Superintegrability on the two-dimensional hyperboloid

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In this work we examine the basis functions for classical and quantum mechanical systems on the two-dimensional hyperboloid that admit separation of variables in at least two coordinate systems. We present all of these cases from a unified point of view. In particular, all of the special functions that arise via variable separation have their essential features expressed in terms of their zeros. The principal new results are the details of the polynomial bases for each of the nonsubgroup bases, not just the subgroup spherical coordinate cases, and the details of the structure of the quadratic symmetry algebras. © *1997 American Institute of Physics.* [S0022-2488(97)03610-4]

I. INTRODUCTION

In a previous article[1] we have studied the so-called superintegrable Hamiltonian systems in two-dimensional Euclidean space and the two-dimensional sphere. In that article we showed how to compute the basis functions for all the bound state wave functions for all the possible coordinate systems for which separation was possible. We recall that the notion of a superintegrable system relates to a potential for which the solution via separation of variables is possible in more than one coordinate system.[2–4] In two dimensions there remains the case of the two-dimensional hyperboloid. Here we investigate this case, making suitable use of what we already know from Refs. 5, 6 and earlier papers such as Refs. 7–16. The main emphasis will be on those features that are new, together with the role that the quadratic algebra has in these computations. There are four potentials listed in Ref. 6 that permit multiple variable separation on the hyperboloid, but we limit our attention to the first three, since the remaining potential does not give rise to bound states. (In distinction to the case for Euclidean space and the sphere, a complete listing of distinct superintegrable systems on the two-dimensional hyperboloid has not yet been achieved. For some additional superintegrable systems, see Ref. 5.) We study each of the three cases systematically. We will consider two examples of expansions relating bases associated to different coordinate systems. In the first example we use the structure of the quadratic algebra relations to compute an expansion; in the second we compute directly.

For each of the superintegrable systems we observe that, for the discrete spectrum of the quantum mechanical Hamiltonian, one can consider this operator as acting on a space of polynomials in appropriate variables.[17] Each eigenvalue is multiply degenerate. However, every separable coordinate system gives rise to an orthonormal basis of polynomial eigenfunctions in this space and breaks the degeneracy. These bases are characterized as simultaneous eigenfunctions of second-order symmetry operators for the Hamiltonian.[18] We show that under commutation the symmetry operators close to form a quadratic algebra,[11] and we determine the structure of that algebra. The superintegral systems are of two types: the ''normal type'' in which the original Hamiltonian is diagonalized, and the ''conformal type'' in which the Hamiltonian is modified by multiplying the eigenvalue equation by a function and considering the energy as fixed. The

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TABLE I. Separable coordinates for three superintegrable systems.

Potential $V(\omega)$	Coordinate system
$V_1(\omega)=\frac{\alpha}{2}\frac{\omega_2^2+\omega_3^2}{\omega_1^2}+\frac{1}{2}\left[\frac{k_2^2-\frac{1}{4}}{\omega_2^2}+\frac{k_3^2-\frac{1}{4}}{\omega_3^2}\right]$	Spherical Equidistant Elliptic Hyperbolic
$V_2=-\frac{\alpha\omega_1}{\sqrt{\omega_2^2+\omega_3^2}}+\frac{1}{4\sqrt{\omega_2^2+\omega_3^2}}\left(\frac{k_1^2-\frac{1}{4}}{\sqrt{\omega_2^2+\omega_3^2}+\omega_2}+\frac{k_2^2-\frac{1}{4}}{\sqrt{\omega_2^2+\omega_3^2}-\omega_2}\right)$	Spherical Elliptic–parabolic Elliptic II Semihyperbolic
$V_3=\frac{1}{2}\left[\frac{\alpha}{(\omega_1-\omega_2)^2}+\Omega^2\,\frac{1+4\omega_3^2}{(\omega_1-\omega_2)^4}-\lambda\,\frac{\omega_3}{(\omega_1-\omega_2)^3}\right]$	Horicyclic Semicircular–parabolic

modified equation is then interpreted as the eigenvalue equation for a Hamiltonian on a conformal Euclidean space with a ''charge'' as the eigenvalue.

II. SOLUTIONS OF THE SCHRÖDINGER EQUATION

The two-dimensional hyperboloid is characterized via the Cartesian coordinates $\omega_1,\omega_2,\omega_3$, where $\omega_1^2-\omega_2^2-\omega_3^2=1$, $\omega_1>1$. The notion $\omega_1>1$ means that we consider only one sheet of the double-sheet hyperboloid. Throughout this paper we will consider the Schrödinger equation on the hyperboloid in the form ($\hbar=m=1$)

$$H\Psi\equiv(-\tfrac{1}{2}\Delta_{\rm LB}+V)\Psi=E\Psi, \tag{1}$$

where V is the potential function and the Laplace–Beltrami operator $\Delta_{\rm LB}$ can be written as

$$\Delta_{\rm LB}=K_3^2+K_2^2-M_1^2 \tag{2}$$

Here K_3,K_2,M_1 generate the Lie algebra $so(2,1)$,[18]

$$K_3=w_1\partial_{w_2}+w_2\partial_{w_1},\quad K_2=w_1\partial_{w_3}+w_3\partial_{w_1},\quad M_1=\omega_2\partial_{\omega_3}-\omega_3\partial_{\omega_2}. \tag{3}$$

We now consider the potentials V for which (1) is superintegrable, see Table I.

A. First potential

The first potential considered in the paper[6] is the singular oscillator potential,

$$V_1=\frac{\alpha}{2}\,\frac{\omega_2^2+\omega_3^2}{\omega_1^2}+\frac{1}{2}\left[\frac{k_2^2-\frac{1}{4}}{\omega_2^2}+\frac{k_3^2-\frac{1}{4}}{\omega_3^2}\right], \tag{4}$$

where α and $k_{2,3}$ are constants with $\alpha>0$, and $k_{2,3}>0$. The corresponding Schrödinger equation admits separable solutions in four coordinate systems: spherical, equidistant, elliptic, and hyperbolic. Unlike the corresponding oscillator system on the two-dimensional sphere and Euclidean space, the potential (4) on the hyperboloid has a discrete and a continuous spectrum. The bound state solutions can be found as follows.

1. Spherical coordinates

Here,

$$\omega_1=\cosh\tau,\quad \omega_2=\sinh\tau\cos\varphi,\quad \omega_3=\sinh\tau\sin\varphi \tag{5}$$

$(\tau>0,\varphi\in[0,2\pi))$ and the potential (4) reads as

$$V_1(\tau,\varphi)=\frac{\alpha}{2}+\frac{1}{2}\left[\frac{1}{\sinh^2\tau}\left(\frac{k_2^2-\frac{1}{4}}{\cos^2\varphi}+\frac{k_3^2-\frac{1}{4}}{\sin^2\varphi}\right)-\frac{k_1^2-\frac{1}{4}}{\cosh^2\tau}\right], \tag{6}$$

where $k_1=\sqrt{\alpha+\frac{1}{4}}$. Choosing the wave function Ψ in the form

$$\Psi_1(\tau,\varphi)=(\sinh\tau)^{-1/2}S(\tau)\Phi(\varphi), \tag{7}$$

we go to the system of coupled differential equations in the Pöschl–Teller form

$$\frac{d^2S}{d\tau^2}+\left[\tilde{E}+\frac{k_1^2-\frac{1}{4}}{\cosh^2\tau}-\frac{\lambda^2-\frac{1}{4}}{\sinh^2\tau}\right]S=0, \tag{8}$$

$$\frac{d^2\Phi}{d\varphi^2}+\left[\lambda^2-\left(\frac{k_2^2-\frac{1}{4}}{\cos^2\varphi}+\frac{k_3^2-\frac{1}{4}}{\sin^2\varphi}\right)\right]\Phi=0, \tag{9}$$

where $\tilde{E}=2E+\alpha-\frac{1}{4}$ and λ is a spherical separation constant. The solution of the equation (9) normalized in the region $\varphi\in[0,\pi/2]$ has the form[2]

$$\begin{aligned}\Phi(\varphi)=\Phi_m^{(\pm k_3,\pm k_2)}(\varphi)=&\sqrt{\frac{2(2m+1\pm k_2\pm k_3)m!\,\Gamma(1+m\pm k_2+k_3)}{\Gamma(1+m\pm k_2)\Gamma(1+m\pm k_3)}}\\&\times(\sin\varphi)^{1/2\pm k_3}(\cos\varphi)^{1/2\pm k_2}P_m^{(\pm k_3,\pm k_2)}(\cos 2\varphi),\end{aligned} \tag{10}$$

with $m\in\mathbf{N}$, where $P_n^{(\alpha,\beta)}$ is the Jacobi polynomial.[19] The separation constant is quantized as

$$\lambda=(2m\pm k_2\pm k_3+1). \tag{11}$$

The positive sign at $k_{2,3}$ has to be taken if $k_{2,3}>1/2$ and both positive and negative signs must be taken if $0<k_{2,3}<1/2$, so the function Φ has a finite form.

The second (modified) Pöschl–Teller equation (8) was investigated by Frank and Wolf[20] and Barut *et al.*[21] The bound state wave function is given by

$$S(\tau)\equiv S_n^{(\lambda,k_1)}(\tau)=N_n^{(\lambda,k_1)}(\sinh\,\tau)^{1/2+\lambda}(\cosh\,\lambda)^{2n-k_1+1/2}{}_2F_1(-n,k_1-n;1+\lambda;\tanh^2\,\tau)$$
$$=\frac{n!\Gamma(1+\lambda)}{\Gamma(1+\lambda+n)}\,N_n^{(\lambda,k_1)}(\sinh\,\tau)^{1/2+\lambda}(\cosh\,\tau)^{1/2-k_1}P_n^{(\lambda,-k_1)}(\cosh 2\tau), \tag{12}$$

with $n=0,1,...[\frac{1}{2}(k_1-\lambda-1)]$ and a bound state solution is possible only for $k_1-\lambda>1$. The wave function $S_n^{(\lambda,k_1)}(\tau)$ satisfies the orthogonality relation

$$\int_0^\infty S_n^{(\lambda,k_1)}(\tau)S_{n'}^{(\lambda,k_1)}(\tau)d\tau=\delta_{nn'},$$

and the normalization factor has the form

$$N_n^{(\lambda,k_1)}=\frac{1}{\Gamma(1+\lambda)}\,\sqrt{\frac{2(k_1-\lambda-2n-1)\Gamma(1+\lambda+n)\Gamma(k_1-n)}{\Gamma(k_1-\lambda-n)n!}}. \tag{13}$$

Here, and in the remainder of this paper, the bound state wave functions are normalized so that they form an orthonormal set with respect to the invariant measure on the hyperboloid (which in spherical coordinates is $\sinh\,\tau\,d\tau\,d\varphi$).

The normalized total wave function $\Psi_1(\tau,\varphi)\equiv\Psi_{nm}(\tau,\varphi;k_1,\pm k_2,\pm k_3)$ is given by (7), (10), (12), and (13). The quantized energy is

$$E_N=-\frac{1}{2}\left[(2N-k_1\pm k_2\pm k_3+2)^2-\frac{1}{4}\right]+\frac{\alpha}{2}, \tag{14}$$

where $N=m+n$ is the principal quantum number and the bound states occur for

$$0\leqslant N\leqslant\tfrac{1}{2}(k_1\mp k_2\mp k_3-2). \tag{15}$$

Each solution $\Psi_1(\tau,\varphi)$ satisfies the eigenvalue equation

$$L_1\Psi_1\equiv\left[M_1^2-(\omega_2^2+\omega_3^2)\left(\frac{k_2^2-\frac{1}{4}}{\omega_2^2}+\frac{k_3^2-\frac{1}{4}}{\omega_3^2}\right)\right]\Psi_1=-4\left(m+\frac{1\pm k_2\pm k_3}{2}\right)^2\Psi_1, \tag{16}$$

and L_1 commutes with H.

2. Equidistant coordinates

Here,

$$\omega_1\cosh\,a\,\cosh\,b,\quad \omega_2=\cosh\,a\,\sinh\,b,\quad \omega_3=\sinh\,a$$

$[\tau_{1,2}\in(-\infty,\infty)]$ and the potential V_1 has the form

$$V_1(a,b)=\frac{\alpha}{2}-\frac{1}{2}\left[\frac{1}{\cosh^2\,a}\left(\frac{k_1^2-\frac{1}{4}}{\cosh^2\,b}-\frac{k_2^2-\frac{1}{4}}{\sinh^2\,b}\right)-\frac{k_3^2-\frac{1}{4}}{\sinh^2\,a}\right].$$

After putting

$$\Psi_2(a,b)=(\cosh\,a)^{-1/2}S(a)S(b), \tag{17}$$

we go to the two modified Pöschl–Teller equations,

$$\frac{d^2S}{da^2}+\left[\tilde{E}+\frac{\mu^2-\frac{1}{4}}{\cosh^2 a}-\frac{k_3^2-\frac{1}{4}}{\sinh^2 a}\right]S=0, \tag{18}$$

$$\frac{d^2S}{db^2}+\left[-\mu^2+\left(\frac{k_1^2-\frac{1}{4}}{\cosh^2 b}-\frac{k_2^2-\frac{1}{4}}{\sinh^2 b}\right)\right]S=0, \tag{19}$$

where μ is the equidistant separation constant. The normalized for $a,b\in[0,\infty)$ bound state wave functions have the form

$$S(b)\equiv S_{n_1}^{(\pm k_2,k_1)}(b)=\sqrt{\frac{2(k_1\mp k_2-1-2n_1)\Gamma(k_1-n_1)n_1!}{\Gamma(k_1\mp k_2-n_1)\Gamma(1\pm k_2+n_1)}}\times(\sinh b)^{1/2\pm k_2}(\cosh b)^{1/2-k_1}P_{n_1}^{(\pm k_2,-k_1)}(\cosh 2b), \tag{20}$$

$$S(a)\equiv S_{n_2}^{(\pm k_3,\mu)}(a)=\sqrt{\frac{2(\mu\mp k_3-1-2n_2)\Gamma(\mu-n_2)n_2!}{\Gamma(\mu\mp k_3-n_2)\Gamma(1\pm k_3+n_2)}}\times(\sinh a)^{1/2\pm k_3}(\cosh a)^{1/2-\mu}P_{n_2}^{(\pm k_3,-\mu)}(\cosh 2a) \tag{21}$$

and the separation constant is

$$\mu=(k_1\mp k_2-2n_1),$$

with $n_1=0,1,...[\frac{1}{2}(k_1\mp k_2-1)]$ and $n_2=0,1,...[\frac{1}{2}(\mu\mp k_3-1)]$. The bound state energy E_N is given by (14), where now the principal quantum number N is $N=n_1+n_2$ and satisfies the equation (15). The total wave function $\Psi_2(a,b)\equiv\Psi_{n_1n_2}(a,b;k_1,\pm k_2,\pm k_3)$ satisfies

$$L_3\Psi_2\equiv\left[-K_3^2-(\omega_1^2-\omega_2^2)\left(\frac{k_1^2-\frac{1}{4}}{\omega_1^2}-\frac{k_2^2-\frac{1}{4}}{\omega_2^2}\right)\right]\Psi_2=-4\left(n_1+\frac{1-k_1\pm k_2}{2}\right)^2\Psi_2, \tag{22}$$

where $[L_3,\ H]=0$.

We can introduce also the second equidistant system of coordinates by the interchange $\omega_2\leftrightarrow\omega_3$. Then the total bound state wave function $\Psi_3(\tilde{a},\tilde{b})\equiv\Psi_{\tilde{n}_1,\tilde{n}_2}(\tilde{a},\tilde{b};k_1,\pm k_2,\pm k_3)$ may be obtained from (20) and (21) by the interchange $k_2\leftrightarrow k_3$ and satisfies

$$L_2\Psi_3=\left[-K_2^2-\left(\omega_1^2-\omega_3^2\right)\left(\frac{k_1^2-\frac{1}{4}}{\omega_1^2}-\frac{k_3^2-\frac{1}{4}}{\omega_3^2}\right)\right]\Psi_3=-\left(\tilde{n}_1+\frac{1-k_1\pm k_3}{2}\right)^2\Psi_3, \tag{23}$$

where $[L_2,\ H]=0$.

We can call these three bases that are the eigenfunctions of the Hamiltonian for the potential V_1 and of each of the operators (L_1,L_2,L_3) the *fundamental bases* for the potential of a singular oscillator on the two-dimensional hyperboloid. Note also that

$$L_1+L_2+L_3=2H-(2k_1^2+k_2^2+k_3^2-1). \tag{24}$$

3. Elliptic coordinates of type one

For the remaining elliptic-type coordinates it is convenient to introduce the quantities

$$s_1^2=\frac{(\mu-e_j)(\nu-e_j)}{(e_j-e_k)(e_j-e_l)},\quad i,j,k=1,2,3,\quad \text{and}\quad \neq, \tag{25}$$

For elliptic coordinates of type one, the coordinates on the hyperboloid are given by

$$\omega_1^2=s_1^2,\quad \omega_2^2=-s_2^2,\quad \omega_3^2=-s_3^2,$$

where $e_1<e_2<\nu<e_3<\mu$. Rather than express the wave functions in the separable coordinates μ, ν, we will use a Niven equations approach,[22] in which the separated solutions are expressed in Cartesian coordinates and the critical information is the location of the zeros of these solutions. The bound state wave functions are

$$\Psi_4=(\Pi_{l=1}^3\omega_l^{k_l+1/2})\Pi_{j=1}^q\left[\frac{\omega_1^2}{\theta_j-e_1}-\frac{\omega_2^2}{\theta_j-e_2}-\frac{\omega_3^2}{\theta_j-e_3}\right], \tag{26}$$

where the θ_j satisfy

$$\frac{k_1+1}{\theta_j-e_1}+\frac{k_2+1}{\theta_j-e_2}+\frac{k_3+1}{\theta_j-e_3}+\sum_{j\neq m}\frac{2}{\theta_m-\theta_j}=0.$$

These solutions satisfy the eigenvalue equations

$$(e_3L_3+e_2L_2+e_1L_1)\Psi_4=\left(\sum_{i,j,l\neq}\left[-2k_i(e_j+e_l)+e_ik_jk_l-4e_ie_j(k_l+1)\sum_{m=1}^q\frac{1}{\theta_m-e_l}\right]\right.$$
$$\left.-\frac{3}{2}(e_1+e_2+e_3)\right)\Psi_4. \tag{27}$$

4. Elliptic coordinates of type two

The coordinates on the hyperboloid are given by

$$\omega_1^2=s_2^2,\quad \omega_2^2=-s_1^2,\quad \omega_3^2=-s_3^2,$$

where $\nu<e_1<e_2<e_3<\mu$. The bound state wave functions are

$$\Psi_5=(\Pi_{l=1}^3\omega_l^{e_l+1/2})\Pi_{j=1}^q\left[-\frac{\omega_2^2}{\theta_j-e_1}+\frac{\omega_1^2}{\theta_j-e_2}-\frac{\omega_3^2}{\theta_j-e_3}\right], \tag{28}$$

where $\epsilon_1=k_2$, $\epsilon_2=k_1$, $\epsilon_3=k_3$. The θ_j satisfy

$$\frac{\epsilon_1+1}{\theta_j-e_1}+\frac{\epsilon_2+1}{\theta_j-e_2}+\frac{\epsilon_3+1}{\theta_j-e_3}+\sum_{j\neq m}\frac{2}{\theta_m-\theta_j}=0.$$

The eigenvalue equation is

$$(e_3L_3+e_2L_1+e_1L_2)\Psi_5=\left(\sum_{i,j,l\neq}\left[-2\epsilon_i(e_j+e_l)+e_i\epsilon_j\epsilon_l-4e_ie_j(\epsilon_l+1)\sum_{m=1}^{q}\frac{1}{\theta_m-e_l}\right]\right.$$
$$\left.-\frac{3}{2}(e_1+e_2+e_3)\Psi_5.\right. \tag{29}$$

5. Quadratic algebra relations

Here, we define the operator R by the equality

$$R=[L_1,\ L_2]=[L_3,\ L_1]=[L_2,\ L_3], \tag{30}$$

and can verify that R satisfies

$$[L_j,R]=-4\{L_j,L_m-L_l\}+8[(\tfrac{3}{2}-k_m^2)L_m-(\tfrac{3}{2}-k_l^2)L_l]+8(k_l^2-k_m^2),$$

where j,m,l is an even permutation of 1,2,3. In addition,

$$\begin{aligned}R^2=&\tfrac{8}{3}\{L_1,L_2,L_3\}-16(1-k_1^2)L_1^2-16(1-k_2^2)L_2^2-16(1-k_3^2)L_3^2+\tfrac{52}{3}(\{L_1,L_2\}+\{L_2,L_3\}\\&+\{L_1,L_3\})+\tfrac{1}{3}((60-176k_1^2)L_1+(60-176k_2^2)L_2+(60-176k_3^2)L_3)+64k_1k_2k_3\\&+48(k_1k_2+k_2k_3+k_1k_3)+\tfrac{32}{3}(k_1+k_2+k_3),\end{aligned} \tag{31}$$

where $\{A,B\}=AB+BA$ and

$$\{A,B,C\}=ABC+ACB+BCA+BAC+CAB+CBA.$$

B. Second potential

The singular Coloumb–Kepler potential is

$$V_2=-\frac{\alpha\omega_1}{\sqrt{\omega_2^2+\omega_3^2}}+\frac{1}{4\sqrt{\omega_2^2+\omega_3^2}}\left(\frac{k_1^2-\frac{1}{4}}{\sqrt{\omega_2^2+\omega_3^2}+\omega_2}+\frac{k_2^2-\frac{1}{4}}{\sqrt{\omega_2^2+\omega_3^2}-\omega_2}\right), \tag{32}$$

where $\alpha>0$ and $k_{1,2}>0$. The Schrödinger equation for this potential can be solved for the bound state wave functions in four coordinate systems: spherical, elliptic–parabolic, elliptic II, semihyperbolic.

1. Spherical coordinates

In coordinates (5) we have

$$V_2(\tau,\varphi)=-\alpha\coth\tau+\frac{1}{8\sinh^2\tau}\left(\frac{k_1^2-\frac{1}{4}}{\cos^2(\varphi/2)}+\frac{k_2^2-\frac{1}{4}}{\sin^2(\varphi/2)}\right). \tag{33}$$

Choosing the wave functions in form,

$$\Psi_1=(\sinh\tau)^{-1/2}S(\tau)\Phi_m^{(\pm k_2,\pm k_1)}(\varphi/2), \tag{34}$$

where the wave function $\Phi_m^{(\pm k_2,\pm k_1)}(\varphi/2)$ normalized for $\varphi\in[0,\pi]$ is given by (10) and $m\in\mathbf{N}$. After separation of variables we come to the equation

$$\frac{d^2S}{d\tau^2}+\left[\widetilde{E}+2\alpha\coth\tau-\frac{\lambda^2-\frac{1}{4}}{\sinh^2\tau}\right]S=0, \tag{35}$$

where $\widetilde{E}=2E-\frac{1}{4}$. This equation was first introduced by Manning and Rosen[23] to study the vibrations of diatomic molecules.

For the bound state wave function normalized in region $\tau\in[0,\infty)$, we have

$$\begin{aligned}S(\tau)&\equiv S_{nm}(\tau)\\&=\frac{2^{m+1\pm k_1/2\pm k_2/2}}{\Gamma(2m+2\pm k_1+k_2)}\left[\frac{(\sigma_N^2-N^2)}{N}\frac{\Gamma(n+2m+2\pm k_1\pm k_2)}{n!}\frac{\Gamma(\sigma_N+m+1\pm k_1/2\pm k_2/2)}{\Gamma(\sigma_N-m\mp k_1/2\mp k_2/2)}\right]^{1/2}\\&\quad\times(\sinh\tau)^{m+1\pm k_1/2\pm k_2/2}e^{-\tau(\sigma_N-n)}\\&\quad\times{}_2F_1\left(-n,m+1\pm\frac{k_1}{2}\pm\frac{k_2}{2}+\sigma_N;2m+2\pm k_1\pm k_2;\frac{2}{1+\coth\tau}\right),\end{aligned}$$

where $\sigma_N=\alpha/N$ and $N=n+m+1\pm k_1/2\pm k_2/2$. The quantisation condition for the energy is

$$E_N=-\frac{N^2-\frac{1}{4}}{2}-\frac{\alpha^2}{2N^2}, \tag{36}$$

and the bound states occur for

$$0\leqslant n+m\leqslant\left[\sqrt{\alpha}\mp\frac{k_1}{2}\mp\frac{k_2}{2}-1\right]. \tag{37}$$

The solutions $\Psi_1\equiv\Psi_{nm}(\tau,\varphi;\pm k_1\pm k_2)$ are eigenfunctions of the symmetry operator

$$L=M_1^2-\frac{1}{2}\sqrt{\omega_2^2+\omega_3^2}\left(\frac{k_1^2-\frac{1}{4}}{\sqrt{\omega_2^2+\omega_3^2}+\omega_2}+\frac{k_2^2-\frac{1}{4}}{\sqrt{\omega_2^2+\omega_3^2}-\omega_2}\right), \tag{38}$$

with eigenvalue $-[m+1/2(1\pm k_1\pm k_2)]^2$.

2. Semihyperbolic coordinates

Here

$$\omega_1=\frac{1+xy}{\sqrt{(y^2-1)(x^2-1)}},\quad \omega_2=\frac{y-x}{\sqrt{(y^2-1)(x^2-1)}},\quad \omega_3=\frac{2\sqrt{xy}}{\sqrt{(y^2-1)(x^2-1)}},$$

where $x,y\in[0,1)$. [Note that these coordinates are connected with corresponding coordinates in Ref. 6 by the transformation $\mu_1=2x/(x^2-1)$ and $\mu_2=2y/(y^2-1)$]. The potential V_2 now has the form

$$V_2(x,y)=-\alpha\,\frac{xy+1}{x+y}+\frac{1}{8}\,\frac{(x^2-1)(y^2-1)}{x+y}\left(\frac{k_1^2-\frac{1}{4}}{y}+\frac{k_2^2-\frac{1}{4}}{x}\right).$$

The computation of the bound state wave functions proceeds as follows. The Schrödinger equation has the form

$$\sqrt{x(x^2-1)}\,\frac{\partial}{\partial x}\,\sqrt{x(x^2-1)}\,\frac{\partial\Psi}{\partial x}+\sqrt{y(y^2-1)}\,\frac{\partial}{\partial y}\,\sqrt{y(y^2-1)}\,\frac{\partial\Psi}{\partial y}+\Bigg[2E\left(\frac{x}{x^2-1}+\frac{y}{y^2-1}\right)$$
$$+\alpha\left(\frac{x^2+1}{x^2-1}+\frac{y^2+1}{y^2-1}\right)-\frac{xy-1}{4}\left(\frac{k_1^2-\frac{1}{4}}{y}+\frac{k_2^2-\frac{1}{4}}{x}\right)\Bigg]\Psi=0. \tag{39}$$

Choosing the wave function in the form $\Psi_2(x,y)=X(x)Y(y)$, after the separation of variables we have two equations:

$$x(x^2-1)\left[\frac{d^2X}{dx^2}+\frac{1}{2}\left(\frac{1}{x}+\frac{1}{x-1}+\frac{1}{x+1}\right)\frac{dX}{dx}\right]$$
$$+\left[\frac{1}{2}\,(2\alpha+\lambda)+\frac{2\alpha}{x^2-1}-\frac{1}{4}\left(k_1^2-\frac{1}{4}\right)x+\frac{k_2^2-\frac{1}{4}}{4x}+\frac{2Ex}{x^2-1}\right]X=0,$$
$$y(y^2-1)\left[\frac{d^2Y}{dy^2}+\frac{1}{2}\left(\frac{1}{y}+\frac{1}{y-1}+\frac{1}{y+1}\right)\frac{dY}{dy}\right] \tag{40}$$
$$+\left[\frac{1}{2}\,(2\alpha-\lambda)+\frac{2\alpha}{y^2-1}-\frac{1}{4}\left(k_2^2-\frac{1}{4}\right)y+\frac{k_1^2-\frac{1}{4}}{4y}+\frac{2Ey}{y^2-1}\right]Y=0,$$

where λ is a semihyperbolic separation constant. The first of these equations can be solved as follows. Looking for solutions of the form

$$X=x^{(1/2)(1/2\pm k_2)}(x-1)^{-(1/4)(2p+1\pm k_1\pm k_2)+\alpha(2p+2\pm k_1\pm k_2)^{-1}}$$
$$\times(x+1)^{-(1/4)(2p+1\pm k_1\pm k_2)-\alpha(2p+2\pm k_1\pm k_2)^{-1}}\Pi_{j=1}^{p}(x-\xi_1), \tag{41}$$

we see that the zeros ξ_i satisfy

$$\frac{1\pm k_2}{\xi_i}+(1\pm k_1)\xi_i-2\sum_{j\neq i}\frac{\xi_i\xi_j-1}{\xi_i-\xi_j}-\frac{2\alpha}{p+1\pm k_1/2\pm k_2/2}=0,\quad i=1,...,p. \tag{42}$$

A similar solution for the equation in y can be obtained via the substitution

$$Y=y^{(1/2)(1/2\pm k_1)}(y-1)^{-(1/4)(2p+1\pm k_1\pm k_2)+\alpha(2p+2\pm k_1\pm k_2)^{-1}}$$
$$\times(y+1)^{-(1/4)(2p+1\pm k_1\pm k_2)-\alpha(2p+2\pm k_1+k_2)^{-1}}\Pi_{j=1}^{p}(y-\zeta_i). \tag{43}$$

The zeros satisfy

$$\frac{1\pm k_1}{\zeta_i}+(1\pm k_2)\zeta_i-2\sum_{j\neq i}\frac{\zeta_i\zeta_j-1}{\zeta_i-\zeta_j}-\frac{2\alpha}{p+1\pm k_1/2\pm k_2/2}=0,\quad i=i,\ldots,p. \tag{44}$$

The separation constant λ has the value

$$\lambda = 2(1 \pm k_1)(\xi_1 + \cdots + \xi_p) - 4\alpha \frac{p \pm k_2/2 \mp k_1/2}{2p + 2 \pm k_1 \pm k_2}$$
$$= -2(1 \pm k_2)(\zeta_1 + \cdots + \zeta_p) + 4\alpha \frac{p \pm k_1/2 \mp k_2/2}{2p + 2 \pm k_1 \pm k_2}. \tag{45}$$

The equality of these two expressions can be seen by noting that the equations for the zeros can be obtained from each other via the substitutions $k_1 \leftrightarrow k_2, \xi_i \leftrightarrow 1/\zeta_i$. The symmetry operator with eigenvalue λ is

$$M = 2A_2 + \frac{\omega_1 \omega_3^2}{2\sqrt{\omega_2^2 + \omega_3^2}} \left[\frac{k_1^2 - \frac{1}{4}}{(\sqrt{\omega_2^2 + \omega_3^2} + \omega_2)^2} - \frac{k_2^2 - \frac{1}{4}}{(\sqrt{\omega_2^2 + \omega_3^2} - \omega_2)^2} \right], \tag{46}$$

where A_2 is a component of the two-dimensional Runge–Lenz vector,[24]

$$A_i = \frac{1}{2}\{K_i, M_1\} + \frac{\alpha \omega_i}{\sqrt{\omega_2^2 + \omega_3^2}}, \quad i = 2,3.$$

3. Elliptic parabolic coordinates

Here we have

$$\omega_1 = \frac{\cosh^2 a + \cos^2 \theta}{2 \cosh a \cos \theta}, \quad \omega_2 = \frac{\sinh^2 a - \sin^2 \theta}{2 \cosh a \cos \theta}, \quad \omega_3 = \tanh a \tan \theta,$$

where $a > 0$, $\theta \in (-\pi/2, \pi/2)$. The potential in these coordinates has the form

$$V_2(a,\theta) = -\alpha \frac{\cosh^2 a + \cos^2 \theta}{\cosh^2 a - \cos^2 \theta} + \frac{1}{2} \frac{\cosh^2 a \cos^2 \theta}{\cosh^2 a - \cos^2 \theta} \left(\frac{k_1^2 - \frac{1}{4}}{\sinh^2 a} + \frac{k_2^2 - \frac{1}{4}}{\sin^2 \theta} \right).$$

Putting for the wave function $\Psi_3(a,\theta) = S(a)\Phi(\theta)$, after separation of variables we get two equations in Pöschl–Teller form:

$$\frac{d^2 S}{da^2} + \left[-\mu^2 + \frac{\nu^2 - \frac{1}{4}}{\cosh^2 a} - \frac{k_1^2 - \frac{1}{4}}{\sinh^2 a} \right] S = 0, \tag{47}$$

$$\frac{d^2 \Phi}{d\theta^2} + \left[\mu^2 - \left(\frac{\beta^2 - \frac{1}{4}}{\cos^2 \theta} + \frac{k_2^2 - \frac{1}{4}}{\sin^2 \theta} \right) \right] \Phi = 0, \tag{48}$$

where μ is a elliptic parabolic separation constant and

$$\beta = \sqrt{\tfrac{1}{4} - 2(E + \alpha)}, \quad \nu = \sqrt{\tfrac{1}{4} - 2(E - \alpha)}.$$

The separation constant is quantized as

$$\mu = (2n_1 + \beta \pm k_2 + 1) = (\nu \mp k_1 - 1 - 2n_2),$$

and introducing the quantum number $N = n_1 + n_2 + 1 \pm k_1/2 \pm k_2/2$ we go to the energy quantization (36) and condition (37) for $n_1 + n_2$.

The total bound state wave function $\Psi_3(a,\theta) \equiv \Psi_{n_1 n_2}(a,\theta)$ satisfies the orthogonality relation

$$\int_0^\infty \int_0^{\pi/2} \Psi_{n_1 n_2}(a,\theta)\Psi_{n_1' n_2'}(a,\theta)\, \frac{\cosh^2 a - \cos^2 \theta}{\cosh^2 a \cos^2 \theta}\, da\, d\theta = \delta_{n_1 n_1'}\delta_{n_2, n_2'},$$

and has the form

$$\Psi_{n_1 n_2}(a,\theta) = \sqrt{\frac{\sigma_N^2 - N^2}{(2\sigma_N + 2n_1 - 2n_2 \pm k_2 \mp k_1)N}}\, S_{n_2}^{(\pm k_1,\nu)}(a)\Phi_{n_1}^{(\pm k_2,\beta)}(\theta), \tag{49}$$

where the wave function $\Phi_{n_1}^{(\pm k_2,\beta)}(\theta)$ is given by (10) and

$$S_{n_2}^{(\pm k_1,\nu)}(a) = \sqrt{\frac{2(\nu \mp k_1 - 1 - 2n_2)\Gamma(\nu - n_2)n_2!}{\Gamma(\nu \mp k_1 - n_2)\Gamma(1 \pm k_1 + n_2)}}\, (\sinh a)^{1/2 \pm k_1}(\cosh a)^{1/2-\nu} P_{n_2}^{(\pm k_1, -\nu)} \times (\cosh 2a). \tag{50}$$

The elliptic parabolic wave function (49) also satisfies the operator equation,

$$\Lambda \Psi_{n_1,n_2} = \left(2L - M - 2H - \frac{k_1^2 + k_2^2 - \frac{1}{2}}{2}\right)\Psi_{n_1 n_2} = \left(\sigma_N + n_1 - n_2 \pm \frac{k_2}{2} \mp \frac{k_1}{2}\right)^2 \Psi_{n_1 n_2}. \tag{51}$$

4. Modified elliptic coordinate systems

$$\omega_1 = \cosh f \sqrt{\frac{(\rho_1 - a_3)(\rho_2 - a_3)}{(a_3 - a_2)(a_3 - a_1)}} + \sinh f \sqrt{-\frac{(\rho_1 - a_2)(\rho_2 - a_2)}{(a_2 - a_1)(a_2 - a_3)}},$$

$$\omega_2 = \sinh f \sqrt{\frac{(\rho_1 - a_3)(\rho_2 - a_3)}{(a_3 - a_2)(a_3 - a_1)}} + \cosh f \sqrt{-\frac{(\rho_1 - a_2)(\rho_2 - a_2)}{(a_2 - a_1)(a_2 - a_3)}},$$

$$\omega_3 = \sqrt{-\frac{(\rho_1 - a_1)(\rho_2 - a_1)}{(a_1 - a_2)(a_1 - a_3)}},$$

where

$$\sinh f = \sqrt{\frac{a_1 - a_2}{a_2 - a_3}}, \quad \cosh f = \sqrt{\frac{a_1 - a_3}{a_2 - a_3}},$$

and $a_3 < a_2 < \rho_2 < a_1 < \rho_1$. The solution for the bound state wave functions is facilitated by making the change of variables,

$$a_2 = a_1 + \frac{1}{4}(A_+ + A_-)^2, \quad a_3 = a_1 + \frac{1}{4}(A_+ - A_-)^2,$$

$$\rho_j = a_1 + \frac{1}{4}(A_+^2 + A_-^2) + \frac{1}{4}A_+ A_-\left(R_j + \frac{1}{R_j}\right), \quad j = 1,2.$$

In terms of these coordinates, the coordinates on the hyperboloid are given by

$$\omega_1=-\frac{R_1R_2+1}{2\sqrt{R_1R_2}},\quad \omega_2=-\frac{(A_+^2+A_-^2)(R_1R_2+1)+2A_+A_-(R_1+R_2)}{2(A_+^2-A_-^2)\sqrt{R_1R_2}},$$

$$\omega_3=\frac{i\sqrt{(A_+R_1+A_-)(A_+R_2+A_-)(A_-R_1+A_+)(A_-R_2+A_+)}}{(A_+^2-A_-^2)\sqrt{R_1R_2}}.$$

The corresponding bound state solutions are

$$\Psi_4=(\Pi_{l=1}^3U_l^{k_l+1/2})\Pi_{j=1}^g\left(\frac{U_1^2}{\theta_j+\Lambda_-}+\frac{U_2^2}{\theta_j+\Lambda_+}+\frac{U_3^2}{\theta_j}\right), \tag{52}$$

where

$$U_1^2=\frac{(R_1+\Lambda_-)(R_2+\Lambda_-)}{\Lambda_-^2-1},\quad U_2^2=\frac{(R_1+\Lambda_+)(R_2+\Lambda_+)}{\Lambda_+^2-1},\quad U_3^2=R_1R_2,$$

and $\Lambda_+=A_+/A_-$, $\Lambda_-=A_-/A_+$. The zeros θ_j satisfy

$$\frac{k_1+1}{\theta_m+\Lambda_-}+\frac{k_2+1}{\theta_m+\Lambda_+}+\frac{\sqrt{\frac{1}{4}+2(E-\alpha)}+1}{\theta_m}+\sum_{j\neq m}\frac{2}{\theta_m-\theta_j}=0. \tag{53}$$

The symmetry operator characterizing this coordinate system is

$$L'=(\Lambda_++\Lambda_-)(2L-H)-2(\Lambda_+-\Lambda_-)M, \tag{54}$$

with eigenvalues

$$\begin{aligned}\mu=&2[\Lambda_-(k_2+\sqrt{\tfrac{1}{4}+2(E-\alpha)})+\Lambda_+(k_1+\sqrt{\tfrac{1}{4}+2(E-\alpha)})+[\Lambda_+k_1+\Lambda_-k_2]\\&\times\sqrt{\tfrac{1}{4}+2(E-\alpha)}]+\frac{3}{2}(\Lambda_++\Lambda_-)+\sqrt{\tfrac{1}{4}+2(E-\alpha)}\Bigg[4\Lambda_+(k_1+1)\\&\times\sum_{m=1}^n\frac{1}{\theta_m+\Lambda_-}+4\Lambda_+(k_2+1)\sum_{m=1}^n\frac{1}{\theta_m+\Lambda_+}\Bigg]-4\sqrt{\tfrac{1}{4}+2(E-\alpha)}\sum_{m=1}^n\frac{1}{\theta_m}.\end{aligned} \tag{55}$$

5. Quadratic algebra relations

The commutation relations for the quadratic algebra with basis L, M, and H are, for $R=[L,M]$,

$$\begin{aligned}[R,M]=&-24L^2+3M^2+16HL+(10-4(k_1^2+k_2^2))L+(k_2^2-k_1^2)M\\&-(1+2(k_1^2+k_2^2))H+\tfrac{1}{2}(k_1^2+k_2^2)-\tfrac{1}{4}-8\alpha^2,\end{aligned} \tag{56}$$

$$[R,L]=-2\{L,M\}+(k_1^2-k_2^2)L+M+\tfrac{1}{2}(\tfrac{1}{4}+\alpha)(k_1^2-k_2^2),$$

$$\begin{aligned}R^2=&-16H^2L+16L^3+16L^2H-\tfrac{2}{3}\{M,M,L\}+8H\{L,M\}+8(k_1^2+k_2^2-6)L^2-\tfrac{11}{3}M^2-\tfrac{44}{3}H^2\\&-\tfrac{44}{3}HL+\tfrac{44}{3}ML+(2(k_1^2+k_2^2)-1)\{L,M\}+(\tfrac{41}{12}-\tfrac{19}{3}(k_1^2+k_2^2)+4\alpha^2-4k_1^2k_2^2)L\\&+(\tfrac{10}{3}-8(k_1^2+k_2^2)+(k_1^2-k_2^2)(k_1^2-k_2^2)+4\alpha^2))H+(-\tfrac{11}{6}+\tfrac{11}{3}(k_1^2+k_2^2)\end{aligned}$$

$$+\tfrac{88}{3}\,\alpha(k_2^2-k_1^2))M+\alpha^2(2(k_1^2+k_2^2)-\tfrac{1}{3})+\tfrac{1}{2}\,(k_1^2-k_2^2)(2(k_1^2+k_2^2)-1)\alpha$$
$$-\tfrac{1}{16}+\tfrac{7}{12}\,(k_1^2+k_2^2)-\tfrac{3}{4}\,(k_1^4+k_2^4)-\tfrac{13}{6}\,k_1^2k_2^2.$$

C. Third potential

Here

$$V_3=\frac{1}{2}\left[\frac{\alpha}{(\omega_1-\omega_2)^2}+\Omega^2\,\frac{1+4\omega_3^2}{(\omega_1-\omega_2)^4}-\lambda\,\frac{\omega_3}{(\omega_1-\omega_2)^3}\right].$$

The corresponding Schrödinger equation admits separable solutions in two coordinate systems.

1. Horicyclic coordinates

$$\omega_1=\frac{x^2+y^2+1}{2y},\quad \omega_2=\frac{x^2+y^2-1}{2y},\quad \omega_3=\frac{x}{y},$$

where $y>0$, $x\in(-\infty,\infty)$.

2. Semicircular parabolic coordinates

$$\omega_1=\frac{(\xi_1-\xi_2)^2+\frac{1}{4}}{2\sqrt{-\xi_1\xi_2}},\quad \omega_2=\frac{(\xi_1-\xi_2)^2-\frac{1}{4}}{2\sqrt{-\xi_1\xi_2}},\quad \omega_3=-\frac{\xi_1+\xi_2}{2\sqrt{-\xi_1\xi_2}},$$

where $\xi_1>0,\xi_2<0$.

In the case of horicyclic coordinates for the potential V_3, we get

$$V_3(x,y)=\tfrac{1}{2}[\alpha y^2+\Omega^2y^2(y^2+4x^2)-\lambda y^2x],$$

and the Schrödinger equation assumes the form

$$y^2\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)\Psi+[2E-\alpha y^2-\Omega^2(y^2+4x^2)y^2+\lambda xy^2]\Psi=0. \tag{57}$$

Putting $\Psi_1(x,y)=\psi_1(x)\psi_2(y)$, we have the separation equations in the form

$$\frac{d^2\psi_1}{dx^2}+(-\epsilon+\lambda x-4\Omega^2x^2)\psi_1=0, \tag{58}$$

$$\frac{d^2\psi_2}{dy^2}+\left[(\epsilon-\alpha)-\Omega^2y^2-\frac{A^2-\frac{1}{4}}{y^2}\right]\psi_2=0, \tag{59}$$

where ϵ is the horicyclic separation constant and $2E=\frac{1}{4}-A^2$. Solving the first equation we have

$$\psi_1(x)\equiv\psi_{n_1}=\left(\frac{2\Omega}{\pi}\right)^{1/4}\frac{1}{\sqrt{2^{n_1}n_1!}}\,e^{-\Omega(x-\lambda/8\Omega^2)^2}H_{n_1}\left(\sqrt{2\Omega}\left[x-\frac{\lambda}{8\Omega^2}\right]\right),$$

and the separation constant is

$$\epsilon=2\Omega(2n_1+1)-\frac{\lambda^2}{16\Omega^2}.$$

For the second equation we get

$$\psi_2(y)\equiv\psi_{n_2}(y)=\sqrt{\frac{2}{\Omega}\,\frac{n_2!\Gamma(A+1)}{\Gamma(A)\Gamma(1+n_2+A)}}\,(\Omega y^2)^{A+1/2}e^{-(\Omega/2)y^2}L^A_{n_2}(\Omega y^2),$$

and

$$A=\frac{\lambda^2}{32\Omega^3}-\frac{\alpha}{2\Omega}-2-2N,$$

where $N=n_1+n_2$ is the principal quantum number. Here $H_n(x)$ is a Hermite polynomial and $L^A_m(z)$ a Laguerre polynomial.[19,25] The discrete spectrum occurs for

$$0\leqslant N\leqslant\frac{1}{4\Omega}\left(\frac{\lambda^2}{16\Omega^2}-\alpha\right).$$

The total bound state wave function $\Psi_1(x,y)\equiv\Psi_{n_1n_2}(x,y)$ is normalized by the condition

$$\int_{-\infty}^{\infty}dx\int_0^{\infty}\frac{dy}{y^2}\,\Psi_{n_1n_2}(x,y)\Psi_{n_1'n_2'}(x,y)=\delta_{n_1n_1'}\delta_{n_2n_2'}.$$

The symmetry operator L_1 with eigenvalue ϵ is

$$L_1\Psi_{n_1n_2}(x,y)\equiv\left[(K_2-M_1)^2-4\Omega^2\,\frac{\omega_3^2}{(\omega_1-\omega_2)^2}+\lambda\,\frac{\omega_3}{\omega_0-\omega_1}\right]\Psi_{n_1n_2}(x,y)$$
$$=\epsilon\Psi_{n_1n_2}(x,y)=\left[\frac{\lambda^2}{16\Omega^2}-2\Omega^2(2n_1+1)\right]\Psi_{n_1n_2}(x,y). \tag{60}$$

Let us consider the semicircular parabolic coordinates. For the potential V_3 we obtain

$$V_3(\xi_1,\xi_2)=-\xi_1\xi_2[8\alpha+16\lambda(\xi_1+\xi_2)+128\Omega^2(\xi_1^2+\xi^2+\xi_1\xi_2)].$$

The Schrödinger equation in semicircular parabolic coordinates has the form

$$\frac{2\xi_1\xi_2}{\xi_2-\xi_1}\left[\left(\xi_1\,\frac{\partial^2}{\partial\xi_1^2}+\frac{1}{2}\,\frac{\partial}{\partial\xi_1}\right)-\left(\xi_2\,\frac{\partial^2}{\partial\xi_2^2}+\frac{1}{2}\,\frac{\partial}{\partial\xi_2}\right)\right]\Psi-\xi_1\xi_2[8\alpha+16\lambda(\xi_2+\xi_1)$$
$$+128\Omega^2(\xi_1^2+\xi_2^2+\xi_2\xi_1)]\Psi=E\Psi. \tag{61}$$

Choosing the wave function as $\Psi_2=\psi_1(\xi_1)\psi_2(\xi_2)$ we obtain the two identical separation equations in the form

$$\xi\,\frac{d^2\psi}{d\xi^2}+\frac{1}{2}\,\frac{d\psi}{d\xi}+\left(\mu+\frac{E}{\xi}-4\alpha\xi-8\lambda\xi^2-64\Omega^2\xi^3\right)\psi=0,$$

where $\psi=\psi_{1,2}$, $\xi=\xi_{1,2}$ and μ is the separation constant. The symmetry operator with eigenvalue μ is

$$
\begin{aligned}
L_2\Psi_2 &\equiv \frac{1}{\xi_1-\xi_2}\left[\xi_2\left(\xi_2\frac{\partial^2}{\partial\xi_2^2}+\frac{1}{2}\frac{\partial}{\partial\xi_2}\right)-\xi_1\left(\xi_2\frac{\partial^2}{\partial\xi_2^2}+\frac{1}{2}\frac{\partial}{\partial\xi_2}\right)\right]\Psi_2\\
&\quad+[4\alpha(\xi_1+\xi_2)+8\lambda(\xi_1\xi_2+\xi_1^2+\xi_2^2)+64\Omega^2(\xi_1+\xi_2)(\xi_1^2-\xi_2^2)]\Psi_2\\
&\equiv\left[4\{K_3,K_2-M_1\}-2\alpha\frac{\omega_3}{\omega_1-\omega_2}-\lambda\frac{4\omega_2^3+1}{2(\omega_1-\omega_2)^2}-4\Omega^2\frac{\omega_3(2\omega_3^2+1)}{(\omega_1-\omega_2)^3}\right]\Psi_2=\mu\Psi_2.
\end{aligned}
\tag{62}
$$

The bound state solutions to these equations are of the form

$$
\Psi_2(\xi_1,\xi_2)=\exp\left[-4\Omega(\xi_2^2+\xi_1^2)-\frac{\lambda}{2\Omega}(\xi_1+\xi_2)\right](\xi_2\xi_1)^Q\Pi_{j=1}^N(\xi_1-b_j)(\xi_2-b_j), \tag{63}
$$

where $Q=-\alpha/4\Omega+\lambda^2/64\Omega^3-N-\frac{3}{4}$. If we make this substitution, the zeros b_j satisfy

$$
\sum_{j\neq l}\frac{b_l}{b_l-b_j}-2-4N-\frac{\alpha}{\Omega}+\frac{\lambda^2}{16\Omega^3}-\frac{2\lambda}{\Omega}b_l-32\Omega b_l^2=0, \tag{64}
$$

and the eigenvalues μ have the form

$$
\mu=-32\Omega\sum_{l=1}^N b_l+\frac{\lambda}{2\Omega}-\frac{\alpha\lambda}{\Omega^2}-\frac{\lambda^3}{32\Omega^4}.
$$

The energy is quantized according to the condition $E=-Q(2Q-1)$.

3. Quadratic algebra relations

$$
\begin{aligned}
&[L_{12},L_1]=-8\lambda L_1-16\Omega^2L_2+8\alpha\lambda,\\
&[L_{12},L_2]=96L_1^2+128\alpha L_1+8L_2-256\Omega^2H-96\Omega^2+32\alpha^2,\\
&L_{12}^2=64L_1^3+512\Omega^2HL_1+8\lambda\{L_1,L_2\}+128\alpha L_1^2+16\Omega^2L_2^2\\
&\qquad+16\alpha\lambda L_2+704\Omega^2L_1+64\alpha^2L_1-32\lambda^2H-12L^2+512\alpha\Omega^2,
\end{aligned}
\tag{65}
$$

where $L_{12}=[L_1,L_2]$.

The quadratic algebra contains useful information in its relations. Indeed, if we consider the orthonormal bound state wave functions φ_m and ψ_n that satisfy $L_1\varphi_m=\lambda_m\varphi_m$ and $L_2\psi_p=\rho_p\psi_p$, then the matrix with elements D_{mn} where $L_2\varphi_m=\Sigma_{n=1}^N D_{mn}\varphi_n$ can readily be determined as follows. The quadratic algebra relations imply that

$$
(\lambda_m-\lambda_p)^2D_{mp}=-16\Omega^2D_{mp}+8\lambda(\alpha-\lambda_m)\delta_{mp}, \tag{66}
$$

$$
\sum_{s=1}^N D_{sp}D_{ms}(\lambda_p+\lambda_m-2\lambda_s)=8D_{mp}+(96\lambda_m^2+128\alpha\lambda_m-256\Omega^2E+96\Omega^2+32\alpha^2)\delta_{mp}, \tag{67}
$$

where $\lambda_n=2\Omega(2n+1)-(\lambda^2/16\Omega^2)$. The first of these conditions implies that the only nonzero components of D_{pq} are $D_{m,m+1}$, $D_{m+1,m}$, and $D_{mm}=8\lambda(\alpha-\lambda_m)$. The second relation implies

$$
\Omega[D_{p-1,p}^2-D_{p+1,p}^2]=12\lambda_p^2+16\alpha\lambda_p-32\Omega^2E+12\Omega^2+4\alpha^2+8\lambda\alpha-8\lambda\lambda_p.
$$

There is no further information available about the matrix elements D_{pq}. Indeed, the matrix elements are essentially determined by these relations. We could also consider the matrix C with elements defined by

$$L_1\psi_p=\sum_{k=1}^{N} C_{pk}\psi_k .$$

The quadratic algebra implies that C satisfies certain relations, such as

$$C_{np}(\rho_p-\rho_n)^2=96\sum_{m=1}^{N} C_{mp}C_{nm}+128\alpha C_{np}+(\rho_n-256\Omega^2E+96\Omega^2+32\alpha^2)\,\delta_{np} .$$

One interesting condition we deduce from this is that

$$96\ \mathrm{Tr}(C^2)+128\alpha\ \mathrm{Tr}(C)+\sum_{n=1}^{N}\rho_n+N(32\alpha^2-256\Omega^2E+96\Omega^2)=0,$$

where Tr is the trace function. We can also introduce the matrices that relate the various bases that we are considering via $\varphi_m=\Sigma_{s=1}^{N}A_{ms}\psi_s$ and $\psi_m=\Sigma_{s=1}^{N}B_{ms}\varphi_s$. We note, in particular, that

$$\sum_{p=1}^{N} D_{mp}A_{ps}=\rho_sA_{ms}, \tag{68}$$

i.e., the vector $v^{(s)}=(A_{1s},\ldots,A_{Ns})$ is an eigenvector of D with eigenvalues ρ_s. In fact, the eigenvalues of D are solutions of the characteristic equation written in continued fraction form,

$$\frac{\beta_N}{\alpha_N}\,\gamma_{N-1}=\beta_{N-1}-\alpha_{N-1}\gamma_{N\text{-}2}]\beta_{N-2}-\alpha_{N-2}\gamma_{N-3}]\cdots]\beta_2-\alpha_2\gamma_1/\beta_1, \tag{69}$$

where $\alpha_n=D_{nn-1}$, $\beta_n=D_{nn}-\rho$, and $\gamma_n=D_{nn+1}$.

III. A DIRECT INTERBASIS EXPANSION

As an example of a direct interbasis expansion we consider the singular oscillator (4). For a fixed value of E_N, we can write the equidistant wave function (17) in terms of the spherical wave function (7) as

$$\Psi_{n_1n_2}=\sum_{m=0}^{n_1+n_2} W_{n_1n_2}^{nm}(k_1,\pm k_2,\pm k_3)\Psi_{nm}, \tag{70}$$

where $n_1+n_2=n+m$. The connection between the equidistant (a,b) and spherical (τ,φ) coordinates is

$$\sinh a=\sinh\tau,\quad \sinh b=\frac{\tanh\tau\cos\varphi}{\sqrt{1-\tanh^2\tau\cos^2\varphi}}. \tag{71}$$

Using spherical coordinates on the left side of expansion (70), then, considering the limit $\tau\to\infty$ in the so-obtained equation and the formula

$$\lim_{x\to1}{}_2F_1(-n,\alpha;\beta;x)\to\frac{\Gamma(\beta)\Gamma(\beta-\alpha+n)}{\Gamma(\beta-\alpha)\Gamma(\beta+n)}, \tag{72}$$

we see that dependence on τ cancels on both sides of (70). Now using the orthogonality condition of the angular wave function (10), we find the following expression for the interbasis coefficients $W_{n_1 n_2}^{nm}$:

$$W_{n_1n_2}^{nm}=(-1)^m\sqrt{\frac{\Gamma(k_1-n)\Gamma(1+m\pm k_2\pm k_3)m!n_1!}{\Gamma(1+m\pm k_2)\Gamma(1+m\pm k_3)n_2!n!}}\,A_{n_1n_2}^{nm}B_{n_1n_2}^{nm}, \tag{73}$$

where

$$\begin{aligned}A_{n_1n_2}^{nm}&=\sqrt{(k_1\mp k_2-1-2n_1)(2m\pm k_2\pm k_3+1)\Gamma(k_1\mp k_2-1-n_2-2n_1)}\\&\quad\times\sqrt{\frac{\Gamma(k_1-n_1)\Gamma(k_1\mp k_2\mp k_3-1-n_2-2n_1)\Gamma(k_1\mp k_2\mp k_3-1-n-2m)}{\Gamma(k_1\pm k_3+n_2)\Gamma(k_1\mp k_2-n_1)\Gamma(k_1\pm k_2+n_1)\Gamma(2+n+2m\pm k_2\pm k_3)}}\end{aligned}$$

and

$$B_{n_1n_2}^{nm}=\frac{1}{2}\int_0^{\pi/2}(\sin\varphi)^{1+2n_2\pm k_3}(\cos\varphi)^{1\pm k_2}P_{n_1}^{(k_1\pm k_2-1-2n_1,\pm k_2)}(\cos 2\varphi)P_m^{(\pm k_3,\pm k_2)}(\cos 2\varphi)\,d\varphi. \tag{74}$$

The integral B_{n_1,n_2}^{nm} can be evaluated by means of [25]

$$\begin{aligned}&\int_{-1}^{1}(1-x)^\tau(1+x)^\beta P_n^{(\alpha,\beta)}(x)P_m^{(\rho,\beta)}(x)\,dx\\&\quad=\frac{2^{\beta+\tau+1}\Gamma(\alpha-\tau+n)\Gamma(\beta+1+n)}{m!n!\Gamma(1+\rho)\Gamma(\alpha-\tau)}\,\frac{\Gamma(1+m+\rho)\Gamma(1+\tau)}{\Gamma(2+n+\beta+\tau)}\\&\qquad\times{}_4F_3\left(\left.\begin{matrix}-m,1+m+\beta+\rho,1+\tau,1+\tau-\alpha\\1+\rho,2+n+\beta+\tau,1-n+\tau-\alpha\end{matrix}\right|1\right).\end{aligned}$$

We thus obtain

$$\begin{aligned}W_{n_1n_2}^{nm}&=\frac{(-1)^m\Gamma(k_1\mp k_2\mp k_3-1-n-m)}{\Gamma(2+n+m\pm k_2\pm k_3)\Gamma(1\pm k_3)}\sqrt{\frac{(k_1\mp k_2-1-2n_1)(2m\pm k_2\pm k_3+1)}{\Gamma(k_1\mp k_2\mp k_3-1-n_2-2n_1)}}\\&\quad\times\sqrt{\Gamma(1+n_1\pm k_2)\Gamma(k_1-n_1)\Gamma(1+n_2\pm k_3)\Gamma(k_1\mp k_2\mp k_3-1-n-2m)}\\&\quad\times\sqrt{\frac{\Gamma(k_1\mp k_2-1-n_2-2n_1)\Gamma(1+m\pm k_2\pm k_3)\Gamma(1+m\pm k_3)\Gamma(k_1-n)}{\Gamma(k_1\mp k_2-n_1)\Gamma(2+n_1+n_2+m\pm k_2\pm k_3)\Gamma(1+m\pm k_2)n_1!n_2!n!m!}}\\&\quad\times{}_4F_3\left(\left.\begin{matrix}-m,1+m\pm k_2\pm k_3,1+n_2\pm k_3,2+n_2+2n_1\pm k_2\pm k_3-k_1\\1\pm k_3,2+n_1+n_2\pm k_2\pm k_3,2+n_1+n_2\pm k_2\pm k_3-k_1\end{matrix}\right|1\right),\end{aligned}\tag{75}$$

a closed form expression for the interbasis expansion coefficients. Note that the hypergeometric function ${}_4F_3$ is of the Saalschutzian type.[19,26]

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