A STUDY OF KNOT-GRAPHS

by

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ABSTRACT

This thesis is an account of studies made of knot projections, the tools used being graph-theoretic methods and extensions of them. The basic tool is the adjacency matrix. In fact several kinds of adjacency matrix are defined, for knot diagrams which are either nonoriented or are given orientations which reflect certain topological properties of the knots concerned.

First the properties of knot-graph adjacency matrices are studied. In particular, a matrix equation is derived which is shown to be satisfied by the α- and β-adjacency matrices of all knot-graphs; other matrices appearing in the equation are studied.

Examination of types of walk that can be made on a knot-graph leads to a definition of walk-groups and groupoids on knot-graphs. Relations between the Cayley diagrams of walk-groups and certain other knots which are derived from the diagrams lead to interesting correspondences between knots, both finite and infinite.

An extensive study of the spectra of knot-graph adjacency matrices is presented. Methods for obtaining spectra are given, both for general cases and for certain classes of knots.

The concept of balanced orientations of knot-graphs is introduced; and numbers of rooted directed spanning trees of oriented knot-graphs are studied. It is shown that one type of tree number is a knot invariant, and evidence for the invariance of another one is given. A general inequality relating the two types of number is conjectured. The first tree number has several topological interpretations, and these are discussed. Tables of the numbers are given for knots of orders $n = 3,\dots,10$; and methods for calculating them for various knot classes are developed.

A vertex deletion theorem relating tree numbers of three knot-graphs is proved. Families of twins which arise from vertex deletions in knot-graphs are defined and studied. It is shown that by repeatedly applying the deletion operation on a given alternating knot-graph, one is led to a certain distribution of 'twists', which is given the name 'twist spectrum'. Twist spectra have many interesting topological properties: they are topological invariants, and their moments are powerful knot discriminants; they distinguish between a knot-graph and its mirror image and hence can be used to investigate amphicheirality.

The twist spectra for prime alternating 1-links up to order $n = 9$ and for 2-links up to order $n = 8$ are tabulated. A general algorithm for computing a twist spectrum is given, and formulae are obtained for twist spectra of members of certain knot-classes.

FOREWORD AND ACKNOWLEDGEMENTS

The study of knots which I have carried out over the last six years has been very much a personal exploration: a following of tastes and fancies. It began when I was shown a diagram of a trefoil, and was asked what a graph-theorist would say about such a diagram. I am still trying to answer the question!

I recognised at the outset that general graph-theorists might regard the studies as too particular to interest them; and algebraic topologists might be uninterested because direct study of knot-diagrams is long out of fashion. I realised, too, that most of the objects I chose to study would not be knot invariants. I decided to be disinterested in the uninterest of others and to continue undaunted. I soon discovered graphical measures and relationships from the diagrams which excited my interest, and study of them induced a deep fascination for knot theory that is still with me.

I am pleased to have found several general properties of knot-graph adjacency matrices. In particular the general equation satisfied by α- and β-matrices of all knot-graphs is attractive, and in the setting of 4-regular graphs it should interest graph theorists. Whilst it may be claimed that the equation is an immediate consequence of standard labelling of a knot-graph and a theorem on multiplication of adjacency matrices, that fact and the discovery of the form of the equation needs considerable insight. One might somewhat analagously say that the general algebraic equation of a circle is an immediate consequence of Pythagoras' theorem and the Cartesian axes system; yet more than two thousand years separated the two discoveries.

The notions of walk-groups introduced in chapter four are not likely to be of wide interest, although the group classes I have defined may be worthy of further study. As with many other study directions I have followed, I was keen to explore consequences of the fact that just one permutation, say U, together with the common permutation $P = (123...n)$, will characterise a given 1-link alternating knot-graph. The fact that the group $<U,P>$ had an interpretation in terms of walks on the knot-graph seemed to me to make the group worth studying. The relating of certain of these groups with Cayley diagrams, which in turn give rise to beautifully symmetric knot-graphs, was an appealing flight of fancy for me. I intend to develop further the study of sequences of knot-graphs and Cayley graphs begun in chapter four, and hope to achieve completeness in the description of sequences involving planar Cayley graphs.

The technique of looking at the spectra of graph adjacency matrices, and studying relations between spectral coefficients and other measures of the graphs, is one which has attracted the attention of many graph theorists, both pure and applied, in recent years. The material of chapter five represents my attempts to develop the technique for studying knot-graphs. I make no apology for the discursive style used in presenting the material. Not to have given simple diagrams and examples alongside the exposition would have been to deny the reader the chance to share the pleasures I have felt in observing diagram symmetries and changes corresponding to algebraic measures and operations. The moving spirit behind the work has been fuelled by such pleasures.

After examining many parameters derived from unoriented knot-graphs, studies of which are not reported in this thesis, I finally started looking at balanced oriented knot-graphs, and stumbled upon directed spanning tree numbers. I noticed the 1-1 correspondence of values for one of them with Alexander's first torsion number, and felt sure that I had discovered a useful knot invariant which could be computed directly from a knot-graph adjacency matrix. My search for a proof of the equivalence of the two measures (mine and Alexander's) , and the subsequent development of these studies, is described in chapters six and seven.

I believe that the methods of proof of invariance, the work on other tree numbers and a general inequality relating them, and the ideas of families of twins and twist spectra, should hold some interest to modern knot theorists. I have presented several conjectures which seem to me important and well worthy of attempts to prove or disprove them.

I would like to record my thanks to my chief supervisor, Professor A.Zulauf, for his constant encouragement and for the great amount of time he has given to following up my work. He has always been ready to discuss my problems; and almost invariably I have left a discussion with a mental blockage removed, a new line of thought to pursue, or an obstacle to progress surmounted.

My second supervisor is Dr.K.Broughan; he is the person who showed me the trefoil graph and began it all. I thank him for his support through the years, bolstering my enthusiasm whenever it flagged. I offer him the thesis as part answer to his question: What would a graph theorist say about a knot-graph?

I give thanks, too, to the following persons who gave me inspiration, time and friendship during my year of study leave in 1981/1982.

Professor B.Treybig, Texas A. and M. University, and his two Ph.D. students, for the stimulating six-week knot-theory seminar we shared.

Professor J.Eells, Warwick University, whose post-graduate course on Homology Theory I attended. He showed me the mountain tops, whilst I laboured on the slopes!

Dr.C.Leedham-Green, Queen Mary's College, for kindly computing the walk-group characteristics reported in chapter four.

Finally I wish to thank my daughter and son-in-law, who have given me so much help in preparing the thesis for computer type-setting; and my wife, for her patience and support throughout my long years in the knotting wilderness.

GLOSSARY OF SYMBOLS

Symbol	Use	Page
K, K_1, K_2, etc.	knots or knot-graphs	1
$K = [k_{ij}]$	knot adjacency matrix	9
K^*	related alternating knot-graph	7
Aut(K)	automorphism group	58
$Q(K), Q = [q_{ij}]$	Kirchoff matrix for K	123
(K_i, K_i')	pair of twins from $\partial_i(K)$	126
$\textcircled{H}_{K,i}$	set of all r.d.s.t.(i) on K	154
F(K)	family of all twin-pairs of K	165
$\bar{K}$	mirror-image, or obverse, of K	174
K_i^+, K_i^-	twin-pair from K	175
$\delta_{ij}(K)$	deletion of i, then j, from K	176
K^{++} etc.	second-generation twins	176
$\#$	knot composition ($K_1 \# K_2$)	2
	number of ...	4
	parameter of a T-diagram	20
$\alpha\beta$-, $\beta\alpha$-, $\alpha\alpha$-, $\beta\beta$-	edge types	5
$1, \ldots, n$	vertex labels	7
$d^-(v), d^+(v)$	in-degree, out-degree of vertex v	8
bo	balanced orientation	8
bao	balanced alternating orientation	8
bno	balanced nonalternating orientation	8
J_α	α-adjacency matrix (α-matrix)	9
J_β	β-adjacency matrix (β-matrix)	9
J	$\beta\alpha$-adjacency matrix; α-adjacency matrix	11;43
J^*	α-matrix of non-standard labelled knot	18
$\langle J_\alpha, J_\beta \rangle, \langle J_\alpha, J_\beta \rangle^{(r)}$	commutators	34,36,37
det(J)	determinant of J	34
$tr\langle J, J' \rangle$	trace of $\langle J, J' \rangle$	36
J_{2r}	element of sequence of α-matrices	44
Spec(J)	spectrum of J	83

J^T	transpose of J	113
A	αα-adjacency matrix	11
B	ββ-adjacency matrix	11
$M^+(M^-)$	matrices setting nonpositive	
	(nonnegative) elements to zero	12
μ-link	knot with μ components	14
	(we use 'link' to mean 'component')	2
T	T-diagram	14
	torus adjacency matrix	36
	set of words generated by U,V,U^{-1},V^{-1}	50
T_n	torus knot-graphs, n crossings	60
T(u)	twist function, or spectrum	170
μ,f,ρ,n,#	parameters of a T-diagram	20
$n^{\alpha\alpha},n^{\beta\beta},n^{\alpha\beta}$	numbers of αα, ββ, αβ (or βα) edges in an n-gon	24
$N^{\alpha\alpha},N^{\beta\beta},N^{\alpha\beta}$	total numbers of αα,ββ,αβ (or βα) edges in a	
	knot-graph	24
P_1,P_2	permutation matrices, $J_\alpha = P_1 + P_2$	27
P	permutation (matrix) (12...n); $P_2 = PP_1$	27
P(λ)	characteristic polynomial of matrix J	83
$P_{mn}(\mu)$	characteristic polynomial of rational knot mn	96
P	number of positive regions in K	177
C	matrix C = I + P	28
	clump set	54
.m.	merge operation	29
$\underline{m}$,M	merge vector, matrix	30
S	≡ <J,J'>	37
D^k	$T^k - P_1'T^kP_1$	37
4-valent graph	4-regular graph	41
U,V	permutation matrices, with	
	V = PU or V = UP'; $J_\alpha = U + V$	42
‖*‖	sum of the elements of matrix *	47

$E(T)$	walk semi-group	51
$E(T)/(\sim)$	set of equivalence classes under $\sim$	52
$\tilde{E}$	the set $E(T)/(\sim)$,	52
	the elementary walk-group	52
$W = \langle U,V \rangle$	the group of permutation matrices generated by U,V	52
Q_1, Q_2	clumped permutations	53
C	clump set	54
C_n	Cyclic group, order n	58
$\|C_n\|$	number of clumped n-permutations	77
$\|C_n^r\|$	number of clumped n-permutations with r zeros	77
$\underline{c}$	spokes vector	147
X	set $\{1,2,...,n\}$	54
$\textcircled{H}$	mapping $X \to C$	54
	mapping $G \to G'$ (two walk-groups)	56
	homomorphism $D \to S$	72
$\approx$	group isomorphism	57
$\subseteq$	subgroup	57
D_n	dihedral group, order n	58
A_n	alternating group on n symbols	58
S_n	permutation group on n symbols	58
W_j	walk-group Cayley graphs	61
10*	Conway's notation for knot 10_{123}	64
(A)	sequence of knot-graphs K_0, W_1, ...	61
$D(1,m,n)$	Van Dyck group	72
$F(2,6)$	Fibonacci group	73
S_n^{n-r}	Stirling's number of the second kind	79
$P_{\mathcal{J}}^*(\mu)$	$\|\mu I+J\|$	83
$m(\lambda_i)$	multiplicity of eigenvalue λ_i	84
$a_o, ..., a_n$	coefficients of characteristic polynomial	87
$U_n^{(2)}(x)$	Chebyshev polynomial of the second kind	93
$U_n^{(1)}(x)$	Chebyshev polynomial of the first kind	94

n_a, n_b, etc.	numbers of various cases in U'	108
χ', χ	characteristic polynomials, with subscripts na or a	111
χ_{ij}	cofactor of element ij of matrix for χ	112
χ_G	characteristic polynomial of granny knot	115
χ_S	characteristic polynomial of square knot	115
$\underline{x}_i$	characteristic vector	121
τ	tree number for a bao knot-graph	123
$\tilde{\tau}$	tree number for a bno knot-graph	123
adj(Q)	adjoint matrix of Q	124
δ_i	deletion operator (see also page 168)	125
r.d.s.t.(R)	rooted directed spanning tree, root R	131
$\textcircled{H}_i$	set of r.d.s.t.(i)	133
W_T	product of weights on arcs of tree T	133
$\Delta(t)$	Alexander polynomial	133
$H_1(\Sigma_2)$	homology group	134
Σ_2	2-fold branched cyclic cover	134
bmo	balanced mixed orientation	138
τ_m	tree number for knot-graph with bmo	138
1*(t)	Conway's notation for knot-graph formed from tangle t	141
W	wheel knot	147
S_π	$\tau(m,n,\ldots,t)$	145
$A \uplus B$	disjoint union of sets A,B	156
$t_1 t_2, t_1+t_2$, etc.	Conway's notation	161
$\Lambda = \{1,\ldots,n\}$	index set	165
S_{r-1}	union set of all first-generation twins of K_r	167
W_{r-1}	set of 1- and 2-link alternating knot-graphs	167
$\mu_0, \mu_1, \mu_{[1]}, \mu_2$	moments of twist spectra	171
u^i	an i-twist	170
T(u)	twist function, or spectrum	170
$b, b^+, b^-; t, t^+, t^-$	minimum and maximum degrees of spectra	175
r	range t-b of spectrum	175

S^{++} etc.	spectra of second-generation twins	176
max(S)	t of S	176
min(S)	b of S	176
n-twist	twist with n crossings	169
n-loop	loop with n crossings	169
$U^{(b)}$	$u^0 + u^1 + \dots + u^{b-1}$	185
$U^{(-b)}$	$u^0 + u^{-1} + \dots + u^{-(b-1)}$	

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CHAPTER 1

INTRODUCTION

1.1 OBJECTIVES

This thesis is an account of studies made of knot projections, the tools used being graph-theoretic methods and extensions of them. The basic tool is the adjacency matrix. In fact, several kinds of adjacency matrix are defined for knot diagrams which are either nonoriented or are given orientations which reflect certain topological properties of the knots concerned.

The *main objectives* of the study are:

(i) to examine properties of, and relationships between, these adjacency matrices;

(ii) to compute and study graphical measures, related diagrams and functions which are derived from knot-graphs and their adjacency matrices: examples are T-diagrams, word diagrams, characteristic polynomials, walk-groups, and spanning tree numbers;

(iii) to investigate the properties of the measures and other objects, and, where appropriate, to obtain formulae for calculating them for various knot classes; to determine to what extent each measure or object is an invariant of knot-type, and a discriminator between knot-types.

1.2 THE ENTITIES STUDIED

The basic objects to which we have applied our graph-theoretic methods of study are diagrams which may be drawn to represent the sets of points obtained when knots are projected into a plane. To make these notions mathematically precise, we present a number of definitions (following CROWELL and FOX, 1977) which are the ones normally used to define knots. We shall not enlarge upon them, but illustrative diagrams will be given where it seems necessary or appropriate.

Definitions

(i) K is a ***knot*** if there exists a homeomorphism of the unit circle C into 3-dimensional space R^3 whose image is K.

(ii) Two knots K_1 and K_2 are ***equivalent*** if there exists a homeomorphism of R^3 onto itself which maps K_1 onto K_2.

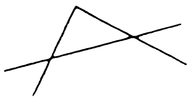
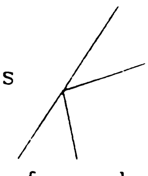
(iii) Equivalent knots are said to be of the same ***type***, and each equivalence class of knots is a ***knot type***. Knots equivalent to the circle $x^2+y^2=1$, $z=0$ are known as ***unknots***, and are said to be ***trivial***; they constitute the ***trivial type***.

(iv) A ***polygonal knot*** is one which is the union of a finite number of closed straight-line segments called ***edges***, whose endpoints are the ***vertices*** of the knot. (N.B. we shall use the terms 'edge' and 'vertex' in a different sense later, when we define 'knot-graph'.)

(v) A knot is ***tame*** if it is equivalent to a polygonal knot; otherwise it is ***wild***.

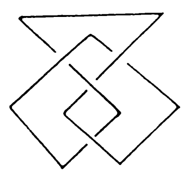
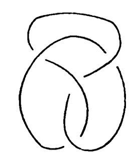
(vi) A knot is usually specified by a ***knot-projection***. Normally the knot is projected into the xy-plane by the parallel projection $P:R^3 \rightarrow R^3$ defined by $P(x,y,z) = (x,y,0)$. A point p of the image PK is called a ***multiple point*** if the inverse image $P^{-1}p$ contains more than one point of K.

(vii) A polygonal knot K is in ***regular position*** if both (a) the only multiple points of K are double points and there is a finite number of them, and (b) no double point is the image of any vertex of K: for example,

 is a permissible crossing, whereas  is not.

(viii) Each double point of the projected image of a polygonal knot in regular position is the image of two points of the knot. The one with the larger z-coordinate is called an *overcrossing* and the other is the corresponding *undercrossing*.

(ix) The term ***knot-diagram*** will refer to the plane diagram used to depict a projection of a knot in regular position: we shall not retain the polygonal nature of the knot, as the following diagrams show.

For example, the 4-knot polygonal diagram  will always be drawn as 

(x) Any knot in regular position has an infinite number of projections, each having a knot-diagram with a certain number of crossings in it. Let n be the number of crossings in a knot-diagram. Then clearly ***minimal n*** is a knot invariant. We call a projection of a knot in regular position whose knot-diagram has minimal n a ***minimal projection***. The knot-diagram is then said to have *order n*.

(xi) If two (or more) knots are tied in the same piece of string, the result is a ***composite knot***: the notation we use for the composition of knots K_1 and K_2 is $K_1 \# K_2$. If a knot is not composite it is *prime*.

(xii) An ***m-link*** is the union of m mutually disjoint simple closed curves in R^3. Note that a 1-link is a knot. It is customary to speak of an m-link as being a knot with ***m components***, but 'component' has a different meaning in graph theory. For that reason we shall use the term ***link*** instead of 'component'.

(xiii) A knot is said to be ***alternating*** if it has a minimal projection such that the overcrossings and undercrossings alternate around the knot-diagram. Otherwise it is ***nonalternating***.

1.3 COMPILATIONS OF KNOT TABLES

The most obvious problem in the theory of knots is that of determining how many knot-types there are for a given n (henceforth we shall speak of the number of knots of order n, though strictly we shall be referring to knot-types). The solution of this problem was deemed by Gauss to be an important goal, but he did not publish any work on it. He did, however, produce an analytic formulation of 'linking number'(*GAUSS*,1833), which is a tool basic to knot theory and other branches of topology. Listing made the first contribution to the problem (*LISTING*,1847), but it wasn't until the last quarter of the nineteenth century that it began to be tackled seriously. Three men, Tait, Kirkman and Little, succeeded in finding mathematical methods for defining and classifying knots, producing tables that were substantially complete and correct for $n = 3,\ldots,10$.

Tait, independently of Listing, obtained the fundamental propositions for knots, and found the knot-types for orders $n = 3,\ldots,7$ (*TAIT*,1877,*a*,*b*). Kirkman, using different methods, published the forms of knots of orders 8 and 9 (*KIRKMAN*,1885,*a*). Immediately afterwards Tait made use of Kirkman's work to extend his own census of knots to those orders; and between them they eliminated a number of errors in their tables.

In 1885 both Little and Kirkman were able to publish censuses of knots of order 10 *(LITTLE,1885,a; KIRKMAN,1885,b)*. Tait also worked on the polyhedra discovered by Kirkman, and produced his tables for $n = 10$ in the same year (*TAIT*,1885,*d*). Their methods were becoming more refined as the work progressed, and they were finally able to resolve a large number of the alternating 11-crossing knots. Kirkman provided Little with a manuscript of 1581 polyhedral drawings, from which he distinguished 357 different knot-types.

There the matter rested for some eighty years. Then in 1970 a paper by J.H.Conway described a notation whereby knots could be defined in terms of their manner of construction from so-called 'tangles'. Rules were given for determining equivalences between knots. And a listing of knot-types in this notation was given for the following: all the prime alternating and nonalternating knots for $n = 3,\ldots,11$; the 2-links up to $n = 8$; the 2-, 3-, and 4-links for $n = 9$; all links for $n = 10$. In addition, for most of the knots in his tables Conway gave values for a number of classical, and also new, invariants which he had obtained by his methods *(CONWAY,1970)*.

Conway's methods were much simpler than previous ones, and lent themselves to programming for computer use. Later we use his notation to define certain knot classes, and obtain formulae related to them (in Chapters 6 and 7); we give a brief description of as much of the notation as we need in Appendix III of Chapter 6.

In 1974, K.A.Perko completed the classification by knot-type of the 10-crossing knots tabled by Tait and Little. He gave diagrams of the 165 knots of that order, and proved them to be prime *(PERKO,1974)*. Conway claims to have a proof that the table is now complete.

Later Perko used his knot invariants to check Conway's list of 11-crossing knots, and decided that a further 195 should be added to the 357 determined by Little. The total for $n = 11$ thus now stands at 552; Perko does not claim completeness, however. Perko also produced knot diagrams for the additional 195 knots.

We have produced a computerised data-base, which stores in coded form all of the knots determined by Conway and Perko. Within this thesis our examples and computed measures are mainly for knots of order $n \leq 10$. For these knots we have used the diagrams tabulated in Rolfsen's book 'Knots and Links' *(ROLFSEN,1976)*. They were compiled by J.Bailey and drawn by A.Roth. They are ordered in the same way as in *ALEXANDER and BRIGGS,1927*; and they use the labelling notation n_i, which means 'the ith knot of order n'. As well as a diagram, these tables give the Alexander polynomial and Conway's notation for each knot.

The numbers of prime knots of order n, for n = 3,...,10, as tabulated in *ROLFSEN* are:

number of crossings n	0	2	3	4	5	6	7	8	9	10*
# 1-links	0	0	1	1	2	3	7	21(3)	49(8)	165(43)
# 2-links	1	1	0	1	1	3	8(2)	16(2)	61(19)	-
# 3-links	0	0	0	0	0	3(1)	1	10(4)	21(9)	-
# 4-links	0	0	0	0	0	0	0	3(2)	1	-

N.B. entries are totals, of all alternating and nonalternating prime knots. In brackets are given the numbers of nonalternating prime knots.

* Rolfsen gives 166 10-knots, but Perko shows that two of these are equivalent.

Recently we have communicated with M.Thistlethwaite (London Polytechnic) who has obtained by computer methods a list of 2176 12-crossing knots (DOWKER and THISTLETHWAITE, 1982). He claims that, subject to programming errors, this is complete for n = 12. In 1982 D.Seal of Cambridge University was also obtaining the 12-crossing knots, but by different computing methods; comparison of the two lists will provide checks.

As regards the purely combinatoric problem of counting the number of knot-types for a given n, we know of only one result in this direction. Tait, in his early work, showed that an alternating knot is determined by a single permutation P which has the property that $P(i) \neq i+1(mod\ n)$ for $i = 1,\ldots,n$. And that a given knot corresponds to an equivalence class of these permutations (they are now known as menage permutations). In *GILBERT, 1956* a formula is given for computing $T(n)$, the number of equivalence classes with given n, hence providing upper bounds to the numbers of alternating knots on n-crossings. Of course, we must realise that these classes of permutations include not only all the cases for all links and all composed knots; but also those which do not lead to a planar knot diagram. Thus the bounds are very poor ones. A brief table, giving the prime alternating knot totals for comparison, follows.

n	3	4	5	6	7	8	9	10
$T(n)$	1	2	5	20	87	616	4843	44128
# prime knots	1	1	2	3	7	18	41	122

Obtaining good bounds, even for 'simple' knot classes, seems to be a very difficult problem indeed.

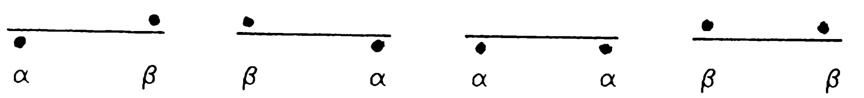
1.4 KNOT GRAPHS

Our work towards the achievement of the objectives listed in 1.1 requires first a careful definition of how knot-diagrams are to be regarded as graphs.

Definitions

(i) Let the crossings (double points) of a knot-diagram be labelled 1,2,...,n and called *vertices*; and call the segments between crossings *edges*. The set of vertices, and the set of edges, constitute what we shall call the ***associated graph*** of the knot.

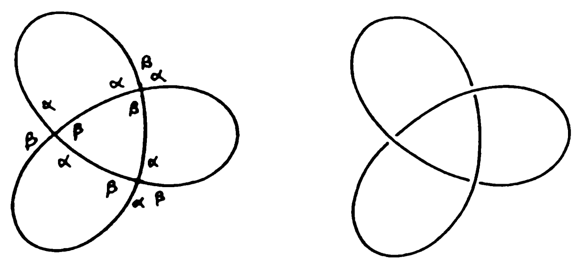
(ii) If an edge is adjacent to an overcrossing at one of its ends, we shall say it has an *α-end* there; similarly, if an edge is adjacent to an undercrossing, we shall say it has a *β-end*. Then, depending upon the kinds of ends an edge has, there are four types of edge (assuming an ordering of end vertices). The following diagrams represent these types:



We shall call these types αβ-, βα-, αα-, ββ-edges respectively.

If this information is added to an associated graph, we call the result a *knot-graph*.

(iii) When sketching a knot-graph it is customary to indicate an α-end by continuing an edge through the end vertex; and leaving a break before the vertex at a β-end. Thus the following diagrams are equivalent knot-graphs:



Note that there are four edges adjacent to each vertex (i.e. a knot-graph is 4-*regular*, or *regular of degree* 4). And once the α-, β- nature of one end is specified, the nature of the other three ends is known. This is true, of course, whether the knot is alternating or nonalternating.

(iv) It will often be convenient to use descriptive phrases such as 'moving along an edge', 'touring the knot', 'taking an α-step', 'orienting along the string'; the meaning of these should be perfectly clear, when the context is taken into account. The following definitions provide some precision for this language.

(v) A *step* is a movement along one edge; thus we may have an αα-step, a ββ-step, an αβ-step, or a βα-step, according to the types of end at the initial and terminal vertices.

A step *towards* an α-end is an *α-step* (thus either an αα-step or a βα-step is an α-step);

A step *towards* a β-end is a β-step (either a ββ- or an αβ-step);

Each step has a unique *successor* 'along the string'; an αα- or βα-step must be followed by an αα- or αβ-step, and an αβ- or ββ- step must be followed by a βα- or ββ-step.

(vi) A *walk of length l* is a sequence of l consecutive steps, no edge being used more than once and each edge being the successor of the preceding one. A *closed* walk is a walk of length ≥ 1 which ends at the starting point.

(vii) A *tour* is a closed walk such that the first step is the successor to the last one. If a closed walk is not a tour, it can be lengthened to a tour by a second closed walk. Because of planarity†, a closed walk which is not a tour

† Gauss stated this without proof. *TAIT,a(1877)* gives a proof, as does *NAGY,(1927)*.

has odd length; a tour has even length.

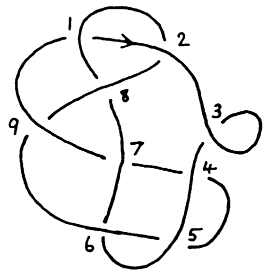
We note the following correspondences:

graph	digraph	knot-graph
edge	arc	step
simple chain	simple path	walk
simple circuit	simple cycle	closed walk,tour

It must be pointed out that terminology in Graph Theory is not yet standard. Variations in the use of such words as 'simple', 'circuit', 'cycle', and 'tour' often occur. Our uses of 'walk' and 'tour' are specialised for our purposes.

Examples: (see the diagram below)

Movements along the string from 1→2, 2→3, 3→3, 3→4 are an $\beta\alpha$-step, $\alpha\alpha$-step, $\alpha\beta$-step, $\beta\alpha$-step respectively.From 4→5 there is an $\alpha\alpha$-step, and also a $\beta\beta$-step on the lower string. Edge (2,8), regarded as a $\beta\alpha$-step,has the unique successor (8,9) regarded as an $\alpha\beta$-step.



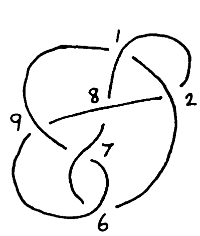
If we begin at vertex 1, and move 1→2 (as indicated by the arrow on the diagram), 2→3, 3→3, 3→4, 4→5 we have made a walk from 1 to 5 of length 5. Continuing the walk thus: 5→6, 6→7, 7→8, 8→1, we complete a closed walk from 1 to 1. Note that the length of this is 9; it is not a tour, since the successor is the $\alpha\beta$-step 1→2, which is not the $\beta\alpha$-step 1→2 with which the walk began.

If we continue the walk again, we proceed 1→2, 2→8, 8→9, 9→6, 6→5, 5→4, 4→7, 7→9, 9→1 (which has the first step 1→2 as successor). We have made two closed walks from 1, which combined make a tour of the knot-graph of length 9+9 = 18 (i.e. an even length).

(viii) A *region* (or *face*) is a complementary domain of the associated graph. If a region has n edges as its boundary, it will be called an n*-gon*.

(ix) *Simple transformations* of a knot-graph are (i) 'removing a loop by untwisting' and (ii) 'removing a 2-gon which has $\alpha\alpha$- and $\beta\beta$-edges by sliding out the flap'. We shall call these operations *untwisting* and *unflapping* respectively. The following diagram shows the above knot-graph after the loop at vertex 3 and the 2-gon (or *flap*) at (4,5) have both been removed. Note that we also delete all the vertices at which these operations take place.

Example regions on this new knot-graph are: (1,2) and (6,7) are 2-gons, (6,7,9) and (7,8,9) are 3-gons, (2,8,7,6) is a 4-gon.
The exterior region (1,2,6,9) is also a 4-gon.



Two basic properties of knot-graphs

(i) *Number of edges* Let a knot-graph have n vertices and m edges. Then since the knot-graph is 4-regular, we have $m = 2n$ (using the well-known theorem that the sum of the degrees of vertices is equal to twice the total number of edges).

(ii) *Number of regions (or faces)* Using the result of (i), together with Euler's relation $n - m + f = 2$ where f is the total number of regions (faces) in the knot-graph, we easily find that $f = n+2 = (m+4)/2$.

Example : for the knot-graph shown immediately above,

$n = 6$, $m = 12$, $f = n+2 = 8$ (exterior region is counted 1).

1.5 LABELLING OF KNOT-GRAPHS

The n vertices of a knot-graph may be given labels (for example: $\{1,2,\ldots,n\}$, $\{x_1,x_2,\ldots,x_n\}$) in $n!$ different ways. A number of interesting properties of knot-graphs quickly reveal themselves when the following labelling method is applied. Begin at any vertex, and label it 1. Carry out a tour, commencing with a step in any chosen direction; and as the vertices are encountered, label alternate ones 2, 3, 4, etc.

If the knot is a 1-link, then the last but one vertex will take the label n, and the whole knot-graph will then be labelled. The proof that this must be so follows from the fact that one can never encounter a previously labelled vertex during a tour; if one did, a closed walk with odd length would be completed, and that would be impossible since double-steps are always taken. When, however, the tour is completed, n double-steps will have been taken (since $m = 2n$) and therefore the digits $1,2,\ldots,n$ will have been used as labels.

If the knot is a μ-link, $\mu>1$, the above procedure will succeed in applying labels $1,2,\ldots,n_1$, $n_1<n$, to one of the component links. To continue the labelling, continue with the tour (i.e. begin to repeat it) until an unlabelled vertex is encountered. Label that vertex n_1+1, then set off on a new tour, on a second link, in either of the two available directions, again labelling vertices alternately n_1+2, n_1+3, etc.

The general rule for finding a new starting point, after ν links have been toured, and $\nu<\mu$, is : choose as starting point for the next tour the first unlabelled vertex encountered if all the previous tours are toured in proper sequence.

The above method must eventually supply labels $1,\ldots,n$ to the n vertices of the μ-link. Such a labelling will be called a ***standard labelling*** of the knot-graph. Note that there are always several possible ones, because of the freedoms of choice of direction at the starts of the tours.

1.6 THE RELATED ALTERNATING KNOT-GRAPH, K^*

Let K be any knot-graph which has been given a standard labelling. We define the ***related alternating knot-graph*** (or simply the ***related graph***) K^*, to be that standardly labelled alternating knot-graph which is obtained from K by changing over- and under-crossings so that each tour used in the labelling begins with a $\beta\alpha$-step and then has $\alpha\beta$- and $\beta\alpha$-steps alternating throughout. Note that $K^* \neq K$ unless K is an alternating knot-graph with the first step used in its labelling being a $\beta\alpha$-step.

1.7 ORIENTING KNOT-GRAPHS

If a sense, or direction, is given to an edge, the result is called an *arc*. An arrow is used to show the direction. By an orientation of a knot-graph we mean an assignment of a direction to every edge of the graph. Since an oriented graph is called a *digraph*, we may speak of knot-digraphs; but usually we shall not use this term.

There are $2n$ edges in a knot-graph, and each edge may be directed in either of 2 ways. Hence a knot-graph has 2^{2n} possible orientations.

In this study we find that the orientations which are of most interest and topological use are those for which at every vertex two arrows point towards, and two away from, the vertex. In a digraph the number of arcs directed towards a vertex v is called the *in-degree of* v, denoted by $d^-(v)$. Similarly the number of arcs directed away from v is its *out-degree*, denoted by $d^+(v)$.

Definitions

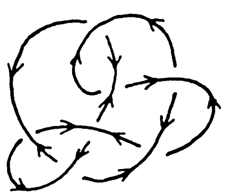
(i) A *balanced orientation* (*bo*) of a knot-graph is an orientation such that $d^-(v) = d^+(v) = 2$ for every vertex v.

(ii) A *balanced alternating orientation* (*bao*) is a *bo* such that if two arcs are adjacent along the string then their arrows point in opposite directions.

(iii) A *balanced nonalternating orientation* (*bno*) is a *bo* such that if two arcs are adjacent along the string then their arrows point in the same directions.

Examples - two balanced orientations of knot 7_6

bao



bno

Note that in the *bao* example, all the arrows are determined by α-steps; whereas in the *bno* case they are determined alternately by α-step, β-step, as the knot-graph is toured.

It is clear that reversing the arrow in every arc of the two diagrams would produce another *bao* and another *bno* for the 7_6 knot-graph.

It is equally clear that for any knot-graph (1-link, without loops) there exist just two *bao* s and two *bno* s. This is not to say that there are only four balanced orientations for any knot-graph of this type. It is possible for some vertices to have a balanced alternating orientation on their surrounding edges, whilst the remaining vertices have a balanced nonalternating orientation around them. We may call such an orientation on a knot-graph a ***mixed balanced orientation***.

CHAPTER 2

ADJACENCY MATRICES

A number of adjacency matrices will now be defined, and their properties examined. The definitions will apply to all graphs obtained from regular plane projections of tame knots (including links, unless otherwise stated), alternating or nonalternating. In all cases we refer to knot-diagrams with n crossings labelled $1,2,\ldots,n$.

In subsequent sections we study problems related to touring a knot-graph, and make use of certain diagrams defined from the adjacency matrices or from the ordering of the vertices as a tour is made.

2.1 THE ADJACENCY MATRIX OF THE ASSOCIATED GRAPH

Denoting the associated graph (1.4(i)) of a knot-diagram by K, we define its adjacency matrix thus:

The *adjacency matrix* of K is the $n \times n$ matrix K whose elements are

k_{ij} = number of edges which have
initial and terminal vertices i,j respectively,
for $i = 1,\ldots,n$ and $j = 1,\ldots,n$, with $i \neq j$,

k_{ii} = 2 if there is a loop at i,
0 otherwise.

Elementary properties of the matrix K

The following properties of K are immediately obvious from considerations of regular knot projections:

(i) K is symmetric;

(ii) If $n \geq 3$, each k_{ij} has value 0,1 or 2;

(iii) $k_{ij} = 2$ implies a 2-gon in the graph if $i \neq j$, and a loop if $i = j$;

(iv) All row and column sums equal 4;

(v) The sum of all elements of K is $4n$.

2.2 THE α- AND β-ADJACENCY MATRICES

From each vertex of a knot-graph there are four steps (1.4(v)) each being either an α-step or a β-step. (A loop counts as being both an α- and a β-step.)

We define two matrices with reference to these outward steps thus:

(i) The α-matrix, J_α:
the ijth element of J_α is 1 if there is an α-step from i to j and 0 otherwise

(ii) The β-matrix, J_β:
the ijth element of J_β is 1 if there is a β-step from i to j and 0 otherwise.

Elementary properties of J_α and J_β

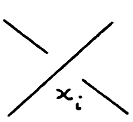
If J_α and J_β are obtained from any knot-graph, they each have the following properties:

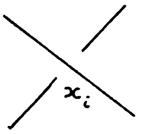
(i) there are exactly two 1s in each column;

(ii) the sum of their elements is $2n$, where n is the number of crossings;

(iii) each row has 0,1,2,3 or 4 1s in it; the total of elements in row i from both matrices is 4; if the knot-graph is alternating there are exactly two 1s in each row of both matrices.

Proofs:

(i) At each vertex, there is a crossing such as 

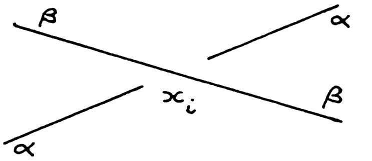
or 

In every case, two edges are incident *into* and *over* at x_i (causing two 1s to occur in column i of J_α) and two edges are incident *into* and *under* at x_i (causing two 1s to occur in column i of J_β).

(ii) follows from (i).

(iii) Four edges leave every vertex x_i; each element contributes an element 1 or 0 in row i of J_α depending on whether or not its end furthest from x_i passes over or under at the adjacent vertex. If the contribution is in J_α, it must be 0 in J_β, and vice versa.

If the knot-graph is alternating, then at every vertex we have the following arrangement of edges:



Then the ith rows of both J_α and J_β have exactly two 1s in them.

Propositions

(i) For any knot-graph, $J_\alpha + J_\beta = K$.

(ii) For alternating knots, $J_\alpha = J'_\beta$.

(iii) $J_\alpha \neq J'_\beta$ for any non-alternating knot which has at least one $\alpha\alpha$-edge which is not an edge of a 2-gon. (N.B. the presence of an $\alpha\alpha$-edge implies the presence of a $\beta\beta$-edge and vice-versa: see the appendix to this chapter.)

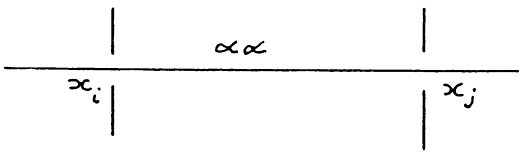
(iv) $\mathrm{tr}(J_\alpha^2) = \mathrm{tr}(J_\beta^2)$ for any knot-graph.

Proofs:

(i) If there is no edge from x_i to x_j, then $(J_\alpha)_{ij} = (J_\beta)_{ij} = (K)_{ij} = 0$. If there is an edge, then its type is one of the four types $\alpha\alpha$, $\alpha\beta$, $\beta\alpha$, $\beta\beta$. It contributes 1 to $(K)_{ij}$, whichever type it is; and to $(J_\alpha)_{ij}$ or to $(J_\beta)_{ij}$ according as its edge-type ends in α or in β. Hence $(J_\alpha)_{ij} + (J_\beta)_{ij} = (K)_{ij}$, for all i,j.

(ii) If an $\alpha\beta$-edge joins x_i to x_j, it contributes 1 to $(J_\beta)_{ij}$. The same edge can be described as a $\beta\alpha$-edge joining x_j to x_i, which contributes 1 to $(J_\alpha)_{ji}$.

(iii) Consider an $\alpha\alpha$-edge in the knot-graph, joining x_i to x_j. If it is not an edge of a 2-gon, there must be a portion of the knot-graph as in the diagram below.



Then in this case $(J_\alpha)_{ij} = 1$, whilst $(J_\beta)_{ji} = 0$.

(iv) Suppose an $\alpha\alpha$-edge joins x_i to x_j. It will contribute 1 to both $(J_\alpha^2)_{ii}$ and $(J_\alpha^2)_{jj}$, because of the two-step paths $x_i \to x_j \to x_i$ and $x_j \to x_i \to x_j$ respectively. Now each loop in the graph contributes 1, and each alternating 2-gon contributes two 1s, to the leading diagonal of J_α^2. No other combinations of edges cause positive contributions.

Hence

$$\mathrm{tr}(J_\alpha^2) = 2(\#\ \alpha\alpha\text{-edges}) + \#\ \text{loops} + 2(\#\ \text{alternating 2-gons}).$$

Similarly

$$\mathrm{tr}(J_\beta^2) = 2(\#\ \beta\beta\text{-edges}) + \#\ \text{loops} + 2(\#\ \text{alternating 2-gons}).$$

We show at the end of this chapter (see the appendix) that the number of $\alpha\alpha$-edges is equal to the number of $\beta\beta$-edges in any knot-graph; and this completes the proof. //

2.3 THE VERTEX/EDGE ADJACENCY MATRICES

It is possible to describe the relationships between the crossing points (vertices) and their adjacent edges by means of three [0,1] matrices, which we shall call J, A, B and which respectively define the $\beta\alpha$, $\alpha\alpha$, $\beta\beta$ edges.

Thus:

Matrix J:

$$(J)_{ij} = \begin{cases} 1 & \text{if there is a } \beta\alpha\text{-edge between } x_i \text{ and } x_j\ , \\ 0 & \text{otherwise.} \end{cases}$$

Matrix A:

$$(A)_{ij} = \begin{cases} 1 & \text{if there is an } \alpha\alpha\text{-edge between } x_i \text{ and } x_j\ , \\ 0 & \text{otherwise.} \end{cases}$$

Matrix B:

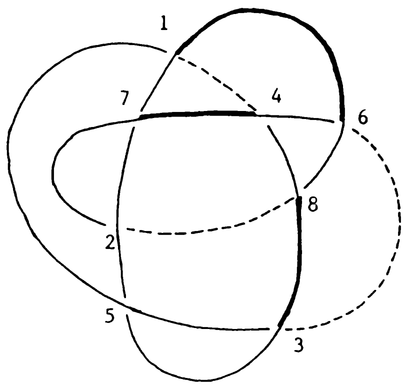
$$(B)_{ij} = \begin{cases} 1 & \text{if there is a } \beta\beta\text{-edge between } x_i \text{ and } x_j\ , \\ 0 & \text{otherwise.} \end{cases}$$

Note that J' defines the $\alpha\beta$-edges, and that A and B are symmetric matrices. If the knot-graph is alternating, both A and B are null. J obviously cannot be null.

Example: knot 8_{20}

J ($\beta\alpha$–edges)

$$\begin{bmatrix} 0&0&0&0&1&0&0&0 \\ 0&0&0&0&0&0&1&0 \\ 0&0&0&0&1&0&0&0 \\ 0&0&0&0&0&0&0&1 \\ 0&1&1&0&0&0&0&0 \\ 0&0&0&1&0&0&0&0 \\ 0&1&0&0&0&0&0&0 \\ 1&0&0&0&0&1&0&0 \end{bmatrix}$$



A ($\alpha\alpha$-edges, heavy)

$$\begin{bmatrix} 0&0&0&0&0&1&0&0 \\ 0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&1 \\ 0&0&0&0&0&0&1&0 \\ 0&0&0&0&0&0&0&0 \\ 1&0&0&0&0&0&0&0 \\ 0&0&0&1&0&0&0&0 \\ 0&0&1&0&0&0&0&0 \end{bmatrix}$$

B ($\beta\beta$-edges, dotted)

$$\begin{bmatrix} 0&0&0&1&0&0&0&0 \\ 0&0&0&0&0&0&0&1 \\ 0&0&0&0&0&1&0&0 \\ 1&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0 \\ 0&0&1&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0 \\ 0&1&0&0&0&0&0&0 \end{bmatrix}$$

2.4 RELATIONSHIPS BETWEEN THE MATRICES OF 2.2 AND 2.3

Recall from 2.1 that K is the adjacency matrix of the associated graph of a knot-projection; and from 2.2 that J_α and J_β are respectively the α- and β-adjacency matrices.

Relationships between these matrices and J, A and B, which follow directly from their definitions, are:

(i) $K = J_\alpha + J_\beta$

(ii) $J_\alpha = J + A$ and $J_\beta = J' + B$

(iii) $K = J + J' + A + B$.

Note that the pair (J_α, J_β) can be obtained from the triple (J, A, B), and vice-versa, by means of the following equations:

Given (J, A, B)

$$J_\alpha = J + A \,,$$
$$J_\beta = J' + B$$

Given (J_α, J_β)

$$A = (J_\alpha - J'_\beta)^+ = (J_\beta - J'_\alpha)^-$$
$$B = (J_\alpha - J'_\beta)^- = (J_\beta - J'_\alpha)^+$$

(where M^+ means the matrix obtained from M by retaining all positive elements and setting all other elements to zero: and M^- is the matrix obtained similarly from $-M$).

Finally:

$$J = J_\alpha - A = J_\alpha - (J_\alpha - J'_\beta)^+$$

Proofs:

Both $J_\alpha = J + A$ and $J_\beta = J' + B$ follow from the definitions of the various matrices.

Then $J_\alpha - J_\beta' = A - B$ (B is symmetric); and since A and B have no 1 entries in common positions, if we set all the -1 elements (from $-B$) to zero, we obtain A. That is, $A = (J_\alpha - J'_\beta)^+$.

Similarly we may derive the other formulae. //

2.5 AN ALGORITHM FOR TOURING A 1-LINK KNOT-GRAPH

The following algorithm is a method for determining from the matrices J, A, B the sequence of crossing-points which are met when a 1-link knot-graph is toured, starting at an arbitrary point and always proceeding 'along the string' (i.e. without turning to right or to left when passing through a vertex).

Step 1:

Choose a vertex, and decide whether to begin the tour by taking first an α-step, or first a β-step (there may not be a choice).

Suppose we begin by taking an α-step from vertex i.

Then we look in row i of matrices J and A, and note the column (say the jth, of J) where first a 1 appears. (N.B. if the ith rows of both J and A are found to be null, the first step must be a β- one.

Take the first step, from vertex x_i to vertex x_j.

Steps 2,3,...,2n:

All subsequent steps are uniquely determined by the following rules:

Follow a step determined from J by one determined from J' or A ;

Follow a step determined from A by one determined from J' or A ;

Follow a step determined from J' by one determined from J or B ;

Follow a step determined from B by one determined from J or B .

For example, to continue from the end of Step 1 above, we must determine the second step from either J' *or* A, since the first step was determined using the ijth element of J. Since we arrived at x_j by an α-step, we can only continue (without turning left or right) by means of either an $\alpha\beta$-edge or an $\alpha\alpha$-edge. One of these must occur; and it will be indicated by a 1 in the jth row of either J' or A. If there are two 1s in these rows (there cannot be more than two), one of them will be in a position ji, indicating a step back to x_i. In order to advance, we discard the jith element, and take the step indicated by the other 1 in the jth rows.

Since each advancing step (after the first) is uniquely determined by the above rules, $2n-1$ of them will produce a sequence of vertices which define a tour of the knot-graph which visits each vertex twice, traverses each edge once, and always proceeds 'along the string'.

Example, using knot 8_{20}

The diagram and adjacency matrices of a knot-graph of knot 8_{20} are given in section 2.3 above.

Suppose we decide to start the tour at vertex x_1 and make the first step an $\alpha\alpha$-step to vertex x_6. This uses the element 1 in the (1,6) position of matrix A.

The following scheme shows how the tour of the knot-graph proceeds. It may be checked by following it round the diagram given in section 2.3.

$$1 \underset{A}{\overset{\alpha\alpha}{\rightarrow}} 6 \underset{J'}{\overset{\alpha\beta}{\rightarrow}} 8 \underset{B}{\overset{\beta\beta}{\rightarrow}} 2 \underset{J}{\overset{\beta\alpha}{\rightarrow}} 7 \underset{A}{\overset{\alpha\alpha}{\rightarrow}} 4 \rightarrow \cdots \rightarrow 7 \underset{J}{\overset{\beta\alpha}{\rightarrow}} 1$$

Note that if the knot-graph is alternating, A and B are null and the tour is determined by alternate use of J and J'. Furthermore, we show later in Chapter 3, section 3.5, that in the case of an alternating knot-graph, an even simpler algorithm is available making use of permutations.

Touring a μ-link, with $\mu > 1$

The algorithm given above, if started on a μ-link having $\mu > 1$, would produce a tour round one of the links. The particular link toured would be determined by the chosen starting vertex and first step.

2.6 DIAGRAMS DERIVABLE FROM THE ADJACENCY MATRICES

In this section we describe two kinds of diagram which are obtainable from the matrices J_α and J_β, both of which have interesting properties of their own as well as providing insights into properties of their knot-graphs.

The first is obtained directly from a J_α matrix of an alternating knot-graph by 'joining up the 1s'; we shall call this a *T-diagram* of the knot-graph.

The second is obtained by arranging the ordered vertices of a tour of the knot-graph around a circle; we shall call this a *circular-word diagram* of the knot. It will be useful for studying both alternating and nonalternating knot-graphs.

2.6.1 *T*-diagrams of alternating knot-graphs

If the α-matrix is taken from an alternating knot-graph which has been given a standard labelling, and the centres of the cells containing 1s are joined by horizontal and vertical lines (one in each row and column), one or more closed rectilinear diagrams appear within the matrix. The following two pages show several of these diagrams, for knots and links of small order: in fact we have removed the 0s, and replaced all the 1s by dots which then form the corners of the rectilinear diagrams.

(i) Definitions

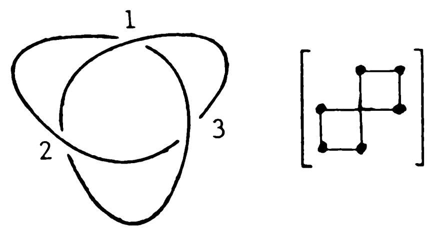
A dot will be called a *corner* of the rectilinear diagram to which it belongs. If the T-diagram consists of only one closed rectilinear diagram, it will be called *connected*. A self-intersection point of a non-simply closed rectilinear diagram will be called an *intersection*.

Unfortunately, T-diagrams cannot provide knot-invariants since they are heavily dependent upon the labelling chosen for the knot-graph. Nevertheless they have many interesting properties which reflect properties of the knot-graphs from which they come. Some of these will be described here, and others later in chapter 3.

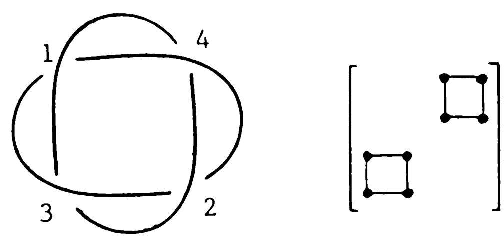
T-Diagrams for Various Knots and Links

(i) <u>The Torus knot-graphs</u> (knots 3_1, 4_1^2, 5_1, 6_1^2, 7_1, 8_1^2)

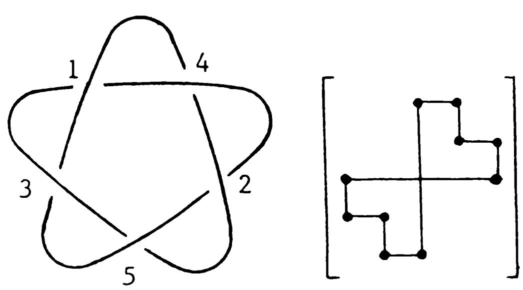
<u>$n = 3$</u>



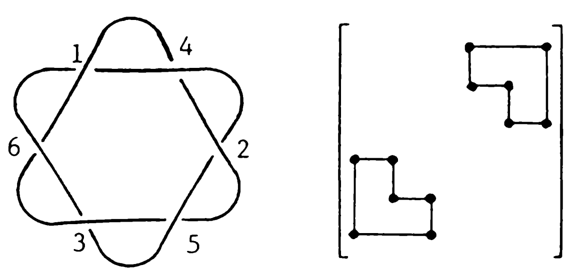
<u>$n = 4$</u>



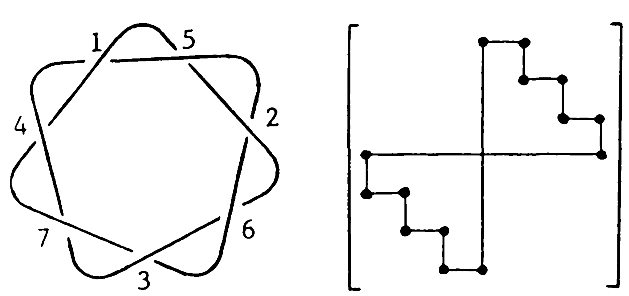
<u>$n = 5$</u>



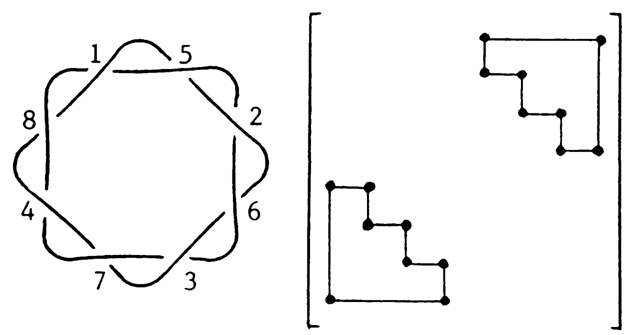
<u>$n = 6$</u>



<u>$n = 7$</u>

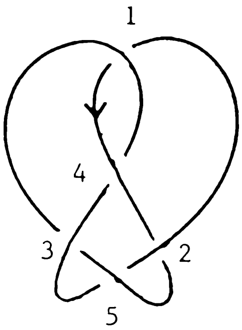


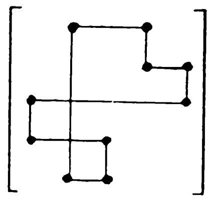
<u>$n = 8$</u>



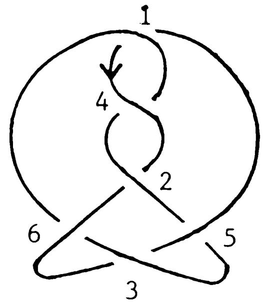
(ii) Comparisons of knot 5_2 and knot 6_2^2

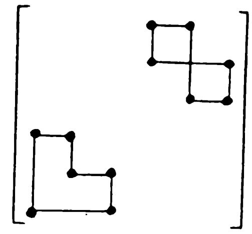
5_2



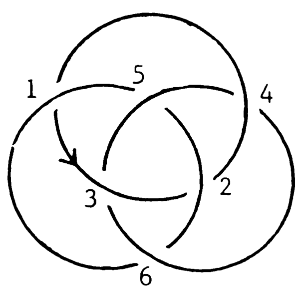


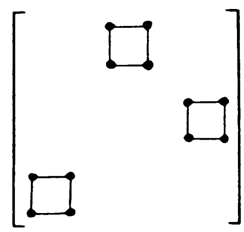
6_2^2





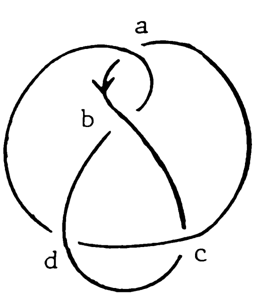
(iii) The Borromean rings (symbol of the Medici: knot 6_2^3)



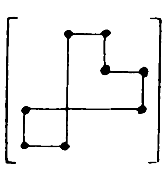


(iv) Moving the position of label 1, on knot 4_1:

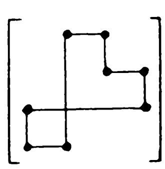
4_1



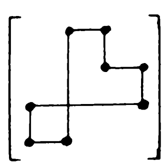
$a \equiv 1$



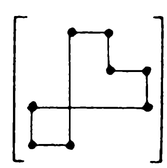
$b \equiv 1$



$c \equiv 1$



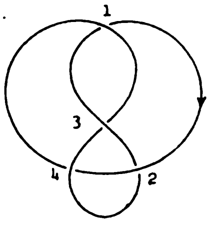
$d \equiv 1$

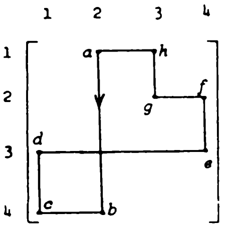


(ii) Tour of a knot-graph having a connected T-diagram

By proceeding around the rectilinear diagram, and noting the numbers of the rows and columns visited successively, we are provided with the sequence of vertex labels on a tour of the knot-graph. In this way a tour of the T-diagram corresponds to a tour of the knot-graph (hence the name 'T-diagram').

An example will clarify the procedure: we give below the knot-graph and T-diagram for knot 4_1. The corners of the T-diagram have been labelled in the order of a tour around its perimeter.





The following table sets out the coordinates of the corners as the T-diagram is toured, and immediately below is shown the corresponding steps in the tour of the knot-graph. It will be seen how the vertex labels for the tour of the knot-graph are obtainable from the coordinates of the corners.

T-diagram:	a	b	c	d	e	f	g	h
Coordinates:	(1,2)	(4,2)	(4,1)	(3,1)	(3,4)	(2,4)	(2,3)	(1,3)
Knot-graph:	$1\xrightarrow{\alpha}2$	$2\xrightarrow{\beta}4$	$4\xrightarrow{\alpha}1$	$1\xrightarrow{\beta}3$	$3\xrightarrow{\alpha}4$	$4\xrightarrow{\beta}2$	$2\xrightarrow{\alpha}3$	$3\xrightarrow{\beta}1$

The proof that this correspondence between perimeter of T-diagram and tour of knot-graph always holds follows directly from the definition of the α-matrix (2.2) and the fact that the β-matrix is the transpose of the α-matrix when the knot-graph is alternating.

(iii) T-diagrams for knots and links

The T-diagrams for knots (i.e. 1-links) are immediately distinguishable from those for μ-links with $\mu > 1$, for the diagrams of the latter are separated into μ non-intersecting rectilinear diagrams.

The various example T-diagrams shown earlier demonstrate this fact admirably. We will state it as a theorem:

Theorem 1

Let an alternating μ-link have a knot-graph K with standard labelling, and let T be the associated T-diagram. Then T consists of μ mutually disjoint closed rectilinear diagrams.

Proof:

By virtue of the standard labelling, if K has n vertices in total, n_1 are labelled $1,2,\ldots,n_1$ in alternating positions on the first link, n_2 are labelled n_1+1, $n_1+2,\ldots,n_1+n_2$ on the second, and so on, with $n_1+n_2+\ldots+n_\mu = n$. (Note that the ith link has n_i vertices labelled $\nu+1,\nu+2,\ldots,\nu+n_i$ in alternating positions, where $\nu = \sum_{j=1}^{n-i} n_j$, and a further n_i vertices which receive labels which are assigned whilst labelling the other links which intersect with the ith one.)

We have shown in (i) above that if T is connected, its perimeter corresponds to a tour of the whole knot-graph. So, in the case $\mu = 1$, T consists of one closed rectilinear diagram.

In the case that $\mu > 1$, consider the first labelled link. If we begin at vertex 1 and tour the link, we meet in turn the vertices $1, v_1, 2, v_2, 3, v_3, \ldots, n_1, v_{n_1}, 1$, where some of the v_i belong to the set $\{1, 2, \ldots, n_1\}$ in the case that the link crosses itself at one or more points, and the rest belong to the set $\{n_1+1, n_1+2, \ldots, n\}$.

By the same arguments used in (i), the point $(1, v_1)$ will be the corner of a rectilinear diagram on the T-diagram, and a tour from corner to corner of the rectilinear diagram will be $(1,v_1),(2,v_1),(2,v_2),(3,v_2),\ldots, (n_1,v_{n_1})$.

Since the first coordinate of all these corners is a member of the set $\{1,\ldots,n_1\}$, the entire diagram is confined to the first n_1 rows of the matrix. And since the horizontal line joining $(1,v_{n_1})$ to $(1,v_1)$ belongs to T, and closes this particular rectilinear diagram, the first labelled link corresponds to a closed rectilinear diagram confined to the first n_1 rows of the J_α-matrix.

Similarly we can show that the second link corresponds to a closed rectilinear diagram confined to the rows $n_1+1,\ldots,n_1+n_2$ (and therefore disjoint from that of the first link). And so on, for links $3,4,\ldots,\mu$, proving the theorem.

(iv) On the rank of J_α

The following algebraic property of α-matrices is easily shown to be true with the help of T-diagrams.

Theorem 2

Let an alternating μ-link have a knot-graph K with an $n \times n$ α-matrix J_α.

Then $\text{rank}(J_\alpha) < n$ (i.e. $\det(J_\alpha) = 0$) if and only if at least one of its links has a total number of vertices (counting self-crossings twice) which is a multiple of 4.

Proof:

The conditions of the theorem do not require K to have a standard labelling. If it hasn't, however, we apply the following lemma.

Lemma

Let J_α and $J_\alpha^\bullet$ be the α-matrices of a knot-graph given respectively a non-standard and a standard labelling. Then $\det(J_\alpha) = \det(J_\alpha^\bullet)$.

Proof of lemma

A change of labels of n vertices corresponds to a permutation of the integers $1,2,\ldots,n$.

Let P be the $n \times n$ permutation matrix corresponding to the change of K to a standard labelling. It is easy to show that

$$J_\alpha = PJ_\alpha^\bullet P' ,$$

from which

$$\begin{aligned}\det(J_\alpha) &= \det(P)\det(J_\alpha^\bullet)\det(P') \\ &= \det(J_\alpha^\bullet)\end{aligned}$$

since $\det(P) = \pm 1$ and $\det(P) = \det(P')$.

Returning to the original theorem, we may now assume without loss of generality that K has a standard labelling. So, by theorem 1, its T-diagram consists of μ mutually disjoint closed rectilinear diagrams.

Suppose, again without loss of generality, that the first labelled link has a total number of crossings $2n_1$ which is a multiple of 4 (so n_1 is even). Then, as described in theorem 1, the corners of the closed rectilinear diagram corresponding to this link have coordinates

$$(1,v_1),\ (2,v_1),\ (2,v_2),\ (3,v_2),\ (3,v_3),\qquad ,\ (1,v_{n_1})$$

Recall that each corner corresponds to a 1 in the matrix J_α, and that there are just two 1s in each row and each column. Let C_i denote the ith column of J_α. Consideration of the coordinates just listed for the first link shows that

$$C_{v_1} - C_{v_2} + C_{v_3} - \qquad + C_{v_{n_1-1}} - C_{v_{n_1}} = \underline{0}$$

(Note that if n_1 is not even, we cannot find a vanishing linear combination of these columns.)

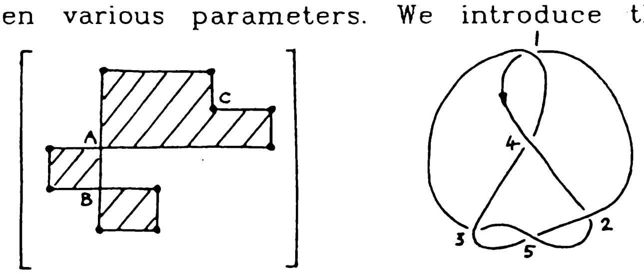
Hence $\quad \mathrm{rank}(J_\alpha) < n$ and $\det(J_\alpha) = 0$

The condition is also sufficient, for suppose that $\det(J_\alpha) = 0$. This implies that there is a vanishing linear combination of columns of J_α. And by theorem 1, the matrix J_α has a set of columns with 1s appearing only in the first n_1 rows, a second set with 1s appearing only in the rows n_1+1 to n_2, and so on. Therefore a linear combination of the columns in at least one of these μ sets must vanish if $\det(J_\alpha)$ is to equal zero. And since in any particular set of columns there are just two 1s in each row and each column, in order to vanish the combination must be arrangeable in the form $(C_{i_1} - C_{i_2} + C_{i_3} - \cdots - C_{i_{n_j}})$.

Thus n_j must be even; and the jth link, corresponding to this rectangular diagram, has $2n_j$ crossings.

(v) On parameters of T-diagrams

Next we look at some of the salient features of a T-diagram, and prove some relationships between various parameters. We introduce the parameters through an example.



A T-diagram for knot 5_2

Note that the T-diagram has 3 *faces* (the shaded regions), 10 *corners* (the dots), 2 *intersections* (labelled A,B) and 1 *reentrant* point (labelled C).

Theorem 3

Let a T-diagram, derived from an alternating μ-link with standard labelling and n vertices, have:

μ	rectilinear diagrams having respectively $n_1, n_2, \cdots, n_\mu$ corners
f_i	faces in the ith diagram, $(i = 1, \cdots, \mu)$
ρ_i	reentrant points in the ith diagram
n_i	corners in the ith diagram
$\#_i$	intersections in the ith diagram .

Then:

(a) $f = \# + \mu$,

(b) $n = \# + \rho + 2\mu$,

where $f = \sum_i f_i$, $\# = \sum_i \#_i$

and $\rho = \sum_i \rho_i$, $i = 1,\ldots,\mu$.

Proof:

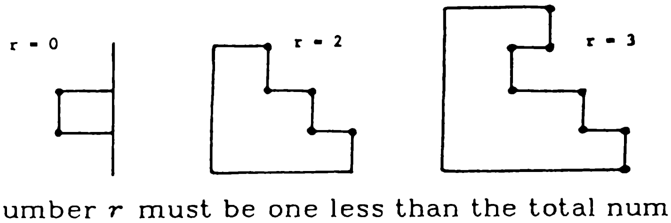
Consider the ith rectilinear diagram. By theorem 1 it is disjoint from all the others. Because of the standard labelling, all except one of the vertical sides of the diagram (representing β-steps on the knot-graph) are from one row to the next. The exception is that line which rises through all n_i rows, representing the last β-step back to the starting vertex of a tour of the knot-graph. It is this line which contains all the intersection points of the ith rectilinear diagram. The faces are formed successively to the right and left of this line, or vice versa.

Now, since the top and bottom faces have one intersection point each, and all the others have two, we obtain immediately

$$2\#_i = 2f_i - 2(\text{each intersection appears in two faces}).$$

Dividing by 2, and summing over i, gives (a).

To prove (b), we note that the number r of reentrant points in a face is equal to the number of horizontal steps in the face not including the first and the last. The figure below shows three possible faces, to illustrate this fact.



The number r must be one less than the total number of unit vertical steps in the face, i.e. $(v-1)$ say. Summing over all the faces in the rectilinear diagram, we obtain

$$\rho_i = \sum r = \sum (v-1)$$
$$= \sum v - f_i$$

since we sum over f_i faces.

Now $\sum v$ is the total number of vertical steps in the ith rectilinear diagram, which is $\frac{n_i-2}{2}$. Inserting this, and summing over i = 1, ,μ, we obtain

$$\rho = \sum_i \rho_i = \sum_i (\frac{n_i-2}{2}) - f$$

$$= n - \mu - f \quad (\text{since} \sum_i n_i = 2n)$$

which gives

$$n = \rho + \mu + f$$

$$= \# + \rho + 2\mu \quad (\text{using } (a)).$$

(vi) Some observations and questions about T-diagrams

Referring to the pages of example T-diagrams given at the beginning of this section, we make a few observations and pose a few problems.

T-diagrams of torus knot-graphs

The diagrams shown cover the cases n = 3,...,8. It can be seen that when:

n is odd

the torus T-diagram is connected (μ = 1) and has two faces (f = 2). There is just one intersection ($\#$ = 1), and there are $\rho = n-3$ reentrant points.

n is even

$$\mu = 2, f = 2, \# = 0; \quad f_1 = 1 = f_2, \#_1 = 0 = \#_2; \quad \rho = n-4$$

Comparing the T-diagrams for n = 3 and n = 4 (call them T_3 and T_4 for convenience) we see that if the two faces of T_3 are 'split apart', and corners added as needed, diagram T_4 is obtained. A similar observation can be made about diagrams T_5, T_6, and about T_7, T_8; indeed, about any pair T_{2n-1}, T_{2n}. In fact, the effect of inserting an additional 2-gon into the knot-graph corresponding to T_{2n-1} is to split apart the two faces of T_{2n-1} and reflect them in the leading diagonal of a $2n \times 2n$ matrix, producing T_{2n} .

A similar kind of comparison may be made between the T-diagrams of knots 5_2 and 6_2^2 (see the diagrams). Again a 2-gon has been inserted into 5_2 to produce 6_2^2.

It is clear that we can pose many questions about what happens to the geometry of T-diagrams when transformations of the knot, such as insertions of 2-gons, are made.

The following subsection discusses a combinatoric question.

How many T-diagrams has a given knot-graph?

From the definitions of T-diagrams and standard labelling, at first sight it would appear there are $4n$ different standard labellings, since each vertex can be chosen to assign the label 1, and from that vertex the labelling can then continue in any one of four directions.

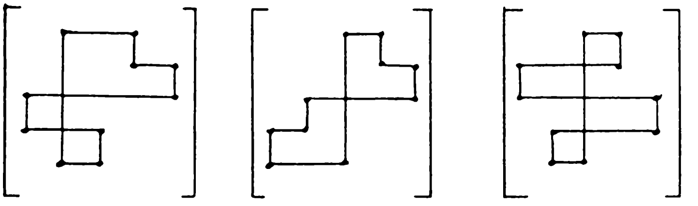
However, on examining the diagrams for torus knots (see the page of T-diagrams) we quickly see that varying the standard labellings cannot change their T-diagrams. In other words, torus knots have the property that they have only one T-diagram.

Have any other knots got this property? By inspection of figures (iv) on the diagram page we see that knot 4_1 appears to have it; indeed that is the case, for all its standard labellings lead to the same T-diagram.

Some questions that arise now are: Can we find some algebraic conditions on J_α for its T-diagram to be invariant under standard labelling changes? Can we find formulae or algorithms to determine how many different T-diagrams a given knot-graph has? How many knot- or link-graphs are there with an invariant T-diagram, given the values of n and μ? How many are there which have a given number τ distinct T-diagrams?

Instead of requiring invariance over standard labellings, we could require a less stringent form of invariance, such as 'invariance up to symmetry transformations within the matrix'. For example, with knot 5_2 (see diagram (ii)), there are only three distinct T-diagrams, as shown below, if we allow the diagrams to be rotated in the plane and turned over.

Knot 5_2
T-diagrams



These questions are discussed further in section 3.4, and some partial answers are obtained.

Other questions can be asked which relate to the parameter # (number of intersections in the T-diagram). For example: Can we classify all knot-graphs which have at least one T-diagram with # = 0?

It is clear that answers to all the above questions relate in one way or another to the amount of symmetry that exists in the knots concerned. Indeed, one might take as a measure of symmetry of a knot the number of distinct T-diagrams (up to given allowable symmetry operations) that its knot-graph possesses. The matter is complicated, however, by the fact that most knots have several non-isomorphic knot-graphs.

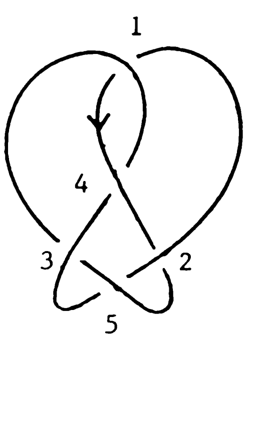
The conjecture that the number of distinct T-diagrams associated with a knot is an invariant of knot-type is unlikely to be true.

2.6.2 Circular-word diagrams (1-link knots)

The diagrams to be defined below have proved to be useful tools for several of our purposes. They also suggest many interesting problems of geometric and combinatoric natures, but we shall not pursue these in this thesis.

If a knot-graph is given a labelling, and an Eulerian tour of the graph is made, then the ordered vertices visited can be written in a string to form the so-called *word* of the knot (*PENNEY*, 1966). Note that the label of each vertex will appear twice in the word - once when the tour underpasses the vertex and once when it overpasses it. We may distinguish between the two occurrences by writing a superscript of -1 above the label when an underpass occurs (and sometimes a superscript +1 is written when an overpass occurs).

Example: knot 5_2



Starting the tour underneath at vertex 1, and proceeding in the direction of the arrow, we obtain the word:

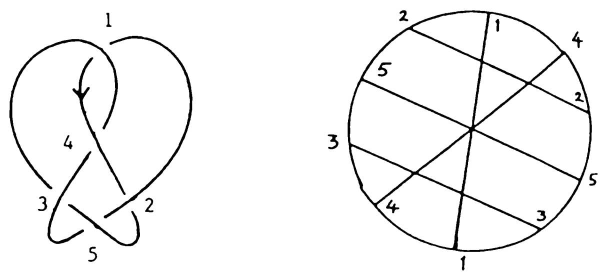
$1^{-1}42^{-1}53^{-1}14^{-1}35^{-1}2$
(and back to 1^{-1})

Definition

The *circular-word diagram* of an n-crossing, 1-link knot is a plane circle with the symbols of the word of the knot placed in order around the circumference, at equal angular distances of $\frac{\pi}{n}$. Those symbols with a -1 superscript are placed inside the circle; the others are placed outside the circle.

Noting that any circular-word diagram has on its circumference two 1s, two 2s,...,two ns, and we may join all pairs of common symbols by chords, it is clear that interesting geometrical objects can be so obtained. We give only two examples below, and give brief comments upon them. Later, however (e.g. in 3.2), we shall make use of circular-word diagrams to prove theorems about adjacency matrices of knot-graphs.

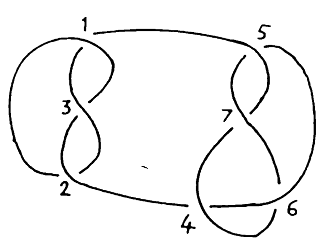
Example 1: knot 5_2



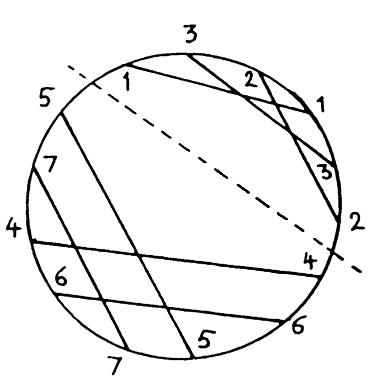
Comments: interesting visual comparisons between the circular diagram and the knot-graph can be made. We note, for example, that chords 22, 55 and 33 are parallel; it is easy to show that the two adjacent vertices of a 2-gon have chords which are $\frac{\pi}{n}$ of arc apart at their ends and so are either parallel or they intersect (as do the pair 11, 44 in the diagram). We can show that two chords arising from a 2-gon are parallel if in the tour of the knot-graph the edges of the 2-gon are traversed in opposite directions.

Note the symmetries in the circular diagram, and compare them with those of the knot-graph.

Example 2: composite knot, $3_1 \# 4_1$



Comments: note how the dotted line on the circular diagram partitions the sets of chords which belong to the two knots involved in the composition knot. It may be shown that under certain conditions partitioning of the circular-word diagram characterizes composite knots.



APPENDIX

THE NUMBERS OF $\alpha\alpha$ AND $\beta\beta$ EDGES IN A KNOT-GRAPH

In Chapter 1 we defined the possible types of edge that can occur in a knot-graph, namely $\alpha\beta$, $\beta\alpha$, $\alpha\alpha$, and $\beta\beta$. In this Appendix we prove several results about the various numbers of these types that can occur in any given knot-graph. We shall not distinguish between $\alpha\beta$ and $\beta\alpha$ edges.

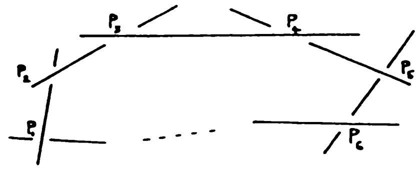
An alternative proof is given for the main result.

Lemma

In the boundary of any region of a knot-graph, the numbers of $\alpha\alpha$ edges and $\beta\beta$ edges are equal.

Proof:

The diagram below shows the first five edges of an n-gon region. P_1P_2, P_2P_3, P_4P_5 are $\alpha\beta$-edges; P_3P_4 is an $\alpha\alpha$-edge; P_5P_6 is a $\beta\beta$-edge; and so on. Let $n^{\alpha\alpha}$, $n^{\beta\beta}$, $n^{\alpha\beta}$ be respectively the numbers of $\alpha\alpha, \beta\beta$, and $\alpha\beta$ edges in the n-gon.



Let us think of an $\alpha\alpha$-edge as being marked with an α at both ends, a $\beta\beta$-edge with a β at both ends, and an $\alpha\beta$-edge with an α at the overcrossing end and a β at the undercrossing end.

Then the total number of α marks in the n-gon is

$$n^{\alpha} = 2n^{\alpha\alpha} + n^{\alpha\beta} \qquad (1)$$

And the total number of β marks in the n-gon is

$$n^{\beta} = 2n^{\beta\beta} + n^{\alpha\beta} \qquad (2)$$

But at each vertex of the n-gon there occurs exactly one α mark and one β mark (on the n-gon itself), one on each of the two edges which meet and cross there.

Therefore $2n = n^{\alpha} + n^{\beta}$, and $n^{\alpha} = n^{\beta} = n$

Equating (1) and (2) now gives $n^{\alpha\alpha} = n^{\beta\beta}$ as required.

Theorem

In any knot-graph, the number of $\alpha\alpha$-edges equals the number of $\beta\beta$-edges.

Proof:

Let $N^{\alpha\alpha}$ and $N^{\beta\beta}$ be the total numbers of $\alpha\alpha$- and $\beta\beta$-edges respectively in the knot-graph; and let $N^{\alpha\beta}$ be the total of the $\alpha\beta$- and the $\beta\alpha$-edges.

If m is the total number of edges in the knot-graph, and n the total number of crossings, we know that $m = 2n$ (since the graph is regular of degree 4, and the sum of all vertex degrees equals $2m$).

Let the ith region contain $n_i^{\alpha\alpha}$, $n_i^{\beta\beta}$, $n_i^{\alpha\beta}$ respectively of the three types of edge; and number the regions 1 to $n+2$. Then:

$$m = 2n = \frac{1}{2}\sum_{i=1}^{n+2}(n_i^{\alpha\alpha} + n_i^{\beta\beta} + n_i^{\alpha\beta}) \ , \tag{3}$$

since each edge appears in just two regions.
We see that

$$N^{\alpha\alpha} = \frac{1}{2}\sum_{i=1}^{n+2} n_i^{\alpha\alpha} \quad \text{and} \quad N^{\beta\beta} = \frac{1}{2}\sum_{i=1}^{n+2} n_i^{\beta\beta}$$

Therefore $N^{\alpha\alpha} = N^{\beta\beta}$, using the lemma only. //
Using (3) gives

$$N^{\alpha\alpha} = \frac{1}{2}(4n - N^{\alpha\beta}) = 2n - \frac{1}{2}N^{\alpha\beta} \tag{4}$$

Corollaries

1. $N^{\alpha\beta}$ is even for any knot graph (using (4)) .
2. An odd n-gon has at least one edge of type $\alpha\beta$ or $\beta\alpha$ (using the lemma).

Alternative method of proof of the theorem that $N^{\alpha\alpha} = N^{\beta\beta}$

This proof uses the fact that from the associated graph of any regular knot-projection can be constructed an alternating knot-graph.

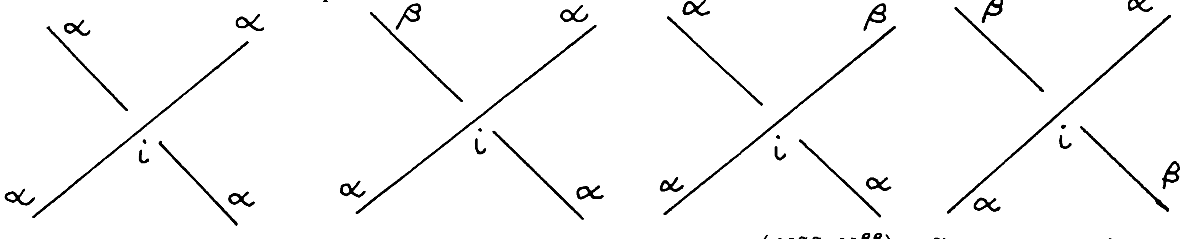
Given a knot-graph K; then if K is alternating there is nothing to prove, since then $N^{\alpha\alpha} = 0 = N^{\beta\beta}$.

If K is nonalternating, consider its associated graph and construct from it an alternating knot-graph $\tilde{K}$, which has, of course, no $\alpha\alpha$ or $\beta\beta$ edges.

It is clear that $\tilde{K}$ differs from K only in that at certain vertices, say $P_1, P_2, \ldots, P_r$, the over- and under-passes are reversed. So we can recover K from $\tilde{K}$ by successively reversing the over- and under-passes at $P_1, P_2, \ldots, P_r$. If we can show that at each reversal the changes in both $N^{\alpha\alpha}$ and $N^{\beta\beta}$ are equal, and that that must be so whatever the situation at a vertex undergoing reversal, then the theorem will be proven.

We can effect the proof by checking all possible cases (strictly there are 32, but many are equivalent).

The cases are easily drawn, and checked by systematically changing the α, β symbols on the ends adjacent to the vertex being reversed. We will show only four cases, for examples, thus:



It may be seen that the nett changes in $(N^{\alpha\alpha}, N^{\beta\beta})$ after reversals are respectively (0,0), (-1,-1), (1,1), (-2,-2) for the cases shown. Note that the nett changes are equal within each pair; and this is true in every type of reversal. //

CHAPTER 3

PROPERTIES OF $\alpha-$ AND $\beta-$ ADJACENCY MATRICES

The adjacency matrices J_α and J_β were defined in Chapter 2, section 2.2 . We will now derive a number of general properties of these matrices.

In the following two sections we shall refer only to knot-graphs of 1-links. At the end of these sections we shall briefly indicate how the results generalise to μ-links with $\mu > 1$.

3.1 ALTERNATING KNOT-GRAPHS (1-LINK, NO LOOPS, n VERTICES)

J_α is a bipermutation matrix

The α-adjacency matrix of an alternating knot-graph must contain exactly two 1s in each of its rows and columns. This follows from the fact that there are two α-steps away from, and two towards, every vertex of an alternating knot-graph.

Referring back to section 2.5 , where a tour of a T-diagram from a 1-link knot-graph is described, we see that we can split J_α into two [0,1] permutation matrices P_1 and P_2 such that $J_\alpha = P_1 + P_2$, in the following way.

Take the 1 from the first corner (row 1) of the T-diagram and place it in P_1 (in the same relative position, of course). Then move around the closed rectilinear diagram, placing the 1s from the successive corners into $P_2, P_1, P_2, \ldots$ *etc.* until all the $2n$ 1s have been placed. Then place zeros in all the unfilled places of P_1 and P_2.

Given the starting point, and the labelling, this produces the two permutation matrices required. Thus J_α is a bipermutation matrix.

Relationship between P_1 and P_2 (standard labelling of K)

If the knot-graph has been given a standard labelling, then P_1 and P_2 are related by one of the following two equations:

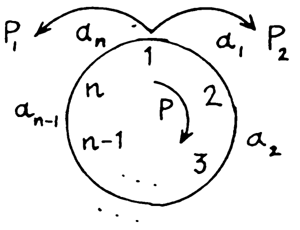
$$\begin{aligned} &\textit{either} \quad P_2 = PP_1 \\ &\quad\textit{or} \quad P_2 = P_1P' \end{aligned} \tag{1}$$

where P represents the permutation $(123 \cdots n)$, and P' its inverse. Note that $P' = (n, n-1, \cdots 321) = P^{-1}$.

N.B. We shall use various representations or notations for permutation matrices, choosing at any point the most convenient to write. In all cases the corresponding matrix forms are to be understood.

Proof of (1)

Let us assume that the standard labelling leads to the following circular word diagram (see 2.6.2). The J_α matrix then has entries of 1s as follows:



row 1, columns a_n and a_1
row 2, columns a_1 and a_2
row 3, columns a_2 and a_3
.... and so on, to ....
row n, columns a_{n-1} and a_n

Applying the T-diagram algorithm to split J_α places 1-entries in P_1 and P_2 as follows (assuming $a_n > a_1$):

	P_1	P_2
row 1	col a_n	col a_1
row 2	col a_1	col a_2
row 3	col a_2	col a_3
...	and so on	...

Thus the two permutations in J_α are

$$P_1 = \begin{pmatrix} 1 & 2 & 3 & & n \\ a_n & a_1 & a_2 & & a_{n-1} \end{pmatrix}$$

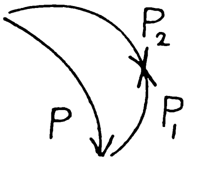
$$P_2 = \begin{pmatrix} 1 & 2 & 3 & & n \\ a_1 & a_2 & a_3 & & a_n \end{pmatrix}.$$

It can now be checked immediately that $P_2 = PP_1$, where $P = (123 \cdots n)$.

This is the first equation of (1). Had we chosen to begin the splitting algorithm by placing a 1 in (row 1, col a_1) of P_1, the second equation of (1) would have resulted. We could then, of course, have simply switched subscripts on P_1 and P_2 and obtained the first equation. In future we shall always assume that the first one holds.

Note the arrows placed on the above word diagram. They indicate that the P_1 maps can be read anti-clockwise (from inside the circle outwards) and the P_2 maps clockwise. The clockwise arrow inside the circle indicates that P maps $1 \to 2$, $2 \to 3$, and so on.

A simple vector diagram (below left) indicates the relationship



$P_2 = PP_1$.

This kind of diagram can be useful when studying walks on knot-graphs.

We have proved the following:

$$\begin{aligned} J_\alpha &= P_1 + P_2 \\ &= P_1 + PP_1 \\ &= (I + P)P_1, \quad \text{where } I \text{ is the } n \times n \text{ identity matrix.} \end{aligned} \tag{2}$$

We note from (2) that the single permutation P_1 is sufficient to define the knot-graph. For if we know P_1 we get J_α using (2); and then $J_\beta = J_\alpha'$; hence all adjacency relationships in the knot-graph are determined.

Theorem 1

Let K be a knot-graph from a 1-link, loopless alternating knot, the graph having a standard labelling. Then the adjacency matrices J_α ,J_β satisfy the following equations:

$$J_\alpha J_\alpha = J_\alpha J'_\alpha = CC' \qquad (3)$$

where $C = I + P$. (I is the $n \times n$ identity matrix, and P is the permutation represented by $(123 \cdots n)$.)

Proof:

$J_\beta = J'_\alpha$ for any alternating knot-graph.

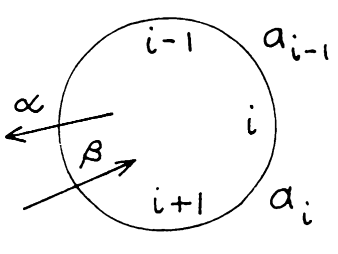
And by (2) above,

$$J_\alpha J'_\alpha = (I + P)P_1 . P'_1(I + P)'$$
$$= CC', \text{ since } P_1 P'_1 = I$$

Graph theoretic proof of (3)

We can obtain (3) immediately, by considering the graph-theoretic interpretation of the product of the α- and β-adjacency matrices. This is as follows:

The ijth element of $J_\alpha J_\beta$ equals the number of possible walks of length 2 which begin at vertex i and proceed to vertex j by means of an α-step followed by a β-step. Looking at the relevant portion of a word-diagram (see below left) we see that the only possible walks of this kind which can be made from i are:



$$i \xrightarrow{\alpha} a_{i-1} \xrightarrow{\beta} i-1$$
$$i \xrightarrow{\alpha} a_i \xrightarrow{\beta} i+1$$
$$i \xrightarrow{\alpha} a_{i-1} \xrightarrow{\beta} i$$
$$i \xrightarrow{\alpha} a_i \xrightarrow{\beta} i$$

Thus in matrix $J_\alpha J_\beta$ there are 1s in positions $(i,i-1)$, $(i,i+1)$, a 2 in position (i,i), and 0s elsewhere; with $i = 1,2, \cdots ,n \ (mod\ n)$.

Therefore

$$J_\alpha J_\beta = 2I + P' + P$$
$$= (I + P)(I' + P')$$
$$= CC'$$

This second proof makes very clear the fact that (3) is a direct consequence of the standard labelling of the alternating knot-graph.

3.2 NONALTERNATING KNOT-GRAPHS (1-LINK, NO LOOPS, n VERTICES)

In this section we show how theorem 1 generalises to a form which is satisfied by the adjacency matrices of nonalternating knot-graphs. First we need a result on the merging of a pair (J_α, J_β).

Merging a pair of adjacency matrices

Suppose we are given a nonalternating knot-graph K, which has a standard labelling and adjacency matrices K, J_α and J_β.

The standard labelling used on K corresponds with that of an alternating knot-graph on the same vertices and edges. Let this graph be called A and have adjacency matrices $(J_\alpha^\bullet, J_\beta^\bullet)$. We know, by definition of the various adjacency matrices, that

$$K = J_\alpha + J_\beta = J_\alpha^\bullet + J_\beta^\bullet \tag{4}$$

Next we show that $J_\alpha^\bullet$ can be constructed from (J_α, J_β) by a suitable *merging* of columns, an operation which we shall formally denote by

$$J_\alpha^\bullet = J_\alpha . m . J_\beta \tag{5}$$

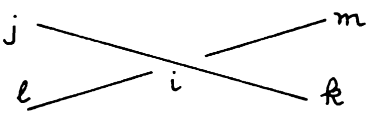
The following lemma ensures that the operation can always be carried out.

Lemma

Given any column in $J_\alpha^\bullet$. The same column must occur, in the same position, in either of J_α or J_β.

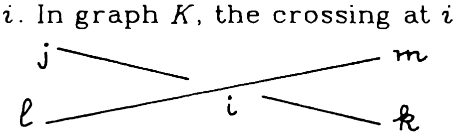
Proof:

Consider the i^{th} column of $J_\alpha^\bullet$; it contains two 1s, and its other elements are 0s. Suppose the 1s are elements ji and ki of $J_\alpha^\bullet$. This means that in the graph A , at vertex i is the arrangement of arcs as shown below in figure (a):



Note that i is α-adjacent to both j and k.

Now, in the corresponding nonalternating knot-graph K, the labels $ijklm$ do not differ, but any or all of the crossings at the five vertices may stay the same or be reversed (i.e. changed from under to over, or vice versa). However, for our purposes the crossing states at $jklm$ are immaterial; we need only consider the position at i. In graph K, the crossing at i may be as in figure (a) or as in figure (b) :



If it is as in figure (a), then the ith column of J_α = ith col. of $J_\alpha^\bullet$. If it is as in figure (b), then the ith column of J_β = ith col. of $J_\alpha^\bullet$.

This same argument holds for $i = 1,2, \cdots, n$, so the lemma is proved. //

Next we give an *algorithm* which will enable us to decide which columns have to be taken from the matrices J_α and J_β to carry out the merging operation.

Step 1

From (J_α, J_β) we can determine the matrices J, A, B, as shown in chapter 2.

Step 2

Using J, A, B we can tour the knot-graph A, as shown in chapter 2.

Step 3

The tour of Step 2 provides the word of the knot-graph. From this we can draw the circular-word diagram. Because the knot is nonalternating, the letters of the word will not go alternately inside the circle and out; occasionally there will be successions of them inside, and occasionally successions outside.

Step 4

Consider double-steps of a tour, beginning at 1 (inside) and proceeding clockwise round the circle; we can write these as follows:

$$(1a_1 2),(2a_2 3),(3a_3 4), \quad ,(na_n 1).$$

Then, if a_i lies outside the circle, take the a_ith column from J_α; whilst if a_i lies inside the circle, take the a_ith column from J_β.

Placing all the extracted columns together, in the same order as taken, into an $n \times n$ matrix produces J^*_α, the α-matrix of a related alternating knot-graph (there are two; the other is its mirror image).

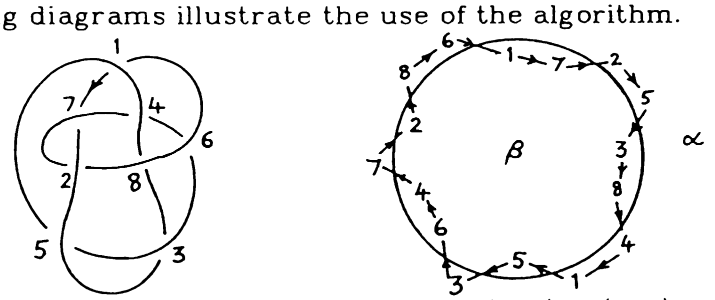
Proof:

It is clear that the set of triples $(1a_1 2)$,$(2a_2 3)$, etc. can always be formed, since there are $2n$ edges in the knot-graph and the labelling is standard.

Further, if a triple is $(ja_i(j+1))$ and a_i is outside the circle, both $j \to a_i$ and $(j+1) \to a_i$ are α-steps, so the a_ith column of J_α has 1s in positions ja_i and $(j+1)a_i$, and zeros elsewhere. Similarly, if a_i is inside the circle, $j \to a_i$ and $(j+1) \to a_i$ are β-steps, and the a_ith column of J_β has 1s in the required positions.

Finally, it is clear that if we used the newly formed J^*_α to produce a circular-word diagram, now the triples would be (say) $(1a_1^* 2)$, $(2a_2^* 3)$, ... , $(na_n^* 1)$, and all the a_i^*, $i = 1,\ldots,n$ would lie outside the circle; the other digits, $1,2\ldots,n$, would lie inside the circle; and the resulting diagram would correspond to an alternating knot-graph. //

The following diagrams illustrate the use of the algorithm.



Triples	(172)	(253)	(384)	(415)	(536)	(647)	(728)	(861)
Columns	7	5	8	1	3	4	2	6
Adj.matrix	β	α	β	α	α	β	β	α

Merge vector: $\underline{m} = (0,1,0,1,1,0,0,1)$

Figure: Knot 8_{20} ; *determining its merge vector*

Definitions

(i) The *merge vector* is a $1 \times n$ vector $\underline{m} = (m_1, m_2, \cdots, m_n)$ where

$$m_i = \begin{cases} 1 \text{ if } a_i \text{ (see Step 4 above) lies outside the circle} \\ 0 \text{ if } a_i \text{ lies inside the circle} \end{cases}$$

(ii) The *merge matrix* is the $n \times n$ diagonal matrix

$$M = diag\ \{ m_1, m_2, \ldots, m_n \}$$

(iii) The *merge operator* is defined by the action of the above algorithm on a pair (J_α, J_β) of adjacency matrices. We shall use the symbol $.m.$ for the binary operator 'merge'. Thus we can write

$$J_\alpha^\bullet = J_\alpha.m.J_\beta$$

(iv) We shall use $\underline{\overline{m}}$ as the vector obtained from $\underline{m}$ by changing 1s to 0s and vice-versa. And $\overline{M}$ will mean the diagonal matrix having $\underline{\overline{m}}$ as leading diagonal and 0s elsewhere.

The following results can now be obtained.

Theorem 2

(i) $\overline{M} = I - M$, where I is the identity matrix
(ii) $M^2 = M$; $\overline{M}^2 = \overline{M}$; $M = M'$; $\overline{M} = \overline{M}'$
(iii) $M\overline{M} = \underline{0}$; $\overline{M}M = \underline{0}$
(iv) $J_\alpha^\bullet = J_\alpha.m.J_\beta = J_\alpha M + J_\beta \overline{M}$
(v) $J_\beta^\bullet = MJ'_\alpha + \overline{M}J'_\beta$
(vi) $J_\beta^\bullet = J_\alpha \overline{M} + J_\beta M$
(vii) $J_\alpha \overline{M} - MJ'_\alpha = \overline{M}J'_\beta - J_\beta M$
(viii) $(J_\alpha - MJ'_\alpha)\overline{M} = \overline{M}J'_\beta \overline{M}$
$\overline{M}(J'_\beta - J_\beta M) = \overline{M}J'_\alpha \overline{M}$
(ix) $MJ_\beta \overline{M} = M[K - (J_\alpha M + J'_\alpha)]$
(x) $\overline{M}J_\alpha M = \overline{M}[K - (J_\beta \overline{M} + J'_\beta)]$

Proofs:

(i), (ii) and (iii) follow from the definitions of M and $\overline{M}$.

(iv) Suppose $\underline{m}$ has 1s in r positions and 0s in the remainding $n-r$ positions. Let us suppose, without loss of generality, that the first r elements are 1s.

Then $J_\alpha M$ is an $n \times n$ matrix whose first r columns are the same as in J_α, and the rest are $\underline{0}$s. And $J_\beta \overline{M}$ is an $n \times n$ matrix with its last $n-r$ columns as in J_β and the rest $\underline{0}$s. So

$$J_\alpha^\bullet = J_\alpha.m.J_\beta = J_\alpha M + J_\beta \overline{M}$$

as required by $.m.$.

(v) $J_\beta^\bullet = J_\alpha^{\bullet\prime} = MJ'_\alpha + \overline{M}J'_\beta$, since $M, \overline{M}$ are symmetric.

(vi) K is the adjacency matrix of the associated graph, so $K = J_\alpha^\bullet + J_\beta^\bullet = J_\alpha + J_\beta$ (see chapter 2).

Therefore

$$\begin{aligned} J_\beta^\bullet &= K - J_\alpha^\bullet \\ &= J_\alpha + J_\beta - (J_\alpha M + J_\beta \overline{M}), \qquad \text{using } (iv) \\ &= J_\alpha(I - M) + J_\beta(I - \overline{M}) \\ &= J_\alpha \overline{M} + J_\beta M \end{aligned}$$

(vii) From (v) and (vi), $(J_\alpha \overline{M} - MJ'_\alpha) = (\overline{M}J'_\beta - J_\beta M)$ as required.

(viii) Multiply (vii) on the right by $\overline{M}$, and use (ii) and (iii). Multiply (vii) on the left by $\overline{M}$, and use (ii) and (iii).

N.B. Multiplying by M from the left, and from the right, gives respectively

$$MJ_\beta M = M(J'_\alpha - J_\alpha \overline{M})$$

and $$MJ'_\alpha M = (\overline{M}J'_\beta - J_\beta)M$$

(ix) and (x) are obtained from (iv) and (v), using $K = J_\alpha^\bullet + J_\beta^\bullet$

The most surprising result is perhaps that the two forms of $J_\beta^\bullet$ (i.e. (v) and (vi)) exist for knot-graphs. This leads to the following:

Corollary

(a)
$$(J_\alpha .m. J_\beta)' = J_\beta .m. J_\alpha = J_\beta^\bullet \qquad (6)$$

and the attractive

(b)
$$K = J_\alpha .m. J_\beta + J_\beta .m. J_\alpha \ , \qquad \{\text{using } (4)\}. \qquad (7)$$

Theorem 3

Given any loopless, 1-link knot-graph, having adjacency matrices (J_α, J_β), and merge matrix M.

Then

$$J_\alpha M J'_\alpha + J_\beta \overline{M} J'_\beta = CC' \ , \qquad (8)$$

where $C = I + P$.

Proof:

(i) If the knot-graph is alternating, then $M = I$, $\overline{M} = \underline{0}$, and $J'_\alpha = J_\beta$. Then we have to prove that $J_\alpha J_\beta = CC'$. But this is theorem 1.

(ii) If the knot-graph is nonalternating, we can merge J_α and J_β , and obtain $J_\alpha^\bullet, J_\beta^\bullet$ for the corresponding alternating knot-graph.

Now $(J_\alpha^\bullet, J_\beta^\bullet)$ satisfy theorem 1, so

$$\begin{aligned} J_\alpha^\bullet J_\beta^\bullet &= (J_\alpha M + J_\beta \overline{M})(J_\alpha M + J_\beta \overline{M})' \\ &= (J_\alpha M + J_\beta \overline{M})(MJ'_\alpha + \overline{M}J'_\alpha) \\ &= J_\alpha M J'_\alpha + J_\beta \overline{M} J'_\beta \\ &= CC' , \qquad \text{using theorems 1 and 2} \end{aligned}$$

Note : Using the alternative form of $J_\beta^\bullet$ (thm.2(vi)) gives

$$(J_\alpha M + J_\beta \overline{M})(J_\alpha \overline{M} + J_\beta M) = CC'$$

i.e. $$(J_\alpha .m. J_\beta)(J_\beta .m. J_\alpha) = CC' \qquad (9)$$

Examples; Generalisation To Links

(i) *On alternating μ-links*

In the figure below we show the graphs of knots 4_1 and 4_1^2 (with standard labellings), the latter being a 2-link. Then we give the products of their α- and β-matrices.

Note that the product in each case can be expressed as a matrix sum $2I + P + P'$, where $P = (1234)$ in the 1-link case 4_1 and $P = (12)(34)$ in the 2-link case 4_1^2. And that in each case the matrix sum factorises to CC', where $C = I + P$.

$$\begin{array}{ccc} J_\alpha & J_\beta & CC' \end{array}$$

$$\begin{bmatrix} 0&1&1&0\\ 0&0&1&1\\ 1&0&0&1\\ 1&1&0&0 \end{bmatrix} \begin{bmatrix} 0&0&1&1\\ 1&0&0&1\\ 1&1&0&0\\ 0&1&1&0 \end{bmatrix} = \begin{bmatrix} 2&1&0&1\\ 1&2&1&0\\ 0&1&2&1\\ 1&0&1&2 \end{bmatrix}$$

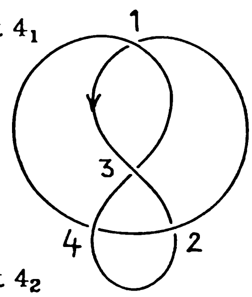
$$= \quad 2I + P + P' ; \quad P = (1234)$$

$$\begin{bmatrix} 0&0&1&1\\ 0&0&1&1\\ 1&1&0&0\\ 1&1&0&0 \end{bmatrix} \begin{bmatrix} 0&0&1&1\\ 0&0&1&1\\ 1&1&0&0\\ 1&1&0&0 \end{bmatrix} = \begin{bmatrix} 2&2&0&0\\ 2&2&0&0\\ 0&0&2&2\\ 0&0&2&2 \end{bmatrix}$$

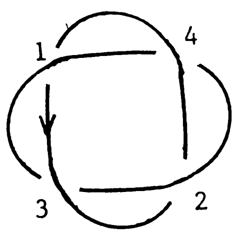
$$= \quad 2I + P + P' ; \quad P = (12)(34)$$

Demonstrations of $J_\alpha J_\beta = CC'$

Knot 4_1



Knot 4_2



The matrix product for knot 4_1 illustrates the results proved in section 3.1 for alternating 1-link knot-graphs. And the product equation given for knot 4_1^2 suggests that theorem 1 can be generalised to alternating 2-links and possibly further to μ-links, with the only difference being in the form that P (and hence C) takes. This is in fact the case. It can be shown that if an alternating μ-link (with $\mu > 1$) is given a standard labelling, then its α-adjacency matrix can be expressed as the sum of two permutation matrices, thus:

$$J_\alpha = P_1 + P_2,$$

with $P_2 = PP_1$, where P has μ-cycles; i.e. P has the form

$$(12 \cdots n_1)(n_1+1, \cdots ,n_1+n_2) \cdots (n - n_\mu + 1, \cdots ,n),$$

where $n_1, n_2, \cdots, n_\mu$ are the numbers of labels placed on the links $1, \cdots, n_\mu$ as the standard labelling of the knot-graph is carried out.

Then Theorem 1 holds for alternating μ-links, with the matrix C (and hence C') being in block-diagonal form. The blocks are of orders $n_1, n_2, \ldots, n_\mu$, and they correspond to the cycles in P.

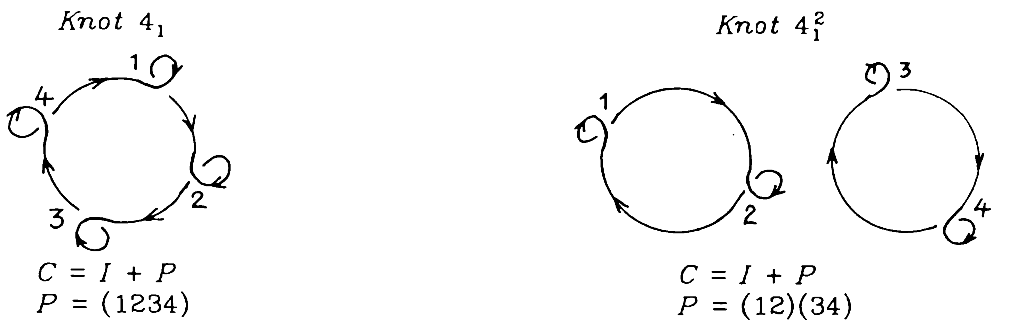
Three elementary properties of C matrices are noted next, with indications of their relationships to knots;

(a) Since $C = I + P$ we easily confirm that $C + C' = CC'$; and also that $P'C = CP' = C'$ and that $PC' = C'P = C$

(b) $CC' - 2I = P + P'$ and so is a bipermutation matrix. It is the adjacency matrix of the torus knot-graph. Thus through theorem 1 all knots are related to the torus knots.

(c) $C = I + P$ is itself a bipermutation matrix; and it is the adjacency matrix of the knot (unknot) which for 1-link knots is S^1 with n loops made in it, as shown in the diagram below left. Thus through theorem 1, all alternating 1-links may be thought of as related to a pair of S^1s (one for each of C and C') with n loops in each (the $2n$ loops then correspond to the $2n$ edges in the knot-graph of the 1-link).

In the μ-link case, C represents μ S^1s having $n_1,\ldots,n_\mu$ loops respectively in them. The figure below right illustrates this for the case of 2-link 4_1^2 .



Knot-graphs from C matrices

(ii) *Nonalternating μ-links*

The result of theorem 3 for nonalternating 1-links may be generalised in similar fashion, with C taking block-diagonal form when $\mu > 1$.

3.4 FURTHER PROPERTIES OF ADJACENCY MATRICES

We now study the determinant and inverse of adjacency matrices of alternating knot-graphs. We then obtain several results about the commutator $<J_\alpha, J_\beta> \equiv J_\alpha J_\beta - J_\beta J_\alpha$ and about the so-called r^{th} power commutator $<J_\alpha, J_\beta>^{(r)} \equiv (J_\alpha J_\beta)^r - (J_\beta J_\alpha)^r$

In this section, all adjacency matrices are derived from alternating knots of one link, except where variations are expressly described. And for convenience the subscripts α, β will be dropped; we shall write J for J_α and J' for J_β.

The determinant of J

The 1-link case

We showed in 3.1 that if J is the adjacency matrix of a loopless, 1-link alternating knot-graph, with standard labelling, then $J = (I + P)P_1$. I is the identity matrix, P is the permutation (matrix) $(123\ldots n)$, and P_1 is a permutation matrix.

$$\begin{aligned}\text{Hence} \quad \det(J) &= \det(I + P) \times \det(P_1) \\ &= [\det(I) + \det(P)] \times \det(P_1) \qquad (10)\end{aligned}$$

N.B. The operation of taking a determinant distributes over $(I + P)$ in the case that P has no cycles, which is the case with a 1-link knot-graph.

Now, a glance at a matrix of P shows the following:

$$\det(P) = \begin{cases} +1 & \text{if } n \textit{ is odd} \\ -1 & \text{if } n \textit{ is even} \end{cases}$$

And since $\det(I) = 1$ we have

$$\det(J) = \begin{cases} 2 \times \det(P_1) = \pm 2, \text{ depending on the parity of } P_1, \text{ if } n \text{ is odd} \\ 0, \text{ if } n \text{ is even} \end{cases} \qquad (11)$$

The μ-link case

If the knot is a μ-link, then the matrix P is block diagonal, with blocks C_i (say) of orders $n_1, n_2, \ldots, n_m$ (see the example with $m = 2$ below).

Figure: examples of the matrix P

$$\begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix} \qquad \qquad \left[\begin{array}{cc|ccc} 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 \end{array}\right]$$

Case: $m = 1, n = 4$ Case: $m = 2, \; P = (12)(345) = dg[C_1, C_2]$

Now it can be shown that

$$\det(I + P) = \prod_{i=1}^{m} (1 + \det(C_i))$$

Hence $\det(I + P) = 0$ if any of the integers $n_1, \ldots, n_m$ is even.

If all the integers $n_1, \ldots, n_m$ are odd, then $\det(I + P) = 2^m$, so:

$$\det(J) = \begin{cases} \pm 2^m, \text{ if all } n_i \text{ are odd,} \\ 0 \text{ , otherwise} \end{cases} \qquad (12)$$

The inverse of J

Theorem 4

Let J be the α-matrix of a loopless, 1-link alternating knot-graph with standard labelling on n vertices. Then

(i) J is singular if n is even

(ii) $J^{-1} = \frac{1}{2} P'_1 \sum_{i=1}^{n} (-1)^{i+1} P'^i$ if n is odd.

Proof:

(i) It was shown in (11) that $\det(J) = 0$ if n is even.

(ii) Since by (2) $J = (I + P)P_1$, we find by direct multiplication that with n odd:

$$J \frac{1}{2} P'_1 \sum_{i=1}^{n} (-1)^{i+1} P'^{i}$$

$$= \frac{1}{2}(I + P)(P' - P'^2 + P'^3 - \qquad - P'^{n-1} + P'^{n})$$

$$= \frac{1}{2}[P' - P'^2 + P'^3 - \qquad - P'^{n-1} + P'^{n} + I - P' + P'^2 - $$

$$- P'^{n-2} + P'^{n-1}]$$

$$= I, \quad \text{since } P'^{n} = I$$

Similarly, $\frac{1}{2} P'_1 (\sum_{i=1}^{n} (-1)^{i+1} P'^{i})(I + P)P_1 = \frac{1}{2} P'_1 \, 2IP_1 = I$ //

Any matrix $2J^{-1}$ has all its elements ± 1, with $\frac{n}{2}$ -1s in each row and column.

The inverse of J appears to have no useful interpretation or application in the study of knot-graphs.

The commutator $<J_\alpha, J_\beta>$

We define the commutator by

$$<J_\alpha, J_\beta> \equiv J_\alpha J_\beta - J_\beta J_\alpha$$
$$= JJ' - J'J \quad \text{for an alternating knot-graph.} \tag{13}$$

Theorem 5

Let $<J, J'>$ be the commutator of a 1-link alternating knot-graph. Then:

(i) $<J, J'>$ is symmetric;

(ii) $<J, J'>$ has in each of its rows

either all 0s,

or one -1, one +1, and 0s elsewhere,

or two -1s, two +1s, and 0s elsewhere.

(iii) $\det <J, J'> = 0$

(iv) $\mathrm{tr} <J, J'> = 0$

Proof:

(i) is trivial;

(ii) since $J = (I + P)P_1$, a little algebra gives

$$<J, J'> = T - P'_1 T P_1, \tag{14}$$

where $T = P + P'$, which is the adjacency matrix of the torus knot-graph with standard labelling. Now T has two 1s in each row and column; and $P'_1 T P_1$ is a permutation of both the rows and the columns of T, so it has two 1s in each row. Result (ii) follows from this.

(iii) From (ii), the sum of the n columns of $\langle J, J'\rangle$ is $\underline{0}$; hence $\det\langle J, J'\rangle = 0$.

(iv) Since $\mathrm{tr}(P'_1 T P_1) = \mathrm{tr}(T)$ the result follows from (14).

Use of the commutator:

It is clear that the vanishing of $\langle J, J'\rangle$ indicates a certain high degree of symmetry in the corresponding knot-graph. It is easy to show that the commutator vanishes if and only if the permutations P and P_1 commute. In the torus knot-graph $P_1 = P$, so the torus commutator vanishes; alternatively, working directly from (14) and setting $P_1 = P$ we get

$$T - P'_1 T P_1 = T - P'(P + P')P = T - (P + P') = T - T = \underline{0}$$

Indeed, we can see from (14) that we can regard the action of P_1 (the permutation which defines a given alternating knot-graph) as causing a distortion of the torus matrix, and then $\langle J, J'\rangle$ measures in some way the distortion of the given knot from the torus knot.

To get a numerical measure of the amount of distortion we can count the number of non-zero elements in $\langle J, J'\rangle$; or take a quarter of that number, since $\langle J, J'\rangle$ is symmetric, and there are always equal numbers of +1s and -1s.

Denoting the latter (i.e. one-quarter the number of non-zero elements in the commutator) by s, and writing $S \equiv \langle J, J'\rangle$, we find:

Theorem 6: $s = \frac{1}{4}\, trace\,(S^2)$

Proof: Since S is symmetric, $S^2 = SS'$. And SS' has $\sum_i \sum_j s_{ij}^2$ for trace, which counts those non-zero elements s_{ij} which are +1 or -1.

The rth power commutator

Interesting formulae can be developed for differences of powers of JJ' and $J'J$, as the following theorems and discussion shows.

Theorem 7(a)

Let J be the α-adjacency matrix of an alternating knot-graph, with $J = (I + P)P_1 = CP_1$.
Define the rth-power commutator by $\langle J, J'\rangle^{(r)} \equiv [(JJ')^r - (J'J)^r]$. Let $T = P + P'$. Then:

$$\langle J, J'\rangle^{(r)} = (2I + D)^{(r)}\,, \tag{15}$$

where the R.H.S. has to be interpreted as follows:

first $(2I + D)^r$ is expanded in the normal way, into $r+1$ terms

then in each term the D^k is replaced by $(T^k - P'_1 T^k P_1)$

Note that with $T^0 \equiv I$ we obtain $D^0 = 0$.

Proof:

$$JJ' = CP_1 P'_1 C' = CC' = 2I + T$$

and $J'J = P'_1 C' C P_1 = 2I + P'_1 T P_1$

Therefore

$$<J,J'>^{(r)} = (2I + T)^r - (2I + P'_1TP_1)$$

$$= \sum_{k=0}^{r} \binom{r}{k} 2^{r-k} T^k - \sum_{k=0}^{r} \binom{r}{k} 2^{r-k} (P'_1TP_1)^k ,$$

$$= \sum_{k=0}^{r} \binom{r}{k} 2^{r-k} (T^k - P'_1T^kP_1) ,$$

since $(P'_1TP_1)^k = P'_1T^kP_1$,

$$= (2I + D)^{(r)} . \qquad //$$

Examples

(i) $<J,J'>^{(1)} = (2I + D)^{(1)}$

$= 2I^1D^0 + D^1 = D$, since $D^0 = 0$.

(ii) $<J,J'>^{(2)} = (2I + D)^{(2)}$

$= (2I)^2D^0 + 2(2I)^1D^1 + D^2$

$= 0 + 4(T - P'_1TP_1) + (T^2 - P'_1T^2P_1)$

We can replace the entity D in (15) by a series which involves only powers of P and allows for easier interpretation later, thus:

Theorem 7(b)

$$<J,J'>^{(r)} = [2I + (X + X^{-1})]^{(r)} , \tag{16}$$

where the R.H.S. is to be interpreted as follows:

first expand $[2I + (X + X^{-1})]^r$ in powers of $(X + X^{-1})$;

next expand each power $(X + X^{-1})^j$ in the normal way, and reduce indices (e.g. the third term in the expansion is $\binom{j}{2}X^{j-2}X^{-2} = \binom{j}{2}X^{j-4}$); and

finally each power X^s is replaced by $(P^s - P'_1P^sP_1)$ for each value of s which occurs.

The result expressed in $\sum\sum$ form is:

$$<J,J'>^{(r)} = \sum_{k=0}^{r} \binom{r}{k} 2^{r-k} \sum_{s=0}^{k} \binom{k}{s} (P^{k-2s} - P'_1P^{k-2s}P_1) \tag{17}$$

Proof:

Using symbols defined in theorem 7(a), we have

$$D^k = T^k - P'_1T^kP_1$$

$$= (P + P')^k - P'_1(P + P')^kP_1$$

Expanding and combining terms gives

$$D^k = X^k + kX^{k-2} + \binom{k}{2} X^{k-4} + \quad\quad + X^{-k},$$

where $X^s \equiv (P^s - P'_1 P^s P_1)$, and $P^{-t} = P'.P' \cdot \quad \cdot P'$ to t factors.

$$= (X + X^{-1})^k \tag{18}$$

(16) follows immediately from theorem 7(a). To obtain (17) we use line 5 of the proof of theorem 7(a). //

Notes:

(i) It is seen from (17) that in order to compute the commutators for an alternating knot-graph defined by permutation matrix P_1, one has simply to compute the differences of powers of P and their conjugates with respect to P_1, and then make the double summation.

(ii) It is interesting to speculate on a comparison of the two equations

$$(J.J')^r = [2I + (P + P')]^r \quad \text{(from theorem 1)}$$

and $\langle J,J'\rangle^{(r)} = [2I + (X + X^{-1})]^{(r)}$ which is (16).

Corollaries

(i) $\langle J,J'\rangle^{(r)} = 0$ for any r if P and P_1 commute.
Proof: If $PP_1 = P_1P$, then $P^jP_1 = P_1P^j$; using this in (17) gives the result.

(ii) If $X^i = X^j$, then P_1 and $(P^i - P^j)$ commute.
Proof: The result follows from use of the definition of X^s, from (18), in $X^i - X^j = 0$.

(iii) The first commutator can be calculated directly from the matrices P and P_1 thus:

$$\langle J,J'\rangle = (P - P_1PP_1) + (P' - P_1P'P_1). \tag{19}$$

Proof: using $D = X + X^{-1}$, as in (18). //

We have said nothing so far about the commutator for nonalternating knot-graphs. One result is contained in the following theorem.

Theorem 8

Let (J_α, J_β) be the adjacency matrices of a nonalternating knot-graph, and let K be a corresponding alternating knot-graph (i.e. one with the same associated graph) having adjacency matrices $(\tilde{J},\tilde{J}')$.
Let $\Delta_\alpha = \tilde{J} - J_\alpha$ and $\Delta_\beta = \tilde{J}' - J_\beta$.
Then: $\langle J_\alpha, J_\beta\rangle = \langle K, \Delta_\alpha\rangle$, where K is here used to denote the adjacency matrix of the undirected associated graph.

Proof:

From definitions (chapter 2) we have $K = \tilde{J} + \tilde{J}' = J_\alpha + J_\beta$.

Therefore $\Delta_\alpha + \Delta_\beta = \underline{0}$

$$\text{so } \Delta_\alpha = -\Delta_\beta. \tag{20}$$

Then by definition,

$$\begin{aligned}<J_\alpha, J_\beta> &= J_\alpha J_\beta - J_\beta J_\alpha \\ &= (\mathcal{J} - \Delta_\alpha)(\mathcal{J}' - \Delta_\beta) - (\mathcal{J}' - \Delta_\beta)(\mathcal{J} - \Delta_\alpha) \\ &= K\Delta_\alpha - \Delta_\alpha K = <K, \Delta_\alpha> \qquad //\end{aligned}$$

Interpretation of the commutators

We described earlier how the first commutator could serve as a measure of symmetry of a knot-graph. We now give a graphical interpretation of commutators, which serves to give insight into the matrix results of the previous theorems.

$$(J_\alpha J_\beta)^r = J_\alpha J_\beta J_\alpha J_\beta \cdots J_\alpha J_\beta \equiv A \quad \text{,say}$$

and $(J_\beta J_\alpha)^r = J_\beta J_\alpha J_\beta J_\alpha \cdots J_\beta J_\alpha \equiv B$,say.

Therefore $<J_\alpha, J_\beta>^{(r)} = A - B$: and the elements of A and B are interpretable as follows:

A_{ij} is the number of walks, of length $2r$, which may be made on the knot-graph by following edges which are alternately of types $\beta\alpha$, $\alpha\beta$, each walk starting at vertex i with a $\beta\alpha$-edge and ending at vertex j. Note that edges may occur more than once in a walk: and that in an alternating knot-graph all such walks are traced 'along the string'.

B_{ij} is similarly the number of walks of length $2r$, from vertex i to vertex j and with alternating types of edge, but this time beginning with an $\alpha\beta$-edge.

The proof of the above assertions requires only a simple variation of a well-known theorem on powers of adjacency matrices counting walks in graphs.

Then:

$$<J_\alpha, J_\beta>_{ij}^{(r)} = \begin{cases} 0 & \text{if there is no walk } i \text{ to } j, \\ 0 & \text{if there is a walk of type A, and also one of type B, } i \text{ to } j, \\ 1 & \text{if there is a walk of type A, and none of type B, } i \text{ to } j, \\ -1 & \text{if there is no walk of type A, and one of type B, } i \text{ to } j, \\ & \text{and so on} \end{cases}$$

3.5 APPLICATIONS

In this section we discuss several applications or developments making use of the formulae and theorems on adjacency matrices which were found in the earlier sections.

Touring algorithms revisited

In chapter 2 an algorithm for touring a knot-graph was given in terms of the adjacency matrices J, A and B. It was noted that in the case that the knot-graph is alternating, the algorithm uses J and J' alternately to determine a sequence of vertices for an 'along the string' tour.

Now that we know from 3.2(2) that for an alternating knot-graph $J = (I + P)P_1 = P_1 + PP_1$, we can give a much simpler algorithm for touring such a graph. For convenience we shall use the notation $U \equiv P_1$ and $V \equiv PP_1$, so $J = U + V$. Then the algorithm is:

Algorithm

Provide the knot-graph with a standard labelling, and hence obtain the permutation matrices U and V for the α-steps. Note that the β-steps are given by U' and V'.

Choose a vertex to start from.

The first step may be an α-step in one of two directions; or it may be a β-step, again in one of two directions. Thus there are four tours of the graph, starting from that vertex.

The vertices for the four tours are given directly from the following sequences of permutations:

α-step starts

U,V',U,V',...,U,V' ($2n$ steps, U and V' alternate)

V,U',V,U',...,V,U' ($2n$ steps, V and U' alternate)

β-step starts

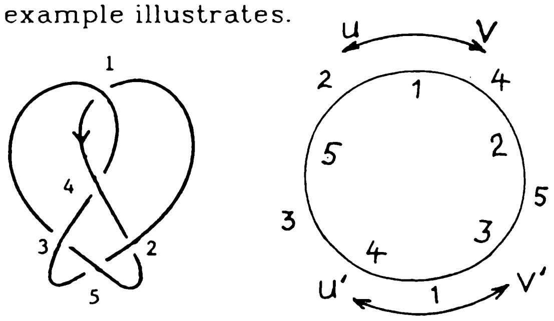
U',V,U',V,...,U',V ($2n$ steps, U' and V alternate)

V',U,V',U,...,V',U ($2n$ steps, V' and U alternate)

Proof:

A general proof using the circular-word diagram of the knot-graph is elementary. The following example illustrates.

Example



The 5_2 *knot, with its circular-word diagram*

The permutations

U = (124)(35)	U' = (142)(35)
V = (143)(25)	V' = (134)(25)

The four tours from vertex 1

(U,V') :	1,2,5,3,4,1,3,5,2,4,(1)
(V,U') :	1,4,2,5,3,1,4,3,5,2,(1)
(U',V) :	1,4,3,5,2,1,4,2,5,3,(1)
(V',U) :	1,3,5,2,4,1,2,5,3,4,(1)

General result for regular 4-valent graphs

It would seem worth pointing out here that many results we found above for knot-graphs relied on the fact that knot-graphs are regular of degree four. For example, the basic facts about the adjacency matrix of a 4-regular graph can be expressed in the following theorem, which we shall give without proof since the various items in it have already been proved with reference to knot-graphs and only have to be re-worded appropriately to apply to general 4-regular graphs.

Theorem 9

Given a regular 4-valent graph on n vertices, which is labelled and then has adjacency matrix K.

There are two bipermutation matrices X,Y, and a permutation matrix Q, such that:

$$Q'KQ = X + X' = Y + Y' \tag{i}$$

$$Q'KK'Q = (X + X')^2 = (Y + Y')^2 \tag{ii}$$
$$= (CU)^2 = [(CU) + (CU)'](CU)' + CC' \quad ,$$

where $X = U + PU$, with U a permutation and P a permutation of the type $(123 \cdot\cdot n_1)(n_1+1, \cdot\cdot\cdot, n_1+n_2)\cdot\cdot(\quad\cdot n)$.

Also $Y = U + U'P'$, and $C = I + P$, with I the $n \times n$ identity matrix.

Note that Q is the permutation which converts the given labelling of the graph to a standard one. And that X,X' and Y,Y' induce respectively balanced alternating and balanced nonalternating orientations, on the graph.

We have considered ways of exploiting this theorem to study regular graphs of valencies other than 4, but will not discuss these here.

T-diagrams revisited

In chapter 2 the concept of a T-diagram was introduced, and several propositions about them were proved.

One question that was raised was how many distinct T-diagrams there might be for a given knot-graph (distinct up to certain symmetry operations on the $n \times n$ square containing the diagram).

We are now in a position to say more about T-diagrams, using the fact that $J = U + PU$ *or* $U + UP'$ and other results proved in this chapter.

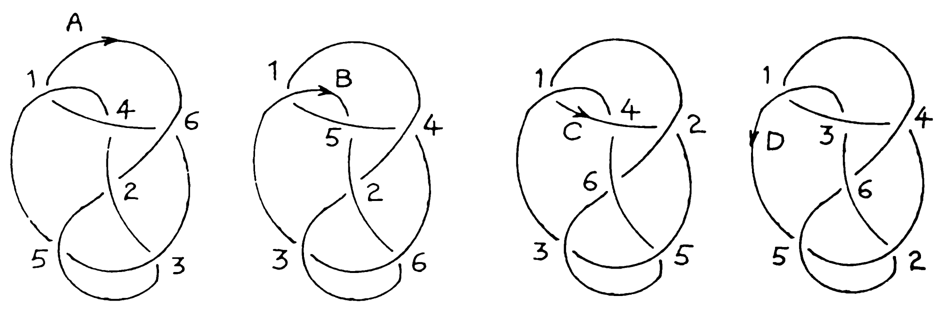
(i) **A counter-example**

We first provide a counter-example to a conjecture.

The diagram below shows the four standard labellings associated with starting at a chosen vertex 1 of a knot-graph from the knot 6_3. The corresponding pairs of permutations (U,V), where $V = PU$ in cases A and C, and $V = UP'$ in cases B and D, with $P = (123456)$, are given. Finally the four T-diagrams are shown.

It is clear, by inspection of the T-diagrams, that no symmetry operation on the squares containing the diagrams will transform any two of the four into each other.

This, then, is a counter-example to the conjecture that there is always some symmetry relationship between the four T-diagrams associated with any given vertex of a knot-graph.



U = (142635)   (152463)   (124)(356)   (143)(265)
V = (164)(253)   (145)(236)   (146253)   (136425)

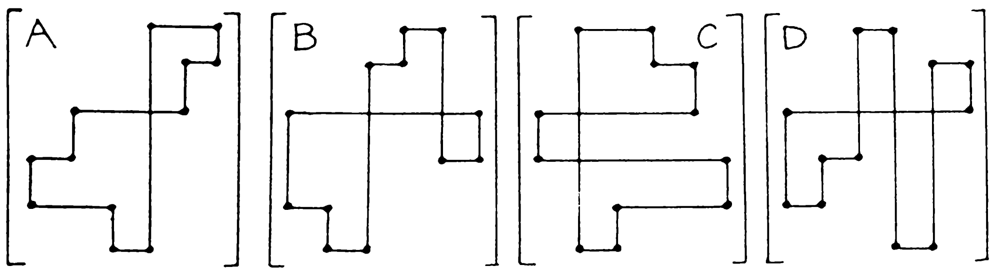
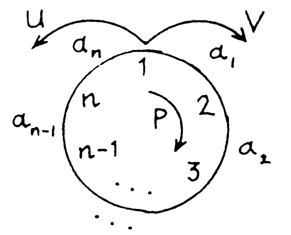


Figure: four standard labellings from vertex 1 of knot 6_3, *and the corresponding T-diagrams*

(ii) **Moving the starting vertex**

We next obtain formulae for relating the J matrices obtained by starting a labelling from a given vertex with those obtained by starting from another vertex.

Consider a general circular-word diagram for a given knot-graph which has been given a standard labelling (see the diagram below):



The α-adjacency matrix is given by $J = U + V$, where

$$U = \begin{pmatrix} 1 & 2 & 3 & & n \\ a_n & a_1 & a_2 & & a_{n-1} \end{pmatrix}$$

$$\text{and } V = \begin{pmatrix} 1 & 2 & 3 & & n \\ a_1 & a_2 & a_3 & & a_n \end{pmatrix}$$

If the starting vertex is moved along the first arc to a_1, and a new standard labelling of the knot-graph is made from there, the effect is to change $a_1 \to 1$, $a_2 \to 2$, $a_3 \to 3$, and so on. But this is precisely the permutation V'. Hence the new α-matrix is obtained from the old by permuting the rows and the columns thus:

$$\begin{aligned} J_{new} &= VJ_{old}V' \\ &= V(U + V)V' \\ &= (VUV') + V = VC' \ , \end{aligned}$$

since $P' = UV'$, and $C' = P' + I$

Call this result $\quad J_2 = VC'$ (1)

Moving the starting vertex one arc further, and relabelling the knot-graph, is equivalent to permuting the vertices by 2→1, 3→2, 4→3, , i.e. by P'. The new matrix this time is:

$$J_3 = VJ_2V' = PJ_1P',$$

where J_1 is the original α-matrix. Thus

$$J_3 = PUP' + PVP' = CVP' \tag{2}$$

Continuing around the knot, moving the starting vertex, we obtain a sequence of $2n$ α-matrices, given by the following formulae:

$$\left.\begin{aligned} J_{2r} &= (P^{r-1}VP'^{r-1})C' && \text{for } r = 1,2,3, \cdots ,n \\ J_{2r+1} &= C'(P^r VP'^r) && \text{for } r = 0,1,2, \quad ,n-1 \end{aligned}\right\} \tag{3}$$

It would appear that we now have $8n$ possible α-matrices, since we could set off from vertex 1 in any one of 4 directions, and produce $4 \times 2n$ matrices by means of formulae (3). But we reduce the total number to $4n$ by observing that any sequence of $2n$ matrices produced by travelling in one direction must overlap (and equal, term by term) one of the other three sequences, produced in the other directions. An example will make this clear.

Let the two sequences of matrices produced from the knot graph shown below, by proceeding in the A and the B directions as indicated, be labelled:

$$A_1, A_2, A_3, A_4, A_5, A_6, A_7, A_8, A_9, A_{10}$$
$$B_1, B_2, B_3, B_4, B_5, B_6, B_7, B_8, B_9, B_{10}$$

in the orders in which they are produced.

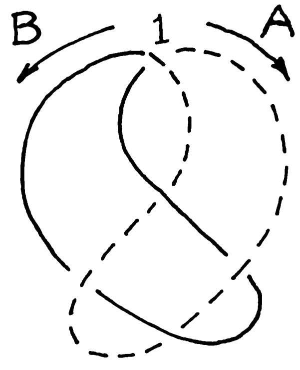
But if we trace the dotted line from 1, in the A direction, we see that $A_6 = B_1$, $A_7 = B_2$ and so on. Thus the sequences overlap, and provide equal sets of matrices, as follows:

$$A_1, A_2, A_3, A_4, A_5, A_6, A_7, A_8, A_9, A_{10}$$
$$B_6, B_7, B_8, B_9, B_{10}, B_1, B_2, B_3, B_4, B_5$$

Similarly, the other two sequences from vertex 1 must provide two equal sets of matrices.

And this overlapping of sequences must occur no matter what is the common starting vertex, in any knot-graph.

Thus all $4n$ adjacency matrices, and hence the $4n$ T-diagrams of a knot-graph, can be obtained by determining just two permutations, say V_1 and V_2, from two standard labellings made from a given starting vertex and then applying formulae (3) to them.



(iii) The number of distinct T-diagrams

Although new insights have been provided through the above examples and formulae for the $4n$ T-diagrams, the answer to the question 'How many distinct T-diagrams has a given knot-graph?' has not been advanced far.

CHAPTER 4

WALKS, GROUPS AND GROUPOIDS ON KNOT-GRAPHS

In a multi-(di-) graph any sequence of consecutive edges(arcs), where in the directed case we require the edge orientations to have the same sense from edge to edge , is called a *walk*. The number of edges (arcs) in a walk is called the *length of the walk*. Note that a walk may pass through the same edge (arc) more than once.

It is clear that a study of the types of walk and computation of the numbers of walks of given types on knot-graphs will yield summary information about the structure of knot-graphs. Such information need not, however, lead us to useful topological invariants of the knots themselves.

In this chapter we obtain various formulae for describing and counting walks on knot-graphs, and then discuss a *walk-group* which arises naturally out of these studies. The automorphism group of the adjacency matrix is also introduced.

4.1 WALKS ON KNOT-GRAPHS

The principal tool for studying walks on graphs is the adjacency matrix; the following well-known theorem gives a general formula for the number of walks from vertex i to vertex j of a given length n, in terms of that matrix.

4.1.1 Theorem 1 (e.g in *DEO*,*p*. 222)

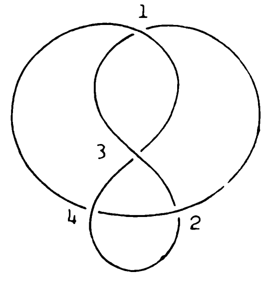
If A is the adjacency matrix of a connected graph G, then the ij-th element of A^n is the number of walks which have initial vertex i, final vertex j, and length n.

Proof: by induction on n . //

We can apply this theorem directly to the associated graph of a knot-graph, with K (see 2.1) being the adjacency matrix. When we pass to the knot-graphs themselves, however, we shall want to work in terms of *both* adjacency matrices J_α and J_β (see 2.2), and so we shall introduce a generalization of theorem 1.

First let us apply the theorem to an example, viz. the associated graph of the
knot 4_1.

Knot 4_1



Adjacency matrix

$$K = \begin{bmatrix} 0 & 1 & 2 & 1 \\ 1 & 0 & 1 & 2 \\ 2 & 1 & 0 & 1 \\ 1 & 2 & 1 & 0 \end{bmatrix}$$

Powers of the adjacency matrix

$$K^2 = \begin{bmatrix} 6 & 4 & 2 & 4 \\ 4 & 6 & 4 & 2 \\ 2 & 4 & 6 & 4 \\ 4 & 2 & 4 & 6 \end{bmatrix} \qquad K^3 = \begin{bmatrix} 12 & 16 & 20 & 16 \\ 16 & 12 & 16 & 20 \\ 20 & 16 & 12 & 16 \\ 12 & 20 & 16 & 12 \end{bmatrix} \qquad K^4 = \begin{bmatrix} 72 & 64 & 56 & 64 \\ 64 & 72 & 64 & 56 \\ 56 & 64 & 72 & 64 \\ 64 & 56 & 64 & 72 \end{bmatrix}$$

Discussion

This is not a good example from which to divine generalities, since knot 4_1 has a number of special properties not possessed by most other knots. We may note immediately that K is not only symmetric (as are all unoriented graph matrices) but also circulant, with top row having pattern (a,b,c,b). Very few underlying graphs of knots have the latter property. It is not difficult to see that all powers of K must have the same property and top row pattern. Indeed, the elements of the top row (a_r,b_r,c_r,b_r) of K^r may be quickly calculated for $r=2,3,\ldots$, using recurrence equations

$$\left.\begin{aligned} a_r &= 2(b_{r-1} + c_{r-1}) \\ b_r &= a_{r-1} + 2b_{r-1} + c_{r-1} \\ c_r &= 2(a_{r-1} + b_{r-1}) \end{aligned}\right\} \quad \text{with } a_1=0,\ b_1=1,\ c_1=2$$

It is not our purpose, however, to discuss particular methods for computing powers of K for special cases of knot-graph, but to interpret Theorem 1 in an example. To that end we make the following remarks.

Remarks

1. The *total* number of walks of length r in the graph is $||K^r||$ (where $||*||$ means the sum of all elements in the matrix $*$). In the example, these totals are 16,64,256,1024 for r = 1,2,3,4 respectively.

Proposition:

If K is the adjacency matrix of any n-vertex underlying graph, the total number of walks of length r in the graph is $4^r n$

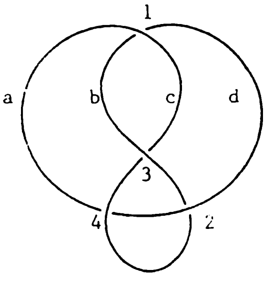
Proof:

Let i be any vertex. Then since the graph is 4-regular, there are 4 walks of length 1 which *start* at i: suppose they end at vertices i_1, i_2, i_3, i_4 (pairs of which may be coincident). Since each of vertices i_1, i_2, i_3, i_4 is adjacent to 4 edges, any walk of length 1 which began at i may be extended to a walk of length 2 in 4 ways: hence the total number of walks of length 2, which begin at i, is 4^2. By induction on r, we prove that there are 4^r walks of length r which begin at i. And this is true whichever of the n vertices we select for i. Hence the proposition is true. (N.B. Another proof is given in 4.1.3.)

2. We note that K^r is symmetric, since K is (thus$(K^r)' = (K')^r = K^r$). Therefore column sums are equal to row sums (all equal to 4^r): so 4^r different walks of length r *terminate* at each vertex.

3. The elements of the leading diagonal indicate the number of *closed* walks, or *circuits* which have length r and start and finish at a given vertex. For example, from element (1,1) of K^2 we deduce that there are six walks of length 2 which begin at vertex 1 and end at vertex 1. The six walks are (using edge labels as given in the diagram below):

$a4a,\ b3b,\ c3c,\ d2d,\ b3c,\ c3b.$



We note that the diagonal elements are all the same for any given power of K. The conjecture that this must always be the case is, however, false. A counter-example comes from knot 5_2 which has four 6s and an 8 on the leading diagonal of K^2.

4.1.2 Use of Matrices J_α, J_β

Since $K = J_\alpha + J_\beta$ for any knot-graph, we can take $K^r = (J_\alpha + J_\beta)^r$ and interpret terms of the expansion in terms of walks composed of sequences of α-steps and β-steps on the graph.

Thus for $r = 2$,

$$K^2 = J_\alpha^2 + J_\alpha J_\beta + J_\beta J_\alpha + J_\beta^2$$

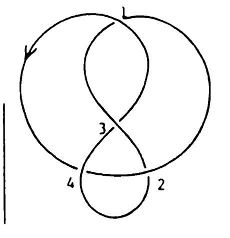
Interpretations of terms are as follows:

$(J_\alpha^2)_{ij}$	=	number of walks consisting of an α-step followed by a second α-step,starting at crossing i and finishing at crossing j
$(J_\alpha J_\beta)_{ij}$	=	number of walks of type [(α-step)(β-step)] from crossing i to crossing j
$(J_\beta J_\alpha)_{ij}$	=	number of walks of type [(β-step)(α-step)] from crossing i to crossing j
$(J_\beta^2)_{ij}$	=	number of walks of type [(β-step)(β-step)] from crossing i to crossing j

Note that whenever a step is taken, it may be on either the under-pass or the over-pass from that particular crossing. In other words, 'switching from string to string at a crossing' is allowed.

Similarly, higher powers of K will be sums of terms which are strings, or words, consisting of products of powers of J_α and J_β. Elements of these terms count the numbers of the walks in the graph, between two given vertices, which have the type which corresponds to the string. For example, $(J_\alpha J_\beta^2 J_\alpha^3 \ldots J_\beta\, J_\alpha^2)_{ij}$ is the number of walks of type $[\alpha\beta\beta\alpha\alpha\alpha \cdots \beta\alpha\alpha]$ from crossing i to crossing j.

Example (Knot 4_1 again)



$$K^2 \quad = \quad J_\alpha J_\alpha \quad + \quad J_\alpha J_\beta \quad + \quad J_\beta J_\alpha \quad + \quad J_\beta J_\beta$$

$$\begin{vmatrix} 6&4&2&4 \\ 4&6&4&2 \\ 2&4&6&4 \\ 4&2&4&6 \end{vmatrix} = \begin{vmatrix} 1&2&1&0 \\ 0&1&2&1 \\ 1&0&1&2 \\ 2&1&0&1 \end{vmatrix} + \begin{vmatrix} 2&1&0&1 \\ 1&2&1&0 \\ 0&1&2&1 \\ 1&0&1&2 \end{vmatrix} + \begin{vmatrix} 2&1&0&1 \\ 1&2&1&0 \\ 0&1&2&1 \\ 1&0&1&2 \end{vmatrix} + \begin{vmatrix} 1&0&1&2 \\ 2&1&0&1 \\ 1&2&1&0 \\ 0&1&2&1 \end{vmatrix}$$

Remarks

1. The totals $||*||$ are equal, for each of the four matrices on the right. Thus whatever type of walks of length 2 we consider, there are 16 of that type in the knot-graph. In general, for any knot-graph (alternating or nonalternating) the total number of walks of a particular type, and length r, is $4^r n / 2^r = 2^r n$.
2. $(J_\beta J_\beta) = (J_\alpha J_\alpha)'$. This is true for any alternating graph , for then $J_\beta = J_\alpha'$.
3. $\mathrm{tr}(J_\alpha J_\beta) = \mathrm{tr}(J_\beta J_\alpha)$. This is true for any pair of matrices. More generally, for matrices $A_1, A_2, \quad , A_n$, we have

   $$\mathrm{tr}(A_1 A_2 \quad A_n) = \mathrm{tr}(\text{any circular permutation of } A_1 A_2 \cdots A_n).$$

   Thus, in terms of walks, the total number of closed walks of a particular type is equal to the total number of closed walks of any type which is a circular permutation of the steps of the particular one.

4.1.3 Use of the permutation matrices, U and V

Alternating knot-graphs

We have seen that when a knot-graph is alternating, $J_\alpha = J'_\beta (\equiv J)$, and $J = U + V$, with $V = PU$, where U,V are permutation matrices, and P is (for 1-links) the permutation $(123 \cdots n)$ in matrix form.

Then terms in K^r may be expressed as products of powers of U, V, U', V'. For example, $K^2 = (J + J')^2 = (U + V + U' + V')^2$ has sixteen terms, four from each of the product terms JJ, JJ', $J'J$, $J'J'$.

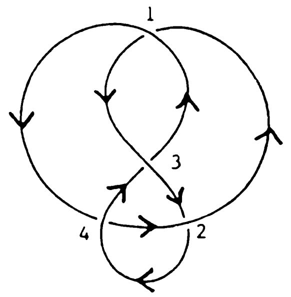
Let us take two of these product terms, and analyse the types of walk which they count and represent. Thus:

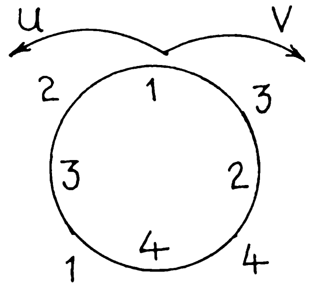
$$JJ = (U + V)(U + V) = UU + UV + VU + VV$$

Each term represents a two step $[\alpha\alpha]$ walk. Referring to the circular-word diagram of the 4_1 knot-graph below, we can distinguish the four kinds of $[\alpha\alpha]$ walk.

The two permutations are

$$U = \begin{pmatrix} 1&2&3&4 \\ 2&3&4&1 \end{pmatrix} \quad \text{and} \quad V = \begin{pmatrix} 1&2&3&4 \\ 3&4&1&2 \end{pmatrix}$$





We have 'sensed' the knot-graph, placing arrows on edges so that consecutively they point '*along the string*'. Notice that always an α-step chosen from U is *against* the string orientation; whereas one chosen from V is with the string orientation. So the four types of $[\alpha\alpha]$ walk are represented by the JJ terms as shown in the following table:

		Second step orientation	
First step orientation	$[\alpha\alpha]$	along	against
	along	VV	VU
	against	UV	UU

In all cases, before making the second step there has to be a switch from an overpass to the corresponding underpass. So that in none of the four cases does the walk proceed entirely along the same portion of the string.

The second example we shall take is JJ':

$$JJ' = (U + V)(U' + V') = UU' + UV' + VU' + VV';$$

all terms on the right hand side represent (count) walks of type $[\alpha\beta]$, and they may be categorized thus:

		Second step orientation	
First step orientation	$[\alpha\beta]$	along	against
	along	VU'	VV'
	against	UU'	UV'

In all these types, both steps take place on the same portion of the string. In UU' and VV' the steps are actually on the same edge, the first step being along the edge and the second in reverse direction along the same edge. It is with VU' and UV' that the two steps cover an entire over-pass (i.e. two consecutive edges, 'passing over' a crossing point). We have already exploited this idea, when we derived an algorithm for touring the graph 'along the string'.

The truth of proposition 4.1.2 (Remark 1) is immediately deducible from :

$$||K^r|| = ||(J+J')^r|| = ||\Sigma A_1 A_2 \cdots A_r||$$

where $A_i \varepsilon \{J, J'\}$, and the summation extends over 2^r terms. Each term $A_1 A_2 \cdots A_r$, when multiplied out in terms of U, V, U', V' has 2^r terms, each of which is a product of permutation matrices and therefore reduces to a single permutation matrix P_r say. Now $||P_r|| = n$, therefore $||K^r|| = 2^r . 2^r . n = 4^r . n$ as before.

It is now clear that certain structural properties of a knot-graph will be reflected in properties of products of the permutation matrices U and V and their inverses. For a given knot-graph, any possible type of walk on it may be represented by an element of T, the set of all words generated by the symbols

U, V, U^{-1}, V^{-1}. For example, the word $U^2V^{-1}VUV^3U^{-1}U^2$ represents a walk of type $[\overleftarrow{\alpha}\overleftarrow{\alpha}\overleftarrow{\beta}\overrightarrow{\alpha}\overleftarrow{\alpha}\overrightarrow{\alpha}\overrightarrow{\alpha}\overrightarrow{\alpha}\overrightarrow{\beta}\overleftarrow{\alpha}\overleftarrow{\alpha}]$, the arrows indicating whether a step is against or along a given consecutive string orientation (c.f the 4_1 example above).

Strictly, a string of U, V, U^{-1}, V^{-1} powers represents not one walk but a set of n walks, all taking place simultaneously around the knot-graph. All these walks, however, are of the same type: in future we shall say that an element of T 'represents a walk-type'.

If we examine the example word given above, we see that at two points a reduction of the element (i.e. of the *word*) can be made, if we set $V^{-1}V = I$ and $U^{-1}U = I$: that is, if we treat words as products of permutations, and use the identity matrix I as we normally use the emptyword. This is a sensible thing to do, for it would appear to lead to no loss of information about the structure of the knot-graph; because if $V^{-1}V$ occurs in one word, it means that all n walks that are simultaneously occurring have taken one β-step along an edge (with the V^{-1}) and then an α-step back *along the same edge* (with the V), and so $V^{-1}V$ essentially contributes nothing to the walk-type.

Further reductions in words of T will occur when the syllables of a subword multiply to become I. For example, let us suppose that in the word described above U and V are permutations such that $UV^2=I$. Then the word reduces as follows:

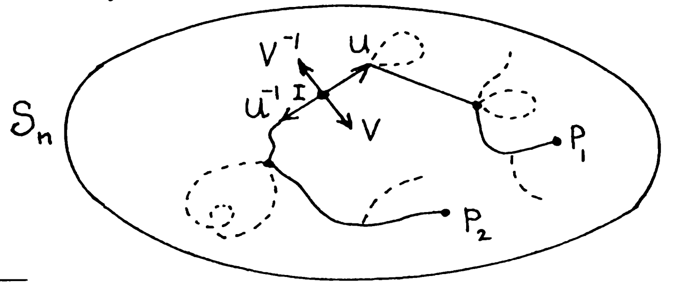
$$U^2(V^{-1}V)UV^3(U^{-1}U)U \rightarrow U^3V^3U$$
$$= U^2(UV^2)VU \rightarrow U^2VU$$

Definition:

Any word from T, from which all subwords which multiply to I by syllable multiplication have been removed, is called an *elementary*† *walk-type*. We shall call the set of all elementary walk-types (with I) the *walk semi-group*, $E(T)$.

It is evident that the operation of multiplication of two elements of $E(T)$, which consists of juxtaposition followed by any possible reduction by subwords multiplying to I, is associative; and I is both a left and a right identity. Hence $E(T)$ is a *semi-group*.

We can picture the reduction process and the semi-group by considering trajectories from I in space S_n of all permutations of 1,2,...n. We give two examples of walk-types; if the dotted portions are removed they become elementary walk-types. One of the walk-types begins with U and proceeds to point P_1: the other begins with U^{-1} and proceeds to P_2. (The final position is the permutation obtained when all syllables in the word are multiplied out as permutations.)



† In graph-theory, an *elementary walk* is one that never visits a vertex, nor traverses an arc, more than once.

We see at once that the semi-group $E(T)$ has finite order (since $||S_n|| = n!$; and since there is no back-tracking or looping occurring in a reduced walk, there must be a finite number of trajectories which can begin at I and end at some given permutation point P_i).

However $E(T)$ is not a group; for in general each of its members have several inverses. For example, as we shall see later (in 4.2), in any alternating knot-graph with standard labelling on n crossings the walk-type UV^{-1} has at least two inverses, viz. VU^{-1} and $(UV^{-1})^{n-1}$, since $UV^{-1} = P^{-1}$ and $P^n = I$.

We obtain a group from $E(T)/(\sim)$, the set of equivalence classes under the relation $\sim$, where $E_i \sim E_j$ if both E_i and E_j are elementary walk-types which have the same end-point. More precisely:

Let $E(T)/(\sim)$ have $m+1$ equivalence classes; denote these by $[E_i]$, and their set by $\tilde{E} = \{[I],[E_1],...,[E_m]\}$. Thus $E_i \varepsilon E(T)$ is an elementary walk-type, representing the equivalence class $[E_i]$. Define multiplication of elements of $\tilde{E}$ by $[E_i][E_j] = [E_iE_j]$, where E_iE_j means juxtaposition followed by reduction as described for the semi-group $E(T)$. Let the inverse of $[E_i]$ be $[E_i]^{-1} \equiv [E_i^{-1}]$, where E_i^{-1} is the elementary walk-type obtained from E_i by reversing the order of its syllables and changing the sign of the exponent of each syllable.

Then $\tilde{E}$, with this multiplication and inverses, is a group, which we shall call the *elementary walk-group* of the knot-graph. (Note that it is not a group of elementary walks on the graph itself, but a group of elementary walk-types viewed as trajectories from I in the S_n space.) Later, in 4.2, we give examples of these groups and discuss their properties in relation to knot-graphs.

Proposition

The elementary walk-group of a knot-graph is isomorphic to the group generated by U, V with ordinary matrix multiplication.

Proof :

There is a natural map from $\tilde{E}$ to the group of permutation matrices $W = \langle U,V \rangle$ namely $\Theta : \tilde{E} \to W$ which maps $[E_i] \to P_i$ where $P_i \varepsilon W$ is the end-point (permutation) common to all the elementary walk-types in the equivalence class $[E_i]$ (recall that an end-point is equal to the product of some sequence of powers of U, V, U^{-1}, V^{-1} and hence is a member of W). The mapping is a homomorphism, since

$$\Theta([E_i][E_j]) = \Theta([E_iE_j]) = P_iP_j = \Theta([E_i])\Theta([E_j]);$$

and it is one-to-one, since any member of $\tilde{E}$ maps to just one element of W, and vice-versa, by definition of $\sim$.

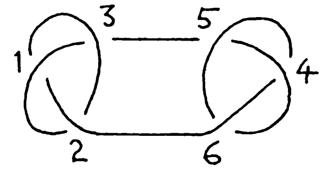
4.1.4 NonAlternating Knot-graphs

When one considers walk-types on nonalternating knot-graphs, one quickly finds that the situation is much more complex, since the adjacency matrices are no longer bipermutations. One can, for example, attempt a development parallel to that given above for alternating knot-graphs; one is then led to a concept which we have called *clumped permutation*, and an elementary walk groupoid which is generated by products of clumped and ordinary permutations. We have not carried our studies of these groupoids very far, so we will merely introduce the concepts involved by means of an example, and leave it at that.

The smallest nonalternating † knot is the square knot, which is the composition of two forms of the trefoil. It will serve for our example; below is a diagram of it, followed by the α- and β-adjacency matrices.

$$K = J_\alpha + J_\beta$$

$$J_\alpha = \begin{bmatrix} & \textcircled{1} & 1 & & & \\ 1 & & \textcircled{1} & & & 1 \\ \textcircled{1} & & & & & \\ & & & & 1 & \textcircled{1} \\ & & & \textcircled{1} & & \\ & 1 & & 1 & \textcircled{1} & \end{bmatrix} \qquad J_\beta = \begin{bmatrix} & 1 & \textcircled{1} & & & \\ \textcircled{1} & & & & & \\ 1 & \textcircled{1} & & & 1 & \\ & & & & \textcircled{1} & 1 \\ & & 1 & 1 & & \textcircled{1} \\ & & & \textcircled{1} & & \end{bmatrix}$$



As with alternating knot-graph matrices, we can split these matrices in two; but instead of sums of two permutations U, V, uniquely determined, we obtain the sum of a permutation P (say) and a 'clumped permutation' Q (say). We indicate the P permutations in both J_α and J_β by circling elements: note that we begin construction of a P permutation by encircling those 1's which stand alone in their rows.

The sums are as follows, using ordinary permutation notation for convenience:

Ordinary **Clumped**

$$J_\alpha = \overset{P_1}{\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 2 & 3 & 1 & 6 & 4 & 5 \end{pmatrix}} + \overset{Q_1}{\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 3 & (16) & 0 & 5 & 0 & (24) \end{pmatrix}}$$

$$J_\beta = \overset{P_2}{\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 3 & 1 & 2 & 5 & 6 & 4 \end{pmatrix}} + \overset{Q_2}{\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 2 & 0 & (15) & 6 & (34) & 0 \end{pmatrix}}$$

We see immediately that the splittings are not unique. For example, J_α splits also thus:

$$J_\alpha = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 3 & 6 & 1 & 5 & 4 & 2 \end{pmatrix} + \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 2 & (13) & 0 & 6 & 0 & (45) \end{pmatrix}$$

They are never unique in the case of composite knots. Nor is it true that the splitting is unique in prime nonalternating knots: indeed, the very first prime nonalternating knot in Alexander and Brigg's knot-tables, viz 8_{19}, provides an example of this non-uniqueness.

Many interesting remarks could be made about the various splittings obtainable from a given knot-graph, but we shall not pursue that topic here.

Continuing our study of walk-types, we see now that counting walk-types of length r involves the following powers:

$$K^r = (J_\alpha + J_\beta)^r = (P_1 + Q_1 + P_2 + Q_2)^r.$$

† The square knot has an alternating form; that fact does not concern us here.

Expansion of the right hand side of the expression gives a sum of terms which are products of powers of P_1, Q_1, P_2, Q_2. The question now arises whether we can carry out a procedure with these walk-types analogous to the one which we used above for the reduction of powers of permutations U and V. Can we sort walk-types into equivalence classes, and thereby obtain a finite algebraic system under some kind of multiplication of the classes? The answer is yes, but the system obtained is not a group. We shall call it the ***elementary walk-groupoid***.

First we define 'clumped permutations' and show how they may be multiplied. We shall continue to use a common permutation notation; but it must be remembered that we could represent any clumped permutation by a [0,1]-matrix, and that the multiplication operation given below is equivalent to ordinary matrix multiplication.

Definitions

(i) A *clumped permutation* is a mapping Θ of $X = 1,2,3,\ldots,n$ into the ***clump set***

$C = \{0,1, \quad ,n,(12),(13), \cdots ,(56),(123), \cdots ,(456), \cdots ,(123 \cdots n)\};$

$\Theta : X \to C$

with the properties that

$$\Theta(x_i) = x'_i,\ \Theta(x_j) = x'_j \Rightarrow x'_i \neq x'_j$$

unless $i = j$ or both equal 0;

$$\text{and } \bigcup_{i=1}^{n} x'_i - \{0\} = X$$

(here $x'_i \in C$ is regarded as the set of digits defining it).

(ii) Consider the set of n images (of $\Theta(X)$). We say Θ is of *clumping type* $0^{\varepsilon_0}1^{\varepsilon_1}2^{\varepsilon_2}3^{\varepsilon_3} \cdots n^{\varepsilon_n}$ if the set of images comprises

ε_0	0's,
ε_1	single digits (of X),
ε_2	pairs $(x_i x_j)$,
ε_3	triples $(x_i x_j x_k)$, etc.

(Note that $\sum_{i=0}^{n} \varepsilon_n = n = |\Theta(X)|$. We may omit any type syllable having $\varepsilon_i = 0$.)

Example ($n = 6$)

Clumped Permutation	**Clumping type**
$\Theta_1 = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 0 & 0 & 0 & (123) & 0 & (456) \end{pmatrix}$	$0^4 1^0 2^0 3^2 4^0 5^0 6^0 \sim 0^4 3^2$
$\Theta_2 = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 2 & (45) & 0 & 6 & 0 & (13) \end{pmatrix}$	$0^2 1^2 2^2$

(iii) Multiplication of Θ_i and Θ_j is defined as for matrix multiplication of the two [0,1]-matrices which are equivalent to Θ_i, Θ_j respectively.

Example $(n=6)$

We may work directly with the clumped permutations, thus (with Θ_1, Θ_2 as above):

$$\Theta_1\Theta_2 = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 0 & 0 & 0 & (123) & 0 & (456) \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 2 & (45) & 0 & 6 & 0 & (13) \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 0 & 0 & 0 & (245) & 0 & (136) \end{bmatrix}$$

Thus, for example, we find $\Theta_1\Theta_2(4)$ as follows:

$$\Theta_1(4) = (123)\ ,\ \text{then}\ \Theta_2((123)) = \bigcup\{\Theta_2(1),\Theta_2(2),\Theta_2(3)\} - \{0\}$$

(digits placed in (,) brackets)

$$= (245)$$

It should be clear from this example how we can multiply any two clumped permutations together; and that the result will be a clumped permutation. It may be noted that an ordinary permutation is a clumped permutation of clumping type 1^n.

Let C_n be the set of all clumped permutations of $X=1,2,\ldots n$. Let C_n^r, $r=0,1,\ldots,n-1$, be the subclass of C_n whose members have $\varepsilon_0=r$. Then $C_n^0 = S_n$, the symmetry group: and C_n, together with its multiplication table, is a groupoid. The order of C_n is $|C_n| = n^n$, a result we prove in the appendix to this chapter, where also we give the complete groupoids for $n=2$ and 3.

Now we see that for nonalternating knot-graphs on n-crossings, walk-types can be regarded as trajectories in the finite space C_n of clumped permutations; and they are generated by products of powers of P_1,Q_1,P_2,Q_2, where $K = P_1+Q_1+P_2+Q_2$. They can be reduced, so that no back-tracking or looping occurs; and equivalence classes of reduced walk-types are obtained under the relation 'equal end-point clump permutation'.

The set of the end-points of the equivalence classes, with the multiplication as given above for clumped permutations, forms an ***elementary walk-groupoid*** which is a subgroupoid of C_n. An elementary walk-groupoid for the square knot has 58 elements; we give a list of these in the appendix. We note that they are generated by $\langle P_1,P_2,Q_1\rangle$ with Q_2 being a consequence. We conjecture that Q_2 is always a consequence of $\langle P_1,P_2,Q_1\rangle$ in nonalternating knot groupoids. We conjecture also that if P'_1,P'_2,Q'_1,Q'_2 is another splitting of K into clumped permutations, then the two elementary walk-groupoids $\langle P_1,P_2,Q_1\rangle$ and $\langle P'_1,P'_2,Q'_1\rangle$ are isomorphic. In the appendix we give an example of a second splitting for the square knot.

We have not studied these groupoids further. Now we return to the groups of alternating knot-graphs, and study some of their properties.

4.2 GROUPS ON KNOT-GRAPHS

In this Section we continue the study of the elementary walk-groups, which were defined in 4.1.3 for alternating knot-graphs and shown to be isomorphic to the groups generated by the pairs (U, V) of permutation matrices derived from the α-adjacency matrices of the graphs.

4.2.1 Independence of Labelling

Let an alternating knot-graph have n crossings labelled $1,2,\ldots,n$, α-adjacency matrix $J = U + V$, and elementary walk-group $G<U,V>$. Suppose the knot-graph is re-labelled, the new labelling being a permutation Q of the old. We shall show that the new elementary walk-group $G'<U',V'>$ is isomorphic to the old one, thus:

The new adjacency matrix is $J' = U' + V'$ (here a prime does not signify transposition),

$$\text{with } J' = Q^{-1}JQ = Q^{-1}UQ + Q^{-1}VQ$$

So we can take $U' = Q^{-1}UQ$ and $V' = Q^{-1}VQ$, and using these obtain a natural mapping Θ from G to G'. Thus if $A = U^{i_1}V^{j_1}U^{i_2}V^{j_2}\cdot$ is an element of G, $\Theta(A)$ is to be the element A' in G' obtained by replacing U, V in A by $Q^{-1}UQ$, $Q^{-1}VQ$ respectively wherever they occur. Then, since $(Q^{-1}UQ)^i = Q^{-1}U^iQ$, we find that

$$\begin{aligned} A' &= (Q^{-1}U^{i_1}Q)(Q^{-1}V^{j_1}Q) \\ &= Q^{-1}U^{i_1}V^{j_1}\cdots Q \\ &= Q^{-1}AQ \end{aligned}$$

Clearly Θ is a one-one mapping. And if $A, B \in G$, we have

$$\begin{aligned} \Theta(AB) &= Q^{-1}(AB)Q \\ &= (Q^{-1}AQ)(Q^{-1}BQ) \\ &= \Theta(A)\Theta(B) \end{aligned}$$

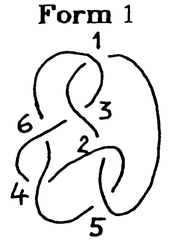
So Θ is an isomorphism: and therefore elementary walk-groups are independent (up to isomorphism) of labelling.

4.2.2 Non-invariance of elementary walk-groups.

In general, a knot has several knot-groups, and the elementary walk-groups for these are not isomorphic. It is conjectured, however, that all the elementary walk-groups arising from a given knot are subgroups of one another. An example of this is the following. It treats two forms of the 2-link 6^2_3.

Knot 6^2_3

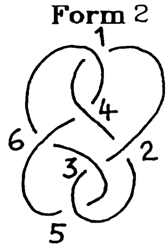
Form 1



$U_1 = (1\ 3)(2\ 5)(4\ 6)$

$V_1 = (1\ 5\ 4\ 2\ 3\ 6)$

Form 2



$U_2 = (1\ 2\ 4)(3\ 5\ 6)$

$V_2 = (1\ 4\ 6)(2\ 5\ 3)$

The elementary walk-groups are given below. Form 1 has 48 elements and form 2 has 12 elements.

$G_1 < U_1, V_1 >$ elements

(1)(2)(3)(4)(5)(6)	(1523)(46)	(156234)	(13)(25)(4)(6)
(13)(25)(46)	(14)(26)(35)	(1)(2)(35)(46)	(12)(36)(45)
(154236)	(1624)(3)(5)	(145263)	(14)(26)(3)(5)
(143)(265)	(1)(2)(36)(45)	(136254)	(15)(23)(46)
(12)(35)(46)	(1523)(4)(6)	(12)(3)(4)(5)(6)	(16)(24)(35)
*(134)(256)	(1426)(35)	(165)(243)	(156)(234)
(163245)	(12)(34)(56)	(134256)	(165243)
(1624)(35)	(1325)(46)	(12)(3)(46)(5)	(12)(3456)
(1)(2)(3456)	(143265)	(154)(236)	(1325)(4)(6)
(15)(23)(4)(6)	(136)(254)	(1)(2)(3)(46)(5)	(16)(24)(3)(5)
(1426)(3)(5)	(12)(35)(4)(6)	(1)(2)(34)(56)	(1)(2)(3654)
(12)(3654)	*(163)(245)	(16)(24)(35)	(145)(263)

The two asterisked elements may be compared with U_2 and V_2. If $\alpha = (1\ 3\ 4)(2\ 5\ 6)$ and $\beta = (1\ 6\ 3)(2\ 4\ 5)$ are both relabelled by permutation (1)(2 5 3 4 6), they become

$$\alpha' = (1\ 4\ 6)(5\ 3\ 2) = V_2 \quad \text{and} \quad \beta' = (1\ 2\ 4)(3\ 5\ 6) = U_2$$

Hence

$$G_2 < U_2, V_2 > \approx G < \alpha, \beta > \subseteq G_1 < U_1, V_1 >.$$

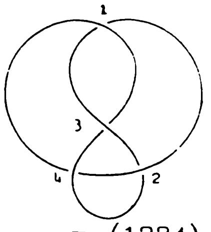
$G_2 < U_2, V_2 >$ elements and group table

Element	×	I	α	β	γ	δ	A	B	C	D	E	F	G
(1)(2)(3)(4)(5)(6)	I	I	α	β	γ	δ	A	B	C	D	E	F	G
U = (124)(356)	α	α	β	I	B	D	δ	F	E	A	G	γ	C
U^2 = (142365)	β	β	I	α	F	A	D	γ	G	δ	C	B	E
V = (146)(253)	γ	γ	A	C	δ	I	E	β	B	F	α	G	D
V^2 = (164)(235)	δ	δ	E	B	I	γ	α	C	β	G	A	D	F
(1)(26)(34)(5)	A	A	C	γ	β	F	I	G	α	E	D	δ	B
(15)(26)(3)(4)	B	B	δ	E	D	α	G	I	F	γ	β	C	A
(123)(456)	C	C	γ	A	G	E	F	δ	D	I	B	β	α
(132)(465)	D	D	G	F	α	B	β	E	I	C	δ	A	γ
(136)(254)	E	E	B	δ	C	G	γ	D	A	α	F	I	β
(163)(245)	F	F	D	G	A	β	C	α	γ	B	I	E	δ
(15)(2)(34)(6)	G	G	F	D	E	C	B	A	δ	β	γ	α	I

The above walk-groups were calculated 'by-hand'; clearly the study of such groups will be much facilitated if a computer is programmed to produce them. In the next section we give the elementary walk-groups computed from one knot-graph (form) of each of the first 32 prime alternating knots. It is of interest to note that among these 32 knots many have only one form. (*TAIT*, 1898), gives only one form for 18 of them. If the form is an invariant of the knot, then so is the walk-group obtained from it.

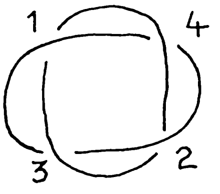
Before presenting these walk-groups, we give an example of two knots which have the same walk-group. The knots are 4_1 and 4_1^2, a prime knot and a prime 2-link respectively.

Knot 4_1



$U = (1234)$
$V = (13)(24)$
$G<U,V> = \{I, U, V, UV\}$
$= \{I, U, U^2, U^3\}$
$= C_4$ (*cyclic, order* 4)

Knot 4_1^2



$U = (1423)$
$V = (1324)$
$G<U,V> = \{I, U, V, U^2\}$
$= \{I, U, U^2, U^3\}$
$= C_4$

4.2.3 The first 32 prime alternating knots

The following information was obtained using the software package 'CAYLEY'; we are indebted to Dr C. Leedham-Green, Queen Mary's College, London University, for carrying out the work and advising us on the results.

The table gives the generators, the elementary walk-groups and their orders, with some information about the 'unusual' ones, for $n=3,\ldots,8$.

In the right-hand columns, for $n=3,\ldots,7$ are shown the automorphism groups with generators.

Notation: Elementary walk-group $G=<U,P>$ where $P=(123\cdots n)$.

Automorphism group $\mathrm{Aut}(K) = \{Q \mid Q^{-1}JQ=J,\ Q\in S_n,\ J=U+PU\}$

C_n = Cyclic group, order n

D_n = Dihedral group, order n (not $2n$)

A_n = Alternating group on n symbols

S_n = Permutation group on n symbols

Knot	Generator U	Walk Group	Order	Aut(k)	Generators
3_1	(123)	C_3	3	D_6	(13),(12)
4_1	(1234)	C_4	4	C_4	(1234)
5_1	(13524)	C_5	5	D_{10}	(12)(35),(12345)
5_2	(134)(25)	S_5	120	C_2	(13)(45)
6_1	(14)(26)(35)	$G(6_1)$	48	C_2	(14)(25)(36)
6_2	(136245)	S_6	720	I	identity
6_3	(142635)	S_5	120	I	identity
7_1	(1473625)	C_7	7	D_{14}	(17)(26)(35),(27)(36)(45)
7_2	(145)(27)(36)	A_7	2520	C_2	(14)(23)(57)
7_3	(14)(25736)	S_7	5040	C_2	(17)(26)(35)
7_4	(15)(247)(36)	A_7	2520	C_2	(16)(25)(34)
7_5	(13725)(46)	S_7	5040	I	identity
7_6	(1253746)	A_7	2520	I	identity
7_7	(164)(25)(37)	A_7	2520	I	identity
8_1	(15)(28)(37)(46)	$G(8_1)$	64		
8_2	(14837256)	S_8	40320		
8_3	(1357)(26)(48)	$G(8_3)$	32		
8_4	(1584)(27)(36)	S_8	40320		
8_5	(13825746)	$G(8_5)$	192		
8_6	(138256)(47)	S_8	40320		
8_7	(125)(36847)	S_8	40320		
8_8	(168375)(24)	S_8	40320		
8_9	(1625)(3847)	S_8	40320		
8_{10}	(13825)(467)	S_8	40320		
8_{11}	(158347)(26)	S_8	40320		
8_{12}	(13485267)	$G(8_{12})$	336		
8_{13}	(152846)(37)	S_8	40320		
8_{14}	(134826)(57)	S_8	40320		
8_{15}	(13857)(264)	S_8	40320		
8_{16}	(13857)(246)	S_8	40320		
8_{17}	(1385)(2746)	S_8	40320		
8_{18}	(1357)(2468)	C_8	8		

Further information is given below about walk-groups designated $G(*)$.

In all cases $G=\langle U,P\rangle$, **where** $P=(123\cdots n)$.

Additional Information

Of all the 32 elementary walk-groups, only six are not either C_n, A_n, S_n. These six are as follows:

$G(6_1)$ - order 48, group of isometries of the cube (including reflection) acting on the six faces.

$G(6_3)$ - S_5, acts transitively on 6 symbols

$G(8_1)$ - order 64, metabelian

$G(8_3)$ - order 32, metabelian

$G(8_5)$ - order 192, derived length 4, centre order 2, derived subgroup has index 2

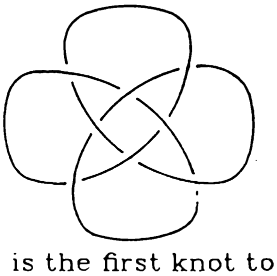
$G(8_{12})$ - order 336, is extension of the simple group PSL(2,7) (e.g. 2x2 matrices over 7 elements) by an outer automorphism of order 2.

Remarks

(i) The cyclic groups arise from knots 3_1, 4_1, 5_1, 7_1, 8_{18}. We define K_n to be a *simple knot-graph* if its walk-group is C_n. Clearly the torus knots T_n, n odd, form a subclass of $S(K_n)$, the class of all simple knot-graphs: the first three are 3_1, 5_1, 7_1, which have generators (U,P) satisfying $U=P$, $U^3=P$, $U^5=P$ respectively: in general, $U^{n-2}=P$, for T_n with n odd.

Knot 4_1 (Listing's figure-of-eight knot) also has $U = P$.

The remaining example from the table is 8_{18}:



This knot is remarkable in many ways. It is the first knot to have no 2-gons in its graph; the first 1-link basic polyhedron (see appendix III of chapter 6); and it displays beautiful symmetry. For this simple knot-graph, $U = P^2$.

Evidently simple knot-graphs are characterised by the property that $U^i = P$ or $U = P^j$ for some i,j (then a Tietze operation removes a generator from $G<U,V>$). A study of $S(K_n)$ and its sub-classes should prove worthwhile. Interesting questions are : Have all members of S a single knot-graph form (in which case their walk-group is an invariant)? Can construction methods be developed for defining and counting sub-classes of S? Some discussion on these classes occurs in 4.2.4, Example 2.

(ii) Knots having A_n as walk-group only occur in the $n=7$ set. Why? Is this a property restricted to knots of odd order, or of prime order only?

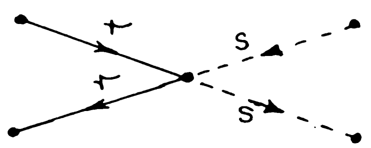
(iii) The number of knots in the list having a walk-group S_n was surprisingly high. It appears that most knots (i.e. a very high proportion for $n \geq 8$) will have sufficient asymmetry for the number of walk-types to be the maximum $n!$. This, of course, focusses great interest on the relatively few which do not have this property.

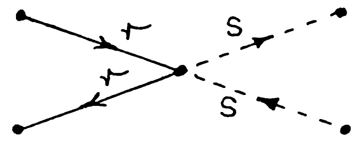
(iv) The automorphism groups (on the meagre evidence attained so far) seem to be either D_{2n} (for torii T_n, n odd), C_2, or I, with one exception, viz. 4_1 which has $U = P$ and $\mathrm{Aut}(4_1) = C_4$. A theoretical study of $\mathrm{Aut}(K)$ will probably yield precise results, making further computations either simple or unnecessary.

4.2.4 Cayley Colour Graph

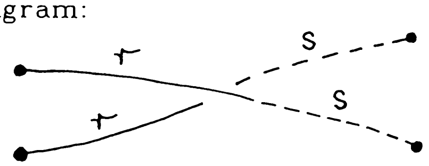
We have drawn Cayley colour graphs of certain walk-groups which are planar (definition: a *group is planar* if its Cayley colour graph is planar) and have noted a number of remarkable connections between such colour graphs and knot-graphs.

For example, if the knot-graph is not simple (see remark (i), 4.2.3), the walk-group has two generators, (r,s) say, (irreducible) and the Cayley graph is regular of degree 4. The coloured arcs which represent the generators will be adjacent to each vertex in one of the following manners:





Hence we can convert the Cayley graph into an alternating knot-graph if we disregard the r,s orientations and introduce over- and under- crossings in the way indicated by the diagram:



We proceed thus from vertex to vertex, touring the graph until the conversion is completed. Interesting questions arising now are: How does the α-orientation compare with that of the Cayley graph (is it bao, bno, neither)? What is the walk-group of this new knot? What is its Cayley graph like? Can a second knot-graph be generated from it, and so on?

To formalize this inquiry, we introduce a flow-chart with notation for the sequence. Let the initial knot-graph be K_0, and denote its walk-group Cayley graph by W_1; then the knot-graph formed from W_1 may be denoted K_1, and its walk-group Cayley graph by W_2; and so on. We then have the sequence:

Initial knot-graph K_0	$\rightarrow$	Cayley graph W_1	$\rightarrow$	Derived knot-graph K_1	$\rightarrow$	Cayley graph W_2	$\rightarrow$	(A)

The basic question is :

What kinds of sequence (or double-sequences $(\{K_i\},\{W_j\})$) can result?

We have one well-known result in group theory to guide us (*WHITE*, 1973, *p.* 70):

Theorem

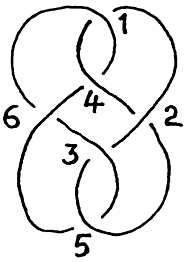
The finite group G is planar if and only if

$$G = G_1 \times G_2 \text{ where } G_1 = C_1 \text{ or } C_2, \text{ and } G_2 = C_n, D_n, S_4, A_4, \text{ or } A_5$$

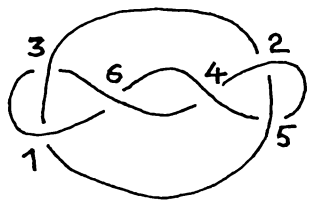
We will now give some examples of K_0, which have planar Cayley graphs of their walk-groups. Their (A) sequences are derived and discussed.

Example 1 : Knot 6_3^2

This knot has two forms, with knot graphs



I



II

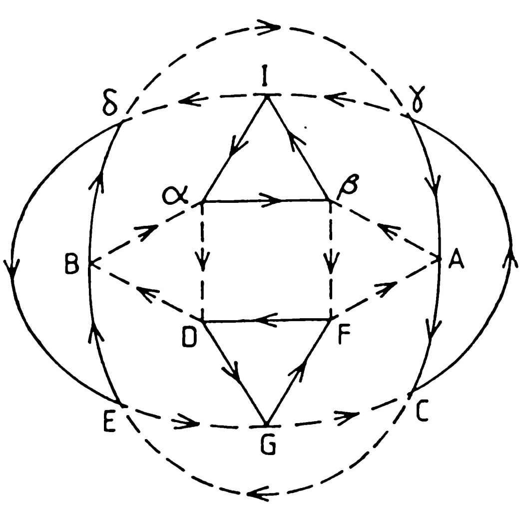
Group G_I is of order 12 whilst G_{II} is of order 48. Using knot-graph I as K_0, we find the following group table and Cayley graph.

Group table: $G_I < U_I, P >$, U_I = (1 2 4)(3 5 6), P = (1 2 3)(4 5 6)

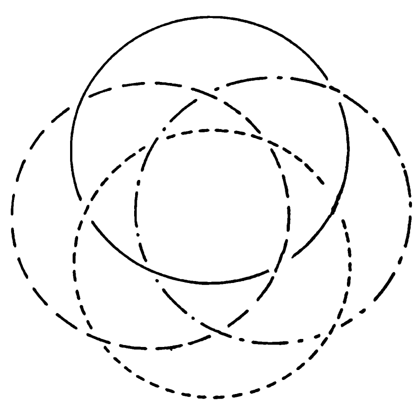
×	I	α	β	γ	δ	A	B	C	D	E	F	G	Elements
I	I	α	β	γ	δ	A	B	C	D	E	F	G	I
α	α	β	I	B	D	δ	F	E	A	G	γ	C	U
β	β	I	α	F	A	D	γ	G	δ	C	B	E	U^2
γ	γ	A	C	δ	I	E	β	B	F	α	G	D	$V = PU$
δ	δ	E	B	I	γ	α	C	β	G	A	D	F	$V^2 = V'$
A	A	C	γ	β	F	I	G	α	E	D	δ	B	VU
B	B	δ	E	D	α	G	I	F	γ	β	C	A	UV
C	C	γ	A	G	E	F	δ	D	I	B	β	α	$VU^2 = P$
D	D	G	F	α	B	β	E	I	C	δ	A	γ	U^2V^2
E	E	B	δ	C	G	γ	D	A	α	F	I	β	V^2U
F	F	D	G	A	β	C	α	γ	B	I	E	δ	U^2V
G	G	F	D	E	C	B	A	δ	β	γ	α	I	UV^2U

We have shown this group to be simply isomorphic with A_4, the alternating group of four symbols. Hence $G_I = C_1 \times A_4$ so its Cayley colour graph is planar. We show the Cayley graph W_1 on the left, and its derived knot-graph K_1 on the right. We drew W_1 using generators $< U_I = \alpha, \delta >$.

Cayley graph W_1 (of 6_3^2)



Derived Knot-graph K_1

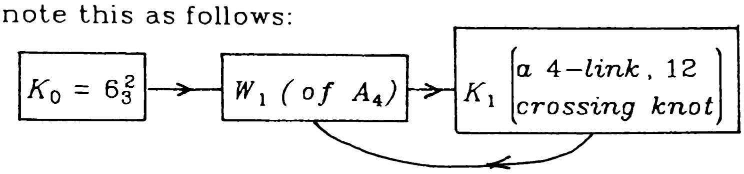


We note that K_1 is a 4-link; it has no 2-gons, eight 3-gons, four 4-gons. As it has no 2-gons, it is an example of Conway's basic polyhedra (Appendix III, ch.6).

We note also that if the orientation on W_1, though balanced, is transferred to K_1 it is neither alternating nor non-alternating.

We have shown that the elementary walk-group for K_1 is also isomorphic with the alternating group on four symbols, i.e. A_4. Hence the sequence (A) will now repeat itself indefinitely.

We may denote this as follows:



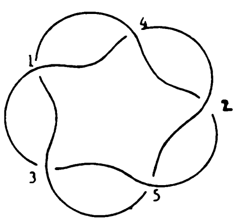
Example 2: Simple Knot-graphs (walk-groups C_n)

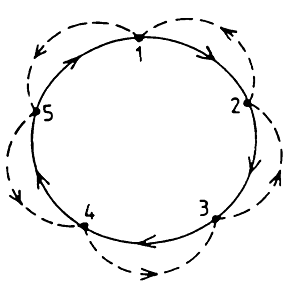
We have already remarked that these groups have only one generator. However, they derive from $G<U,P>$, where one generator becomes redundant owing to the existence of a relation $U^i = P$ or $U = P^j$. Hence *both* generators U and P have the same cyclic Cayley graph. Let us see if it makes sense to generalise Cayley graphs (in our context at least!) by saying that the Cayley graph of a simple knot-graph is to be *two* n-cycles fitted together. In the case $U^i = P$ we shall fit them by identifying points $1, \cdots, n$ directly; in the case $U = P^j$ we shall identify points of the cycles of U alternately inside and outside one of the n-cycles (which has been labelled $1, 2, \cdots, n$ clockwise).

Thus for 5_1 and 8_{18} respectively:

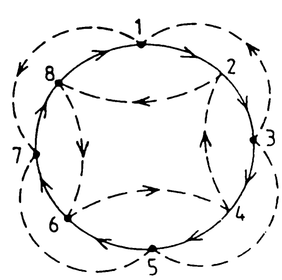
Knot 5_1: $U = (1\ 3\ 5\ 2\ 4)$
$P = (1\ 2\ 3\ 4\ 5)$
$U^3 = P$(also $U = P^2$)

Knot 8_{18} $U = (1\ 3\ 5\ 7)(2\ 4\ 6\ 8)$
$P = (1\ 2\ 3\ 4\ 5\ 6\ 7\ 8)$
$U = P^2$

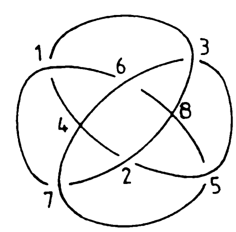




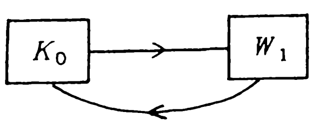
Cayley graph



Cayley graph



With this somewhat arbitrary way of fitting two Cayley graphs of one generator each together, we find that the derived knot-graphs are, in both cases, the same as the initial knot-graphs. Thus the (A) sequences are of type



, in these cases.

In the event that the order of the permutation U is n, the same as the order of P, we can fit the two Cayley cycles together in the manner of Cayley's original definition. We will use knot 5_1 again as an example, and it will be seen that further simple knot-graphs are obtainable from the result. We first list the powers of P and U, and then draw the two cycle graph as Cayley intended:

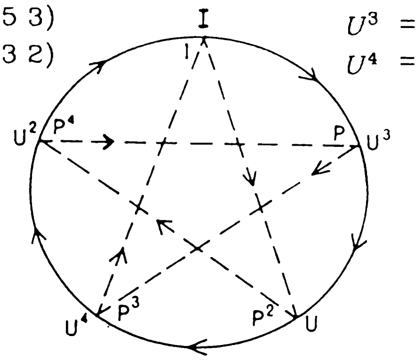
$P^5 = I = (1)(2)(3)(4)(5)$ $\qquad$ $U^5 = I$

$P = (1\ 2\ 3\ 4\ 5)$ $\qquad$ $U = (1\ 3\ 5\ 2\ 4) = P^2$

$P^2 = (1\ 3\ 5\ 2\ 4)$ $\qquad$ $U^2 = (1\ 5\ 4\ 3\ 2) = P^4$

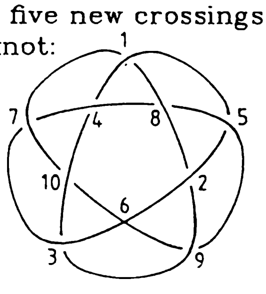
$P^3 = (1\ 4\ 2\ 5\ 3)$ $\qquad$ $U^3 = (1\ 2\ 3\ 4\ 5) = P$

$P^4 = (1\ 5\ 4\ 3\ 2)$ $\qquad$ $U^4 = (1\ 4\ 2\ 5\ 3) = P^3$



Cayley Graph

It is seen that the Cayley graph is non-planar, and so it cannot provide a new 5-knot. If, however we ignore this setback, and introduce five new crossings (where pairs of dotted lines cross) we obtain the following 10-knot:

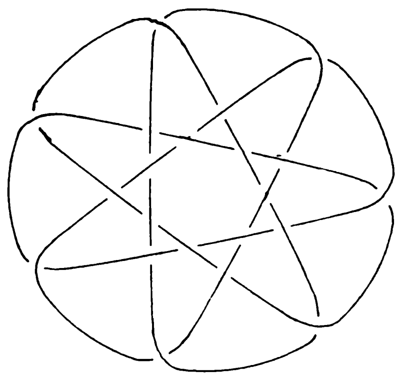


This is another basic polyhedron! It has no 2-gons. It is knot 10_{123} of A.&B. tables, and Conway's knot 10^*.

On investigation we find that for 10^*, $U = (1\ 7\ 3\ 9\ 5)(2\ 8\ 4\ 10\ 6)$ and $U = P^6$. (Though $U^i \neq P$ for any i). Hence we have discovered another simple knot-graph. And, moreover, we have paired in a unique (?) way the knots 5_1 (torus) and 10^* (basic polyhedron); the former has the maximum number of 2-gons possible (i.e. n) whilst the latter has the minimum number (i.e. zero).

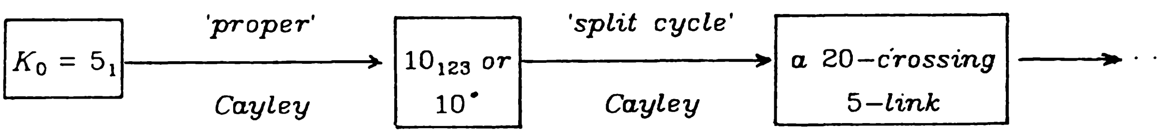
It is clear that this procedure with any torus knot-graph T_n, n odd, will produce a pair of simple knot-graphs; the basic polyhedron will be formed from an n-point start within a circle.

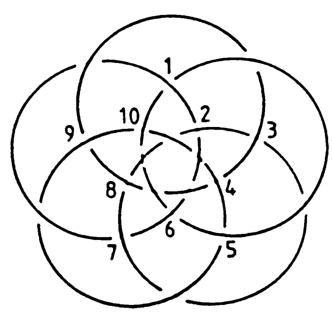
The example for $n = 7$ is



This is a $3n = 21$-crossing knot, with 14 3-gons, 7 4-gons, 2 7-gons. The relation between U and P is $U = P^3$ (also $U^5 = P$). Of course there is nothing original in the drawing of these beautifully symmetric knots. We believe, however, that their classification and construction, via walk-groups, and Cayley graphs, may be new.

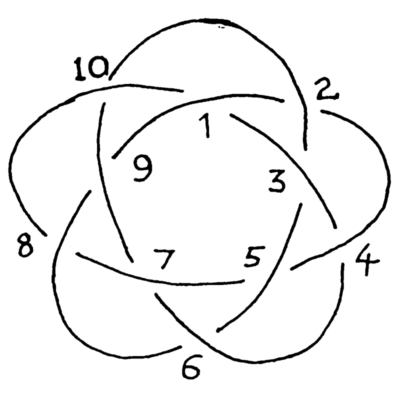
Returning to the construction we made of knot 10_{123} (from 5_1, previous page) we shall examine whether we can proceed from that to another knot via sequence (A). We found that U has two equal length cycles, so $U^5 = I$; if we use the method we used for 8_{18} to fit the two Cayley cycle graphs together we obtain yet another basic polyhedron, with 20 crossings. It is also a simple knot-graph. Hence the sequence has continued, entirely with simple graphs; and it seems likely that we can continue the sequence indefinitely in this manner. The sequence $\{K_i\}$ and knot diagram follow:





The production of 8_{18} using $U = P^2$ and the split cycle Cayley combination (2 pages back) suggests yet another method for producing simple knot graphs, all having $U = P^2$, this time for a sequence of $n = 8, 10, 12, \cdots$ crossing knots. We draw an $\frac{n}{2}$-gon inside the P-circle, and $\frac{n}{2}$ 3-gons outside it, like leaves.

The result for $n = 10$ is:



But this knot is again 10_{123}, which we produced earlier by a sequence beginning with 5_1!

Our final comments on simple knot-graphs are that the α-adjacency matrices in the two cases are:

Case 1	$U^i = P$	$J = U + PU$
		$= U + U^{i+1} = U(I + U^i)$
Case 2	$U = P^j$	$J = P^j + P^{j+1} = P^j(I + P)$

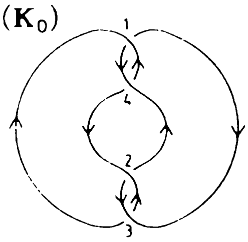
Example 3 : Knots with dihedral walk-groups (D_n, order n)

Dihedral groups are planar, and we may enquire whether any knots have dihedral walk-groups. The following are examples of such knots.

(i) **Chain Bracelets**

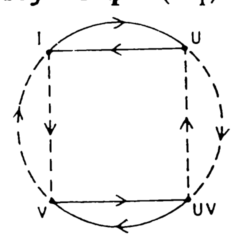
(a)

Knot 4_1^2 (K_0)



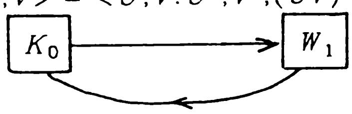
$U = (1\ 3)(2\ 4)$
$V = (1\ 4)(2\ 3)$
$U^2 = V^2 = I$
$(UV)^2 = I$

Cayley Graph (W_1)



Walk-group: $G<U,V> = <U,V: U^2, V^2, (UV)^2> \approx D_4$

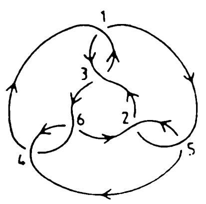
Sequence (A)



(K_1 is the same knot as K_0).

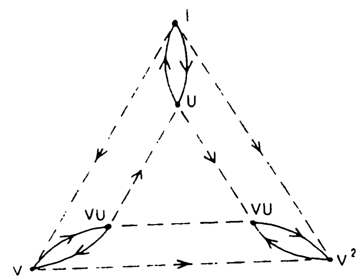
(b)

Knot 6_1^3



$U = (1\ 3)(2\ 5)(4\ 6)$
$V = (1\ 5\ 4)(2\ 3\ 6)$
$U^2 = I$
$V^3 = I$
$(UV)^2) = I$

Cayley Graph (W_1)



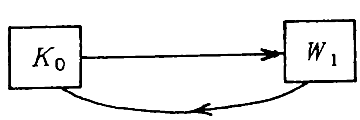
Walk-group: $G<U,V> \approx D_6$

Sequence (A)

(K_1 is the same knot as K_0).

(c) m-**link chain bracelet** ($2m$ crossings)

It is clear that the $(2m)_1^m$ knot has D_{2m} as walk-group, with a sequence (A) of



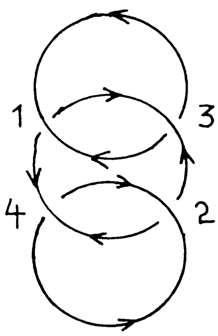
In words, we have the proposition:

> The m-link chain bracelet has walk-group isomorphic to the dihedral group D_{2m}; and its Cayley colour graph is isomorphic to the knot-graph of the chain bracelet.

(ii) **Open Chains**

It turns out that open chains also have dihedral walk-groups, though not necessarily ones that are isomorphic to themselves. We give one example:

3-link Open Chain

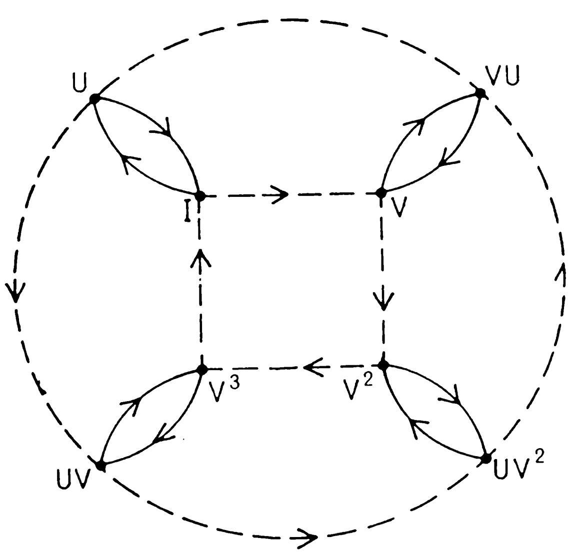


Walk-Group
$G<U,V>$

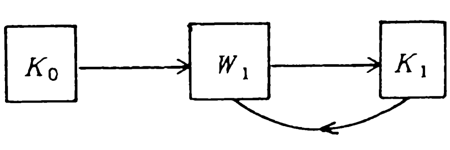
	Elements	*Orders*
	I	1
U =	(1 3)(2 4)	2
	(1 2)(3 4)	2
	(1 4)(2 3)	2
	(1 2)	2
	(3 4)	2
V =	(1 4 2 3)	4
	(1 3 2 4)	4

We see that the walk-group G is non-commutative, and has order 8.

Of the five groups of order 8, only D_8 and Q_8 are noncommutative. The information on element orders identifies G as being the dihedral group D_8. It s Cayley colour graph is:



It can be seen that the knot derivable from the Cayley graph is the 4-link chain bracelet. Hence the sequence (A):



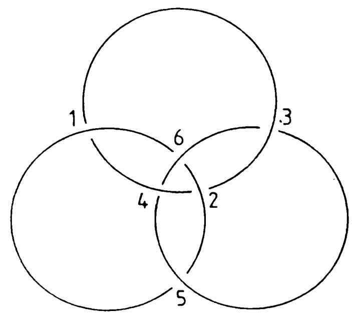
Conjecture:

For $m > 2$, the m-link open chain has a walk-group with Cayley graph knot an $(m+1)$-link chain bracelet.

N.B. We note in passing that the walk-groups distinguish the knots of (a) and (d) (i.e. 4_1^2 and the 3-link open chain), whereas they both have the same τ-value (v. Ch. VI) and spectra (v. Ch. V).

Example 4: A knot with $C_2{\times}C_3$ walk-group

The historic Borromean rings, much used in magic and heraldry, has a $C_2{\times}C_3$ walk-group.



Knot 6_2^3 : **Borromean rings**

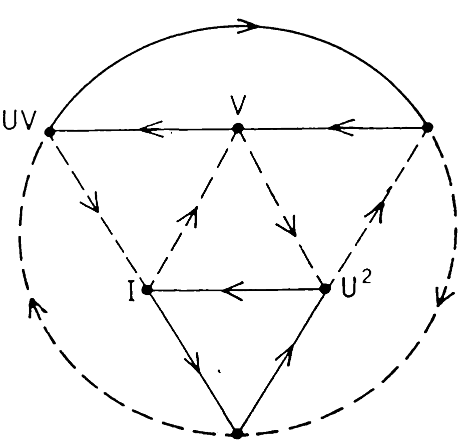
(if any link is removed, the remaining two are unlinked; there is a general construction for linking n rings, $n-1$ of which are unlinked if any one is removed).

$G<U,V>$	*Elements*	*Orders*
	I	1
	$U=(1\ 3\ 5)(2\ 4\ 6)$	3
	$V=(1\ 4\ 5\ 2\ 3\ 6)$	6
	$U^2=(1\ 5\ 3)(2\ 6\ 4)$	3
	$UV=(1\ 6\ 3\ 2\ 5\ 4)$	6
	$U^2V=(1\ 2)(3\ 4)(5\ 6)$	2

Commutative
Order of $G = 6$

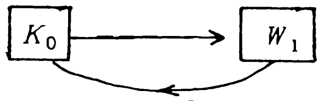
Group: $C_2{\times}C_3$

An interesting remark is that two Cayley graphs may be drawn, using different pairs of generators; and one has a derived knot ($\approx C_2{\times}C_3$) whereas the other does not admit of one because of arrow directions. This is a curious phenomenon that requires further study. We illustrate thus:



Cayley graph : <U,V>

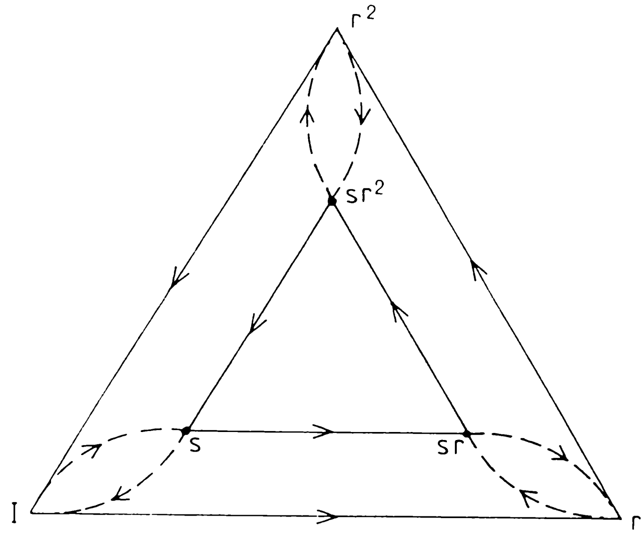
The knot-graph derived from this is the Borromean rings. So sequence (A) is, $K_0 = 6_2^3$:



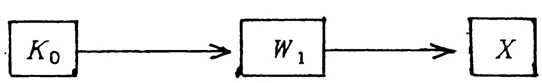
If we now take as generators $r = U$, $s = U^2V$, we get the presentation

$$G<r,s : r^3 = s^2 = srs^{-1}r^{-1} = I>$$

The Cayley graph for this is:



It looks like the Cayley graph drawn earlier for D_6: but the arrows on the two triangles are now in the same directions rather than opposing. And on trying to draw a knot-graph in the manner of this section, with bao orientation, we find that we cannot. So with *this* Cayley graph, the sequence (A) is



!

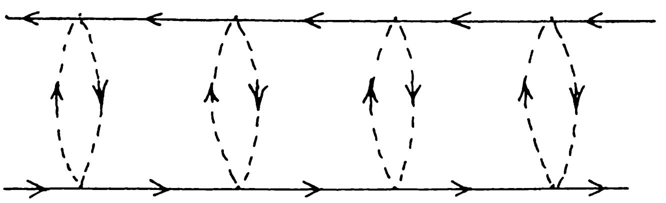
Example 5 : Wild Knots

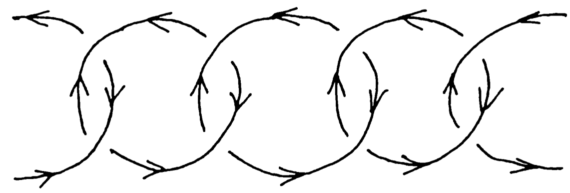
Definition:

A knot is ***tame*** if it is equivalent to a polygonal knot (i.e. one which is the union of a finite number of closed straight-line segments). Otherwise it is ***wild***.

Essentially, knots with finite numbers of crossings (on at least one of their knot-graphs) are tame. In this example we show a number of wild knots which are derived from Cayley colour graphs which represent groups of infinite order.

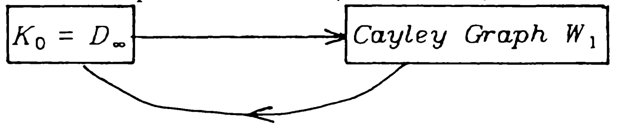
(i) **The dihedral group** D_∞



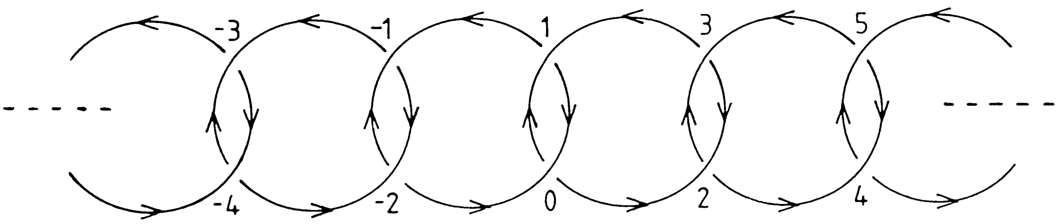


Cayley Graph: $G<r,s:s^2=(rs)^2=I>$ **Derived Knot** : **infinite chain bracelet (bao)**

This is just an extension of Ex.3(i)(a),(b),(c); the sequence for this too, if we formalise the idea of infinite permutation (see below), will be :



The necessary development of infinite permutations is:



With this labelling, the generators of the walk-group are as follows:

$U = \cdots(-3\ -4)(-1\ -2)(0\ 1)(2\ 3)(4\ 5)$

$$V = \begin{pmatrix} \cdot & -4 & -3 & -2 & -1 & 0 & 1 & 2 & 3 & 4 & \cdots \\ \cdots & -2 & -5 & 0 & -3 & 2 & -1 & 4 & 1 & 6 & \end{pmatrix};$$

In mapping notation: $\begin{Bmatrix} U : n \to n+1 \\ V : n \to n+(-1)^n 2 \end{Bmatrix}$, with $n \in Z$

Now $UV = \cdot(-3\ -2)(-1\ 0)(1\ 2)(3\ 4)$, so $(UV)^2 = I$.

With this formalization, then, we have a presentation

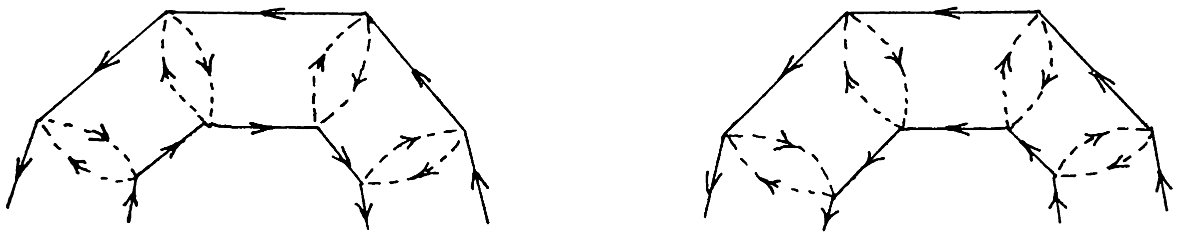
$G<U,V : U^2 = (UV)^2 = I>$ of D_∞, the infinite dihedral group

Note: We may compare the following developments from 'tame' to 'wild':

Circle	$\longrightarrow$	line	(circle of infinite radius)
Bracelet	$\longrightarrow$	chain	(with infinite number of links, closing to a bracelet at infinity)
D_n	$\longrightarrow$	D_∞	

(ii) **Cayley graphs for groups $C_2 \times C_n$, $C_2 \times C_\infty$ and $C_\infty \times C_\infty$**

We saw in example 4 that a Cayley graph for $C_2 \times C_3$ could be obtained from one of D_6 by reversing the arcs around one 3-gon. This idea generalises to $C_2 \times C_n$ thus :



(*a*) (*b*)

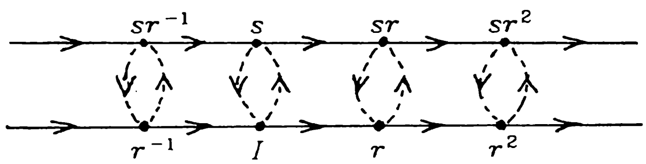
(*a*) Cayley Graph for D_{2n}

(*b*) Cayley Graph for $C_2 \times C_n$

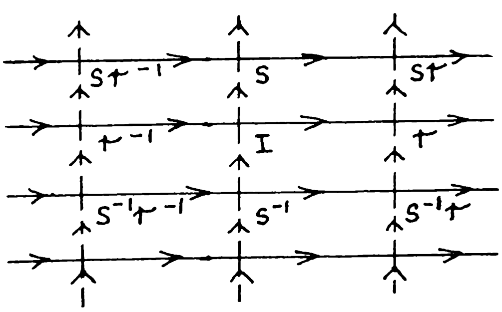
$G_{(b)} < r, s : r^n = s^2 = srs^{-1}r^{-1} = I >$

If we attempt to draw a knot from the oriented graph of (b), we find we cannot produce one that is either bno or bao. In the case of $C_2 \times C_3$ we found another presentation, using generators U, V, and obtained the bno version of the Borromean rings. The natural question is whether there are other presentations of $C_2 \times C_4$, $C_2 \times C_5$ etc. which admit of a bo knot-graph. We have not yet answered that question; but see (iii) below.

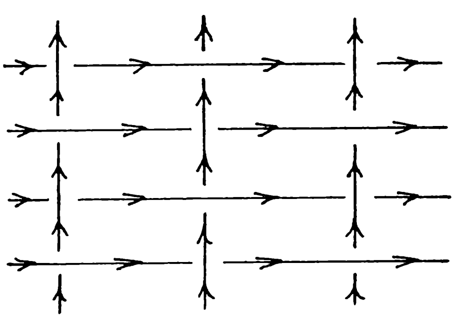
From the graph of D_∞ we can obtain the 'double-cyclic' group $C_2 \times C_\infty$, which has the following Cayley graph:



We cannot produce a bo knot-graph from this. If, however, we generalize still further to $C_\infty \times C_\infty \approx < r, s : srs^{-1}r^{-1} >$, we obtain the two graphs:



Cayley Graph



An infinite bno knot-graph

(iii) Infinite knot-graphs from Cayley graphs with two generators

(a) The Van Dyck group $D(4,4,2)$

We have just obtained the simple lattice knot-graph, with bno orientation, from the Cayley graph of $C_\infty \times C_\infty$. We now show that the Van Dyck group $D(4,4,2)$ provides the same lattice knot but with a bao orientation.

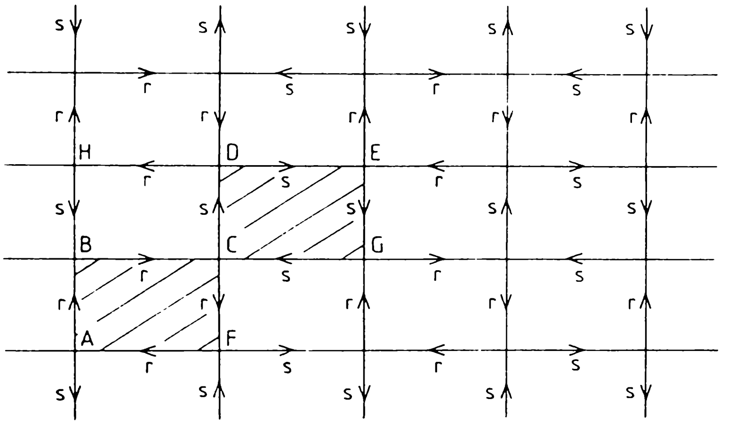
Definition:

The Van Dyck group $D(l,m,n)$ is the group

$$<r,s : r^l = s^m = (rs)^n = I>, \quad l,m,n \ integers > 0$$

Graph for $D(4,4,2)$

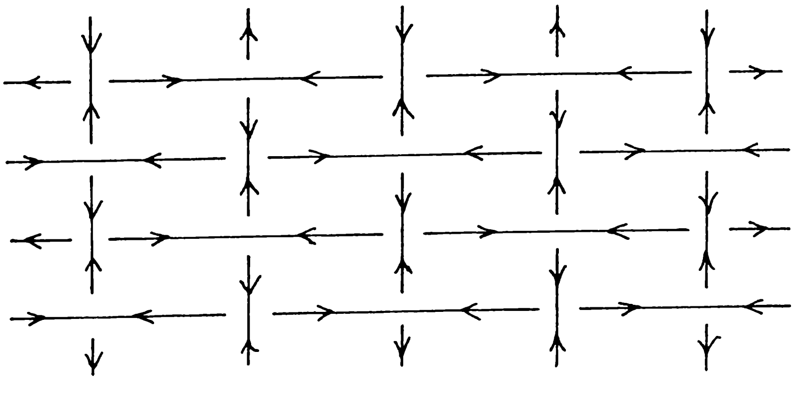
The group is $<r,s : r^4, s^4, (rs)^2>$, and the following graph on the lattice in R^2 is such that each point in the graph is fixed by a permutation represented by a relation.



Focussing on the two shaded squares we observe the relations $r^4 = I$ (from any vertex of $ABCF$) and $s^4 = I$ (from any vertex of $CDEG$). And square $BCDH$ gives an example of relation $(rs)^2 = I$.

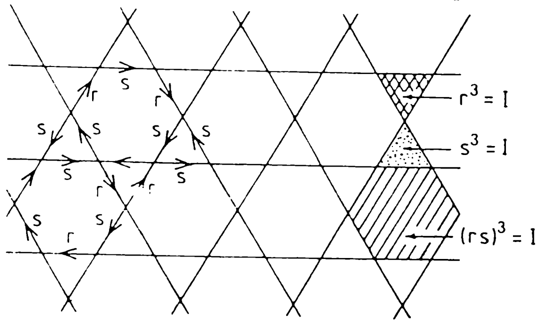
We can demonstrate a homomorphism $\Theta: D \rightarrow S_v$ where D is the Van Dyck group and S_v is the set of permutations of all vertices in the plane lattice. (It is clearly an infinite group, since the pattern $ABCDE = r^2s^2$ repeats ad infinitum, indicating an infinite cyclic subgroup.)

Thus the walk-group of the knot-graph shown below, with bao orientation, is isomorphic to the Van Dyck group $D(4,4,2)$.

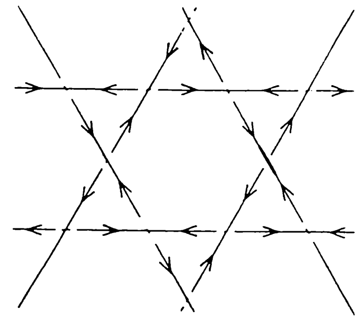


(b) **The Van Dyck group** $D(3,3,3)$

The group is $<r,s : r^3, s^3, (rs)^3>$, and the following graph shows how a homomorphism between D and S_v may be established (c.f. example (a) above). The graph on the right is the corresponding bao knot-graph.



Group Graph
Hexagons: $(rs)^3 = I$
Triangles: $r^3 = I = s^3$
(rs^{-1}) *generates infinite subgroup*



Corresponding knot-graph
with
bao orientation

(c) **The Fibonacci group** $F(2,6)$

$$F(2,6) = <a,b,c,d,e,f \mid ab=c,\ bc=d,\ cd=e,\ de=f,\ ef=a,\ fa=b>$$

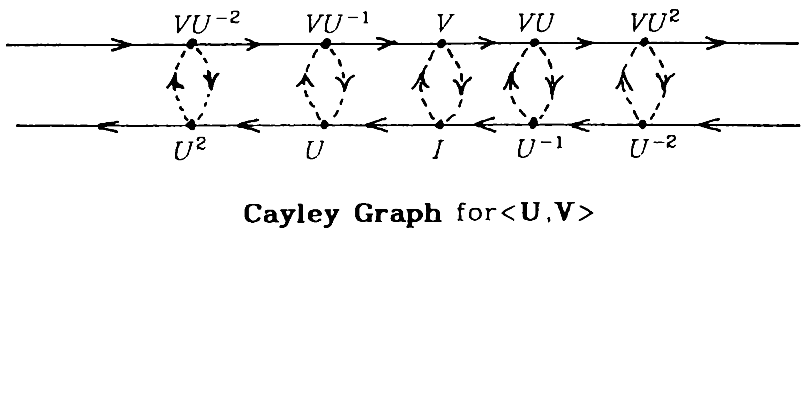
A representation of this group can be given in terms of infinite permutations, with two generators, thus:

Consider the permutations

$$\left.\begin{array}{l} \pi_a : n \rightarrow\ \ n+1 \\ \pi_b : n \rightarrow -n \\ \pi_c : n \rightarrow -n-1 \\ \pi_d : n \rightarrow\ \ n-1 \\ \pi_e : n \rightarrow -n-2 \\ \pi_f : n \rightarrow -n-1 \end{array}\right\} \text{ for all } n \ \varepsilon Z = \{0, \pm 1, \pm 2, \cdots\}$$

The relations of $F(2,6)$ are all preserved under multiplication of these permutations (e.g. $\pi_e \pi_f : n \rightarrow n+1 = \pi_a$, $\pi_f \pi_a : n \rightarrow -n = \pi_b$, etc.). And by Tietze operations the generator set can be reduced to two, say a, b. Hence an epimorphism can be demonstrated between $F(2,6)$ and the infinite group $<\pi_a, \pi_b>$.

Setting $\pi_a \equiv U$, and $\pi_b \equiv V$, we obtain a Cayley colour graph for the latter group thus:



Cayley Graph for**<U,V>**

Note that

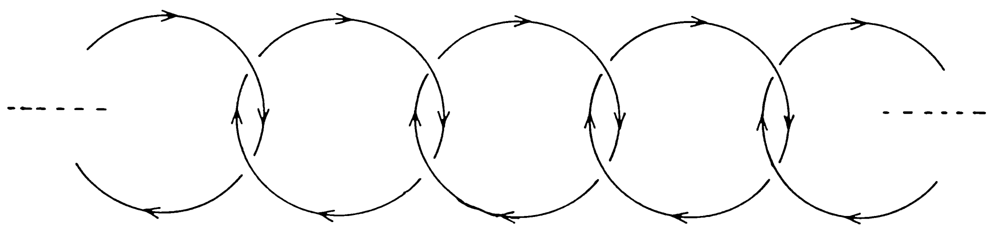
	. .	-3	-2	-1	0	1	2	3	. . .
U =		-2	-1	0	1	2	3	4	. .
V =		3	2	1	0	-1	-2	-3	
UV =		2	1	0	-1	-2	-3	-4	

$$\text{with } UV = (0\ -1)(1\ -2)(2\ -3)$$

Hence $V^2 = (UV)^2 = I$.

We may compare the Cayley graph with that for D_∞ in example 5(i); and also compare the presentations for the two groups.

The derived oriented knot-graph, for the present example is:



Infinite Chain Bracelet (bno)

Thus we have demonstrated relationships between D_∞, $F(2,6)$, and the elementary walk-groups obtained from the infinite chain bracelet with balanced alternating (respectively, nonalternating) orientations.

Example 6: The groups S_4, A_4 and A_5

We know from the theorem on planar groups that S_4, A_4, and A_5 have planar Cayley graphs. We have already given an example of a knot (viz. 6_3^2) having an elementary walk-group isomorphic to A_4; its derived knot-graph was a 4-link, but its orientation (if taken from the Cayley graph) was neither bno or bao.

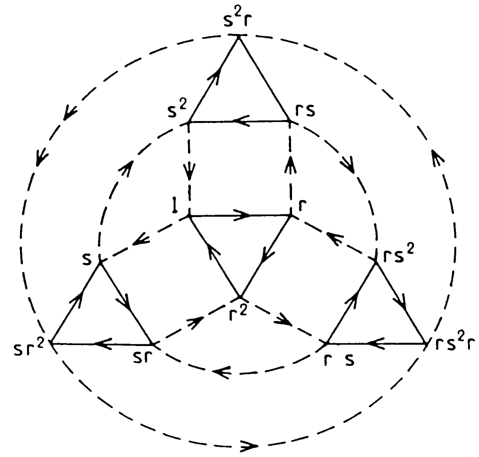
In this section we first give an example of an A_4 group with a two generator presentation. This provides a Cayley graph whose derived knot-graph bears the same bao orientation.

(i) **The Van Dyck group** $D(3,3,2)$

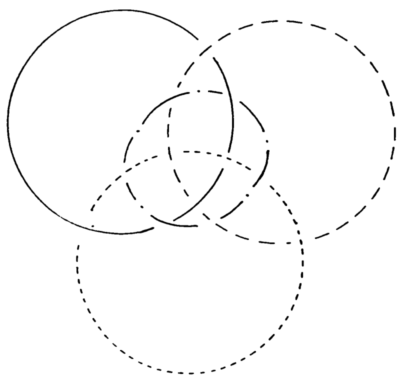
The group is

$$G\langle r, s : r^3,\ s^3,\ (rs)^2\rangle \approx A_4$$

Thus there are 24 vertices in the Cayley graph, and the graph is as follows:



Cayley Graph D(3,3,2)



Derived Knot-Graph K_1

If this knot-graph is given an α-orientation, the resulting bao is the same as in the Cayley graph above. This 4-link knot may not be transformed into the one represented by the knot-graph derived from the Cayley graph for the walk-group of knot 6_3^2 (see Example 1), even though they both have the same distributions of n-gons.

So we have found two different 4-links, both related to knot 6_3^2 in that they all have walk-groups isomorphic to A_4; one of which has a knot-graph (bao) which is isomorphic to the Cayley graph of the Van Dyck group $D(3,3,2)$.

(ii) **The Van Dyck groups** $D(2,3,4)$ **and** $D(2,3,5)$

It is known that $D(l,m,n)$ is finite iff $1/l + 1/m + 1/n > 1$ (theorem, Miller); so for cases $l=2$, $m=3$, $n=3,4,5$, only $D(2,3,3)$ (which is isomorphic to the case $D(3,3,2)$ treated above), $D(2,3,4)$ and $D(2,3,5)$ are finite.

It may be shown that

$$D(2,3,4) \approx S_4 \qquad \text{and} \qquad D(2,3,5) \approx A_5$$

Since these two graphs are planar, with two generators, their Cayley graphs may be drawn and knot graphs derived from them. Thus we may obtain examples of planar knot-graphs having walk-groups isomorphic to S_4 and A_5, on 24 and 60 crossings respectively (provided we find them to have bao or bno orientations). We have not yet drawn these knot-graphs.

Summary

In this section (4.2.4) we have studied relationships between elementary walk-groups, their Cayley colour graphs, and knot-graphs derived from the Cayley graphs.

Using a well-known theorem which gives a characterization of planar groups, we have been able to discover and classify many knots having knot-graphs with planar walk-groups. It would seem possible, at least for tame knots, to give a complete classification of knots with planar walk-groups; and we have gone some way towards presenting this.

Many interesting questions for further study relate to the varieties of balanced orientations which arise; and to the types of sequence (A) (i.e. $\boxed{K_0} \longrightarrow \boxed{W_1} \longrightarrow \boxed{K_1} \longrightarrow \cdots$) which are possible. Our examples have laid the foundations for answering these questions.

The introduction of infinite groups into the discussion, with the new problems and wild knots that they point to, broadens the general interest of the subject considerably. Comparisons and contrasts with infinite number series (e.g when the infinite chain bracelet with its various balanced orientations is considered) abound.

One final comment is that all knot-graphs having planar elementary walk-groups have great symmetry and consequently beautiful diagrams.

APPENDIX

GROUPOIDS OF CLUMPED PERMUTATIONS

In 4.1.4 the notions of *clumped permutations* and *elementary walk-groupoid* were introduced. C_n was defined to be the set of all clumped permutations of $X = \{1,2, \cdots ,n\}$: and C_n^r, $r=0, \quad ,n-1$, to be the subclass of C_n whose members have $\varepsilon_0 = r$. In this Appendix we give the groupoids C_2 and C_3, and obtain formulae for the orders $|C_n^r|$ and $|C_n|$. We also give two elementary walk-groupoids for the square knot.

(1) The Groupoid C_2

Clumped permutations

$n=2$	X	1	2
C_2^0	I	1	2
	a	2	1
C_2^1	b	0	(12)
	c	(12)	0

Groupoid Table

X	I	a	b	c
I	I	a	b	c
a	a	I	c	b
b	b	b	b	b
c	c	c	c	c

(2) The Groupoid C_3

Clumped Permutations

$n=3$	X	1	2	3
C_3^0	I	1	2	3
	a	1	3	2
	b	2	1	3
	c	2	3	1
	d	3	1	2
	e	3	2	1
C_3^2	p	0	0	(123)
	q	0	(123)	0
	r	(123)	0	0

	X	1	2	3	X	1	2	3
C'_3	f	0	(2 3)	1	f'	0	1	(2 3)
	g	0	(1 3)	2	g'	0	2	(1 3)
	h	0	(1 2)	3	h'	0	3	(1 2)
	i	1	0	(2 3)	i'	(2 3)	0	1
	j	2	0	(1 3)	j'	(1 3)	0	2
	k	3	0	(1 2)	k'	(1 2)	0	3
	l	1	(2 3)	0	l'	(2 3)	1	0
	m	2	(1 3)	0	m'	(1 3)	2	0
	n	3	(1 2)	0	n'	(1 2)	3	0

Groupoid Table (C_3)

	X	I	a	b	c	d	e	f	g	h	f'	g'	h'	i	j	k	i'	j'	k'	l	m	n	l'	m'	n'	p	q	r	
	I	I	a	b	c	d	e	f	g	h	f'	g'	h'	i	j	k	i'	j'	k'	l	m	n	l'	m'	n'	p	q	r	
	a	a	I	c	b	e	d	f'	g'	h'	f	g	h	l	m	n	l'	m'	n'	i	j	k	i'	j'	k'	q	p	r	
C_3^0	b	b	d	I	e	a	c	i'	j'	k'	i	j	k	f'	g'	h'	f	g	h	l'	m'	n'	l	m	n	p	r	q	
	c	c	e	a	d	I	b	i	j	k	i'	j'	k'	l'	m'	n'	l	m	n	f'	g'	h'	f	g	h	r	p	q	
	d	d	b	e	I	c	a	l'	m'	n'	l	m	n	f	g	h	f'	g'	h'	i'	j'	k'	i	j	k	q	r	p	
	e	e	c	d	a	b	I	l	m	n	l'	m'	n'	i'	j'	k'	i	j	k	f	g	h	f'	g'	h'	r	q	p	
	f	f	f	g	g	h	h	q	q	q	q	q	q	f	g	h	f'	g'	h'	f	g	h	f'	g'	h'	q	q	p	
	g	g	h	f	h	f	g	f'	g'	h'	f	g	h	q	q	q	q	q	q	f'	g'	h'	f	g	h	q	p	q	0
	h	h	g	h	f	g	f	f	g	h	f'	g'	h'	f'	g'	h'	f	g	h	q	q	q	q	q	q	p	q	q	in
	f'	f'	f'	g'	g'	h'	h'	p	p	p	p	p	p	f'	g'	h'	f	g	h	f'	g'	h'	f	g	h	p	p	q	1st
	g'	g'	h'	f'	h'	f'	g'	f	g	h	f'	g'	h'	p	p	p	p	p	p	f	g	h	f'	g'	h'	p	q	p	posn
	h'	h'	g'	h'	f'	g'	f'	f'	g'	h'	f	g	h	f	g	h	f'	g'	h'	p	p	p	p	p	p	q	p	p	
	i	i	i	j	j	k	k	p	p	p	p	p	p	i	j	k	i'	j'	k'	i	j	k	i'	j'	k'	p	p	r	
	j	j	k	i	k	i	j	i'	j'	k'	i	j	k	p	p	p	p	p	p	i'	j'	k'	i	j	k	p	r	p	0
C_3^1	k	k	j	k	i	j	i	i	j	k	i'	j'	k'	i'	j'	k'	i	j	k	p	p	p	p	p	p	r	p	p	in
	i'	i'	i'	j'	j'	k'	k'	r	r	r	r	r	r	i'	j'	k'	i	j	k	i'	j'	k'	i	j	k	r	r	p	2nd
	j'	j'	k'	i'	k'	i'	j'	i	j	k	i'	j'	k'	r	r	r	r	r	r	i	j	k	i'	j'	k'	r	p	r	posn
	k'	k'	j'	k'	i'	j'	i'	i'	j'	k'	i	j	k	i	j	k	i'	j'	k'	r	r	r	r	r	r	p	r	r	
	l	l	l	m	m	n	n	q	q	q	q	q	q	l	m	n	l'	m'	n'	l	m	n	l'	m'	n'	q	q	r	
	m	m	n	l	n	l	m	l'	m'	n'	l	m	n	q	q	q	q	q	q	l'	m'	n'	l	m	n	q	r	q	0
	n	n	m	n	l	m	l	l	m	n	l'	m'	n'	l'	m'	n'	l	m	n	q	q	q	q	q	q	r	q	q	in
	l'	l'	l'	m'	m'	n'	n'	r	r	r	r	r	r	l'	m'	n'	l	m	n	l'	m'	n'	l	m	n	r	r	q	3rd
	m'	m'	n'	l'	n'	l'	m'	l	m	n	l'	m'	n'	r	r	r	r	r	r	l	m	n	l'	m'	n'	r	q	r	posn
	n'	n'	m'	n'	l'	m'	l'	l'	m'	n'	l	m	n	l	m	n	l'	m'	n'	r	r	r	r	r	r	q	r	r	
	p	p	p	p	p	p	p	p	p	p	p	p	p	p	p	p	p	p	p	p	p	p	p	p	p	p	p	p	
C_3^2	q	q	q	q	q	q	q	q	q	q	q	q	q	q	q	q	q	q	q	q	q	q	q	q	q	q	q	q	
	r	r	r	r	r	r	r	r	r	r	r	r	r	r	r	r	r	r	r	r	r	r	r	r	r	r	r	r	

(3) Formulae for $|C_n^r|$ and $|C_n|$

(i) $|C_n^r| = (n)_{n-r}.S_n^{n-r}$, where S_n^{n-r} is Stirling's Number of the second kind.

Proof:

An element of C_n^r may be constructed by the following sequence of tasks:

(a) Choose the r zero-positions: there are $\binom{n}{r}$ possible ways;

(b) Partition $\{1,\ldots,n\}$ into $(n-r)$ clumps : there are S_n^{n-r} possible ways;

(c) Place the clumps into $n-r$ nonzero-positions: $(n-r)!$ ways;

Total number of ways to perform the sequence:

$$\begin{aligned} |C_n^r| &= \binom{n}{r}.S_n^{n-r}.(n-r)! \\ &= n(n-1)\cdot\quad(n-r+1)S_n^{n-r} \\ &= (n)_{n-r}.S_n^{n-r} \end{aligned}$$

(ii) Then $|C_n| = \sum_{r=0}^{n-1} |C_n^r| = \sum_{r=0}^{n-1} (n)_{n-r} S_n^{n-r} = n^n$

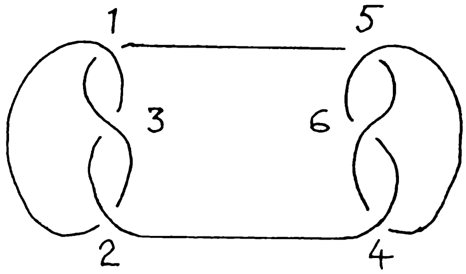
Proof:

RIORDAN,p. 99, gives an enumerator for the number of ways of putting n different objects into m cells, with p cells occupied: the number is $(m)_p S_n^p$, and it is the coefficient of $\frac{t^n}{n!}$ in the expansion of $\binom{m}{p}(e^t-1)^p$. Setting $m = n$, $p = n-r$, we find that $|C_n^r|$ is the coefficient of $\frac{t^n}{n!}$ in the expansion of $\binom{n}{n-r}(e^t-1)^{n-r}$. Therefore, the required result is the coefficient of $\frac{t^n}{n!}$ in $\sum_{r=0}^{n-1}\binom{n}{n-r}(e^t-1)^{n-r}$.

But we may obtain the result in a very simple direct way thus: We can place each of the digits 1, $\cdot$,n in any one of n positions when forming a clumped permutation. Hence there are n^n clumped permutations.

(4) Examples of elementary walk-groupoids for the square knot

We give below two possible walk-groupoids derived from a nonalternating diagram of the square knot. We show two splittings of K, and tabulate all 58 elements of the groupoid arising from the first one.



$K = J_\alpha + J_\beta$

$$J_\alpha = \begin{bmatrix} & & 1 & & & \\ 1 & & 1 & 1 & & \\ 1 & 1 & & & & \\ & 1 & & & 1 & 1 \\ & & & & & 1 \\ & & & 1 & 1 & \end{bmatrix} \qquad J_\beta = \begin{bmatrix} & 1 & 1 & & 1 & \\ & & 1 & & & \\ 1 & 1 & & & & \\ & & & & & 1 \\ 1 & & & 1 & & 1 \\ & & & 1 & 1 & \end{bmatrix}$$

(i) **First Splitting** : $K = (P_1 + Q_1) + (P_2 + Q_2)$, where

	1	2	3	4	5	6
P_1	3	1	2	5	6	4
Q_1	0	(34)	1	(26)	0	5
P_2	2	3	1	6	4	5
Q_2	(35)	0	2	0	(16)	4

Note that $Q_2 = P_1(Q_1P_2)^4$: i.e., Q_2 is a consequence of $<P_1, P_2, Q_1>$.

(ii) **Second Splitting** : $K = (P'_1 + Q'_1) + (P'_2 + Q'_2)$, where

	1	2	3	4	5	6
P'_1	3	4	1	2	6	5
Q'_1	0	(13)	2	(56)	0	4
P'_2	5	3	2	6	1	4
Q'_2	(23)	0	1	0	(46)	5

The groupoid $<P'_1, Q'_2, P'_2>$ arising from this splitting has 58 elements.

(iii) **The elements of the groupoid** $<P_1, Q_1, P_2>$

The following is a table showing all the 58 elements of the groupoid $<P_1, Q_1, P_2>$. Below it are set out relations indicating how they were computed; they occur largely in sets of three.

1	2	3	4	5	6
3 2 0	1 3 (34)	2 1 1	5 6 (26)	6 4 0	4 5 5
1 (34) 0	0 1 (34)	(34) 0 1	0 5 (26)	5 (26) 0	(26) 0 5
0 3 (25)	(25) 0 3	3 (25) 0	(14) 0 6	0 6 (14)	6 (14) 0
0 0 (126)	(126) 0 0	0 (126) 0	(345) 0 0	0 0 (345)	0 (345) 0
0 (34) 5	5 0 (34)	(34) 5 0	1 0 (26)	0 (26) 1	(26) 1 0
0 0 (134)	(134) 0 0	0 (134) 0	(256) 0 0	0 0 (256)	0 (256) 0
0 (25) 6	6 0 (25)	(25) 6 0	3 0 (14)	0 (14) 3	(14) 3 0
0 0 (235)	(235) 0 0	0 (235) 0	(146) 0 0	0 0 (146)	0 (146) 0
0 (16) 4	4 0 (16)	(16) 4 0	2 0 (35)	0 (35) 2	(35) 2 0
0 5 (26)	(26) 0 5	5 (26) 0	(34) 0 1	0 1 (34)	1 (34) 0
0 0 (256)	(256) 0 0	0 (256) 0	(134) 0 0	0 0 (134)	0 (134) 0

(the table is continued on the next page)

1	2	3	4	5	6
0 6 (14)	(14) 0 6	6 (14) 0	(25) 0 3	0 3 (25)	3 (25) 0
0 0 (146)	(146) 0 0	0 (146) 0	(235) 0 0	0 0 (235)	0 (235) 0
0 4 (35)	(35) 0 4	4 (35) 0	(16) 0 2	0 2 (16)	2 (16) 0
0 (26) 1	1 0 (26)	(26) 1 0	5 0 (34)	0 (34) 5	(34) 5 0
0 (14) 3	3 0 (14)	(14) 3 0	6 0 (25)	0 (25) 6	(25) 6 0
0 (35) 2	2 0 (35)	(35) 2 0	4 0 (16)	0 (16) 4	(16) 4 0
1	2	3	4	5	6

Note that

$$G = \{P_1, P_2, Q_1, A_1, A_2, A_3, B_1, B_2, B_3, \cdots, T_1, T_2, T_3, I\}$$

with

$$A_1 = P_1Q_1,\ A_2 = P_1A_1,\ A_3 = P_1A_2;$$
$$B_1 = Q_1P_1,\ B_2 = P_1B_1,\ B_3 = P_1B_2;$$
$$C_1 = Q_1P_2,\ C_2 = P_1C_1,\ C_3 = P_1C_2;$$
$$D_1 = Q_1Q_1,\ D_2 = P_1D_1,\ D_3 = P_1D_2;\ etc.$$

CHAPTER 5

SPECTRA OF KNOT-GRAPH MATRICES

One of the main objects of this study is to determine properties of adjacency matrices of knot-graphs which are related to structural properties of the graphs. It follows that we shall be interested primarily in those properties of an adjacency matrix which are invariant to the labellings of the vertices of the graph (and therefore to row-column permutations of the matrix). One important set of such properties is that derived from the spectrum of the matrix.

Much research into spectra of graphs has taken place in the last fifteen years or so, and the monograph *Spectra of Graphs*, *(CVETKOVIC,1980)* provides an excellent review of this. We shall be drawing on this research to derive results relating specifically to knot-graphs; but as well, because of the special nature of knot-graph adjacency matrices, we are able to develop new methods for studying their spectral properties.

5.1 THE SPECTRA TO BE STUDIED

In the monograph *(CVETKOVIĆ,p.24,1980)* are defined several different characteristic polynomials derivable from adjacency matrices of graphs.

We shall deal with only the following:

$$P_M(\lambda) \equiv |\lambda I - M| \,, \tag{5.1}$$

where M is an adjacency matrix.

For the alternating knot-graphs we study properties of $P_J(\lambda)$; and for the nonalternating knot-graphs we study the pair $[P_{J_\alpha}(\lambda), P_{J_\beta}(\lambda)]$ arising from each graph. We shall also discuss properties of $P_K(\lambda)$, and of $P_A(\lambda)$, where $K = J_\alpha + J_\beta$, and A is the *bno* matrix $U + V'$ (3.5).

For convenience in calculations we shall often first obtain the form

$$P_J{}^*(\mu) \equiv |\mu I + J|, \tag{5.2}$$

and then convert using

$$P_J(\lambda) = (-1)^n P_J{}^*(-\lambda). \tag{5.3}$$

(N.B. We shall often omit the superscript * when obtaining form (5.2). It will be clear from the context which polynomial is being derived.)

The solutions of $P_J(\lambda) = 0$ are the eigenvalues of J: together they comprise the *spectrum of* J. We shall use either of the following notations for this:

(i)

$$\mathrm{Spec}(J) = [\lambda_1, \lambda_2, \quad , \lambda_n]; \tag{5.5}$$

(ii)

$$\mathrm{Spec}(J) = \begin{vmatrix} \lambda_1 & \lambda_2 & & \lambda_r \\ m(\lambda_1) & m(\lambda_2) & & m(\lambda_r) \end{vmatrix}, \tag{5.6}$$

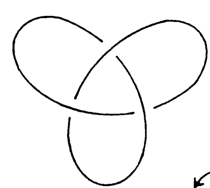
where $\lambda_1>\lambda_2>\cdots>\lambda_r$ and $m(\lambda_i)$is the multiplicity of λ_i.

If $\lambda_i \in \mathrm{Spec}(J)$, then non-zero vectors $\underline{x}$ such that $J\underline{x} = \lambda_i\underline{x}$ will be called (right-) *eigenvectors of* J (or eigenvectors of the knot-graph oriented by J).

5.2 SIMPLE EXAMPLES OF KNOT-GRAPH SPECTRA

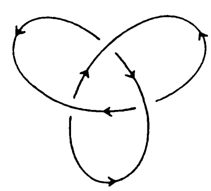
(i) **The trefoil, 3_1**

$$K = \begin{bmatrix} 0 & 2 & 2 \\ 2 & 0 & 2 \\ 2 & 2 & 0 \end{bmatrix}, \qquad J = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix} \qquad A = \begin{bmatrix} 0 & 2 & 0 \\ 0 & 0 & 2 \\ 2 & 0 & 0 \end{bmatrix}$$



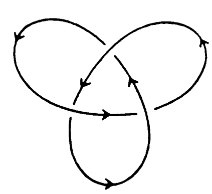
Undirected $\quad P_K(\lambda) = (\lambda-4)(\lambda+2)^2$

$$\mathrm{Spec}(K) = \begin{bmatrix} 4 & -2 \\ 1 & 2 \end{bmatrix}$$



Directed (bao) $\quad P_J(\lambda) = (\lambda-2)(\lambda+1)^2$

$$\mathrm{Spec}(J) = \begin{bmatrix} 2 & -1 \\ 1 & 2 \end{bmatrix}$$



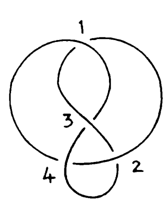
Directed (bno) $\quad P_A(\lambda) = \lambda^3-8$

$\mathrm{Spec}(A) = [2, 2w, 2w^2]$

where $w = \exp(2\pi i/3)$

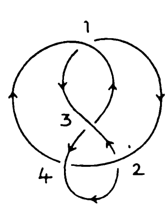
(ii) **Listing's knot, 4_1**

$$K = \begin{bmatrix} 0 & 1 & 2 & 1 \\ 1 & 0 & 1 & 2 \\ 2 & 1 & 0 & 1 \\ 1 & 2 & 1 & 0 \end{bmatrix}, \qquad J = \begin{bmatrix} 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 \end{bmatrix}, \qquad A = \begin{bmatrix} 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \end{bmatrix}$$



$P_K(\lambda) = (\lambda-4)\lambda(\lambda+2)^2$

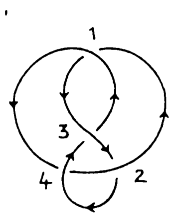
$$\mathrm{Spec}(K) = \begin{bmatrix} 4 & 0 & -2 \\ 1 & 1 & 2 \end{bmatrix}$$



$P_J(\lambda) = (\lambda-2)\lambda(\lambda^2+2\lambda+2)$

$\mathrm{Spec}(J) = [2, 0, -1\pm i]$

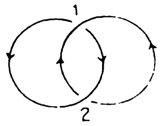
$P_A(\lambda) = P_J(\lambda)$, since $A = J'$



(iii) **Chains 2_1^2 and 4_1^3 ; and Link 4_1^2**

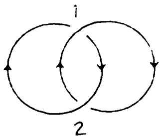
Knot 2_1^2

(a)



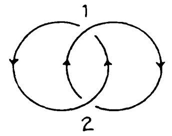
$$J = \begin{bmatrix} 0 & 2 \\ 2 & 0 \end{bmatrix} \qquad P_J(\lambda) = (\lambda-2)(\lambda+2)$$
$$\text{Spec}(J) = [2, -2]$$

(b)



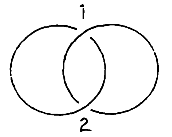
$$A_1 = \begin{bmatrix} 0 & 2 \\ 2 & 0 \end{bmatrix} \qquad P_{A_1}(\lambda) = P_J(\lambda)$$

(c)



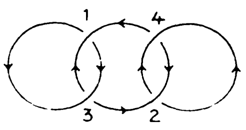
$$A_2 = \begin{bmatrix} 0 & 2 \\ 2 & 0 \end{bmatrix} \qquad P_{A_2}(\lambda) = P_J(\lambda)$$

(d)



$$K = \begin{bmatrix} 0 & 4 \\ 4 & 0 \end{bmatrix} \qquad P_K(\lambda) = (\lambda-4)(\lambda+4)$$
$$\text{Spec}(K) = [4, -4]$$

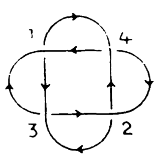
Chain 4_1^3



$$J = \begin{bmatrix} 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 2 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \end{bmatrix} \qquad P_J(\lambda) = \lambda^2(\lambda-2)(\lambda+2)$$
$$\text{Spec}(J) = [2, 0, 0, -2]$$

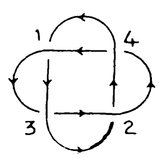
Link 4_1^2

(a)



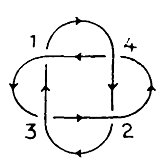
$$J = \begin{bmatrix} 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \end{bmatrix} \qquad P_J(\lambda) = \lambda^2(\lambda-2)(\lambda+2)$$
$$\text{Spec}(J) = [2, 0, 0, -2]$$

(b)



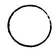
$$A_1 = \begin{bmatrix} 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 2 \\ 0 & 2 & 0 & 0 \\ 2 & 0 & 0 & 0 \end{bmatrix} \qquad P_{A_1}(\lambda) = (\lambda-2)(\lambda+2)(\lambda^2+4)$$
$$\text{Spec}(A_1) = [2, \pm 2i, -2]$$

(c)

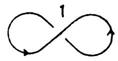


$$A_2 = \begin{bmatrix} 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \end{bmatrix} \qquad P_{A_2}(\lambda) = P_J(\lambda)$$

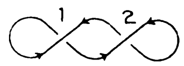
(iv) **Some unknots**



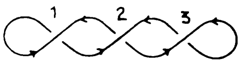
$P_J(\lambda) \equiv \ 0$



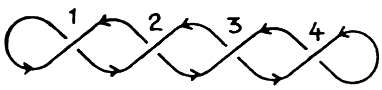
$P_J(\lambda) = \lambda-2$
Spec(J) = [2]



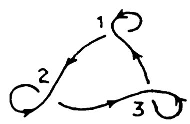
$P_J(\lambda) = \lambda(\lambda-2)$
Spec(J) = [2,0]



$P_J(\lambda) = (\lambda-2)(\lambda-1)(\lambda+1)$
Spec(J) = [2, 1, −1]

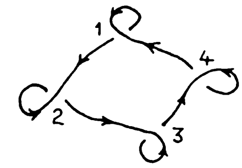


$P_J(\lambda) = \lambda(\lambda-2)(\lambda^2-2)$
Spec(J) = [2, 0, $+\sqrt{2}, -\sqrt{2}$]



$P_J(\lambda) = (\lambda-1)^3-1 = \lambda^3-3\lambda^2+3\lambda-2$
Spec(J) = $[1+w_0,\ 1+w_1,\ 1+w_2]$
where w_i is a cube root of unity, i = 0,1,2

$P_J(\lambda) = (\lambda-1)^4-1 = \lambda^4-4\lambda^3+6\lambda^2-4\lambda$
Spec(J) = $[1+w_0,\ 1+w_1,\ 1+w_2,\ 1+w_3]$,
$w_i = \sqrt[4]{1}$



5.2.1 Discussion on the examples

(1) Examples (i) and (ii) give the polynomials and spectra for the first two prime knots, the trefoil and Listing's 4_1 knot.

Note that K has a characteristic root $\lambda = 4$ in both; and J and A have a root $\lambda = 2$ in both.

In Listing's knot $P_J(\lambda) = P_A(\lambda)$; this is not, however, true in general.

Note that the K matrix is always symmetric, and so all of the characteristic roots of K are real. Whereas in $P_J(\lambda)$ and $P_A(\lambda)$ complex roots may arise.

(2) Figure (iii) shows various chains of two components, and one of three components; and chain 4_1^3 is to be compared with link 4_1^2. Note that all forms of orientation of the 2-component chains lead to the same characteristic polynomial.

Note that 4_1^3 and 4_1^2 are cospectral (i.e. each has the same J-spectrum).

We shall meet with cospectral knot-graphs again, in 5.7.3 when composed knots are discussed, and the granny and square knots are found to have cospectral forms. However, we have not found cospectrality among any prime knots, for n = 3 *to* 10 (which is not to say, of course, that it may not exist for higher values of n, or for different forms of the knots studied). We have only found it in these very special, highly symmetric, simple chains, and in compositions.

The various (three) forms of orientation for the 2-link 4_1^2 show that it is possible to have $P_J = P_{A_2}$; it is evident that this will always occur in an even torus link, if one component is oriented in the opposite sense to the other.

(3) Figures (iv) show seven different forms of the unknot. General classes of unknot forms are treated later, in section 5.4.

(4) It may be noted that all real roots of spectra for J matrices lie in the interval [-2, 2]. And that imaginary characteristic roots occur with some knot-graphs.

5.3 SOME GENERAL RESULTS ON KNOT-GRAPH SPECTRA

A number of general theorems given in (*CVETOVIĆ*,1980) and (*BIGGS*,1974) on the spectra of graphs may be applied to K and J matrices, giving results directly for underlying graphs and directed knot-graphs respectively. Other results follow after slight modification or specialization of the theorems, taking care of the special nature of knot-graphs. We list these results next, and follow them with an example.

We shall use the following notation :

K is the adjacency matrix of an undirected associated graph

J is the α-adjacency matrix of an alternating knot-graph (regarded as the adjacency matrix of a multidigraph);

K,J will also on occasion refer to the corresponding graphs;

$P_K(\lambda)$, $P_J(\lambda)$ will denote their characteristic polynomials; the polynomials will be denoted by $a_0\lambda^n + a_1\lambda^{n-1} +$ $\quad$ $+ a_{n-1}\lambda + a_n$ in both cases.

Results

(1) K is an undirected connected multigraph, which is regular of valency 4. This implies that Spec(K) has only real characteristic roots λ_i , with $|\lambda_i| \leq 4$ for all i (*BIGGS*,*p*. 14)

(2) K has a characteristic root $\lambda=4$ of multiplicity 1, having corresponding characteristic vector 1 (i.e. a vector of 1's). (*BIGGS*,*p* 14)

(3) $-a_1 = \sum\lambda_i = \mathrm{tr}(K) =$ twice the number of loops in the underlying graph (a loop on x_i corresponds to the element $K_{ii} = 2$)

(4) (Results for J, corresponding to (1),(2),(3) : note that J is not in general symmetric; and a directed loop on x_i corresponds to $J_{ii} = 1$)

   (a) Spec(J) has one root $\lambda=2$ with multiplicity 1 and vector $\underline{1}$; the other roots are not necessarily real; $|\lambda_i| \leq 2$

   (b) $-a_1$ = number of loops in the directed knot-graph

   (c) If J has no loops, $\sum\lambda_i = 0$.

Propositions

(5) If n is even, J has a characteristic root $\lambda=0$.

Proof: we showed in 3.3.1 that $|J| = 0$ when n is even; then $a_n = 0$, so λ is a factor of the characteristic polynomial.

(6) If a knot-graph has no **nugatory crossings, then:**

$$-a_2 = \text{number of 2-gons in the knot-graph}$$

Proof: $a_2 = \sum$ (principal minors of order 2 in J).

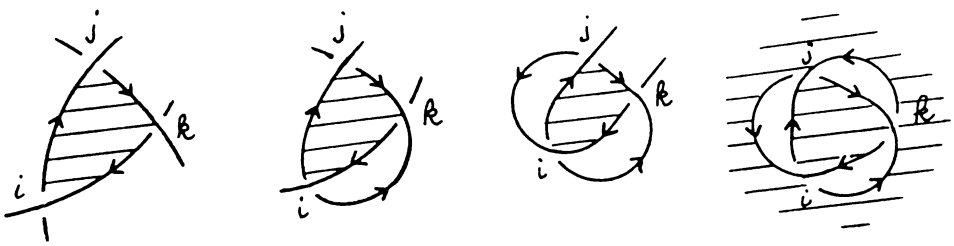
The only such principal minors in J which are non-zero are those of type $\begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix}$: suppose one of these corresponds to rows and columns i,j. Then vertices i and j are joined by a 2-gon, thus : 

(7) If a knot-graph has no nugatory crossings, then $-a_3$ = number of 3-gons in the knot-graph.

Proof: $a_3 = \sum$(principal minors of order 3 in J).

The only such minors in J which are non-zero are the following (underneath each is shown a corresponding portion of a knot-graph) :

(a) $\begin{array}{c|ccc|} & i & j & k \\ i & 0 & 1 & 0 \\ j & 0 & 0 & 1 \\ k & 1 & 0 & 0 \end{array}$ (b) $\begin{vmatrix} 0 & 1 & 1 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{vmatrix}$ (c) $\begin{vmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 0 & 0 \end{vmatrix}$ (d) $\begin{vmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{vmatrix}$



(e),(f) are similar to (b); (g),(h) are similar to (c); plus all transposes of all types, when different. All 3-gons are shown shaded.

Note that cases (c),(g),(h) indicate that the knot is a composition of a trefoil with some other knot; and case (d) only occurs when the knot actually is the trefoil.

In all cases except (d), the value of the principal minor is -1, and the proposition follows.

In case (d), the minor has value -2; and since the trefoil has two 3-gons, the proposition holds.

A number of authors have shown how values of the coefficients of a_i of the characteristic polynomial of a multi-digraph (J, say) can easily be computed if the set of all directed cycles of the knot-graph is known. The fundamental theorem on this matter is called *the coefficients theorem for digraphs*. We next state this theorem, and as an example apply it to knot 5_2.

Theorem 1 (*CVETKOVIĆ*,*p* 32)

Let $P_J(\lambda)$ be the characteristic polynomial $\sum_{i=0}^{n} a_i \lambda^{n-i}$ of a knot-graph oriented by α-adjacency matrix J.

Then

$$a_i = \sum_{L \in L_i} (-1)^{p(L)} \quad (i = 1, 2, \cdots, n)$$

where L_i is the set of all linear directed subgraphs L of J with exactly i vertices; $p(L)$ denotes the number of components of L (i.e. the number of cycles of which L is composed).

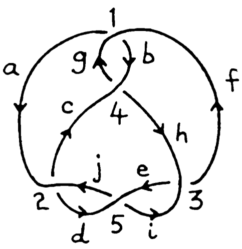
An alternative form, which we apply in our example below, is :

> The coefficient a_i depends only on the set of all linear directed subgraphs L of J having exactly i vertices, the contribution of L to a_i being +1 if L contains an even, and -1 if L contains an odd, number of cycles.

Example

(we label all edges, using a, ,j, in order to keep track of them)

Knot 5_2



$$P_J(\lambda) = \begin{vmatrix} \lambda & a & 0 & b & 0 \\ 0 & \lambda & 0 & c & d \\ f & 0 & \lambda & 0 & e \\ g & 0 & h & \lambda & 0 \\ 0 & j & i & 0 & \lambda \end{vmatrix}$$

(after replacing each of a, ,j by -1)

Expanding the determinant and collecting terms we obtain :

$$P_J(\lambda) = 1\lambda^5 + 0\lambda^4 + (-jd - ei - bg)\lambda^3 + (acg + bhf)\lambda^2$$
$$+ (-jche - achf - adif + bgei + bgdj)\lambda + (-acgei - bhfdj).$$

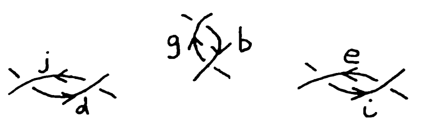
The coefficients of λ^{n-1} indicate just those subgraphs, on exactly i vertices, which have a nonzero contribution to make to the coefficient of a_i. Using the letters a, $\cdot\cdot$,j, to guide us, we sketch below all the associated subgraphs, showing how they are indeed made up of cycles.

$a_0 \equiv 1$

a_1 = no. of directed loops = 0

$-a_2$ = no. of directed 2-gons

or $a_2 = (-1)^1 + (-1)^1 + (-1)^1$

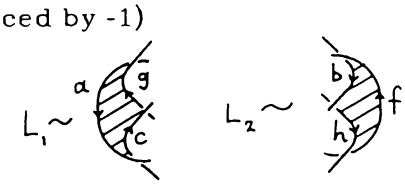


$-a_3$ = no. of directed 3-gons

($acg = bhf = -1$ when the letters are replaced by -1)

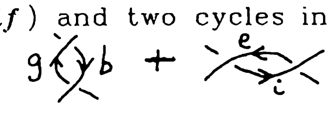
or $a_3 = (-1)^1 + (-1)^1$,

there being one cycle in each of L_1 and L_2



$a_4 = (-1)^1 + (-1)^1 + (-1)^1 + (-1)^2 + (-1)^2$

there being one cycle in each of the first three subgraphs ($jche$, $achf$, $adif$) and two cycles in $bgei$ and $bgdj$. For example $bgei$ ~



$a_5 = (-1)^2 + (-1)^2$,

$= +2$

since each of the subgraphs $acgei$ and $bhfdj$ has two cycles

Finally, then, for the knot-graph 5_2,

$$P_J(\lambda) = \lambda^5 - 3\lambda^3 - 2\lambda^2 - \lambda + 2$$

The converse problem, of deducing structural properties of a knot-graph when the characteristic polynomial is given is much more difficult (apart from the obvious inferences that can be drawn from a_1, a_2, a_3). This problem is treated in Chapter 3 of *CVETKOVIĆ*; we will not pursue it here in relation to knot-graphs.

However, in 5.5 we present an algorithm for determining the coefficients a_i directly from permutations U and P (recall $J = U + PU$), and obtain further properties of the polynomial coefficients.

5.4 SPECTRA FOR SIMPLE CLASSES OF KNOT-GRAPHS

In some simple cases it is possible to obtain recurrence relations for expanding $|\lambda I - J_n|$, where J_n is the adjacency matrix for the n^{th} member of a class of knot-graphs. Then general formulae for the spectra of members of the class can be derived. We shall derive such formulae for the following classes:

(i) the unknot with n loops

(ii) the unknot with n twists (see 5.2 (iv) for examples of (i) and (ii));

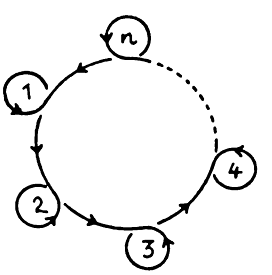
(iii) the torii knot-graphs, on n crossings ;

(iv) knot-graphs from Conway's rational tangles ;

(v) a sub-class of knot-graphs which are covered by 2-gons.

5.4.1 The unknot with n loops

We give the result first in the case that all the loops are arranged so that the *knot-graph is alternating*, thus :



The characteristic polynomial is given by

$$P_J(\lambda) = \begin{vmatrix} \lambda-1 & -1 & & & & \\ & \lambda-1 & -1 & & & \\ & & \lambda-1 & -1 & & \\ & & & \lambda-1 & -1 & \\ & & & & & -1 \\ -1 & & & & & \lambda-1 \end{vmatrix}$$

$$= (\lambda - 1)^n - 1$$

Therefore

$$\mathrm{Spec}(J_n) = \{1+w^{j-1} \mid j=1,\ \cdot\ ,n\} \text{ where } w = \exp\left[\frac{2\pi i}{n}\right],\ i = \sqrt{-1};$$

since $(\lambda-1)^n-1 = 0$ has roots $\lambda_j = 1 + \sqrt[n]{1}$, i.e. $\lambda_j = 1$ plus an n^{th} root of unity

In 5.2(iv) is shown the cases n=3,4. Note that the formula holds also for n=1,2 (see the 'twist' diagrams in 5.2(iv)).

If the loops are arranged so that parts of the knot-graph alternate, and others do not, a general result can still be obtained, thus:

Proposition

Let K be an unknot-graph being a circle with n loops in it, arranged in such a way that the graph has n_1 $\alpha\beta$-edges (apart from the n loops), n_2 $\alpha\alpha$-edges and n_2 $\beta\beta$-edges, with $n_1+2n_2 = n$. Then if J is the α-adjacency matrix of the unknot-graph:

$$P_J(\lambda) = (\lambda-2)^{n_2}(\lambda-1)^{n_1}\lambda^{n_2}, \quad n_2 > 0$$

So

$$\mathrm{Spec}(J) = \begin{bmatrix} 2 & 1 & 0 \\ n_2 & n_1 & n_2 \end{bmatrix}$$

Note:

(i) For this class, clearly $\mathrm{Spec}(J_\alpha) = \mathrm{Spec}(J_\beta)$.

(ii) It is striking that this result does not specialize to the previous one when $n_2 = 0$

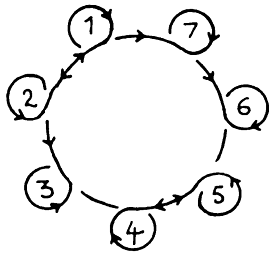
(iii) It is also striking that now all the roots are real, and all belong to the set $\{0,1,2\}$; whereas in the fully alternating case, all except $\lambda = 2$, and $\lambda = 0$, are complex, and all roots are distinct

A proof will not be given, but an example with n=7, n_1=3, n_2=2 will demonstrate the result.

$$|\lambda I - J| = \begin{vmatrix} \lambda-1 & -1 & & & & & -1 \\ -1 & \lambda-1 & -1 & & & & \\ & & \lambda-1 & & & & \\ & & & \lambda-1 & -1 & & \\ & & & -1 & \lambda-1 & & \\ & & & & & \lambda-1 & \\ & & & & & -1 & \lambda-1 \end{vmatrix}$$

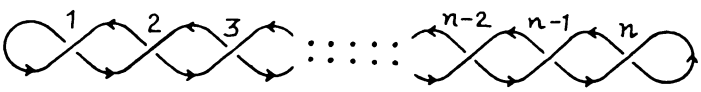
So $$P_J(\lambda) = [(\lambda-1)^2-1]^2(\lambda-1)^3$$
$$= (\lambda-2)^2(\lambda-1)^2\lambda^2,$$

i.e. $$\mathrm{Spec}(J) = \begin{bmatrix} 2 & 1 & 0 \\ 2 & 3 & 2 \end{bmatrix}$$



5.4.2 The unknot with n twists

The first examples of 5.2 (iv) were of the circular unknot with 0,1,2,3 and 4 twists in it. The general case is indicated as below:



The characteristic polynomial is given by

$$P_{J_n} = |\lambda I - J_n| = \begin{vmatrix} \lambda-1 & -1 & & & & & & \\ -1 & \lambda & -1 & & & & & \\ & -1 & \lambda & -1 & & & & \\ & & -1 & \lambda & -1 & & & \\ & & & \ddots & \ddots & \ddots & & \\ & & & & -1 & \lambda & -1 & \\ & & & & & -1 & \lambda & -1 \\ & & & & & & -1 & \lambda-1 \end{vmatrix}$$

Let us denote the $(n-1)\times(n-1)$ principal minor, shown dotted, by M_{n-1}; and then expand the whole determinant beginning with the top row.

This gives:

$$P_{J_n}(\lambda) = (\lambda-1)M_{n-1} - M_{n-2} \tag{5.7}$$

But expanding a determinant of type M_n we find that

$$M_n = \lambda M_{n-1} - M_{n-2} \tag{5.8}$$

Using this in (5.5) gives

$$P_{J_n}(\lambda) = M_n - M_{n-1} \tag{5.9}$$

We note that M_n retains its essential 'type' when $n\geq3$; for $n=1,2$ we must use

$$M_1 = \lambda-1 \text{ and } M_2 = \begin{vmatrix} \lambda & -1 \\ -1 & \lambda-1 \end{vmatrix} = \lambda^2-\lambda-1 \tag{5.10}$$

If we define $M_0\equiv1$, this together with 5.8, 5.9, 5.10 enable $P_{J_n}(\lambda)$ to be calculated easily for $n=1,2$,

By eliminating M_i from (5.8) and (5.9) we can obtain a single recurrence for $P_{J_n}(\lambda)$, namely:

$$P_{J_n}(\lambda) = (1+\lambda)(P_{J_{n-1}} - P_{J_{n-2}}) + P_{J_{n-3}}(\lambda) \tag{5.11}$$

This is correct for $n\geq4$; the polynomials for $n<4$ are as follows:

$$P_{J_1}(\lambda) = \lambda-2 \ ;$$

$$P_{J_2}(\lambda) = \lambda(\lambda-2) \ ;$$

$$P_{J_3}(\lambda) = (\lambda-2)(\lambda-1)(\lambda+1)$$

General Solution

The general solution of recurrence equation (5.11) is as follows (*MUIR*,1923)

$$P_J(\lambda) = \prod_{k=1}^{n} (\lambda + 2cos(k\pi/n)) \tag{5.12}$$

$$= (-1)^n (2-\lambda) U_{n-1}^{(2)}(-\lambda/2) \text{ . where} \tag{5.13}$$

$$U_n^{(2)}(x) = \frac{sin[(n+1)arccosx]}{\sqrt{1-x^2}} \tag{5.14}$$

the Chebyshev polynomial of the second kind.

Then

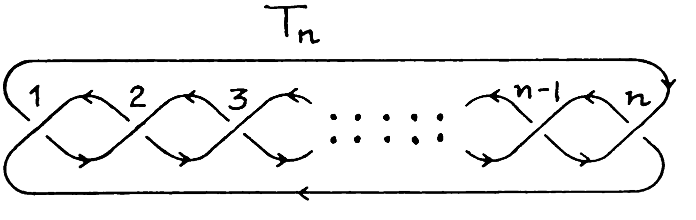
$$\text{Spec}(J) = \{-2cos\frac{k\pi}{n} \mid k=1, \cdots, n\} \tag{5.15}$$

Chebyshev polynomials occur again, in the solutions for other classes of knots. In 5.4.4 we will use a generating function method, to demonstrate how they arise.

It may be noted that all the eigenvalues for the knot-graphs in class 5.4.1 (i.e. for unknots which are a circle with n loops, all alternating) are complex (except $\lambda=2$; and $\lambda=0$, n even). Whereas for the class of this section, the unknots with n twists, all eigenvalues are real: this is true for all n, since the matrix J_n is symmetric for all n.

5.4.3 The torii knot-graphs, on n crossings

The torus knot-graph with n double points may be drawn as follows



In the notation of *CONWAY*,1970, it is denoted simply by n, and formed from the integral tangle $n = 1 + 1 + \cdots + 1$. We note that T_n is a 1-link when n is odd, and a 2-link when n is even. The characteristic polynomial of its α-adjacency matrix is obtained from :

$$P_{T_n}(\lambda) = \begin{vmatrix} \lambda-1 & -1 & & & -1 \\ -1 & \lambda & -1 & & \\ & -1 & \lambda & -1 & \\ & & \ddots & \ddots & \ddots \\ -1 & & & -1 & \lambda \end{vmatrix}$$

Note that the matrix is a circulant one. Expansions for this are well-known. We note that the determinant is the same as that required for a circuit graph C_n, a solution for which is given in (*CVETOVIĆ*, *p*. 72) as follows:

$$P_{T_n}(\lambda) = -2+\sum_{k=0}^{[n/2]}(-1)^k \frac{n}{n-k}\binom{n-k}{k}\lambda^{n-2k} \tag{5.16}$$

$$= 2[\, U_n^{(1)}\left[\frac{\lambda}{2}\right] - 1]\,, \tag{5.17}$$

where

$$U_n^{(1)}(x) = cos(n\ arccos\ x)\,, \tag{5.18}$$

the Chebyshev polynomial of the first kind

The spectrum is

$$\text{Spec}(T_n) = \{2\cos\frac{2\pi i}{n},\ i=1,2,\cdots,n\} \tag{5.19}$$

Examples 5.2(i) and 5.2(ii) show the cases $n=3$ and 4.

We have obtained the following recurrence equation for $P_{T_n}(\lambda)$:

Setting

$$X_n \equiv P_{T_n}(\mu) \equiv |\mu I + J|,$$

then

$$X_n = \mu X_{n-1} - X_{n-2} + (-1)^{n-1}2(2+\mu) \tag{5.20}$$

with

$$X_3 = \mu^3-3\mu+2 \text{ and } X_4 = \mu^4-4\mu^2.$$

Our method for obtaining this result was the same as that described in full, for another class of knots, in 5.4.4 below : so we shall not go through it here.

We may note that $\mu = -2$ is always a characteristic root of knot-graph matrices, since every column of $(\mu I + J)$ sums to $\mu+2$. We may factor out $(\mu+2)$ through the recurrence equation (5.20), leaving

$$g_n = \mu g_{n-1} - g_{n-2} + (-1)^{n-1}2 \tag{5.21}$$

with

$$g_3 = \mu^2-2\mu+1 \text{ and } g_4 = \mu^3-2\mu^2,$$

where g_n is the reduced polynomial of $P_{T_n}(\mu)$.

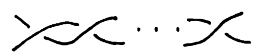
Having got $X_n(\mu)$ from 5.21, in the usual way we obtain

$$P_{T_n}(\lambda) = (-1)^n X_n(-\lambda)$$

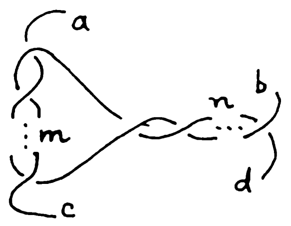
5.4.4 A knot class, using Conway's rational tangles

In this sub-section we obtain formulae for the general characteristic polynomial for a class of knots which can be defined in terms of Conway's notation (*CONWAY*, 1970) for combining integral tangles. In knot-theory members of this class which are formed from only two integral tangles are generally known as *twist-knots*.

Definitions (see (*CONWAY*, 1970) or chapter 6, appendix III, for details)

$m = 1 + 1 + \quad + 1$ ~ 

is an **integral tangle**

$(mn) = m\,0 + n$ ~ 

is a 2-part **rational tangle**

$((mn)p) = (mn)0 + p$ ~ produces a 3–part rational tangle .

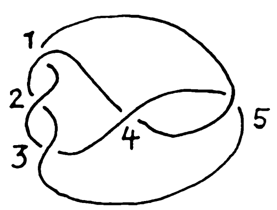
It is clear how this method of combining integral tangles may be extended indefinitely, to $((\cdots (mn)\, p \ldots s\ )\ t\ )$. It is customary to drop the brackets, and simply write $mnp \cdots st$ to denote the general rational tangle.

If the ends a,b are joined, and ends c,d are joined, a *rational knot* results

(i) **Characteristic polynomial of mn (the twist knots)**

Example

Knot 32



$$\left[\begin{array}{ccc|cc} \mu & 1 & & & 1 \\ 1 & \mu & 1 & & \\ & 1 & \mu & 1 & \\ \hline 1 & & & \mu & 1 \\ & & 1 & 1 & \mu \end{array}\right]$$

$(\mu I + J)$

$$P_J(\mu) = |\mu I + J|$$
$$= \mu^5 - 3\mu^3 + 2\mu^2 - \mu - 2$$

General Case

Knot mn

$$P_{mn}(\mu) = \left|\begin{array}{ccccc|cccc}
\mu & 1 & & & & & & & 1 \\
1 & \mu & 1 & & & & & & \\
 & 1 & \mu & 1 & & & & & \\
 & & \ddots & \ddots & 1 & & & & \\
 & & & 1 & \mu & 1 & & & \\
\hline
1 & & & & 1 & \mu & 1 & & \\
 & & & & & 1 & \mu & 1 & \\
 & & & & & & \ddots & \ddots & 1 \\
 & & & & 1 & & & 1 & \mu
\end{array}\right| \quad \text{(Diagram above)}$$

The following theorem gives the spectrum for this case.

Theorem 2

Let A_k be the $k \times k$ tridiagonal matrix having leading diagonal elements μ, and 1's on the diagonals above and below the leading diagonal, and zero's elsewhere. Then:

(1) $P_{mn}(\mu) = A_m A_n + 2[(-1)^n A_{m-1} + (-1)^m A_{n-1}] + A_{m-2}A_{n-2}$
(Then $P_{mn}(\lambda) = (-1)^{m+n} P_{mn}(-\lambda)$)

(2) The spectrum of the mn graph is $\{\lambda_i \mid i=1,\cdots,m+n\}$ where $\lambda_i = -2\cos\Theta_i$ and Θ_i is a root of the following equation:

$$(-1)^m \cos(m-1)\Theta + (-1)^n \cos(n-1)\Theta - \sin(m-1)\Theta \sin(n-1)\Theta$$
$$= (-1)^m \cos(m+1)\Theta + (-1)^n \cos(n+1)\Theta + \sin(m+1)\Theta \sin(n+1)\Theta,$$

with $\sin^2\Theta \neq 0$

Proof :

(1) Applying Laplacian expansion (*AITKEN*, *p*. 78) to the determinant $P_{mn}(\mu)$, taking the sum of the products of minors from the first m rows and their cofactors from the last n rows, we find that there are only six non-vanishing terms. Before attaching signs, these reduce to

$$A_m A_n, \quad (-1)^{n-1}A_{m-1}A_0, \quad (-1)^{m-1}A_0(-1)^{n-1}A_{n-1},$$
$$A_0 A_{n-1}, \quad (-1)^{m-1}A_{m-1}A_0, \quad \text{and} \quad (-1)^{m-1}A_{m-2}(-1)^{n-1}A_{n-2}$$

Attaching signs as required to turn minors into cofactors, then adding and collecting terms, we obtain the required result. ($A_0 \equiv 1$, $A_1 = \mu$, $A_2 = \mu^2 - 1$).

(2) We first use a generating function method to obtain expressions for A_k, thus:

$$A_k = \left|\begin{array}{ccccc}
\mu & 1 & & & \\
1 & \mu & 1 & & \\
 & 1 & \mu & \ddots & \\
 & & \ddots & \ddots & 1 \\
 & & & 1 & \mu
\end{array}\right| = \mu A_{k-1} - A_{k-2} \quad \text{(Expand by row 1)} \qquad (5.22)$$

Thus the A_k satisfy the recurrence relation

$$A_{k+2} = \mu A_{k+1} - A_k, \quad \begin{cases} k=0,1,\cdots \\ A_0 \equiv 1, \ \ A_1=\mu \end{cases} \tag{5.23}$$

Let $Z(t) \equiv \sum_{k=0}^{\infty} A_k t^k$ be a generating function for A_k.

Multiplying 5.23 by t^{k+2} and summing gives:

$$\sum_{k=0}^{\infty} A_{k+2}t^{k+2} = \mu \sum_{k=0}^{\infty} A_{k+1}t^{k+2} - \sum_{k=0}^{\infty} A_k t^{k+2}$$

That is,

$$(Z(t)-1-\mu t) = \mu t(Z(t)-1)-t^2 Z(t)$$

from which

$$Z(t) = [1-t(\mu-t)]^{-1} \tag{5.24}$$

For any given value of μ, a value for t can be chosen so that $0<t<\mu$, and $t(\mu-t)<1$, and the right hand side of 5.24 can be expanded as the power series

$$Z(t) = \sum_{i=0}^{\infty} t^i \mu^i (1-t/\mu)^i$$

Expanding the binomial, then picking the coefficient of t^r from the double sum, we obtain

$$\begin{aligned} A_r &= \sum_{j=0}^{[n/2]} \mu^{r-2j}(-1)^j \binom{r-j}{j} \qquad (5.25) \\ &= U_n^{(2)}(\mu/2) \quad \text{(Chebyshev polynomial, 2nd kind)} \\ &= \frac{sin[(r+1)cos^{-1}(\mu/2)]}{\sqrt{1-(\mu/2)^2}} \quad \text{(see (5.13))} \end{aligned}$$

Using this result in (1) above (first part of theorem) we get, with $\mu = 2cos\,\Theta$:

$$A_r = \frac{sin(r+1)\Theta}{\sin\Theta},$$

then

$$\begin{aligned} \sin^2\Theta P_{mn}(\mu) &= sin(m+1)\Theta sin(n+1)\Theta + 2[(-1)^n sinm\,\Theta\sin\Theta \\ &+ (-1)^m \sin n\,\Theta\sin\Theta] + sin(m-1)\Theta\sin(n-1)\Theta \end{aligned} \tag{5.26}$$

Setting $P=(m+1)\Theta$, $Q=(n+1)\Theta$, $R=(m-1)\Theta$, $S=(n-1)\Theta$, we see that when m,n are both odd, and $\sin^2\Theta \neq 0$, $P_{mn}(\mu)=0$ when

$$2\sin m\,\Theta\sin\Theta + 2\sin n\,\Theta\sin\Theta = sinPsinQ + sinRsinS$$

And using the identity $2sinAsinB \equiv cos(A-B) - cos(A+B)$, the left-hand side becomes: $cosR - cosP + cosS - cosQ$.

Rearrangement of these terms then gives

$$cosR + cosS - sinRsinS = cosP + cosQ + sinPsinQ \quad , \tag{5.27}$$

which is the desired result in the case that both m, n are odd.

There are three more cases, depending on the respective parities of m, n. The corresponding equations for these are:

m **odd,** n **even**

$$-cosR + cosS - sinRsinS = -cosP + cosQ + sinPsinQ$$

m **even,** n **odd**

$$cosR - cosS - sinRsinS = cosP - cosQ + sinPsinQ$$

m, n **both even**

$$-cosR + cosS + sinRsinS = cosP + cosQ - sinPsinQ$$

We may express all four cases together, thus :-

$$(-1)^m cosR + (-1)^n cosS - sinRsinS$$
$$= (-1)^m cosP + (-1)^n cosQ + sinPsinQ \tag{5.28}$$

which is the desired result

(ii) **A subclass of mn knots**

If special cases of (m, n) are taken, spectra for subclasses of the rational knots mn may be determined. One example will suffice to illustrate this.

Let m, n be both odd and $m = n-2$. This defines the class $\{53, 75, 97, \cdots\}$ of mn knots. Applying 5.28 with

$$P = (m+1)\Theta = (n-1)\Theta = S, \quad Q = (m-1)\Theta = (n-3)\Theta$$
$$R = (n+1)\Theta, \; S = (n-1)\Theta,$$

we obtain:

$$cos(n+1)\Theta + cos(n-1)\Theta - sin(n+1)\Theta sin(n-1)\Theta$$
$$= cos(n-1)\Theta + cos(n-3)\Theta + sin(n-1)\Theta sin(n-3)\Theta$$

Then

$$cos(n+1)\Theta - cos(n-3)\Theta = sin(n-1)\Theta[sin(n+1)\Theta + sin(n-3)\Theta] \quad ,$$
$$2sin(n-1)\Theta sin(-2\Theta) = sin(n-1)\Theta[2sin(n-1\Theta cos 2\Theta]$$

The spectrum is therefore given by $m+n = 2(n-1)$ values of $\lambda_i = -2cos\,\Theta_i$, where

$$\begin{aligned} \text{either} \quad sin(n-1)\Theta &= 0 \\ \text{or} \quad sin(n-1)\Theta &= -tan 2\Theta \\ \text{with} \quad \sin^2\Theta &\neq 0 \end{aligned}$$

(iii) **The 3-part rational knots, *mnp***

We may extend the results of (i) to 3-part rational knots, using the same methods as there.

With the same notation as used in (i), we may obtain the following equation for $P_{mnp}(\mu) = |\mu I + J|$, again using Laplacian expansion on the first m rows

$$\begin{aligned} P_{mnp}(\mu) &= A_m A_n A_p + [(-1)^{p+n} A_m A_{n-1} + (-1)^m A_p A_{n-1}] \\ &+ 2(-1)^{n-1} A_{m-1} A_{p-1} + [(-1)^{m+p} A_{n-1} A_{p-2} + (-1)^p A_{n-1} A_{m-2}] \\ &+ (-1)^{m+n-1}[A_n + (-1)^{p-1} A_{n-2}] - A_{m-2} A_{n-2} A_{p-2} \end{aligned} \tag{5.29}$$

It would appear to be possible, with a good deal of perseverance, to continue this extension to obtain expressions for $P_{mnpr}(\mu)$ and beyond in terms of A_i; perhaps a general expansion is obtainable. We have not attempted this; but we have displayed the matrix $[\mu I + J]$ in general form, and obtained recurrence equations for the case $m = n = p = \ldots = 2$. A brief description of this work now follows.

(iv) **The general case; and the case 222...**

The matrix $[\mu I + J]$ in the general case $mnpq \cdots$ can be written in block form thus (all elements not shown are zeros):

$$\begin{bmatrix} A_m & S & & & & Q \\ P & A_n & S & & & \\ Q & 0 & A_p & S & & \\ & Q & 0 & A_q & \ddots & \\ & & \ddots & \ddots & \ddots & S \\ & & & Q & R & A \end{bmatrix},$$

where A_i is an $i \times i$ matrix of type

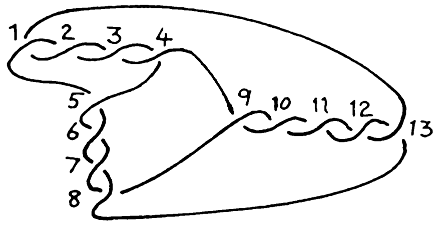
$$\begin{bmatrix} \mu & 1 & & & \\ 1 & \mu & 1 & & \\ & 1 & \mu & 1 & \\ & & \ddots & \ddots & 1 \\ & & & 1 & \mu \end{bmatrix},$$

and P, Q, R, S are rectangular matrices all of whose elements are zeros except one 1 in a corner thus

$$P = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ - & - & - & - \\ 0 & 0 & \cdots & 0 \end{bmatrix}, \quad Q = \begin{bmatrix} 0 & 0 & \cdots & 1 \\ 0 & 0 & \cdots & 0 \\ - & - & - & - \\ 0 & 0 & \cdots & 0 \end{bmatrix}, \quad R = \begin{bmatrix} 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ - & - & - & - \\ 0 & 0 & \cdots & 1 \end{bmatrix}, \quad S = \begin{bmatrix} 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ - & - & - & - \\ 1 & 0 & \cdots & 0 \end{bmatrix}$$

Of course, in order for this matrix to occur, the labelling of crossings must be a continuation of that used in (i) for the case mn. Rather than give a precise definition of this labelling we present an example of a 3-part rational knot which should suffice to make the labelling method clear. $P_{mnp}(\mu) = |\mu I + J|$ does not depend upon the labelling method

Example: Knot 445



Finally, we give a brief description of recurrence equations which may be used to obtain $P(\mu)$ for the cases 22, 222, 222, etc. These were more fully explained in (*TURNER et al*, 1979). The first step is to apply Laplacian expansion by the top two rows of the transpose of the general matrix shown above, with $m=n=p=\ldots=2$, $|A_2| = \mu^2 - 1$; note that $P=P', Q'=S, R'=R, S'=Q$.

Denoting $|\mu I+J|$ by C, and its cofactors by C_{ij}, the expansion results in the following:

$$C = |\mu I+J| = \mu^2 C_{12} - \mu(C_{15} + C_{23})+(C_{13}+C_{25}-C_{12}-C_{35}). \tag{5.30}$$

Each of the six cofactors may be obtained by a recurrence relation. We show how to obtain the first and then list the remaining five.

(1)

$$C'_{12} = \begin{vmatrix} A_2 & & S & & & \\ Q & A_2 & & S & & \\ & Q & A_2 & & \ddots & \\ & & \ddots & \ddots & & S \\ & & & Q & \ddots & R \\ & & & & Q & A_2 \end{vmatrix}$$

Let us denote this determinant by E_m, where m is the number of rows in it.

Expanding in terms of the top two rows we get:

$$E_m = (\mu^2 - 1)E_{m-2} - \mu C'_{15} \ , \tag{5.31}$$

where C'_{15} is the cofactor of C_{12} with the form -

$$C'_{15} = \left|\begin{array}{cccc|cccc} 1 & \mu & 1 & 0 & 0 & 0 & 0 & \\ 0 & 1 & \mu & 0 & 1 & 0 & 0 & (0\,s) \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 & \\ & & & \mu & 0 & 0 & 1 & \\ \hline & & & 1 & & & & \\ & & & 0 & & E_{m-4} & & \\ & & & & & & & \end{array}\right|$$

$$= \left|\begin{array}{c|cccc} \mu & 0 & 0 & 1 & 0 \\ \hline 1 & & E_{m-4} & & \\ 0 & & & & \end{array}\right|$$

which by further expansion reduces to

$$C'_{15} = \mu E_{m-4} - \left| \begin{array}{c|cccccc} \mu & 0 & 0 & 1 & 0 & & \\ \hline 1 & & E_{m-8} & & & & \\ 0 & & & & & & \end{array} \right|$$

Thus, if we denote by F_m the determinant

$$\left| \begin{array}{c|cccccc} \mu & 0 & 0 & 1 & 0 & \cdots & \\ \hline 1 & & E_m & & & & \\ 0 & & & & & & \end{array} \right| \tag{5.32}$$

we have

$$F_m = \mu E_{m-4} - F_{m-8} \tag{5.33}$$

Collecting (5.31), (5.32) and (5.33) together we see that

$$C_{12} = E_m = (\mu - 1)E_{m-2} - \mu F_{m-4} \tag{5.34}$$

$$\text{and } F_m = \mu E_{m-4} - F_{m-8}$$

After computing F_m and E_m for the smallest necessary cases, we can use (5.34) to compute C_{12} for increasing cases of the 2222 · knot-graph.

Equations for the other cofactors in (5.30) are as follows.

(2)

$$C_{13} = \left| \begin{array}{cc|cccccc} 1 & 1 & 0 & 0 & 1 & 0 & & \\ 0 & \mu & 0 & 0 & 1 & 0 & \cdots & \\ \hline 0 & 1 & & E_{m-2} & & & & \\ & 0 & & & & & & \end{array} \right| = F_{m-2} \text{ ,}$$

(3) $C_{15} = F_{m-4}$

(4)

$$C_{23} = K_{m-2}, \text{ where } K_m = \left| \begin{array}{cccc|cc} 0 & 0 & 1 & \cdots & 0 & 0 \\ \hline & & & & \vdots & \vdots \\ & & & & 0 & 0 \\ & & H_m & & 1 & 0 \\ & & & & 0 & 0 \\ & & & & 0 & 1 \\ \hline 0 & 0 & & 0 \quad 1 & \mu & 1 \end{array} \right|$$

$$\text{with } H_m = \left| \begin{array}{cccccc} A_2 & & S & & & \\ Q & A_2 & & S & & \\ & Q & A_2 & & & \\ & & Q & A_2 & & \end{array} \right| \text{, and } K_m = (\mu^2-1)\mu K_{m-2} + \mu K_{m-4}$$

(5) $C_{25} = (2\mu^2-1)K_{m-4} + \mu^2 K_{m-6}$

(6) $C_{35} = L_m = \mu K_{m-2} - L_{m-2}$, where

$$L_m = \left|\begin{array}{cc|cccccc|cc} \mu & 0 & 1 & 0 & & & & & 0 & 0 \\ 1 & 1 & 0 & 0 & & & & & & \\ 0 & \mu & 0 & 0 & 1 & 0 & \cdots & & & \\ \hline 0 & 1 & & & & & & & 1 & 0 \\ & 0 & & & H_{m-2} & & & & 0 & 0 \\ & & & & & & & & 0 & 1 \\ \hline 0 & 0 & 0 & 0 & & & 0 & 1 & \mu & 1 \end{array}\right|$$

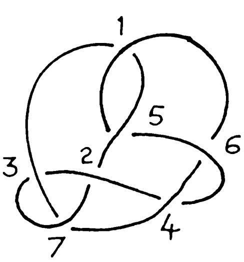
5.5 A GENERAL ALGORITHM FOR THE CHARACTERISTIC POLYNOMIAL OF A PRIME ALTERNATING KNOT

In the preceding section we have given methods and formulae for the characteristic polynomial of a number of special knot-graphs and their classes. We now turn to a description of a general algorithm which will provide the coefficients of the characteristic polynomial of any alternating knot-graph. We have not programmed the algorithm for computer use, and indeed we do not spell out all the details which would have to be dealt with if we did. We give the algorithm here because it provides insight into the way in which the polynomial is related to the structure of the adjacency matrix, and provides us with means for proving theorems about the polynomial coefficients. In particular it shows how the coefficients can be obtained directly from U and P, the two permutations whose interaction provides a definition of the knot projection (3.1(2)).

We will first give a detailed example of the algorithm in action, on knot 7_6, and follow this with a flow diagram of the algorithm. Finally we will give some general results about characteristic polynomial coefficients for alternating knot-graphs.

5.5.1 The algorithm in action

Example: Knot 7_6



The α-adjacency matrix is

$$J = U + PU ,$$

where U,P are the permutation matrices whose cyclic forms are

$$U = (1374256) , \text{ and } P = (1234567)$$

The characteristic polynomial (μ-form) is

$$\begin{aligned} P_J(\mu) = |\mu I + J| &= |\mu I + U + PU| \\ &= |\mu U' + I + P|\,|U'| \\ &= |M + C|\,|U'| \end{aligned}$$

where M is the matrix obtained by replacing all 1s in U' by μ, and $C = I + P$. Since $|U'|$ is ± 1, we can therefore obtain the characteristic polynomial by expanding $|M + C|$ and then multiplying by +1 or -1 according to the parity of U'.

Now a determinant of $A = [a_{ij}]$ is defined to be

$$|A| = \sum \pm a_{1\alpha} a_{2\beta} \qquad a_{n\gamma},$$

the summation, of $n!$ terms, being extended over all permutations $(\alpha\beta \cdots \gamma)$ of the second suffices of the elements a_{ij}, with the sign + or - being affixed according as a permutation is even or odd.

In the case of $|M + I + P|$, then, the only non-vanishing terms of the sum are those which are $\pm a_{1\alpha} a_{2\beta} \cdots a_{n\gamma}$ with no $a_{i\varepsilon} = 0$, and every $a_{i\varepsilon}$ being either a 1 drawn from I or from P, or a μ drawn from M. Such a term, including s μ-elements say, will have value $\pm\mu^s$, and thus will contribute ± 1 to the coefficient of μ^s in the polynomial $P_J(\mu)$.

The algorithm proceeds by finding directly how many permutations can be made up from the two permutations U' and P; such a permutation will contain s μ-elements, with s ranging from 0 to n. If $s<n$, the remaining elements ($n-s$ of them) will consist of r elements from P, and the remaining $(n-s-r)$ will be diagonal 1s from I.

To illustrate, we give below the determinant $|M + I + P|$ for knot 7_6 ; in it we have inserted the symbols a,b,c,d,e,f,g in the places where μ occurs. This allows us to keep track of different permutations involving different combinations of the μs.

$$\begin{vmatrix} 1 & 1 & 0 & 0 & 0 & a & 0 \\ 0 & 1 & 1 & b & 0 & 0 & 0 \\ c & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & d \\ 0 & e & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & f & 1 & 1 \\ 1 & 0 & g & 0 & 0 & 0 & 1 \end{vmatrix}$$

N.B. $U' = (1652473)$, hence a is in position (1,6), b is in position (2,4), and so on.

We next list, for $s = 0,\ldots,7$, all possible 'mixtures' of permutations made from U' and P. We mark with an asterisk over an integer those elements drawn from U'. Explanations on how the mixtures are discovered will follow the table.

No. of U' elements involved		Permutation	Parity	Contribution to $\vert U' \vert P_J(\mu)$
s = 0	-	I	+	+1
		P = (1234567)	+	+1
s = 1	a	(1*67)	+	
	b	(2*45671)	-	
	c	(3*12)	+	
	d	(4*7123)	+	
	e	(5*234)	-	
	f	(6*5)	-	
	g	(7*3456)	+	$+1\mu$
s = 2	bd	(2*4*71)	-	
	be	(2*45*)	+	
	dg	(4*7*3)	+	
	a,e	(1*67)(5*234)	+.-	
	c,f	(3*12)(6*5)	+.-	
	d,f	(4*7123)(56*)	+.-	$-2\mu^2$
s = 3	agc	(1*67*3*)	-	
	bgc	(2*4567*3*1)	+	
	a,be	(1*67)(2*45*)	+.+	
	f,bd	(6*5)(2*4*71)	-.-	
	f,dg	(6*5)(4*7*3)	-.+	$+1\mu^3$
s = 4	afec	(1*6*5*23*)	+	
	afed	(1*6*5*234*7)	+	
	bcdg	(2*4*7*3*1)	+	$+3\mu^4$
s = 5	afebd	(1*6*5*2*4*7)	-	
	agc,be	(1*67*3*)(2*45*)	-	
	bdgc,f	(2*4*7*3*1)(56*)	-	$-3\mu^5$
s = 6	-	-		$0\mu^6$
s = 7	abcdefg	U' =(1*6*5*2*4*7*3*)	+	$+1\mu^7$

And so, collecting the terms,

$$P_J(\mu) = \mu^7-3\mu^5+3\mu^4+\mu^3-2\mu^2+\mu+2.$$

Therefore, finally,

$$P_J(\lambda) = (-1)^7 P_J(-\lambda)$$
$$= \lambda^7-3\lambda^5-3\lambda^4+\lambda^3+2\lambda^2+\lambda-2,$$

which is the characteristic polynomial of J_α for knot 7_6.

Explanations

(i) There are no 'mixtures' of I and P, since P is a single cycle on the integers $1,2,\ldots,n$.

(ii) For s = 1, each element of U' determines a mixture, using it (the element) together with $r \leq n-1$ elements of P.

Suppose $U' = \{ \binom{i}{\lambda_i} \}$. Then the mixed permutation associated with i is $(i^*, \lambda_i, \lambda_i+1, \lambda_i+2, \ldots, i-1)$, where $\binom{\lambda_i}{\lambda_i+1}, \binom{\lambda_i+1}{\lambda_i+2}, \quad , \binom{i-1}{i}$ are all elements of P.

(iii) The mixture-permutations are found in the following order.

Stage A

First , all single cycles which begin with 1^* are discovered. Thus for 7_6 we first obtain (1^*67), then $(1^*67^*3^*)$,then $(1^*6^*5^*23^*)$, then $(1^*6^*5^*234^*7)$.

Briefly, the technique for finding these cycles is as follows: (1^*67) is automatic, since after the step $(1^*6\ldots$ we continue using elements $\binom{i}{i+1}$ from P until we complete the cycle; thus $(1^*671$ gives us (1^*67) . Then for $s = 2$ (i.e. to include two elements of U') we try $(1^*6^*56\ldots$, which is seen to fail since we repeat the 6 before returning to 1. So we must try $(1^*67^*3456\ldots$, which fails because 6 is repeated before 1 is. Proceeding in this way we find there are no cycles which start with 1^* and include just two elements of U'. Putting $s = 3$, we begin again with $(1^*6^*5^*2345\ldots$ and reject this. Then we examine $(1^*6^*56\ldots$ and reject it. And so the cycle cannot begin with 1^*6^*. We next try $(1^*67^*3^*)$, and accept this. Then we try, and reject, $(1^*67^*34^*7\ldots$ And so on.

It is clear that all these single cycles contribute to $|M|$, since they combine with elements $\binom{i}{i}$ from I, to make a full permutation on 7 integers. Thus (1^*67) becomes $(1^*67)(2)(3)(4)(5)$, which is a permutation of even parity, and $s = 1$, so its contribution to $|M|$ is $+\mu$.

Second , all single cycles which begin with 2^* and do not include 1^* are found, using the techniques described above.

Third , all single cycles which begin with 3^* and do not include 1^* or 2^* are found.

And so on. Clearly the amount of searching becomes rapidly less as s increases. There are no solutions possible when $s = n-1$ or n.

Stage B

Referring back to the table, we see that at the end of Stage A the permutations for $s = 0$ and $s = 1$ have all been found; but for $s = 2,3,\ldots$ only the single cycles are known.

Stage B consists in combining non-intersecting single cycles, to build up all possible cases for $s = 2,3,\ldots$ Thus all $s = 1$ cycles must be taken in pairs; if a pair is non-intersecting, it constitutes a permutation for the case $s = 2$; i.e. it has two asterisked elements in total. For example, (3^*12) and (6^*5) combine to make the permutation $(123^*)(4)(56^*)(7)$, which contributes $-\mu^2$ to $|M|$. An example for $s = 3$ is $(1^*67)(2^*45^*)(3)$, contributing $+\mu^3$ to $|M|$.

Stage C

This consists simply in determining the parity of all the permutations in the table, and then totalling their contributions to $|M|$.

Finally we multiply by $|U'|$, and the μ-characteristic polynomial is obtained.

5.5.2 Summary flow diagram

START

↓

Determine $J, U, U', |U'|$. Elements $\binom{i}{\lambda_i}$ of U' appearing in a cycle will be marked with a * superscript over the i .

↓

Stage A Discover all single cycles beginning with i^*. Discover all single cycles beginning with i^* and not using $1^*, 2^*, \ldots, (i^*-1)$, for $i = 2, \ldots, n$.

↓

Stage B Discover all possible combinations of the single cycles found in Stage A, which do not intersect and which have s asterisked integers, for $s = 2, \ldots, n-2$.

↓

Stage C Determine the parity of all single and multiple cycles (with any necessary elements $\binom{i}{i}$ inserted from I to complete the n-permutations). Sum all the contributions to $|M|$. Multiply by $|U'|$.

5.5.3 Comments and propositions

The algorithm given above showed that the coefficients of the characteristic polynomial of an alternating knot-graph can be determined directly from examination of the interactions between its α-step permutation $U' = \{\binom{i}{\lambda_i}\}$ and the permutation $P = \{\binom{i}{i+1}\}$

Comments (re the characteristic polynomial $\sum_{i=0}^{n} a_i \lambda^{n-i}$)

(1) In view of theorem 1, any comment made about the coefficients a_r of the characteristic polynomial of an oriented knot-graph can be interpreted in terms of directed cycles on subgraphs defined by r-subsets of the vertices.

It is evident from the sparseness of the matrix $M+I+P$ that we can expect the coefficients, and hence the numbers of directed cycles in the graph, to be generally quite small. In section 5.6 there is given a table of the characteristic polynomials for all prime alternating knots for $n=3,4,\ldots 8$. A glance at this will tell that $|a_r|$ rarely rises above n, and indeed it is usually less than $\frac{n}{2}$.

(2) The constant term a_n is given by $|U'|\ |I+P| = |U'|\ (1+|P|)$; and is therefore 0 when n is even (since then $|P| = -1$) and ± 2 when n is odd.

(3) $a_0 = 1$, arising from $|M| = \pm\mu^n$

(4) $a_1 = 0$; this is always true, because a permutation cannot be constructed from $n-1$ of the nonzero elements of M together with a 1 from either of I or P. The n^{th} element which completes the permutation must be the one remaining μ from M; this leads to a term μ^n, and never allows a term μ^{n-1} to occur. ($a_1 = 0$ also follows, of course, from the fact that there are no loops, so the trace of J is zero)

(5) a_2 is always negative. This follows immediately from proposition 5.3(6), namely that $a_2 = -(\#$ of 2-gons$)$. It leads to the suggestion that there is an interesting dualism between # of 2-gons and # of mixed (U',P) permutations with $s = n-2$ elements of U', and further, that all such mixed permutations are of odd parity. Similar remarks can be made about a_3, also negative by 5.3(7), and its relationship with the # of mixed (U',P) permutations having $s = n-3$.

(6) Because of the nature of $I+P$, in relation to $U' = \{ \binom{i}{\lambda_i} \}$, it is possible to make certain general statements about the coefficient of μ, namely a_{n-1}. Thus:

Proposition (i)

Let $(\, i^{*}\lambda_i \ldots)$ be the single cycle, $s=1$, determined by element $\binom{i}{\lambda_i}$ (see Stage A, first step of the algorithm). Then the contribution of this cycle to the coefficient a_{n-1} is given by

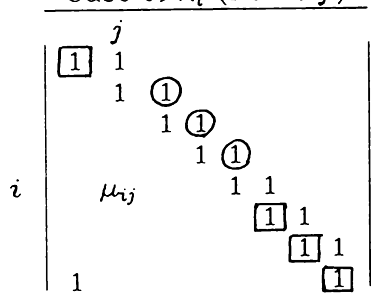
$$(-1)^{i-\lambda_i} \quad \text{if } i>\lambda_i$$
$$\text{and} \quad (-1)^{\lambda_i - i - 1} \quad \text{if } i<\lambda_i$$

(NB. for a knot-graph $\lambda_i \neq i$ or $i+1$)

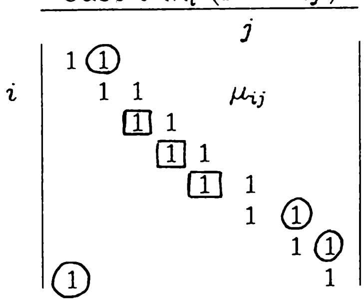
Proof:

Below are diagrams for the two cases, showing how a single element μ_{ij} determines which elements of I and P must be taken with it. Note that μ_{ij} corresponds to element $\binom{i}{\lambda_i}$ of permutation U'.

Case $i>\lambda_i$ (i.e. $i>j$)



Case $i<\lambda_i$ (i.e. $i<j$)



The circled 1s are taken from P, whereas the squared 1s are taken from I. The mixed permutations obtained are:

$$i>j \qquad\qquad i<j$$

$$\begin{bmatrix} 1 & 2 & 3 & .. & j & j+1 & .. & i & i+1 & .. & n \\ 1 & 2 & 3 & .. & j+1 & j+2 & .. & j & i+1 & .. & n \end{bmatrix} \quad \begin{bmatrix} 1 & 2 & 3 & .. & i-1 & i & i+1 & .. & j-1 & j & .. & n \\ 2 & 3 & 4 & .. & i & j & i+1 & .. & j-1 & j+1 & .. & 1 \end{bmatrix}$$

Parities: $(-1)^{i-j}$ $\qquad$ $(-1)^{j-i-1}$

Hence proof.

Proposition (ii)

The coefficient a_{n-1} is odd when n is odd,

and a_{n-1} is even when n is even.

Proof:

Let n_a = no. of cases in U' such that $i>j$
and n_b = no. of cases in U' such that $i<j$

Let n_a^+ = no. of cases such that $(i-j)$ is even,
n_a^- = no. of cases such that $(i-j)$ is odd,
n_b^+ = no. of cases such that $(j-i-1)$ is even,
n_b^- = no. of cases such that $(j-i-1)$ is odd,

Then in $P_J(\mu)$, the coefficient of μ is a_{n-1} and $a_{n-1} = (n_a^+ + n_b^+) - (n_a^- + n_b^-)$.
Suppose a_{n-1} is even, when n is odd. Then we have

$$(n_a^+ + n_b^+) - (n_a^- + n_b^-) = a_{n-1}, \text{ even}$$

and $$(n_a^+ + n_b^+) + (n_a^- + n_b^-) = n, \text{ odd.}$$

Subtracting: $$-2(n_a^- + n_b^-) = a_{n-1} - n$$

i.e. even = odd

hence contradiction, so a_{n-1} is odd.

Similarly we can show that a_{n-1} is even when n is.

Further comments on a_{n-1}

(i) The conjecture that $|a_{n-1}| < n$ (it can only be $= n$ if all the single cycles with $s=1$ have the same parity, which seems unlikely) is false. Counter-examples are knots 4_1, 7_1, 8_1, 8_3, etc.

(ii) When $n=8$, $|a_7|$ has the following distribution of values in the 18 knot-graphs:

$\lvert a_7 \rvert$	: 0	4	8
frequency	: 7	8	3

(see the table, in 5.6)

The reason why they are all multiples of 4 is not yet clear.

(7) **Comment on topological invariance**

Since many knots give rise to several knot-graphs which are non-isomorphic, it is clear that the characteristic polynomial of an adjacency matrix is not a topological knot-invariant.

5.6 THE CHARACTERISTIC POLYNOMIALS OF KNOT-GRAPHS FROM THE ALEXANDER AND BRIGGS TABLES ($n = 3, \cdots, 8$)

We have prepared computer programs to compute knot-graph characteristic polynomials from their adjacency matrices. The tables below give the polynomial coefficients for the first 32 prime alternating knots, i.e. for $n = 3, \cdots, 8$. The knot-graphs from which they were derived are those drawn in *ROLFSEN*, 1976. The numbering of the knots is as given in the *ALEXANDER* and *BRIGGS* (1927) listing.

Characteristic Polynomials: $a_0\lambda^n + a_1\lambda^{n-1} + \cdots + a_{n-1}\lambda + a_n$

$n = 3, \cdots, 7$ **Coefficients**						
Knot No.	$a_0 = 1$ and $a_1 = 0$ *always*					
	a_2	a_3	a_4	a_5	a_6	a_7
3_1	-3	-2				
4_1	-2	-4	0			
5_1	-5	0	5	-2		
5_2	-3	-2	-1	+2		
6_1	-4	-2	3	2	0	
6_2	-3	-3	1	2	0	
6_3	-2	-4	-1	2	0	
7_1	-7	0	14	0	-7	-2
7_2	-5	-2	6	6	-3	-2
7_3	-5	0	4	-2	3	2
7_4	-4	-2	2	4	1	-2
7_5	-4	-1	2	0	-1	2
7_6	-3	-3	1	2	1	-2
7_7	-2	-5	0	5	-1	-2

$n=8$ Coefficients							
Knot No.	$a_0=1$ and $a_1=0$ *always*						
	a_2	a_3	a_4	a_5	a_6	a_7	a_8
8_1	-6	-2	10	8	-4	-8	0
8_2	-5	-3	7	9	-4	-4	0
8_3	-6	0	10	-4	-4	8	0
8_4	-5	-2	6	5	0	-4	0
8_5	-5	-2	8	2	-4	0	0
8_6	-5	-1	7	-1	-4	4	0
8_7	-4	-3	3	6	0	0	0
8_8	-4	-3	4	5	0	-4	0
8_9	-4	-2	2	2	4	0	0
8_{10}	-4	-2	3	2	0	0	0
8_{11}	-4	-2	3	2	0	0	0
8_{12}	-4	-2	4	2	-4	0	0
8_{13}	-3	-4	0	9	0	-4	0
8_{14}	-3	-4	2	5	0	-4	0
8_{15}	-3	-3	0	5	0	-4	0
8_{16}	-2	-5	-1	5	4	-4	0
8_{17}	-2	-4	-3	6	0	0	0
8_{18}	0	-8	-2	0	12	-8	0

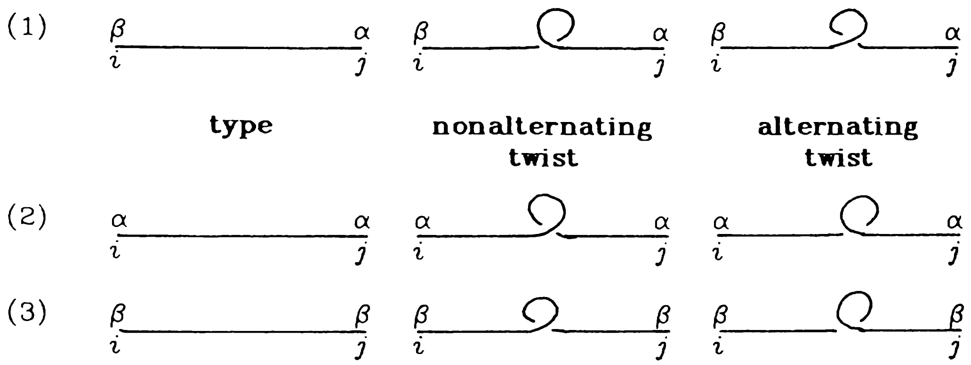
5.7 SOME OPERATIONS ON KNOT-GRAPHS: EFFECTS ON SPECTRA

In this section we study a number of operations on knot-graphs. The first is the elementary *Alexander move* which consists of twisting a loop into an edge; this creates a new crossing in the knot-graph, and of course the characteristic poynomial is changed: we investigate this change. The second is an operation which, by cutting and rejoining two edges after crossing them, creates a new crossing-point and a new 2-gon. The third is to compose two knots, by tying them one after the other in the same piece of string; we shall investigate different ways of doing this, and the different characteristic polynomials that result.

5.1.1 Twisting a loop into an edge

Let G be a knot-graph (it may be nonalternating) and let J be its α-adjacency matrix, with $P_J(\mu) \equiv |\mu I + J|$ its μ-characteristic poynomial as usual.

Then an edge (i,j) may be any of three types, viz. $\beta\alpha$ (or $\alpha\beta, \alpha\alpha$, or $\beta\beta$: and a twist may create a loop in an edge in one of two ways. The following diagrams show all the possible combinations:



For case (1) we shall obtain equations for the μ-characteristic polynomial after inserting the loop in terms of μ and the original poynomial. For simplicity we shall write X' and X for the two polynomials (after and before twisting); and use subscripts na and a to mean nonalternating and alternating respectively.

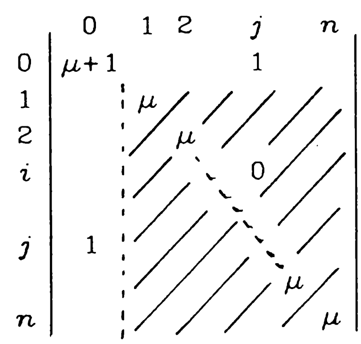
For cases (2) and (3) we shall just state results.

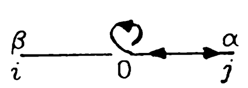
Case (1)

Nonalternating Twist

Without loss of generality, we may label the new crossing vertex 0, and then the $(n+1)$-square determinant to be evaluated will be as follows:

$X_{na}' \;=\; |\mu I + J'| \;=$





The $n \times n$ shaded matrix is the same as for χ *except that* element ij is 0 instead of 1.

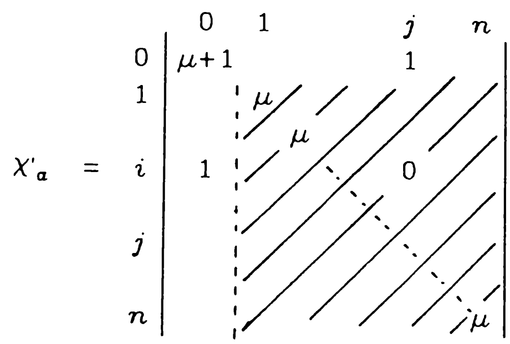
Expanding the determinant simultaneously by first row and column (*AITKEN*, *p.* 74, *Cauchy expansion*) we obtain

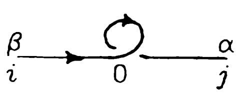
$$\chi'_{na} = (\mu+1)(\chi-\chi_{ij}) - \chi_{jj} \qquad (5.35)$$

where χ_{ij} and χ_{jj} are cofactors of elements ij, jj respectively of χ : we have used $\chi_{11}=\chi-\chi_{ij}$.

Alternating twist

Working as above, but for an alternating twist we find:





Shaded matrix as for χ, but with element $ij=0$ instead of 1.
Expansion by row 0 and column 0 gives:

$$\begin{aligned}\chi'_a &= (\mu+1)(\chi-\chi_{ij})-\chi_{ij} \\ &= (\mu+1)\chi-(\mu+2)\chi_{ij}\end{aligned} \qquad (5.36)$$

Corollaries

(i)
$$\frac{\chi'_{na}+\chi'_a}{2} = (\mu+1)(\chi-\chi_{ij}) - \frac{\chi_{ij}+\chi_{jj}}{2} \qquad (5.37)$$

(ii)
$$\chi'_{na}-\chi'_a = \chi_{ij}-\chi_{jj} \qquad (5.38)$$

(iii) Since every knot-graph adjacency matrix, whether alternating or not, has every column sum equal to 2, every characteristic polynomial has a root $\mu = -2$. Therefore, $\chi'_{na}(-2) = 0 = \chi'_a(-2)$, and by 5.38 :

$$\chi_{ij}(-2) = \chi_{jj}(-2) \quad \text{for any } ij \text{ such that } i \to j \text{ is a } \beta\alpha\text{-edge.} \qquad (5.39)$$

(iv) A knot with a twist in its ij-edge has a characteristic root $\lambda = 1$ if and only if
   (a) $\chi_{jj}(-1) = 0$ if the knot is nonalternating
   (b) $\chi_{ij}(-1) = 0$ if the knot is alternating

   Proof: follows from (5.35) and (5.36), inserting $\mu = -\lambda = -1$

(v) For any alternating knot-graph, $\chi_{ii}(-2)$ = constant for $i=1,2,\cdot \quad ,n$

Proof: With an alternating knot $J_\beta = J_\alpha^T$, where ^{T} means *transpose*. Applying the case(1) methods to J_β, inserting a twist into the same edge ij, and following through to the analogue of corollary (ii), we obtain

$$\chi'_{n\alpha.J_\beta} - \chi'_{\alpha.J_\beta} = \chi^T_{ji.J_\alpha} - \chi^T_{ii.J_\alpha}$$

Then since $\mu = -2$ is a characteristic root of both polynomials on the L.H.S, we obtain

$$0 = \chi^T_{ji.J_\alpha}(-2) - \chi^T_{ii.J_\alpha}(-2)$$

or $\quad \chi_{ij}(-2) = \chi_{ii}(-2)$, for J_α

Therefore, using (5.38), we have

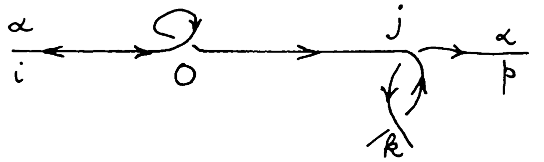
$$\chi_{ii}(-2) = \chi_{jj}(-2) \tag{5.40}$$

And since the knot-graph is connected, with an $\alpha\beta$-edge adjacent to each vertex, (5.40) holds for $i,j \in \{1, \cdot \quad ,n\}$

Case (2)

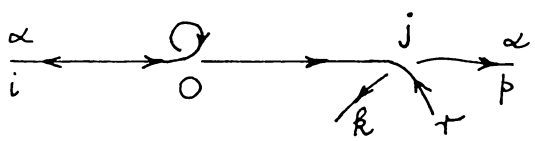
Two possibilities, depending on j-adjacencies.

(i)



$$\chi' = (\mu+1)(\chi+\chi_{ki}+\chi_{ik}+\chi_{ip}+\chi_{ii}) - (\chi_{ij}-\chi_{ij:ji}) - \chi_{ii} \tag{5.41}$$

(ii)



$$\chi' = (\mu+1)(\chi+\chi_{ri}+\chi_{ik}+\chi_{ip}+\chi_{ii}) - (\chi_{ij}-\chi_{ij:ji}) - \chi_{ii} \tag{5.42}$$

And similarly when the twist is made the other way.

Case (3)

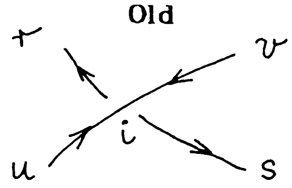
For both types of twist, made in a $\beta\beta$-edge,

$$\chi'_{J_\alpha} = (\mu+1)\chi_{J_\alpha} \tag{5.43}$$

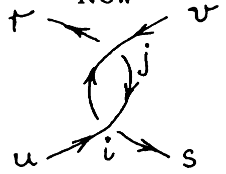
5.7.2 Creating a new 2-gon

The following diagrams and portions of determinants $|\mu I + J_\alpha|$ are self-explanatory.

Old



New



Edges (i,r) and (i,v) in 'Old' have been cut and rejoined to form the new crossing j and 2-gon (i,j) in 'New'.

Old $\chi(\mu)$

	i	r	s	u	v	$\cdot\ \cdot$
i	μ	1	1			
r						
s						
u	1					
v	1					

New $\chi'(\mu)$

	j	i	r	s	u	v	$\cdot\ \cdot$
j	μ	1	1				
i	1	μ	0	1			
r							
s							
u		1					
v	1	0					

The elements in the shaded areas are unchanged by the creation of j and the 2-gon. Using Cauchy expansions we find:

$$\chi = \mu\chi_{ii} - \chi_{ur} - \chi_{us} - \chi_{vr} - \chi_{vs} \qquad (5.44)$$

then (after some rearrangement)

$$\chi' = \mu(\chi + \chi_{ur} + \chi_{vr} + \chi_{vs}) - (\chi_{ii} + \chi_{ir} + \chi_{vr} + \chi_{vi}) + 2\chi_{ir;vi} \qquad (5.45)$$

Here, $\chi_{ir;vi}$ means the cofactor of element vi within the matrix of the cofactor χ_{ir}

Another equation for χ', entirely in terms of second-order cofactors of χ, can be obtained by applying Laplacian expansion to the determinant $\chi'(\mu)$, operating with rows j and i. The result is:

$$\chi' = (\mu^2-1)\chi_{ii;jj} + \mu(\chi_{ii;vr} - \chi_{ii;us}) + (\chi_{ii;ur} - \chi_{ii;vs}) \qquad (5.46)$$

If we use χ^- to denote the matrix obtained from χ by striking out the i^{th} row and i^{th} column, (5.46) takes on the neater appearance

$$\chi' = (\mu^2-1)\chi^-_{jj} + \mu(\chi^-_{vr} - \chi^-_{us}) + (\chi^-_{ur} - \chi^-_{vs}) \qquad (5.47)$$

Cofactors related to a crossing i

It is worth recording here the following relationship between cofactors related to any crossing i which has alternating adjacencies. Referring to the 'Old' matrix above, we see that

$$\chi = \mu\chi_{ii} + \chi_{ir} + \chi_{is} \quad \text{(expanding by row } i\text{)}$$

and also

$$\chi = \mu\chi_{ii} + \chi_{ui} + \chi_{vi} \quad \text{(expanding by column } i\text{)}$$

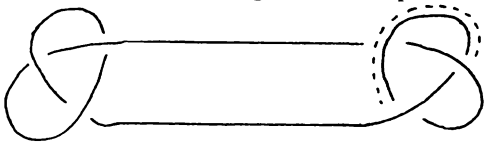
Therefore

$$\chi_{ir} + \chi_{is} = \chi_{ui} + \chi_{vi} \qquad (5.48)$$

5.7.3 Composition of Two Knots (knot-graphs)

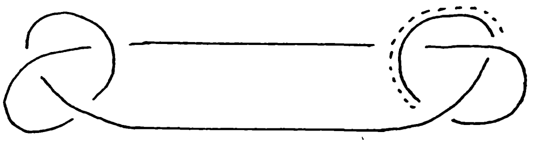
We will begin with an example showing how two trefoils can be tied in a length of string in a number of ways, some of which will represent the same knot-types, others distinct ones. All, however, lead to different characteristic polynomials, pointing up the difficulties of using such polynomials as a means for studying knot properties.

A Two trefoils of the same type, composed to make a fully alternating knot. (The **granny** knot: the dotted edges are explained later)



Characteristic Polynomial: $P_{J_\alpha}(\mu) = (\mu+2)(\mu-1)^2(\mu^2-1)\mu$ (5.49)

B A trefoil composed with its mirror image forming a non-alternating knot with a $\beta\beta$- and an $\alpha\alpha$- edge.(The **square** knot)



Characteristic Polynomials: $P_{J_\alpha}(\mu) = (\mu+2)(\mu-1)^2(\mu^2-2)\mu$ (5.50)

$$P_{J_\beta}(\mu) = P_{J_\alpha}(\mu)$$

N.B. this last is not generally true; it is so here because of symmetry.

Notes

(i) Denoting the two polynomials by χ_G and χ_S respectively, we see that

$$\chi_G(\mu^2-2) = \chi_S(\mu^2-1) \quad (5.51)$$

(ii) The trefoil knot has a polynomial $\chi_T = (\mu+2)(\mu-1)^2$. Using this together with (5.49) and (5.50) we get the interesting combination

$$\chi_G - \chi_s = \mu\chi_T, \quad (5.52)$$

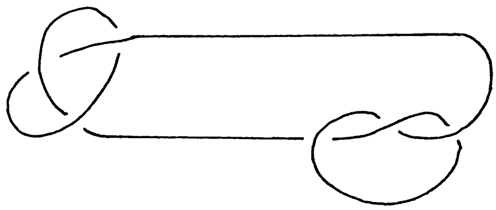
$$\text{with} \begin{cases} \chi_G = \mu(\mu^2-1)\chi_T \\ \chi_S = \mu(\mu^2-2)\chi_T \end{cases}$$

(iii) Graphs A and B are not from knots of the same type (*FOX*,*p*. 131). It may be seen that the left-hand trefoil in B is the mirror image of the left-hand one in A, whereas their righthand ones are the same. And since a trefoil is not amphicheiral, knot A cannot be transformed into knot B, without cutting the string.

The characteristic polynomials χ_G and χ_S do distinguish their diagrams; whereas their fundamental groups and Alexander matrices, derived from the diagrams are isomorphic or identical (*FOX*,*p* 131).

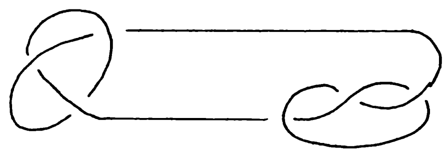
We can, however, continue the story by transforming diagram A to A', a nonalternating graph, and B to B', an alternating graph, simply by 'lifting the two dotted flaps out of the page, pulling them downwards,and dropping them'. Thus:

A'



$$\chi'_G = \mu(\mu^2-2)\chi_T = \chi_S \qquad (5.53)$$

B'

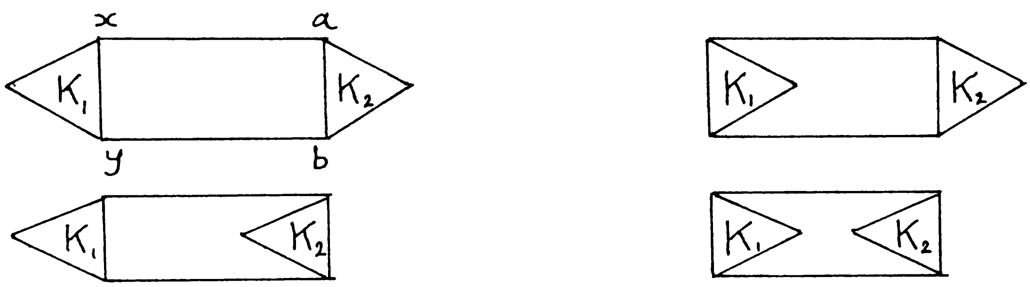


$$\chi'_S = \mu(\mu^2-1)\chi_T = \chi_G \qquad (5.54)$$

Clearly A' represents the same knot as A (the Granny); and B' represents the same knot as B (the Square). Yet their J_α characteristic polynomials are '*opposite in pairs*' as shown in the following table:

	Graph and Polynomial	
Granny Knot	(A,χ_G)	(A',χ_S)
Square Knot	(B,χ_S)	(B',χ_G)

It is evident that we can associate with a composed knot not one but many characteristic polynomials, derived from the various forms which the composite knot-graph can take. To give a brief indication of the complexity that arises in the study of composite knot-graphs, we show four possible ways (in obvious diagrammatic form) of composing two knot-graphs K_1 and K_2:



To complicate matters almost endlessly, we may now do any one of the following:

- Replace K_1 by its mirror-image
- Replace K_2 by its mirror-image
- Replace K_1, or K_2 by any other of the knot projections which the knots they represent have
- Recompose by joining x to b, and y to a.
- Recompose by joining two other edges from K_1,K_2

As we have seen in the example of two composed trefoils, two characteristic polynomials arose from the granny knot, and two from the square knot, and the two were the same (all other types of composition of two trefoils lead to one or other of these polynomials). But in general the situation is much more complex, with many different polynomials arising from a given pair of knots (K_1, K_2).

In spite of this complexity, interesting general results may be obtained, such as the following:

Theorem 2

Let K_1 and K_2 be any two knot-graphs, having α-adjacency matrices J_1, J_2 respectively. Denote by X_1, X_2 the two μ-characteristic polynomials :

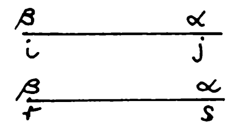
That is

$$X_1 = |\mu I + J_1| = |A|$$

and

$$X_2 = |\mu I + J_2| = |B|, \text{ say}$$

The two knots are to be composed, by cutting and joining the $\beta\alpha$-edges; say those edges are e_{ij} in K_1, and e_{rs} in K_2.



Two types of join are possible, viz:

(i) **alternating** , obtained by joining i to s and r to j;

(ii) **nonalternating** , obtained by joining i to r and j to s;

Let the μ-characteristic polynomials for the two composed knot-graphs which result be named X_a, and X_{na} respectively.

Then:

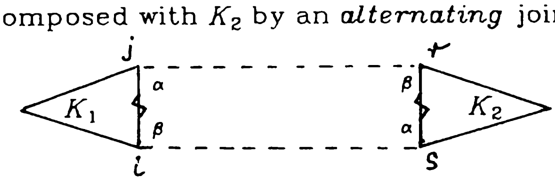
(a) $X_a = X_1 X_2 - (X_1 B_{rs} + X_2 A_{ij})$

(b) $X_{na} = X_a - \begin{vmatrix} A_{jj} & B_{rs} \\ A_{ij} & B_{ss} \end{vmatrix}$, where

A_{ij} etc. are cofactors, and the expression $|*|$ in (b) is to be expanded as for a determinant.

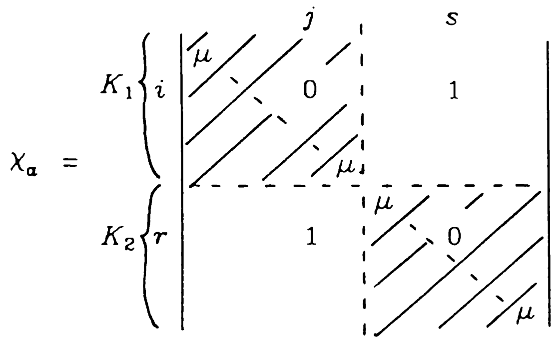
Proof:

(a) A diagram showing K_1 composed with K_2 by an *alternating* join is as follows:

$K_1 \# K_2 =$ 

Edges e_{ij} in K_1, and e_{rs} in K_2, have been cut, and rejoined as shown by dotted lines.

Assuming n_1, n_2 crossings in K_1 and K_2 respectively, χ_a is an (n_1+n_2)-square determinant with the following form.



Top shaded matrix is A except for element ij, which was formerly 1, and is now 0.

Lower shaded matrix is B, except for element rs, changed from 1 to 0.

All elements in the bottom left and top right matrices are 0, except for $is = 1$ and $rj = 1$.

Using Laplacian expansion, in terms of n-square matrices from the first n_1 rows, and their n_2-square cofactors, we obtain

$$\chi_a = (\chi_1 - A_{ij})(\chi_2 - B_{rs}) + (-1)^{j+s} D_1 D_2 \qquad (5.55)$$

where the last term needs explanation thus:

(1) Its sign is $(-1)^x$, where

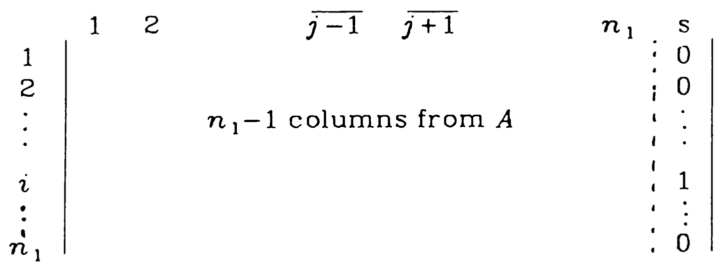
$$x = (1+2+ \cdots +j+ \cdots +n_1)+(1+2+ \cdots +\overline{j-1}+\overline{j+1}+ \cdots +n_1+s)$$

$$= 2k+j+s$$

Therefore

$(-1)^x = (-1)^{j+s}$.

(2) D_1 is the determinant obtained from the indicated rows and columns



If the s column is moved to position j, by making n_1-j column transpositions (and therefore changing the sign by $(-1)^{n_1-j}$) the resulting determinant is simply the cofactor A_{ij}, obtained by expanding by column j.

That is, $D_1 = (-1)^{n_1-j} A_{ij}$

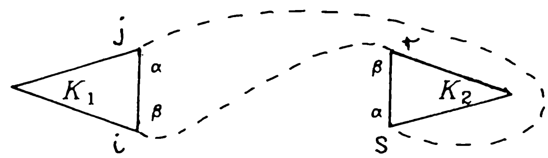
(3) Similarly, we can show that $D_2=(-1)^{s-n_1+1}B_{rs}$

Collecting the results of (1),(2),(3) together, we see that the last term of (5.55) is

$$(-1)^{j+s+n_1-j+s-n_1+1}A_{ij}B_{rs} \;=\; -A_{ij}B_{rs}$$

Putting this in (5.55) we obtain the desired result for (a).

(b) The diagram for composing K_1 and K_2 by a *nonalternating* join is as follows:



[We note in passing that if we flip K_2 over, about an axis rs, a new *alternating* knot-graph will result. This is the subject of Theorem 4 below.]

Proceeding as for (a), but omitting details, we find

$$\chi_{na} = \begin{array}{c} \\ i \\ \\ j \\ \\ r \\ \\ s \end{array}\left|\begin{array}{c:c} \begin{matrix} \mu & 0 & & \\ & \ddots & & \\ & & \ddots & \\ & & & \mu \end{matrix} & 1 \\ \hdashline 1 & \begin{matrix} \mu & & & \\ & \ddots & & 0 \\ & & \ddots & \\ & & & \mu \end{matrix} \end{array}\right|$$

$$= (\chi - A_{ij})(\chi_2 - B_{rs}) - A_{jj}B_{ss}$$

$$= \chi_a - (A_{jj}B_{ss} - A_{ij}B_{rs})\,. \qquad //$$

Example

Let us compose the trefoil with itself, on any pair of edges. Then:

$$\chi_1 = \chi_2 = \begin{vmatrix} \mu & 1 & 1 \\ 1 & \mu & 1 \\ 1 & 1 & \mu \end{vmatrix} = \mu^3 - 3\mu + 2$$

$$A_{ij} = B_{rs} = 1-\mu$$

$$A_{jj} = B_{ss} = \mu^2 - 1$$

Therefore, from (a)

$$\begin{aligned} \chi_a &= (\mu^3-3\mu+2)^2 - (\mu^3-3\mu+2)[(1-\mu)+(1-\mu)] \\ &= (\mu^3-3\mu+2)[\mu^3-3\mu+2-2+2\mu] \\ &= \mu(\mu+2)(\mu-1)^2[\mu^2-1] \quad \text{(granny knot)} \end{aligned}$$

and using (b):

$$\begin{aligned} X_{na} &= X_a - [(\mu^2-1)^2-(1-\mu)^2] \\ &= X_a - [\mu^2-1+1-\mu][\mu^2-1-1+\mu] \\ &= X_a - \mu(\mu-1)^2(\mu+2) \\ &= \mu(\mu-1)^2(\mu+2)[\mu^2-2] \quad \text{(square knot)} \end{aligned}$$

Theorem 3

Let K_1 and K_2 be alternating knot-graphs, composed as described in Theorem 2.

Then

$$X_\alpha^\beta = X_\alpha^\alpha$$

and

$$X_{n\alpha}^\beta - X_{n\alpha}^\alpha = \begin{vmatrix} A_{jj} & B_{rr} \\ A_{ii} & B_{ss} \end{vmatrix},$$

where the superscripts α,β refer to the μ-polynomials computed from α- and β-adjacency matrices respectively

Proof:

The proof follows the same lines as for Theorem 2, and makes use of the fact that $J_\beta = J'_\alpha$

Note that $X_{n\alpha}^\beta = X_{n\alpha}^\alpha$ if $A_{jj}B_{ss} = A_{ii}B_{rr}$

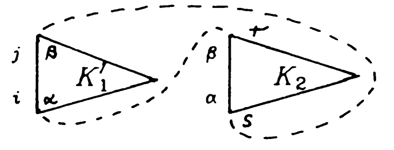
Theorem 4 (Refer back to the diagram and note in part (b) of Theorem 2)

Let K_1 and K_2 be alternating knot-graphs, composed by a nonalternating join as shown above. Then the results of flipping K_1 about axis ij, and of flipping K_2 about axis rs, are shown diagrammatically below. The formulae for the new μ-characteristic polynomials are:

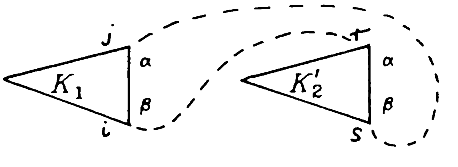
(i) after the K_1 flip: $X'_1 - X_a = A_{ij} - A_{ji}$, and

(ii) after the K_2 flip: $X'_2 - X_a = B_{rs} - B_{sr}$

Diagrams



(i)



(ii)

Proof : Similar to those for previous theorems. **N.B.** Before the flips are made, the graphs are nonalternating: to compare χ' with the nonalternating polynomial in the two cases, we use Theorem 2(b), to obtain

$$\text{(i)}\quad \chi_1' - \chi_{na} = \begin{vmatrix} A_{jj} & B_{rs} \\ A_{ij} & B_{ss} \end{vmatrix} + A_{ij} - A_{ji}$$

$$\text{(ii)}\quad \chi_2' - \chi_{na} = \begin{vmatrix} A_{jj} & B_{rs} \\ A_{ij} & B_{ss} \end{vmatrix} + B_{rs} - A_{sr} \qquad (5.56)$$

5.8 CHARACTERISTIC VECTORS

So far we have said very little about the characteristic vectors of adjacency matrices of knot-graphs. We have not made much study of them, so we can only present one or two results which follow easily from elementary properties of the matrices.

5.8.1. Examples : the trefoil, and Listing's knot

(i) **The Trefoil**

$$P_J(\lambda) = \begin{vmatrix} \lambda & -1 & -1 \\ -1 & \lambda & -1 \\ -1 & -1 & \lambda \end{vmatrix} = (\lambda-2)(\lambda+1)^2$$

$$\text{Spec}(J) = \begin{pmatrix} -1 & 2 \\ 2 & 1 \end{pmatrix}$$

Characteristic vectors: $\underline{x}$ such that $J\underline{x} = \lambda\underline{x}$

$\lambda = 2$ $\quad \underline{x}_1 = (1,1,1)'$
$\lambda = -1$ $\quad \underline{x}_2 = (1,\alpha,-(1+\alpha))'$, any α

(ii) **Listing's 4-knot**

$$P_J(\lambda) = \begin{vmatrix} \lambda & & -1 & -1 \\ -1 & \lambda & & -1 \\ -1 & -1 & \lambda & \\ & -1 & -1 & \lambda \end{vmatrix} = \lambda(\lambda-2)(\lambda^2+2\lambda+2)$$

$$\text{Spec}(J) = \begin{pmatrix} -1-i & -1+i & 0 & 2 \\ 1 & 1 & 1 & 1 \end{pmatrix}$$

Characteristic Vectors

$\lambda = 2$ $\quad \underline{x}_1 = (1,1,1,1)'$
$\lambda = 0$ $\quad \underline{x}_2 = (1,-1,1,-1)'$
$\lambda = -1+i$ $\quad \underline{x}_3 = (1,-i,-1,i)'$
$\lambda = -1-i$ $\quad \underline{x}_4 = (1,i,-1,-i)'$

Notes:

Both the trefoil and Listing's knot have only one characteristic polynomial, for minimal regular projections.

Observe the form of characteristic vector for $\lambda=2$ (common to all alternating knot-graphs); and for $\lambda=0$ (common pattern for all even alternating knot-graphs).

Observe that $\underline{x}_1$ is orthogonal to all other vectors; and in (ii) $\underline{x}_2$ is orthogonal to $\underline{x}_1$, $\underline{x}_3$, $\underline{x}_4$.

5.8.2 Propositions

(i) **Proposition**

All alternating (loopless) knot-graph matrices have left and right characteristic vectors of $(1,1,\ldots,1)$ and $(1,1,\ldots,1)'$ respectively.

Proof: J has just two 1s in each row, and in each column; so both $J\underline{x} = \lambda\underline{x}$ and $\underline{y}J = \mu\underline{y}$ have the characteristic root 2. With this root, the characteristic vectors are as proposed.

(ii) **Proposition**

All alternating (loopless) knot-graph matrices of even order n have a characteristic vector consisting of a permutation of $\frac{n}{2}$ 1s and $\frac{n}{2}$ -1s.

Proof: When n is even, $|J| = 0$ (see 3.4) and so λ is a factor of the characteristic polynomial; therefore $\lambda = 0$ is a characteristic root. With just two 1s in each row of J, solution of $J\underline{x} = \underline{0}$ leads to $\underline{x}$ being as proposed.

(iii) **Proposition**

Let J be the adjacency matrix of an alternating knot with standard labelling.

Let $\underline{x}_i, \underline{x}_j$ be two characteristic vectors of J which are orthogonal.

Then $\underline{x}'_i M\underline{x}_j = \underline{0}$, where $M = U'TU$, T being the torus adjacency matrix $(P+P')$, and U being the permutation matrix such that $J = U+PU$ (see 3.1).

Proof: $J\underline{x}_i = \lambda_i\underline{x}_i$ implies $\underline{x}'_i J' = \lambda_i\underline{x}'_i$, therefore $\underline{x}'_i J'J\underline{x}_j = \lambda_i\lambda_j\underline{x}'_i\underline{x}_j = 0$ in view of the orthogonality of $\underline{x}_i$, $\underline{x}_j$

$$\begin{aligned} \text{But} \qquad \underline{x}'_i J'J\underline{x}_j &= \underline{x}'_i(U'(2I+T)U)\underline{x}_j \qquad \text{(see 3.1)} \\ &= 2\underline{x}'_i\underline{x}_j + \underline{x}'_i U'TU\underline{x}_j \\ &= \underline{x}'_i M\underline{x}_j \\ \text{Therefore} \qquad \underline{x}'_i M\underline{x}_j &= 0 \end{aligned}$$

CHAPTER 6

ROOTED DIRECTED SPANNING TREES AND TREE NUMBERS

In this chapter we shall study two tree numbers, which are counts of rooted directed spanning trees of alternating knot-graphs which have been given balanced orientations.

We shall obtain various methods for computing the numbers, and examine their topological properties.

6.1 ROOTED DIRECTED SPANNING TREES

If a vertex v_i of a digraph is designated 'root', then the number of trees which are directed away from v_i and which span the digraph may be counted. We shall show that for a knot-graph with balanced orientation this number is independent of root vertex; hence for any given bao or bno knot-graph there is a unique *tree number* that may be associated with it.

Definitions

Let K be a knot-graph with bao or bno (v. 1.7). Its *tree number* is the number of rooted directed trees which span it, any vertex being taken as root.

Notation: We use the symbol τ for the tree number of a bao knot-graph, and, and $\tilde{\tau}$ for the tree number of a bno knot-graph. Symbols $\tau(K)$ and $\tilde{\tau}(K)$ will also be used.

Computation of a tree number: A theorem from graph theory is available which gives a matrix method for computing the tree number of a digraph. It makes use of the Kirchoff matrix (sometimes called the inward degree matrix) of the oriented knot-graph. This is defined as follows:

Let K be a labelled oriented knot-graph with n vertices (it may have loops and parallel edges), and let $d^-(v_i)$ denote the in-degree of the ith vertex. Then the *Kirchoff matrix* is $Q(K)$ or $Q = [q_{ij}]$, an $n \times n$ matrix whose elements are

$$q_{ij} = \begin{cases} d^-(v_i)\ , & \text{if } i = j \\ -x_{ij}\ , & \text{if } i \neq j\ , \end{cases}$$

where x_{ij} is the number of arcs directed from v_i to v_j.

(N.B. A loop at v_i adds 1 to both the in-degree and out-degree of v_i.)

The theorem for computing a tree number is then:

Theorem 1.

The value of the (i,i)th cofactor of the Kirchoff matrix of K is equal to the number of directed spanning trees of K which are rooted at vertex v_i.

A proof of the theorem may be found in (*DEO*, 1974).

Independence of choice of root vertex

The following theorem establishes the uniqueness of a tree number for a given knot-graph with a balanced orientation.

Theorem 2

Let G be a knot-graph with balanced orientation, and $Q(G)$ its Kirchoff matrix. Let Q_{ij} denote the (i,j)th cofactor of Q. Then $Q_{ij} = \lambda$, some fixed constant, for all $i = 1, \cdots, n$; $j = 1, \quad, n$.

Proof (extending an argument in BIGGS, 1974)

In view of the balanced orientation of G, and the definition of Q, all column sums and row sums of Q are zero.

Hence $\det(Q) = 0$

So $Q.\mathrm{adj}(Q) = \det(Q).I = 0$.

Furthermore, it is easy to show that with a balanced orientation G has at least one rooted directed spanning tree; so $\mathrm{rank}(Q) = n-1$.

Therefore in the space of solutions of $Q.\underline{X} = \underline{0}$ (where $\underline{X}$ and $\underline{0}$ are $n \times 1$ vectors), each column of adj(Q) belongs to the kernel of Q; and the kernel is a [1]-space.

Now the unit column vector $\underline{U} = (1, \cdots, 1)'$ belongs to this kernel, since $Q.\underline{U} = 0$ (row sums of Q are zero).

Therefore the kernel of Q is spanned by $\underline{U}$, and solutions of $Q.\underline{X} = \underline{0}$ are given by $\underline{X} = \lambda.\underline{U}$ where λ is a constant.

Let $\underline{Q_i}$ denote the i^{th} column of adj(Q); i.e. $\underline{Q_i} = (Q_{i1}, Q_{i2}, \quad, Q_{in})'$. Then $Q.\underline{Q_i} = \underline{0}$ for each i; so $\underline{Q_i} = \lambda.\underline{U}$; and therefore $Q_{i1} = Q_{i2} = \cdots = Q_{in} = \lambda$, for $i = 1, \cdots, n$.

Similarly $\underline{U}'$ is a solution of $\underline{X}.Q' = \underline{0}$ ($\underline{X}$ and $\underline{0}$ are now row-vectors) implying that $Q_{1j} = Q_{2j} = \quad = Q_{nj} = \mu$ (say), for $j = 1,\ldots,n$.

But $Q_{ii} = Q_{jj}$ when $i = j$; *so* $\lambda = \mu$, and all cofactors have equal value.

Corollaries:

(i) Using the tree number theorem given above we see that the number of rooted directed spanning trees on G, rooted at v_i, is the same for $i = 1,2,\ldots,n$, since $Q_{11} = Q_{22} = \quad = Q_{nn}$.

(ii) If G has a bao, then $\mathrm{adj}(Q) = \tau U$; and if G has a bno, then $\mathrm{adj}(Q) = \tilde{\tau} U$, where U is the $n \times n$ matrix with each element unity.

6.2 VALUES OF τ AND $\tilde{\tau}$

In Appendix I we tabulate the tree numbers for the prime alternating knots with orders $n = 3,\ldots,10$ for 1-links, and with orders $n = 2,\ldots,9$ for 2-links.

These values were calculated from the knot-graph adjacency matrices, using the Kirchoff matrix theorem.

For the purposes of comment and preliminary discussion we give here the values of τ and $\tilde{\tau}$ for the first fourteen prime 1-link knots.

Knot	3_1	4_1	5_1	5_2	6_1	6_2	6_3	7_1	7_2	7_3	7_4	7_5	7_6	7_7
n	3	4	5		6			7						
τ	3	5	5	7	9	11	13	7	11	13	15	17	19	21
$\tilde{\tau}$	4	5	16	8	9	20	20	64	18	32	16	64	24	25

Observations

(i) τ is an odd integer for all 14 knots. For a given n it increases monotonically, in steps of 2 except for one case; it appears to be an excellent discriminator.

(ii) $\tilde{\tau}$ takes both even and odd values, and it fluctuates considerably in the given ordering. Different knots have equal values of $(n,\tilde{\tau})$.

(iii) The pair $(\tau,\tilde{\tau})$ discriminates all 14 knots.

Conjectures

Several conjectures about τ and $\tilde{\tau}$ spring to mind from the above observations. We deal with some of these in section 6.4. In particular we prove that τ is a knot invariant, and is an odd integer when K is a prime alternating 1-link.

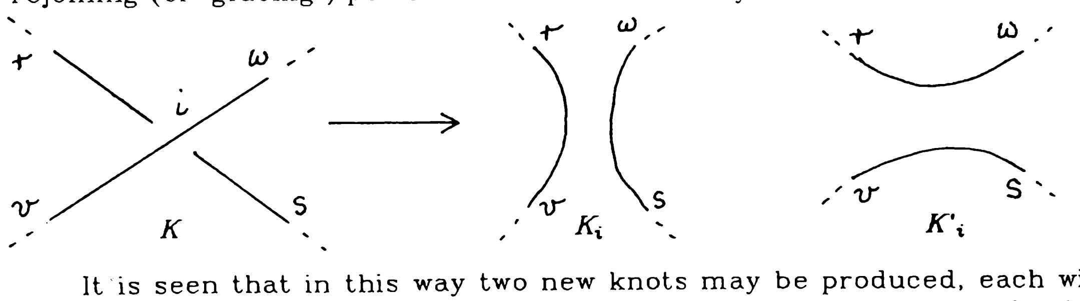
First we prove a vertex deletion theorem which we make much use of later.

6.3 A VERTEX DELETION OPERATION AND THEOREM

In this section we shall be referring to knot-graphs, but writing as if they are actual knots, made of string and lying on a plane. It should be evident that the operations defined can be made mathematically precise.

The deletion operator δ_i

In the following diagram, the L.H. figure shows the vertex i of a given knot-graph, together with its edges and four adjacent vertices v,w,r,s. The figures to its right show the two possible results of upper and lower strings at i) and rejoining (or 'glueing') pairs of cut ends in new ways.



It is seen that in this way two new knots may be produced, each with one fewer vertex than the original knot. It may happen that in the new knots there exist nugatory crossings; i.e. ones which can be removed either by untwisting a loop or by twisting a portion of the knot through 180° out of the plane and back. Let us require that after a new knot is formed all nugatory crossings be removed. (Later, in chapter 7, we shall remove this requirement.)

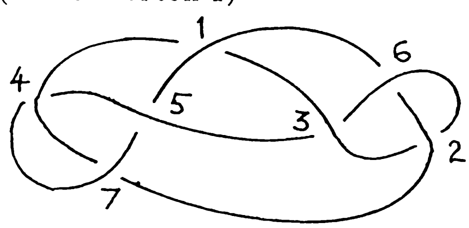
We shall call the whole set of operations described above the *deletion operation*, and use the symbols

$$\delta_i(K) \rightarrow (K_i, K'_i)$$

to describe it. Here δ_i represents the deletion operator; K the knot-graph from which vertex i is deleted; and K_i, K'_i the pair of possible knot-graphs that can result. We may call these last a pair of 'twins' derived from K.

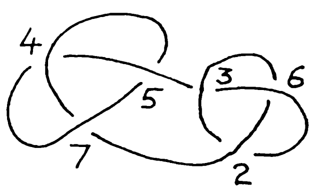
Next we give an example, which will serve to illustrate both the operation and the subsequent discussions.

Example: Knot 7_7 (delete vertex 1)

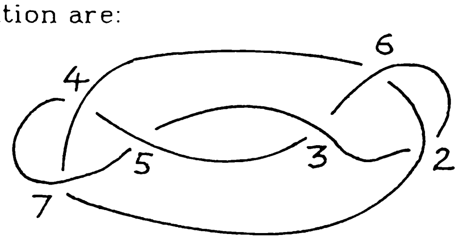


The twins resulting from the deletion operation are:

K_1 =



K'_1 =



Note that K_1 is a 1-link. It is the composition of two trefoils, namely $3_1 \# 3_1$. And K'_1 is a 2-link, namely knot 6^2_3.

Thus $\delta_1(7_7) \rightarrow (3_1 \# 3_1, 6^2_3)$.

It is easy to show that if K is a 1-link knot, the twins from any deletion operation on K (i.e. on any vertex) must be such that one is a 1-link and the other is a 2-link. We cannot produce identical twins!

In the case that K is a μ-link, with $\mu > 1$, there are two possibilities:

(i) If i is a crossing point of the two links, the twins are both $(\mu-1)$-links.

(ii) If i is a self-crossing point, one twin is a μ-link and the other a $(\mu+1)$-link.

We shall not prove these assertions, since they were known to Tait.

Our purpose in introducing the deletion theorem is to exploit its relationship with the tree numbers of K and its twins. We discovered the following theorem, and found both a graph and a matrix proof for it; later we found in the literature a connection between Euler tours and tree numbers (described in (DEO, 1974)). The connection enabled us to find a simple proof of the theorem, which we now give. The other two proofs are given in Appendix II.

Theorem 3 (deletion theorem)

Let K be a loopless alternating knot-graph, with n vertices and balanced orientation. Let i be any vertex, and τ be the number of rooted directed spanning trees on K. Apply the deletion operation at i, and let the resulting twins be K_i and K'_i. Let the respective tree numbers of the twins be τ_i and τ'_i. Then $\tau = \tau_i + \tau'_i$

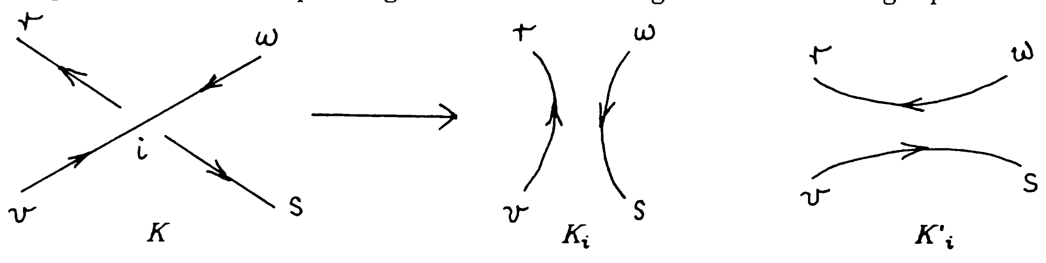
Proof:

In (*AARDENNE–EHRENFEST*,1951) it is shown that there is a 1-1 correspondence between the set of rooted directed spanning trees (any given root vertex) of an Euler digraph and the set of its Euler ditours (directed tours). This leads to the following formula for the number of Euler ditours in an Euler digraph:

$$\tau \times \prod_{i=1}^{n} [d^{-}(v_i)-1]\,! \quad \text{(see DEO, Thm. 9-13, p. 226).}$$

Now, in a balanced knot-digraph the in-degree $d^{-}(v_i)$ is 2 at every vertex. Therefore the number of Euler ditours in a knot-digraph equals τ in the bao case, and $\tilde{\tau}$ in the bno case.

To complete the proof of the deletion theorem, we must show that $\varepsilon = \varepsilon_i + \varepsilon'_i$, where ε is the number of Euler ditours in K, and $\varepsilon_i, \varepsilon'_i$ are the numbers of Euler ditours in the twins. We redraw the diagram given at the beginning of this section, placing arrows on the edges as for a bao graph:



Consider the set E of Euler ditours of K. We can partition this to $\{E_1, E_2\}$ where E_1 is the set of ditours each of which uses the double-steps $[vi,ir]$ and $[wi,is]$, and E_2 is the set which uses the double-steps $[vi,is]$ and $[wi,ir]$. It is clear that E_1 and E_2 are disjoint. And by reference to the diagrams on the right-hand side, we see that the members of E_1 are in 1-1 correspondence with the Euler ditours of K_i; and those of E_2 are in 1-1 correspondence with the Euler ditours of K'_i. // (The same argument holds for the bno case, with $\tilde{\tau}$. See Appendix II for comments on this case.)

6.4 SOME THEOREMS ON THE TREE NUMBERS

Before considering general topological properties of τ and $\tilde{\tau}$ (e.g. knot invariance), we shall give several theorems about the measures, which may be proved by arguing directly from adjacency matrices or T-diagrams, or from the knot-diagrams themselves.

Theorem 4

Given a knot-graph K on n vertices, with a balanced orientation (either bao or bno) and adjacency matrix J. Then:

(i) tree number $= \frac{1}{n^2}\det(U + 2I - J)$, where U is the $n \times n$ matrix with each element unity;

(ii) tree number $= \chi'(2)$, where $\chi(\lambda)$ is the characteristic polynomial of J, and the prime indicates its first derivative.

Proofs: may be found in (*BIGGS*,*pp.* 35,36).

Theorem 5

Let K be an alternating n-crossing μ-link knot-graph.

Then τ is odd if $\mu = 1$. If $\mu > 1$, τ is even if one of the links passes through $4i$ vertices (some may be visited twice).

(N.B. A topological proof of the first part is contained in Theorem 12 below).

Proof:

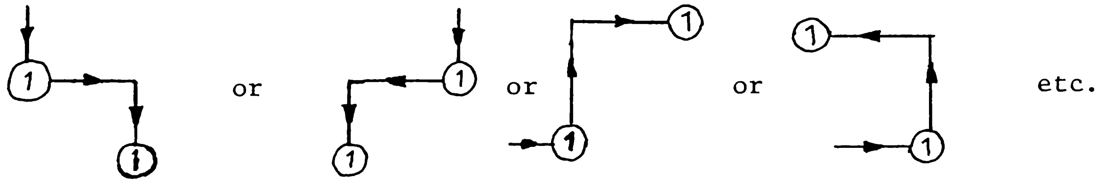
(i) **Case $\mu = 1$**

We first show that any principal minor of the α-adjacency matrix J has value +1 or -1 when $\mu = 1$.

Consider the T-diagram formed from J. When we take the principal minor J_{11} (for example) from it, we strike out the first row and column, thereby removing one horizontal edge (with two corner 1s) and one vertical edge from the rectangular T-diagram.

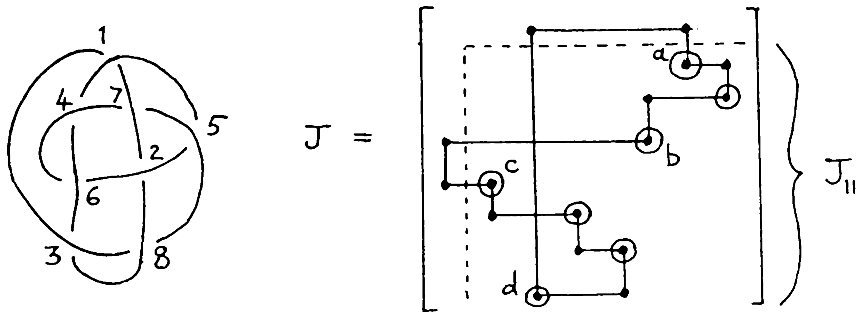
Let us now encircle any 1 in J_{11} which is the only 1 in its row or column, and proceed to tour the T-diagram (either direction may be taken). We are to encircle the 1s on every alternate corner as long as we remain inside J_{11}; after leaving J_{11} (on row 1 or column 1) we proceed on the T-diagram until we re-enter J_{11}, then we encircle the first 1 we meet. Proceeding thus, we complete the tour. We claim that (a) the [0,1] matrix obtained from J_{11} by replacing all uncircled 1s by 0s is a permutation matrix, and (b) the permutation matrix is unique.

Proof (of the claim) : Every horizontal and vertical edge of the T-diagram is traversed in the tour, and just one 1 is encircled in every row and every column except the two struck out. Hence the encircled 1s determine a permutation matrix in J_{11}; hence (a). The tour always leaves J_{11} from an encircled 1, since encircling always proceeds, when within J_{11}, according to patterns such as the following:



(We have assumed a standard labelling in the above diagrams, but we need not have done so.) And by construction, when the tour re-enters J_{11} the first 1 is encircled. Since we began the tour by encircling one of these 1s, it is immaterial which direction we take around the T-diagram, and which 1 we begin with; thus the $(n-1)$-permutation matrix of encircled 1s is uniquely determined.

An example will help to make the above explanations clear. We show below knot 8_{17} and its T-diagram from the given labelling. Note that the tour may begin at any of a, b, c or d, and proceed in either direction; the same permutation matrix results in each case.



We must now show that we cannot construct another permutation matrix from the elements of J_{11}. Suppose we begin a tour by encircling one of the 1s which does not appear in the above permutation. We must then proceed as before, encircling every other 1 we reach on the tour (in order to ensure only one 1 in each row and column).

But eventually we must arrive at a 1 that is the only one of its row or column; and we cannot encircle it, since we encircled the previous one. Hence we cannot construct any other permutation matrix in J_{11}.

Thus we have shown that $\det(J_{11}) = \pm 1$.

Finally we examine the characteristic polynomial of J_{11}, which is $\det(\lambda I - J_{11})$, to prove that τ is odd. Call it $\chi_1(\lambda)$. The constant term of this is $\det(J_{11})$, which we have just shown to be ± 1. Now $\tau = Q_{11} = \chi_1(2)$, which is the sum of powers of 2 plus ± 1; hence τ is odd. //

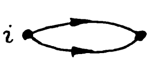
(ii) **Case $\mu > 1$ and one link meets $4i$ vertices.**

In the case of there being several links, we use an argument similar to that used in theorem 2 of chapter 2. With a standard labelling, the T-diagram will contain a rectangular diagram which has an even number of rows and columns, corresponding to the link which meets $4i$ vertices. This will be contained completely within some $(n-1)$-principal minor of J, which minor will therefore equal zero (see the theorem in chapter 2). Using the corresponding minor from $Q = 2I - J$ to compute τ (suppose it is $\det(Q_{ii})$) we see that $\tau = \chi_i(2)$; but this is even, being a sum of powers of 2. //

Theorem 6

Let K be an alternating knot-graph with a bno orientation. Then $\tilde{\tau}$ is even if it contains at least one 2-gon oriented as  .

Proof:

Let the vertices of the 2-gon be i, j, thus: i  j

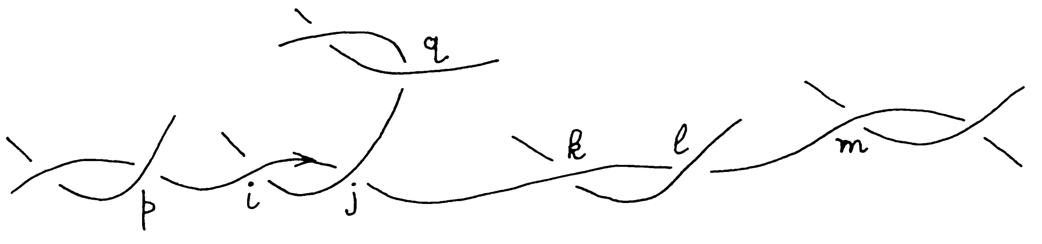
Then the ith row of some $(n-1)$-principal minor of $Q = 2I - J$ has only two non-zero elements, namely -2 and 2. Hence 2 is a factor of that minor, and so $\tilde{\tau}$ is even. //

Theorem 7

Let K be an alternating 1-link knot-graph each of whose vertices are adjacent to just one 2-gon. Then $\tau = \tilde{\tau}$ for K.

Proof:

We show below a portion of such a knot-graph. We first prove that every 2-gon in the graph must be a cycle when K is given a bno.



Suppose we sense the graph by beginning at i and placing an arrow on (ij) as shown, then one on (jk) in the same direction, and so on. Note that now (jk) takes an arrow in an α-step direction; then so must (lm), and so on, for all the edges which connect the 2-gons. Hence all edges within the 2-gons must take β-step directions; then all 2-gons are cycles.

It is now easy to see that the adjacency matrix for this bno graph is the same as would result if the graph were given a bao orientation (suitably chosen from the two available). It follows that $\tau = \tilde{\tau}$. //

Examples

We have studied knot-graphs of the type defined in Theorem 7, and call them 'minimally strongly covered by 2-gons' -graphs (msc2-graphs) (*TURNER*, *EDSON* and *CHAN*, 1978). Examples are relatively rare, for given n. For example, among the prime 1-links with $n \leq 10$ there are only four, namely 4_1, 8_{12}, 10_{35} and 10_{58}.

We have found several methods for constructing families of msc2 knots for higher values of n, but were unable to make any progress in counting the families.

For comments on knot-graphs having $\tau = \tilde{\tau}$ see 6.6(ii).

Theorem 8

Given an alternating knot-graph K which is the composition of two alternating knots K_1 and K_2 (i.e. $K = K_1 \# K_2$), then:

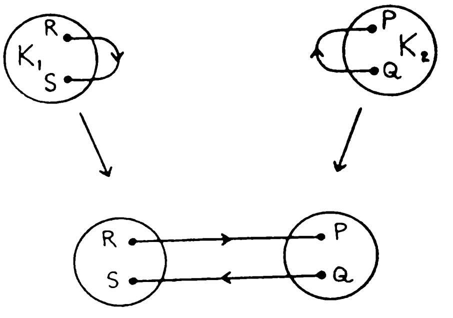
$$\tau = \tau_1 \times \tau_2 \qquad (1)$$

$$\text{and} \quad \tilde{\tau} = \tilde{\tau}_1 \times \tilde{\tau}_2 \qquad (2)$$

using an obvious notation.

Proof:

We give a graphic argument which serves for both (1) and (2). The following diagrams show how the composition of K_1 and K_2 takes place. In order for $K_1\#K_2$ to be alternating, the directions in edges RS and PQ must be as shown, before they are cut and rejoined as RP and QS.



Denote $\tau_1 = \#\text{r.d.s.t.}(R) = \tau(K_1)$, and τ_2 similarly for K_2 using root P.

If we take a r.d.s.t.(R) and join it to a r.d.s.t.(P) by the new edge $\vec{RP}$, it will become a r.d.s.t.(R) in $K_1\#K_2$ if $\vec{RS}$ was not used in the tree rooted at R. If $\vec{RS}$ was used, then we must add $\vec{QS}$ to the joined trees to make an r.d.s.t.(R) in $K_1\#K_2$. Thus, since every tree in K_1 can be joined in this way with every tree in K_2, to form a r.d.s.t. of the composed knot, and no others can be formed, we have $\tau = \tau_1 \times \tau_2$.

Similarly we can show that $\tilde{\tau} = \tilde{\tau}_1 \times \tilde{\tau}_2$. //

Definitions (following CONWAY, 1970)

From any oriented knot-graph K (bao or bno) we can obtain three others by simple geometric operations: reflecting in a mirror gives the *obverse* of K; reversing the arrow in every arc gives the *reverse* of K; and doing both operations gives the *inverse* of K.

Theorem 9

The values of τ and $\tilde{\tau}$ in a knot-graph are unchanged by any of the operations which produce the obverse, reverse or inverse of the knot-graph.

Proof:

The α-adjacency matrix of a knot-graph is either left unchanged or else transposed by the operations. //

6.5 TOPOLOGICAL PROPERTIES OF τ AND $\tilde{\tau}$

In Appendix I we have given tables of $(\tau,\tilde{\tau})$, for prime knots up to $n = 10$ crossings; they show how the numbers vary and serve as discriminators between the knots.

Their real value for discriminating knots depends, of course, on whether or not they are knot-invariants. It is this question we address now. The main result of this section is to show that τ is a knot-invariant; and there is strong evidence that $\tilde{\tau}$ is too. A topological proof for τ, due to Crowell, is outlined; and our graph proof is given.

The following preliminary results are easy to prove using the knot-graph adjacency matrices.

Theorem 10

Let the tree numbers be calculated from an alternating knot-graph. Then:

(i) τ (and $\tilde{\tau}$) are independent of the labelling used on the knot-graph.

(ii) τ (and $\tilde{\tau}$) are invariant to removal of nugatory crossings (i.e. loops or other crossings that can be removed by twisting a portion of the knot-graph through 180° out and back into the plane).

Proof:

(i) If Q and Q' are the in-degree matrices of the knot-graph under two different labellings, then there is a permutation matrix P such that $Q' = P^{-1}QP$. The result follows.

(ii) Consideration of the in-degree matrices, before and after removing a loop for example, and choosing appropriate $(n-1)$ minors, gives the result. //

We conjectured early in our studies of tree numbers that both τ and $\tilde{\tau}$ are knot invariants. We took the sets of nonisomorphic knot-graphs given by Tait for the ten knots of fewer than 9 crossings that have different graph forms; and in all cases we found that $(\tau,\tilde{\tau})$ was invariant to the changes of form.

Later we noticed that the τ-values for $n = 3,\ldots,9$ were identical with the values of a certain knot invariant discovered by ALEXANDER, 1922 and tabulated by him (in ALEXANDER, 1928) using a method working directly from the knot-projection. We should say that the values were identical except for that of knot 9_{36}. However, using Alexander's own method we calculated the value of his invariant and found his tabled value to be in error; it now equalled the τ-value, so the two parameters were indeed in 1-1 correspondence in this range!

The Alexander invariant is a certain group torsion number. He showed that if the space of a knotted curve be covered by an n-sheeted 'Riemann 3-spread' (the 3-dimensional analogue of a Riemann surface) with a branch curve of order $n-1$ covering the knot itself, then the topological invariants of the covering spread will also be invariants of the knot. He calculated the Betti numbers and coefficients of torsion of the covering spreads determined by some of the simpler knots and found these invariants sufficient, in the cases examined, to distinguish one knot type from another.

The torsion numbers referred to were rediscovered as knot invariants by K.Reidemeister (see REIDEMEISTER, 1926) who derived them from a study of the space complementary to the knot, and then identified them with the invariants of the Riemann covering spreads.

We sought to prove the equivalence of the torsion number and our tree number by establishing equivalences between the in-degree matrix method of calculating τ and the various methods given in knot-theory literature for computing the torsion number. We failed to do this, but we did find a graph proof of the invariance of τ, which we describe below. Later, almost by chance, we found a theorem which enabled us to connect the two numbers in a paper by R.H.Crowell. The paper (CROWELL, 1959) is entitled 'Genus of alternating link types', and the theorem is proved only as a step to his main results on genus. We have found no references to it elsewhere.

In his introduction to the theorem he states: "The connection between knot theory and graph theory is provided by a combinatorial theorem which relates certain minor determinants of a matrix of values assigned to the edges of an oriented graph to the maximal rooted trees of the graph." (These last are what we call rooted directed spanning trees.)

As we shall see, Crowell in fact proves more than we actually need. To obtain his theorem he uses a generalisation of the in-degree matrix theorem which we used to compute τ ; we must explain briefly this generalised matrix-tree theorem.

Consider the set Θ_i of all directed spanning trees rooted at i in an oriented graph K; and suppose that a symbol or weight has been assigned to each arc of K. Then if $T \varepsilon \Theta_i$, and W_T is the product of all the weights on the arcs of T, the matrix-tree theorem shows that $\sum_{T\varepsilon\Theta_i} W_T$ is equal to the ith principal minor of a certain matrix which may be called the 'generalised in-degree matrix'. Note that if all the weights are set to 1, the result reduces to the original in-degree matrix-tree theorem.

Theorem 11 (Crowell's theorem, with outline of proof)

Crowell takes an alternating knot-graph, and defines on it an alternating orientation and an assignment of weights which assigns either $+1$ or $-t$ to each arc of the knot-graph.

He is then able to show that the generalised in-degree matrix is equal to the transpose of the reduced Alexander matrix derived from the Wirtinger presentation of the fundamental group of the knot, which is determined from the given knot-projection.

Taking any $(n-1)$-principal minor (they are all equal) of both these matrices gives the result $\Delta(t) = \sum_{T\varepsilon\Theta_i} W_T$, where $\Delta(t)$ is the reduced Alexander polynomial of the knot-type.

Corollary

The tree number τ is a knot-invariant, for setting $t = -1$ in Crowell's theorem gives $|\Delta(-1)| = \sum_{T\varepsilon\Theta_i} 1 = \tau$: and Alexander's polynomial is a knot-invariant.

At this point we can make the connection back to Alexander's torsion numbers, and also to our theorem that τ is odd for a 1-link, by means of the following theorem:

Theorem 12 (proved in ROLFSEN, (1976), p.212)

Let the 2-fold branched cyclic cover Σ_2 of the knot have homology group $H_1(\Sigma_2)$. Then H_1 is finite, with order $|\Delta(-1)|$ which is always odd.

Because of the fundamental nature of Crowell's theorem, and because we wish to refer to our methods later, we give next our graphical proof of the invariance of τ. The strategy is to appeal to a statement made by Tait to the effect that if an alternating knot has two different plane projections, then a sequence of flyping operations will transform one projection into the other.

If this statement is true, it is sufficient for us to show that τ is invariant to a flyping operation on the knot-graph; this we shall do.

With regard to Tait's statement, we shall call it a conjecture; his arguments for it may be found in TAIT,b,pp.289-91,1877. We give below a shorter, but less rigorous, argument taken from LITTLE,a,p.38,1885.

We remark that Dowker and Thistlethwaite (1982), in computing their list of the 12-crossing alternating knots, relied on Tait's conjecture to remove certain duplications from the list.

LITTLE,a,(1885), on Tait's conjecture: "If a ten-fold knot be placed upon a plane in such a way as to have but ten crossings the eye will project it upon the plane in a form which will be found among the 364 above obtained. If the knot gives more than one form it will be possible to obtain any other of its forms by one or more turnings over of restricted portions of the knot while the remainder is held fixed. Now the string cannot issue from the portion of the knot that is turned at more than four points, for in that case the turning would introduce consecutive overs, and one or more additional crossings; the portion of the knot that is turned must therefore be wholly between two parts of the given knot form and in turning it we untwist two of the strings at one point and twist two at another, the result being simply to change the position of a single bond from one end of the connection to the other. The class of the form is therefore not changed, and all the forms of any knot belong to the same class."

Although the terms 'part' and 'bond' have special meanings (Tait's and Little's method consisted of taking pairs of partitions of $2n$ and considering how they may be 'bonded' to produce a knot-form) the description of 'flyping' (turning over of a restricted portion of the knot whilst the remainder is held fixed) and how equivalent forms are produced is abundantly clear in the above passage.

Theorem 13

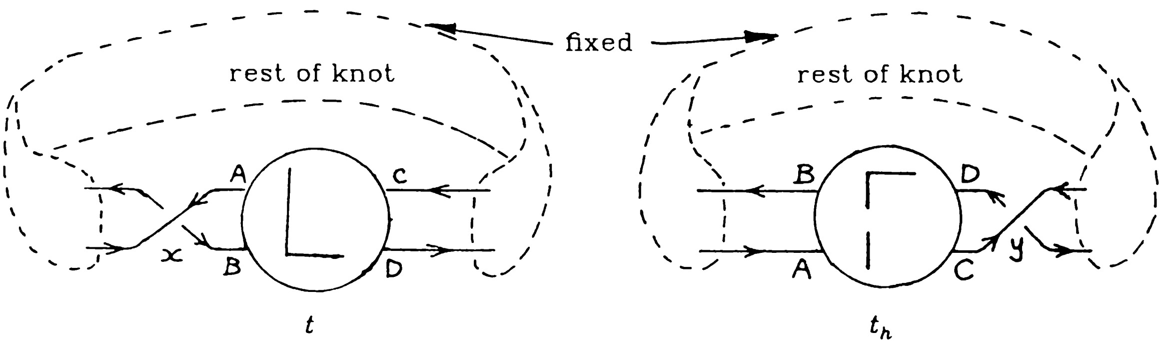
The tree number τ of an alternating knot-graph is an invariant of knot-type.

Proof:

We have already shown that τ is invariant up to relabelling of the knot-graph (and hence up to isomorphic forms of knot-graph); and that removing loops or other nugatory crossings does not change it.

Assuming the truth of Tait's conjecture, we must show that τ is invariant to a flyping operation in order to complete the proof.

The following diagrams show the knot-graph with a portion singled out for flyping on the left; and on the right the resulting knot-graph after flyping has taken place. We have used Conway's notation (see Appendix III) to depict the two 'flyped' portions; t and t_h are the names given to the two tangles involved.



(i) Before flyping, (K) *(ii) After flyping, (Q)*

Note that directions must alternate on the four arcs around an alternating tangle t, otherwise $1*t$ would not be an alternating knot.

The strategy of the proof is to apply the deletion operation to both diagrams, and then show that $\tau_i + \tau'_i$ is the same in both cases.

Diagram (i) δ_x produces the pair (K_x, K'_x), where

K_x =

K'_x =

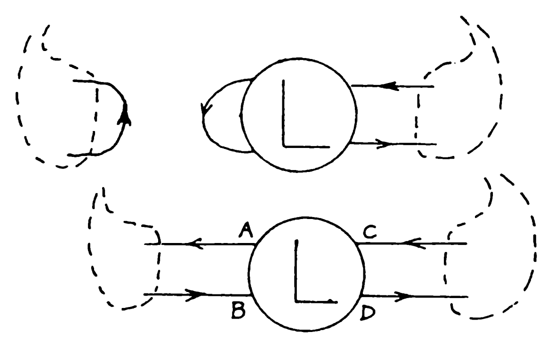
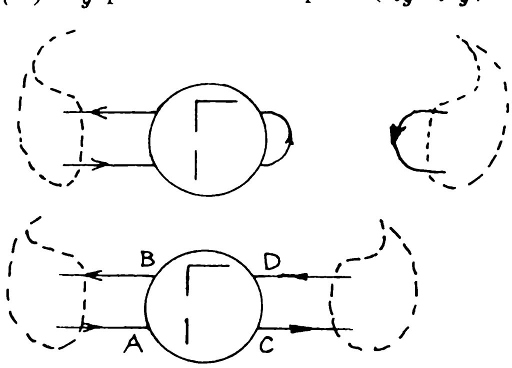


Diagram (ii) δ_y produces the pair (Q_y, Q'_y), where

Q_y =

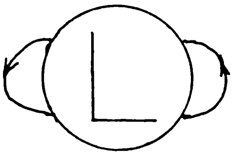
Q'_y =



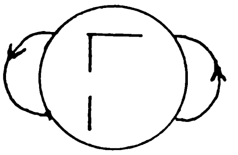
Equalities

The two knots (a) and (b) shown below are the same knots, viewed from different sides, so of course they have the same τ value.

(a)



(b) (knot (a), turned over about a WE-axis)



N.B. The arrows are not 'fixed' in the string; they indicate the α-directions of the two outer edges.

Now K_x is the composition knot (a) # *'rest of knot'*; and Q_y is the composition knot (b)# *'rest of knot'* (it is immaterial which particular arcs are joined during composition).

Therefore, by theorem 8 of 6.4,

$$\begin{aligned}\tau(K_x) &= \tau((a)) \times \tau(\text{'rest of knot'}) \\ &= \tau((b)) \times \tau(\text{'rest of knot'}) \\ &= \tau(Q_y).\end{aligned}$$

To complete the proof we have to show that $\tau(K'_x)$ is equal to $\tau(Q'_y)$.

Now if the tangle being flyped is an integer tangle (CONWAY, 1970) (see Appendix III for notation and examples), t and t_h are precisely the same tangle, and then the two knots K'_x and Q'_y are the same knots, and so have equal τ-values. The same is true if t has the form $n0$.

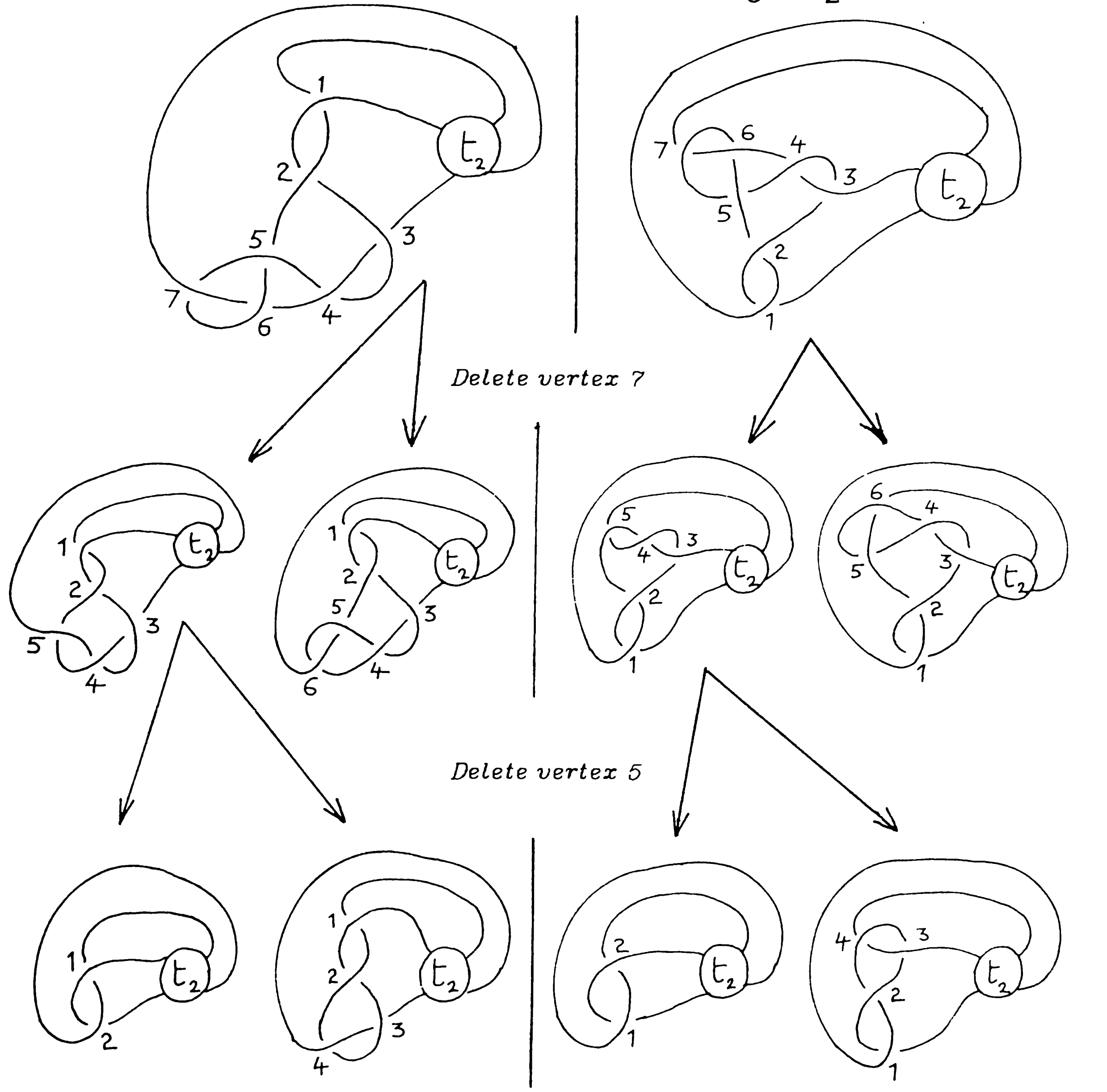
If t is not an integer tangle and is not a form $n0$, we can repeatedly apply the deletion operation to a vertex of it, then to one of the vertices in each of the first pair of twins, and so on, until we have a collection of reduced knot-graphs (i.e. reductions of K'_x), all of which have integer or $n0$ tangles in the place where t was. Note that in this reduction process, once we reach a twin which is an integer or $n0$ tangle, we must not reduce it further. Nor must we ever delete a vertex which disconnects the tangle.

If we now perform the deletion operations to the t_h part of knot-graph Q'_y, operating on the same sequence of vertices as was taken in t, we shall end up with precisely the same collection of reduced knot-graphs from Q'_y (N.B. an integer or $n0$ tangle turned over is exactly the same as before).

Using Theorem 8 on the whole collection of reduced knot-graphs, and summing τ-values, completes the proof. //

(see below for an example of the process in the last part of this proof.)

Example : (two generations only) t is knot 7_6 ; t_2 is 'rest of knot'



One equivalent pair, with $t^{(i)}$ a 20 tangle, has been reached. It should be evident that the process can proceed to the position claimed in the theorem.

6.6 INEQUALITIES FOR τ AND $\tilde{\tau}$

One observation that may be made at once from the tables of tree numbers (Appendix I) is that for all knots examined $\tau \le \tilde{\tau}$.

We have made many attempts to prove this inequality to be true for all knot-graphs, but have not succeeded in finding a convincing proof. We have also studied bounds for τ and $\tilde{\tau}$. Brief notes on this work now follow.

(i) We have shown that $n \le \tau$. The proof is elementary, but requires rather a lot of technical detail. $\tau = n$ only in the cases of torus knots and links.

(ii) We showed in 6.4 (theorem 7) that $\tau = \tilde{\tau}$ if the knot-graph is minimally strongly covered by 2-gons (i.e. msc2). This is not a necessary condition, however. Counter-examples are:

  (a) knot 10_{13}, which has a knot-graph with six 2-gons and $\tau = 53 = \tilde{\tau}$;

  (b) all torus knot-graphs with n even (2-links) and the bno chosen to make the 2-gons all cycles.

  Indeed, given an msc2 knot-graph, one can convert any of its 2-gons into a chain of 2-gons (cut through a vertex, insert twists, rejoin); the result is a knot-graph with increased n but which is not msc2 and has $\tau = \tilde{\tau}$.

  In all these types of graph with $\tau = \tilde{\tau}$, the adjacency matrices are identical, because the directed 2-gons are cycles and all other arcs are sensed equally.

  Open questions are: What are necessary and sufficient conditions for $\tau = \tilde{\tau}$? Is it possible to have $\tau = \tilde{\tau}$ with non-identical adjacency matrices?

(iii) We show in the next section (6.7) that the torus 2-links (n even) have $\tilde{\tau} = 2^{n-1}$ if the bno is such that the 2-gons are not cycles. It is easy to show that the maximum $\tilde{\tau}$ occurs when a maximum number of non-cyclic 2-gons occur; and this is the torus case just mentioned. Hence $\tilde{\tau} \le 2^{n-1}$ (and $\tau \le 2^{n-1}$ of course).

(iv) Two strategies used to try to prove that $\tau \le \tilde{\tau}$, neither of which has yielded a conclusive proof, are:

  (a) We proved that all 4-regular graphs must have at least one 2-gon or one 3-gon. We tried using the vertex deletion theorem and induction. We showed that if a knot has a 2-gon, and deletion of a vertex yields twins with 2-gons, the inequality holds. But the induction part fails if one of the twins has no 2-gon. Assuming only a 3-gon has not yet yielded a proof.

  (b) We have discovered a useful mapping of a spanning tree of a bao knot-graph into a spanning tree of the same knot-graph with bno. But our attempts to show that the mapping is 1-1 and into break down.

(v) For tree numbers of knot-graphs with mixed balanced orientations (bmo; see 1.7) we have shown that, if τ_m is such a number and $\tau \le \tilde{\tau}$, then $\tau \le \tau_m$ for any bmo.

  It is not true, however, that $\tau \le \tau_m \le \tilde{\tau}$. A counter-example to this may be found in knot 4_1, which has a bmo such that $\tau = \tilde{\tau} = 5$ and $\tau_m = 6$.

Summary and comment

We can state that: $n \le \textit{any tree number} \le 2^{n-1}$

We strongly believe that for all knot-graphs: $\tau \le \tilde{\tau}$

The questions examined here are known as 'arrow diagram problems'. There is a large literature on them, particularly in relation to physical models of molecules and their reaction processes. Much of the theory developed for these models is concerned with oriented plane lattices, with $n \to \infty$. A search of this literature has not brought to light results on our questions about inequalities for the tree numbers of knot-graphs.

We are devising computer algorithms to help study arrow problems on knot-graphs.

One other question of interest is: How many balanced orientations does a knot-graph have (i.e. alternating, nonalternating and mixed)? The number could possibly be a knot-invariant.

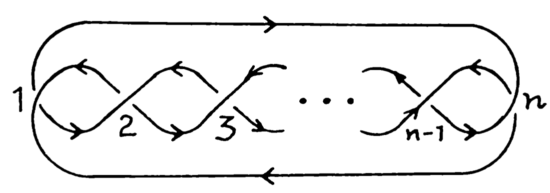
6.7 CALCULATION OF TREE NUMBERS FOR VARIOUS KNOT-CLASSES

In this final section of chapter 6 we develop general formulae for computing tree numbers for members of several classes of knot-graphs.

The Tori Knot-Graphs

First we obtain formulae for τ and $\tilde{\tau}$ in a torus knot-graph with n crossings. To obtain the first we use the deletion theorem; for the second we expand the in-degree matrix.

The bao case



The n-torus graph (T_n) with bao

Applying the deletion operator to vertex 1 gives a difference equation:

thus $\delta_1(T_n) \to (0_1, T_{n-1})$

therefore $\tau_n = 1 + \tau_{n-1}$

The solution of this is:

$$\tau_n = n \quad , \text{ for } n \geq 2$$

The bno case

There are several possible ways to give a torus knot-graph a balanced nonalternating orientation; we consider these in turn.

If *n is even*, T_n is a 2-link, and there are four ways of assigning the bno. Two of them lead to the same adjacency matrix as for the bao case; then $\tilde{\tau} = n$.

In the other two cases, and also for odd n, the bno makes all the 2-gons cycles, and the in-degree matrix (when the knot-graph is labelled as shown above) is:

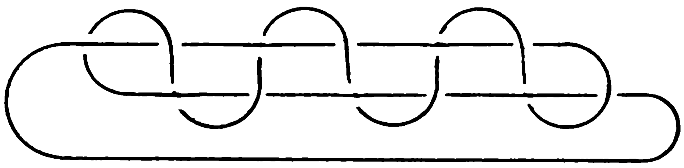
$$Q_n = \begin{bmatrix} 2 & -2 & & & \\ & 2 & -2 & & \\ & & 2 & -2 & \\ .. & & & & .. \\ -2 & & & & 2 \end{bmatrix}$$

whence $\tilde{\tau}_n = Q_{n.11} = 2^{n-1}$

Multiple Figure-Of-Eight

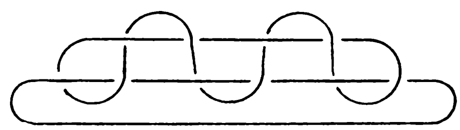
The figure-of-eight knot is a common name for Listing's knot (i.e. 4_1). We may tie a number ν of these in a string, producing a knot with $n = 4\nu$ crossings; the following example shows the manner of tying when $\nu = 3$. The result is known as a multiple figure-of-eight. We will denote the general case by F_n, n = 4,5,12,...

Knot F_{12}



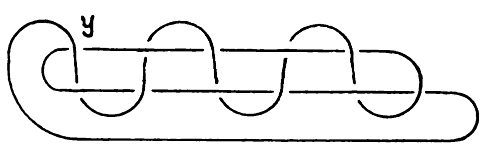
Applying the deletion operation at point x gives:

$\delta_x(F_{12})$ → $K' =$



(a 2-link)

$\delta_x(F_{12})$ → $K =$



We find K' an even more attractive knot than F_n, in view of its symmetry. It is a 2-link, which we shall denote by G_n, and call the $\nu \frac{1}{2}$ figure-of-eight link.

Applying the deletion operation to vertex y of K in the example gives:

$\delta_y(K)$ → $K' = G_{10}$

$\delta_y(K)$ → $K = F_8$

Hence $\tau(F_{12}) = 2\tau(G_{10}) + \tau(F_8)$

It is clear that in general

$$\tau(F_n) = 2\tau(G_{n-2}) + \tau(F_{n-4})$$

We may now deal with G_n in the same way, and show that:

$$\tau(G_n) = 2\tau(F_{n-2}) + \tau(G_{n-4})$$

Combining these two results gives:

(i) $\tau(F_n) = \frac{1}{2}(\tau(G_{n+2}) - \tau(G_{n-2})) = 6\tau(F_{n-4}) - \tau(F_{n-8})$

(ii) $\tau(G_n) = \frac{1}{2}(\tau(F_{n+2}) - \tau(F_{n-2})) = 6\tau(G_{n-4}) - \tau(G_{n-8})$

A corollary of (i) is that $\tau(G_{n+2}) - \tau(G_{n-2})$ is not divisible by 4, since $\tau(F_n)$ is always odd.

Table of values of τ for multiple figure-of-eight knots

ν $(n=4\nu)$	1	2	3	4	5	6	7	8	9	10
$\tau(F_n)$	5	-	29	-	169	-	985	-	5741	-
$\tau(G_n)$	-	12	-	70	-	408	-	2378	-	13860

Conway's Classes (Rational and Pretzel Knots)

We next obtain general formulae for two knot-classes arising out of Conway's notation for combining tangles (CONWAY, 1970). A brief description of the notation is given in Appendix III.

We shall need the following theorems.

Theorem 1

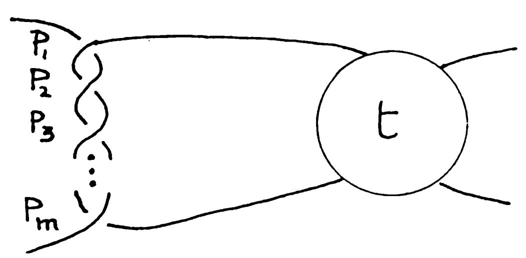
Let m be an integral tangle other than 0. Denote the knot-graphs $1^*(mt)$, $1^*(t\,0)$ and 1^*t by K, K_0, K_1 respectively.

Then $\tau(K) = m\tau(K_1) + \tau(K_0)$.

(Similarly, if $K = 1^*(tn)$ and $t \neq 0$, $\tau(K) = n\tau(K_1) + \tau(K_0)$; with the same method of proof.)

Proof:

The diagram below shows K. Applying the deletion operation and theorem first to P_1 in K, then to P_2 in $1^*((m-1)t)$, and so on to $P_3, \cdots, P_m$ gives the theorem.



Dealing with the tangle $0 = $) (

We define $\tau(0_1) \equiv 1$.

By direct calculation from the knot-graph of the n-twist we find that $\tau(0n) = \tau(n0) = 1$

Theorem 2 (formulae for $\tau(mn)$ and $\tilde{\tau}(mn)$)

Let mn denote the rational knot $1^*(mn)$.

Then:

(i) $\tau(mn) = mn + 1$;

(ii) $\tilde{\tau} = \begin{cases} mn + 1, & \text{if both } m \text{ and } n \text{ are even} \\ 2^{even-1} \times (odd + 1), & \text{if } m,n \text{ is even,odd or } odd,even \end{cases}$

If both m and n are odd, the knot is a 2-link which can be oriented in four different ways. The two possible results for $\tilde{\tau}$ are: $\tilde{\tau}(mn) = 2^{a-1}(b+1)$ if there are $(a-1)$ 2-gons like  , and $(b-1)$ like  , with $(a,b) = (m,n)$ or (n,m).

Proof:

(i) Applying theorem 1 to $1^*(mn)$ gives the result.

(ii) There are four cases to consider, as indicated in the following table:

(m,n)	even	odd
even	a	b
odd	c	d

Case (a) If $1^*(mn)$ is given a bno, then all 2-gons are cyclic; and the α-edges are all α-directed (or all β-directed). Then the adjacency matrix is the same as (or the transpose of) that for the bao case. Hence $\tilde{\tau} = \tau$.

Case (b) (case (c) is treated similarly)

Examination of a diagram for this case shows that m(even) 2-gons are non-cyclic, and n(odd) 2-gons are cyclic. Labelling the vertices suitably leads to the following in-degree matrix for the knot-graph: (N.B. we give the matrix for knot (43), as a convenient example)

$$Q = \begin{bmatrix} 2 & & & & -1 & & -1 \\ -2 & 2 & & & & & \\ & -2 & 2 & & & & \\ & & -2 & 2 & & & \\ & & & -1 & 2 & -1 & \\ & & & & -1 & 2 & -1 \\ & & & -1 & & -1 & 2 \end{bmatrix}$$

Applying Laplace's expansion method to Q_{11}, using its first $m-1$ rows, gives:

$$\tilde{\tau}(m,n) = 2^{m-1}(n+1)$$

Case (c) gives $2^{n-1}(m+1)$.

Case (d) When m,n are both odd, the knot is a 2-link. The strings may be oriented in such ways that either there are $m-1$ noncyclic 2-gons and $n-1$ cyclic 2-gons, or vice versa. In either case the formula as in (b) is obtained from the in-degree matrix Q.

Formulae for the class of rational knots $1^*(mnp...rst)$

Each of the letters $m, n, ..., t$ denotes an integral tangle. As usual, we shall omit $1^*(\ )$ when convenient.

Applying theorems 1 and 2 as required we obtain:

$$\tau(0) = 1$$

$$\tau(m) = m \quad (1)$$

$$\tau(mn) = mn + 1 \quad (2)$$

$$\begin{aligned}\tau(mnp) &= \tau((mn)p) \\ &= p.\,\tau(mn) + \tau((mn)0) \\ &= p.\,\tau(mn) + \tau(m) \\ &= p.\,(mn+1) + m \\ &= mnp + m + p \quad (3)\end{aligned}$$

$$\tau(mnp \qquad rst) = t.\,\tau(mnp \qquad rs) + \tau(mnp \qquad r) \quad (4)$$

Example:

$$\begin{aligned}\tau(3113) &= 3.\tau(311) + \tau(31) \\ &= 4.\tau(31) + 3.\tau(3) \\ &= 4.(3.1 + 1) + 3.3 \\ &= 25\end{aligned}$$

For a given number of crossings, a large proportion of knots belong in this class (Conway calls them *rational* knots.) It is seen that τ can be computed very easily by the above relations. For a given number of tangles, explicit formulae can be obtained for τ in terms of the separate integral tangles involved, by repeated use of the formula. We have already given these for 1, 2 and 3 tangles. For 4 and 5 they are

$$\tau(m_1m_2m_3m_4) = m_1m_2m_3m_4 + [m_1m_2 + m_1m_4 + m_3m_4] + 1 \quad (5)$$

$$\begin{aligned}\tau(m_1m_2m_3m_4m_5) &= m_1m_2m_3m_4m_5 \\ &\quad + [m_1m_2m_3 + m_1m_2m_5 + m_1m_4m_5 + m_3m_4m_5] \\ &\quad + [m_1 + m_2 + m_3] \quad (6)\end{aligned}$$

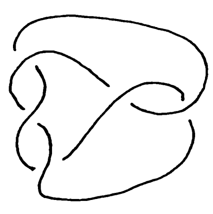
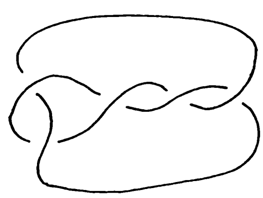
Corollary : (relative numbers of 1-links and 2-links). Since τ is odd for a 1-link and even for a 2-link (there are no knots in this class with more than 2 links), we can decide from the above formulae which strings of integral tangles will produce a 1-link and which a 2-link rational knot. The following table illustrates the cases mn and mnp.

Table: 1-link and 2-link class members

Knot mn	m	n	τ = mn+1	type
	even	even	odd	1-link
	even	odd	odd	1-link
	odd	even	odd	1-link
	odd	odd	even	2-link

Knot mnp	m	n	p	mnp	m+p	τ	type
	e	e	e	e	e	e	2-link
	e	e	o	e	o	o	1-link
	e	o	e	e	e	e	2-link
	e	o	o	e	o	o	1-link
	o	e	e	e	o	o	1-link
	o	e	o	e	e	e	2-link
	o	o	e	e	o	o	1-link
	o	o	o	o	e	o	1-link

It appears that among mn knots there will be 3 times as many 1-links as 2-links; and among mnp knots 5:3 times as many. But the matter is not so simple as that, because different permutations of the same set of integral tangles might produce homeomorphic knots. Further, combinations of different sets of integral tangles might produce equivalent knots. One immediate result in this direction is that $\tau(mn) = \tau(nm)$, $\tau(mnp) = \tau(pnm)$, $\tau(mnpr) = \tau(rpnm)$, $\tau(mnprs) = \tau(srpnm)$, as can be seen from our formulae. For example:

32 =  and 23 = 

have $\tau(32) = 7 = \tau(23)$. It is not immediately obvious that the first knot can be transformed into the other (equality of τ values does not guarantee it). In fact, it cannot! But one can be twisted into the obverse of the other; and we showed earlier that $\tau(K) = \tau(\text{obv}(K))$. (32 is not an *amphicheiral* knot, which means that it cannot be transformed into 23, its obverse).

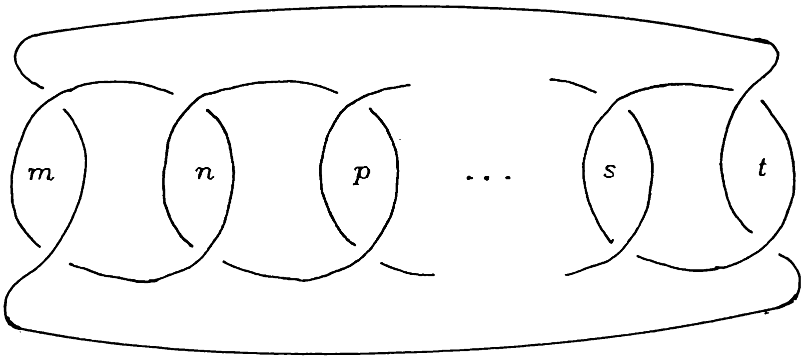
The 1- and 2-links listed in Alexander and Briggs tables of type 1*(mn) with $m + n \leq 10$, are:

1-links: 22, 32, 42, 52, 43, 62, 44, 72, 63, 54, 82, 64

2-links: 33, 53, 73.

The knot-class $1^*(m,n,p,\cdots,s,t)$, each tangle being integral

Conway uses the symbols m, n, p, ... , s, t to denote the combination of tangles $m0 + n0 + p0 + \quad + s0 + t0$. In the knot literature, a knot of this type is called a *Pretzel knot*. We give a diagram below, in which we use the symbol 'n' to mean a tangle consisting of n twists. All joins are alternating.



Formula for τ

Applying theorem 1, regarding the Pretzel as being $1^*(mT)$ where T is the tangle $n,p,...,s,t$, we obtain:

Theorem 3

$$\tau(m,p,...,s,t) = m.\tau(n,p,...,s,t) + \tau(n0 \# p0 \# \quad \# s0 \# t0)$$
$$= m.\tau(n,p,...,s,t) + np...st \qquad \text{(i)}$$

which leads to

$$\tau(m,n,p,...s,t) = s_\pi, \qquad \text{(ii)}$$

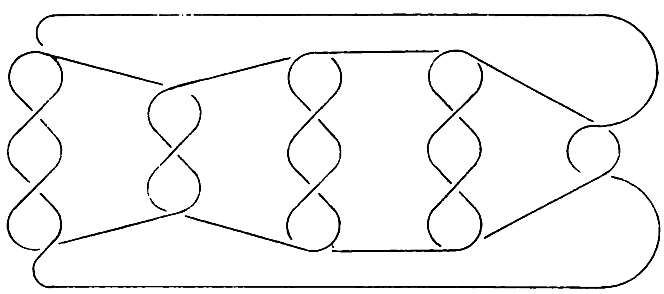
where s_π is the sum of products of all combinations of $(\nu-1)$ integers chosen from the set $\{m, n, p, \cdots, s, t\}$.

Proof: (i) uses theorem 1; (ii) follows by repeated use of (i). //

Example:

The pretzel knot shown is 1*(4, 3, 4, 4, 2) = P, say.

$\tau(P) = 4.\tau(3, 4, 4, 2) + 3.4.4.2$
$= 4.[3.\tau(4, 4, 2) + 32] + 96$
$= 4.[3.[4.\tau(4,2) + 8] + 32] + 96$
$= 4.[3.[4 \times 6 + 8] + 32] + 96$
$= 608$



Or, by (ii),

$\tau = s_\pi$
$= 4.3.4.4 + 4.3.4.2 + 4.4.4.2 + 4.3.4.2 + 3.4.4.2$
$= 608$

Notes

(i) One has to be careful with the last step; thus

$$\begin{aligned}\tau(m,n) &= m.\tau(n0) + n \\ &= m.\tau(0_1) + n \\ &= m + n \\ (&= \tau(n,m)).\end{aligned}$$

(ii) The next two formulae are:

$$\begin{aligned}\tau(m,n,p) &= m.n + n.p + p.m \\ (&= \tau(p,m,n) \\ &= \tau(n,m,p) \\ &= etc.)\end{aligned}$$

$$\begin{aligned}\tau(m,n,p,s) &= m.n.p + m.n.s + m.p.s + n.p.s \\ (&= \tau(s,p,n,m) \\ &= etc.)\end{aligned}$$

(iii) τ is invariant to permutation of positions of the constituent integral tangles.

Many more formulae could be developed from our theorems to enable us to compute τ for knots defined in different ways by Conway's tangle calculus. One simple example will make the point; then we shall not pursue the matter further.

Example:

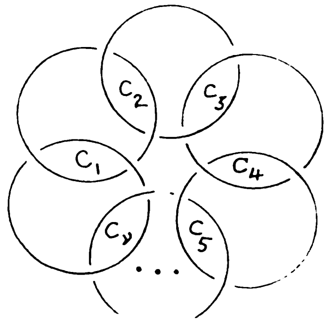
Let $K = 1^*(mn, p)$ with m, n and p integral, then

$$\begin{aligned}\tau(K) &= p.\tau((mn).0) + \tau(mn) \\ &= p.\tau(m) + m.n + 1 \\ &= p.m + m.n + 1\end{aligned}$$

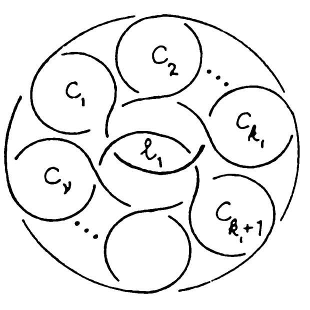
(drawing a diagram shows that $mn,p = m\overline{n + p}$

therefore $\tau(mn,p) = m.(n + p) + 1$

The Wheel, Or Double Circle With Chords Added



We call the knot illustrated on the left a *Wheel W* having *spokes vector* $\underline{c} = (c_1, c_2, \quad , c_\nu)$, where each c_i is an integral tangle with c_i crossings. It might be called a double circle with spokes, or a chain bracelet or a flower. After applying theorem 1 to it, we discovered that $\tau(W) = s_\pi$, the same as for the pretzel knot! Indeed, on inspection one can see easily how to transform one into the other; they are the same knot. However if we treat the knot as a 'double circle', the notion of inserting 'chords' (same as 'spokes') into the inner circle arises.



The next diagram shows a double circle with ν spokes and 1 chord k_1, (length l_1) inserted, maintaining the alternating nature of the knot, we can describe the configuration by writing

$$W[\underline{c}^1 \underset{\substack{k_1 \\ l_1}}{|} \underline{c}^2],$$

indicating that $\underline{c}$ has been partitioned by the chord k_1 into $\underline{c} = (\underline{c}^1, \underline{c}^2)$ where $\underline{c}^1 = (c_1, \cdots, c_{k_1})$ and $\underline{c}^2 = (c_{k+1}, \cdots, c_\nu)$.

Applying the deletion theorem l_1 times to the crossings in the chord we find:

$$\tau(W) = l_1.s_\pi(\underline{c}) + s_\pi(\underline{c}^1) \times s_\pi(\underline{c}^2)$$

Note that if $l_1 = 0$ (i.e. no crossings in k_1) the result is $s_\pi(\underline{c}^1) \times s_\pi(\underline{c}^2)$, the same as for the composition of two wheels $W(\underline{c}^1) \# W(\underline{c}^2)$. It is easy to see that the knot is indeed the combination of two such wheels, when $l_1 = 0$.

Special case

Let $c_1 = c_2 = \quad = c_\nu = c$. Then it is found from the formula that $\tau(W) = l_1.\nu.c^{\nu-1} + k_1.(\nu-k_1).c^{\nu-2}$

Generalisation

So long as there are sufficient spokes that disjoint knot-graphs do not result we can insert a vector $\underline{k} = (k_1, k_2, \quad , k_i)$ of the chords into the inner circle; this leads to a chorded wheel knot

$$W[\underline{c}^1 \underset{\substack{k_1\\ l_1}}{|} \underline{c}^2 \underset{\substack{k_2\\ l_2}}{|} \underline{c}^3 \underset{\substack{k_3\\ l_3}}{|} \qquad \underset{\substack{k_i\\ l_i}}{|} \underline{c}^{i+1}].$$

Applying the deletion theorem to each chord in turn we can arrive at a formula for W. For two- and three-chord wheel knots the formulae are:

$$\tau(W[\underline{c}^1 \underset{\substack{k_1\\ l_1}}{|} \underline{c}^2 \underset{\substack{k_2\\ l_2}}{|} \underline{c}^3]) = l_1.l_2.s_\pi(\underline{c}) + s_\pi(\underline{c}^1).s_\pi(\underline{c}^2).s_\pi(\underline{c}^3)$$
$$+ [l_1.s_\pi(\underline{c}^1 \cup \underline{c}^2).s_\pi(\underline{c}^3) + l_2.s_\pi(\underline{c}^1).s_\pi(\underline{c}^2 \cup \underline{c}^3)]$$

$$\tau(W[\underline{c}^1 \underset{\substack{k_1\\ l_1}}{|} \underline{c}^2 \underset{\substack{k_2\\ l_2}}{|} \underline{c}^3 \underset{\substack{k_3\\ l_3}}{|} \underline{c}^4]) = l_1.l_2.l_3.s_\pi(\underline{c}) + s_\pi(\underline{c}^1).s_\pi(\underline{c}^2).s_\pi(\underline{c}^3).s_\pi(\underline{c}^4)$$
$$+ [l_1.l_2.s_\pi(\underline{c}^{123})s_\pi(\underline{c}^4) + l_1.l_3.s_\pi(\underline{c}^{12}).s_\pi(\underline{c}^{34})$$
$$+ l_2.l_3.s_\pi(\underline{c}^1).s_\pi(\underline{c}^{234})] + [l_1.s_\pi(\underline{c}^{12}).s_\pi(\underline{c}^3).s_\pi(\underline{c}^4)$$
$$+ l_2.s_\pi(\underline{c}^1).s_\pi(\underline{c}^{23}).s_\pi(\underline{c}^4) + l_3.s_\pi(\underline{c}^1).s_\pi(\underline{c}^2).s_\pi(\underline{c}^{34})],$$

where we have used the notation $\underline{c}^{123} \equiv \underline{c}^1 \cup \underline{c}^2 \cup \underline{c}^3$; note that the order of the elements in a $\underline{c}$ vector does not matter. We could contract the notation still further, by writing $s_\pi(\underline{c}^{123}) = s^{123}$ and $l_1.l_2 = l_{12}$, e.g. Then with 3 chords:

$$\begin{aligned}\tau(W[\underline{c}, \underline{k}]) &= l_{123}.s^0 + s^1.s^2.s^3.s^4 \\ &+ \{l_{12}.s^{123}.s^4 + l_{13}.s^{12}.s^{34} + l_{23}.s^1.s^{234}\} \\ &+ \{l_1.s^{12}.s^3.s^4 + l_2.s^1.s^{23}.s^4 + l_3.s^1.s^2.s^{34}\}\end{aligned}$$

No doubt it would be possible to find a general formula,
when $\underline{k} = (k_1, k_2, \ldots, k_r)$; but we haven't pursued the matter thus far.

APPENDIX I

TABLES OF THE TREE NUMBERS τ AND $\tilde{\tau}$

The tree numbers in the following tables were computed from the inward degree matrices of the knot-graph; i.e. for each adjacency matrix J, the values of a principle minor of $2I - J$ was computed.

All the tree numbers are of *prime alternating* knots.

The table contents are as follows:

I.1 $(n, \tau, \tilde{\tau})$ for the 1-links, with $n = 3, \ldots, 8$

I.2 $(n, \tau, \tilde{\tau})$ for the 1-links, with $n = 9$

I.3 $(n, \tau, \tilde{\tau})$ for the 1-links, with $n = 10$

I.4 $(n, \tau, \tilde{\tau}, \tilde{\tau}_2)$ for the 2-links, with $n = 3, \ldots, 9$

Appendix I. 1

THE TREE NUMBERS $(\tau, \tilde{\tau})$
OF THE PRIME ALTERNATING 1-LINKS (n = 3, ..., 8)

n = 3			n = 4			n = 5		
KNOT	τ	$\tilde{\tau}$	KNOT	τ	$\tilde{\tau}$	KNOT	τ	$\tilde{\tau}$
3_1	3	4	4_1	5	5	5_1	5	16
						5_2	7	8

n = 6			n = 7			n = 8		
KNOT	τ	$\tilde{\tau}$	KNOT	τ	$\tilde{\tau}$	KNOT	τ	$\tilde{\tau}$
6_1	9	9	7_1	7	64	8_1	13	13
6_2	11	20	7_2	11	18	8_2	17	80
6_3	13	20	7_3	13	32	8_3	17	17
			7_4	15	16	8_4	19	36
			7_5	17	64	8_5	21	80
			7_6	19	24	8_6	23	36
			7_7	21	25	8_7	23	80
						8_8	25	36
						8_9	25	80
						8_{10}	27	80
						8_{11}	27	40
						8_{12}	29	29
						8_{13}	29	40
						8_{14}	31	40
						8_{15}	33	48
						8_{16}	35	84
						8_{17}	37	84
						8_{18}	45	85

All these knots for $n \leq 7$ are discriminated by (n, τ).
All these knots for $n = 8$ are discriminated by $(\tau, \tilde{\tau})$.

Appendix I.2

THE TREE NUMBERS (τ, $\tilde{\tau}$) OF THE PRIME ALTERNATING 1-LINKS (n = 9)

n = 9								
KNOT	τ	$\tilde{\tau}$	KNOT	τ	$\tilde{\tau}$	KNOT	τ	$\tilde{\tau}$
9_1	9	256	9_{15}	39	44	9_{29}	51	104
9_2	15	16	9_{16}	39	128	9_{30}	53	100
9_3	19	128	9_{17}	39	100	9_{31}	55	100
9_4	21	48	9_{18}	41	64	9_{32}	59	104
9_5	23	24	9_{19}	41	45	9_{33}	61	104
9_6	27	40	9_{20}	41	96	9_{34}	69	109
9_7	29	128	9_{21}	43	48	9_{35}	27	28
9_8	31	40	9_{22}	43	100	9_{36}	37	96
9_9	31	128	9_{23}	45	64	9_{37}	45	49
9_{10}	33	64	9_{24}	45	96	9_{38}	57	80
9_{11}	33	96	9_{25}	47	56	9_{39}	55	64
9_{12}	35	44	9_{26}	47	100	9_{40}	75	112
9_{13}	37	64	9_{27}	49	100	9_{41}	49	61
9_{14}	37	45	9_{28}	51	96			

All these 9-knots are discriminated by the pair (τ, $\tilde{\tau}$).

Appendix I.3

THE TREE NUMBERS $(\tau, \tilde{\tau})$ OF THE PRIME ALTERNATING 1-LINKS (n = 10) KNOT NOTATION $10_i : i = 1, \cdots, 123$

n = 10											
KNOT			KNOT			KNOT			KNOT		
i	τ	$\tilde{\tau}$	i	τ	$\tilde{\tau}$	i	τ	$\tilde{\tau}$	i	τ	$\tilde{\tau}$
1	17	17	32	57	72	63	57	80	94	71	336
2	23	320	33	69	160	64	51	320	95	91	176
3	25	25	34	65	80	65	63	160	96	93	129
4	27	52	35	37	52	66	75	192	97	87	96
5	33	320	36	49	49	67	63	76	98	81	176
6	37	144	37	51	60	68	57	76	99	81	336
7	43	60	38	53	68	69	87	124	100	65	336
8	29	144	39	59	72	70	67	116	101	85	112
9	39	320	40	61	160	71	77	116	102	73	164
10	45	60	41	71	120	72	73	160	103	75	164
11	43	68	42	81	120	73	83	120	104	77	336
12	47	144	43	73	116	74	63	80	105	91	136
13	53	53	44	79	120	75	81	125	106	75	336
14	57	160	45	89	125	76	57	144	107	93	136
15	43	144	46	31	320	77	63	144	108	63	164
16	47	72	47	41	320	78	69	112	109	85	336
17	41	320	48	49	320	79	61	320	110	83	136
18	55	72	49	59	192	80	71	192	111	77	176
19	51	160	50	53	160	81	85	128	112	87	340
20	35	52	51	67	160	82	63	336	113	111	184
21	35	52	52	59	160	83	83	168	114	93	169
22	45	160	53	73	96	84	87	164	115	109	144
23	49	144	54	47	144	85	57	336	116	95	340
24	59	160	55	61	80	86	85	168	117	103	184
25	65	57	56	65	160	87	81	164	118	97	340
26	65	160	57	79	160	88	101	129	119	101	184
27	61	160	58	65	65	89	99	128	120	105	128
28	71	160	59	75	120	90	77	164	121	115	188
29	53	72	60	85	125	91	73	336	122	105	185
30	63	116	61	33	144	92	89	176	123	121	341
31	67	80	62	45	320	93	67	164			

The following pairs of knots have equal values of $(\tau, \tilde{\tau})$: $(10_{20}, 10_{21})$, $(10_{17}, 10_{47})$, $(10_{12}, 10_{54})$, $(10_{24}, 10_{52})$, $(10_{26}, 10_{56})$.

Appendix I.4

THE TREE NUMBERS (τ, $\tilde{\tau}$, $\tilde{\tau}_2$) OF THE PRIME ALTERNATING 2-LINKS ($n = 2, \ldots, 9$)

n = 2				n = 4				n = 5				n = 6			
KNOT 2^2	τ	$\tilde{\tau}_1$	$\tilde{\tau}_2$	KNOT 4^2	τ	$\tilde{\tau}_1$	$\tilde{\tau}_2$	KNOT 5^2	τ	$\tilde{\tau}_1$	$\tilde{\tau}_2$	KNOT 6^2	τ	$\tilde{\tau}_1$	$\tilde{\tau}_2$
1	2	2	2	1	4	4	8	1	8	10	10	1	6	6	32
												2	10	16	16
												3	12	12	16

n = 7				n = 8				n = 8				n = 8			
KNOT 7^2	τ	$\tilde{\tau}_1$	$\tilde{\tau}_2$	KNOT 8^2	τ	$\tilde{\tau}_1$	$\tilde{\tau}_2$	KNOT 8^2	τ	$\tilde{\tau}_1$	$\tilde{\tau}_2$	KNOT 8^2	τ	$\tilde{\tau}_1$	$\tilde{\tau}_2$
1	14	18	40	1	8	8	128	6	20	20	24	11	28	48	64
2	18	20	40	2	16	24	64	7	30	48	50	12	32	48	48
3	16	18	18	3	22	22	64	8	34	50	50	13	40	52	52
4	16	40	40	4	24	32	64	9	28	28	48	14	36	40	52
5	20	24	40	5	26	32	48	10	32	50	50				
6	24	42	42												

n = 9															
KNOT 9^2	τ	$\tilde{\tau}_1$	$\tilde{\tau}_2$	KNOT 9^2	τ	$\tilde{\tau}_1$	$\tilde{\tau}_2$	KNOT 9^2	τ	$\tilde{\tau}_1$	$\tilde{\tau}_2$	KNOT 9^2	τ	$\tilde{\tau}_1$	$\tilde{\tau}_2$
1	20	26	160	12	50	60	160	23	36	38	160	34	50	82	168
2	28	30	160	13	24	160	160	24	54	62	96	35	46	82	168
3	30	34	72	14	36	96	160	25	48	58	58	36	48	82	82
4	24	72	72	15	40	80	80	26	52	60	80	37	48	168	168
5	32	36	160	16	44	48	80	27	40	72	72	38	60	88	88
6	36	72	80	17	36	40	72	28	44	72	56	39	54	88	168
7	44	80	80	18	48	80	80	29	44	96	160	40	50	56	168
8	34	36	72	19	26	72	160	30	52	64	96	41	56	68	168
9	40	80	80	20	34	40	160	31	40	168	168	42	66	92	170
10	24	26	26	21	38	80	160	32	56	84	84				
11	46	58	80	22	46	80	96	33	56	82	82				

Only the pair (9^2_9, 9^2_{15}) have the same triple (40, 80, 80).

APPENDIX II

TWO PROOFS OF THE DELETION THEOREM

In section 6.3, theorem 3, we proved that the sum of the tree numbers of a pair of twins, obtained by deleting a vertex from a knots-graph with a balanced orientation, is equal to the tree number of the original knot-graph. We gave a proof which used a theorem of Aardenne-Ehrenfest and de Bruijn (1951) which related the number of distinct Euler chains to the tree number.

In this Appendix we give two other proofs, the first using a graph-theoretic method, and the second working directly with the adjacency matrices of the knot-graphs concerned.

II(a) A GRAPH-THEORETIC PROOF

Before giving the proof we will re-state the theorem to establish the symbols to be used. We shall treat the alternating knot case here: but the same method may be used for the nonalternating case.

Theorem (deletion theorem)

Let K be an alternating knot-graph with a balanced alternating orientation. Let i be any vertex of K, and τ be the number of directed spanning trees which are rooted at i.

Let (K_i, K'_i) be the pair of twins resulting from the deletion of vertex i. Then if τ_i, τ'_i are the tree numbers of K_i, K'_i respectively, $\tau = \tau_i + \tau'_i$.

Proof:

Recall that for any alternating knot-graph, the number τ is independent of which vertex is chosen as root (see section 6.3). We must distinguish three cases, namely

(1) the deleted vertex i is adjacent to two 2-gons;

(2) the deleted vertex i is adjacent to one 2-gons;

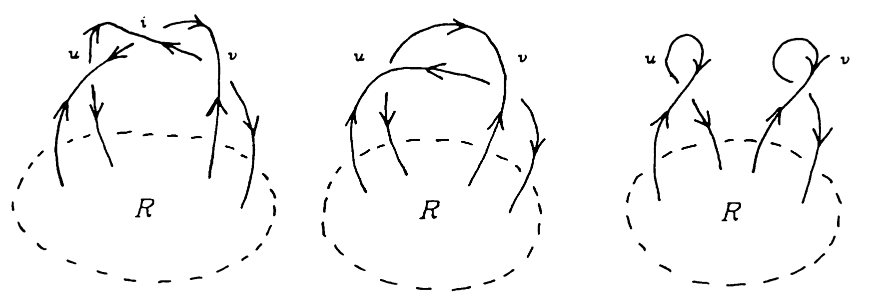
(3) the deleted vertex i is adjacent to no 2-gons.

(An example to illustrate the method of proof for case (1) is given after the conclusion of the proof.)

Case (1)

The arrangements of vertices and arcs of the knot-graphs K, K_i, K'_i are as shown below.

Knot-graph K | Knot-graph K_i | Knot-graph K'_i (leaving loops in)



$R \equiv$ rest of knot

Note that we have left the loops in the knot-graph K'_i. Theorem 10 tells us that τ is invariant to addition or deletion of loops. Consider the following sets:

$\Theta_{K,i}$ consisting of all r.d.s.t.(i) on K;

$\Theta_{K_i,u}$ consisting of all r.d.s.t.(u) on K_i;

$\Theta_{K'_i,v}$ consisting of all r.d.s.t.(v) on K'_i.

Note that no member of $\Theta_{K_i,u}$ can contain the arc $[v, u]$, since u is the root.

We shall demonstrate a one-one mapping from $\Theta_{K,i}$ to $\Theta_{K_i,u} \cup \Theta_{K_i',v}$, as follows.

Partition $\Theta_{K,i}$ into disjoint sets T_1, T_2, T_3 where

T_1 contains those r.d.s.t.(i) which use both arcs $[i, u]$ and $[i, v]$,

T_2 contains those which use arc $[i, v]$ but do not use $[i, u]$,

T_3 contains those which use $[i, u]$ but do not use $[i, v]$.

(a) Select any r.d.s.t.(v),say $t \in \Theta_{K'_i,v}$, and attach an arc $[i, v]$ to it. This produces a directed tree on n vertices which is rooted at i and is isomorphic to just one element of T_2. Therefore $|T_2| \geq |\Theta_{K'_i,v}| = \tau'_i$.

Similarly, if we take any r.d.s.t.(i) from T_2 and remove arc $[i, v]$, we obtain an r.d.s.t.(v) which is isomorphic to a single member of $\Theta_{K_i,v}$. Hence $|T_2| \leq |\Theta_{K'_i,u}|$; therefore $|T_2| = \tau'_i$.

(b) The set $\Theta_{K_i,u}$ may be partitioned into sets U_1 and U_2, where

members of U_1 use arc $[u, v]$, and

members of U_2 do not use arc $[u, v]$.

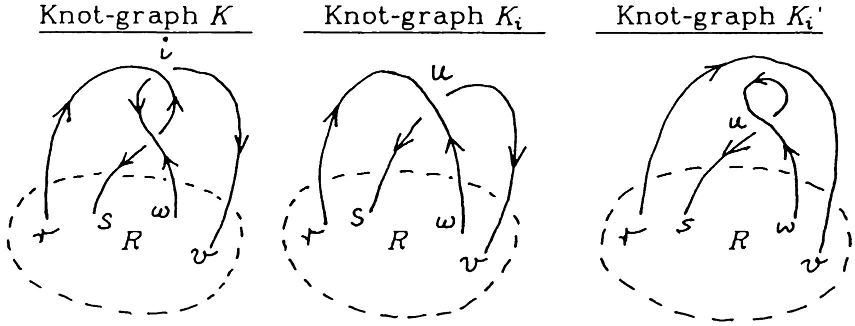
Select any r.d.s.t.(u), say $t \in U_1$. Attach the arcs $[i, u]$ and $[i, v]$ to it, and remove $[u, v]$. The result is an r.d.s.t.(i) which is isomorphic to a single member of the set T_1. Therefore $|U_1| \leq |T_1|$. Carrying out the operations in reverse order on any member of T_1 produces an isomorph of a member of U_1, therefore $|T_1| \leq |U_1|$; so $|T_1| = |U_1|$.

(c) Finally, select any r.s.d.t.(u) from U_2. Attach $[i, u]$ to it, and thereby obtain an isomorph of some member of T_3. The inverse operations on a member of T_3 leads similarly to a unique member of U_2. Hence $|U_2| = |T_3|$.

Therefore

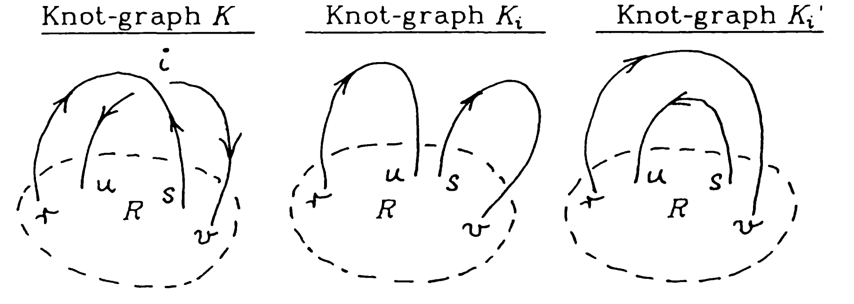
$$\begin{aligned}\tau &= |\Theta_{K,i}| \\ &= (|T_1| + |T_3|) + |T_2| \\ &= (|U_1| + |U_2|) + \tau_i' \\ &= \tau_i + \tau_i'\end{aligned}$$

Case (2) diagrams: i is adjacent to one 2-gon



The proof for case (2) is exactly the same as for case(1).

Case (3) : the deleted vertex i is not adjacent to a 2-gon.



Proof:

As for case(1), partition $\Theta_{K,i}$ into the same class $\{T_1, T_2, T_3\}$.

We may partition $\Theta_{K_i.u}$ and $\Theta_{K_i'.v}$ as follows:

$$\Theta_{K_i.u} = A \dot{\cup} B$$

where A contains only trees with $[s, v]$, and B contains only trees without $[s, v]$.

$$\Theta_{K_i'.v} = C \dot{\cup} D,$$

where C contains only trees with $[s, u]$, and D contains only trees without $[s, u]$.

Now set T_1 is in 1-1 correspondence with the union of set A (attaching arcs $[i, u]$ and $[i, v]$ to a member of A, and deleting $[s, v]$, gives a member of T_1) and set C (attaching arcs $[i, u]$ and $[i, v]$ to a member of C, and deleting $[s, u]$, gives a member of T_1).

T_2 is in 1-1 correspondence with set D (attaching $[i, v]$ to a member of D gives a member of t_2). T_3 is in 1-1 correspondence with set B (attaching $[i, u]$ to a member of B gives a member of T_3).

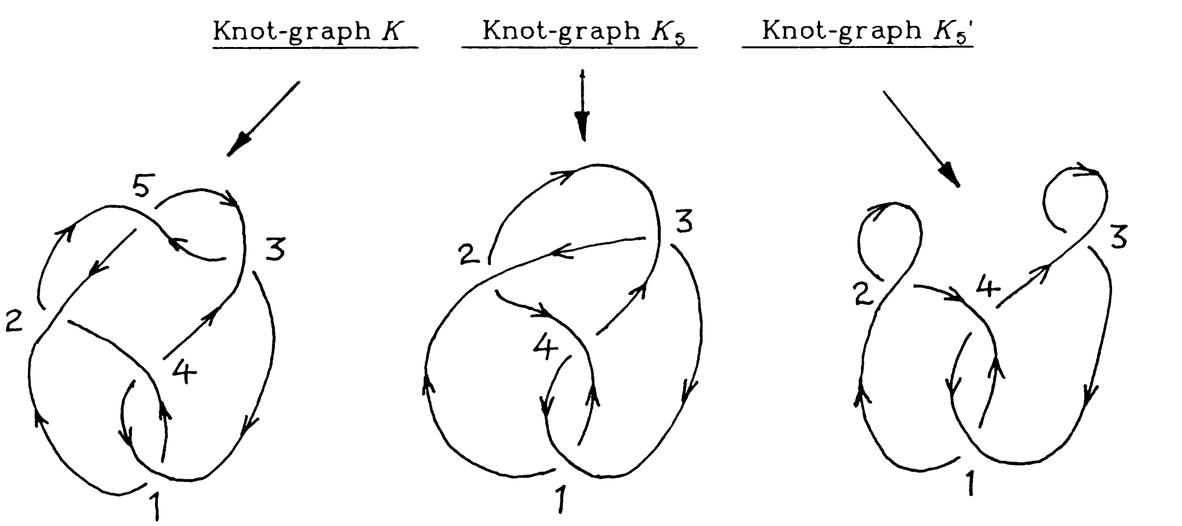
All the above operations invert and have unique inverses. Therefore

$$\begin{aligned} \tau &= |T_1| + |T_2| + |T_3| \\ &= (|A| + |C|) + |D| + |B| \\ &= |\Theta_{K_i.u}| \end{aligned}$$

$$\rightarrow |\Theta_{K_i'.v}| = \tau_i + \tau_i'$$

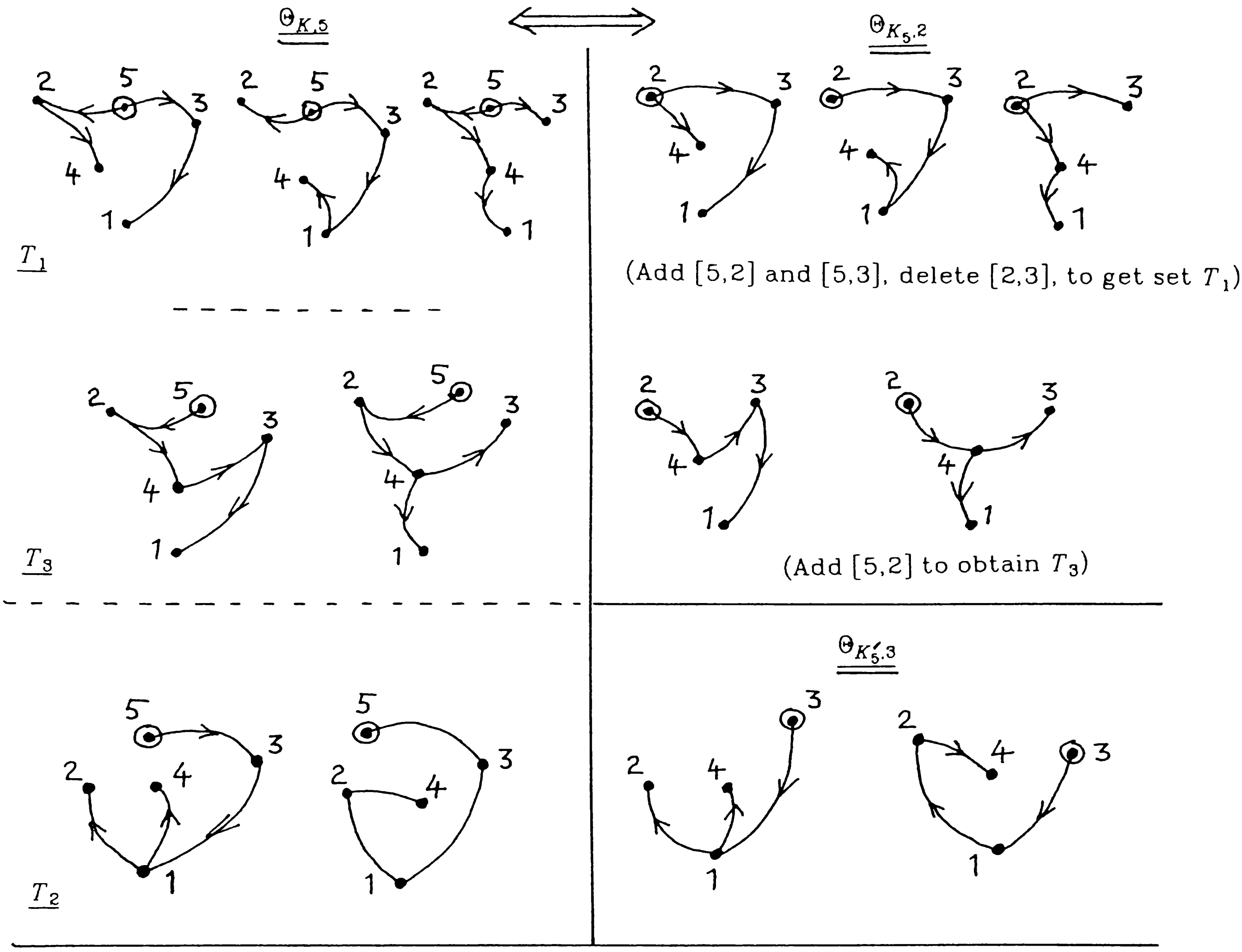
Example of case(1): Knot 5_2

The vertex to be deleted is $i = 5$, having two adjacent 2-gons; its adjacent vertices are $u = 2$ and $v = 3$.



On the next page we show the sets T_1, T_2, and T_3 of rooted directed spanning trees of K, together with the corresponding sets obtained from the twins.

Sets of rooted directed spanning trees



Result:

$$|\Theta_{K.5}| = |\Theta_{K_5.2}| + |\Theta_{K_5'.3}| \quad , \qquad \text{i.e. } \tau = \tau_5 + \tau_5' = 5 + 2 = 7$$

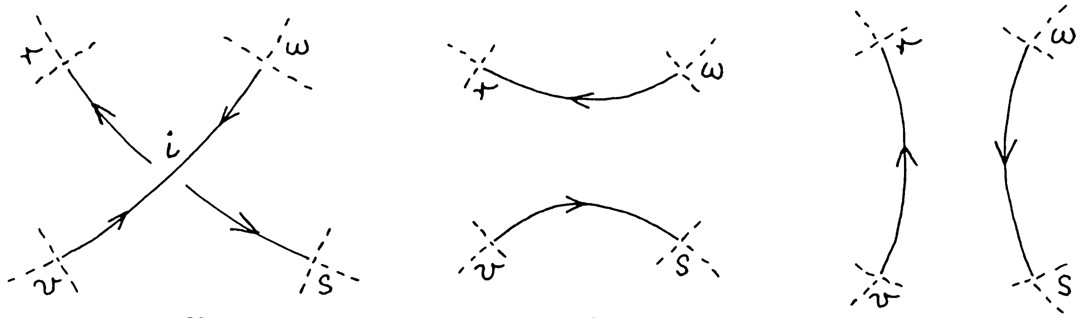
II(b) A MATRIX PROOF

We next give a matrix proof of the deletion theorem, for the case of knot-graphs with balanced alternating orientations. The same proof can be used to show that the theorem holds when the balanced orientation is nonalternating (and then one of the twins has an 'extra' non-labelled crossing).

Many attractive results can be derived in terms of submatrices of Q, related to deletion operations on the knot-graph. One such is given in (iii) below, without proof.

(i) The alternating case (bao)

The diagram below shows only the portions of the knot-graph K, and of the twins K',K'', which are adjacent to the vertex i which is being deleted.



K $\qquad$ K' $\qquad$ K''

Knot-graph $\qquad$ Twins, after deleting vertex i

Let the Kirchoff matrix of K be $Q = [q_{ij}]$, and the α*-adjacency* matrix be $J = [a_{ij}]$. We may suppose, without loss of generality, that the vertices are labelled such that the first five rows and columns of Q (and J) correspond to i, v, w, r, s in that order.

Then we have:

$$Q = 2I - J = \begin{array}{c} \\ i \\ v \\ w \\ r \\ s \\ \vdots \end{array} \begin{array}{ccccc} i & v & w & r & s \\ 2 & q_{12} & q_{13} & q_{14} & q_{15} \\ q_{21} & 2 & q_{23} & q_{24} & q_{25} \\ q_{31} & q_{32} & 2 & q_{34} & q_{35} \\ q_{41} & q_{42} & q_{43} & 2 & q_{45} \\ q_{51} & q_{52} & q_{53} & q_{54} & 2 \\ \cdots & \cdots & \cdots & \cdots & \end{array}$$

with $q_{21} = -1$, $q_{31} = -1$, $q_{14} = -1$, $q_{15} = -1$, and all other elements labelled q, in the first row and first column, being 0.

Denoting the Kirchoff matrices of the two twins K' and K'' by Q' and Q'' respectively, we have:

$$Q' = \begin{array}{c} \\ v \\ w \\ r \\ s \\ \vdots \end{array} \begin{array}{ccccc} v & w & r & s & \cdots \\ 2 & q_{23} & q_{24} & q_{25}-1 & \\ q_{32} & 2 & q_{34}-1 & q_{35} & \cdots \\ q_{42} & q_{43} & 2 & q_{45} & \cdots \\ q_{52} & q_{53} & q_{54} & 2 & \cdots \\ \cdots & \cdots & \cdots & & \cdots \end{array}$$

and

$$Q'' = \begin{array}{c} \\ v \\ w \\ r \\ s \\ \vdots \end{array} \begin{array}{ccccc} v & w & r & s & \cdots \\ 2 & q_{23} & q_{24}-1 & q_{25} & \\ q_{32} & 2 & q_{34} & q_{35}-1 & \\ q_{42} & q_{43} & 2 & q_{45} & \\ q_{52} & q_{53} & q_{54} & 2 & \\ \cdots & \cdots & \cdots & & \end{array}$$

We may find the tree numbers τ, τ', τ'' by computing any n-cofactor from Q, and any (n-1)-cofactor from each of Q' and Q''.

We show below how the result is obtained directly by taking the $(v,i)^{th}$ cofactor from Q, and expanding it by it's top row.

$$\tau = - \begin{vmatrix} q_{12} & q_{13} & q_{14} & q_{15} \\ q_{32} & 2 & q_{34} & q_{35} \\ q_{42} & q_{43} & 2 & q_{45} \\ q_{52} & q_{53} & q_{54} & 2 \\ \dots & \dots & \dots & \end{vmatrix}$$

$$= \quad (v,i)^{th} \quad \textit{cofactor of } Q$$

$$= + \begin{vmatrix} q_{32} & 2 & q_{35} \\ q_{42} & q_{43} & q_{45} \\ q_{52} & q_{53} & 2 \\ \dots & \dots & \end{vmatrix}$$

$$+ (-) \begin{vmatrix} q_{32} & 2 & q_{34} \\ q_{42} & q_{43} & 2 \\ q_{52} & q_{53} & q_{54} \\ \dots & \dots & \dots \end{vmatrix}$$

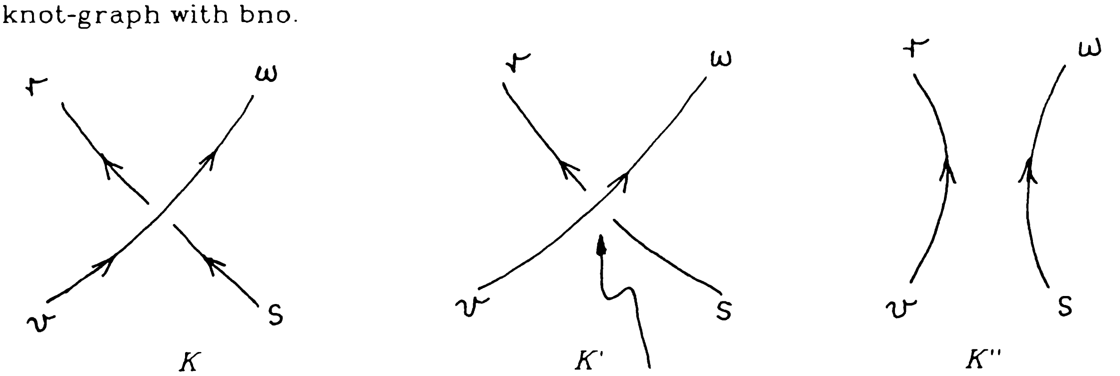
on expanding by the top row, and noting that $q_{12} = 0$ (no arc from i to v in K), $q_{13} = 0$, $q_{14} = -1$, $q_{15} = -1$.

But the two determinants on the R.H.S. are respectively the cofactor of q_{24} in Q' and of q_{25} in Q''.

Hence $\tau = \tau' + \tau''$.

(ii) The nonalternating case (bno)

The diagram below shows what happens when vertex i is deleted from a knot-graph with bno.



K K' K''

(the crossing now is not a vertex)

Note that after i is deleted there are still two ways in which the strings can be rejoined, to leave bno graphs; but one leaves a graph with an 'extra' crossing, not to be regarded as a vertex.

By redefining elements of Q in the proof given in (i) above, we could similarly prove that

$$\tilde{\tau}(K) = \tilde{\tau}(K') + \tilde{\tau}(K'').$$

(iii) Corollary

We give just one attractive matrix result which relates the tree numbers in special cases. Many more may be found.

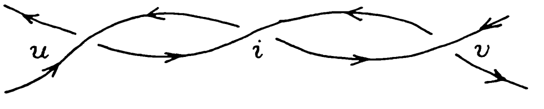
Let K, a bao knot-graph, contain a pair of adjacent 2-gons as shown below.

Let the common vertex i be deleted.

Then:

$$(\tau', \tau'') = \frac{1}{2}(\tau \pm Q_{iuv.iuv}),$$

where $Q_{iuv.iuv}$ is the cofactor obtained from Q by deleting the rows and columns i, u, v.



The proof requires only selection and manipulation of appropriate cofactors from Q.

APPENDIX III

CONWAY'S NOTATION FOR KNOT CONSTRUCTION

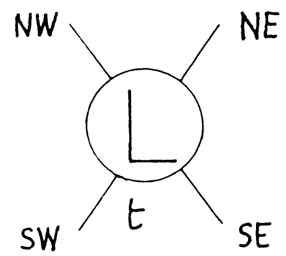
In (*CONWAY*,1970) is given a notation which describes knots in a way that reflects their construction from what he calls *tangles*. He gives definitions and rules for manipulation and equivalences of tangles; and he explains how knot-tables can be constructed by their use.

In this thesis we have used the notation in chapters 6 and 7, in order to defined certain knot classes for which we could obtain general formulae for knot parameters. We present here just enough of the notation to enable those passages to be understood.

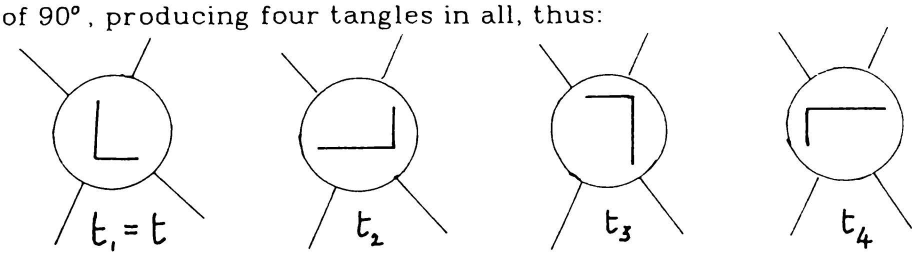
III.1 NOTATION FOR TANGLES

Definitions: A *tangle* is a portion of a knot-diagram from which there emerge just four edges pointing in the directions NW, NE, SW, SE. The NW edge is called the *leading* string of the tangle, and the NW-SE axis its *principal diagonal*.

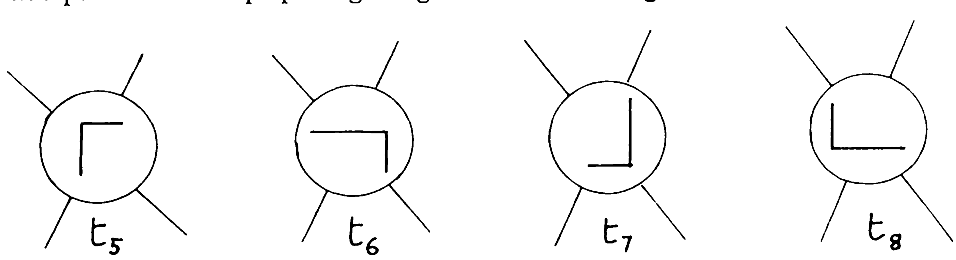
Diagrammatic representation: A typical tangle t is represented diagrammatically by a circle containing an L-shaped symbol, with the letter t placed nearby. Thus:



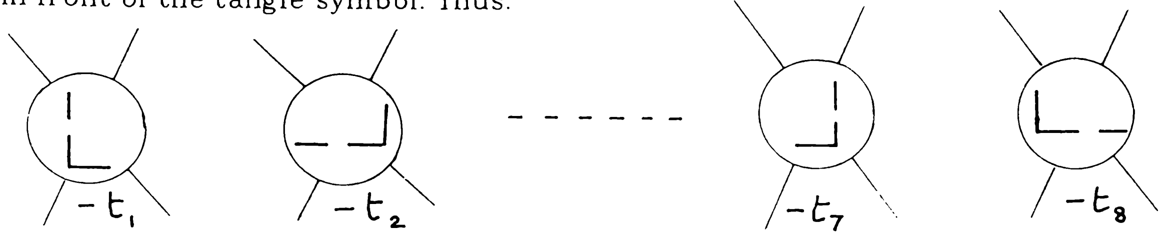
Operations on tangles: A tangle t may be rotated in the plane through multiples of 90°, producing four tangles in all, thus:



Each of these may now be reflected in a plane through W-E and perpendicular to the plane of the paper, giving four more tangles, thus:

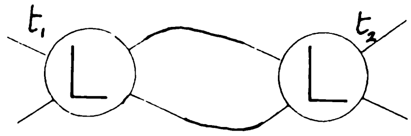


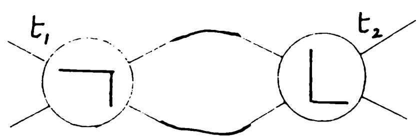
Notice that in all the above eight tangles, the 'front', or 'top', of t has been preserved. If now each is reflected in the plane of the paper, taking reflection of this kind is denoted by making a 'break' in the L-sign, and placing a minus sign in front of the tangle symbol. Thus:



With these sequences of operations, sixteen distinct tangles can be obtained from t.

Combinations of tangles: Only two methods of combining tangles will be given here, namely $t_1 + t_2$ and t_1t_2; diagrams will suffice to define them.

$t_1 + t_2 \equiv$ 

$t_1t_2 \equiv$ 

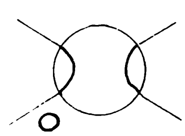
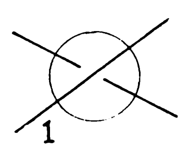
Extension to combinations of more than two tangles is carried out in a natural way. We indicate how it is to be done by means of brackets: if $m, n, p, \ldots, s, t$ are tangles, then

$$m + n + p + \qquad + s + t \equiv ((\ldots(m + n) + p) + \ldots + t)$$

$$\text{and} \quad mnp\ldots st \equiv ((\ldots(mn)p\ldots s)t) \quad .$$

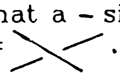
Clearly brackets are fully associative in the former, but only associated to the left in the latter.

Basic and integral tangles:

Basic tangles are  , and 

Tangles formed from these, using the above two combining operations, are called *algebraic*. In particular, the following algebraic tangles are called *integral tangles*:

$$n = 1 + 1 + \qquad + 1 \quad \text{and} \quad \bar{n} = -n = \bar{1} + \bar{1} + \qquad + \bar{1}.$$

Recall that a – sign indicates reflection in the plane of the paper, and note that $\bar{1} \equiv -1 =$  .

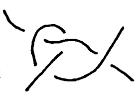
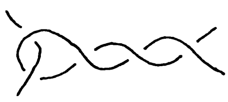
Examples:

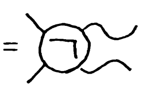
2 = 1+1 =  ; 3 =  ; 4 =  , etc.

Rational tangles:

Applying the second operation to integral tangles produces what are called *rational tangles*.

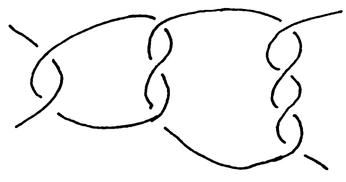
Examples:

21 =  ; 211 =  ; $2\bar{3}$ = 

Note that $t_1 0$ = , therefore $t_1 t_2 = t_1 0 + t_2$ (using the definition of $t_1 t_2$).

The comma notation:

It is convenient to use the notation $(a, b, \ldots, r)$ for the tangle $a0 + b0 + \quad + r0$.

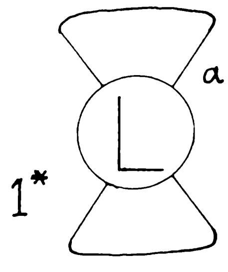
Example: (2,3,4) ≡ 20 + 30 + 40 = 

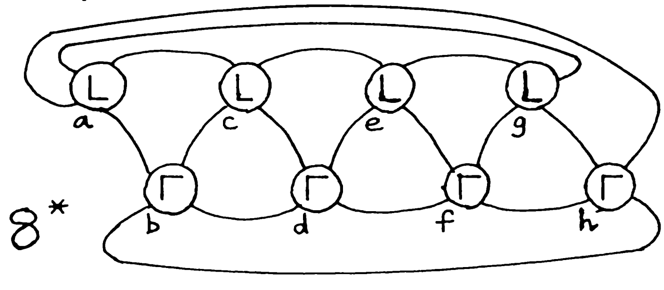
III.2 NOTATION FOR KNOTS

Definition: An edge-connected 4-valent planar graph is called a ***polyhedron***. A polyhedron without any 2-gon is called a ***basic polyhedron***.

Fact: Any knot-diagram K can be obtained by substituting algebraic tangles for the vertices of some basic polyhedron P.

Conway lists the eight basic polyhedra which are needed to construct all prime knots for $n = 3, \ldots, 11$ crossings, showing how algebraic tangles are to be substituted at their vertices. We give only the first two:

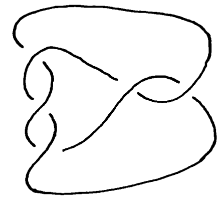




Knot-diagrams obtained from these basic polyhedra would be described 1^*a (or $1^*(a)$, or just a), and $8^*a.b.c.d.e.f.g.h.$ respectively.

Examples:

$\bar{2}$ =  , so $1*(\bar{2})$ = 

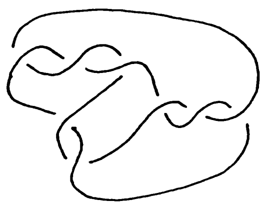
32 =  , so 1*(32) = 32 = 

III.3 KNOT CLASSES

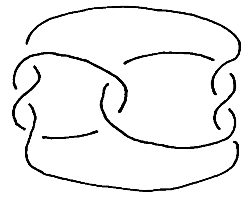
In chapters 6 and 7 we obtain formulae for certain parameters of the knot-diagrams obtained by inserting the rational tangle into the basic polyhedron $1^{\bullet}$; similarly we insert the tangle $m,n,p,\ldots,r,s,t$ in the first polyhedron.

Examples

(*i*) $1^{\bullet}(323)$ $\equiv 323 =$



(*ii*) $1^{\bullet}(3,\bar{2},3)$ $\equiv 3,\bar{2},3 =$



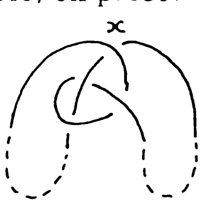
Notes: Members of the class $(m,n,p,\ldots,t)$ are known in the knot literature as *pretzel knots* .

The class of torus knots is a subclass of the first class: thus $1^{\bullet}(3) = 3_1$, the trefoil, and etc.

III.4 THE FLYPING OPERATION

In (TAIT,b,(1877),p.290) is described an operation called ***flyping***, whose nearest English phrase equivalent Tait gives as 'turning outside in'. He gives the following example, on p.289:

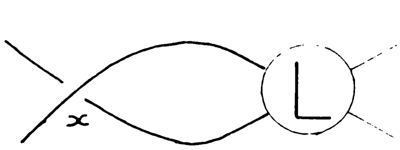
(*i*)



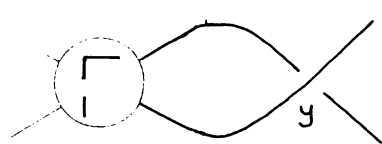
(*ii*)

He says: 'It may be seen that by twisting, the dotted parts being held fixed, either of these may be changed into the other, or changed to its own reverse (as from left to right).' (N.B. twisting from (i) to (ii) involves the loss of vertex x and the introduction of vertex y.)

Conway's system makes use of flyping equivalences, which in tangle representation is expressible thus:



$\approx$



$1 + t$ $\qquad$ $t_h + 1$

where t_h is obtained from t by rotation about the W-E axis. The untwisting of x in $1 + t$, by rotating tangle t, produces the new crossing y to the right of t_h.

CHAPTER 7

FAMILIES OF TWINS; TWIST SPECTRA

In this chapter we study what happens when the deletion operator is applied to each vertex in turn of a given alternating knot-graph K.

We find that a family of twins results, and we investigate properties of such families.

Later we apply the deletion operator repeatedly to a knot-graph and the descendant twins, until a point is reached when a collection of so-called 'twists' is obtained. The distribution of twists is called the 'twist spectrum' of the knot. It has many interesting topological properties, a study of which ends the chapter.

7.1 THE DELETION FAMILY OF A KNOT

We begin by defining families of twins from knot-graphs.

Definitions

Let K be an alternating knot-graph with labelling index set $\Lambda = \{1,...,n\}$. Applying the deletion operation to vertex $i \in \Lambda$ gives (see 6.3) $\delta_i(K) \to (K_i, K'_i)$, where the pair of knot-graphs K_i and K'_i have been called the *twins* produced by the operation.

Denote the twin-pair (K_i, K'_i) by T_i.

Then we define:

$$F(K) = \{T_i \mid i \in \Lambda\}$$

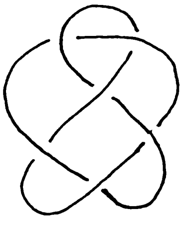
We call $F(K)$ *the family* of the knot K.

Notes

(i) We have remarked earlier that the members of a twin-pair cannot be identical, since always one is a 1-link and the other a 2-link.

(ii) We could have called $F(K)$ the 'first-generation family of K', and gone on to define second- and higher-generation families. We do not want to introduce a more complex notation at this stage, however.

Example



If we take the knot-graph shown for knot 6_2, and make the deletion operation on each of its vertices, we obtain six pairs of twins; but only three of these pairs are distinct. They, and their frequencies, are as follows.

Family $F(6_2)$: $\{(3_1, 5_1^2), (5_1, 3_1 \# 2_1^2), (5_2, 4_1^2)\}$

The frequencies of the twin-pairs (from left to right) are 3, 1, 2 .

In the following table we give the families of the first seven prime alternating knots, together with the frequencies of occurrence of the individual twin-pairs. (These were computed from the knot-diagrams given in ROLFSEN. The question of invariance of knot families will be discussed later.) The families for the knots on $n = 7,8$ crossings, again from the diagrams in ROLFSEN, are tabulated in Appendix I.

Table: Families of twins; prime 1-links; $n = 3,4,5,6$

Knot K	Twin-pairs K_i	K'_i	frequencies f
3_1	0_1	2_1^2	3
4_1	3_1	2_1^2	4
5_1	0_1	4_1^2	5
5_2	4_1	2_1^2	3
	3_1	4_1^2	2
6_1	5_1	4_1^2	2
	5_2	2_1^2	4
6_2	3_1	5_1^2	3
	5_1	$3_1\#2_1^2$	1
	5_2	4_1^2	2
6_3	4_1	5_1^2	4
	5_2	$3_1\#2_1^2$	2

Observations and conjectures

There are many interesting observations that can be made on the families of twins tabled above for $n = 3,4,5,6$ and extended to $n = 8$ in Appendix I. We present three of these, with brief discussions on the possibilities for generalisations.

Observation (i)

A particular pair of twins (K_1,K_2) may arise several times in the family of a given knot-graph K.

Remarks: The frequencies of occurrence of particular twins obviously bears some relation to the knot structure. The number of different twin-pairs occurring in a given family must have a bearing on structure, too. Study of these frequencies may be worthwhile.

Observation (ii)

In no case do we observe (K_1,K_3) also being twins from K, with $K_3 \neq K_2$. Nor do we ever observe (K_1,K_2) occurring as twins of a knot-graph other than K.

Remarks: Bearing in mind that two members of a twin-pair are very close in structure, and that their τ-values sum to $\tau(K)$, it is not surprising that we do not observe these phenomena. Particularly as we have only examined knots of small order. Further, it is one thing to say they are extremely unlikely, and quite another to prove that they can never occur.

We will turn these observations into conjectures.

Given an alternating knot-graph K. Let (K_1,K_2) be a twin-pair from K.

CONJECTURES:

A. If (K_1,K_3) is another twin-pair from K, then $K_3 = K_2$.
B. (K_1,K_2) cannot be a twin-pair from a knot-graph other than K.

We have not succeeded in constructing a general proof of these conjectures. Indeed, beyond a certain value of n we believe them to be false. At the end of section 7.2.3 we describe a method which could be used to discover counter-examples.

Observation (iii)

Consider, for example, the set of all twin-pairs arising from the prime 1-link 6-knots (i.e. 6_1, 6_2, and 6_3). Taking the union of all the 1-link twins we obtain the set: $\cup K_i = \{3_1, 4_1, 5_1, 5_2\}$; and the union of the 2-link twins is $\cup K'_i = \{2_1^2, 4_1^2, 5_1^2, 3_1 \# 2_1^2\}$.

We see that, in the families of the 6-knots there occurs at least once every knot of fewer than 6 crossings (with the exception of the unknots 0_1 and 0_1^2).

A similar result holds for $n = 7$ (the unknot 0_1 is included when n is odd) and 8, as inspection of the tables in Appendix I shows. This appears to us at once remarkable and obvious!

We next state the general result as a theorem and give a proof by induction.

Theorem 1

Let K_r be the set of all prime alternating 1-link knot-graphs of order r. Let S_{r-1} be the union set of all first generation twins derived from members of K_r.

Let W_{r-1} equal the set of all 1- and 2-link alternating knot-graphs (including composites) of order n with $2 \leq n \leq r-1$.

Then

$$S_{r-1} = \begin{cases} W_{r-1} & \text{if } r \text{ is even,} \\ W'_{r-1} = W_{r-1} \cup \{0_1\} & \text{if } r \text{ is odd.} \end{cases}$$

Proof:

We know the theorem is true for $r = 3$ and 4, by inspection.

Suppose it is true for $r-1$ but not for r (and r is odd). Then there is a knot-graph $\kappa \in W'_{r-1}$, with fewer than r crossings which does not have an immediate parent in K_r; (i.e. κ has $m < r$ crossings and is not a member of S_{r-1}. We shall deduce a contradiction.

Let $r - m = w$. We examine cases for w. We know that $\kappa \neq 0_1$, since the torus $T_r \in K_r$.

(i) Suppose that $w = 1$. But then we can insert a 2-gon in place of any vertex in κ in such a way that the result is a knot-graph which is a member of K_r: and this knot-graph has a twin pair (obtained by deleting the newly introduced vertex) one of which is κ ; a contradiction.

(ii) Suppose that $w > 1$. Then by the inductive hypothesis κ has a parent in K_{r-1}; suppose this parent is V.

Now κ is obtained by deleting a vertex of V. We can insert a 2-gon into V, in place of that vertex, in such a way that the resulting knot-graph is a member of K_r which has a pair of twins (κ, V); again we have a contradiction.

In the case that r is even, we know that 0_1 is not a member of S_{r-1}, since 0_1 can only be a twin of an odd torus knot-graph.

The above argument may again be applied, leading to the contradiction.

Hence proof.

One question that must be asked is whether a deletion family is a knot invariant. Is it possible that two non-isomorphic knot-graphs, both of the same knot, have the same family of twins?

CONJECTURE C: the deletion family of twins from an alternating knot K is an invariant for K.

We believe this conjecture to be false; see the Corollary , last page of section 7.2.3, for evidence of this.

7.2 THE TWIST SPECTRUM OF A KNOT

7.2.1 Definition Of Twist Spectra

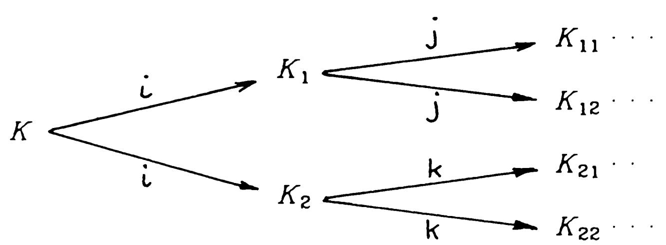
Modified δ-operation

We first modify the deletion operation which was defined in 6.3. The symbols $\delta_i(K)$ are now to mean the following:

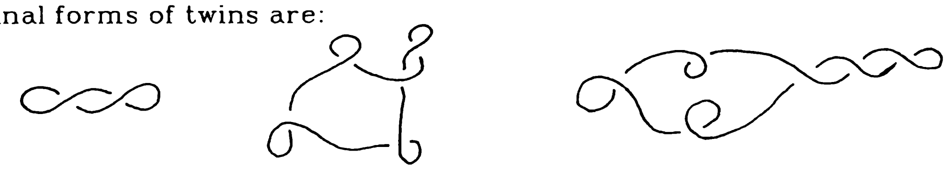
Vertex i is to be deleted from the alternating knot-graph K, and twins formed in the two ways defined in 6.3, *but* any nugatory crossings occurring in K_i or K_i' are not to be removed by untwisting operations. Further, no vertex i may be deleted if a twin with disconnected knot-graph would be produced. This means that applying δ to a nugatory crossing is not permitted.

Repeated reduction of K

The following tree-diagram shows how we can apply δ to K, then to each of the twins, and so on. We indicate the vertices deleted by placing letters $i, j, k, \cdots$ on arcs, and now we use multiple subscripts to designate twins of successive generations. We shall call this a *reduction tree* for K.



The reduction process is to continue until every twin has all nugatory crossings; these twins form the leaf nodes of the reduction tree. The tree will not be complete binary, because as soon as a twin is a final form, that branch of the tree ends; but the tree is binary full, since every node is either a leaf, or it has two arcs from it pointing to the two twins produced from that deletion. Examples of the final forms of twins are:



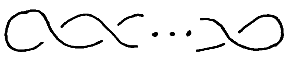
The question arises as to the nature of the set of final forms; that is, what properties has the set of twins at the leaf nodes of the reduction tree. And whether or not this set is independent of the order of the vertex reductions.

We next give terminology for the final forms, and then study properties of sets of them. The uniqueness and invariance of these sets are proved at the end of this subsection.

Definitions

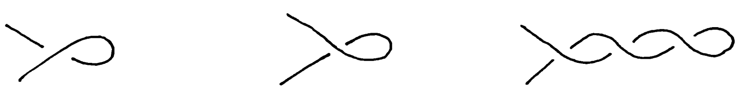
(i) An n-twist is an unknot of the form , having

n alternating crossings. In Conway's notation, it is the knot $1^*(n0)$. If n is negative, the twist is the mirror image of the $|n|$-twist,

that is  .

If $n = 0$, we have the unknot with no crossings.

(ii) Similarly we shall refer to positive and negative n-loops, examples being the following 1-loop, $\bar{1}$-loop, and $\bar{3}$-loop:



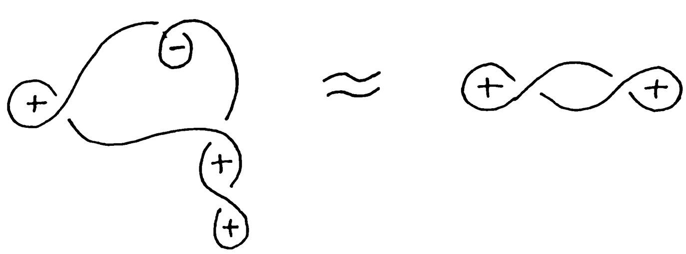
Notes

An infinite number of equivalent forms of a given n-twist are possible; we shall normally think of them in their *minimal form*, as shown in (i). Examples are:

Note that two loops of opposite sign cancel one another out. And a loop turned over in the plane does not change sign.

To determine the sign of a loop, place an arrow on the α-step round the loop. If the step is anticlockwise the loop is positive; if clockwise the loop is negative.

With this convention we can label a twist with signs thus: in all but one of the loops and 2-gons of the twist place the same sign, that being + if an end loop is positive and - otherwise (both end-loops must have the same sign). For example:



Final forms of twins

The final forms of all twins arising from modified δ-reduction of an alternating knot K are twists. Let them all be placed in minimal twist form.

Definitions

(i) The symbol u^i will represent an i-twist (i may be positive or negative; and u may be regarded as a dummy variable).

(ii) The collection of minimal final forms of twins, partitioned into sets of equal value i-twists, is called the *twist spectrum* of K.

(iii) If the signs are ignored, we call this the ***absolute twist spectrum***; if the signs are taken account of, we call it the *signed twist spectrum*. Since we do not discuss the absolute twist spectrum in this thesis, we shall use only the term 'twist spectrum' and the fact that it is signed must be understood.

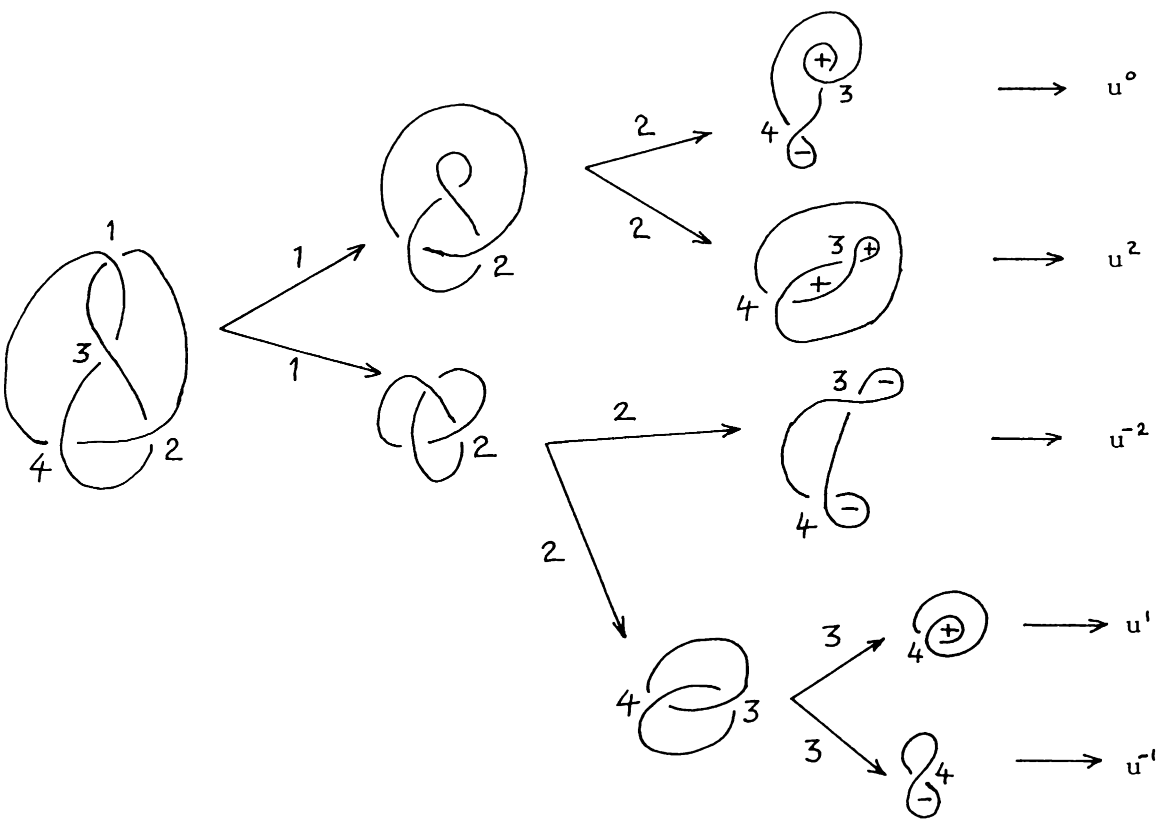
(iv) The symbol ν_i will denote the frequency of occurrence of i-twists in the spectrum.

(v) The *twist function* of a nontrivial alternating knot-graph is the polynomial $T(u) = \Sigma \nu_i u^i$, the summation being over all i-values occurring in the spectrum. N.B. It will sometimes be convenient to refer to $T(u)$ as the twist spectrum itself. And often we shall write the frequencies ν_i as a vector of detached coefficients, marking where the ν_0 occurs in the vector with an underline.

(vi) We define the twist function of a twist u^t to be $\sum_{i=0}^{t} \nu_i u^i$, with $\nu_t = 1$ and $\nu_i = 0$ if $i \neq t$. Using detached coefficients, the spectrum is either $(\underline{0}0 \cdots 1)$ or $(10 \cdots 0\underline{0})$.

Example

We show below a complete reduction of Listing's 4_1 knot, and obtain its twist spectrum.



Spectrum: $T(u) = u^{-2} + u^{-1} + u^{0} + u^{1} + u^{2}$

Vector of coefficients: $(11\underline{1}11)$

7.2.2 Properties of twist spectra

Now that twist spectra of alternating knots have been defined, we may examine some of their properties. We shall leave the important questions of uniqueness and invariance of a knot spectrum until we have proved some basic propositions about spectra.

First we give a table showing the spectrum frequencies, and four moments of the spectrum about the origin $i = 0$, of the first prime knots. These are for the signed twist spectra; and they are chosen for each knot such that the mean twist is nonnegative (see property (v) below for an explanation of this). The moments are the following:

$$\text{Torsion } (\tau) = \Sigma \nu_i \quad \text{(see property (i))}$$

$$\text{Twist moment} = \Sigma i \nu_i$$

$$\text{Modulus moment} = \Sigma |i| \nu_i$$

$$\text{Twist inertia} = \Sigma i^2 \nu_i$$

Tables of the twist spectra are given in Appendix II for knots of orders up to nine.

	Spectral coefficients (ν_i)			Torsion (τ)	Twist moment	Modulus moment	Twist inertia
Knot	... −1	0	123 ...	μ_0	μ_1	$\mu_{[1]}$	μ_2
3_1	1	0	11	3	2	4	6
4_1	11	1	11	5	0	6	10
5_1	1	0	1111	5	9	11	31
5_2	11	1	211	7	4	10	20
6_1	11	1	2211	9	10	16	40
6_2	11	2	2221	11	13	19	49
6_3	122	3	221	13	0	18	38

Table of twist spectra, for knots of orders $n = 3,\ldots,6$

Observations on the twist spectra and their moments

The following observations are made directly from the twist spectra we have computed for knots of orders $n = 2,\ldots,9$. Some observations lead to propositions proved or discussed later. The others are comments on points of interest.

(i) The sum of the frequencies of a spectrum equals the bao spanning tree number of the knot-graph.

(ii) The first and the last frequency is always unity, and their twist indices are always of opposite sign. It seems clear that no alternating knot in minimal form can have all twists of the same sign, otherwise the knot would surely be unknotted! (See the corollary to theorem 2 for a proof of this.)

(iii) The degree (or range) of a spectrum is equal to the order of the knot.

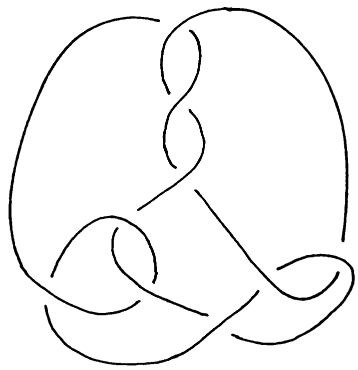
(iv) Amphicheiral knots have spectra which are symmetric about $i = 0$.

(v) τ is a good discriminator of knots. We have remarked on this in chapter 6. The modulus moment and the twist inertia are better discriminators, however. For example, amongst the eighteen 8-crossing knots there are five pairs of knots with the same value of τ; two pairs have the same modulus moment; and no two have the same twist inertia.

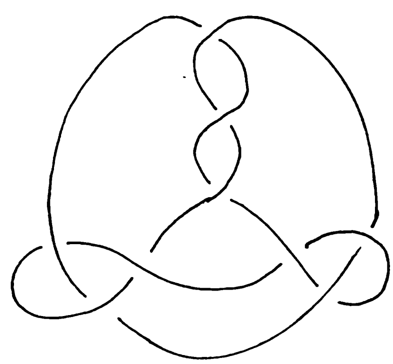
For the forty-one 9-crossing knots the twist inertia is still easily the best discriminator, although now three pairs have the same value for this.

For $n = 2,\dots,9$, no two knots have all four moments the same. Only one pair has three the same, namely 9_{24} and 9_{37}. These two have almost identical spectra, thus: $(1247\underline{7}87531)$ for the former and $(1347\underline{7}87431)$ for the latter. Their four moments, and knot-diagrams are given below. Notice how very similar in structure the two knot-graphs are; they have the same n-gon distribution.

Knot 9_{24} (45, 29, 79, 211)



Knot 9_{37} (45, 23, 79, 211)



(vi) The conjecture that no two knots are isospectral we believe to be false, and discuss it later.

Some theorems and properties of twist spectra

In each of the following results and proofs the uniqueness and invariance of a twist spectrum are assumed. These two properties are proved later.

(i) If $T(u) = \Sigma \nu_i u^i$ is the twist spectrum of K, then

$$T(1) = \Sigma \nu_i = \tau\,, \tag{1}$$

the spanning tree number of K with balanced alternating orientation.

This follows from the fact that any n-twist has $\tau = 1$ (for $n = 0$ the twist is 0_1, and we have defined $\tau(0_1) \equiv 1$); and by the vertex deletion theorem, the tree number of a knot-graph is the sum of the tree numbers of any pair of its first generation twins. Working back through a reduction tree, from the n-twist leaves through all the generations to the root K, applying the theorem at each node, we arrive at result (1). //

Here is yet another interpretation of τ; it is seen to be, in a sense, the total amount of twist in the knot. In the table given above, and in the tables of Appendix II, we have called it the ***torsion*** of the knot, which seems more appropriate than 'tree number'. (Recall, however, that τ has already been identified as a torsion number of a group of a covering space of K ; the two uses must not be confused.)

(ii) Let (K_1, K_2) be a pair of twins of K. Then

$$\text{Spec}(K) = \text{Spec}(K_1) + \text{Spec}(K_2) \tag{2}$$

Proof:

In the reduction tree rooted at K, K_1 and K_2 are roots of two disjoint trees, say T_1 and T_2. The set of leaf nodes (n-twists) of T_1 constitutes the twist spectrum of K_1, and that of T_2 constitutes the spectrum of K_2 But the disjoint union of these two sets constitutes the twist spectrum of K. Hence proof. //

(iii) If an n-loop is made in any edge of a knot-graph K, the resulting knot-graph (say K') has twist spectrum given by

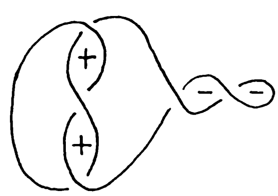
$$\text{Spec}(K') = u^n \times \text{Spec}(K) \tag{3}$$

(N.B. trailing zero frequencies are to be maintained; see after the proof for an example calculation.)

Proof:

Suppose a deletion sequence on K leads to a reduction tree T. Then performing exactly the same deletion sequence on K' leads to a tree T' whose node knot-graphs are the same as those of T except that every one of them has an additional n-twist in one of its edges. This is true of the leaf-node twists; hence every leaf-node twist of T' is u^n times the corresponding one in T. Hence proof. //

Example calculation (see the knot on the right)



We require $\text{Spec}(K) = u^{-2}\text{Spec}(3_1)$.

Using detached coefficients the calculation is:

$$(10\underline{0})(1\underline{0}11) = (0\underline{0}00) + (00\underline{0}0) + (101\underline{1})$$
$$= (101\underline{1}00)$$

Thus u^1 and u^2 are retained in the spectrum, but with zero coefficients.

(iv) Let $K = K_1 \# K_2$ be an alternating composition of two alternating knots. Then

$$\text{Spec}(K) = \text{Spec}(K_1) \times \text{Spec}(K_2) \tag{4}$$

Proof:

We can perform a deletion process on K_2, say, leaving K_1 untouched, until no further vertices of K_2 can be deleted. At this point we have a collection of leaf-node knot-graphs each of which is K_1 plus an n-loop in one of its arcs. Applying result (3) to each of these leads to (4). //

N.B. If K_1 were shrunk to a point in each of the leaf-node n-twists, the result would constitute the twist spectrum of K_2.

(v) Let $\bar{K}$ be the mirror-image (obverse) of K. Then

$$\mathrm{Spec}(\bar{K}) = \text{'}reverse\text{'}\ \mathrm{Spec}(K)$$
$$= \Sigma \nu_i u^{-i} \qquad (5)$$

Proof:

Reduction of the knot-graph of $\bar{K}$ leads to a collection of n-twists which is in 1-1 correspondence with the spectrum of K, with corresponding n-twists being mirror images of one another. //

(vi) If K is an amphicheiral knot then its twist spectrum is symmetric with respect to $i = 0$:

$$i.e.\ \nu_i = \nu_{-i} \text{ for each } i. \qquad (6)$$

Proof:

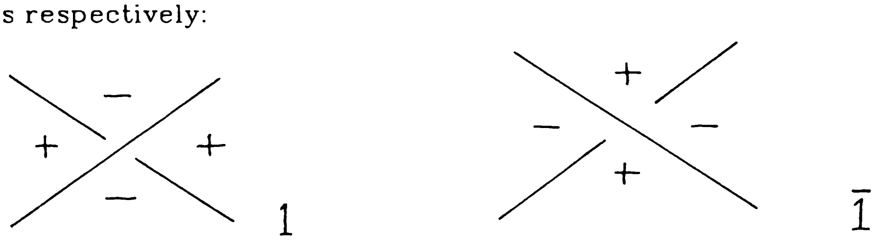
If $K \equiv \bar{K}$, then using (5) we have $\mathrm{Spec}(K) = \mathrm{Spec}(\bar{K})$ = 'reverse' $\mathrm{Spec}(K)$. Hence $\nu_i = \nu_{-i}$ for each i. //

(vii) We conjecture the converse of (6) to be true, namely that if a knot has symmetric twist spectrum it is amphicheiral. But we have not proved it! All those knots we have found so far with symmetric spectra have been shown to be amphicheiral by other means. We have asterisked them in the tables of Appendix II.

Before proving the next theorem we need to show how a knot-graph may be sensed in a special way, and to establish some terminology.

Sensing a knot-diagram

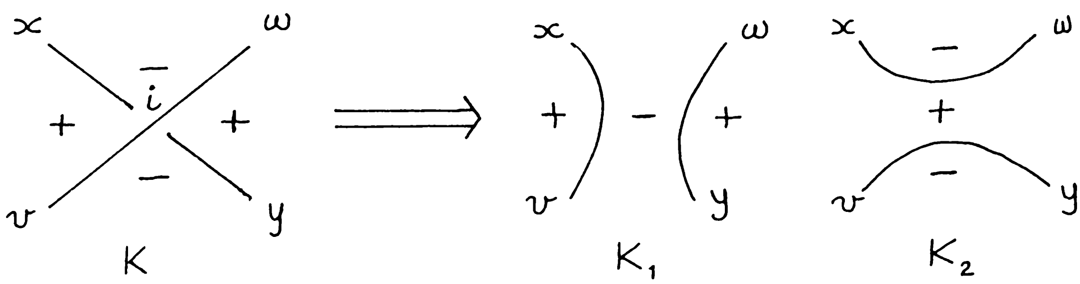
We may place a + or a - sign in each region of a knot-graph in such a way that at every crossing the signs alternate around it. There are two ways: we shall choose the one which produces the following arrangements at +1 crossings and -1 crossings respectively:



It is well-known that this can always be done on alternating knot-graphs.

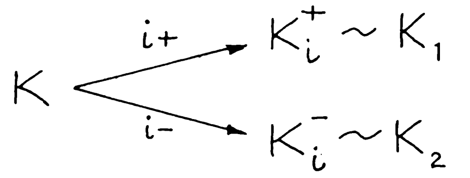
Terminology and definitions

(i) Below we show a sensed crossing, followed by the results of carrying out the δ_i-operation upon it. The first result will be in twin K_1 say, of the knot-graph K, and the second will be in twin K_2.



It may be seen that K_1 contains one fewer - region and the same number of + regions as does K. Whereas K_2 contains one fewer + region and the same number of - regions as K.

We shall denote this situation by



These observations have important implications for the twist spectrum of K, as the theorem after (iv) will show.

(ii) Let S, S_i^+, S_i^- be the spectra of the knot-graphs K, K_i^+, K_i^- respectively.

(iii) Let b('bottom') and t('top') be the minimum degree and maximum degree respectively of the spectrum of K (if $K \equiv 0_1$, i.e. an n-twist, either $b = 0$ and $t = n$ or $b = -n$ and $t = 0$).

Let b_i^+, t_i^+ and b_i^-, t_i^- be the corresponding degrees for the spectra S_i^+ and S_i^-.

(iv) Define the *range* r of a spectrum to be the difference $t - b$.

Theorem 2

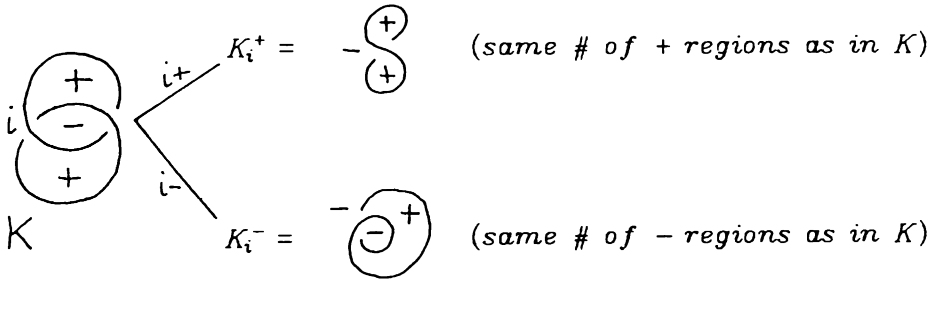
Let K_n be an alternating knot-graph; and let operation δ_i be applied to any vertex i of it. Then, for $n \geq 2$:

$$t = t^+ > t^-, \text{ and } b = b^- < b^+$$

Proof:

By induction on n, as follows:

When $n = 2$ the only nontrivial knot is 2_1^2, which has twist spectrum $u^{-1} + u^{+1}$. Hence $b = -1$, and $t = +1$. For (i) we have to choose any vertex i, and apply δ_i, thus:



Then S_i^+ has $b_i^+ = 0$, $t_i^+ = +1$,

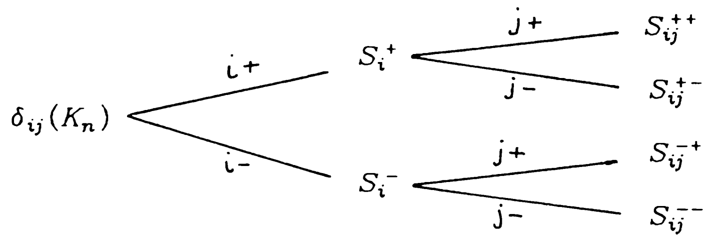
and S_i^- has $b_i^- = -1$, $t_i^- = 0$, so the theorem holds for this knot-graph.

Similarly, by direct examination of knot-graphs, we find that knots of order $n = 3$ and 4 satisfy the theorem.

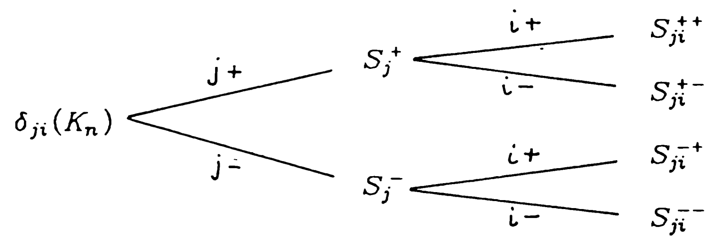
Suppose the theorem holds for $n = 2,3,\ldots,n-1$.

Consider deletions of two crossings i and j from knot-graph K_n, $n > 4$, carried out (A) in the order i,j, and (B) in the order j,i. Let us assume first that no nugatory crossings arise in any of the deletions. We may draw tree diagrams showing how the various spectra arise as follows:

(A)



(B)



Using similar notation for the second generation knot-graphs as for their spectra, examination of the operations carried out shows that the following pairs of knot-graphs are equivalent:

$$(K_{ij}^{++}, K_{ji}^{++}), \quad (K_{ij}^{+-}, K_{ji}^{-+}), \quad (K_{ij}^{-+}, K_{ji}^{+-}), \quad (K_{ij}^{--}, K_{ji}^{--})$$

Let us now use the notation $\max(S) \equiv t$ and $\min(S) \equiv b$ for the highest and lowest degrees of a spectrum S. Then, by the induction hypothesis,

$$\max(S_{ij}^{++}) > \max(S_{ij}^{+-}).$$

And also

$$\max(S_{ij}^{++}) = \max(S_{ji}^{++}) > \max(S_{ji}^{+-}) = \max(S_{ij}^{-+}).$$

Similarly

$$\min(S_{ij}^{--}) < \min(S_{ij}^{-+}) \ ;$$

and also

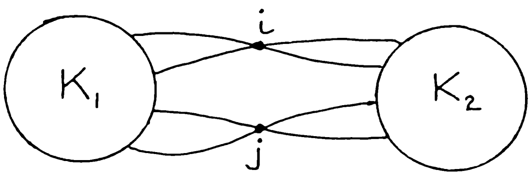
$$\min(S_{ij}^{--}) < \min(S_{ij}^{+-})$$

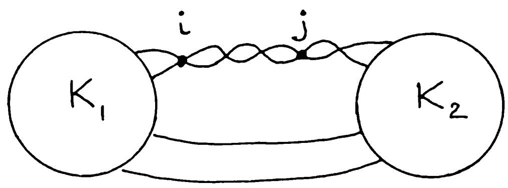
Then the pair of spectra

$$S_i^+ = S_{ij}^{++} + S_{ij}^{+-}$$
$$S_i^- = S_{ij}^{-+} + S_{ij}^{--}$$

satisfy the conditions of the theorem for δ_i applied to K_n. Hence proof.

In the cases that nugatory crossings arise when i or j are deleted, then either one can still proceed as above, or else one can only proceed to the second generation along one branch of the reduction trees in both (A) and (B). Example graphs in which the latter situation arises are:





Appealing to results (3) and (4) proved above, the same spectral arguments can be developed to cover these cases. We shall not give the details. //

Corollary

The extreme frequencies of the twist spectrum of a knot-graph with no nugatory crossings are unities; i.e. $\nu_b = \nu_t = 1$.

If the knot-graph has nugatory crossings, the spectral frequency vector may or may not have trailing zeros at one end. The extreme frequencies after trailing zeros are removed are unities.

Proof :

Let K_n have no nugatory crossings. Extending a reduction tree of K_n, proceeding as in (A) of the theorem, on the upper branch we obtain a sequence:

$$S_n \xrightarrow{i+} S_i^{+} \xrightarrow{j+} S_{ij}^{++} \xrightarrow{k+} S_{ijk}^{+++}$$

This ends when it arrives at a leaf-node twist. This has a twist spectrum u_i with $\nu_i = 1$; and by the theorem and the manner of obtaining the spectrum of K_n from all nodes in the reduction tree, this twist is the maximum one in the spectrum of K_n, and it is the only such one.

A similar argument applied to the lower branch of the reduction tree shows that $\nu_b = 1$ in the spectrum of K_n.

If K_n has nugatory crossings, it can be expressed as the composition of an m-twist and a knot-graph with no nugatory crossings. applying the above argument to the latter, then using proposition (iii) proved above, we get the result required. //

Theorem 3

Let the knot-graph K_n be sensed as described above; let P be the number of + regions, and N the number of - regions in it. Then:

$t = P-1$, and $b = 1-N$

Proof :

Consider again the upper branch of the extended reduction tree for K_n. We have just shown that its leaf node is the twist u^t. But by the manner of construction of this twist, (no + region of K_n has been lost; whereas all - regions have been deleted) the twist must be such that $t = P-1$, where P is the number of + regions in K_n.

Similarly the lower branch of the reduction tree tells us that $b = -(N-1)$.

Corollary (i)

The range of the spectrum of K_n is $r = n$.

Proof: The total number of regions in the knot-graph is $n+2$. Then $r = t-b = P+N-2 = n$. //

Corollary (ii)

t and b are always of opposite sign if the knot is nontrivial (see the comment in observation (*ii*)) and contains no m-loops.

Proof: P and N are both greater than 1 in such knots. //

An algorithm for computing the twist spectrum

Using the results of the above theorems and corollaries, we can devise an algorithm which computes the twist spectrum sequentially as it produces the successive sets of twins in a reduction tree.

Algorithm

Preliminary note: Whenever a knot-graph (a twin in the reduction tree) has several m_i-loops in its arcs, compute m, the algebraic sum of the m_i-values.

Step 1

Sense the parent knot-graph K, and count P and N; if m_i-loops exist, determine m.

$$\text{Compute } t = \begin{cases} P-1 & \text{if } m \geq 0 \\ P-1-|m| & \text{if } m < 0 \end{cases}$$

$$\text{and } b = \begin{cases} -(N-1) & \text{if } m \leq 0 \\ -(N-1)+m & \text{if } m > 0 \end{cases}$$

Then set $\nu_b = 1$, and $\nu_t = 1$ in Spec(K).

Step 2

First generation: Choose any nonnugatory vertex i, and apply δ_{i+} and δ_{i-}. For each of the twins K^+, K^- compute their m-value; then b^+ for K^+ and t^- for K^- (same formulae as given in Step 1). Then set $\nu_{b^+} = 1$ and $\nu_{t^-} = 1$ in Spec(K).

Step 3

Second generation: for the twins of K^+ compute b^{++} and t^{+-}, and set corresponding frequencies to 1 in Spec(K).

Continue in this way, using always b from the positive twin and t from the negative twin. With any branch, stop when a leaf-node is reached (do not compute b or t for that leaf-node).

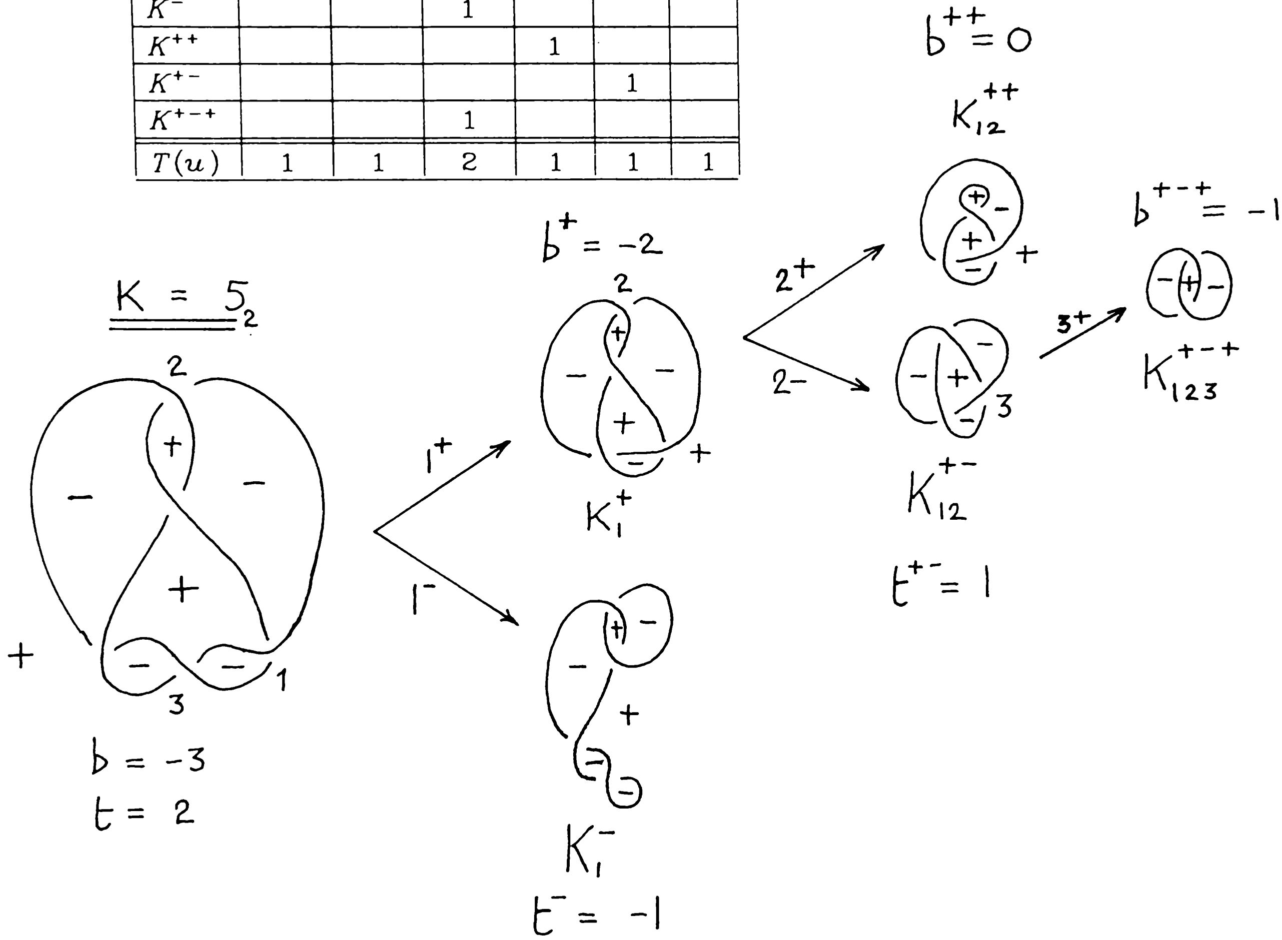
Spec(K) will be complete when all the nodes of the reduction tree (minus the leaf-nodes) have been treated in this way.

Example

The following table and diagrams illustrate the algorithm in action.

Knot 5_2: computation of its spectrum

	u^{-3}	u^{-2}	u^{-1}	u^{0}	u^{1}	u^{2}
K	1					1
K^{+}		1				
K^{-}			1			
K^{++}				1		
K^{+-}					1	
K^{+-+}			1			
$T(u)$	1	1	2	1	1	1



On amphicheirality

The above theorems provide a method for testing for amphicheirality. We showed earlier that if a knot is amphicheiral its twist spectrum is symmetric about $i = 0$. Supposing K is in minimal form, we know that for t and b to be equal it must be that $P = N$. This implies that no knot of odd order can be amphicheiral, a result which is in the literature (although a very recent publication gives it as a conjecture).

If we could prove the conjecture that a knot with symmetric twist spectrum is amphicheiral, we would have a direct graphical way of determining amphicheiral knots from knot-diagrams.

Further remarks on amphicheirality are given in the Summary.

Uniqueness of the twist spectrum

In the previous sections we have tacitly assumed that any knot-graph possesses just one twist spectrum.

The need for proof of uniqueness derives from the fact that in general a knot-graph has several different first generation twin-pairs; and each member of these pairs also has several twin-pairs; and so on. Hence for any knot-graph K_n a large number of different reduction trees can be constructed. Uniqueness of spectrum demands that the sets of leaf-node twists from all these trees are identical.

We now establish this uniqueness.

Theorem

Let K be an alternating knot-graph with n-vertices. Then all reduction trees formed from K by δ-operations give the same twist spectrum for K.

Proof:

The proof is by induction on n.

For $n = 2$, the knot 2_1^2 has only two reduction trees, both leading to the twist spectrum $(1\underline{0}1)$: so the theorem is true for $n = 2$.

Suppose it is true for $n = 2,\ldots,n-1$, and not for n. Let K be a failing case.

Then K has two distinct twin-pairs, say (K_1,K_2) and (K_1',K_2') such that:

$$\mathrm{Spec}_1(K) = \mathrm{Spec}(K_1) + \mathrm{Spec}(K_2) \;,$$

$$\mathrm{Spec}_2(K) = \mathrm{Spec}(K_1') + \mathrm{Spec}(K_2') \;,$$

and $\mathrm{Spec}_1(K) \neq \mathrm{Spec}_2(K)$

(Note that $\mathrm{Spec}_1(K)$ and $\mathrm{Spec}_2(K)$ are uniquely defined, since all the twins involved are of order $< n$, and the induction hypothesis holds for them.)

But $\delta_1(K)$ marks the beginning of a reduction tree for K; and $\delta_2(K)$ marks the beginning of another. Let us continue these reduction trees to their second generations (if possible), forming $\delta_{12}(K)$ branches in the first tree and $\delta_{21}(K)$ branches in the second.

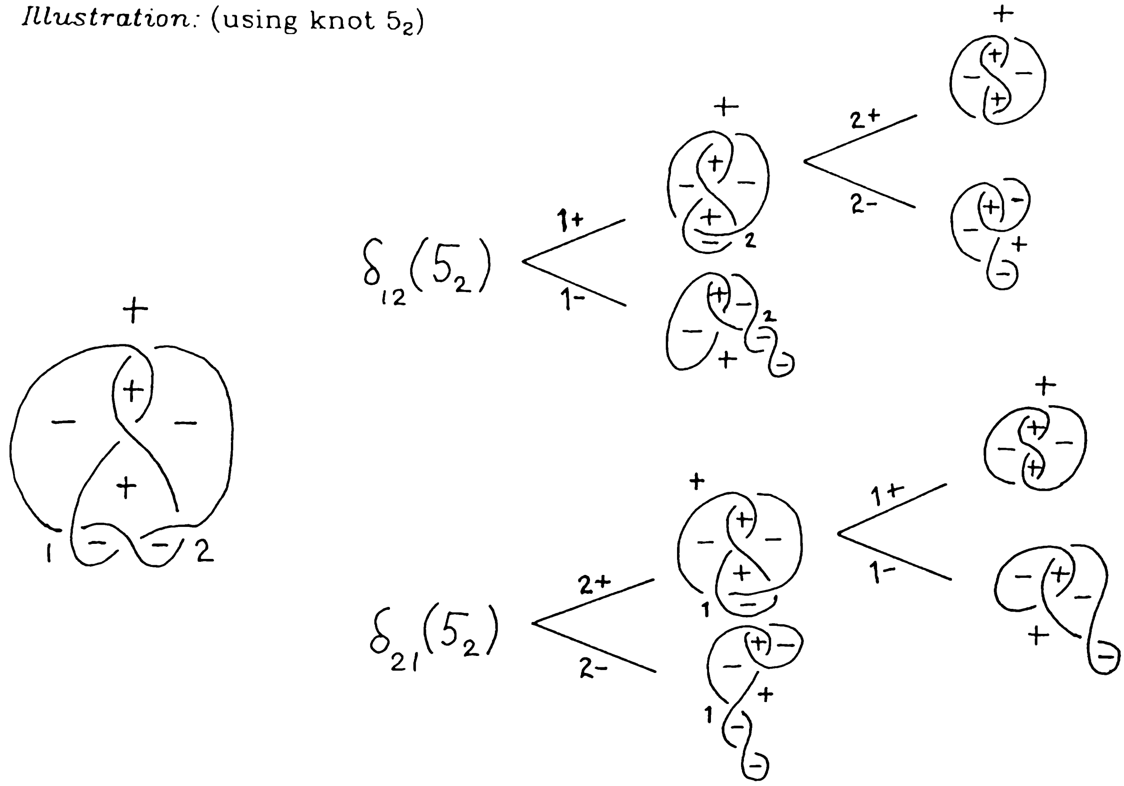
There are three cases to consider (see the discussion in the proof of theorem 2, where diagrams of possible cases were given): but in all the cases the end results (i.e. the leaf node sets) can be shown to correspond in so far as the consequent production of twist spectra is concerned. The same knots occur in the two sets, but m-loops may be formed in different arcs of them; or else composed knots may occur, composed on different pairs of arcs) .

The sets placed in correspondence either contain four second-generation twins, or contain one first-generation twin and two second-generation ones, depending on whether or not the second δ-operations are permissible. (We give an illustration below, of the latter case, to help clarify this point.)

$$\begin{aligned} \text{Then}\quad \mathrm{Spec}_1(K) &= \textstyle\sum \text{ of spectra of the } \delta_{12}(K) \text{ leaf nodes} \\ &= \textstyle\sum \text{ of spectra of the } \delta_{21}(K) \text{ leaf nodes} \\ &= \mathrm{Spec}_2(K) \end{aligned}$$

This is a contradiction, so the theorem is true. //

Illustration: (using knot 5_2)



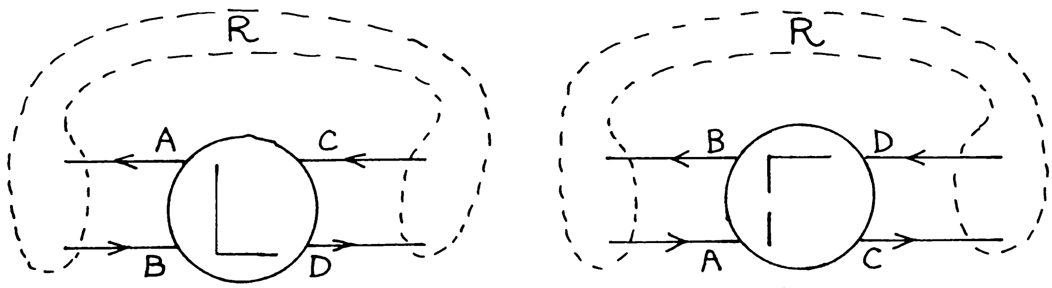
Note the correspondences of leaf-node knots; in the second and third cases from each reduction tree a 1-loop and a 2-loop respectively occur on different arcs, but this doesn't affect the final spectra of the cases.

Invariance of the twist spectrum

We can show that the twist spectrum is a knot-invariant in a similar way to that in 6.5 which showed τ to be invariant. The demonstration again rests on Tait's conjecture that, if a knot has two non-isomorphic knot-graphs there is a sequence of flyping operations which will transform one form to the other.

So we have to show that a twist spectrum is invariant to a flyping operation.

The proof follows exactly the same lines as in 6.5, up to the point where it has to be shown that knots (a) and (b) below have equal spectra.

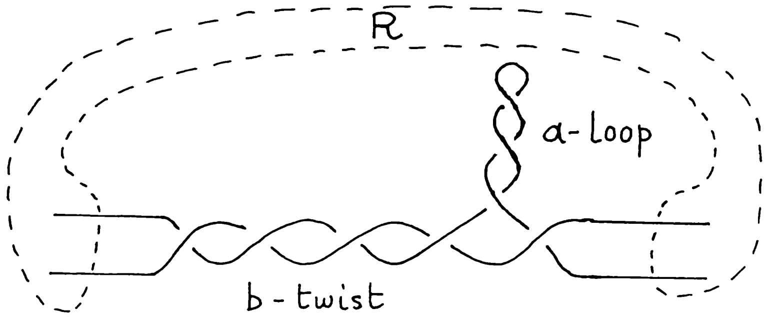


(a) (b)

Suppose we keep the four strings fixed on either side of both tangles. Then label the vertices of both tangles so that corresponding vertices (pairs that mapped from one to the other when (a) was flyped to form (b)) take the same label.

Then carry out the δ-reduction process on each, using the same sequence of vertices in each, until all those that can be deleted without disconnecting the tangle have been.

The result will be a set of knot-graphs of the following form produced from (a): (variations of this occur; different α-loops, etc.)



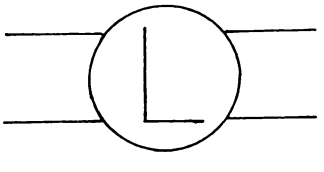
And the resulting knot-graphs from (b) will be in 1-1 correspondence with these but will have their α-loops pointing in opposite directions. N.B. Turning a knot over in the plane does not change the sense of any of its regions; so the sense of all these twist and loop forms will be the same.

If we now continue reducing the knot-graphs, forming reduction trees for the whole graphs and using the same sequences of vertices for deletions, the result will be two identical spectra. The α-loops will occur in each and every one of the end-forms, and their whereabouts on them is immaterial.

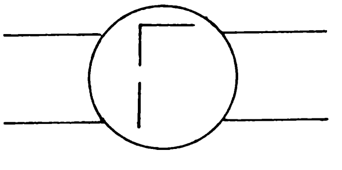
Isospectrality

Although we have not yet discovered two knot-graphs which we have shown to be non-equivalent whilst having the same twist spectrum (isospectral), we believe the proof given above for invariance supplies a method for constructing such knot-graphs at will.

Referring to diagrams (a) and (b) in the above section, supposing that the tangles



and

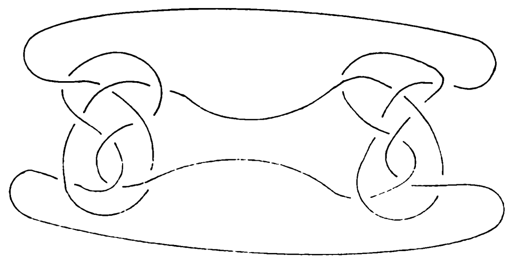


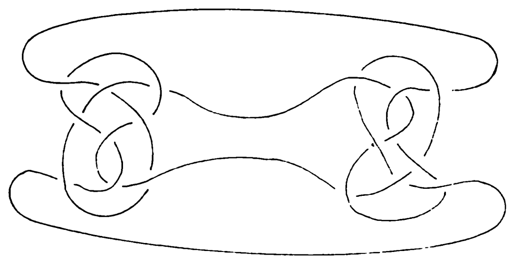
are inequivalent, it would seem that the whole knots would be inequivalent. Yet, as we have shown, they have the same twist spectrum.

In (CONWAY, 1970, p.333) is stated that if the above two tangles are rational they are equivalent. That is, one can be transformed into the other by making a sequence of Alexander's elementary transformations whilst holding the four arcs fixed. So, if we are to construct isospectral nonequivalent knot-graphs we must not use rational tangles in our suggested mode of construction.

Below we give a diagram of two 18-crossing knots. The first is formed by placing a 9_{34} knot-graph (an 8*20 in Conway's notation, and not rational) alongside another 9_{34} knot-graph and double-composing them. Below this we have again double-composed two 9_{34} knot-graphs, but this time one is overturned with respect to the other. We claim that these two knot-graphs are isospectral but not equivalent.

We have not the computer programs available (nor the time, currently, to do the job without them!) to test our claim.



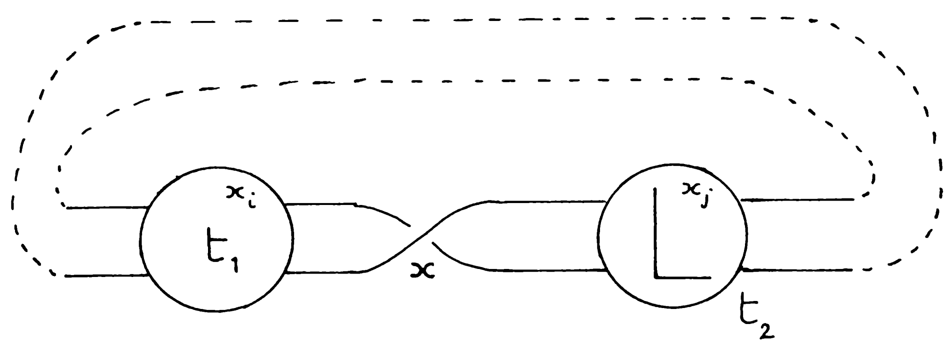


The method described above for examining the question of isospectrality can be used to study the conjectures A,B, and C of 7.1, which refer to families of twin-pairs of a given knot K. We give one suggestion, as a corollary stemming from the method, on how to construct a counter-example to conjecture C.

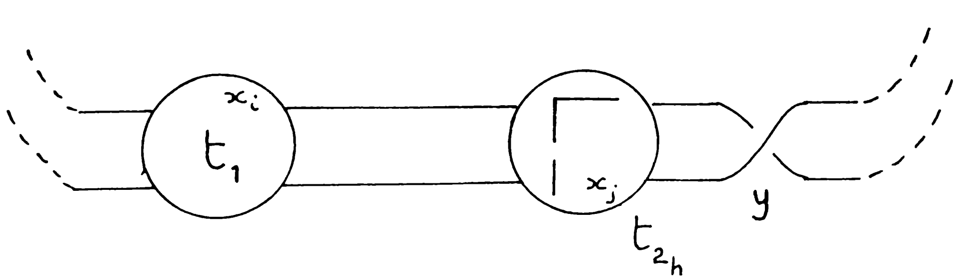
Corollary

Conjecture C states that the family of twins of a given alternating knot K is an invariant for K.

The following diagrams show an alternating knot formed from three tangles, namely $K = t_1 + x + t_2$, labelled (A); and the same knot after flyping to t_1 + flip(t_2) + y, labelled (B).



(A)



(B)

The twin-pairs obtained by deleting x_i from (A) and (B) will be the same. Similarly when deleting x_j from (A) and (B).

But it would seem that the twin-pair obtained by applying δ_x to (A) is not the same as that obtained by applying δ_y to (B), unless $t_2 \equiv \mathrm{flip}(t_2)$ or $t_1 \equiv \mathrm{flip}(t_1)$.

Now all knots for $n \leq 7$ are rational, and $t \equiv \mathrm{flip}(t)$ if t is a rational tangle. Hence when $n \leq 14$ conjecture C will be true; but when $n > 14$ it should be possible to construct counter-examples to conjecture C by taking knot-graphs as in (A) with both t_1 and t_2 being non-rational tangles.

7.2.3 The spectral function $T(u)$ for members of Conway's classes

We finally obtain general formulae for the twist spectra of knots in the classes arising from Conway's notation, which were defined in 6.7. These classes are of the rational and the Pretzel knots.

The formulae are obtained by repeatedly applying the deletion operation, and using the fact that $\mathrm{Spec}(K) = \mathrm{Spec}(K_1) + \mathrm{Spec}(K_2)$, where (K_1, K_2) are twins produced from K.

We shall not go into full details. We will give diagrams and formulae for the first few, and indicate how the sequence proceeds.

Class (i) The rational knots (abcd...)

Recall that $a, b, \ldots$ denote integer tangles, and $\bar{a}, \bar{b}, \ldots$ denote their obverses, or mirror images.

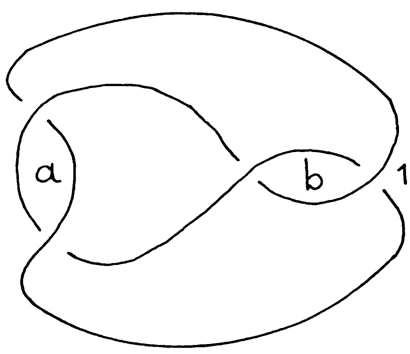
The twist spectra of knots $1{*}a$ and $1{*}\bar{a}$ are respectively:

$$\mathrm{Spec}(a) = u^{-1} + u^{1} + u^{2} + \qquad + u^{a-1}$$

and $$\mathrm{Spec}(\bar{a}) = u^{1} + u^{-1} + u^{-2} + \qquad u^{1-a}$$

since all twists in $1{*}\bar{a}$ have the opposite sign to the corresponding ones in $1{*}a$.

The knot-graph for (ab) is shown below.



Applying the deletion operation b times, from vertex 1 leftwards, gives the following formula:

$$\mathrm{Spec}(ab) = U^{(b)}\mathrm{Spec}(\bar{a}) + u^{-a} \quad , \qquad (1)$$

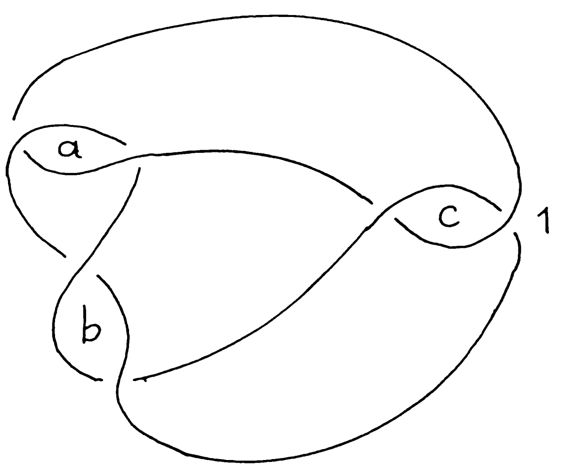
where here we use a notation

$$U^{(b)} \equiv u^{b-1} + u^{b-2} + \qquad + u^{0} \quad ,$$

and later we use similarly

$$U^{(-b)} \equiv u^{-(b-1)} + u^{-(b-2)} + \qquad + u^{0}$$

The knot-graph for (abc) is as below.



Again by repeatedly deleting the vertices of c, starting from vertex 1, we find the spectrum formula:

$$\mathrm{Spec}(abc) = U^{(c)}\mathrm{Spec}(\bar{a}\bar{b}) + u^{-b}\,\mathrm{Spec}(a) \qquad (2)$$

The pattern is now clear; and we can apply (2) repeatedly until (1) is reached in order to obtain Spec($abcd$), Spec($abcde$), etc.

In (TURNER, 1984) the sequence of knots (1), (11), (111), ... is defined to be the Fibonacci class of knots, since their spanning tree numbers (τ) follow the Fibonacci sequence 1,2,3,5,8,... As an example of the above method we will compute the twist spectra of the first four knots in this sequence.

Thus: (frequencies are shown after → as detached coefficients)

$\mathrm{Spec}(1) = u^{-1}$ → $1\underline{0}$
$\mathrm{Spec}(11) = U^{(1)}\mathrm{Spec}(\bar{1}) + u^{-1} = u^{0}(u^{1}) + u^{-1}$ → $1\underline{0}1$
$\mathrm{Spec}(111) = U^{(1)}\mathrm{Spec}(\overline{11}) + u^{-1}\mathrm{Spec}(1)$ → $11\underline{0}1$
$\mathrm{Spec}(1111) = U^{(1)}\mathrm{Spec}(\overline{111}) + u^{-1}\mathrm{Spec}(11)$ → $11\underline{1}11$

Note that the general recurrence rule for this knot sequence can be written (using $\underline{n}$ to mean (111...1)):

$$\text{Spec}(\underline{n}) = \text{Spec}(\underline{n-1}) + u^{-1}\text{Spec}(\underline{n-2}) \ ,$$

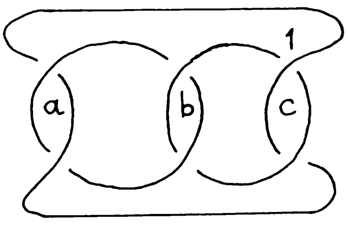
with Spec(1) and Spec(11) as given above.

This may be compared with the relation $F_n = F_{n-1} + F_{n-2}$, with $F_1 = 1$, $F_2 = 2$, which generates the Fibonacci numbers.

An examination of the pyramid of spectral coefficients (c.f. Pascal's triangle) for the sequence of Fibonacci knots should yield interesting relationships.

Class (ii) The pretzel knots (a,b,c,...)

We show the knot (a,b,c) and indicate how its twist spectrum may be obtained.



Applying the deletion operation at vertex 1, then again and again until all vertices of (c) have been deleted, we follow the sequence:

$$\begin{aligned} \text{Spec}(a,b,c) &= \text{Spec}(a,b,c-1) + u^{-(c-1)}\text{Spec}(a,b) \\ &= \qquad etc \\ &= \text{Spec}(\bar{a}\#\bar{b}) + U^{(-c)}\text{Spec}(a,b) \\ &= \text{Spec}(\bar{a})\times\text{Spec}(\bar{b}) + U^{(-c)}\text{Spec}(\overline{a+b}) \qquad (1) \end{aligned}$$

Similarly we can obtain

$$\text{Spec}(a,b,c,d) = \text{Spec}(\bar{a})\times\text{Spec}(\bar{b})\times\text{Spec}(\bar{c}) + U^{(-d)}\text{Spec}(a,b,c) \qquad (2)$$

and so on.

We give one example, and use detached coefficients for the working.

Example: Knot (3,3,2) ≡ 8_5

$$\begin{aligned} \text{Spec}(3,3,2) &= \text{Spec}(\bar{3})\times\text{Spec}(\bar{3}) + U^{(-2)}\text{Spec}(\bar{6}) \\ &= (11\underline{0}1)\times(11\underline{0}1) + (1\underline{1})(11111\underline{0}1) \\ &= (1212\underline{2}01) + (122221\underline{1}1) \\ &= (123433\underline{3}11) \end{aligned}$$

APPENDIX I

FAMILIES OF TWIN-PAIRS

PRIME 1-LINK KNOTS, $n = 7$

Knot K	Twin-pairs K_i	K'_i	frequencies f
7_1	0_1	6^2_1	7
7_2	5_1	6^2_1	2
	6_1	2^2_1	5
7_3	3_1	6^2_2	4
	6_1	4^2_1	3
7_4	$3_1 \# 3_1$	6^2_1	1
	6_2	4^2_1	6
7_5	4_1	6^2_3	3
	5_2	6^2_2	2
	6_2	$3_1 \# 2^2_1$	2
7_6	5_2	6^2_3	2
	6_1	$4_1 \# 2^2_1$	1
	6_2	5^2_1	2
	6_3	$3_1 \# 2^2_1$	2
7_7	$3_1 \# 3_1$	6^2_3	1
	6_2	$4_1 \# 2^2_1$	2
	6_3	5^2_1	4

PRIME 1-LINK KNOTS. $n = 8$

Knot K	Twin-pairs K_i	K'_i	frequencies f
8_1	7_1	6^2_1	2
	7_2	2^2_1	6
8_2	3_1	7^2_1	5
	7_1	$5_1\# 2^2_1$	1
	7_2	6^2_1	1
8_3	7_3	4^2_1	8
8_4	5_1	7^2_1	3
	7_1	$3_1\# 4^2_1$	1
	7_4	4^2_1	1
8_5	5_1	7^2_1	6
	7_4	6^2_1	6
8_6	5_2	7^2_3	3
	7_3	6^2_2	2
	7_5	$3_1\# 2^2_1$	3
8_7	4_1	7^2_2	4
	6_1	7^2_1	2
	7_2	$3_1\# 4^2_1$	1
	7_3	$5_1\# 2^2_1$	1
8_8	6_1	7^2_3	2
	6_2	7^2_1	2
	7_2	$5_2\# 2^2_1$	1
	7_6	$3_1\# 2^2_1$	3
8_9	5_2	7^2_2	6
	7_3	$3_1\# 4^2_1$	2
8_{10}	5_2	7^2_5	3
	6_1	7^2_2	2
	6_2	7^2_4	2
	7_5	$5_1\# 2^2_1$	1

Knot K	Twin-pairs K_i	K'_i	frequencies f
8_{11}	$3_1\# 3_1$	7^2_2	2
	7_3	$5_2\# 2^2_1$	1
	7_5	6^2_2	2
	7_6	5^2_1	3
8_{12}	7_5	6^2_3	4
	7_6	$4_1\# 2^2_1$	4
8_{13}	$3_1\# 4_1$	7^2_1	1
	6_2	7^2_2	2
	7_4	$5_2\# 2^2_1$	1
	7_5	$3_1\# 4^2_1$	1
	7_7	5^2_1	3
8_{14}	$3_1\# 4_1$	7^2_3	1
	6_3	7^2_2	2
	7_5	$5_2\# 2^2_1$	1
	7_6	6^2_3	2
	7_7	$4_1\# 2^2_1$	2
8_{15}	6_3	7^2_8	4
	7_6	$5_2\# 2_{1sup} 2$	2
	7_7	6^2_3	2
8_{16}	6_2	7^2_6	4
	7_4	7^2_5	1
	7_6	7^2_4	2
	7_7	7^2_1	1
8_{17}	6_3	7^2_6	4
	7_5	7^2_5	2
	7_6	7^2_2	2
8_{18}	7_7	7^2_8	8

APPENDIX II

THE SIGNED SPECTRA OF ALTERNATING KNOTS

Definition of twist-spectrum (see 7.2)

twist-value (i)	...	-3	-2	-1	0	1	2	3	...
frequency (ν_i)	...	ν_{-3}	ν_{-2}	ν_{-1}	ν_0	ν_1	ν_2	ν_3	...

The tables give the frequencies, followed by the spectrum moments:

μ_0	=	$\Sigma\, \nu_i$	=	τ, the *torsion* or tree number
μ_1	=	$\Sigma\, i\nu_i$	=	the first twist moment, or *mean twist*
$\mu_{[1]}$	=	$\Sigma\, \lvert i \rvert \nu_i$	=	the *modulus twist moment*
μ_2	=	$\Sigma\, i^2 \nu_i$	=	the second twist moment, or *twist inertia*

The asterisked knots are amphicheirals: they have symmetric twist spectra.

TABLE A1: THE ALTERNATING 1-LINKS; n = 3, ..., 7

KNOT n_j		Twist-spectrum frequencies ν_i			torsion (τ)	twist moment	modulus moment	twist inertia
n	j	... -1	0	1 2 3 ...	μ_0	μ_1	$\mu_{[1]}$	μ_2
3	1	1	0	1 1	3	2	4	6
4	*1	1 1	1	1 1	5	0	6	10
5	1	1	0	1 1 1 1	5	9	11	31
	2	1 1	1	2 1 1	7	4	10	20
6	1	1 1	1	2 2 1 1	9	10	16	40
	2	1 1	2	2 2 2 1	11	13	19	49
	*3	1 2 2	3	2 2 1	13	0	18	38
7	1	1	0	1 1 1 1 1 1	7	20	22	92
	2	1 1	1	2 2 2 1 1	11	18	24	74
	3	1 1 2	2	3 2 1 1	13	7	21	51
	4	1 1	2	3 2 3 2 1	15	26	32	100
	5	1 1 3	3	3 3 2 1	17	9	27	65
	6	1 2 3	3	4 3 2 1	19	24	30	70
	7	1 2 3	4	4 3 3 1	21	13	33	79

TABLE A2: THE ALTERNATING 1-LINKS; n = 8

KNOT n_j		Twist-spectrum frequencies ν_i			torsion (τ)	twist moment	modulus moment	twist inertia
n	j	… -1	0	1 2 3 …	μ_0	μ_1	$\mu_{[1]}$	μ_2
8	1	1 1	1	2 2 2 2 1 1	13	28	34	126
	2	1 1	2	2 3 3 2 2 1	17	56	44	164
	•3	1 1 2 3	3	3 2 1 1	17	0	28	72
	4	1 1	2	3 3 3 3 2 1	19	43	46	181
	5	1 1	3	3 3 4 3 2 1	21	46	49	190
	6	1 1 3 4	4	4 3 2 1	23	3	37	91
	7	1 2 2	4	4 4 3 2 1	23	25	43	123
	8	1 2 3	4	4 5 3 2 1	25	26	46	128
	•9	1 2 3 4	5	4 3 2 1	25	0	0	100
	10	1 3 3	4	4 5 4 2 1	27	27	51	141
	11	1 2 3 5	5	4 4 2 1	27	1	43	105
	•12	1 2 4 5	5	5 4 2 1	29	0	46	110
	13	1 2 3	5	5 5 4 3 1	29	34	54	154
	14	1 3 4 5	6	5 4 2 1	31	3	49	119
	15	1 3 4 6	6	5 5 2 1	33	2	52	124
	16	1 3 4	6	6 6 5 3 1	35	37	63	173
	•17	1 3 5 6	7	6 5 3 1	37	0	58	138
	•18	1 4 6 7	9	7 6 4 1	45	0	70	166

TABLE A3: THE ALTERNATING 1-LINKS; $n = 9$

KNOT n_j (n)	(j)	Twist-spectrum frequencies ν_i (... -1)	(0)	(1 2 3 ...)	torsion (τ) μ_0	twist moment μ_1	modulus moment $\mu_{[1]}$	twist inertia μ_2
9	1	1	0	1 1 1 1 1 1 1 1	9	35	37	205
	2	1 1	1	2 2 2 2 2 1 1	15	40	46	200
	3	1 1 2	2	3 3 3 2 1 1	19	30	44	150
	4	1 1 2 3	3	4 3 2 1 1	21	11	39	111
	5	1 2	2	4 3 4 3 2 1 1	23	53	61	241
	6	1 1 3	3	4 5 4 3 2 1	27	46	62	210
	7	1 2 3 4	5	5 4 3 1 1	29	11	51	139
	8	1 2 3	4	5 5 5 3 2 1	31	48	68	224
	9	1 1 3	4	5 5 5 4 2 1	31	54	70	236
	10	1 1 3 5	5	6 5 4 2 1	33	23	59	161
	11	1 2 3	4	6 5 5 4 2 1	33	53	73	241
	12	1 2 3 5	6	6 5 4 2 1	35	20	62	170
	13	1 2 4 5	6	7 5 4 2 1	37	19	65	175
	14	1 2 3	5	6 6 6 4 3 1	37	63	83	279
	15	1 2 3 6	6	7 6 4 3 1	39	26	70	192
	16	1 1 4	4	7 6 6 6 3 1	39	73	91	309
	17	1 2 4	5	6 7 6 4 3 1	39	64	86	284
	18	1 2 4 6	6	7 5 4 2 1	38	18	66	176
	19	1 3 4 6	7	7 6 4 2 1	41	17	71	189
	20	1 2 4	5	7 7 6 5 3 1	41	69	91	301
	21	1 3 4 6	7	8 6 4 3 1	43	22	76	206
	22	1 3 4	5	6 8 6 5 4 1	43	73	99	333
	23	1 3 5 6	8	8 6 5 2 1	45	19	77	203
	24	1 2 4 7	7	8 7 5 3 1	45	29	79	211
	25	1 3 6 7	7	8 7 5 2 1	47	18	82	212
	26	1 3 4 7	8	8 7 5 3 1	47	26	82	220
	27	1 3 5 7	8	9 7 5 3 1	49	25	85	225
	28	1 4 4 9	8	8 9 4 3 1	51	22	88	230
	29	1 3 5	7	9 8 8 6 4 1	52	85	113	371
	30	1 3 4 8	8	8 8 6 4 1	51	34	92	250
	31	1 4 5 8	10	9 8 6 3 1	55	26	94	248
	32	1 4 6 9	10	10 9 6 3 1	59	26	100	258
	33	1 3 6 9	10	11 9 7 4 1	61	37	105	275
	34	1 4 7 10	12	12 10 3 4 1	64	22	102	258
	35	1 1	3	4 3 5 4 3 2	26	65	71	277
	36	1 2 4	5	6 6 6 4 2 1	37	57	79	255
	37	1 3 4 7	7	8 7 4 3 1	45	23	79	211
	38	1 4 6 8	10	10 8 6 3 1	57	25	97	253
	39	1 3 6 8	9	10 8 6 3 1	55	28	94	244
	40	1 4 8 11	13	13 11 9 4 1	75	40	126	322
	41	1 3 5	7	8 8 8 5 3 1	49	75	103	329

TABLE B: THE ALTERNATING 2-LINKS; n = 2, ..., 8

KNOT n_j^2 n	j	Twist-spectrum frequencies ν_i ... -1	0	1 2 3 ...	torsion (τ) μ_0	twist moment μ_1	modulus moment $\mu_{[1]}$	twist inertia μ_2
2	*1	1	0	1	2	0	2	2
4	1	1	0	1 1 1	4	5	7	15
5	1	1 1	2	1 2 1	8	5	11	23
6	1	1	0	1 1 1 1 1	6	14	16	56
	*2	1 1 2	2	2 1 1	10	0	14	30
	3	1 1 3	2	2 2 1	12	1	17	35
7	1	1 1	2	2 3 2 2 1	14	24	30	94
	2	1 2 2	4	3 3 2 1	18	10	28	68
	3	1 2 2	3	3 3 1 1	16	7	25	59
	4	1 1	2	3 2 3 3 1	16	30	36	116
	5	1 2 3	4	3 4 2 1	20	11	31	73
	6	1 3 3	5	4 4 3 1	24	13	37	87
8	1	1	0	1 1 1 1 1 1 1	8	27	29	141
	2	1 1 2	2	3 3 2 1 1	16	17	31	89
	3	1 1 3	3	4 4 3 2 1	22	26	42	120
	4	1 1 3	4	4 4 4 2 1	24	29	45	129
	5	1 2 3	4	5 4 4 2 1	26	28	48	134
	6	1 2 2	3	4 3 3 1 1	20	19	37	103
	7	1 2 4	4	6 5 4 3 1	30	34	56	156
	*8	1 3 4 6	6	6 4 3 1	34	0	54	130
	9	1 2 4	4	5 5 4 2 1	28	29	51	139
	10	1 2 4	5	6 5 5 3 1	32	37	59	165
	11	1 1 4	3	6 4 4 4 1	28	38	56	164
	12	1 2 4 6	5	6 4 3 1	32	3	51	121
	13	1 3 5 7	7	7 5 4 1	40	3	63	149
	14	1 3 4	6	7 5 6 3 1	36	39	65	179

SUMMARY

In the Foreword we gave many indications of how we believe our work relates to other mathematical fields, in particular to graph theory and algebraic topology. We explained our motives for pursuing the various directions of study; and we pointed to the discoveries which we believe to be new and worthy of attention by other mathematicians.

In the Contents, after each chapter heading is an ordered list of the topics covered in that chapter; so the thesis is descriptively summarised there.

Consequently this summary will be brief. We will review our methods of modelling knots, mention the main objects studied and results achieved, and close by discussing some possible uses of our interesting discovery, the twist spectrum.

If one is to study knots by means of graph theory, one has to find ways of adding information to a plane projection of a knot which will turn it into a useful graph model of that knot. We discovered that standard labelling of the vertices, and marking of edges with α and β symbols to distinguish over- and under-crossings, provided a satisfactory model from which useful knot-graph measures could be computed. Most of the measures reported on in chapters two to five are related to walks on the knot-graph, and are derived from adjacency matrix manipulations or equations.

Of particular note in chapter three are the discovery of the (U,P) structure of the α- and β- adjacency matrices derived from alternating knots, and the equation $(J_\alpha.m.J_\beta)(J_\beta.m.J_\alpha) = CC'$ satisfied by matrices from general knot-graphs. The touring algorithms we describe are simple consequences of the structure of the matrices.

The work on walk-groups and groupoids is summarised at the end of chapter four. In that chapter we report on studies of relationships between elementary walk-groups, their Cayley colour graphs, and knot-graphs derived from the Cayley graphs; and we pose several interesting questions about these classes and sequences. Many examples of knot-graphs having planar elementary walk-groups are given, and they and their Cayley graphs are found to have beautiful symmetries.

Chapter five contains a detailed study of the spectra of knot-graph adjacency matrices. Whilst we knew at the outset that a characteristic polynomial is not a knot-invariant, we felt it worthwhile to study the graph information contained in the polynomial coefficients; to see how they related to the (U,P) structure of the matrices; and to observe how they changed when knot transformations were carried out.

Apart from the objects described in this thesis, we have computed and studied the properties of many other knot-graph measures. For example, in knot-graphs with given characteristics we have counted n-gons, centres and peripheral points, euler tours and hamiltonian cycles; and we have defined α- and β-distances and studied the corresponding radii and diameters of the knot-graphs. None of these studies led to useful knot parameters. It was not until we looked at knot-graphs with balanced orientations did we discover a good knot-invariant. The arrows placed on the edges did not add new information to the model, since

for bao and bno orientations the direction information is contained in the J_α and J_β matrices. Examining parameters of di-graphs, however, opened new lines of thought.

In chapter six we describe how we discovered the rooted directed spanning tree numbers, and we report on our studies of their properties. The vertex deletion operation we introduce there is not new; indeed, a 'switching operation' (Umschaltung) at crossings of directed curves is a fundamental topological procedure which goes back to Gauss. However, its relationship to spanning trees of knot-graphs and their twins does not seem to have been exploited in the literature. We believe that at least our graphic methods for studying properties of tree numbers, and our formulae for calculating them for various knot-classes, are new. The invariance of τ, for example, was shown by Crowell in 1959; but our method for showing invariance, based on the deletion theorem and Tait's flyping conjecture would seem to be interesting and valuable.

The conjecture that τ is the smallest spanning tree number possible for a given knot-graph with a balanced orientation is likely to be true. We would like to have found a proof for this; one always seemed tantalisingly close. If proven, it is a good general result for 4-regular graphs.

The concept of a twist spectrum of an alternating knot seems to us to be an important one, and we hope it is new. A twist spectrum appears to encapsulate the essence of a knot in an appealing and tractable mathematical form. We thought at first that isospectrality would never occur, and hence that the twist spectrum would be a long-sought 'magic bullet' of knot-theory - a mathematical expression that would discriminate all alternating knots. Now, however, we believe that isospectrality will occur, but not among knots with fewer than fifteen crossings. If this is so (and we give evidence for it in chapter seven), the twist spectrum goes a long way further than other known invariants in providing a single discriminator of alternating knots. Usually it is only necessary to calculate one or two spectral parameters in order to distinguish between two knots.

Further, twist spectra provide a very simple, unified method for testing for nonamphicheirality. For example, by merely counting + and − regions on the knot-graphs of the 123 alternating 10-crossing knots, one finds that 86 of these cannot have a symmetric twist spectrum and therefore cannot be amphicheiral. If we proceed to the next one or two generations of twins from the remaining 10-knots (applying our algorithm for computing the spectra) we quickly reduce the possible candidates for amphicheirality to 13. These 13 knots have symmetric spectra; and they are the ones stated to be amphicheiral in CONWAY, 1970. Classical methods for checking amphicheirality are much more complex. For example, Perko (in *PERKO*,*b*,1979) checks the amphicheirality of these 13 knots, and the nonamphicheirality of the others, by a variety of methods. He makes use of several knot invariants (mainly linking numbers between branch curves) associated with homomorphisms of a knot group on dihedral, alternating and symmetric groups.

We have tried to extend the concept of twist spectrum to nonalternating knot-graphs, but without much success. If a consistent method were to be found for doing this, the method and the resulting spectrum would surely shed light on the fundamental problem of beknottedness.

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Errata and Addenda

(page,line)	
(12,10)	first 1 should be 0
(28,8)	$J_\alpha J_\alpha$ should be $J_\alpha J_\beta$
(36,8)	$n/2$ should be $(n-1)/2$
(38,2)	$(2I + P'_1TP_1)$ should be $(2I + P'_1TP_1)^r$
(38,5)	this line should be attached to the end of line 4
(40,4*)	3.2(2) should be 3.1(2)
(46,11)	insert at end of para ', for any particular knot may give rise to several nonisomorphic minimal knot-graphs.'
(50,1)	transpose 3 and 4 inside the circle
(51,23)	extensive changes made to the definition paragraph: see below
(52,1)	delete the first two paragraphs
(52,16)	replace 'semi-group' by 'elements of'
(53,1)	delete 'The smallest' and insert 'A 6-crossing'
(56,9*)	'knot-groups' should be 'knot-graphs'
(60,9*)	'torii' should be 'torus knots'
(63,9)	insert a backward arc between the last two boxes
(89,17)	λ^{n-1} should be λ^{n-i}
(90,16)	'torii' should be 'torus'
(93,12*)	'torii' should be 'torus'
(96,16)	change 'matrix' to 'determinant'
(98,19)	change 53,75,97 to 35,57,79
(98,5*)	insert) after $n-1$
(110,6*)	'poynomial' should be 'polynomial'
(111,1)	5.1.1 should be 5.7.1
(111,13)	'poynomial' should be 'polynomial'
(115,1*)	insert 'CROWELL and ' before 'FOX'
(117,3)	insert 'in minimal form' after 'trefoils'
(128,8)	change 'principal' to '$(n-1)$-square'
(131,8)	insert a line (above 'Denote') thus: Let r.d.s.t.(v) mean 'directed spanning tree rooted at vertex v'.
(134,6*)	after 13 insert '(graphical proof, subject to Tait's conjecture)
(138,7)	after $n \leq \tau$ insert 'for minimal knot-graphs'
(139,5)	'tori' should be 'torus'
(140,13)	5 should be 8
(140,14)	insert x on F_{12}–diagram at left uppermost vertex
(145,13*)	insert ', where ν is the number of twist-tangles involved'
(146,1*)	insert:) at end
(149,5)	change 'a principle minor' to 'an $(n-1)$-square minor'
(167,15)	delete this line
(167,25)	delete all the proof, and see below for insertions
(180,11)	should be Theorem 4
(188,13)	for knot 8_5, change 7_1^2 to 7_3^2
(188,21)	for knot 8_{15}, change $2_{1sup}2$ to 2_1^2
(191,8*)	for knot 34, should be 1 4 8 10 .. 12 .. 12 10 7 4 1 .. 69 32 116 298
(191,7*)	for knot 35, should be 1 1 .. 3 .. 4 3 5 4 3 2 1 .. 27 72 78 326

Changes to be made on page 51:

Delete the heading 'Definition' at mid-page, and the paragraph of definitions, and the paragraph following that. Replace this by three definitions, as follows:

Definitions:

(*i*) A power of U, or of V, will be called a *syllable*; and a collection of adjacent syllables will be called a *subword*.

(*ii*) Any word from T, from which all subwords which multiply to I by syllable multiplication (i.e. ordinary matrix multiplication) have been removed, is called an *elementary walk−type*. We shall denote the set of all elementary walk-types (with I) by $E(T)$.

(*iii*) Each syllable in an elementary walk-type represents a permutation matrix. The result of multiplying these matrices together is a permutation matrix. We define the permutation which this last permutation matrix represents to be the *end−point* of the elementary walk-type.

In the first line of the final paragraph, delete 'semi-group' and replace it by the phrase 'elements of $E(T)$'.

Changes to be made to page 167.

There is an error in part (*ii*) of the proof of Theorem 1. Delete all of the theorem and proof, and make the following changes. (167,12) After 'result' insert 'appears to' and remove the s from 'holds'. (167,13,14) Change 'as' to 'by', and delete 'shows'. Delete the sentence 'This ... obvious!' (167,15) Delete this line, and then insert the following:

What appears to be at once remarkable and obvious is found to be untrue, when one considers the *types* of composite knot-graph that are found amongst the twins. One does not find *all composite* 1-link or 2-link alternating knot-graphs (as well as all the prime alternating ones) of order less than r occurring amongst the first generation of twins of the prime alternating 1-links of order r.

For example, for $r = 7$ the granny knot $T_3\#T_3$ occurs twice (as a twin from both 7_4 and 7_7); but the square knot $T_3\#\bar{T}_3$ does not occur at all. Whilst among the twins of the order 8 knot-graphs, only the square knot occurs, having parent 8_{11}. Again amongst the order 8 knots, only one type of composition of 3_1 and 4_1 occurs (two types exist).

If we leave aside the question of occurrence of composite twins, we can prove a weaker statement about the union of the families of twins, as follows.

Theorem 1

Let K_r be the set of all prime alternating 1-link knot-graphs (in minimal form) of order r. Then the union set of all the first-generation twins derived from the members of K_r includes all 1- and 2-link prime alternating knot graphs (in minimal form) of order m with $2 \le m \le r-1$. If r is odd, the unknot 0_1 is also included.

Proof: (by construction)

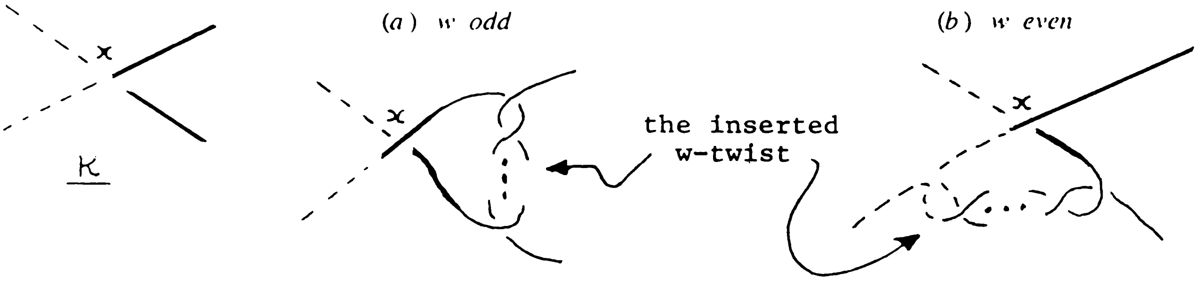
Let κ be a prime alternating knot-graph with m crossings and $2 \le m \le r-1$.

We will show how to construct a 1-link with r crossings which has κ as one of its twins. There are two cases to consider, namely κ an alternating 1-link, and κ an alternating 2-link.

(i) κ *an alternating* $1-link$

We construct a knot-graph K, having r crossings, from κ in of two ways depending on the parity of $w = r - m$.

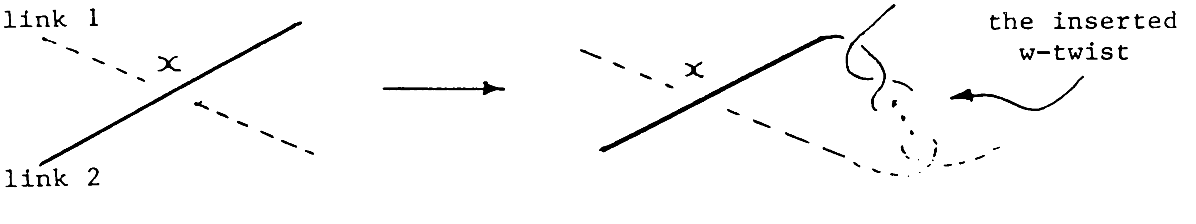
The technique is to insert a w-crossing twist between two adjacent edges of κ. The two edges must be chosen so that the result is an alternating 1-link; we choose them as indicated in the diagrams below. Note that x is any crossing in κ. The dotted edges indicate the start and finish of one closed walk starting at x; the thickened edges indicate the start and finish of the other closed walk in κ (see page 6).



The result in both (a) and (b) is an alternating knot-graph $K \in K_r$. Deleting any vertex of the w-twist leaves κ as one of the resulting twins.

(ii) κ *an alternating* $2-link$

For this case, a vertex where the two links cross is found, and the w-twist inserted across a pair of edges which are adjacent to the vertex but belong one to each link. The result is a 1-link parent of κ, with r crossings as desired. The following diagrams illustrate:



Finally, if r is odd, the null knot 0_1 is a twin of the torus T_r. //