)\}$$

and $B_n(r) \subseteq C(K)$ by, $B_n(r) := \{f \in C(K) : p_n(f) \leq r\}$. (Note: each $B_n(r)$ is τ_p-closed and convex.) We shall also let $\sigma : \mathbb{N} \to \mathbb{N}$ be any mapping onto $\mathbb{N}$ such that for each $n \in \mathbb{N}$, $\sigma^{-1}(n)$ is cofinal in $\mathbb{N}$. We now proceed inductively.

Step 1: Choose $f_1 \in \varphi(V_0)$, with $V_0 := M$ and let $L_1 := \langle f_1 \rangle = \{f_1\}$ (i.e., the lattice generated by f_1). Then put $s_1 := \sup\{\min\{p_{\sigma(1)}(f - l) : l \in L_1\} : f \in \varphi(V_0)\}$. If $s_1 = \infty$, then choose a non-empty open subset $V_1 \subseteq \overline{V}_1 \subseteq V_0$ such that

$$\operatorname{diam}(V_1) < 1 \quad \text{and} \quad \varphi(V_1) \cap [L_1 + B_{\sigma(1)}(1)] = \emptyset.$$

Note: It is possible to do this by Lemma 2.2 since φ is a minimal usco, $L_1 + B_{\sigma(1)}(1)$ is τ_p-closed and $\varphi(V_0) \nsubseteq L_1 + B_{\sigma(1)}(1)$. Otherwise (i.e., $s_1 < \infty$ and $\varphi(V_0) \subseteq L_1 + B_{\sigma(1)}(s_1)$) one of the two cases must hold: (a) $s_1 = 0$ or (b) $0 < s_1 < \infty$. In case (a) $\varphi(V_0) \subseteq L_1 + B_{\sigma(1)}(s_1)$ and so there exists a non-empty open subset $V_1 \subseteq \overline{V}_1 \subseteq V_0$ such that

$$\operatorname{diam}(V_1) < 1 \quad \text{and} \quad \varphi(V_1) \subseteq l_1 + B_{\sigma(1)}(s_1) \text{ for some } l_1 \in L_1.$$

In case (b) we may choose a non-empty open subset $W \subseteq V_0$ such that

$$\varphi(W) \cap [L_1 + B_{\sigma(1)}(s_1/2)] = \emptyset.$$

Note: It is possible to do this by Lemma 2.2 since φ is a minimal usco, $L_1 + B_{\sigma(1)}(s_1/2)$ is τ_p-closed and $\varphi(V_0) \nsubseteq L_1 + B_{\sigma(1)}(s_1/2)$. Now by Lemma 2.18 there exists a non-empty open subset $V_1 \subseteq \overline{V}_1 \subseteq W \subseteq V_0$ such that

$$\operatorname{diam}(V_1) < 1 \quad \text{and} \quad \varphi(V_1) \subseteq l_1 + B_{\sigma(1)}(s_1) \text{ for some } l_1 \in L_1.$$

In both cases (a) and (b) $\inf\{\min\{p_{\sigma(1)}(f - l) : l \in L_1\} : f \in \varphi(V_1)\} \geq s_1/2$.

In general, suppose that we have completed the first n steps of the induction. Then we will have constructed the sets: $V_n \subseteq \overline{V}_n \subseteq V_{n-1} \subseteq \cdots V_2 \subseteq \overline{V}_2 \subseteq V_1$ such that $\operatorname{diam}(V_j) < 1/j$ for all $1 \leq j \leq n$ and finite lattices $L_j := \langle f_1, f_2, \ldots, f_j \rangle$ with $f_j \in \varphi(V_{j-1})$ for all $1 \leq j \leq n$ and extended real numbers $\{s_j : 1 \leq j \leq n\}$ defined by, $s_j := \sup\{\min\{p_{\sigma(j)}(f - l) : l \in L_j\} : f \in V_{j-1}\}$ such that either;

(i) $\varphi(V_j) \cap [L_j + B_{\sigma(j)}(1)] = \emptyset$ (in the case $s_j = \infty$) or

(ii) $\inf\{\min\{p_{\sigma(j)}(f - l) : l \in L_j\} : f \in \varphi(V_j)\} \geq s_j/2$ and $\varphi(V_j) \subseteq l_j + B_{\sigma(j)}(s_j)$ for some $l_j \in L_j$ (in the case $0 \leq s_j < \infty$).

Step $(n+1)$**:** Choose $f_{n+1} \in \varphi(V_n)$ and let $L_{n+1} := \langle f_1, f_2, \ldots, f_{n+1} \rangle$ (i.e., the lattice generated by $\{f_1, f_2, \ldots, f_{n+1}\}$; which is a finite set). Then put

$$s_{n+1} := \sup\{\min\{p_{\sigma(n+1)}(f - l) : l \in L_{n+1}\} : f \in \varphi(V_n)\}.$$

If $s_{n+1} = \infty$ then choose a non-empty open set $V_{n+1} \subseteq \overline{V}_{n+1} \subseteq V_n$ such that

$$\operatorname{diam}(V_{n+1}) < 1/(n+1) \quad \text{and} \quad \varphi(V_{n+1}) \cap [L_{n+1} + B_{\sigma(n+1)}(1)] = \emptyset.$$

Note: It is possible to do this by Lemma 2.2 since φ is a minimal usco, $\varphi(V_n) \not\subseteq L_{n+1} + B_{\sigma(n+1)}(1)$ and $L_{n+1} + B_{\sigma(n+1)}(1)$ is τ_p-closed. Otherwise, (i.e., $s_{n+1} < \infty$ and $\varphi(V_n) \subseteq L_{n+1} + B_{\sigma(n+1)}(s_{n+1})$) one of the two cases must hold: (a) $s_{n+1} = 0$ or (b) $0 < s_{n+1} < \infty$. In case (a) $\varphi(V_n) \subseteq L_{n+1} + B_{\sigma(n+1)}(s_{n+1})$ and so by Lemma 2.18 there exists a non-empty open subset $V_{n+1} \subseteq \overline{V}_{n+1} \subseteq V_n$ such that

$$\operatorname{diam}(V_{n+1}) < 1/(n+1) \quad \text{and} \quad \varphi(V_{n+1}) \subseteq l_{n+1} + B_{\sigma(n+1)}(s_{n+1}) \text{ for some } l_{n+1} \in L_{n+1}.$$

In case (b) we may choose a non-empty open subset $W \subseteq V_n$ such that

$$\varphi(W) \cap [L_{n+1} + B_{\sigma(n+1)}(s_{n+1}/2)] = \emptyset.$$

Note: It is possible to do this by Lemma 2.2 since φ is a minimal usco, $L_{n+1} + B_{\sigma(n+1)}(s_{n+1}/2)$ is τ_p-closed and $\varphi(V_n) \nsubseteq L_{n+1}+B_{\sigma(n+1)}(s_{n+1}/2)$. Now by Lemma 2.18 there exists a non-empty open subset $V_{n+1} \subseteq \overline{V}_{n+1} \subseteq W \subseteq V_0$ such that

$$\operatorname{diam}(V_{n+1}) < 1/(n+1) \quad \text{and} \quad \varphi(V_{n+1}) \subseteq l_{n+1}+B_{\sigma(n+1)}(s_{n+1}) \text{ for some } l_{n+1} \in L_{n+1}.$$

In both cases (a) and (b) $\inf\{\min\{p_{\sigma(n+1)}(f-l) : l \in L_{n+1}\} : f \in \varphi(V_{n+1})\} \geq s_{n+1}/2$. This completes the induction.

Let $\{t_\infty\} := \bigcap_{n=1}^{\infty} V_n$. We claim that φ is single-valued at t_∞. To justify this assertion let us consider the following argument. Let $f, g \in \varphi(t_\infty)$ and $k \in K$. We need to show that $|f(k) - g(k)| = 0$. To this end, let ε be an arbitrary positive real number less than 1. Since ψ is onto there exists a $p \in P$ such that $k \in \psi(p)$. Let $f_\infty \in \varphi(t_\infty)$ be a τ_p-cluster point of the sequence $(f_n : n \in \mathbb{N})$. The existence of such a cluster point follows from the fact that φ is an usco mapping. Then $f_\infty \in \overline{\{f_n : n \in \mathbb{N}\}}^{\tau_p} \subseteq \overline{L_\infty}^{\tau_p}$, where $L_\infty := \bigcup_{n\in\mathbb{N}} L_n$. Now since $\psi(p)$ is relatively pseudo-compact and L_∞ is a lattice we have by Lemma 2.19 that there exists an $l_\infty \in L_\infty$ and an open set U containing $\psi(p)$ such that $|f_\infty(k') - l_\infty(k')| < \varepsilon/4$ for all $k' \in U$.

Since ψ is a pseudo-usco mapping there exists an $n_0 \in \mathbb{N}$ such that $p_{n_0}(f_\infty - l_\infty) < \varepsilon/4$. Next, if we choose i sufficiently large so that $l_\infty \in L_i$ and $\sigma(i) = n_0$, then at the ith stage of the induction $s_i < \infty$ since $f_\infty \in \varphi(V_i) \cap [L_i + B_{\sigma(i)}(1)] \neq \emptyset$ and so

$$\varepsilon/4 > p_{n_0}(f_\infty - l_\infty) \geq \inf\{\min\{p_{n_0}(f-l) : l \in L_i\} : f \in \varphi(V_i)\} \geq s_i/2.$$

Therefore $0 \leq s_i \leq \varepsilon/2$. Moreover, by the construction we have that

$$f, g \in \varphi(t_\infty) \subseteq \varphi(V_i) \subseteq l_i + B_{n_0}(s_i) \subseteq l_i + B_{n_0}(\varepsilon/2)$$

and so $|f(k) - g(k)| \leq \varepsilon$ and this completes the proof. □

Remark 2.21 *Let us further note that by appealing to the game characterisation of fragmentability given in (Kenderov and Moors, 1996) one can modify the argument in Theorem 2.20 to show the following: "Suppose K is pseudo-countably determined* *and* $B \subseteq C_p(K)$. *If player* Ω *has a strategy* s *in the* $\mathcal{G}(\tau_p)$*-game played on* B *such that for every* s*-play* $\{A_n : n \in \mathbb{N}\}$ *either* (i) $\bigcap_{n=1}^{\infty} A_n = \emptyset$ *or* (ii) *every sequence* $(f_n : n \in \mathbb{N})$ *with* $f_n \in A_n$ *for all* $n \in \mathbb{N}$ *has a* τ_p*-cluster point in* $C_p(K)$*, then* B *is fragmented by a metric* d *such that the* d*-topology is at least as strong as the relative pointwise topology on* B*". In particular, if* B *is game determined in* $C_p(K)$*, then player* Ω *has a winning strategy.*

Corollary 2.22 (Kenderov and Moors, 1996, Theorem 2.2) *If* X *is a weakly countably determined space then* $(X^*, \textit{weak}^*)$ *is fragmentable.*

Proof. The proof follows from the fact that (X^*, weak^*) is a game determined subspace of $C_p(X, \text{weak})$ and the previous remark. □

2.3 Banach space version of weakly Stegall spaces

We say that a Banach space X belongs to *class*($w\tilde{\mathcal{S}}$) if (X^*, weak^*) is a weakly Stegall space. We shall now establish some permanence properties of class($w\tilde{\mathcal{S}}$).

Theorem 2.23 *Let* X *and* Y *be Banach spaces.*

(i) *If* $X \in$ *class*($w\tilde{\mathcal{S}}$) *and* $T : X \to Y$ *is a bounded linear mapping from* X *into a Banach space* Y *with dense range, then* $Y \in$ *class*($w\tilde{\mathcal{S}}$)*. In particular, if* $X \in$ *class*($w\tilde{\mathcal{S}}$)*, then every quotient of* X *is in class*($w\tilde{\mathcal{S}}$).

(ii) *If* $X \in$ *class*($w\tilde{\mathcal{S}}$) *and* Y *is a subspace of* X*, then* $Y \in$ *class*($w\tilde{\mathcal{S}}$).

(iii) *If* $X \in$ *class*($w\tilde{\mathcal{S}}$) *and* $Y \in$ *class*($\tilde{\mathcal{S}}$)*, then* $X \times Y \in$ *class*($w\tilde{\mathcal{S}}$).

Proof.

(i) Let $X \in \text{class}(w\tilde{\mathcal{S}})$ and $T : X \to Y$ be a bounded linear mapping from X into a Banach space Y with dense range. Hence $T^* : (Y^*, \text{weak}^*) \to (X^*, \text{weak}^*)$ is one-to-one and weak*-to-weak* continuous. Hence we conclude by Theorem 2.17 part (ii) that $Y \in \text{class}(w\tilde{\mathcal{S}})$.

(ii) Let $X \in \text{class}(w\tilde{\mathcal{S}})$. Then (B_{X^*}, weak^*) is a member of weakly Stegall spaces. Now $T : (B_{X^*}, \text{weak}^*) \to (B_{Y^*}, \text{weak}^*)$ defined by $T(x^*) := x^*|_Y$ is a perfect mapping (since a continuous function on a compact space is perfect). Thus by Theorem 2.17 part (i) (B_{Y^*}, weak^*) is weakly Stegall. Therefore, by Theorem 2.17 part (iii), (Y^*, weak^*) is weakly Stegall and so we conclude $Y \in \text{class}(w\tilde{\mathcal{S}})$.

(iii) Since (X^*, weak^*) is a weakly Stegall space and (Y^*, weak^*) is a Stegall space, it follows from Theorem 2.17 part (iv) that $X^* \times Y^*$ is weakly Stegall. Since $(X \times Y)^*$ is homeomorphic to $X^* \times Y^*$ in the weak* topology, we conclude that $(X \times Y)^*$ is weakly Stegall, i.e., $X \times Y \in \text{class}(w\tilde{\mathcal{S}})$. □

2.4 Almost weak Asplund spaces

We shall now study a new class of Banach spaces that will help us distinguish the weak Asplund spaces from the Gâteaux differentiability spaces. We say that a Banach space X is an *almost weak Asplund space* if every continuous convex function defined on it is Gâteaux differentiable at the points of an everywhere second category subset of X.

We note here that the definitions of weak Asplund (almost weak Asplund) [Gâteaux differentiability] spaces are equivalent to every continuous convex function defined on a non-empty open convex subset A of a Banach space X being Gâteaux differentiable at the points of a residual (everywhere second category) [dense] subsets of A. This was originally proven in (Asplund, 1968) for the case when X is a weak Asplund space but below we provide a proof for the case when X is an almost weak Asplund space. To do this we need to recall that a real-valued function f defined on a non-empty open subset A of a Banach space X is *Lipschitz* on A if there exists a $K > 0$ such that

$$|f(x) - f(y)| \leq K\|x - y\| \text{ for all } x, y \in A.$$

In order to proceed further we need the notion of a "subdifferential mapping". Let $\varphi : A \to \mathbb{R}$ be a continuous convex function defined on a non-empty open convex subset A of a Banach space X. Then the *subdifferential mapping of* φ is the mapping $\partial\varphi : A \to 2^{X^*}$ defined by,

$$\partial\varphi(x) := \{x^* \in X^* : x^*(y - x) \leq \varphi(y) - \varphi(x) \text{ for all } y \in A\}.$$

As we shall see later, the relationship between weakly Stegall spaces and almost weak Asplund spaces is established through the subdifferential mapping. We shall now give a well-known property of the subdifferential mapping.

Proposition 2.24 (Phelps, 1993) *If φ is a continuous convex function defined on a non-empty open convex subset A of a Banach space X, then the subdifferential mapping $x \mapsto \partial\varphi(x)$ is weak* upper semicontinuous on A and for each $x \in A$, $\partial\varphi(x)$ is non-empty and weak* compact (i.e., the subdifferential mapping $x \mapsto \partial\varphi(x)$ is a weak* usco on A).*

Next we establish the relationship between the subdifferential mapping and Gâteaux and Fréchet differentiability of convex functions that will be needed in the upcoming chapters.

Proposition 2.25 (Giles, 1982) *If φ is a continuous convex function defined on a non-empty open convex subset A of a Banach space X, then it is Gâteaux (Fréchet) differentiable at a point $x \in A$ if, and only if, the subdifferential mapping $x \mapsto \partial\varphi(x)$ is single-valued (single-valued and and norm upper semicontinuous) at x.*

We shall now consider the following lemma which plays a crucial part in the proof of the next theorem.

Lemma 2.26 (Phelps, 1993, Lemma 2.31) *Let f be a continuous convex function defined on a non-empty open convex subset A of a Banach space X and let $x_0 \in A$. Then there exists a neighbourhood U of x_0 in A and a convex Lipschitz function $\tilde{f}$ defined on X such that $\tilde{f}|_U = f$.*

Proof. Given $n \in \mathbb{N}$, $x \in X$ define f_n to be the "inf-convolution" of f and $n\|\cdot\|$:

$$f_n(x) := \inf\{f(y) + n\|x - y\| : y \in A\}.$$

We need to show that $f_n(x) > -\infty$ for all $x \in X$ and for all sufficiently large values of n. To this end, let $x_0 \in A$ and choose $x^* \in \partial f(x_0)$. If $n \geq \|x^*\|$, then for any $x \in X$ and $y \in A$ we have

$$f(y) - f(x_0) \geq x^*(y - x_0) \geq -n\|y - x_0\| \geq -n\|y - x\| - n\|x - x_0\|,$$

so $f_n(x) \geq f(x_0) - n\|x - x_0\|$. Thus we can assume that n is large enough that $f_n(x) > -\infty$ for all values of $x \in X$. It then follows easily from the definition that for $x_1, x_2 \in X$, $0 \leq \lambda \leq 1$ and any $\varepsilon > 0$, we have $\lambda f_n(x_1) + (1 - \lambda) f_n(x_2) \geq f_n(\lambda x_1 + (1 - \lambda)x_2) - \varepsilon$, so f_n is convex. If we take $y = x$ in the definition of $f_n(x)$ we see that $f_n(x) \leq f(x)$ for all $x \in A$.

Moreover, given $u, v \in X$ and $\varepsilon > 0$ we can choose $y \in A$ such that

$$f_n(u) > f(y) + n\|u - y\| - \varepsilon.$$

Since

$$f_n(v) \leq f(y) + n\|v - y\|$$

we have

$$f_n(v) - f_n(u) < f(y) + n\|v - y\| - [f(y) + n\|u - y\| - \varepsilon] \leq n\|v - u\| + \varepsilon,$$

which shows that

$$f_n(u) - f_n(v) \leq n\|u - v\| \text{ for all } u, v \in X.$$

Interchanging u and v proves that f_n has Lipschitz constant n. Since ∂f is locally bounded, there exists a neighbourhood U of x_0 and an $n \in \mathbb{N}$ such that $\partial f(U)$

is contained in the ball nB_{X^*}. Suppose that $x \in U$ and choose any functional $x^* \in \partial f(x)$; then $\|x^*\| \leq n$ and for all $y \in A$

$$f(x) \leq f(y) + x^*(x-y) \leq f(y) + n\|x-y\|,$$

which implies that $f(x) \leq f_n(x)$ and this shows $f = f_n$ in U. □

Theorem 2.27 *A Banach space X is an almost weak Asplund space if, and only if, every continuous convex function defined on a non-empty open convex subset A of X is Gâteaux differentiable at the points of an everywhere second category subset of A.*

Proof. It is immediately clear from the definition that if every continuous convex function defined on a non-empty open convex subset A of X is Gâteaux differentiable at the points of an everywhere second category subset of A then X is an almost weak Asplund space. Let us consider the converse. Let f be a continuous convex function defined on a non-empty open convex subset A of X and let $x_0 \in A$. Then by Lemma 2.26, there exists a neighbourhood U of x_0 and a continuous convex function $\tilde{f}$ defined on X such that $\tilde{f}|_U = f$. The continuous convex function $\tilde{f}$ on X is Gâteaux differentiable at the points of an everywhere second category subset $S \subseteq X$. This implies that f is Gâteaux differentiable at the points of $S \cap U$ which is a second category subset of U and this completes the proof. □

It follows easily from the definition that every weak Asplund space is an almost weak Asplund space and every almost weak Asplund space is a Gâteaux differentiability space. While we show in Chapter 4 that there are examples of almost weak Asplund spaces that are not weak Asplund, the question as to whether the classes of almost weak Asplund spaces and Gâteaux differentiability spaces coincide still remains open.

2.4.1 Establishing the relationship between different classes of Banach spaces

We will now establish the connection between class($w\tilde{\mathcal{S}}$) and almost weak Asplund spaces and also the connection between $\tilde{\mathcal{F}}$, class($\tilde{\mathcal{S}}$) and weak Asplund spaces. But to do this, let us consider the right-hand and left-hand derivatives of convex functions.

Let φ be a continuous convex function defined on a non-empty open convex subset A of a normed linear space X. Then the *right-hand derivative of* φ *at* $x \in A$ *in the direction* $y \in X$ is given by

$$\varphi_+'(x;y) := \lim_{\lambda \to 0^+} \frac{\varphi(x+\lambda y) - \varphi(x)}{\lambda}.$$

Similarly, the *left-hand derivative of* φ *at* $x \in A$ *in the direction* $y \in X$ is given by

$$\varphi_-'(x;y) := \lim_{\lambda \to 0^-} \frac{\varphi(x+\lambda y) - \varphi(x)}{\lambda}.$$

Therefore φ is Gâteaux differentiable at $x \in A$ in the direction $y \in X$ if, and only if, $\varphi_+'(x;y) = \varphi_-'(x;y)$. The following proposition, although elementary is important for our purposes.

Proposition 2.28 (Giles, 1982, p.118) *Let* φ *be a continuous convex function defined on a normed linear space* X *and let* $x \in X$*. If* $y \mapsto \varphi_+'(x;y)$ *is linear on* X *then* φ *is Gâteaux differentiable at* x*.*

Proof. For all $y \in X$, $0 = \varphi_+'(x)(y-y) = \varphi_+'(x)(y) + \varphi_+'(x)(-y)$. So we have, $\varphi_+'(x)(y) = -\varphi_+'(x)(-y) = \varphi_-'(x)(y)$ and thus we conclude that φ is Gâteaux differentiable at x. □

The relationship between class($w\tilde{\mathcal{S}}$) and both almost weak Asplund spaces and Gâteaux differentiability spaces is revealed in the next theorem.

Theorem 2.29 (Moors and Somasundaram, 2002a, Theorem 13) *Every member of class($w\tilde{\mathcal{S}}$) is an almost weak Asplund space. In particular, every member of class($w\tilde{\mathcal{S}}$) is a Gâteaux differentiability space.*

Proof. Suppose $X \in \text{class}(w\tilde{\mathcal{S}})$. Let $\varphi : A \to \mathbb{R}$ be a continuous convex function defined on a non-empty open convex subset A of X and let $\psi : A \to 2^{X^*}$ be a minimal usco on A such that $\psi(x) \subseteq \partial\varphi(x)$ for all $x \in A$. As A is a non-empty open (and hence G_δ) subset of a complete metric space, it follows from Theorem 2.10 that A itself is completely metrizable and since (X^*, weak^*) is a weakly Stegall space, it follows from Theorem 2.11 that there must exist an everywhere second category subset R of A on which ψ is single-valued. Next we let $\sigma : A \to (X^*, \text{weak}^*)$ be any selection of ψ and let $x \in R$. We claim that φ is Gâteaux differentiable at x. For any $y \in X$ and $\lambda > 0$ we have, by the definition of $\partial\varphi$, that

$$\sigma(x)(\lambda y) \leq \varphi(x + \lambda y) - \varphi(x) \leq \sigma(x + \lambda y)(\lambda y).$$

Since by Proposition 2.24, σ is weak* continuous at x we have that,

$$\sigma(x)(y) = \lim_{\lambda \to 0^+} \frac{\varphi(x + \lambda y) - \varphi(x)}{\lambda}.$$

This means by Proposition 2.28 that φ is Gâteaux differentiable at x, with derivative $\sigma(x)$, and this completes the proof. $\square$

We shall now review the connection between $\text{class}(\tilde{\mathcal{S}})$ and weak Asplund spaces.

Theorem 2.30 (Stegall, 1983) *Every member of class($\tilde{\mathcal{S}}$) is weak Asplund.*

The proof of Theorem 2.30 is similar to the proof of Theorem 2.29 and we shall not present it here. The $\text{class}(\tilde{\mathcal{S}})$ spaces are the largest known well-behaved subclass of weak Asplund spaces. Next we review the connection between the classes of fragmentable and Stegall spaces.

Theorem 2.31 (Namioka, 1983) *Every fragmentable space is a Stegall space. In particular, $\tilde{\mathcal{F}} \subseteq$ class($\tilde{\mathcal{S}}$).*

Proof. Let X be a fragmentable topological space and d be a fragmenting metric on X. Let B be a Baire space and $\varphi : B \to 2^X$ be a minimal usco. For each $n \in \mathbb{N}$, consider the set

$$O_n := \bigcup \{\text{open sets } V : d\text{-diam}[\varphi(V)] < 1/n\}.$$

Clearly each O_n is open. We claim that each O_n is dense in B. To this end, let U be a non-empty open subset of B. Since X is fragmented there exists an open set W in X, such that $W \cap \varphi(U) \neq \emptyset$ and d-diam$[W \cap \varphi(U)] < 1/n$. It follows from the characterisation of the minimality of φ given in Lemma 2.3 that there exists a non-empty open set V of U such that $\varphi(V) \subseteq W$. Hence d-diam$[\varphi(V)] < 1/n$ and so $\emptyset \neq V \subseteq O_n \cap U$. This proves that O_n is dense in B. It now follows that φ is single-valued at each point of $\bigcap_{n=1}^{\infty} O_n$; which completes the proof. $\square$

Thus we have the following relationship between the spaces:

$$\tilde{\mathcal{F}} \subseteq \text{class}(\tilde{\mathcal{S}}) \subseteq \text{weak Asplund spaces} \subseteq \text{almost weak Asplund spaces} \subseteq \text{GDS}$$

where the last two implications come directly from the definitions of Gâteaux differentiability spaces (GDS) and almost weak Asplund spaces.

In the next chapter, we will continue our study of weakly Stegall spaces by considering Banach spaces of the form $C(K)$ for a particular class of compact Hausdorff spaces K.

Chapter 3

$C(K_A)$ spaces

In this chapter we review the role of $C(K)$ spaces (Banach spaces of continuous functions defined on a compact Hausdorff space K, endowed with the supremum norm) in distinguishing the classes: $\tilde{\mathcal{F}}$, class($\tilde{\mathcal{S}}$) and weak Asplund spaces. Specifically we shall consider the role that the Kalenda compacta have played in this situation.

In the first section of this chapter we define the Kalenda compacta K_A and give a characterisation of when they belong to the classes of (i) fragmentable spaces, (ii) Stegall spaces and (iii) weakly Stegall spaces. Our goal in the second section is to provide a detailed description of the duals of the $C(K_A)$ spaces. Subsequent to this, we characterise when Banach spaces of the form $C(K_A)$ belong to $\tilde{\mathcal{F}}$ and class($\tilde{\mathcal{S}}$). We then use the characterisation of the $C(K_A)$ spaces to give examples of Banach spaces that are (i) in class($\tilde{\mathcal{S}}$) but not in $\tilde{\mathcal{F}}$, (ii) weak Asplund but not in $\tilde{\mathcal{F}}$ and (iii) weak Asplund but not in class($\tilde{\mathcal{S}}$). In the final section of this chapter, we show that there exists a weakly Stegall compact space K_A such that $(C(K_A)^*, \text{weak}^*)$ is not weakly Stegall.

3.1 Kalenda compacta

We shall study the Kalenda compacta which play a key role in distinguishing the Gâteaux differentiability spaces from the weak Asplund spaces. Let A be an arbitrary subset of $(0,1)$ and let

$$K_A := [(0,1] \times \{0\}] \cup [(\{0\} \cup A) \times \{1\}].$$

If we equip this set with the order topology generated by the lexicographical (dictionary) ordering (i.e., $(s_1, s_2) \leq (t_1, t_2)$ if, and only if, either $s_1 < t_1$ or $s_1 = t_1$ and $s_2 \leq t_2$) then with this topology K_A is a compact Hausdorff space, (Kalenda, 1999, Proposition 2). The family of all the K_A's has now become known as the *Kalenda compacta.* In the special case of $A = (0,1)$, K_A reduces to the well-known "double arrow" space.

Many of the basic properties of the Kalenda compacta may be found in (Kalenda, 1999). In particular, the following results may be found there.

Theorem 3.1 (Kalenda, 1999, Proposition 3) *Let A be an arbitrary subset of* $(0,1)$. *Then the following properties are equivalent:*

(i) *A is countable.*

(ii) *K_A is metrizable.*

(iii) *K_A is fragmentable.*

In order to exploit the special properties of the Kalenda compacta we need to introduce one more definition. A subset A of $\mathbb{R}$ is called *perfectly meager*, if for every perfect set P of $\mathbb{R}$, $P \cap A$ is meager (i.e., first category) in P.

Theorem 3.2 (Kalenda, 1999, Proposition 5) *For any subset A of* $(0,1)$, *the following conditions are equivalent:*

(i) *Every closed subset of K_A contains a dense completely metrizable subspace.*

(ii) *A is perfectly meager.*

If $\mathcal{C}$ is a class of Baire spaces which is closed with respect to taking open subspaces and dense G_δ subsets, then we say that a topological space X is in *Stegall's class with respect to $\mathcal{C}$* if, whenever B is from $\mathcal{C}$ and φ is a minimal usco from B into X, then φ is single-valued at some point of B.

Theorem 3.3 (Kalenda, 2002, Proposition) *Let $\mathcal{C}$ be a class of Baire metric spaces which is closed with respect to taking open subspaces and dense G_δ subsets and suppose that A is an arbitrary subset of* $(0,1)$. *Then the following assertions are equivalent:*

(i) *K_A is in Stegall's class with respect to $\mathcal{C}$.*

(ii) *For any $B \in \mathcal{C}$ and any continuous function $f : B \to A$ the function f has at least one local minimum or local maximum.*

(iii) *For any $B \in \mathcal{C}$ and any continuous function $f : B \to A$ there is a non-empty open set $U \subseteq B$ such that f is constant on U.*

Proof. (i) $\Rightarrow$ (ii). Suppose (i) holds. Let $B \in \mathcal{C}$ and let $f : B \to A$ be a continuous function. In order to obtain a contradiction, let us assume that f has no local extrema. Then the mapping $\varphi : B \to 2^{K_A}$ defined by, $\varphi(t) := \{f(t)\} \times \{0,1\}$ is not only an usco but is in fact a minimal usco. Therefore, since K_A is in Stegall spaces with respect to $\mathcal{C}$ we have our desired contradiction since φ is everywhere two-valued.

(ii) $\Rightarrow$ (iii). Suppose that (ii) holds and that there is some $B \in \mathcal{C}$ and some continuous function $f : B \to A$ that is not constant on any non-empty open subset of B. Fix a metric ρ generating the topology of B. For each $n \in \mathbb{N}$ define,

$$E_n^{\max} := \{b \in B : f(b) = \max\{f(b') : \rho(b,b') < 1/n\}\};$$

$$E_n^{\min} := \{b \in B : f(b) = \min\{f(b') : \rho(b,b') < 1/n\}\}.$$

Then clearly both of the sets $E_n^{\max}$ and $E_n^{\min}$ are closed and $E := \bigcup_{n\in\mathbb{N}}(E_n^{\max} \cup E_n^{\min})$ is the set of all local extrema of f on B. If one of the sets $E_n^{\max}$ or $E_n^{\min}$ has an interior point, then f is constant on a neighbourhood of it. Indeed, if b is an interior point of $E_n^{\max}$ then $B_\rho(b;\delta) \subseteq E_n^{\max}$ for some $0 < \delta < 1/n$. Let $b' \in B(b;\delta)$. Then both $f(b) \geq f(b')$ and $f(b') \geq f(b)$ hold and so $f(b) = f(b')$. This shows that f is constant on $B(b;\delta)$. Hence both of the sets $E_n^{\max}$ and $E_n^{\min}$ are closed and nowhere dense. Therefore E is a first category set and $B' := B \setminus E$ is a dense G_δ subset of B

and so it belongs to $\mathcal{C}$. Thus, by (ii), $f|_{B'}$ has a local extremum at a point b. Then, by continuity of f and density of B' in B, f has a local extremum at b, with respect to B, too. Thus $b \in E$ and hence we have a contradiction.

(iii) $\Rightarrow$ (i). Let $M \in \mathcal{C}$, $\varphi : M \to 2^{K_A}$ be a minimal usco and let $p : K_A \to [0,1]$ be defined by, $p(t,\varepsilon) := t$ (i.e., p is the natural projection). When K_A is given the order topology and $[0,1]$ is given the usual topology, p is continuous. Therefore by Lemma 2.4, the composition mapping $p \circ \varphi : M \to 2^{[0,1]}$ is a minimal usco on M. Since $[0,1]$ is metrizable (and hence in Stegall's class) there exists a dense G_δ subset G of M such that $p \circ \varphi$ is single-valued at each point of G.

Now if $(p \circ \varphi)(m) \subseteq [0,1] \backslash A$ for some $m \in G$ then φ is single-valued at m. Hence we need only consider the case when $(p \circ \varphi)(G) \subseteq A$. Now by the assumption, there exists a non-empty open subset U of G and an element $a \in A$ such that $(p \circ \varphi)(U) = \{a\}$, i.e., $\varphi(U) \subseteq \{a\} \times \{0,1\}$. Now by applying Lemma 2.8 twice we see that the restriction of φ to U is still a minimal usco and since the two point set $\{a\} \times \{0,1\}$ is metrizable (and hence in Stegall's class), we conclude that φ must be single-valued at some points of U and the proof is complete. $\square$

Next we give a characterisation of K_A spaces that are weakly Stegall. This characterisation is achieved by means of "perfect sets". A subset $A \subseteq \mathbb{R}$ is called *perfect* if it is closed and does not have any isolated points.

Theorem 3.4 (Kalenda, 1997, Proposition W3) *Let A be an arbitrary subset of $(0,1)$. Then the following assertions are equivalent:*

(i) *A does not contain any perfect compact sets.*

(ii) *K_A is a weakly Stegall space.*

Proof. (i) $\Rightarrow$ (ii). From Theorem 3.3, it is sufficient to show that for any complete metric space M and any continuous function $f : M \to A$ there is a non-empty open subset $U \subseteq M$ such that f is constant on U. Let (M,ρ) be a complete metric space. In order to obtain a contradiction let us suppose that $f : M \to A$ is not locally constant on any non-empty open subset of M. Let D be the set of all finite sequences of 0's and 1's. We shall inductively (on the length $|d|$ of $d \in D$) define a family $\{C_d : d \in D\}$ of non-empty open subsets of M such that:

(a) $\rho\text{-diam}(C_d) < 1/2^{|d|}$;

(b) $\emptyset = \overline{C}_{d0} \cap \overline{C}_{d1} \subseteq \overline{C}_{d0} \cup \overline{C}_{d1} \subseteq C_d$ for each $d \in D$;

(c) $f(\overline{C}_{d0}) \cap f(\overline{C}_{d1}) = \emptyset$.

Base step: Let $C_\emptyset$ be a non-empty open subset of M with $\rho\text{-diam}(C_\emptyset) < 1/2^0$, where the sequence of length zero is denoted by $\emptyset$.

Assuming that we have already defined the non-empty open sets C_d satisfying the properties (a), (b) and (c) for all $d \in D$ with $|d| \leq n$, we proceed to our inductive step.

Inductive step: Fix $d \in D$ of length n. Then there are two points c_0 and c_1 in C_d such that $f(c_0) \neq f(c_1)$. From the continuity of f we can choose open neighbourhoods C_{d0} of c_0 and C_{d1} of c_1 such that conditions (a), (b) and (c) are satisfied. This completes the induction.

Now for each $n \in \mathbb{N}$, let

$$K_n := \bigcup\{\overline{C}_d : d \in D \text{ and } |d| = n\}$$

and let $K := \bigcap\{K_n : n \in \mathbb{N}\}$. Then K is a perfect compact subset of M and since f is continuous, $f(K)$ is a perfect compact subset of A. This contradicts (i) and therefore f must be constant on some non-empty open subset of M.

(ii) $\Rightarrow$ (i). In order to obtain a contradiction, let us suppose that A does contain a perfect compact set C. We shall consider the identity mapping $f : C \to C$ defined by, $f(x) := x$ for all $x \in C$. Now since C is a perfect set, it does not have any isolated points and we have that f is a continuous nowhere constant function defined on a complete metric space. This contradicts Theorem 3.3 part (iii) and therefore A does not contain any perfect compact sets. □

It has been shown in (Kalenda, 1999), that under some additional set theoretic assumptions, there are examples of Stegall compact spaces which are not fragmentable. We shall now show that the Stegall spaces and the weakly Stegall spaces do not coincide. But first, we need the following definition.

A subset $A \subseteq \mathbb{R}$ is called a *Bernstein set* if neither A nor its complement contains any perfect compact sets, (Oxtoby, 1971, p.23). We note here that a Bernstein set with the Euclidean topology is a Baire space.

Example 3.5 (Kenderov et al., 2001a, Example 2) *Let A be a Bernstein subset of* $(0, 1)$. *Then K_A is a weakly Stegall space but not a Stegall space.*

Proof. Let A be a Bernstein subset of $(0, 1)$. Then A does not contain any perfect compact sets and by Theorem 3.4 we conclude that K_A is a weakly Stegall space.

Now to show that K_A is not a Stegall space, consider the mapping $\varphi : A \to 2^{K_A}$ acting from the Baire space A into K_A defined by, $\varphi(t) := \{(t, 0) \cup (t, 1)\}$ for every $t \in A$. We see that φ is not only an usco but is in fact a minimal usco and it is two-valued everywhere. Thus K_A is not a Stegall space. □

The question as to whether the product of weakly Stegall spaces is a weakly Stegall space which was raised in Chapter 2 remains to be answered. We shall now provide a counter-example to show that weakly Stegall spaces are not closed under products.

Theorem 3.6 (Kalenda, 1997, Example W1) *There exists a set A of* $(0, 1)$ *such that K_A is weakly Stegall but $K_A \times K_A$ is not weakly Stegall.*

Proof. Let B be a Bernstein subset of $(0, 1)$ and let

$$A := [(0, 1/2) \cap B] \cup [1/2 + (0, 1/2) \backslash B]$$

(then A does not contain any perfect compact subsets either). Since A does not contain any perfect compact sets, it follows from Theorem 3.4 that K_A is a weakly Stegall space.

Let $p : K_A \to [0,1]$ be the natural projection. Then $p^{-1} : [0,1] \to 2^{K_A}$ defined by, $p^{-1}(t) := \{(t', \varepsilon) \in K_A : p(t', \varepsilon) = t\}$ is an usco mapping.

Next to show that $K_A \times K_A$ is not weakly Stegall, let us consider the mapping $\varphi : (0, 1/2) \to 2^{K_A \times K_A}$ defined by, $\varphi(x) := p^{-1}(x) \times p^{-1}(x + 1/2)$ for every $x \in (0, 1/2)$. Since p^{-1} is an usco, it follows easily that φ is an usco mapping. It then follows from the characterisation of the minimality of φ given in Lemma 2.3 that φ is a minimal usco. On the other hand, we see that φ is two-valued everywhere on $(0, 1/2)$ and this shows that $K_A \times K_A$ is not a weakly Stegall space. □

It has been shown that under some additional set theoretic assumptions, the classes $\tilde{\mathcal{F}}$, class($\tilde{\mathcal{S}}$), and weak Asplund spaces are distinct. To demonstrate this we need a precise description of the duals of the $C(K_A)$ spaces. In (Kenderov et al., 2001b), the authors give a brief description of these dual spaces, however in the next section, for the sake of posterity, we give a more detailed description of these dual spaces.

3.2 Dual representation of $C(K_A)$ spaces

The first step towards characterising the dual space of $C(K_A)$ is to identify this space with something that looks more like $C[0,1]$. (Note: Riesz's representation theorem gives a description of the dual of $C(K_A)$ in terms of regular Borel measures on K_A. However, for our purposes this representation is not so useful.)

Given a subset A of $(0,1)$ we shall denote by D_A the space of all real-valued functions on $(0,1]$ that have (i) finite right-hand limits at the points of $[0,1)$, (ii) are left continuous at the points of $(0,1]$ and (iii) are continuous at the points of $(0,1)\backslash A$. We shall consider this space endowed with the supremum norm.

Theorem 3.7 (Kenderov et al., 2001b, Theorem 5) *For every set $A \subseteq (0,1)$, D_A is isometrically isomorphic to $C(K_A)$.*

Proof. We define an isometry T from D_A onto $C(K_A)$ by $T(f)((t,0)) := f(t)$ for all $t \in (0,1]$ and $T(f)((t,1)) := \lim_{t' \to t^+} f(t')$ for $t \in \{0\} \cup A$. It follows from

(Fabian, 1997, p.47) that T is in fact an isometry from D_A onto $C(K_A)$. Indeed, it is routine to verify that T is a linear isometry into $C(K_A)$, so it suffices to check that T is onto. To this end, let $g \in C(K_A)$ and define $f : (0,1] \to \mathbb{R}$ by $f(t) := g((t,0))$ for all $t \in (0,1]$. Then $f \in D_A$ and $T(f) = g$. $\square$

We shall now characterise the duals of these spaces in terms of functions of bounded variation. Given bounded functions f and α defined on $(0,1]$ and $[0,1]$ respectively and a partition $P := \{t_k : 0 \leq k \leq n\}$ of $[0,1]$ where

$$0 = t_0 < t_1 < t_2 < \cdots < t_n = 1,$$

the *Riemann-Stieltjes sum of f with respect to α, determined by P*, is the real number

$$S(P, f, \alpha) := \sum_{k=1}^{n} f(t_k) \cdot [\alpha(t_k) - \alpha(t_{k-1})].$$

We say that f is *Riemann-Stieltjes integrable with respect to α* if there exists a real number I such that for every $\varepsilon > 0$ there exists a partition P_ε of $[0,1]$ such that $|S(P, f, \alpha) - I| < \varepsilon$ for all partitions P that refine P_ε. In this case I is denoted by,

$$I := \int_{[0,1]} f(t) \mathrm{d}\alpha(t)$$

and is called the *Riemann-Stieltjes integral of f with respect to α.*

Remark 3.8 *The definition of the Riemann-Stieltjes integral given here differs from the standard definition in the sense that the assigned value of the function f over the subinterval $[t_{k-1}, t_k]$ is always taken to be the evaluation at the right-hand end point. The reason for this is that with the standard definition of the Riemann-Stieltjes integral, not all the functions in $D_{(0,1)}$ are Riemann-Stieltjes integrable.*

For any subset A of $(0,1)$ we shall denote by $BV_A[0,1]$ the space of all real-valued functions of bounded variation on $[0,1]$ that are right continuous at the points of $(0,1)\backslash A$ and map 0 to 0. We will consider this space endowed with the total variation norm, i.e., for each $\alpha \in BV_A[0,1]$,

$$\|\alpha\| := \mathrm{Var}(\alpha) = \sup\{\sum_{k=1}^{n} |\alpha(t_k) - \alpha(t_{k-1})| : \{t_k : 0 \leq k \leq n\} \text{ is a partition of } [0,1]\}.$$

Lemma 3.9 (Uniform approximation lemma)(Kenderov et al., 2001b, Lemma 1) *Let* A *be any dense subset of* $(0,1)$, $f \in D_A$ *and* $\varepsilon > 0$. *Then there exists a partition* $P_\varepsilon := \{t_k : 0 \leq k \leq n\}$ *of* $[0,1]$ *with* $t_k \in A$ *for all* $1 \leq k < n$ *such that* $\|f - f_{P_\varepsilon}\|_\infty < \varepsilon$, *where* $f_{P_\varepsilon} : (0,1] \to \mathbb{R}$ *is defined by,*

$$f_{P_\varepsilon}(t) := \sum_{k=1}^{n} f(t_k) \cdot \chi_{(t_{k-1},t_k]}(t).$$

We can now use this lemma to prove the following theorem.

Theorem 3.10 (Kenderov et al., 2001b, Theorem 1) *Suppose that* $\alpha : [0,1] \to \mathbb{R}$ *has bounded variation and* $f \in D_{(0,1)}$. *Then* f *is Riemann-Stieltjes integrable with respect to* α.

Proof. In order to show that f is Riemann-Stieltjes integrable with respect to α we need only show that for every $\varepsilon > 0$ there exists a partition P_ε of $[0,1]$ such that

$$|S(P_\varepsilon, f, \alpha) - S(P', f, \alpha)| < \varepsilon \quad \text{for all partitions } P' \text{ that refine } P_\varepsilon.$$

An elementary calculation shows that for any $g, g' \in D_{(0,1)}$ and partition P we have that $|S(P, g, \alpha) - S(P, g', \alpha)| \leq \|g - g'\| \cdot \mathrm{Var}(\alpha)$. Therefore, if we fix $\varepsilon > 0$ and choose a partition P of $[0,1]$ such that $\|f - f_P\| < \varepsilon/(\mathrm{Var}(\alpha) + 1)$, then

$$\begin{aligned} |S(P,f,\alpha) - S(P',f,\alpha)| &\leq |S(P,f,\alpha) - S(P,f_P,\alpha)| \\ &+ |S(P,f_P,\alpha) - S(P',f_P,\alpha)| \\ &+ |S(P',f_P,\alpha) - S(P',f,\alpha)| \\ &< 0 + 0 + \varepsilon = \varepsilon \end{aligned}$$

for all partitions P' that refine P. □

Theorem 3.11 (Moors and Sciffer, 2002, Theorem 3) *The dual of* $D_{(0,1)}$ *is isometrically isomorphic to* $BV_{(0,1)}[0,1]$.

Proof. Let us consider the mapping $T : D^*_{(0,1)} \to BV_{(0,1)}[0,1]$ defined by,

$$T(x^*)(0) := 0 \quad \text{and} \quad T(x^*)(t) := x^*(\chi_{(0,t]}) \quad \text{for } 0 < t \leq 1.$$

We claim that T is an isometry from $D^*_{(0,1)}$ onto $BV_{(0,1)}[0,1]$. First we will show that $\|T(x^*)\| \leq \|x^*\|$ for all $x^* \in D^*_{(0,1)}$. To this end, let $x^* \in D^*_{(0,1)}$, $P := \{t_k : 0 \leq k \leq n\}$ be any partition of $[0,1]$ and $\sigma_k := \operatorname{sgn}[T(x^*)(t_k) - T(x^*)(t_{k-1})]$ for each $k \in \{1,2,\ldots,n\}$. We define $y \in B_{D_{(0,1)}}$ by,

$$y(t) := \sum_{k=1}^{n} \sigma_k \cdot (\chi_{(0,t_k]} - \chi_{(0,t_{k-1}]}(t)) = \sum_{k=1}^{n} \sigma_k \cdot \chi_{(t_{k-1},t_k]}(t).$$

Then by the definition of σ_k we have the following:

$$\begin{aligned} \sum_{k=1}^{n} |T(x^*)(t_k) - T(x^*)(t_{k-1})| &= \sum_{k=1}^{n} \sigma_k \cdot [T(x^*)(t_k) - T(x^*)(t_{k-1})] \\ &= \sum_{k=1}^{n} \sigma_k \cdot x^*(\chi_{(t_{k-1},t_k]}) = x^*(y) \leq \|x^*\|. \end{aligned}$$

Thus $\|T(x^*)\| = \operatorname{Var}(T(x^*)) \leq \|x^*\|$. Now we show that $\|x^*\| \leq \|T(x^*)\|$ for all $x^* \in D^*_{(0,1)}$. To achieve this it is sufficient to show that $\|x^*(f)\| \leq \|T(x^*)\| \cdot \|f\|_\infty$ for all $f \in D_{(0,1)}$. In fact, by Lemma 3.9, it is sufficient to show that $\|x^*(f)\| \leq \|T(x^*)\| \cdot \|f\|_\infty$ for all $f \in Y$ where Y is the linear span of all the functions in $D_{(0,1)}$ of the form: $\chi_{(0,x]}$ with $x \in (0,1]$. This follows from a routine calculation. So it remains to show that T is onto. To this end, let $\alpha \in BV_{(0,1)}[0,1]$ and define $x^* \in D^*_{(0,1)}$ by,

$$x^*(f) := \int_{[0,1]} f(t) \mathrm{d}\alpha(t) \quad \text{for all } f \in D_{(0,1)}.$$

Then since

$$\left| \int_{[0,1]} f(t) \mathrm{d}\alpha(t) \right| \leq \|\alpha\| \cdot \|f_\infty\| \quad \text{for all } f \in D_{(0,1)},$$

it follows that x^* is indeed a member of $D^*_{(0,1)}$. Now for each $0 < t \leq 1$,

$$T(x^*)(t) = x^*(\chi_{(0,t]}) = \int_{[0,1]} \chi_{(0,t]} \mathrm{d}\alpha(t) = \alpha(t).$$

This shows that $T(x^*) = \alpha$ and proves that T is onto. □

Theorem 3.12 (Kenderov et al., 2001b, Theorem 2) *Let A be any subset of* $(0,1)$. *Then the dual of D_A is isometrically isomorphic to $BV_A[0,1]$.*

Proof. (Moors and Somasundaram, 2002a, Theorem 7) Consider the mapping $T_A : BV_A[0,1] \to D_A^*$ defined by,

$$T_A(\alpha)(x) := \int_{[0,1]} x(t)\mathrm{d}\alpha(t) \quad \text{for each } x \in D_A.$$

We claim that T_A is an isometry from $BV_A[0,1]$ onto D_A^*. First let us note that T_A really does map into D_A^*. Indeed, for any given $\alpha \in BV_A[0,1]$ it is easy to see that the mapping $x \mapsto T_A(\alpha)(x)$ is linear and it is also reasonable routine to check that $|T_A(\alpha)(x)| \leq \|\alpha\| \cdot \|x\|$ for all $x \in D_A$. Therefore, for each $\alpha \in BV_A[0,1]$, $T_A(\alpha) \in D_A^*$. In fact $\|T_A(\alpha)\| \leq \|\alpha\|$. It should also be clear that T_A is a bounded linear operator. We will now show that T_A is onto. To this end, let $x^* \in D_A^*$.

By the Hahn-Banach extension theorem there exists a $y^* \in D_{(0,1)}^*$ such that $\|x^*\| = \|y^*\|$ and $y^*|_{D_A} = x^*$ (since D_A is a subspace of $D_{(0,1)}$). Now from Theorem 3.11 there exists an $\alpha \in BV_{(0,1)}[0,1]$ such that $y^*(x) = \int_{[0,1]} x(t)\mathrm{d}\alpha(t)$ for all $x \in D_{(0,1)}$ and $\|\alpha\| = \|y^*\|$. Unfortunately, this α may not lie in $BV_A[0,1]$. So we must consider a new function $\tilde{\alpha} : [0,1] \to \mathbb{R}$ defined by,

$$\tilde{\alpha}(t) := \alpha(t) \text{ if } t \in \{0\} \cup A \cup \{1\} \quad \text{and} \quad \tilde{\alpha}(t) := \lim_{t' \to t^+} \alpha(t') \text{ if } t \in (0,1)\backslash A.$$

Then by (Bachman and Narici, 1966, Theorem 13.2) it follows that $\|\tilde{\alpha}\| \leq \|\alpha\|$ and

$$\int_{[0,1]} x(t)\mathrm{d}\tilde{\alpha}(t) = \int_{[0,1]} x(t)\mathrm{d}\alpha(t) = y^*(x) = x^*(x) \quad \text{for all } x \in D_A.$$

Therefore, $T_A(\tilde{\alpha}) = x^*$ and $\|x^*\| = \|T_A(\tilde{\alpha})\| \leq \|\tilde{\alpha}\| \leq \|\alpha\| = \|y^*\| = \|x^*\|$. Hence T_A is an isometry onto D_A^*. □

For a non-empty subset A of $(0,1)$ we shall denote by τ_A the topology (on $BV_A[0,1]$) of pointwise convergence on $A \cup \{1\}$. If A is dense in $(0,1)$, then τ_A is a Hausdorff topology. Moreover, the closed unit ball in $BV_A[0,1]$ (with respect to the total variation norm) is τ_A-compact.

Theorem 3.13 (Kenderov et al., 2001b, Corollary 1) *For a non-empty subset A of $(0,1)$, $(BV_A[0,1], \tau_A)$ is homeomorphic to D_A^* endowed with the weak topology generated by the functions $\chi_{(0,a]}$ with $a \in A \cup \{1\}$. If A is dense in $(0,1)$, then*

τ_A is Hausdorff and the closed unit ball $B_{BV_A[0,1]}$ in $BV_A[0,1]$ with the τ_A-topology is homeomorphic to $(B_{D_A^*}, weak^*)$. In fact the mapping T defined in the previous theorem restricted to the ball $B_{BV_A[0,1]}$, realises such a homeomorphism.

Proof. The proof of the first assertion is based upon the fact that for each $\alpha \in BV_A[0,1]$ and $t \in A \cup \{1\}, T(\alpha)(\chi_{(0,t]}) = \alpha(t)$. The fact that T restricted to $B_{BV_A[0,1]}$, realises a homeomorphism onto $(B_{D_A^*}, \text{weak}^*)$, follows from the fact that on $B_{D_A^*}$ the relative weak* topology and the relative topology generated by the functions $\chi_{(0,t]}, t \in A \cup \{1\}$ coincide (see Lemma 3.9). □

3.3 Distinguishing the classes of $C(K_A)$ spaces

The next result follows directly from Theorem 3.1.

Theorem 3.14 (Kenderov et al., 2001b, Theorem 5) *Let A be an arbitrary subset of $(0,1)$. Then the following assertions are equivalent:*

(i) *A is countable.*

(ii) *$(C(K_A)^*, weak^*)$ is fragmentable.*

Proof. (i) $\Rightarrow$ (ii). If A is countable, it follows from Theorem 3.1 that K_A is metrizable. This in turn implies that $C(K_A)$ is separable and thus $(C(K_A)^*, \text{weak}^*)$ is fragmentable.

(ii) $\Rightarrow$ (i). If $(C(K_A)^*, \text{weak}^*)$ is fragmentable, then K_A is fragmentable as K_A is homeomorphic to a subspace of $(C(K_A)^*, \text{weak}^*)$. The result then follows from Theorem 3.1. □

Equipped with the representation theorem for the dual of $C(K_A)$ we may now present the following theorem which will enable us to distinguish the classes: $\tilde{\mathcal{F}}$, class($\tilde{\mathcal{S}}$) and weak Asplund spaces. The following theorem is a special case of (Kalenda, 2002, Proposition).

Theorem 3.15 *Let $\mathcal{C}$ be a class of Baire metric spaces which is closed with respect to taking open subspaces and dense Baire subsets and suppose that A is dense in $(0,1)$. Then the following assertions are equivalent:*

(i) $(C(K_A)^*, \textit{weak}^*)$ *is in Stegall's class with respect to* $\mathcal{C}$.

(ii) *For any $B \in \mathcal{C}$ and any continuous function $f : B \to A$ there is a non-empty open set $U \subseteq B$ such that f is constant on U.*

Proof. (i) $\Rightarrow$ (ii). If $(C(K_A)^*, \text{weak}^*)$ is in Stegall's class with respect to $\mathcal{C}$, then K_A is in Stegall's class with respect to $\mathcal{C}$ as K_A is homeomorphic to a subspace of $(C(K_A)^*, \text{weak}^*)$. The result then follows from Theorem 3.3.
(ii) $\Rightarrow$ (i). This is shown in (Kenderov et al., 2001b). □

Theorem 3.16 (Kenderov et al., 2001b, Theorem 3) *Let Y be a compact topological space. Then Y belongs to Stegall spaces if, and only if, Y belongs to Stegall spaces with respect to all Baire metric spaces.*

The following remark follows from (Kalenda, 2002, Proposition).

Remark 3.17 $(C(K_A)^*, \textit{weak}^*)$ *is a Stegall space if, and only if, K_A is a Stegall space and for any compact Hausdorff space K, $(C(K)^*, \textit{weak}^*)$ is fragmentable if, and only if, K is fragmentable, (Ribarska, 1987). The corresponding general result for the class of Stegall spaces remains unknown.*

We now show that under some additional set theoretic assumptions the classes: $\tilde{\mathcal{F}}$, class$(\tilde{\mathcal{S}})$ and weak Asplund spaces are distinct. Let $\mathcal{C}$ be a class of Baire metric spaces which is closed with respect to taking open subsets and dense Baire subspaces. We will say that a subset A of $(0,1)$ satisfies *property* $(*)$ *with respect to* $\mathcal{C}$ if for every $B \in \mathcal{C}$ and every continuous function $f : B \to A$ there exists a non-empty open set U of B such that f is constant on U. Of course every countable dense subset of $(0,1)$ has this property. We will be particularly interested in the case when A is uncountable.

We are now in a position to distinguish the $C(K_A)$ spaces.

Corollary 3.18 (Moors and Somasundaram, 2002a, Corollary 1)

(i) *If there is an uncountable subset A of $(0,1)$ that satisfies property $(*)$ with respect to the class of all Baire metric spaces then $C(K_A)$ belongs to class($\tilde{\mathcal{S}}$) but not to $\tilde{\mathcal{F}}$.*

(ii) *If there is an uncountable subset A of $(0,1)$ that satisfies property $(*)$ with respect to the class of all Baire metric spaces of density at most card(A) then $C(K_A)$ is a weak Asplund space that does not belong to $\tilde{\mathcal{F}}$.*

(iii) *If there is a subset A of $(0,1)$ that satisfies property $(*)$ with respect to the class of all Baire metric spaces of density at most card(A), but not property $(*)$ with respect to the class of all Baire metric spaces then $C(K_A)$ is a weak Asplund space that does not belong to class($\tilde{\mathcal{S}}$).*

Proof.

(i) From Theorem 3.14 it follows that $C(K_A) \notin \tilde{\mathcal{F}}$. On the other hand, it is shown in Theorem 3.16 that a Banach space X belongs to class($\tilde{\mathcal{S}}$) if, and only if, (X^*, weak^*) is in Stegall spaces with respect to the class of all Baire metric spaces. The result then follows from Theorem 3.15.

(ii) Again from Theorem 3.14 it follows that $C(K_A) \notin \tilde{\mathcal{F}}$. To show that $C(K_A)$ is weak Asplund, we need Theorem 2.30 that for a Banach space X if (X^*, weak^*) is in Stegall spaces with respect to the class of all Baire metric spaces with density at most equal to the density of X then X is weak Asplund. The result then follows from Theorem 3.15 and the fact that the density of $C(K_A)$ equals card(A).

(iii) As mentioned in part (ii) if $(C(K_A)^*, \text{weak}^*)$ is in Stegall spaces with respect to the class of all Baire metric spaces with density at most equal to the density of $C(K_A)$ then $C(K_A)$ is weak Asplund. The fact that $C(K_A) \notin$ class($\tilde{\mathcal{S}}$) follows directly from Theorem 3.15. □

Corollary 3.19 (Moors and Somasundaram, 2002a, Corollary 2) *Under the assumption that there exists an uncountable dense subset A of $(0,1)$ that satisfies property $(*)$ with respect to the class of all Baire metric spaces of density at most card(A), there exists a weak Asplund space that does not admit a smooth Lipschitz bump function.*

Proof. It follows from Theorem 3.14 that $C(K_A) \notin \tilde{\mathcal{F}}$ and so by (Fosgerau, 1992), where it was shown that every Banach space that admits a smooth Lipschitz bump function has weak* fragmentable dual, $C(K_A)$ does not admit a smooth Lipschitz bump function. However, as in Corollary 3.18 part (ii) $C(K_A)$ is weak Asplund. □

Till this point, we have not dwelt upon the question of whether there are in fact subsets of $(0,1)$ that satisfy any of the hypotheses of Corollary 3.18 or Corollary 3.19. For a discussion on this see (Namioka and Pol, 1992), (Kalenda, 1999), (Kalenda, 2002) and (Kalenda and Kunen, 2003). Let us mention here though that in all cases additional set theoretic assumptions are required.

We note here that the set theoretic assumptions used in (Kenderov et al., 2001b) and (Kalenda, 2002) cannot hold simultaneously. So this naturally raises the question as to whether the classes: $\tilde{\mathcal{F}}$, class($\tilde{\mathcal{S}}$) and weak Asplund spaces are mutually distinct. It has now been shown, assuming the consistency of the existence of a measurable cardinal, that it is consistent to have two Banach spaces X and Y where X is a weak Asplund space that does not belong to class($\tilde{\mathcal{S}}$) and Y belongs to class($\tilde{\mathcal{S}}$) but not to $\tilde{\mathcal{F}}$, (Kalenda and Kunen, 2003).

3.4 An example of a non-weakly Stegall space of the form $C(K_A)$

We saw in Section 3.3 that K_A is fragmentable if, and only if, $(C(K_A)^*, \text{weak}^*)$ is fragmentable and K_A is a Stegall space if, only if, $(C(K_A)^*, \text{weak}^*)$ is a Stegall space. It is somewhat surprising that the analogous result fails for weakly Stegall spaces.

We now provide a counter-example.

Let B be a Bernstein subset of $(0,1)$. We now set (as before)

$$A := [(0,1/2) \cap B] \cup [1/2 + (0,1/2) \backslash B]$$

(then A does not contain any perfect compact subsets either). It then follows from Theorem 3.4 that K_A is a weakly Stegall space.

We will now show that $(C(K_A), \|\cdot\|_\infty)$ is not an almost weak Asplund space (in particular, $(C(K_A)^*, \text{weak}^*)$ is not weakly Stegall), but first we need the following lemmas.

Lemma 3.20 (Moors and Somasundaram, 2003, Lemma 1) *Let* $M : C(K_A) \to \mathbb{R}$ *be defined by,*

$$M(f) := \sup\{f(t,\varepsilon) + f(t+1/2,\varepsilon') : (t,\varepsilon,\varepsilon') \in \Sigma\}$$

where Σ *is given by,*

$$\Sigma := \{(t,\varepsilon,\varepsilon') : 0 \le t \le 1/2 \text{ and } \varepsilon,\varepsilon' \in \{0,1\} \text{ with } (t,\varepsilon) \in K_A \text{ and } (t+1/2,\varepsilon') \in K_A\}.$$

Then M *is a continuous convex function on* $C(K_A)$.

Proof. For each $(t,\varepsilon,\varepsilon') \in \Sigma$, define $x^*_{(t,\varepsilon,\varepsilon')} : C(K_A) \to \mathbb{R}$ by,

$$x^*_{(t,\varepsilon,\varepsilon')}(f) := f(t,\varepsilon) + f(t+1/2,\varepsilon').$$

Then each $x^*_{(t,\varepsilon,\varepsilon')}$ is a continuous linear functional on $C(K_A)$, (i.e., $x^*_{(t,\varepsilon,\varepsilon')} \in C(K_A)^*$) and $\|x^*_{(t,\varepsilon,\varepsilon')}\| = 2$. Now, $M(f) = \sup\{x^*_{(t,\varepsilon,\varepsilon')}(f) : (t,\varepsilon,\varepsilon') \in \Sigma\}$. Therefore, as M is the pointwise supremum of continuous (2-Lipschitz) linear functionals, M is convex and 2-Lipschitz. $\square$

Lemma 3.21 (Moors and Somasundaram, 2003, Lemma 2) $M(f) = \max\{f(t,\varepsilon) + f(t+1/2,\varepsilon') : (t,\varepsilon,\varepsilon') \in \Sigma\}$ *for each* $f \in C(K_A)$.

Proof. We define $p : K_A \to [0,1]$ by $p(t,\varepsilon) := t$ (i.e., p is the natural projection). When K_A is given the order topology and $[0,1]$ is given the usual topology, p is continuous. Hence $p^{-1} : [0,1] \to 2^{K_A}$ defined by, $p^{-1}(t) := \{(t',\varepsilon) : p(t',\varepsilon) = t\}$ is an usco mapping. So for each $f \in C(K_A)$ the mapping $T_f : [0,1/2] \to 2^{\mathbb{R}}$ defined by,

$$T_f(t) := f(p^{-1}(t)) + f(p^{-1}(t+1/2))$$

is an usco mapping, since by Lemma 2.4 and Lemma 2.5, continuous images and sums of usco mappings are again usco mappings. Thus, $T_f([0,1/2])$ is a compact subset of $\mathbb{R}$ and so,

$$\begin{aligned} M(f) &= \sup\{T_f(t) : t \in [0,1/2]\} \\ &= \max\{T_f(t) : t \in [0,1/2]\} \\ &= \max\{f(t,\varepsilon) + f(t+1/2,\varepsilon') : (t,\varepsilon,\varepsilon') \in \Sigma\}. \end{aligned}$$

This completes the proof. □

We will say that $(t,\varepsilon,\varepsilon') \in \Sigma$ *supports* M *at* f if $M(f) = f(t,\varepsilon) + f(t+1/2,\varepsilon')$. Let us now observe (using the notation of Lemma 3.20) that if $(t,\varepsilon,\varepsilon')$ supports M at f then,

$$x^*_{(t,\varepsilon,\varepsilon')} \in \partial M(f) := \{x^* \in C(K_A)^* : x^*(g) - x^*(f) \le M(g) - M(f) \quad \text{for all } g \in C(K_A)\}.$$

Since M is convex, it follows from Lemma 2.25 that M is Gâteaux differentiable at f if, and only if, $\partial M(f)$ is a singleton.

Lemma 3.22 (Moors and Somasundaram, 2003, Lemma 3) *Let* $S : C(K_A) \to 2^{[0,1/2]}$ *be defined by,*

$$S(f) := \{t \in [0,1/2] : (t,\varepsilon,\varepsilon') \text{ supports } M \text{ at } f \text{ for some } \varepsilon,\varepsilon' \in \{0,1\}\}.$$

Then S *is a minimal usco.*

Proof. Let $f \in C(K_A)$ and let T_f be defined as in Lemma 3.21, then

$$S(f) = \{t \in [0,1/2] : T_f(t) \cap [M(f),\infty) \neq \emptyset\}.$$

Therefore, $S(f)$ is a non-empty closed (and hence compact) subset of $[0,1/2]$. Consider $f \in C(K_A)$ and U an open subset of $[0,1/2]$ containing $S(f)$. Let

$$M_{[0,1/2]\backslash U}(f) := \max\{T_f(t) : t \in [0,1/2]\backslash U\} < M(f).$$

Choose $0 < \varepsilon < [M(f) - M_{[0,1/2]\backslash U}(f)]/4$. Then an easy calculation shows that,

$$M_{[0,1/2]\backslash U}(g) < M(g)$$

for all $g \in B(f;\varepsilon)$ and so $S(g) \subseteq U$ for all $g \in B(f;\varepsilon)$. This shows that S is upper semicontinuous. Next we show that S is minimal. To this end, let $U \subseteq C(K_A)$ and $W \subseteq [0,1/2]$ be open sets such that $S(U) \cap W \neq \emptyset$. Pick $f \in U$ such that $S(f) \cap W \neq \emptyset$ and let $t_0 \in S(f) \cap W$.

If $S(f) = \{1/2\} \subseteq W$ then the upper semicontinuity of S yields that $S(V) \subseteq W$ for some non-empty open set $V \subseteq U$ and the proof is complete. Consider the case when $S(f) \neq \{1/2\}$. Then take $t_0 \in W \cap (S(f)\backslash\{1/2\})$. Let $p : K_A \to [0,1]$ be the natural projection. There exists a continuous function $h : [0,1] \to [0,1]$ such that $h(t) = 1$ if, and only if, $t = t_0$ and $\text{supp}(h) \subseteq W \cap [0,1/2]$. We claim that for each $\delta > 0$, the mapping $g_\delta : K_A \to \mathbb{R}$ defined by, $g_\delta(x,\varepsilon) := f(x,\varepsilon) + \delta h(p(x,\varepsilon))$ has $S(g_\delta) = \{t_0\} \subseteq W$. Now for $t \neq t_0$, we have

$$\begin{aligned} g_\delta(t_0,\varepsilon) + g_\delta(t_0+1/2,\varepsilon') &= f(t_0,\varepsilon) + \delta h(p(t_0,\varepsilon)) + f(t_0+1/2,\varepsilon') + \delta h(p(t_0+1/2,\varepsilon')) \\ &= f(t_0,\varepsilon) + \delta + f(t_0+1/2,\varepsilon') + \delta h(p(t_0+1/2,\varepsilon')) \\ &> f(t,\varepsilon) + \delta h(p(t,\varepsilon)) + f(t+1/2,\varepsilon') + \delta h(p(t+1/2,\varepsilon')) \\ &= g_\delta(t,\varepsilon) + g_\delta(t+1/2,\varepsilon'). \end{aligned}$$

Therefore for δ sufficiently small, $g_\delta \in U$ and $S(g_\delta) = \{t_0\} \subseteq W$. □

Example 3.23 (Moors and Somasundaram, 2003, Example 3) *$(C(K_A), \|\cdot\|_\infty)$ is not an almost weak Asplund space. In particular, $C(K_A) \notin$ class$(w\tilde{S})$.*

Proof. We show that $C(K_A)$ is not an almost weak Asplund space by showing that the continuous convex function M defined on $C(K_A)$ is non-Gâteaux differentiable at the points of a residual subset of $C(K_A)$. Let $S : C(K_A) \to 2^{[0,1/2]}$ be defined as in Lemma 3.22 and let $G := \{f \in C(K_A) : S(f) \text{ is a singleton}\}$. By Lemma 3.22 and the fact that $[0,1/2]$ is metrizable and hence in Stegall's class, G is a residual (and hence dense) subset of $C(K_A)$. As in Lemma 3.21, let $p : K_A \to [0,1]$ denote the natural projection of K_A onto $[0,1]$ and for each $n \in \mathbb{N}$ let,

$$\begin{aligned} O_{1/n} := \{f \in C(K_A) :\ & \text{diam } [f(p^{-1}(S(f)))] < 1/n \text{ and} \\ & \text{diam } [f(p^{-1}(S(f)+1/2))] < 1/n\}. \end{aligned}$$

Since both mappings $f \mapsto f(p^{-1}(S(f)))$ and $f \mapsto f(p^{-1}(S(f)+1/2)))$ are usco mappings it is easy to see that each $O_{1/n}$ is an open subset of $C(K_A)$. We claim that each $O_{1/n}$ is dense in $C(K_A)$. To justify this assertion let us fix $n \in \mathbb{N}$, $f_0 \in G$ and $\varepsilon > 0$, with the goal of showing that $O_{1/n} \cap B(f_0, \varepsilon) \neq \emptyset$. We shall only consider the case when $S(f_0) := \{t_0\} \subseteq [0,1/2) \backslash A$ as the case when $S(f_0) \subseteq A$ is similar. By the continuity of f_0 there exists a $\delta > 0$ such that,

(i) $|f_0(t,\varepsilon) - f_0(t_0,0)| < 1/4n$ for all $(t,\varepsilon) \in K_A$ with $|t - t_0| < \delta$;

(ii) $|f_0(t+1/2,\varepsilon) - f_0(t_0+1/2,1)| < 1/4n$ for all $(t+1/2,\varepsilon) \in K_A$ with $t \in (t_0, t_0+\delta)$;

(iii) $|f_0(t+1/2,\varepsilon) - f_0(t_0+1/2,0)| < 1/4n$ for all $(t+1/2,\varepsilon) \in K_A$ with $t \in (t_0-\delta, t_0)$.

Since the mapping S is upper semicontinuous there exists a $0 < \delta' < 1/4n$ such that $S(B(f_0;\delta')) \subseteq (t_0-\delta, t_0+\delta)$. Now, if $|f_0(t_0+1/2,0) - f_0(t_0+1/2,1)| < 1/n$, then $O_{1/n} \cap B(f_0;\varepsilon) \neq \emptyset$. So we will assume that $|f_0(t_0+1/2,0) - f_0(t_0+1/2,1)| \geq 1/n$. In fact we will assume that $f_0(t_0+1/2,1) - f_0(t_0+1/2,0) \geq 1/n$. By judiciously choosing a "bump function" $h : K_A \to [0,\infty)$ with $\|h\|_\infty < \delta'$ and

$$\text{supp}(h) := \{(t,\varepsilon) : h(t,\varepsilon) \neq 0\} \subseteq p^{-1}((t_0+1/2, t_0+1/2+\delta))$$

we can assume that $M(f_0) < M(f_0+h)$. In fact with this choice of function we can show that $S(f_0+h) \subseteq (t_0, t_0+\delta)$. Indeed, if $t \in S(f_0+h)$ and $t \leq t_0$ then,

$$(f_0+h)(t,\varepsilon) + (f_0+h)(t+1/2,\varepsilon') = f_0(t,\varepsilon) + f_0(t+1/2,\varepsilon') \leq M(f_0) < M(f_0+h)$$

which contradicts the fact that $t \in S(f_0+h)$. It is now a straight forward calculation to show that,

$$\text{diam }[(f_0+h)(p^{-1}(S(f_0+h)))] < 1/n \text{ and}$$
$$\text{diam }[(f_0+h)(p^{-1}(S(f_0+h)+1/2))] < 1/n.$$

Therefore $(f_0+h) \in O_{1/n}$ and so $B(f_0;\varepsilon) \cap O_{1/n} \neq \emptyset$. We now claim that if $f \in \bigcap_{n=1}^{\infty} O_{1/n}$ then M is not Gâteaux differentiable at f. To see this consider the sets:

$$\Sigma(t) := \{(t', \varepsilon, \varepsilon') \in \Sigma : t' = t\} \quad \text{for all } t \in [0, 1/2].$$

Now $|\Sigma(t)| = 2$ for each $t \in [0,1/2]$ and if $t \in S(f)$ then each member of $\Sigma(t)$ supports M at f, i.e., for each $(t,\varepsilon,\varepsilon') \in \Sigma(t)$, $x^*_{(t,\varepsilon,\varepsilon')}$ is a subgradient of M at f. Therefore M is not Gâteaux differentiable at f. This completes the proof. □

Remark 3.24 *It is an open question to characterise those subsets A of $(0,1)$ for which $C(K_A)$ belongs to class$(w\tilde{S})$.*

Chapter 4

A Gâteaux differentiability space that is not weak Asplund

The main objective of this chapter is to show that the classes of Gâteaux differentiability spaces and weak Asplund spaces are in fact distinct. We saw in Chapter 2 that every member of class($w\tilde{\mathcal{S}}$) is a Gâteaux differentiability space (see Theorem 2.29). However, it is not necessarily true that if X belongs to class($w\tilde{\mathcal{S}}$) then X is weak Asplund. Hence one might hope that there exists a member of class($w\tilde{\mathcal{S}}$) that is not weak Asplund. Indeed, in this chapter we show that there exists a Gâteaux differentiability space that is not weak Asplund by constructing a member of class($w\tilde{\mathcal{S}}$) that is not weak Asplund.

The contents of this chapter are mainly from (Moors and Somasundaram, 2004). We begin by building up the tools that are required to achieve our goal. In the first section of this chapter, we shall introduce a new class of topological spaces known as the "nearly Stegall" spaces and study their permanence properties. In the next section of this chapter, we fulfill the goal of this thesis by showing that there is a member of class($w\tilde{\mathcal{S}}$) that is not weak Asplund.

4.1 Nearly Stegall spaces

Let (M, d) be a complete metric space. We shall denote the space of all continuous functions from $\{0,1\}^{\mathbb{N}}$ into M equipped with the topology of uniform convergence by $C(\{0,1\}^{\mathbb{N}}; M)$. The following notation will be used in the proof of the next lemma.

For each $n \in \mathbb{N}$ and $t \in \{0,1\}^{\mathbb{N}}$ we define, $C_t := \{t' \in \{0,1\}^{\mathbb{N}} : t'|_n = t\}$ and $C_\emptyset := \{0,1\}^{\mathbb{N}}$. Further for each $n \in \mathbb{N}$, we shall let $\Gamma_n := \{\gamma \in C(\{0,1\}^{\mathbb{N}}; M) : \gamma$ is constant on C_t for each $t \in \{0,1\}^{\mathbb{N}}\}$. By a simple compactness argument we see that for each $n \in \mathbb{N}$, $\bigcup_{k \geq n} \Gamma_k$ is dense in $C(\{0,1\}^{\mathbb{N}}; M)$. The following lemma builds the principal tools we shall need in the next section.

Lemma 4.1 (Moors, 2004, Lemma 5) *Let (M, d) be a complete metric space. Then we have the following.*

(i) *If U is a dense open subset of M then $\{\gamma \in C(\{0,1\}^{\mathbb{N}}; M) : \gamma(\{0,1\}^{\mathbb{N}}) \subseteq U\}$ is a dense open subset of $C(\{0,1\}^{\mathbb{N}}; M)$.*

(ii) *If U is a non-empty open subset of M and $g : U \to X$ is a continuous function acting from U into a completely regular space X that is not constant on any non-empty open subset of U, then for each γ belonging to a residual subset of $C(\{0,1\}^{\mathbb{N}}; M)$, $g \circ \gamma$ is one-to-one on $\gamma^{-1}(U)$ and $\gamma^{-1}(U)$ is both open and closed in $\{0,1\}^{\mathbb{N}}$.*

Proof.

(i) We see that $\{\gamma \in C(\{0,1\}^{\mathbb{N}}; M) : \gamma(\{0,1\}^{\mathbb{N}}) \subseteq U\}$ is open in $C(\{0,1\}^{\mathbb{N}}; M)$. So, we only need to show that it is dense in $C(\{0,1\}^{\mathbb{N}}; M)$. For each $n \in \mathbb{N}$, let $\Gamma_n(U) := \{\gamma \in \Gamma_n : \gamma(\{0,1\}^{\mathbb{N}}) \subseteq U\}$. We conclude from the density of U in M and the fact that $\bigcup_{n \geq 1} \Gamma_n$ is dense in $C(\{0,1\}^{\mathbb{N}}; M)$ that $\bigcup_{n \geq 1} \Gamma_n(U)$ is dense in $C(\{0,1\}^{\mathbb{N}}; M)$.

(ii) Let $V := U \cup M \backslash \overline{U}$. Then V is a dense open subset of M. For each $n \in \mathbb{N}$ let

$$\Gamma_n^* := \{\gamma \in \Gamma_n(V) : (g \circ \gamma)(t) \neq (g \circ \gamma)(t') \text{ if } t, t' \in \gamma^{-1}(U) \text{ and } t|_n \neq t'|_n\}.$$

We see that for each $n \in \mathbb{N}$, $\bigcup_{k \geq n} \Gamma_k^*$ is dense in $C(\{0,1\}^{\mathbb{N}}; M)$. Now for each $n \in \mathbb{N}$ and $\hat{\gamma} \in \Gamma_n^*$ choose $r_n(\hat{\gamma}) > 0$ such that the following conditions are satisfied.

(i) $B(\hat{\gamma}(t); r_n(\hat{\gamma})) \subseteq U$ for all $t \in \hat{\gamma}^{-1}(U)$;

(ii) $B(\hat{\gamma}(t); r_n(\hat{\gamma})) \subseteq M \backslash \overline{U}$ for all $t \in \hat{\gamma}^{-1}(M \backslash \overline{U})$;

(iii) $g(B(\hat{\gamma}(t); r_n(\hat{\gamma}))) \cap g(B(\hat{\gamma}(t'); r_n(\hat{\gamma}))) = \emptyset$ for all $t, t' \in \hat{\gamma}^{-1}(U)$ such that $t|_n \neq t'|_n$.

We see that the set

$$G := \bigcap_{n=1}^{\infty} \left(\bigcup_{k=n}^{\infty} \{\gamma \in C(\{0,1\}^{\mathbb{N}}; M) : \text{there exists a } \hat{\gamma} \in \Gamma_k^* \text{ with } \max_{t \in \{0,1\}^{\mathbb{N}}} [d(\gamma(t), \hat{\gamma}(t))] < r_k(\hat{\gamma})\} \right)$$

is residual in $C(\{0,1\}^{\mathbb{N}}; M)$ as it is a countable intersection of dense open sets. It can also be seen that for each $\gamma \in G$, $g \circ \gamma$ is one-to-one on $\gamma^{-1}(U)$. □

If $\mathscr{A}$ is a proper σ-ideal of subsets of $\{0,1\}^{\mathbb{N}}$ and N is a subset of a complete metric space M then we say that N is *$\mathscr{A}$-negligible* if $\gamma^{-1}(N) \in \mathscr{A}$ for each γ belonging to a residual subset R_N of $C(\{0,1\}^{\mathbb{N}}; M)$. We note here that the residual set R_N will in general depend upon the set N.

From Lemma 4.1 we can deduce that for any proper σ-ideal $\mathscr{A}$ of subsets of $\{0,1\}^{\mathbb{N}}$ the $\mathscr{A}$-negligible sets form a proper σ-ideal of subsets of M that contains all the first category subsets of M. Let us also note that a subset N of a complete metric space M that has the Baire property is $\mathscr{A}$-negligible if, and only if, it is of first category. Hence the interesting $\mathscr{A}$-negligible sets are among those sets that are not very topologically respectable.

Given a proper σ-ideal $\mathscr{A}$ of subsets of $\{0,1\}^{\mathbb{N}}$ and a topological space X we shall say that X is *nearly Stegall with respect to $\mathscr{A}$* if for every complete metric space M and minimal usco $\varphi : M \to 2^X$, φ is single-valued except on an $\mathscr{A}$-negligible subset of M. Thus for **any** proper σ-ideal $\mathscr{A}$ of subsets of $(\{0,1\}^{\mathbb{N}}, \tau_p)$ and any topological space X, if X is nearly Stegall with respect to $\mathscr{A}$ then X is weakly Stegall.

4.1.1 Permanence properties of nearly Stegall spaces

Nearly Stegall spaces enjoy several permanence properties like Stegall and weakly Stegall spaces. Below we provide the basic properties of nearly Stegall spaces that will be needed in the proof of Theorem 4.8. We note here that Theorem 4.2 part (ii) can be generalised as in Theorem 2.17 part (iii).

Theorem 4.2 (Moors and Somasundaram, 2004, Theorem 3) *Let* (X, τ) *and* (Y, τ') *be topological spaces and let* $\mathscr{A}$ *denote a proper* σ*-ideal of subsets of* $(\{0,1\}^{\mathbb{N}}, \tau_p)$.

(i) *Let* $f : X \to Y$ *be a perfect mapping onto* Y. *If* X *is nearly Stegall with respect to* $\mathscr{A}$ *then* Y *is nearly Stegall with respect to* $\mathscr{A}$.

(ii) *Let* $\{X_n : n \in \mathbb{N}\}$ *be a cover of* X. *If each* X_n *is a closed subset of* X *and is nearly Stegall with respect to* $\mathscr{A}$, *then* X *is nearly Stegall with respect to* $\mathscr{A}$.

(iii) *If each* $\{X_n : n \in \mathbb{N}\}$ *is nearly Stegall with respect to* $\mathscr{A}$, *then* $\prod_{n=1}^{\infty} X_n$ *is nearly Stegall with respect to* $\mathscr{A}$.

Proof. The proofs of (i) and (ii) are similar to that given in Theorem 2.17. So it remains to prove (iii). Let M be a complete metric space and $\varphi : M \to 2^{\prod_{n=1}^{\infty} X_n}$ be a minimal usco. Denote by π_n the canonical projection from $\prod_{n=1}^{\infty} X_n$ onto X_n. Then $\pi_n \circ \varphi : M \to 2^{X_n}$ is a minimal usco by Lemma 2.4. As each X_n is nearly Stegall with respect to $\mathscr{A}$, $\pi_n \circ \varphi$ is single-valued on R_n which is the complement of $\mathscr{A}$-negligible subsets of M. Clearly $\pi_n \circ \varphi$ is single-valued at the points of $R := \bigcap_{n=1}^{\infty} R_n$. Since $\mathscr{A}$-negligible sets form a proper σ-ideal of subsets of M, we have that $M \backslash R = \bigcup_{n=1}^{\infty} (M \backslash R_n) \in \mathscr{A}$. Therefore, $\prod_{n=1}^{\infty} X_n$ is nearly Stegall with respect to $\mathscr{A}$. □

4.2 An almost weak Asplund space that is not weak Asplund

Let A be an arbitrary subset of $(0,1)$. In the remainder of this chapter, we shall denote by $M_A[0,1]$ the set of all non-decreasing functions in $B_{BV_A[0,1]}$. For any α, β in $M_A[0,1]$ we define,

$$\rho_I(\alpha,\beta) := \sum_{n=1}^{\infty} |(\alpha-\beta)(a_n)|/2^n \quad \text{and} \quad \rho_J(\alpha,\beta) := \sum_{t\in A} |(\alpha-\beta)(t^+) - (\alpha-\beta)(t)|$$

where $a_1 := 1$ and $\{a_n : n \geq 2\} \subseteq A$ is dense in $[0,1]$.

Note: $\{t \in A : |(\alpha-\beta)(t^+) - (\alpha-\beta)(t)| > 0\}$ is at most countable. Then we define $\rho(\alpha,\beta) := \rho_I(\alpha,\beta) + \rho_J(\alpha,\beta)$. With a little thought it should be clear that ρ defines a metric on the set $M_A[0,1]$.

Remark 4.3 *It is not too difficult to check that ρ_I is a continuous pseudo-metric on $M_A[0,1]$, i.e., for each $\alpha \in M_A[0,1]$ and $r > 0$ the set $\{\beta \in M_A[0,1] : \rho_I(\alpha,\beta) < r\}$ is τ_A-open in $M_A[0,1]$. Hence it follows that ρ_I "fragments" $M_A[0,1]$. In particular, this means that for any minimal usco $\varphi : M \to 2^{M_A[0,1]}$ acting from any Baire space M into subsets of $M_A[0,1]$, there is a residual set $R \subseteq M$ such that ρ_I-diam$[\varphi(x)] = 0$ at each point $x \in R$ (see Theorem 2.31). One immediate consequence of this is that for each $x \in R$, we may unambiguously refer to the left-hand and right-hand limits of $\varphi(x)$, since if $\alpha, \beta \in \varphi(x)$, then both the left-hand and right-hand limits of α and β coincide on* $[0,1]$.

Next, we give some technical results that will be needed in our main theorem.

Lemma 4.4 (Kenderov et al., 2001b, Lemma 2) *Let $\varphi : X \to 2^Y$ be a minimal usco acting between topological spaces X and Y and let $f : Y \to \mathbb{R}$ be a continuous function. Then there is a residual set R in X such that the composition mapping $f \circ \varphi : X \to 2^{\mathbb{R}}$ defined by $(f \circ \varphi)(x) := \{f(y) : y \in \varphi(x)\}$ is single-valued at the points of R.*

Proof. Since φ is a minimal usco and f is a continuous function, it follows from Lemma 2.4 that $f \circ \varphi$ is a minimal usco on X. The result then follows from (Fabian, 1997, Theorem 5.1.11). $\square$

Lemma 4.5 (Moors, 2004) *Let M be a complete metric space, $\varphi : M \to 2^{M_A[0,1]}$ be a minimal usco and $g : M \to [0,1]$ be locally constant on M. Then there exists a dense G_δ subset $G \subseteq M$ such that the mapping $T : M \to \mathbb{R}$ defined by, $T(x) := \{\alpha(g(x)) : \alpha \in \varphi(x)\}$ is single-valued at each point $x \in G$.*

Proof. It follows from Remark 4.3 that there exists a residual set $R \subseteq M$ such that $\rho_I\text{-diam}[\varphi(x)] = 0$ at each point $x \in R$. Now let $\mathcal{U} := \{U_\gamma : \gamma \in \Gamma\}$ be a maximal collection of non-empty disjoint open subsets of M such that g is constant on each $U_\gamma, \gamma \in \Gamma$. Such a maximal family is guaranteed by Zorn's lemma. We observe that $U := \bigcup_{\gamma \in \Gamma} U_\gamma$ is dense in M. Now for each $\gamma \in \Gamma$, choose $x_\gamma \in U_\gamma$.

We shall consider the cases when (i) $g(x_\gamma) \in A$ and (ii) $g(x_\gamma) \notin A$. In case (i) for each $\gamma \in \Gamma$, define $t_\gamma := g(x_\gamma)$. Now it follows from Lemma 4.4 that each of the mappings $\hat{t}_\gamma \circ \varphi : M \to 2^{\mathbb{R}}$ defined by, $(\hat{t}_\gamma \circ \varphi)(x) := \{\alpha(t_\gamma) : \alpha \in \varphi(x)\}$ is single-valued on a dense G_δ subset $G_\gamma \subseteq U_\gamma$.

In case (ii), when $g(x_\gamma) \notin A$, the required dense G_δ set will be $G_\gamma := U_\gamma \cap R$, since for any $\alpha, \beta \in \varphi(x)$, α, β are right continuous at $g(x_\gamma)$ and $\alpha(g(x_\gamma)) = \alpha^+(g(x)) = \beta^+(g(x)) = \beta(g(x_\gamma))$. $\square$

Lemma 4.6 (Moors, 2004) *Let M be a complete metric space and $\varphi : M \to 2^{M_A[0,1]}$ be a minimal usco. Then there exist a dense G_δ subset G of M and continuous maps $(g_n : n \in \mathbb{N})$ from G into $[0,1]$ such that, for every $x \in G$*

$$\{t \in [0,1] : 0 < \alpha(t^+) - \alpha(t) \quad \textit{for some}\ \alpha \in \varphi(x)\} \subseteq \{g_n(x) : n \in \mathbb{N}\}.$$

Proof. Let us fix $\varepsilon > 0$, a closed set F and an open set U with $\emptyset \neq F \subseteq U := (a,b) \subseteq [0,1]$. Define

$$\mathcal{Q}_{(\varepsilon,F,U)} := \{\alpha \in M_A[0,1] : \text{there exists a } \xi \text{ in } F \text{ with } \alpha(\xi^+) - \alpha(\xi^-) \geq 2\varepsilon \text{ and}$$
$$\max\{\alpha(\xi^-) - \alpha(a^+) \leq \varepsilon, \alpha(b^-) - \alpha(\xi^+)\} \leq \varepsilon\}.$$

Then $\mathscr{Q}_{(\varepsilon,F,U)}$ is τ_A-closed and for each $\alpha \in \mathscr{Q}_{(\varepsilon,F,U)}$ there is exactly one $\xi := \xi(\alpha) \in F$ with the above property. Moreover, the mapping $\alpha \mapsto \xi(\alpha)$ from $\mathscr{Q}_{(\varepsilon,F,U)}$ into $[0,1]$ is continuous.

By Tietze's extension theorem, this mapping can be extended to a continuous map $\xi_{(\varepsilon,F,U)}$ of $M_A[0,1]$ to $[0,1]$. Since φ is a minimal usco, it follows by Lemma 2.4 that $\xi_{(\varepsilon,F,U)} \circ \varphi : M \to 2^{[0,1]}$ is a minimal usco. Now since $[0,1]$ is metrizable and therefore in Stegall spaces, we have that $\xi_{(\varepsilon,F,U)} \circ \varphi$ is single-valued and continuous at the points of a dense G_δ subset $G_{(\varepsilon,F,U)}$ of M. Let $\mathcal{B}$ be a countable base for the Euclidean topology of open interval on $(0,1)$ and let

$$G := \bigcap \{G_{(\varepsilon,\overline{V},U)} : \varepsilon \in (0,\infty) \cap \mathbb{Q}, \quad V, U \in \mathcal{B} \text{ and } \emptyset \neq V \subseteq \overline{V} \subseteq U\}.$$

Then if we denote by $(g_n : n \in \mathbb{N})$ the functions in

$$\{(\xi_{(\varepsilon,\overline{V},U)} \circ \varphi)|_G : \varepsilon \in (0,\infty) \cap \mathbb{Q}, \quad V, U \in \mathcal{B} \text{ and } \emptyset \neq V \subseteq \overline{V} \subseteq U\}$$

ordered into a sequence then we have the following. If $x \in G, \alpha \in \varphi(x)$ and $t \in [0,1]$ are such that $\alpha(t^+) - \alpha(t) > 0$ then $\{t\} = (\xi_{(\varepsilon,\overline{V},U)} \circ \varphi)|_G(x)$ for some $(\varepsilon, \overline{V}, U)$ and so $t = g_n(x)$ for some $n \in \mathbb{N}$. □

The following remark is a consequence of Remark 4.3 and Lemma 4.6.

Remark 4.7 *Let M be a complete metric space and $\varphi : M \to 2^{M_A[0,1]}$ be a minimal usco. Let $(g_n : n \in \mathbb{N})$ and G be the continuous maps and dense G_δ subset of M given in Lemma 4.6. If $x \in G$ and $\{\alpha(g_n(x)) : \alpha \in \varphi(x)\}$ is a singleton for all $n \in \mathbb{N}$ then ρ_J-diam$[\varphi(x)] = 0$.*

We are now in a position to present our main theorem.

Theorem 4.8 (Moors and Somasundaram, 2004, Theorem 4) *Let $\mathscr{A}$ be a proper σ-ideal of subsets of $\{0,1\}^{\mathbb{N}}$ and let A be any subset of $(0,1)$ such that $\gamma^{-1}(A) \in \mathscr{A}$ for every homeomorphic embedding of $(\{0,1\}^{\mathbb{N}}, \tau_p)$ into $[0,1]$. Then $(BV_A[0,1], \tau_A)$ is nearly Stegall with respect to $\mathscr{A}$. In particular, $(BV_A[0,1], \tau_A)$ is weakly Stegall.*

Proof. First, let us note that by Theorem 4.2 part (ii), we need only show that the closed unit ball $B_{BV_A[0,1]}$ of $BV_A[0,1]$ is nearly Stegall with respect to $\mathscr{A}$. In fact, we need only show that the (τ_A-compact) set $M_A[0,1]$ of all non-decreasing functions in $B_{BV_A[0,1]}$, endowed with the τ_A-topology is nearly Stegall with respect to $\mathscr{A}$. Since if $M_A[0,1]$ is nearly Stegall with respect to $\mathscr{A}$, then by Theorem 4.2 part (iii), $M_A[0,1] \times M_A[0,1]$ is nearly Stegall with respect to $\mathscr{A}$. However, by the Jordan decomposition theorem $B_{BV_A[0,1]} \subseteq \Delta(M_A[0,1] \times M_A[0,1])$, where $\Delta : M_A[0,1] \times M_A[0,1] \to BV_A[0,1]$ is defined by, $\Delta(f,g) := f - g$. Hence the result follows from Theorem 4.2 part (i), since Δ is a perfect mapping.

To this end, let M be a complete metric space and $\varphi : M \to 2^{M_A[0,1]}$ be a minimal usco. We show that ρ-diam$[\varphi(x)] = 0$ at some point $x \in M$. Now since ρ_I fragments $M_A[0,1]$, it follows from Remark 4.3 that there exists a residual set $G' \subseteq M$ such that ρ_I-diam$[\varphi(x)] = 0$ for all $x \in G'$.

Furthermore, let G be the dense G_δ subset of M and $(g_n : n \in \mathbb{N})$ be the continuous maps from G into $[0,1]$ as in Lemma 4.6. Since $C(\{0,1\}^{\mathbb{N}}; G \cap G')$ is a residual subspace of $C(\{0,1\}^{\mathbb{N}}; M)$ and the restriction of φ to $G \cap G'$ is still a minimal usco (see, Lemma 2.8) we can assume without loss of generality that $M = G = G'$.

Thus it is sufficient to show that ρ_J-diam$[\varphi(x)] = 0$ at some point $x \in M$. In fact by Remark 4.7 it is sufficient to show that $\{\alpha(g_n(x)) : \alpha \in \varphi(x)\}$ is a singleton for all $n \in \mathbb{N}$. So we need only show that for each $n \in \mathbb{N}$, the complement of the set

$$G_n := \{x \in M : \{\alpha(g_n(x)) : \alpha \in \varphi(x)\} \text{ is a singleton}\}$$

is $\mathscr{A}$-negligible in M. To this end, let us fix $n \in \mathbb{N}$ and let $\mathscr{G}_n$ be the union of all the open sets of M on which g_n is constant and let $\mathscr{H}_n := M \backslash \overline{\mathscr{G}}_n$. We see by Lemma 4.5 that $\mathscr{G}_n \backslash G_n$ is first category in M and therefore $\mathscr{A}$-negligible. Thus we only need to show that $\mathscr{H}_n \backslash G_n$ is $\mathscr{A}$-negligible since $M \backslash (\mathscr{G}_n \cup \mathscr{H}_n)$ is a closed nowhere dense subset of M and therefore an $\mathscr{A}$-negligible set. Moreover, since $\mathscr{H}_n \backslash G_n \subseteq g_n^{-1}(A) \cap \mathscr{H}_n$, we only need to show that $g_n^{-1}(A) \cap \mathscr{H}_n$ is $\mathscr{A}$-negligible. By Lemma 4.1 the set of all $\gamma \in C(\{0,1\}^{\mathbb{N}}; M)$ for which (i) $\gamma(\{0,1\}^{\mathbb{N}}) \subseteq \mathscr{G}_n \cup \mathscr{H}_n$ and (ii) $g_n \circ \gamma$ is one-to-one on $D_\gamma^n := \gamma^{-1}(\mathscr{H}_n)$ is residual in $C(\{0,1\}^{\mathbb{N}}; M)$. For any such γ we have the following

$$\gamma^{-1}(g_n^{-1}(A) \cap \mathscr{H}_n) = (g_n \circ \gamma)^{-1}(A) \cap \gamma^{-1}(\mathscr{H}_n) = (g_n \circ \gamma)^{-1}(A) \cap D_\gamma^n \in \mathscr{A}$$

since (i) $g_n \circ \gamma$ is one-to-one on the closed and open set D_γ^n and (ii) every continuous one-to-one mapping from a closed and open subset of $\{0,1\}^{\mathbb{N}}$ into $[0,1]$ can be extended to be a homeomorphic embedding of $\{0,1\}^{\mathbb{N}}$ into $[0,1]$. Thus we conclude that $g_n^{-1}(A) \cap \mathscr{H}_n$ is $\mathscr{A}$-negligible in M. Therefore, $\varphi : M \to 2^{M_A[0,1]}$ is single-valued except on an $\mathscr{A}$-negligible subset of M and we conclude that $(M_A[0,1], \tau_A)$ is nearly Stegall with respect to $\mathscr{A}$ and therefore weakly Stegall. □

To apply Theorem 4.8 we need to consider some small subsets of $\mathbb{R}$. The following lemma is due to David Fremlin (private communication).

Lemma 4.9 (Moors and Somasundaram, 2004, Lemma 3) *There exist a proper σ-ideal $\mathscr{A}$ of subsets of $(\{0,1\}^{\mathbb{N}}, \tau_p)$ and an everywhere second category subset A of* $(0,1)$ *such that $\gamma^{-1}(A) \in \mathscr{A}$ for every homeomorphic embedding of* $(\{0,1\}^{\mathbb{N}}, \tau_p)$ *into* $[0,1]$.

Proof. Let κ be the least ordinal of cardinality $2^{\aleph_0}$, let $\{(f_n^\alpha : n \in \mathbb{N}) : \alpha < \kappa\}$ be an enumeration of all the sequences of continuous one-to-one functions from $(\{0,1\}^{\mathbb{N}}, \tau_p)$ into $[0,1]$ and let $\{E^\alpha : \alpha < \kappa\}$ be an enumeration of all the non-meager Borel subsets of $(0,1)$. Inductively, we may choose

$$a_\alpha \in E^\alpha \backslash \{f_n^\beta(x_\beta) : n \in \mathbb{N} \text{ and } \beta < \alpha\} \quad \text{and} \quad x_\alpha \in \{0,1\}^{\mathbb{N}} \text{ such that}$$
$$f_n^\alpha(x_\alpha) \neq a_\beta \text{ for any } n \in \mathbb{N} \text{ and } \beta \leq \alpha.$$

Set $A := \{a_\alpha : \alpha < \kappa\}$. Then A is not meager and for any sequence $(f_n : n \in \mathbb{N})$ of continuous one-to-one functions from $(\{0,1\}^{\mathbb{N}}, \tau_p)$ into $[0,1]$, $(f_n^{-1}(A) : n \in \mathbb{N})$ does not form a cover of $\{0,1\}^{\mathbb{N}}$. So, if we take $\mathscr{A}$ to be the σ-ideal generated by the inverse images, $f^{-1}(A)$, as f runs over all the continuous one-to-one functions from $\{0,1\}^{\mathbb{N}}$ into $[0,1]$, then $\mathscr{A}$ will be a proper σ-ideal of subsets of $(\{0,1\}^{\mathbb{N}}, \tau_p)$ such that $\gamma^{-1}(A) \in \mathscr{A}$ for every homeomorphic embedding of $(\{0,1\}^{\mathbb{N}}, \tau_p)$ into $[0,1]$. □

Remark 4.10 *We note here that there exist other constructions of an everywhere second category subset A of $(0,1)$ under some additional set theoretic assumptions which satisfy the hypotheses given in Theorem 4.8 (see for example, (Somasundaram, 2002)).*

The following result supplies us with some necessary conditions for spaces of the form $C(K_A)$ to be weak Asplund.

Theorem 4.11 (Čoban and Kenderov, 1986) *Let K be a compact Hausdorff space. If $C(K)$ is weak Asplund then every closed subset of K contains a dense completely metrizable subspace.*

Combining Theorem 3.2 and Theorem 4.11, we see that if A is not perfectly meager then $C(K_A)$ is not weak Asplund.

Corollary 4.12 (Moors and Somasundaram, 2004, Corollary 2) *There exists a Banach space $(X, \|\cdot\|)$ such that $(X^*, weak^*)$ is weakly Stegall but $(X, \|\cdot\|)$ is not weak Asplund. In particular, $(X, \|\cdot\|)$ is a Gâteaux differentiability space that is not weak Asplund.*

Proof. Let A be the set constructed in Lemma 4.9 and let $\mathscr{A}$ be the corresponding σ-ideal of subsets of $(\{0,1\}^{\mathbb{N}}, \tau_p)$. Then A satisfies the hypotheses of Theorem 4.8 with respect to $\mathscr{A}$. Hence $(BV_A[0,1], \tau_p)$ is nearly Stegall with respect to $\mathscr{A}$. Therefore, by Theorem 3.13, $(C(K_A)^*, \text{weak}^*)$ is nearly Stegall with respect to $\mathscr{A}$ and so weakly Stegall. On the other hand, if $(C(K_A), \|\cdot\|_\infty)$ is weak Asplund then by Theorem 4.11, every closed subset of K_A contains a dense completely metrizable subspace. However by Proposition 3.2 this implies A is meager (in fact perfectly meager); which it is not. Therefore, $(C(K_A), \|\cdot\|_\infty)$ is not weak Asplund. $\square$

The above corollary answers a question of (Rainwater, 1990). Let f be a continuous convex function defined on a Banach space X. In (Rainwater, 1990) the following question was raised. If X is a Gâteaux differentiability space, is the set of points where f fails to be Gâteaux differentiable necessarily Borel? This question was answered in (Holický et al., 1998) where the authors showed that on every non-separable Hilbert space, there is a continuous convex function such that the set of Gâteaux differentiability points is not Borel. From Corollary 4.12 we know that there exists a continuous convex function such that the set of points where it is Gâteaux differentiable is everywhere second category but nowhere residual. Thus the set of points of Gâteaux differentiability is not a Baire property set.

Chapter 5

Applications of weakly Stegall spaces

Weakly Stegall spaces have applications to other areas of mathematics and the main focus of this chapter is to explore some of these applications. In the first section of this chapter we shall present a selection theorem and then give some applications of it. In the second section, we provide some applications of weakly Stegall spaces and almost weak Asplund spaces in relation to the study of dual differentiability spaces and generic continuity spaces. Finally, in this chapter we provide some more applications of weakly Stegall spaces in connection to the study of the differentiability of Lipschitz functions defined on Banach spaces.

5.1 Selection theorem

We shall begin by giving the following definitions. A set-valued mapping φ acting from a topological space X into non-empty subsets of a topological space Y is said to be *lower semicontinuous* if for each $x_0 \in X$ and every open set W in Y with $\varphi(x_0) \cap W \neq \emptyset$ there exists an open neighbourhood U of x_0 such that $\varphi(x) \cap W \neq \emptyset$ for all $x \in U$. The set-valued mapping φ is said to be *quasi-lower semicontinuous* if for each $x_0 \in X$ and every open set W in Y with $\varphi(x_0) \cap W \neq \emptyset$ there exists an open set U in X such that $x_0 \in \overline{U}$ and $\varphi(x) \cap W \neq \emptyset$ for all $x \in U$.

A commonly occuring notion in the analysis of minimal set-valued mappings is that of a selection. Given a set-valued mapping φ acting from a non-empty set X into a non-empty set Y, a *selection* for φ is a single-valued mapping σ from X into Y such that $\sigma(x) \in \varphi(x)$ for each $x \in X$. Selection theorems provide conditions under which there exists a continuous selection for a set-valued mapping. The best known selection theorem is Michael's selection theorem, (Michael, 1956a) which holds when the domain space is a paracompact topological space, that is, a space which is regular and each open cover of it has an open locally finite refinement, and φ is a lower semicontinuous mapping into non-empty closed convex subsets of a Banach space Y. Michael's second selection theorem, (Michael, 1956b) restricts the domain space but relaxes the conditions on the range space Y to be a complete metric space with φ mapping into closed subsets of Y.

A set-valued mapping $\varphi : X \to 2^Y$ acting from a topological space X into subsets of a topological space Y is called *lower demicontinuous* on X if for every open set V in Y, the interior of the closure of the set $\varphi^{-1}(V) := \{x \in X : \varphi(x) \cap V \neq \emptyset\}$ is dense in the closure of $\varphi^{-1}(V)$, that is, $\text{int}(\overline{\varphi^{-1}(V)})$ is dense in $\overline{\varphi^{-1}(V)}$. When $\{x \in X : \varphi(x) \neq \emptyset\}$ is dense in X, we say φ is *densely defined.* It is clear from the definition that every lower semicontinuous mapping is a lower demicontinuous mapping. Further, from the characterisation of the minimality of usco mappings given in Lemma 2.3, we see that every minimal usco mapping $\varphi : X \to 2^Y$ acting from a topological space X into subsets of a topological space Y is lower demicontinuous on X.

It has been shown in (Čoban et al., 1994) that for every densely defined lower demicontinuous mapping φ acting from a Baire space X into subsets of a monotonely Čech-complete space Y, there exist a dense and G_δ subset $X_1 \subseteq X$ and an usco mapping $G : X_1 \to 2^Y$ such that $G(x) \subseteq \varphi^*(x)$, for every $x \in X_1$, where the mapping $\varphi^* : X \to 2^Y$ is the extension of φ defined by,

$$\varphi^*(x) := \bigcap\{\overline{\varphi(W)} : W \text{ is a neighbourhood of } x\}.$$

Here we proof the above result with the notion of monotone Čech-completeness replaced by the weaker notion of "partition completeness". We shall now state the concept of partition completeness. Let (Y, τ) be a regular topological space, endowed with a pseudo-metric d. A filter-base $\mathcal{F}$ on Y is said to be *d-Cauchy* if for each $\varepsilon > 0$ there exists an $F \in \mathcal{F}$ such that d-diam$(F) < \varepsilon$ and the space itself is said to be *partition complete* if the pseudo-metric d satisfies the following properties:

(i) Every d-Cauchy filter-base $\mathcal{F}$ on Y has a τ-cluster point in Y (i.e., $\bigcap\{\overline{F} : F \in \mathcal{F}\} \neq \emptyset$).

(ii) Y is fragmented by d.

Note: It follows from (i) that in a partition complete space $\bigcap\{\overline{F} : F \in \mathcal{F}\}$ is non-empty and compact for every d-Cauchy filter-base $\mathcal{F}$. The class of partition complete spaces is quite large including all the Čech-complete spaces, that is, completely regular topological spaces that lie as a G_δ subset in some compactification. More details on partition completeness can be found in (Michael, 1991).

Our result also improves the result of (Giles and Moors, 2001) where the authors presented a selection theorem for quasi-lower semicontinuous mappings that map from Baire spaces into subsets of topological spaces that are fragmented by complete metrics. Specifically, we show that for a lower demicontinuous mapping φ with closed graph acting from a Baire space X into a partition complete space Y there exist a dense and G_δ subset $X_1 \subseteq X$ and an usco mapping $G : X_1 \to 2^Y$ such that $G(x) \subseteq \varphi(x)$ for all $x \in X_1$. In addition we show that if the range space Y is partition complete and is a Stegall space then the mapping G may also be assumed to be single-valued on X_1. We also show that if the domain space X is α-favourable and the range space is partition complete and is weakly Stegall then the mapping G is single-valued on an everywhere second category subset of X.

We shall first establish the following lemma which is a key point in the proof of our selection theorem.

Lemma 5.1 (Moors and Somasundaram, 2002b, Lemma 1) *Consider a lower demicontinuous mapping φ acting from a topological space X into subsets of a topological* *space Y. For each pair of non-empty open sets U in X and V in Y, the mapping $\varphi_{(U,V)}$ from U into subsets of V defined by, $\varphi_{(U,V)}(x) := \varphi(x) \cap V$ is a lower demicontinuous mapping on U.*

Proof. The proof of the lemma follows from the fact that for each open set $W \subseteq V$, $\varphi_{(U,V)}^{-1}(W) = \varphi^{-1}(W) \cap U$. $\square$

Theorem 5.2 (Moors and Somasundaram, 2002b, Theorem 1) *Let X be a Baire space and Y be a Hausdorff partition complete space and let φ be a densely defined lower demicontinuous set-valued mapping acting from X into subsets of Y. Then there exist a dense and G_δ subset $X_1 \subseteq X$ and an usco mapping $G : X_1 \to 2^Y$ with $G(x) \subseteq \varphi^*(x)$ for all $x \in X_1$, where the mapping $\varphi^* : X \to 2^Y$ is defined by,*

$$\varphi^*(x) := \bigcap\{\overline{\varphi(W)} : W \textit{ is a neighbourhood of } x\}.$$

In particular, $\{x \in X : \varphi^*(x) \neq \emptyset\}$ is residual in X.

Proof. Let d be the fragmenting pseudo-metric on Y associated with the partition completeness of Y. To prove our theorem we inductively construct a sequence of families of ordered pairs $\mathcal{F}^n := \{(U_\alpha^n, \varphi_\alpha^n) : \alpha \in \Lambda^n\}$ consisting of non-empty open subsets $\{U_\alpha^n : \alpha \in \Lambda^n\}$ of X and densely defined lower demicontinuous mappings $\{\varphi_\alpha^n : \alpha \in \Lambda^n\}$ such that for each $\alpha \in \Lambda^n$, φ_α^n maps U_α^n into subsets of Y.

Base step: Consider $\Lambda^0 := \{\emptyset\}, U_\emptyset^0 := X$ and $\varphi_\emptyset^0 := \varphi$ and define,

$$\mathcal{F}^0 := \{(U_\alpha^0, \varphi_\alpha^0) : \alpha \in \Lambda^0\} \text{ and } W^0 := \bigcup\{U_\alpha^0 : \alpha \in \Lambda^0\} = X.$$

For each $n \in \mathbb{N}$, we require the family $\mathcal{F}^n$ to have the following properties:

(a_n) $U_\alpha^n \cap U_\beta^n = \emptyset$ for each $\alpha \neq \beta$ and for all $\alpha, \beta \in \Lambda^n$;

(b_n) $W^n := \bigcup\{U_\alpha^n : \alpha \in \Lambda^n\}$ is dense in X;

(c_n) $d\text{-diam}[\varphi_\alpha^n(U_\alpha^n)] < \dfrac{1}{n}$ for each $\alpha \in \Lambda^n$;

(d_n) for each $\alpha \in \Lambda^n$ there exists a $\beta \in \Lambda^{n-1}$ such that $U_\alpha^n \subseteq U_\beta^{n-1}$ and $\varphi_\alpha^n(x) \subseteq \varphi_\beta^{n-1}(x)$ for each $x \in U_\alpha^n$.

Step 1: Consider $\mathcal{F}^1 := \{(U_\alpha^1, \varphi_\alpha^1) : \alpha \in \Lambda^1\}$ a family of ordered pairs satisfying the properties $(a_1), (c_1)$ and (d_1) which is maximal with respect to set inclusion. By Zorn's lemma such a maximal family exists. We shall show that $\mathcal{F}^1$ satisfies property (b_1). If $W^1 := \bigcup\{U_\alpha^1 : \alpha \in \Lambda^1\}$ is not dense in X then there exists a non-empty open subset U of X such that $W^1 \cap U = \emptyset$. Since Y is fragmented by d and $\varphi_\emptyset^0$ is densely defined there exists an open set V in Y such that $\varphi_\emptyset^0(U) \cap V \neq \emptyset$ and d-diam$[\varphi_\emptyset^0(U) \cap V] < 1$. By the lower demicontinuity of $\varphi_\emptyset^0$ on U there exists a non-empty open subset U' of U such that $(\varphi_\emptyset^0)_{(U',V)}$ is densely defined and lower demicontinuous on U' by Lemma 5.1. Now $(U', (\varphi_\emptyset^0)_{(U',V)}) \notin \mathcal{F}^1$ and $\{(U', (\varphi_\emptyset^0)_{(U',V)})\} \cup \mathcal{F}^1$ is a family satisfying the properties $(a_1), (c_1)$ and (d_1). This contradicts the maximality of $\mathcal{F}^1$ and hence we may conclude that $\mathcal{F}^1$ satisfies property (b_1).

Assuming that we have constructed the families $\mathcal{F}^k$ in the sequence satisfying the properties $(a_k), (b_k), (c_k)$ and (d_k) up to and including the n^{th} step, we proceed to construct the next step.

Step $(n+1)$**:** Consider $\mathcal{F}^{n+1} := \{(U_\alpha^{n+1}, \varphi_\alpha^{n+1}) : \alpha \in \Lambda^{n+1}\}$ a family of ordered pairs satisfying the properties $(a_{n+1}), (c_{n+1})$ and (d_{n+1}) which is maximal with respect to set inclusion. We shall show that $\mathcal{F}^{n+1}$ satisfies property (b_{n+1}). If $W^{n+1} := \bigcup\{U_\alpha^{n+1} : \alpha \in \Lambda^{n+1}\}$ is not dense in X then there exists a non-empty open subset U of X such that $W^{n+1} \cap U = \emptyset$. Since W^n is dense in X, $W^n \cap U \neq \emptyset$ and so we may assume that $U \subseteq U_\beta^n$ for some $\beta \in \Lambda^n$. Now since Y is fragmented by d and φ_β^n is densely defined there exists an open set V in Y such that $\varphi_\beta^n(U) \cap V \neq \emptyset$ and d-diam$[\varphi_\beta^n(U) \cap V] < 1/(n+1)$. By the lower demicontinuity of φ_β^n on U_β^n there exists a non-empty open subset U' of U such that $(\varphi_\beta^n)_{(U',V)}$ is densely defined and lower demicontinuous on U' by Lemma 5.1. Clearly, $\{(U', (\varphi_\beta^n)_{(U',V)})\} \notin \mathcal{F}^{n+1}$ and $\{(U', (\varphi_\beta^n)_{(U',V)})\} \cup \mathcal{F}^{n+1}$ is a family satisfying the properties $(a_{n+1}), (c_{n+1})$ and (d_{n+1}). This contradicts the maximality of $\mathcal{F}^{n+1}$ and hence we may conclude that $\mathcal{F}^{n+1}$ satisfies property (b_{n+1}). This completes the induction.

Let $X_1 := \bigcap_{n=1}^\infty W^n$. Clearly X_1 is a dense G_δ subset of X and for each $x \in X_1$, there exists a unique $\alpha_n(x) \in \Lambda^n$ such that $x \in U_{\alpha_n(x)}^n$. Therefore we can define a set-valued mapping $\psi : X_1 \to 2^Y$ by,

$$\psi(x) := \bigcap_{n=1}^{\infty} \overline{\varphi^n_{\alpha_n(x)}(U^n_{\alpha_n(x)})}.$$

Clearly, ψ is non-empty and compact-valued since for each $x \in X_1$,

$$\mathcal{F}(x) := \{\varphi^n_{\alpha_n(x)}(U^n_{\alpha_n(x)}) : n \in \mathbb{N}\}$$

is a d-Cauchy filter-base on Y. So to show that ψ is an usco, it remains to show that ψ is upper semicontinuous. To this end, consider $x \in X_1$ and O an open set containing $\psi(x)$. Since $\psi(x)$ is compact it will suffice to show that there exists an open neighbourhood U of x such that $\psi(U) \subseteq \overline{O}$. We claim that for some $n_0 \in \mathbb{N}$, $\varphi^{n_0}_{\alpha_{n_0}(x)}(U^{n_0}_{\alpha_{n_0}(x)}) \subseteq O$, for otherwise, $\mathcal{F}^*(x) := \{\varphi^n_{\alpha_n(x)}(U^n_{\alpha_n(x)}) \setminus O : n \in \mathbb{N}\}$ would be a d-Cauchy filter-base on Y which would have a cluster point in $Y \setminus O$. But this is impossible since,

$$\emptyset \neq \bigcap_{F \in \mathcal{F}^*} \overline{F} \subseteq \bigcap_{F \in \mathcal{F}} \overline{F} = \psi(x) \subseteq O.$$

Therefore there is some $n_0 \in \mathbb{N}$ such that $\varphi^{n_0}_{\alpha_{n_0}(x)}(U^{n_0}_{\alpha_{n_0}(x)}) \subseteq O$ and so,

$$\psi(y) = \bigcap_{n=1}^{\infty} \overline{\varphi^n_{\alpha_n(y)}(U^n_{\alpha_n(y)})} \subseteq \overline{\varphi^{n_0}_{\alpha_{n_0}(y)}(U^{n_0}_{\alpha_{n_0}(y)})} = \overline{\varphi^{n_0}_{\alpha_{n_0}(x)}(U^{n_0}_{\alpha_{n_0}(x)})} \subseteq \overline{O}$$

for all $y \in U^{n_0}_{\alpha_{n_0}(x)} \cap X_1$.

We now define the mapping $G : X_1 \to 2^Y$ by, $G(x) := \psi(x) \cap \varphi^*(x)$ for all $x \in X_1$. We claim that the mapping G is an usco. Obviously G has a closed graph as both ψ and φ^* have closed graphs. Moreover, as $\mathrm{Gr}(G) \subseteq \mathrm{Gr}(\psi)$ and ψ is an usco, we have by Lemma 2.7 that G is also an usco provided we can show that G has non-empty images. So in order to obtain a contradiction, let us suppose that for some $x_0 \in X_1$, $G(x_0) = \emptyset$. This means that the non-empty compact set $\{x_0\} \times \psi(x_0)$ does not intersect the graph of φ^*. Since $\mathrm{Gr}(\varphi^*)$ is a closed subset of $X \times Y$, a straight forward compactness argument shows that there are open sets U of X and V of Y such that $x_0 \in U$, $\psi(x_0) \subseteq V$ and $(U \times V) \cap \mathrm{Gr}(\varphi^*) = \emptyset$. Since $\psi(x_0) \subseteq V$ it follows, as above, that there exists an $n_0 \in \mathbb{N}$ such that $\varphi^{n_0}_{\alpha_{n_0}(x_0)}(U^{n_0}_{\alpha_{n_0}(x_0)}) \subseteq V$ and so,

$$\emptyset \neq \varphi^{n_0}_{\alpha_{n_0}(x_0)}(U^{n_0}_{\alpha_{n_0}(x_0)} \cap U) \subseteq \varphi(U^{n_0}_{\alpha_{n_0}(x_0)} \cap U) \cap V \subseteq \varphi^*(U) \cap V = \emptyset.$$

This gives us the desired contradiction. Therefore G is an usco selection of φ^*. $\square$

Remark 5.3 *In Theorem 5.2,* φ^* *is the unique mapping whose graph is the closure of the graph of* φ *in* $X \times Y$*, endowed with the product topology. In particular, if* φ *has a closed graph then* $\varphi^* = \varphi$*.*

In order to provide some applications of our selection theorem we need the definition of "α-favourability". Let (X, τ) be a topological space. On X we consider the *Choquet-game* played between two players α and β. A *play* $\{(A_n, B_n) : n \in \mathbb{N}\}$ of the game is a decreasing sequence of, alternately chosen, non-empty open subsets $A_n \subseteq B_n \subseteq \cdots B_2 \subseteq A_1 \subseteq B_1$, where the sets A_n are chosen by player α and the sets B_n by player β. Player α is said to have *won* a play of the game if $\bigcap_{n=1}^{\infty} A_n \neq \emptyset$. Otherwise player β is said to have won the play. A *strategy* ζ for player α is a rule that tells him or her how to play (possibly depending on all the previous moves of player β). Since the moves of player α may depend on the moves of player β, we denote the n^{th} move of player α by, $\zeta(B_1, B_2, \ldots, B_n)$. We say that ζ is a *winning strategy,* if using it, player α wins every play, independently of the moves of player β. A topological space (X, τ) is said to be *α-favourable* if player α has a winning strategy in this game.

The following theorem requires the notion of "countable separation" spaces. We say that a subset Y of a topological space (X, τ) has *countable separation in* X if there is a countable family $\{O_n : n \in \mathbb{N}\}$ of open subsets of X such that for every pair $\{x, y\}$ with $y \in Y$ and $x \in X \backslash Y$, $\{x, y\} \cap O_n$ is a singleton for at least one $n \in \mathbb{N}$. For a completely regular topological space (X, τ) we shall simply say that X has *countable separation* if in some compactification bX, X has countable separation in bX. It is shown in (Kenderov and Moors, 1999) that if X has countable separation in one compactification then X has countable separation in every compactification and that every Čech-analytic space has countable separation. The relationship between countable separation spaces and game determined spaces is revealed in the next theorem.

Theorem 5.4 (Kenderov et al., 2001a, Proposition 2) *Every topological space* (X, τ) *with countable separation is game determined.*

Proof. We shall denote by Y some compact space in which X has countable separation and let $\{U_n : n \in \mathbb{N}\}$ be a family of open subsets of Y which separates the points of X from the points of $Y \backslash X$. We will define a strategy s for the player Ω and show that s is a winning strategy in the game $DG(X)$ (see Chapter 2). Suppose the player Σ chooses a non-empty subset A_1 of X. There are two possibilities. Either $A_1 \cap U_1 = \emptyset$ or $A_1 \cap U_1 \neq \emptyset$. In the first case, we put $B_1 := s(A_1) := A_1$ to be the first move of the player Ω. In the second case we take $B_1 := s(A_1)$ some non-empty subset of A_1 which is relatively open in A_1 and $\overline{B_1}^Y \subseteq U_1$ where $\overline{B_1}^Y$ denotes the closure of B_1 in Y. In both cases the set $\overline{B_1}^Y$ is defined in such a way that it either does not intersect U_1 or is entirely contained in U_1.

Proceeding inductively (on the length of the partial plays) we construct the strategy s in such a way that, for every s-play, $p = \{A_n : n \in \mathbb{N}\}$ and every $n \in \mathbb{N}$ one of the following two possibilities hold. Either $\overline{B_n}^Y \cap U_n = \emptyset$ or $\overline{B_n}^Y \subseteq U_n$. The countable separation of X implies that the compact set $\emptyset \neq K(p) := \bigcap_{n=1}^{\infty} \overline{A_n}^Y$ is either entirely contained in X or in $Y \backslash X$. If $K(p) \subseteq Y \backslash X$, then $\bigcap_{n=1}^{\infty} \overline{A_n}^X = K(p) \cap X = \emptyset$. If $K(p) \subseteq X$, then $K(p) = \bigcap_{n=1}^{\infty} \overline{A_n}^X$ and by the compactness of Y, for every open set U containing $K(p)$ there is some $n \in \mathbb{N}$ with $\overline{A_n}^X \subseteq U$. This shows that X is game determined. □

Theorem 5.5 *Let (X, τ) be a game determined topological space. Then X is weakly Stegall if, and only if, every minimal usco mapping $\varphi : A \to 2^X$ acting from an α-favourable space A into X is single-valued at some point of A.*

Proof. Let $\varphi : A \to 2^X$ be a minimal usco acting from an α-favourable space A into X. Let ζ be a winning strategy for the player α in the Choquet-game played on A and let P denote the space of all ζ-plays endowed with the Baire metric d (see Chapter 2). Then it is straight forward that (P, d) is a complete metric space. Let us now define a set-valued mapping $F : P \to 2^X$ acting from the complete metric space P into X by

$$F(p) := \bigcap_{n=1}^{\infty} \varphi(B_n) \text{ where } p := \{B_n : n \in \mathbb{N}\} \text{ is a } \zeta\text{-play in } A.$$

It follows in a similar manner to Lemma 2.9 that the set-valued mapping $F : P \to 2^X$ is minimal. Since F is a minimal non-empty valued mapping acting from a complete metric space P into a game determined space X, it follows from Theorem 2.14 that there is a dense G_δ subset G of P and a minimal usco $\tilde{F} : G \to 2^X$ such that $F(p) \subseteq \tilde{F}(p)$ for all $p \in G$. Now since G is a G_δ subset of a complete metric space it follows from Theorem 2.10 that G is completely metrizable and since X is a weakly Stegall space, we have that $\tilde{F} : G \to 2^X$ is single-valued at some point $p = (B_n : n \in \mathbb{N})$ of G from which it follows that $F(p)$ is single-valued at p. Since $(B_n : n \in \mathbb{N})$ is a ζ-play, $\bigcap_{n=1}^{\infty} B_n$ is non-empty. It now follows that

$$\emptyset \neq \varphi(\bigcap_{n=1}^{\infty} B_n) \subseteq \bigcap_{n=1}^{\infty} \varphi(B_n) \text{ is a singleton.}$$

This shows that φ is single-valued at the points of $\bigcap_{n=1}^{\infty} B_n \subseteq A$. The proof of the reverse implication easily follows from the fact that complete metric spaces are α-favourable. □

The following folklore result is essential for our purposes in Corollary 5.7.

Theorem 5.6 *Let A be an α-favourable space and let R be a residual subset of A. Then R is α-favourable.*

Proof. Let ζ be the strategy for the player α in the Choquet-game played in A and let $R = \bigcap_{n=1}^{\infty} O_n$ where $\{O_n : n \in \mathbb{N}\}$ is a decreasing sequence of dense open subsets of A. We shall construct a strategy ζ' for the player α in the Choquet-game played in R and show that it is a winning one. Let B_1 be a non-empty open subset of R and assign B_1 to be the first move of player β in R. Choose a non-empty open subset $\tilde{B}_1 \subseteq A$ such that $B_1 = \tilde{B}_1 \cap R$. Now $B_1 = (\tilde{B}_1 \cap O_1)$ is a partial ζ-play. Next we define $\tilde{A}_1 := \zeta(\tilde{B}_1 \cap O_1)$ which is non-empty and open in A and let the non-empty open set $A_1 := \zeta'(B_1) := \tilde{A}_1 \cap R$ be the first move of the player α in R. Now let the non-empty open set $B_2 \subseteq A_1$ be the second move of player β in R. Then choose a non-empty open subset $\tilde{B}_2 \subseteq \tilde{A}_1$ such that $B_2 = \tilde{B}_2 \cap R$. We see that $(\tilde{B}_2 \cap O_2) \subseteq \zeta(\tilde{B}_1 \cap O_1)$. Now $(\tilde{B}_1 \cap O_1, \tilde{B}_2 \cap O_2)$ is a partial ζ-play and define

$\tilde{A}_2 := \zeta(\tilde{B}_1 \cap O_1, \tilde{B}_2 \cap O_2)$. Next we define the second move of player α in R to be $A_2 := \zeta'(B_1, B_2) := \tilde{A}_2 \cap R$.

Proceeding like this we will have constructed the sets $B_n = \tilde{B}_n \cap R$ where $(\tilde{B}_1 \cap O_1, \tilde{B}_2 \cap O_2, \ldots, \tilde{B}_n \cap O_n)$ is a partial ζ-play and have defined $\tilde{A}_{n+1} := \zeta(\tilde{B}_1 \cap O_1, \tilde{B}_2 \cap O_2, \ldots, \tilde{B}_n \cap O_n)$ and $A_{n+1} := \zeta'(B_1, B_2, \ldots, B_{n+1}) := \tilde{A}_{n+1} \cap R \subseteq B_n$. Since the space A is α-favourable, it must be the case that $\emptyset \neq \bigcap_{n=1}^{\infty}(\tilde{B}_n \cap O_n) \subseteq \bigcap_{n=1}^{\infty} O_n = R$. Thus we have that $\bigcap_{n=1}^{\infty} B_n = \bigcap_{n=1}^{\infty}(\tilde{B}_n \cap R) = \bigcap_{n=1}^{\infty}(\tilde{B}_n \cap O_n) \neq \emptyset$. This shows that R is α-favourable and the proof is complete. □

With the aid of the Choquet-game, Theorem 5.2 immediately gives rise to the following useful corollary.

Corollary 5.7 (Moors and Somasundaram, 2002b, Corollary 1) *Let X be a Baire (an α-favourable) space and Y be a partition complete space that is a Stegall (weakly Stegall) space. Suppose that $\varphi : X \to 2^Y$ is a densely defined lower demicontinuous mapping with closed graph. Then there exist a residual (everywhere second category) set $X_1 \subseteq X$ and a continuous selection $\sigma : X_1 \to Y$ of φ on X_1.*

Proof. First we shall consider the case when X is a Baire space, Y is partition complete and is a Stegall space. From Theorem 5.2 there exists an usco mapping $G : R \to 2^Y$ acting from a residual subset R of X into Y such that $G(x) \subseteq \varphi(x)$ for all $x \in R$. It follows from Lemma 2.6 that the mapping G contains a minimal usco mapping $S : R \to 2^Y$. Now since the range space Y is a Stegall space the mapping S is single-valued on a residual subset $X_1 \subseteq R$. The restriction of the mapping S to the set X_1 gives rise to the desired selection of φ on X_1.

In the case when the space Y is weakly Stegall and X is α-favourable (see Theorem 5.5) the proof follows in a similar fashion to that shown above except that one requires the additional fact that partition completeness implies game determined and also that a residual subset of an α-favourable space is again α-favourable. □

5.1.1 Applications of the selection theorem

In this section we will use the results obtained in the previous section to prove some new results, but first we need the following definition. We say that a mapping $f : X \to Y$ acting from a topological space X into a topological space Y is *demi-open* if for every open set U in X the set $\mathrm{int}\overline{f(U)}$ is dense in $\overline{f(U)}$. It is easy to verify that $f^{-1} : Y \to 2^X$ is lower demicontinuous on Y if the mapping $f : X \to Y$ is demi-open on X.

Corollary 5.8 (Moors and Somasundaram, 2002b, Corollary 2) *Let $f : X \to Y$ be a demi-open mapping with closed graph acting from a partition complete Stegall (weakly Stegall) space X into a dense subset of a Baire (an α-favourable) space Y. Then there exists a continuous mapping σ from a residual (everywhere second category) subset $Y_1 \subseteq Y$ into X such that $(f \circ \sigma)(y) = y$ for all y in Y_1.*

Proof. Let us consider the inverse mapping $f^{-1} : Y \to 2^X$. This is a densely defined lower demicontinuous mapping with closed graph. Hence from Corollary 5.7, there exist a residual (everywhere second category) subset $Y_1 \subseteq Y$ and a continuous selection $\sigma : Y_1 \to X$ of f^{-1} on Y_1. It immediately follows then that $(f \circ \sigma)(y) = y$ for all $y \in Y_1$. □

Corollary 5.9 (Moors and Somasundaram, 2002b, Corollary 3) *Let $h : G \to K$ be a homomorphism acting from a partition complete group G into a Baire topological group K. If h is demi-open, has a closed graph and dense range then the mapping is open and onto K.*

Proof. The inverse mapping $h^{-1} : K \to 2^G$ is densely defined and lower demicontinuous with closed graph. Hence by Theorem 5.2 the domain of h^{-1} is residual in K, i.e., the range of h is residual in K. However, as $h(G)$ is a subgroup of K it must be the case that $h(G) = K$. To show that h is open it suffices to show that for each non-empty open set U in G, $h(U)$ is somewhere residual in K and this follows by applying Theorem 5.2 to the inverse of the restriction of h to U. □

Remark 5.10 *In topological groups, partition complete groups and Čech-complete groups coincide. So in Corollary 5.9, we could replace partition complete group by Čech-complete group.*

5.2 Applications of weakly Stegall spaces to dual differentiability and generic continuity spaces

In order to proceed further with more applications of weakly Stegall spaces, we need the following definitions. A Banach space X is called a *dual differentiability space* if every continuous convex function defined on a non-empty open convex subset of the dual space possessing a weak* continuous subgradient at the points of a dense G_δ subset of its domain, is Fréchet differentiable on a dense G_δ subset of its domain. Examples of this class of spaces include those spaces with the Radon-Nikodým property and spaces which can be equivalently renormed to be locally uniformly rotund, (Giles et al., 1996).

The following class of Banach spaces is defined in terms of minimal weak* cuscos. We say that a set-valued mapping $\varphi : X \to 2^{Y^*}$ acting from a topological space X into the dual of a Banach space Y is *weak* cusco* if for each $x \in X$, $\varphi(x)$ is non-empty, weak* compact and convex subset of Y and for each weak* open set W in Y, $\{x \in X : \varphi(x) \subseteq W\}$ is open in X. A weak* cusco $\varphi : X \to 2^{Y^*}$ acting from a topological space X into the dual of a Banach space Y is called a *minimal weak* cusco* if its graph does not contain, as a proper subset, the graph of any other weak* cusco defined on X.

A Banach space X is called a *generic continuity space* if every minimal weak* cusco φ acting from a complete metric space M into subsets of the second dual X^{**} for which the set $\{m \in M : \varphi(m) \cap \widehat{X} \neq \emptyset\}$ is residual in M, is single-valued and norm upper semicontinuous at the points of a residual subset of M.

The following proposition shows that the subdifferential of a convex function has a minimality property.

Proposition 5.11 (Phelps, 1993) *If φ is a continuous convex function defined on a non-empty open convex subset A of a Banach space X, then the subdifferential mapping $x \mapsto \partial\varphi(x)$ is a minimal weak* cusco on A.*

Corollary 5.12 (Giles et al., 1996) *Every generic continuity space is a dual differentiability space.*

Proof. Let φ be a continuous convex function defined on a non-empty open convex subset A of X^* such that $G := \{x^* \in A : \partial\varphi(x) \cap \widehat{X} \neq \emptyset\}$ is residual in A. By Proposition 5.11, the subdifferential mapping $x \mapsto \partial\varphi(x)$ is a minimal weak* cusco on A and by Lemma 2.10, A is completely metrizable. Therefore $x \mapsto \partial\varphi(x)$ is single-valued and norm upper semicontinuous at the points of a residual subset R of A. The proof then follows from Proposition 2.25. □

Examples of generic continuity spaces include those spaces with an equivalent locally uniformly rotund norm and those spaces with the Radon-Nikodým property. Banach spaces which are not generic continuity spaces include $l_\infty(\Gamma)$ where Γ is uncountable, (Giles et al., 1996) and $l_\infty(\mathbb{N})$, (Moors and Giles, 1997). The following corollary is a special case of (Moors, 1996, Theorem 1.10).

Corollary 5.13 *A minimal weak* cusco φ acting from a complete metric space M into subsets of the second dual X^{**} of a Banach space X where the set $\{m \in M : \varphi(m) \subseteq \widehat{X}\}$ is everywhere second category in M, is single-valued and norm upper semicontinuous at the points of a dense G_δ subset of M.*

With the aid of Corollary 5.13, we can deduce the following result. Corollary 5.14 and Corollary 5.15 are slight improvements upon (Giles et al., 1996, Corollary 1.6).

Corollary 5.14 *A Banach space X is a dual differentiability space if X^* is an almost weak Asplund space.*

Proof. Let $\varphi : A \to \mathbb{R}$ be a continuous convex function defined on a non-empty open convex subset A of X^* possessing a weak* continuous subgradient at the points of a dense G_δ subset G of A. Since X^* is an almost weak Asplund space there exists an everywhere second category subset S of A on which φ is Gâteaux differentiable. By Proposition 2.25, $\partial\varphi(x) \subseteq \widehat{X}$ for all $x \in G \cap S$. The result then follows from Corollary 5.13 and Proposition 2.25. $\square$

Corollary 5.15 *A Banach space X is a generic continuity space if $(X^*, weak^*)$ belongs to class$(w\tilde{\mathcal{S}})$.*

The proof of Corollary 5.15 follows in a similar manner to the proof of Corollary 5.14.

5.3 Applications of weakly Stegall spaces to the differentiability of Lipschitz functions on Banach spaces

Apart from the above mentioned applications, weakly Stegall spaces also have some applications in connection with the study of the differentiability of Lipschitz functions defined on Banach spaces. Before investigating these applications, we need the following preliminary definitions.

Let f be a locally Lipschitz function defined on a non-empty open subset A of a Banach space X. Then the *Clarke generalised directional derivative of f at $x \in A$ in the direction y*, is given by

$$f^0(x;y) := \limsup_{\substack{z \to x \\ \lambda \to 0^+}} \frac{f(z + \lambda y) - f(z)}{\lambda}.$$

Associated with the Clarke generalised directional derivative is the *Clarke subdifferential mapping*, which is defined by,

$$\partial f(x) := \{x^* \in X^* : x^*(y) \leq f^0(x;y) \text{ for each } y \in X\} \quad \text{for each } x \in A.$$

We shall now present a slightly stronger notion of differentiability. Let f be a locally Lipschitz function defined on a non-empty open subset A of a Banach space X. Then

f is said to be *strictly differentiable at* $x \in A$ *in the direction* y, if

$$\lim_{\substack{z \to x \\ \lambda \to 0^+}} \frac{f(z + \lambda y) - f(z)}{\lambda} \quad \text{exists.}$$

The function f is said to be *strictly differentiable at* $x \in A$, if f is strictly differentiable at x, in every direction $y \in X$. The following lemma gives the relationship between strict differentiability and single-valuedness of the Clarke subdifferential mapping.

Lemma 5.16 (Borwein, 1991, Proposition 3.1) *Let f be a real-valued locally Lipschitz function defined on a non-empty open subset A of a Banach space X. Then $\partial f(x)$ is a singleton if, and only if, f is strictly differentiable at $x \in A$.*

Lemma 5.16 immediately gives rise to the following corollary.

Corollary 5.17 *Let f be a real-valued locally Lipschitz function defined on a non-empty open subset A of a Banach space X that belongs to class($w\tilde{\mathcal{S}}$). If the Clarke subdifferential mapping of f is minimal, then f is strictly differentiable on an everywhere second category subset of A.*

Proof. The proof of this follows directly from Lemma 5.16 and Remark 2.12 since X belongs to class($w\tilde{\mathcal{S}}$). □

In order to proceed further we need to introduce one more definition. Let f be a real-valued locally Lipschitz function defined on a non-empty open subset A of a Banach space X. In the following definition $\nabla f(x)$ represents the Gâteaux derivative of f at x. We say that f is *D-representable on* A if the following conditions hold.

(i) $D := \{x \in A : \nabla f(x) \text{ exists at } x\}$ is dense in A and

(ii) for each dense subset D^* of D we have that $\partial f(x) = \bigcap\{\overline{\text{co}}^{w^*}(\nabla f(V \cap D^*)) : V \text{ is an open neighbourhood of } x\}$.

Next we show that on any Banach space X that belongs to class($w\tilde{\mathcal{S}}$), D-representability may be characterised in terms of sequential limits of Gâteaux derivatives.

Lemma 5.18 (Borwein, 1991, Lemma 1.4 part(b)) *Let X be a Banach space whose dual ball is weak* sequentially compact (that is, every sequence in B_{X^*} has a weak* convergent subsequence) and let $\{A_n : n \in \mathbb{N}\}$ be a decreasing sequence of bounded non-empty subsets of X^*. Then*

$$\bigcap\{\overline{co}^{w^*} A_n : n \in \mathbb{N}\} = \overline{co}^{w^*}\{a \in X^* : a = \textit{ weak}^* - \lim_{n\to\infty} a_n \textit{ and } a_n \in A_n\}.$$

The following theorem shows that Banach spaces that belong to class($w\tilde{\mathcal{S}}$) possess weak* sequentially compact dual balls.

Theorem 5.19 (Giles, 1982, p.203) *Dual balls of Gâteaux differentiability spaces are weak* sequentially compact.*

Lemma 5.20 (Borwein and Moors, 1997, Corollary 2.2) *Let f be a densely Gâteaux differentiable real-valued locally Lipschitz function defined on a non-empty open subset A of a Banach space X. Then f possesses a minimal Clarke subdifferential mapping if, and only if, f is D-representable.*

The following result, which is a slight improvement upon (Borwein and Moors, 1997, Theorem 2.5), provides a sequential characterisation of D-representability.

Theorem 5.21 *Let f be a real-valued locally Lipschitz function defined on a non-empty open subset A of a Banach space X that belongs to class($w\tilde{\mathcal{S}}$). Let $D := \{x \in A : \nabla f(x)$ exists at $x\}$. Then $x \mapsto \partial f(x)$ is a minimal weak* cusco on A if, and only if, for each dense subset D^* of D*

$$\partial f(x) = \overline{co}^{w^*}\{x^* \in X^* : x^* = \textit{ weak}^* - \lim_{x_n\to x} \nabla f(x_n) \textit{ and } x_n \in D^*\}.$$

In particular, f is D-representable if, and only if, $x \mapsto \partial f(x)$ is a minimal weak cusco on A.*

Proof. The proof follows from Theorem 5.19, Lemma 5.18 and Lemma 5.20. □

Appendix A

Main Open Problems

We gather some of the main open problems that arise in the study of weak Asplund spaces.

- The most important open problem in this area is to provide an intrinsic characterisation for the class of weak Asplund spaces. Attempts at characterising these spaces have been going on since the introduction of these spaces in 1968. Even the problem of characterising those compact spaces K for which $C(K)$ belongs to the class of weak Asplund spaces still remains unresolved. It is known that for $C(K)$ to be a weak Asplund space, K must lie somewhere between being fragmentable, (Ribarska, 1987) and having the property that every closed subset of K contains a dense completely metrizable subspace, (Čoban and Kenderov, 1986).

- The problem of characterising those compact spaces K such that $C(K)$ belongs to the class of Gâteaux differentiability spaces, class($\tilde{S}$) and class($w\tilde{S}$) is also interesting and open.

  Several necessary conditions are known for $C(K)$ to be a Gâteaux differentiability space, e.g., K must be sequentially compact and the set of G_δ points is dense in it, (Čoban and Kenderov, 1986).

  Even the simpler problem of characterising those subsets A of $(0, 1)$ for which

$C(K_A)$ belongs to the classes of weak Asplund spaces, Gâteaux differentiability spaces and class$(w\tilde{\mathcal{S}})$ remains open.

We saw in Chapter 4 that there exist some necessary conditions for spaces of the form $C(K_A)$ to belong to the class of weak Asplund spaces. Specifically, A must be perfectly meager for $C(K_A)$ to be weak Asplund, (Kalenda, 1999).

The discussion given above naturally leads us to the following open problem.

- Is $C(K_A)$ a weak Asplund space if, and only if, A is perfectly meager? It follows from Chapter 4 that if A is perfectly meager then $C(K_A)$ is an almost weak Asplund space.

- If X is a weak Asplund (an almost weak Asplund) [Gâteaux differentiability] space must $C((B_{X^*}, \text{weak}^*))$ also be a weak Asplund (an almost weak Asplund) [Gâteaux differentiability] space?

- The following open problem arises from Chapter 4. Is every Gâteaux differentiability space an almost weak Asplund space? The double arrow space, $(C(K_{(0,1)}), \|\cdot\|_\infty)$, mentioned in Chapter 3 is definitely not an almost weak Asplund space as the supremum norm is Gâteaux differentiable only on a first category subset of $C(K_{(0,1)})$, but it might be a Gâteaux differentiability space.

While the class of Stegall spaces and its Banach space counterpart class$(\tilde{\mathcal{S}})$ prove to be very useful in the study of weak Asplund spaces, there are several open problems apart from the above mentioned characterisation problems. We list some of these problems below.

- We saw in Chapter 3 that there are examples of class$(\tilde{\mathcal{S}})$ that do not belong to $\tilde{\mathcal{F}}$, (Kenderov et al., 2001b). This was constructed under additional set theoretic assumptions. Can we prove this result in ZFC? Similarly we saw that there are examples of weak Asplund spaces that do not belong to class$(\tilde{\mathcal{S}})$, (Kalenda, 2002). This was also constructed under different additional set theoretic assumptions. Can this also be proven in ZFC?

We should note here that it is not possible to construct in ZFC an example of a K_A space that belongs to Stegall spaces but which is not fragmentable. If we assume the existence of a precipitous ideal on ω_1, then for every uncountable subset A of $(0,1)$ there exist a Baire metric space B and a continuous function $f : B \to A$ which is nowhere constant. Thus it follows from Chapter 3 that K_A is not a Stegall space whenever A is uncountable.

- Is $(B_{C(K)^*}, \text{weak}^*)$ a Stegall space whenever K is a compact Stegall space?

The analogous results are known for various classes of spaces. For example, it is known that if K is Eberlein, Radon-Nikodým, fragmentable or Gul'ko compact then $(B_{C(K)^*}, \text{weak}^*)$ is Eberlein, Radon-Nikodým, fragmentable and Gul'ko compact respectively, (Fabian, 1997).

We should mention here that K_A belongs to Stegall spaces if, and only if, $(B_{C(K_A)^*}, \text{weak}^*)$ belongs to Stegall spaces. On the other hand, it has been shown that there exists a weakly Stegall space K_A such that $(B_{C(K_A)^*}, \text{weak}^*)$ is not weakly Stegall (see Chapter 3).

Appendix B

Rainwater Seminar

(Presented by Isaac Namioka on October 4, 11, 18, 1983)

The contents of this appendix are reproduced with the permission of the above named author. We note that what we call Stegall spaces and class($\tilde{\mathcal{S}}$) spaces throughout this thesis are the same as the class $\mathcal{S}$ and ($\mathcal{S}$)-spaces respectively in the following article.

B.1 On class $\mathcal{S}$ of Stegall

(After "A class of topological spaces and differentiation of functions on Banach spaces" by C. Stegall.)

Let X and Y be topological spaces. Then a map $F : X \to 2^Y$ is called *upper semicontinuous and compact-valued* (usco) if (i) for each open subset U of Y, $\{x \in X : F(x) \subseteq U\}$ is open in X and (ii) for each $x \in X$, $F(x)$ is a non-void compact subset of Y.

We note a few facts concerning uscos. Let $F : X \to 2^Y$ be an usco, then we let the graph G_F of F be defined by $G_F := \{(x,y) \in X \times Y : y \in F(x)\}$. Then it is easy to see that G_F is a closed subset of $X \times Y$. An usco $F' : X \to 2^Y$ is called a *sub-usco* of F if $F'(x) \subseteq F(x)$ for each $x \in X$. Clearly $G_{F'}$ is a closed subset of G_F. Conversely if C is a closed subset of G_F such that $P_X[C] = X$, then $x \mapsto \{y \in Y : (x,y) \in C\}$ defines a sub-usco of F. (Here $P_X : X \times Y \to X$ is the projection.) It follows that each usco admits a minimal sub-usco.

Note: If $F : X \to 2^Y$ is a minimal usco, so is $F_{|U}$ for each open subset U of X.

A Tychonoff (i.e., completely regular and T_1) space Y is said to be in the *class* $\mathcal{S}$ if, and only if, whenever F is an usco from a Baire space X into subsets of Y, there is a selection $\sigma : X \to Y$ of F which is continuous at each point of a dense G_δ subset of X.

A continuous map $\varphi : S \to T$ is *perfect* if it is onto, closed and $\varphi^{-1}(t)$ is compact for each $t \in T$.

Lemma B.1 *Let $F : X \to 2^Y$ be a minimal usco and let σ be a selection of F. Then $G_F = \overline{\{(x, \sigma(x)) : x \in X\}}$. If in addition σ is continuous at x_0 then $F(x) = \{\sigma(x_0)\}$ and consequently any selection of F is continuous at x_0.*

Proof. It is clear that $G_F = \overline{\{(x, \sigma(x)) : x \in X\}}$ from the minimality of F and the discussion of usco. Hence for each $x_0 \in X$, $F(x_0)$ consists of all possible limits of $\sigma(x)$ as $x \to x_0$ (in the sense of nets). In particular, if σ is continuous at x_0, then $F(x_0) = \{\sigma(x_0)\}$. By the upper semicontinuity of F, for each neighbourhood U of $\sigma(x_0)$ there is a neighbourhood V of x_0 such that $F(x) \subseteq U$ for each $x \in V$. Hence any selection of F is continuous at x_0. ☐

Lemma B.2 *Let $F : X \to 2^Y$ be a minimal usco and let $f : Y \to Z$ be a continuous map. Then $x \mapsto f(F(x))$ is a minimal usco from X into subsets of Z. (We denote $x \mapsto f(F(x))$ by fF.)*

Proof. Note that $G_F \subseteq X \times Y, G_{fF} \subseteq X \times Z$ and $(1 \times f)[G_F] = G_{fF}$. It is clear that fF is an usco. If C is a closed proper subset of G_{fF} such that $P_X[C] = X$ then $(1 \times f)^{-1}[C] \cap G_F$ is a closed proper subset of G_F whose projection on X is X itself. This contradicts the minimality of F. Hence fF is minimal. ☐

Remark B.3 *Lemma B.1 gives the following characterisation of members of $\mathcal{S}$. A Tychonoff space Y is a member of $\mathcal{S}$ if, and only if, whenever X is a Baire space and $F : X \to 2^Y$ is a minimal usco, $\{x \in X : F(x)$ is a singleton$\}$ contains a dense G_δ (i.e., residual) set.*

Theorem B.4 (i) *Suppose $Y_1 \in \mathcal{S}$ and $f : Y_1 \to Y_2$ is perfect, then $Y_2 \in \mathcal{S}$.*

(ii) *If $Z \subseteq Y_1 \in \mathcal{S}$, then $Z \in \mathcal{S}$.*

(iii) *If $Y_n \in \mathcal{S}$ for $n = 1, 2, \ldots$, then $\prod_{n=1}^{\infty} Y_n \in \mathcal{S}$.*

(iv) *If Y is a Tychonoff space, $Y = \bigcup_{n=1}^{\infty} Y_n$ where each Y_n is closed in Y and each $Y_n \in \mathcal{S}$, then $Y \in \mathcal{S}$.*

(v) *If $(Y, \tau) \in \mathcal{S}$ and $\tau_1 \supset \tau$, then $(Y, \tau_1) \in \mathcal{S}$.*

Proof.

(i) Let $F : X \to 2^{Y_2}$ be an usco acting from a Baire space X into subsets of Y_2. Define $G : X \to 2^{Y_1}$ by $G(x) := f^{-1}(F(x))$. Since f is perfect, $G(x)$ is compact for each $x \in X$. If U is an open subset of Y_1, then $V = \{y \in Y_2 : f^{-1}(y) \subseteq U\} = Y_2 \backslash f(Y_1 \backslash U)$ is open. Therefore $\{x \in X : G(x) \subseteq U\} = \{x \in X : F(x) \subseteq V\}$ is open. Hence G is an usco. So there is a selection $\sigma : X \to Y_1$ of G which is continuous at each point of a dense G_δ subset of X. Then $f \circ \sigma$ is a selection of F with the same continuity property.

(ii) Trivial.

(iii) Let $F : X \to 2^{\prod_{n=1}^{\infty} Y_n}$ be an usco where X is a Baire space. We may assume that F is minimal. Let $P_n : \prod Y_n \to Y_n$ be the n-th projection. Then by Lemma B.2, $P_n F$ is a minimal usco acting from X into subsets of Y_n. Then by the above remark, there is a dense G_δ subset A_n of X such that $P_n(F(x))$ is a singleton for each $x \in A_n$. Let $A = \bigcap_{n=1}^{\infty} A_n$. Then A is a dense G_δ set in X and for each $x \in A$, $F(x)$ is a singleton. So by Lemma B.1 (or Remark B.3), $\prod_{n=1}^{\infty} Y_n \in \mathcal{S}$.

(iv) Let $F : X \to 2^Y$ be a minimal usco acting from a Baire space X into subsets of Y and let σ be any selection of F. Let $X_n = \{x \in X : F(x) \cap Y_n \neq \emptyset\}$. Then X_n is a closed subset of X. If one defines $F_n : X_n^0 \to 2^{Y_n}$ by $F_n(x) := F(x) \cap Y_n$, then F_n is an usco. (Therefore if U is an open subset of Y, then

$\{x \in X_n^0 : F_n(x) \subseteq U \cap Y_n\} = \{x \in X : F(x) \subseteq U \cup (Y \backslash Y_n)\} \cap X_n^0$.) Here $X_n^0 = \mathrm{int} X_n$. Since $Y_n \in \mathcal{S}$ and X_n^0 is a Baire space there is a selection $\sigma_n : X_n^0 \to Y_n$ which is continuous at each point of a dense G_δ subset A_n of X_n^0. Since σ_n can be extended to a selection of F by Lemma B.1, σ is continuous at each point of A_n. Hence σ is continuous at each point of $\cup A_n$ and $X \backslash \cup A_n \subseteq \cup (X_n \backslash A_n)$. Since each $X_n \backslash A_n$ is of the first category in X, $\cup A_n$ is residual in X.

(v) Let $F : X \to 2^Y$ be a minimal usco relative to τ_1. Then it is a minimal usco relative to τ by Lemma B.2. If X is a Baire space there is a dense G_δ subset A of X such that $F(x)$ is a singleton for each $x \in A$. This proves that $(Y, \tau_1) \in \mathcal{S}$ (compared for Remark B.3). This concludes the proof of Theorem B.4. □

In order to give some examples we need the following lemmas.

Lemma B.5 *Let $F : X \to 2^Y$ be a minimal usco. Let $(x_0, y_0) \in G_F$ and let U and V be neighbourhoods of x_0 and y_0 respectively. Then there is a non-void open subset W of U such that $F(W) \subseteq V$. (Here $F(W) := \bigcup\{F(x) : x \in W\}$.)*

Proof. We may assume that U and V are open. Then by the minimality of F, $P_X[G_F \backslash (U \times V)] \neq X$. Let $x_1 \notin P_X[G_F \backslash (U \times V)]$. Then $x_1 \in U$ and $F(x_1) \subseteq V$. The conclusion follows from this and the upper semicontinuity of F. □

Corollary B.6 *If (Y, d) is a metric space, then $(Y, d) \in \mathcal{S}$.*

Proof. Let $F : X \to 2^Y$ be a minimal usco acting from a Baire space X into subsets of Y. For $\varepsilon > 0$, let $O_\varepsilon := \bigcup\{U : U \subseteq X, U$ open $d\text{-diam}(F(U)) \leq \varepsilon\}$. We claim that O_ε is dense in X. Let W be an open non-void subset of X and let $x_0 \in W$, $y_0 \in F(x_0)$. Then there is a neighbourhood V of y_0 with $d\text{-diam}(V) \leq \varepsilon$. By Lemma B.5, there is an open subset $W_1 \subseteq W$ such that $W_1 \neq \emptyset$ and $F(W_1) \subseteq V$. Then $W_1 \subseteq O_\varepsilon$ and so $O_\varepsilon \cap W \neq \emptyset$. Let $A = \bigcap_{n=1}^\infty O_{\frac{1}{n}}$. Then $x \in A$ implies $\mathrm{diam}(F(x)) = 0$ and this implies $F(x)$ is a singleton. This proves that $Y \in \mathcal{S}$ (compared with Remark B.3). □

Example B.7 *Let E be a separable Banach space. Then $(E, weak) \in \mathcal{S}$. The reason for this is because since E is separable, E^* is weak* separable. Let D be a countable weak* dense subset of E^*. Then the topology P_D of pointwise convergence on D is a metrizable topology for E. So by Corollary B.6, $(E, P_D) \in \mathcal{S}$. Since $w \supset P_D$, $(E, weak) \in \mathcal{S}$ by Theorem B.4* (v).

We can generalise Corollary B.6 to obtain more interesting examples.

Theorem B.8 *Let (Y, τ) be a Tychonoff space with the following property: There exists a metric d on Y such that for each non-empty subset $A \subseteq Y$ and $\varepsilon > 0$, there is a τ-open subset $U \subseteq Y$ with $A \cap U \neq \emptyset$ and d-diam$(A \cap U) \leq \varepsilon$. Then $(Y, \tau) \in \mathcal{S}$.*

Proof. It is very similar to the proof of Corollary B.6. Let $F : X \to 2^Y$ be a minimal usco acting from a Baire space X into subsets of Y. For $\varepsilon > 0$, let $O_\varepsilon := \bigcup\{W : W \subseteq X, W \text{open and } d\text{-diam}(F(W)) \leq \varepsilon\}$. O_ε is clearly open. We show that O_ε is dense in X. So let V be an open non-void subset of X. Then by the hypothesis, there is a τ-open subset $U \subseteq Y$ such that $U \cap F(V) \neq \emptyset$ and d-diam$(U \cap F(V)) \leq \varepsilon$. Let $y_0 \in U \cap F(V)$. Then there is $x_0 \in V$ such that $y_0 \in F(x_0)$. By Lemma B.5, there is a non-void open set $W \subseteq V$ such that $F(W) \subseteq U$. Then d-diam$(F(W)) \leq \varepsilon$. So $W \subseteq O_\varepsilon$ and hence $V \cap O_\varepsilon \neq \emptyset$. This proves that O_ε is dense in X. Let $A = \bigcap_{n=1}^{\infty} O_{\frac{1}{n}}$. Then A is a dense G_δ set in X and if $x \in A$ then d-diam$(F(x)) = 0$ (i.e., $F(x)$ is a singleton). Hence $(Y, \tau) \in \mathcal{S}$ (compared with Remark B.3). □

Example B.9 (i) *If Y is Eberlein compact, then $Y \in \mathcal{S}$.*

(ii) *If E is a WCG Banach space, then $(E^*, weak^*) \in \mathcal{S}$.*

(iii) *If E is an Asplund space, then $(E^*, weak^*) \in \mathcal{S}$.*

Proof.

(i) Suppose that Y is a w-compact subset of a Banach space E. We apply Theorem B.8 where d is the metric on Y induced by the norm. If $A \subseteq Y$, then there is point x of continuity of $\mathrm{id} : (\overline{A}^w, w) \to (\overline{A}^w, \text{norm})$. Hence given $\varepsilon > 0$, there is a w-open neighbourhood U of x such that $\operatorname{diam}(\overline{A}^w, U) \leq \varepsilon$. Then $A \cap U \neq \emptyset$ and $\operatorname{diam}(A \cap U) \leq \varepsilon$.

(ii) If E is a WCG, then the unit ball of E^* is Eberlein compact in weak* topology. By (i), the unit ball with weak* topology is in $\mathcal{S}$. Hence $(E^*, \text{weak}^*) \in \mathcal{S}$ by Theorem B.4 (iv).

(iii) If E is an Asplund space, then by Namioka-Phelps paper, each bounded subset of E^* has relative weak* open subsets of arbitrarily small diameter. Hence by Theorem B.8, each bounded subset of E^* with weak* topology is in $\mathcal{S}$. Hence $(E^*, \text{weak}^*) \in \mathcal{S}$ by Theorem B.4 (iv) again. □

Now we describe the relevance of $\mathcal{S}$ to the differentiation on Banach spaces. Let E, F be Banach spaces and let U be an open subset of F. A continuous function $\beta : U \to E$ is called *Gâteaux differentiable at* $u_0 \in U$ if for each $k \in F$, the limit

$$\lim_{t \to 0} \frac{\beta(u_0 + tk) - \beta(u_0)}{t} \quad \text{exists in } E.$$

(The limit is taken relative to the norm topology). We shall denote the limit by $\mathrm{d}\beta(u_0)(k)$. Some people (including Bob Phelps) insist that $\mathrm{d}\beta(u_0) : F \to E$ be a bounded linear map.

Theorem B.10 *Let E, F be Banach spaces where $(E^*, \text{weak}^*) \in \mathcal{S}$, let U, V be open subsets of E and F respectively with V convex and let $\beta : U \to V$ be a continuous function which is Gâteaux differentiable at each point of a dense G_δ subset of U (in the norm topology). If $\varphi : V \to \mathbb{R}$ is a continuous convex function, then $\varphi \circ \beta$ is Gâteaux differentiable at each point of a dense G_δ subset of U.*

Proof. For $x \in V$ we call $f \in E^*$ a *subdifferential of φ at x* if $f(h) \leq \varphi(x+h) - \varphi(x)$ for all $h \in E$ such that $x + h \in V$. If we define $\varphi(x + h) := \infty$ if $x + h \notin V$, then $f(h) \leq \varphi(x + h) - \varphi(x)$ holds for all $h \in E$. We shall adopt this convention.

Let $\partial\varphi(x)$ be the set of all subdifferentials of φ at x. The following facts are standard: the map $x \mapsto \partial\varphi(x)$ is an usco from (V, norm) to subsets of (E^*, weak^*)

and it is locally bounded, i.e., for each $x_0 \in V$, there are $M > 0$ and $\delta > 0$ such that $\|x - x_0\| < \delta \Rightarrow \partial\varphi(x) \subseteq \{f \in E^* : \|f\| \leq M\}$. The map $U \mapsto \partial\varphi(\beta(U))$ is clearly an usco. Since (U, norm) is a Baire space, there are a selection $\sigma : U \to E^*$ of $u \mapsto \partial\varphi(\beta(U))$ and a dense G_δ subset A of U such that at each $u \in A$, σ is continuous (norm-to-weak*) and β is Gâteaux differentiable. From the definition of σ we see that

$$(1) \quad \sigma(u)(h) \leq \varphi(\beta(u) + h)) - \varphi\beta(u) \quad \text{for all } u \in U, h \in E.$$

(Note: $\varphi(\beta(u) + h) = \infty$ if $\beta(u) + h \notin V$.)

$$\sigma(u)(-h) \geq \varphi(\beta(u))) - \varphi(\beta(u) + h) \quad \text{or}$$

$$(2) \quad \sigma(u)(h) \geq \varphi(\beta(u)) - \varphi(\beta(u) - h) \quad \text{for all } u \in U, h \in E.$$

Suppose $u_0 \in A$ and $k \in F$. Then by the definition of Gâteaux differentiability, for t sufficiently small

$$(3) \quad \beta(u_0 + tk) - \beta(u_0) = t\mathrm{d}\beta(u_0)(k) + t\varepsilon(t) \text{ where } \|\varepsilon(t)\| \to 0 \text{ as } t \to 0.$$

By (1),(2) and (3) for t sufficiently small,

$$t\sigma(u)(\mathrm{d}\beta(u_0)(k) + \varepsilon(t)) \leq \varphi(\beta(u_0 + tk)) - \varphi(\beta(u_0)) \leq t\sigma(u_0 + tk)(\mathrm{d}\beta(u_0)(k) + \varepsilon(t)).$$

As $t \to 0$, $(\mathrm{d}\beta(u_0)(k) + \varepsilon(t)) \to \mathrm{d}\beta(u_0)(k)$ in norm and $\sigma(u_0 + tk) \to \sigma(u_0)$ in weak* topology and for t sufficiently small, $\|\sigma(u_0 + tk)\| \leq M$ for some $M > 0$. (Note: $\|f_t\| \leq M, f_t \to f_0$ (weak*)$, x_t \to x_0$ (norm) $\Rightarrow f_t(x_t) \to f_0(x_0)$.) Hence $\lim_{t\to 0} \dfrac{\varphi(\beta(u_0 + tk)) - \varphi(\beta(u_0))}{t} = \sigma(u_0)(\mathrm{d}\beta(u_0)(k))$ which shows that $\varphi \circ \beta$ is Gâteaux differentiable at u_0 and $\mathrm{d}(\varphi \circ \beta)(u_0) = \sigma(u_0) \circ \mathrm{d}\beta(u_0)$. (Note: If β is Gâteaux differentiable at u_0 in the strong sense (i.e., $\mathrm{d}\beta(u_0) : F \to E$ is bounded linear) then $\varphi \circ \beta$ is also differentiable at u_0 in the stronger sense). □

The following definition is due to Larman and Phelps. A Banach space E is a *weak Asplund space* if each continuous convex function defined on a convex open subset of E is Gâteaux differentiable at each point of a dense G_δ subset of its domain. The following corollary follows from Theorem B.10 by taking $\beta = \text{id}$.

Corollary B.11 *If E is a Banach space such that $(E^*, weak^*) \in \mathcal{S}$, then E is a weak Asplund space.*

Let us call a Banach space E an *$(\mathcal{S})$-space* if $(E^*, \text{weak}^*) \in \mathcal{S}$. Then we have seen that:

(i) Asplund spaces and WCG Banach spaces are $(\mathcal{S})$-spaces.

(ii) Each $(\mathcal{S})$-space is a weak Asplund space.

Theorem B.12 (i) *If E is an $(\mathcal{S})$-space, then each closed subset of E is an $(\mathcal{S})$-space.*

(ii) *Suppose $T : E \to F$ is a continuous linear map with $\overline{T(E)} = F$. If E is an $(\mathcal{S})$-space, so is F.*

(iii) *Let $E_i (i = 1, 2, \ldots)$ be a sequence of $(\mathcal{S})$-spaces and $1 \leq p < \infty$. Then $l_p(\{E_i\})$ is an $(\mathcal{S})$-space.*

Proof.

(i) Let F be a closed subspace of E. Then the restriction map $R : E^* \to F^*$ sends the unit ball E_1^* onto F_1^*. So (F_1^*, weak^*) is the image of (E_1^*, weak^*) under a perfect map. By Theorem B.4 (i), $(F_1^*, \text{weak}^*) \in \mathcal{S}$ and consequently $(F^*, \text{weak}^*) \in \mathcal{S}$ by Theorem B.4 (iv).

(ii) If $\overline{T(E)} = F$, then $T^* : F^* \to E^*$ is one-to-one and weak*-to-weak* continuous. Since $(E^*, \text{weak}^*) \in \mathcal{S}$, $T^*(F^*) \in \mathcal{S}$ with respect to the relative topology of (E^*, weak^*) (compared for Theorem B.4 (ii)). Hence $(F^*, \text{weak}^*) \in \mathcal{S}$ by Theorem B.4 (v).

(iii) We know that $l_p(\{E_i\})^* \simeq l_q(\{E_i^*\})$ where $\dfrac{1}{p} + \dfrac{1}{q} = 1$ with $1 < q \leq \infty$. The obvious map $(l_q(\{E_i^*\}), \text{weak}^*) \to \prod_{i=1}^{\infty}(\{E_i^*\}, \text{weak}^*)$ is continuous and one-to-one. By hypothesis $(\{E_i^*\}, \text{weak}^*) \in \mathcal{S}$ for each $i \in \mathbb{N}$. Therefore $\prod_{i=1}^{\infty}(E_i^*, \text{weak}^*) \in \mathcal{S}$ by Theorem B.4 (iii). So by Theorem B.4 (ii) and (v), $(l_q(\{E_i^*\}), \text{weak}^*) \in \mathcal{S}$. Hence $l_p(\{E_i\})$ is an $(\mathcal{S})$-space. $\square$

Appendix C

Basic Notations

Here we explain the meaning of the notations that are used throughout this thesis.

The real numbers are denoted by $\mathbb{R}$ and the natural numbers by $\mathbb{N}$.

For a normed linear space $(X, \|\cdot\|)$, we denote by:

B_X - the closed unit ball of X; i.e., $\{x \in X : \|x\| \leq 1\}$;

S_X - the unit sphere of X; i.e., $\{x \in X : \|x\| = 1\}$.

Let $(X, \|\cdot\|)$ denote a real Banach space with norm $\|\cdot\|$. By X^*, we denote the dual of X. That is, X^* is the Banach space of all continuous linear functionals on X endowed with the dual norm $\|x^*\| := \sup\{x^*(x) : x \in B_X\}$, $x^* \in X^*$.

By X^{**}, we denote the second dual of X.

By $\widehat{X}$, we denote the natural embedding of X into X^{**}.

For a non-empty open subset A of a normed linear space $(X, \|\cdot\|)$, we denote by:

2^A - the set of all subsets of A;

$\mathrm{diam}(A)$ - the diameter of A;

$\overline{\mathrm{co}}^{w^*} A$ - the weak* closed convex hull of A;

$\overline{A}$ - the closure of A;

$\mathrm{int}A$ - interior of A.

We use $:=$ when a symbol is being defined and we use italics when we define a word or a phrase.

We use italics in the statements of propositions, lemmas, theorems and corollaries.

C.1 Glossary of definitions

We say that a non-empty subset A of a Banach space X is **dentable** provided it admits slices of arbitrarily small diameter. That is, for every $\varepsilon > 0$ there exists $x^* \in X^*$ and $\alpha > 0$ such that $\operatorname{diam}(S(x^*, A, \alpha)) < \varepsilon$.

A compact space K is called **Eberlein compact** if it is homeomorphic to a weakly compact subset of a Banach space.

A compact space K is called **Gul'ko compact** if the space $C(K)$ endowed with the pointwise topology is $\mathcal{K}$-countably determined.

A Banach space X is said to have the **Radon-Nikodým property** (or RNP for short) if every non-empty bounded subset of X is dentable.

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