http://researchcommons.waikato.ac.nz/

Research Commons at the University of Waikato

Copyright Statement:

The digital copy of this thesis is protected by the Copyright Act 1994 (New Zealand).

The thesis may be consulted by you, provided you comply with the provisions of the Act and the following conditions of use:

- Any use you make of these documents or images must be for research or private study purposes only, and you may not make them available to any other person.
- Authors control the copyright of their thesis. You will recognise the author's right to be identified as the author of the thesis, and due acknowledgement will be made to the author where appropriate.
- You will obtain the author's permission before publishing any material from the thesis.

NUMBER SENSE
IN A SECONDARY SCHOOL CONTEXT

A thesis
submitted in fulfilment
of the requirements for the Degree
of
Doctor of Philosophy
at the
University of Waikato

by

JANE MCCHESNEY

University of Waikato
2004

Abstract

Current mathematics education research in New Zealand is directed at primary school or senior secondary school, in the areas of numeracy, technology innovations, student achievement and teacher professional development. This study investigated junior secondary mathematics lessons as contexts for distributed mathematical activity. Two Year 9 classes and one Year 10 class from the same school constituted case studies; two case studies were lower band and one was middle band. All three teachers continued with their normal mathematics programmes. Using an interpretivist methodology, participant observation was the main method used to collect data of sequential mathematics lessons over five weeks. The data was organised around three domains of classroom activity: classroom social norms, mathematical tools, and interaction. The data analysis focussed on aspects of these domains that afforded and constrained the negotiation of mathematical meaning.

In each classroom context, social norms associated with lesson events were found to afford student participation. Student participation was afforded by interactional spaces; that is, regular opportunities for students to volunteer mathematical contributions and to adopt different roles in joint activity. Interactional spaces provided students with greater access to more diverse mathematical resources. Students activated interactional spaces in different ways with questions, suggestions, explanations, errors and, in one classroom, a discourse of primary school mathematics.

Using a sociocultural perspective, school mathematical artefacts were conceptualised as cognitive mathematical tools. Cognitive tools were identified as representations, thinking devices and analogous models. Tools that were both representations and thinking devices afforded further resources for students, such as mathematical information, cues and other signifiers. The only technological tool in the case study classrooms was the scientific calculator. The roles of the calculator were as a recording device, calculating machine and exploratory object. In these roles, the calculator afforded further resources for students.

The findings in the interaction domain challenge the assumption that traditional classroom mathematics programmes have limited discursive opportunities. Students initiated more focussed negotiations in both whole class and localised situations. In each case study, students and the teacher actively contributed mathematical resources. For these lower and middle ability students, access to mathematical resources was a significant affordance for further mathematical activity. In each classroom context, activity was found to be distributed over social norms, tools and interaction. Number sense activity was distributed according to the adapted 'environment' metaphor (Greeno, 1991). Students negotiated mathematical meanings by activating and using cognitive tools and discursive resources in the number sense 'environment'.

Acknowledgements

Many people have given generously of their time over the duration of this thesis. As Chief Supervisor, Alister Jones provided ongoing direction and guidance. Bronwen Cowie generously contributed as a supervisor, helping me to organise a large amount of data and to find structure for the data analysis. She also shared her insights about mathematics lessons in New Zealand classrooms. As supervisors, Malcolm Carr and Ken Carr guided the early stages of the study, and helped me to clarify my thinking about number sense.

In Hamilton, Margaret Carr has been a constant source of encouragement. She shared her knowledge of sociocultural views of tools and cognition, and helped me to inquire further about distributed activity in classrooms. Members of the Science and Technology Education Research Centre at the University of Waikato have consistently provided a stimulating environment for inquiry and discussion. In the last few years, they have also made me very welcome and provided practical assistance with space and resources. Raewyn Oulton has been particularly accommodating of this distance student who would arrive for a week and then disappear again. Discussions with Gail Cawkwell and Liz McKinley were always valuable and encouraging. I am very appreciative of the generous hospitality of Margaret and Malcolm Carr, Gail and Peter Cawkwell, and Nancy Taylor during my many trips to Hamilton.

At the Christchurch College of Education, Ross Tasker unceasingly supported this study and enthusiastically shared his interest in classroom research. Gillian Tasker extended my reading base and helped me to clarify my thinking about classroom interactive processes. Ross and Gillian's generous hospitality has also contributed to the progress of this study. I am very grateful for the support of Brian Law, my colleague in secondary mathematics education. Brian willingly shared his thoughtful observations of secondary school mathematics and his experiences of teaching diverse students. He has supported this study in numerous ways, not least by clarifying the sporting metaphor. In the last year, Jeff Pepper has also accepted my distracted attention to administrative matters. Jane Abbiss and Myra Kunowski were friends as well as colleagues who discussed ideas and encouraged me to extend my thinking.

I am grateful to the teachers and students in the case study classrooms who shared their mathematical activity with me. Caitriona Cameron generously provided feedback on early drafts as well as meticulous proof reading. Tania Sibley contributed diagrams and formatting. Financial assistance from the Christchurch College of Education Research Fund is gratefully acknowledged.

I warmly thank my family for their encouragement and patience: Shirley, Ian, Ann, Ruth, Joel and Sam McChesney, Caitriona Cameron and Tony Cranshaw. Ann McChesney also provided immediate professional and practical assistance for the OOS. This study could not have happened without their support.

Chapter Outline

1. Introduction

1.1 Introduction to Study 1

1.1.1 School Mathematics 1

1.1.2 Theoretical Framework for the Study 4

1.1.3 Focus of the Study 5

1.2 Towards the Research Focus 7

1.2.1 Site for Research Focus 7

1.2.2 Background of Researcher 8

1.2.3 Outline of Thesis 10

2. Literature Review

2.1 Introduction 11

2.2 Views of Learning 12

2.2.1 Learning as Individual Activity 12

2.2.2 Learning as Situated and Social 13

2.2.3 Learning as Distributed 16

2.3 Recognising the Complexities 18

2.3.1 Metaphors for Learning 19

2.3.2 The Microculture of a Mathematics Classroom 20

2.3.3 A Framework for Distributed Mathematical Activity 24

2.4 Classroom Social Norms 26

2.4.1 Classroom Social Norms 26

2.4.2 Classroom Social Norms for Emergent Classrooms 27

2.4.3 Classroom Social Norms in Traditional Classrooms 30

2.4.4 Classroom Social Norms in New Zealand Classrooms 31

2.4.5 Classroom Social Norms and Student Participation 35

2.4.6 Summary of Classroom Social Norms 37

2.5 Tools 39

2.5.1 Introduction to Tools 39

2.5.2 Tools and School Mathematics 41

2.5.3 Tools in Classroom Mathematical Activity 45

2.5.4 Special Case of Technological Tools 49

2.5.5 Summary of Tools 52
2.6 Interaction 53
2.6.1 Whole Class Interaction in Traditional Classrooms 53
2.6.2 Alternative Interpretations of Teacher Contributions 55
2.6.3 Student Contributions 59
2.6.4 Summary of Interaction 62
2.7 Number Sense 63
2.7.1 Introduction to Number Sense 63
2.7.2 Number Sense in New Zealand 65
2.7.3 A Metaphor for Number Sense 67
2.7.4 The Metaphor Adapted: Number Sense and Tools 68
2.7.5 The Metaphor as Distributed Mathematical Activity 69
2.7.6 Diversity of Number Sense for Learners 71
2.7.7 Classroom Mathematics Lessons 73
2.8 Classroom Negotiation 76
2.8.1 Negotiation of Meaning 76
2.8.2 Negotiation is Distributed 78
2.8.3 Research Questions 79
2.9 Chapter Summary 81

3. Methodology and Methods

3.1 Introduction 82
3.2 Methodology 82
3.2.1 Naturalistic and Interpretive Inquiry 82
3.2.2 Case Study Methods 84
3.2.3 Links with Ethnography 85
3.2.4 Methodology of this Study 85
3.2.5 Trustworthiness of Interpretive Inquiry 86
3.3 Research Design 87
3.3.1 Data Collection Methods 87
3.3.2 Tools for Data Collection 88
3.3.3 Methods used in this Study 89
3.3.4 Research Sites 90
3.4 Ethical Considerations 92

3.4.1 Ethical Consent 92
3.4.2 Informed Consent for this Study 93
3.4.3 Confidentiality 94
3.5 Data Collection 95
3.5.1 Timeline of Data Collection 95
3.5.2 The Data Collected 96
3.5.3 Classroom Role of the Researcher 98
3.5.4 Researching Classroom Lessons 100
3.6 Analysis 101
3.6.1 The Data 101
3.6.2 Transcript Symbols 102
3.6.3 The Lesson Episodes 104
3.6.4 Analysis of Transcript Data 105
3.6.5 Analysis Framework 106
3.6.6 Member Checking 109
3.7 Analysis Categories 110
3.7.1 Background Information 111
3.7.2 The Mathematics Lessons Observed 112
3.7.3 Classroom Curriculum: Analysis of Social Norms Domain 112
3.7.4 Tools: Analysis of Tools Domain 115
3.7.5 Classroom Interaction: Analysis of Interaction Domain 117
3.7.6 Further Analysis 120
3.8 Chapter Summary 120

4. Case Studies

4.1 Introduction 121
4.2 Case Study Lemonwood 1 122
4.2.1 Background Information 122
4.2.2 The Mathematics Lessons Observed 124
4.2.3 Classroom Curriculum 125
4.2.4 Tools 130
4.2.5 Classroom Interaction 137
4.2.6 Lemonwood 1 Summary 168
4.3 Case Study Lemonwood 2 169

4.3.1 Background Information 169
4.3.2 The Mathematics Lessons Observed 171
4.3.3 Classroom Curriculum 172
4.3.4 Tools 177
4.3.5 Classroom Interaction 182
4.3.6 Lemonwood 2 Summary 201
4.4 Case Study Lemonwood 3 202
4.4.1 Background Information 202
4.4.2 The Mathematics Lessons Observed 205
4.4.3 Classroom Curriculum 205
4.4.4 Tools 212
4.4.5 Classroom Interaction 218
4.4.6 Lemonwood 3 Summary 229
4.5 Chapter Summary 230
4.5.1 Overview of Case Study Contexts 230
4.5.2 Case Studies Summary 232

5. Results

5.1 Chapter Introduction 233
5.2 Classroom Social Norms 233
5.2.1 Rituals 234
5.2.2 Roles 237
5.2.3 Overview of Classroom Social Norms 241
5.3 Tools 241
5.3.1 Classroom Mathematical Tools 242
5.3.2 Other Mathematical Tools 247
5.3.3 Technological Tools 248
5.3.4 Overview of Tools 252
5.4 Classroom Interaction 253
5.4.1 Teacher Guided Participation 253
5.4.2 Whole Class Negotiation 255
5.4.3 Student Initiated Negotiation 258
5.4.4 Error Initiated Negotiation 260
5.4.5 Repair Interactions 264

5.4.6 Technology Mediated Negotiation 264
5.4.7 Overview of Classroom Interaction 266
5.5 Chapter Summary 267

6. Discussion and Summary

6.1 Chapter Introduction 268
6.2 The Social Norms Domain 268
6.2.1 Ritualised Participation in Discursive Mathematical Activity 269
6.2.2 Student Contributions were Accorded High Status 270
6.2.3 Synthesis of the Social Norms Domain 272
6.3 The Tools Domain 273
6.3.1 Tools in Classroom Activity 273
6.3.2 Technological Tools 276
6.3.3 Synthesis of the Tools Domain 277
6.4 The Interaction Domain 278
6.4.1 Teacher Contributions 278
6.4.2 Student Contributions 280
6.4.3 Synthesis of the Interaction Domain 282
6.5 Negotiation of Mathematical Meaning 283
6.5.1 Negotiation as Distributed 283
6.5.2 Examples of Distributed Negotiation 285
6.5.3 Number Sense Negotiations 288
6.5.4 Distributed Activity and Mathematical Authority 291
6.6 Limitations of the Methodology 293
6.6.1 Limitations in Research Design 293
6.6.2 Limitations in Analysis 294
6.7 Implications 295
6.7.1 Implications: Social Practices in Mathematics Classrooms 295
6.7.2 Implications: Mathematical Tools and Number Sense 297
6.7.3 Further Implications 299
6.7.4 Issues for Further Research 302
6.8 Summary 304

Appendices 305
Appendix 1 Interview Questions for Teachers 305
Appendix 2 Record of Lemonwood 1 Episodes 306
Appendix 3 Record of Lemonwood 2 Episodes 308
Appendix 4 Record of Lemonwood 3 Episodes 310

References 311

List of Tables and Figures

List of Tables

Table 3.1 Number of Lessons Observed 96
Table 3.2 A Framework for Analysing Transcript Data 107
Table 3.3 Analysis Dimensions and Categories 110
Table 3.4 Roles of the Calculator 116
Table 4.1 Record of Lemonwood 1 Observed Lessons and Episodes 124
Table 4.2 Classroom Mathematical Tools for Lemonwood 1 131
Table 4.3 Categories of Calculator Use for Lemonwood 1 134
Table 4.4 Other Mathematical Tools Identified in Lemonwood 1 137
Table 4.5 Record of Lemonwood 2 Observed Lessons and Episodes 171
Table 4.6 Classroom Mathematical Tools for Lemonwood 2 177
Table 4.7 Categories of Calculator Use for Lemonwood 2 180
Table 4.8 Other Mathematical Tools Identified in Lemonwood 2 182
Table 4.9 Record of Lemonwood 3 Observed Lessons and Episodes 205
Table 4.10 Classroom Mathematical Tools for Lemonwood 3 212
Table 4.11 Categories of Calculator Use for Lemonwood 3 217
Table 4.12 Other Mathematical Tools Identified in Lemonwood 3 218
Table 4.13 Class Size of Case Study Classrooms 231
Table 5.1 Participation Rituals for Mathematical Activity 234
Table 5.2 Teacher Mathematical Values about Learning Mathematics 236
Table 5.3 Contributor Roles of Students to Whole Class Interactions 238
Table 5.4 Classroom Mathematical Tools that were Representations 242
Table 5.5 Classroom Mathematical Tools that were Thinking Devices 244
Table 5.6 Classroom Mathematical Tools that were Analogous Models 245
Table 5.7 Other Mathematical Tools Identified in the Case Study Classrooms 247
Table 5.8 Roles of the Calculator across all Case Study Classrooms 249
Table 5.9 Affordances of Scientific Calculator and Examples of Mediated Activity 251
Table 5.10 Classroom Mathematical Practices Analysed in Chapter 4 255
Table 5.11 Examples of Student Initiated Negotiation 258

Table 5.12 Resultant Shifts in the Discourse from Student Initiated Negotiations 259
Table 5.13 Errors Analysed and Discussed in Case Studies 261
Table 5.14 Technology Mediated Negotiations 265
Table 6.1 Significant Affordances of Classroom Mathematical Tools 273
Table 6.2 Negotiation of the Whoops Rule (Lemonwood 1) 285
Table 6.3 Negotiation of Natalie's Error (Lemonwood 1) 287

List of Figures

Figure 2.1 An Interpretative Framework for Analysing the Classroom Microculture 21
Figure 2.2 Relationship between Domains of Classroom Social Norms, Tools and Interaction. 26
Figure 2.3 The Influence of Participation Social Norms on the Domains of Tools and Interaction. 38
Figure 2.4 The Influence of Tools on the Domains of Social Norms and Interaction 52
Figure 2.5 The Influence of Interaction on the Domains of Social Norms and Tools. 62
Figure 4.1 Signs for Greater and Less Than 139
Figure 4.2 Demonstration of the Number Line Rule 141
Figure 4.3 Introduction of the Whoops Rule 142
Figure 4.4 Teacher Use of Whoops as a Cue 143
Figure 4.5 Student Interpretations of Crash Analogy 144
Figure 4.6 Peer Discussion of Shortcut Rule 148
Figure 4.7 Further Discussion of Shortcut Rule 150
Figure 4.8 Scott's Explanation of his Wicked Number Pairs 151
Figure 4.9 Dean's Rounding Error 154
Figure 4.10 Mathew Initiating a New Example 155
Figure 4.11 Teacher Demonstration of Rounding of Nines 155
Figure 4.12 Dean and the Rounding of Nines 156
Figure 4.13 Negotiation Initiated by Natalie's Error 158
Figure 4.14 Negotiation about Luke's Calculator Result 164

Figure 4.15 Roles during Joint Student Activity 167
Figure 4.16 Kirsty's Identification of Decimal Places 185
Figure 4.17 Melanie's Question about Integers 189
Figure 4.18 Monique and Decimal Magnitude 192
Figure 4.19 Using Fractions to Repair Decimal Multiplication 195
Figure 4.20 Amy, Erin and Louise and Multiplication of Decimals 196
Figure 4.21 Shane's Reasoning and Finn's Explanation 221
Figure 4.22 Tina's, Danny's and Nathan's Methods 223
Figure 4.23 Donna's Conjecture 225
Figure 4.24 Resolution of Donna's Conjecture 226

Chapter 1
Introduction

1.1 Introduction to Study

1.1.1 School Mathematics

Mathematics lessons in many Western societies have been portrayed as dominated by teacher exposition and individual textbook tasks. Students only contribute when called on to answer teacher questions and are passive recipients of school mathematics (Bauersfeld, 1988). In most of the New Zealand Year 9 and 10 classrooms I have visited, however, students were talking together and the classroom atmosphere was relaxed. Mathematics teachers from other countries have often commented to me about how noisy they find New Zealand classrooms, how much the students talk to each other during the lessons, and how students can appear to be 'cheeky' to the teacher. New Zealand mathematics classrooms appear paradoxical, where lessons typically involve solitary worksheets or textbook tasks (Clark, 1997), yet social relationships are valued and student interaction is common.

Evidence of New Zealand student achievement in mathematics is similarly contradictory, at both the primary and secondary school levels. Findings from international studies, such as TIMSS (Garden, 1996, 1997), have raised concern about the mathematical performance of New Zealand students in the 11 to 14 age group. In contrast, the results of the PISA study of mathematical literacy for fifteen year old students indicated New Zealand students were third highest of the countries involved (Ministry of Education, 2001). Within New Zealand, the local National Educational Monitoring Project has also found variation of performance in a range of tasks based on the mathematics curriculum (Crooks and Flockton, 2002; Flockton and Crooks, 1998). In both studies, the variation in performance was found to be statistically significant for particular groups of Year 8 students (Crooks and Flockton, 2002; Flockton and Crooks, 1998). Differences in school mathematics performance have also been identified in the secondary school

(Clark, 1999), and have been evident in national examination results (Blithe, Clark and Forbes, 1993).

Research evidence about the low performance and lack of qualifications of specific groups of students appears to be persistent. The research has identified the under-performing groups as students from lower socioeconomic schools (Crooks and Flockton, 2002; Flockton and Crooks, 1998), and Mäori and Pacific Island students (Clark, 1999; Crooks and Flockton, 2002; Flockton and Crooks, 1998; Forbes, 1995). The New Zealand mathematics curriculum acknowledges groups of under-performing students in two ways. School curriculum and teaching approaches are firstly identified as important influences on student performance.

> In many cases in the past, students have failed to reach their potential because they have not seen the applicability of mathematics to their lives and because they were not encouraged to connect new mathematical concepts and skills to experiences, knowledge, and skills which they already had. (Ministry of Education, 1992, p. 12)

Secondly, the curriculum states that 'lower ability' students "need to have the opportunity to experience a range of mathematics which is appropriate to their age level, interests and capabilities" (Ministry of Education, 1992, p. 12).

This study seeks to investigate classroom experiences for students described as lower ability in mathematics. While this group of students has not been the focus of research in New Zealand, there is some information about general student perceptions of school mathematics. In mathematics survey data from the National Education Monitoring Project, when asked to nominate "up to three things they have trouble with", 45 percent of the Year 8 sample nominated 'Number', closely followed by 41 percent nominating 'Basic facts and tables' (Flockton and Crooks, 1998, p. 62). In the most recent results, students were asked "how much do you like doing maths at school?" and "how good do you think you are at maths?". For liking mathematics at school, 34 percent chose the third or the fourth unhappy face on the four-point scale. 20 percent chose the third or the fourth unhappy face for thinking they were good at maths (Crooks and Flockton, 2002, p. 62). These Year 8 students indicated they had trouble with number topics, and some believed that they were not good at mathematics. When Walls

(2001) tracked ten children through three years of New Zealand primary school mathematics, she found that seven children "talked about loss of confidence and were at varying times quite negative about aspects of mathematics" (Walls, 2001, p. 500). These children still had three years of primary school and at least the same of secondary school mathematics ahead of them. During that time, what will happen to these children, their performance in mathematics and their attitudes to the subject and to themselves as learners?

In New Zealand, there have been studies of individual student experiences of school mathematics (Anthony, 1994, 1996a, 1996b; Walls, 2001) and of individual student mathematical performance or understanding (Irwin, 1999; Young-Loveridge, 2000). Classroom based studies in mathematics have either been at the primary school level (Higgins, 1995; Thomas, 1994) or of curriculum innovations (Anthony and Knight, 1999; Burns-Wilson and Thomas, 1997). Classroom innovations have been studied, such as incorporating technology (Alangui et al, 2002) and adopting a problem solving teaching approach (Holton, Anderson and Thomas, 1997). Data has been collected about student activity, but it has been in terms of their individual mathematical thinking (Anthony, 1998) and not in terms of their activity during mathematics lessons in schools. There has been comparatively little research of the thirteen to fourteen year age group or about mathematical learning over time. Consequently, there is a lack of information about students' classroom experiences in the first two years of secondary school, the years before the national qualifications begin in Year 11. While cohort data about mathematical performance has indicated concern about specific groups of students at primary and secondary school levels, there is a gap in the research about mathematics lessons in typical New Zealand classrooms. In contrast to a focus on individual learning of mathematics, a focus on shared mathematical activity may provide more insight into the nature of classroom mathematical activity. The focus of this research study is on mathematics lessons in typical New Zealand classrooms.

1.1.2 Theoretical Framework for the Study

A distributed perspective connects together significant aspects of classroom mathematical activity. This perspective views mathematical activity as distributed over different domains of classroom lessons. Examples of these different domains are the domain of social practices, such as classroom social norms; the domain of mathematical tools, such as cognitive and technological tools; and the domain of student and teacher interactions. These three domains contribute to classroom mathematical activity. The notion of 'person-plus' considers mathematical activity as distributed or 'stretched-across' student and teacher talk, classroom tasks, and other resources such as materials and cognitive tools (Perkins, 1993; Salomon and Perkins, 1998). This study will focus on these three domains: classroom social norms, mathematical tools and interactions.

Greeno (1997) and Cobb and Bowers (1999), among others, have called for research that encompasses social as well as individual perspectives. Social practices become normative over time and influence classroom mathematical activity. Classroom social norms shape opportunities for, and the quality of, student participation in mathematical activity. This study is specifically interested in classroom social norms that provide opportunities for student participation in mathematical activity. After eight years of primary school mathematics, participation in mathematical activity is influenced by student beliefs about their roles and the teacher's roles. These beliefs, as well as the teacher's beliefs and values, shape when and how students contribute to mathematical activity, and what forms these contributions might take. Classroom social norms influence shared mathematical activity and I propose to examine classroom social norms in typical Year 9 and 10 mathematics classrooms.

Related to the social domain are conversations between classroom participants. How might mathematical conversations be related to the participation of lower ability students? When Greeno stated that "meanings are negotiated in the context of conversations" (Greeno, 1991, p. 210), he highlighted the importance of mathematical conversations in negotiating mathematical meanings. In classroom lessons, mathematical conversations occur among the whole class, or among

groups of students, with or without teacher contributions. While there has been research about questioning patterns (Mehan, 1979), the control of student contributions (Edwards and Mercer, 1987), and talk during small group activity (Barnes, 1999), whole class interactions have received comparatively little attention (Chazan and Ball, 1999). For lower ability students in mathematics, these shared interactions may be particularly significant. This study will focus on lesson conversations among classroom participants during mathematical activity.

Mathematical conversations include communication about mathematical artefacts. By drawing on a sociocultural perspective of cultural tools (Vygotsky, 1978), mathematical artefacts can be reconceptualised as cognitive tools. The use of cognitive tools afford or constrain mathematical activity (Pea, 1993). This study will use affordances and constraints of mathematical tools to examine the roles of tools in classroom activity. Classroom activity involves 'tools in action', including classroom conversations about the tools in the context of activity. Learning mathematics

> is an interpersonal and social enterprise where the learner recreates mathematical structures, where shared meanings are negotiated and refined, and where mathematical structures maintain their context-dependent meanings. (Neyland, 1998, p. 72)

In this study I propose that the three domains of school mathematics activity - social norms, tools and interaction - are significant for shaping the shared negotiation of mathematical meanings in classroom contexts. Social norms shape who contributes to negotiations, the mathematical tools shape the activity, and interactions shape the contributions of tools and other discursive resources. Negotiation of mathematical meaning will be considered as distributed over these three domains.

1.1.3 Focus of the Study

To provide further focus for this study, I will concentrate on one aspect of school mathematics, number sense. Over the past five years, the emphasis on number sense in the existing mathematics curriculum document (Ministry of Education, 1992) has been replaced by a focus on numeracy and the introduction of the

Numeracy Framework and the Numeracy Projects (Ministry of Education, 2002). These projects have not been confined to the primary school, as the Numeracy Exploratory Studies have reached into Years 7 to 10, incorporating the first two years of secondary school. The initiatives have involved considerable money, teacher and parent time. While the implementation of each project has been evaluated (Higgins, 2002; Irwin and Niederer, 2002; Thomas and Ward, 2001), a recent report noted the need for more information about what happens in mathematics lessons such as the effectiveness of various teaching approaches and class organisation (Education Review Office, 2002, p. 37 and 39). This study has an interest in number related issues but will focus on number sense rather than numeracy. This number sense focus will use the metaphor developed by Greeno (1991) to theorise about number sense. His metaphor of an environment described number sense activity as knowing and finding your way around the domain of number. I will discuss, adapt and use this metaphor of number sense in order to focus on distributed mathematical activity in classroom lessons.

Classroom mathematical activity is influenced by school context, curricular focus and the social and material resources of each classroom (Greeno, 1997). A long-term intensive study in England highlighted qualitatively different student learning in two different secondary school contexts (Boaler, 1997a, 1999). The differences in the mathematics learned and valued were linked to the different school curricular approaches as well as setting (subject streaming), gender, and socio economic status (Boaler, 1997a, 1997b, 1997c, 1999). The particular context of the schooling situation was found to be significant (Boaler, 1999). Within the same school, classroom contexts were found to be different and influenced by teachers' beliefs about teaching and learning (Boaler, 1997a). Classroom contexts were found to afford and constrain mathematical activity. This study will investigate student mathematical activity in Year 9 and 10 New Zealand classroom contexts. It is in these first two years of secondary school that the primary school learning of number is revisited and extended.

While studies of experimental classrooms are important, we need to also find out about non-intervention settings. These are the classrooms where teachers, experts in their school situation (Ball, 1997; Ball and Bass, 2000; Putnam and Borko,

2000), experience the dilemmas of teaching diverse students, implementing the curriculum, and other pressures on their daily practice. Research that addressed the dilemmas for teachers in different classroom contexts has been useful for investigating the knowledge-in-action of the teacher (Shulman, 1986). Teachers were recognised as experts in their own classroom contexts with expert knowledge of students, curriculum, and resources (Ball, 1993; Chazan and Ball, 1999; Lampert, 1985, 1990). I propose to take a similar stance and in this study I have adopted an interpretivist approach to the day-to-day complexities of busy classrooms. Taking these aspects together, this research study will focus on number sense in typical New Zealand Year 9 and 10 classrooms.

1.2 Towards the Research Focus

1.2.1 Site for Research Focus

A focus on the ways lower ability students negotiate mathematical meanings in lessons has not been a topic of research in New Zealand or in other countries. This is an important area to investigate because classroom context is linked to the performance of individual students in individual assessment tasks (Boaler, 1997a, 1999). If we are concerned about the performance of all students, we need to find out more about classroom mathematical activity during lessons. In order to understand classroom activity, I propose to explore the relationships between the classroom social norms, the use of mathematical tools, the interactions of students and teachers, and the negotiation of mathematical meaning, in typical classrooms of lower ability students, aged thirteen and fourteen years. I also aim to focus on number sense and how number sense negotiation is distributed between the three domains of the classroom context. The research questions to guide this study are:

1. What classroom social norms were evident in the case study classrooms? Which of these norms related to student participation in mathematical activity?
2. What cognitive tools were evident in the case study classrooms? How were

these tools, and technological tools, used in classroom activity?

3. What was the nature of student and teacher contributions during classroom interactions?
4. How were mathematical meanings negotiated in the case study classrooms? How was number sense distributed during lesson activity?

The first research focus relates to the social norms domain and specifically to opportunities for student participation in mathematical activity. Aspects of social norms that afford or constrain classroom mathematical activity will be investigated. The second research focus relates to the domain of tools, specifically mathematical and technological tools. The roles of the tools will be investigated and, in particular, how tools afford or constrain mathematical activity. The third research focus investigates the domain of interaction during mathematics lessons, and examines the discursive contributions of students and teachers in lesson activity. The three domains are brought together in the final research focus: the negotiation of mathematical meanings in the context of number sense.

This study investigates mathematics lessons in three classrooms in one New Zealand state coeducational secondary school. The school used broad banding to allocate students to classes at the Years 9 and 10 levels; two of the case study classes were designated as lower band and one as middle band. All three teachers carried on with their normal mathematics programmes. The topic for most of the observed lessons was number. Data was collected by participant observation during the last five weeks of a school term. As far as possible, the data tracked a sequence of lessons for each of the three classrooms. The data from each case study was analysed using an interpretivist methodology. Significant categories were further analysed across all three case studies and provided evidence for the research questions.

1.2.2 Background of Researcher

In the very early stages of this study, I piloted some exploratory number sense activities with pairs of students. While I found their talk and activity

fascinating, I was increasingly aware that this process was artificially removed from their classroom experiences. For this age group, the major site of mathematics learning was the school classroom. In New Zealand, mathematics lessons for Years 9 and 10 students typically occur in a classroom with one teacher and about thirty students. While immediately posing challenges for data collection, busy school classrooms provided authentic opportunities for investigating joint mathematical activity. I was interested in classrooms that were neither traditional nor progressive (Boaler, 1997a), nor experimental (Cobb, Yackel and Wood, 1995), but were typical of many New Zealand classrooms. In these 'non-intervention' classrooms, the teacher was assumed to be familiar with current teaching approaches and resources, and classroom lessons were jointly constituted by the teacher, the students, the curriculum and school mathematical resources.

The research questions reflect some of my interests in mathematics education and education in general. An opportunity to visit the CAN project (PrIME, 1989; Shuard, Walsh, Goodwin and Worcester, 1991) resulted in a long-term interest in the use of the simple calculator in primary school mathematics programmes (McChesney, 1995a). This simple technological tool appeared to transform children's mathematical activity and the classroom curriculum. I was curious about how the calculator afforded mathematical thinking in many contexts. This led to an interest in number sense as a way of reconceptualising the learning of number in schools (McChesney, 1995b; McChesney and Biddulph, 1994). Since 1990 I have tracked the debates and research findings of key number sense researchers (McIntosh, Reys and Reys, 1992; Sowder, 1992a, 1992b; Trafton, 1992). I have used Greeno's (1991) metaphor of an environment to gain insights about number processes such as calculation, and this study has provided an opportunity to explore this metaphor more deeply. In secondary mathematics, the innovations of the New Zealand EQUALS network have influenced my use of interactive classroom activities (EQUALS, 1989). My early involvement in educational research was with the Learning in Science Project (Energy) where I experienced working in a community of researchers and was introduced to constructivist theories of learning (Kirkwood et al, 1987). As a research assistant in Project One in Seven, I interviewed students with special needs and observed

them in their classroom situations (McChesney, Mitchell, Hammonds and Weston, 1987). In my classroom teaching, I enjoyed teaching lower and middle band classes in science and mathematics, and was usually allocated these classes as my form class. My present teaching involves working with pre-service secondary teachers in mathematics education, professional education and teaching practice. In each context, a major focus is how classroom interactions and processes contribute to positive teacher-student relationships, and enhance student learning. These are the main aspects of my past and present work that contribute to my approach to this study.

1.2.3 Outline of Thesis

Chapter 2 describes the literature relevant to this study and discusses views of learning that have influenced the research. A framework for analysing the classroom microculture is adapted and identifies the three significant domains for this study. Research associated with each domain is discussed. Key elements of each domain are brought together to consider the negotiation of mathematical meaning with a particular focus on number sense. The negotiation of mathematical meaning is proposed as distributed among social norms, tools and interaction. Chapter 3 outlines the methodology and methods chosen for this study. The process of analysis is described and an analysis framework is introduced. Four dimensions of analysis are used to organise the data and various categories identified within each analysis dimension. Chapter 4 records the analysis of data from each of the three case study classrooms. The data analysis continues in Chapter 5, where case study data from each category is combined and further examined. Chapter 6 discusses the findings of the study in relation to the research questions. This chapter also sets out the implications of the findings and raises questions for further research.

Chapter 2
Literature Review

2.1 Introduction

The research questions for this study arose from an interest in the mathematical learning of lower ability students. Particular perspectives on learning mathematics also shaped the focus of the questions. This chapter presents a review of the literature relevant to these perspectives. A socially situated perspective is outlined and then developed into a distributed perspective for mathematical activity in classrooms. Three domains are identified as significant for participation in mathematical activity: classroom social norms, tools, and interaction. Each domain is discussed in terms of relevant classroom research from New Zealand and international sources. In order to clarify key issues, research from experimental and traditional classes is compared and contrasted. Each domain is also discussed in terms of participation in mathematical activity. Classroom social norms provide a focus on social practices that afford student participation. A distributed perspective provides a focus on cultural tools and the properties of these tools in different contexts. Mathematical tools are discussed in some detail, particularly the roles of tools in mathematical activity. Classroom interaction, between the teacher and students, and between students, is an important component of classroom activity. Research about interactive processes is discussed and illustrates the complexities of joint discursive activity in classrooms.

Number sense is the mathematical focus of this study because it is an important aspect of the school mathematics curriculum in junior secondary school. Number sense is discussed from both a theoretical and a research point of view. A metaphor for number sense is adapted for use in this study, in particular, to include a tools perspective. In the final section of the chapter, the three domains of classroom social norms, tools and interaction, are linked together. I propose and argue for these three domains as a framework for describing distributed mathematical activity. This framework will be used to re-conceptualise number

sense activity in lower and middle band mathematics classes in New Zealand secondary schools.

2.2 Views of Learning

In this section, I discuss four views of learning: individual, situated, social and distributed. These perspectives are briefly outlined and then discussed in terms of learning in school classrooms. Views about relationships between students, teacher and subject matter during lesson activity will be critiqued in terms of contributing to the theoretical basis for this study. During this process, key terms will be clarified and highlighted for further discussion.

2.2.1 Learning as Individual Activity

Learning can be viewed as an individual activity, where the individual learner processes new experiences and constructs knowledge (Hiebert, 1990; Thompson, 2000; von Glasersfeld, 1990). The learner as active is an important tenet of constructivism where "knowledge is not passively received but is actively built up by the cognizing subject" (Wheatley, 1991, p. 10). Students interpret meanings and process information from the external world, and learners "actively make sense of his or her world by constructing internal representations" (Cobb, Yackel and Wood, 1992a, p. 7). For example, when calling up an image of 2+3 = 5, a mental representation might be similar to what is physically written on paper, but what is in the mind is not a mirror of the object. Internal representations "are located in students' heads and external representations are located in the environment" (Cobb, Yackel and Wood, 1992a, p. 7).

These internal representations are linked together in schema considered to be networks of related internal representations (Skemp, 1971; Thomas, 2002). For example, an individual constructs a number schema that can be activated during number or calculation activities (Romberg, 1993). "As new experiences cause students to refine their existing knowledge and ideas, so they construct new knowledge" (Ministry of Education, 1992, p. 12). When individual learners

actively process new experiences through activity in the world, they adapt existing schema or construct new or amended schema. Schema construction can account for the development of knowledge over time, and has been linked to various views of memory (Romberg, 1993). Individual cognition is linked to domain specific knowledge (von Glasersfeld, 1990) and accounts for transfer of knowledge from one situation to another (Ellerton and Clements, 1992; Thompson, 2000). The individual construction of knowledge also accounts for differences in learning such as rate of learning or diversity of knowledge (Hendry, 1996; Ministry of Education, 1992). For these reasons, school curricula in New Zealand accept an individual view of the construction of knowledge (Ministry of Education, 1992, 1993).

An individual view of learning, however, does not adequately account for intersubjectivity (Lerman, 1999), or the role of language in shared experiences (Ellerton and Clements, 1992; Lerman, 1993). The role of communication in learning has been particularly contentious, with claims that ideas cannot be communicated by language (Thompson, 2000; von Glasersfeld, 1990; Wheatley, 1991). In view of these concerns, perspectives that encompass more than the individual learner may provide more useful theories of classroom learning.

2.2.2 Learning as Situated and Social

A situated and social view of learning takes context as its primary focus. In a situated perspective of activity, context encompasses the physical environment (the setting), and the people involved in the setting (the social group), as well as the purposes of the learning and the resources available. A situated perspective has gained attention over the last fifteen years, prompted by studies of activity in situations external to schooling (Lave, 1988; Nunes, Schliemann and Carraher, 1993; Scribner, 1984). Many studies have investigated adult mathematical activity in settings such as the work place (Masingila, 1995; Millroy, 1992; Ueno, 1998) and other activities (Lave, 1988; Lave, Murtaugh and de la Rocha, 1984). One of the early studies investigated in-school and out-of-school calculation strategies of Brazilian street vendors aged 9-15 years

(Nunes et al, 1993). Their calculation strategies for the market and school situations were found to be qualitatively different. In the formal test situation, calculations were based on school algorithmic methods and produced more calculation errors. In the market situation, the calculation strategies involved recomposing numbers for flexible mental calculation. The researchers concluded that not only were the physical settings different, but the two situations had different purposes, different resources, and the tasks were presented in different ways (Nunes et al, 1993). This illustrated that different knowledge was used in different contexts. Knowledge was considered to be constituted by organising the context, that is, the interactions of the participants, the resources used, and the perceived purposes of the activity (Cobb and Bowers, 1999; Greeno, 1997).

One view of context is that a social group forms a community of practice (Lave, 1991; Lave and Wenger, 1991). According to these authors, school contexts cannot be considered as authentic communities of practice because school activity is peculiar to schools and schooling (Lave, 1991; Lave and Wenger, 1991). Genuine participation is not possible due to the institutionalisation of hierarchical practices such as assessment. The "possibility of engaging as a peripheral participant in knowledgeably skilled activity in the classroom", (Lave, 1991, p.78) is severely reduced due to the stratification of students in schools. Consequently, school classrooms are not legitimate communities of practice (Lave, 1991; Lave & Wenger, 1991).

There have been two responses to reconcile the tension between school contexts and communities of practice. One response considered a school as a unique context. Boaler used a situated perspective to examine two school settings with different mathematical cultures (Boaler, 1997a). She found that the different contexts of each school situation profoundly affected the mathematical learning of the students for each school setting. The social groups of both schools were similar in terms of age, gender, and social class. The contextual purposes and the resources were, however, qualitatively different. The differences lay in the mathematics curriculum, the teaching approaches, and the structure of the lessons. Aspects such as the "codes of the mathematics classroom that students

THE UNIVERSITY OF WAIKATO
EDUCATION LIBRARY

lived by" (Boaler, 1999, p. 266), the types of mathematical tasks, and the kinds of assistance from the teacher, influenced students' mathematical activity. These aspects affected student achievement, attitudes towards the subject, and perceptions of themselves as learners (Boaler, 1997a, 1999).

Another response described schools as distinct learning communities (Brown et al, 1993; Rogoff, 1994). Learning communities have particular features where students are considered apprentice learners rather than apprentice scientists, historians, writers, or musicians. Schools are "communities where students learn to learn" (Brown et al, 1993, p.190). 'Learning to learn' redefines the purposes of a mathematics classroom as learning how to learn mathematics, rather than learning how to be a mathematician. In this case, each classroom context can be considered a distinct learning community. In each classroom, learners "re-organize their activity as they participate in the practices of a local community such as that formed by the teacher and students" (Cobb, 1995a, p. 380). Different classrooms involve different social groups, resources and, consequently, interactions between the participants. Different classrooms within the same school constitute legitimate and distinct communities of learners.

By considering classrooms as distinct communities, the focus shifts to the relationships between various aspects of the classroom context. Classroom relationships are often described in terms of class management (Brophy, 1999; Kohn, 1993). A social perspective, however, views classroom social relationships as explicit and implicit practices and conventions (Zevenbergen, 2000). For example, the classroom social group includes not only the participants, but also the relationships between them and how they go about the activity (Ball, 1991; Lerman, 1993, 2000). These practices and conventions, as well as the combination of people, resources, and purposes, constitute different contexts. This is also the case when the setting and curriculum might be the same or very similar. Consequently, two mathematics classrooms with the same year level of students, same teacher and the identical curriculum (on paper), constitute different contexts. The context includes the setting, the physical environment, and the social group, the students and the teacher. The classroom context also includes the social practices based on the relationships between the

social group. I propose to bring a situated and social perspective to this study in order to consider important aspects of classroom activity, namely the social group, the social practices, and activity in different classroom contexts. A limitation of the situated social perspective, however, is that paying attention to the social context neglects the resources of the subject matter and the practices of the subject.

2.2.3 Learning as Distributed

Subject matter is an important constituent of the context of a mathematics classroom. A distributed perspective of learning takes account of how the resources of a subject, such as mathematics, are implicated in the learning. Distributed refers to the notion that cognition is distributed or stretched over people, activity, resources and speech (Salomon, 1993; Salomon and Perkins, 1998). Rather than locating cognition solely in the mind of an individual, or in the practices of a community, cognition is viewed as distributed across the person, the culture, the social world and over time (Cole and Engeström, 1993). Perkins (1993) described distributed cognition as 'person-plus', meaning "person plus surround" (Perkins, 1993, p. 90) where the surround refers to the immediate physical and social resources outside the person.

The concept of cognition being distributed over diverse resources has implications for learning in schools. Firstly, it implicates the artefacts of the subject matter directly in the cognition process. This is particularly relevant to the subject of mathematics, often viewed as intellectual invention rather than physical or artistic expression, personal belief or experience. Mathematical artefacts developed by human invention can be viewed as tools for use in mathematical activity (Cobb, 1995a; Pea, 1993). School mathematics includes mathematical tools and other artefacts such as diagrams and symbols specific to the school context. In school mathematics classrooms, shared and individual mathematical activity involves physical and cognitive resources, in addition to social practices. Much of the activity in mathematics classrooms involves the teacher and students in engagement with mathematical tools. Learning is influenced by students' mastery of these "tools that are specific to the culture"

(Cobb, 1995a, p. 379). A distributed perspective addresses mathematical tools as important constituents of classroom activity.

Secondly, distributed cognition draws on the work on affordances by J.J. Gibson (1979, cited in Pea, 1993). Affordance "refers to the perceived and actual properties of a thing, primarily those functional properties that determine just how the thing could possibly be used" (Pea, 1993, p. 51). Access to physical resources such as technological tools, distributes calculation processes between a person and a calculator. The calculator affords more rapid, and often more precise, access to calculation results. In different activity contexts, different affordances may be identified for the same tool. For example, the calculator affords access to numbers previously stored in the memory function as well as to calculation results using these numbers. Affordances can also be applied to social aspects of mediated activity (Boaler, 1999), such as the organisation of activity and ways of communicating. For example, small group organisation for a task distributes the joint activity over the people in the group and their resources. Small group activity affords a purpose and opportunity to talk about the task, affords more than one person's perspective, and affords the potential to assign different aspects of the task to different people.

Distributed cognition can also be constrained by aspects of an activity. Possible constraints of small groups could be the potential for students to engage more easily in non-task communication, or for one student to take a dominant role to the detriment of other group members. In a study of small groups in mathematics, Stacey (1992) found that constraints included some students dominating the discussion, the neglect of checking solutions, and an apparent preference for simple rather than complex thinking. Affordances and constraints can exist side by side, and depend on the activity context (Boaler, 1999; Pea, 1993). Because affordances can also be applied to social and situated aspects of mediated activity, affordances (and constraints) are useful for analysing situations of school classrooms (Boaler, 1999). A distributed perspective of affordances and constraints, offers a means of understanding relationships between the key aspects of social practices, interactions and the tools of the subject.

A distributed view of mathematics learning encompasses a situated social view but brings tools to the fore. It enables a focus on relationships between tools and interactions of the participants (Resnick, 1991). In this way it may account for ways that aspects of classroom activity are related, such as the participants, their talk, and their use of tools. There has been, however, very little research that has adopted a distributed perspective towards classroom activity, and much of this has explored learning with computer microworlds (Pea, 1993; Salomon, Perkins and Globerson, 1991). By bringing a distributed perspective to New Zealand mathematics classrooms, I hope to find out more about the interactions between students, teachers and tools as lessons unfold over time. In the next section, I elaborate on this perspective by setting out some of the current debates and research in mathematics education.

2.3. Recognising the Complexities

In this study I aim to investigate key features of classroom activity that will provide enough scope and depth of information about lesson activity, but not overwhelm the study. Situated and social views of learning suggest that I need to look at features of social relationships and classroom discourse (Cobb, 2000). I propose to use a distributed perspective to address these complexities, although this has so far had limited application to typical school classrooms. Two other perspectives of classroom learning are relevant for understanding complex classroom settings, both from mathematics education researchers. The first is to use metaphors of participation and acquisition for learning (Sfard, 1998, 2001) as a means of identifying aspects of activity. The second perspective is a framework for classroom microculture, developed by Cobb and his colleagues (Cobb, Gravemeijer, Yackel, McClain and Whitenack, 1997). I will use these perspectives to focus on aspects of distributed classroom activity that might contribute to the negotiation of mathematical meaning.

2.3.1 Metaphors for Learning

Metaphors of acquisition and participation were proposed by Sfard (1998) to describe different orientations to learning. The acquisition metaphor implies an accumulation and possession of knowledge by individual learners. According to Sfard, key terms generated by the acquisition metaphor are knowledge, concept, idea, notion, misconception, meaning, sense, schema, fact, representation, material, and content. Acquiring knowledge portrays the "human mind as a container to be filled with certain materials" (Sfard, 1998, p. 5). This knowledge then belongs to the learner. In addition, terms such as construction, appropriation, and development, belong to an acquisition metaphor to describe learning. Varied construction and appropriation processes among individual students can account for diversity of student mathematical learning. Learners are active constructors and acquire different ideas, processes and meanings at different times and in diverse ways.

The second metaphor used by Sfard was one of participation. This shifted the focus to the process of participation in the practices of a community. Rather than gaining knowledge, the participation metaphor emphasises knowing or coming to know (Sfard, 1998). The participation metaphor is useful for considering joint classroom activity because participation structures and practices influence the mathematical activity of classroom participants (Weade, 1992). Students and teacher in a secondary school mathematics classroom participate in the norms, discourse and practices of a community of learners of mathematics. Students participate with others in shared activities within classroom contexts. This experience of doing and talking mathematics, listening to mathematics, and jointly contributing to the mathematical discourse of a classroom community, constitutes mathematical learning from a participation perspective (Rogoff, 1997). Moreover, a focus on participation avoids the separation of affect and cognition, and "brings social factors to the fore" (Sfard, 1998, p.11). A focus on student participation can encompass social factors in the learning process, and sits within a situated social perspective (Greeno, 1997).

The use of both metaphors addresses concerns about the transfer of learning. Learning transfer across different settings has been used to challenge a situated perspective. An individual cognitive perspective claims that acquired knowledge can be taken from one situation to another, by individuals who then select and use this knowledge for tasks in different contexts (Anderson, Reder and Simon, 1996, 1997). Greeno (1997) suggested that a participation metaphor could reconceptualise transfer of knowledge. Knowing in different situations was viewed as transforming affordances and constraints of the context. The nature of participation was consequently transformed in different situations (Greeno, 1997).

Sfard (1998) claimed that both acquisition and participation perspectives are needed to adequately theorise about learning. The acquisition metaphor is crucial for insights into the development of individual knowledge and for adequately accounting for diversity of performance. Participation, on the other hand, is necessary for adequately describing and theorising the role of the social setting, the nature of inter-subjectivity, and the view that knowledge is evolved in practice rather than contained in people's heads. Participation is a useful metaphor for describing aspects of joint mathematical activity, a focus of this study. In addition, a distributed view of mathematical activity is compatible with a participation perspective. "Participation supports and constrains what students (and as co-participants, their teachers) have an opportunity to learn" (Weade, 1992, p. 93). Participation also includes elements of the individual and the social (Sfard, 1998), an issue for educational research for some time (Damarin, 1999; Driver, Asoko, Leach, Mortimer and Scott, 1994; Greeno, 1997). One approach that has addressed both social and individual perspectives is now discussed in some detail because it has shaped a major part of this study.

2.3.2 The Microculture of a Mathematics Classroom

Over the last fifteen years, a substantial body of mathematics education research has been developed around classroom interventions known as an 'emergent approach'. This approach has been mostly based at Purdue and Vanderbilt Universities in the United States of America, with Cobb providing

continuity from the early to the current projects (Bowers, Cobb and McClain, 1999; Cobb, Gravemeijer et al, 1997; Cobb, Wood, Yackel and McNeal, 1992; Gravemeijer, McClain and Stephan, 1998; McClain and Cobb, 2001; Wood, 1995, 2002; Yackel, Cobb and Wood, 1991). These research projects encompassed both the social nature of learning and an individual psychological approach to mathematical learning in classrooms (Cobb and Yackel, 1998). The social and individual perspectives were viewed as complementary where "individual thought and social and cultural processes are considered to be reflexively related, with neither attributed absolute priority over the other" (Cobb, Gravemeijer et al, 1997, p. 152). Based on the theorising from their research, the researchers developed a framework for analysing learning in social settings such as classrooms. The framework included key aspects of a social perspective to learning and a psychological (individual) perspective to the development of knowledge. The framework is reproduced below.

Social Perspective	Psychological Perspective
Classroom social norms	Beliefs about own role, others' roles, and the general nature of mathematical activity in school
Sociomathematical norms	Mathematical beliefs and values
Classroom mathematical practices	Mathematical interpretations and activity

Figure 2.1 An Interpretative Framework for Analyzing the Classroom Microculture (Cobb, Gravemeijer, Yackel, McClain and Whitenack, 1997, FIG. 8.1 on p. 154)

In the classroom microculture framework, the social perspective column refers to communal classroom processes, and the psychological perspective column relates to aspects of an individual student's activity while participating in these processes. By linking these together, the emergent approach views mathematical learning as both a process of active construction by an individual student, and of enculturation into the norms and practices of a particular class room community (Cobb, Gravemeijer et al, 1997). This framework has been

used to interpret classroom events during teaching experiments based on the instructional design of the emergent approach. The framework sets out three social dimensions that are related to mathematical learning, and these are now briefly outlined.

Classroom Social Norms

Classroom social norms refer to such things as organisational routines, obligations of classroom participants, and expectations about ways of interacting during mathematical activity. When New Zealand teachers refer to routines and expectations, or even class rules of behaviour, these would be subsumed under a heading of classroom social norms. The classroom microculture framework identified the teacher's and student's beliefs about their own roles and others' roles within the classroom setting as the individual psychological correlate of the classroom social norms. Student beliefs have been found to afford or constrain mathematical activity (Bishop, 1991). Cobb, Gravemeijer et al (1997) claimed that the relationship between student beliefs and social norms is direct and reflexive. In other words, student beliefs about their roles within mathematics lessons influenced the normative classroom expectations and vice versa. Classroom social norms include norms for student participation in mathematical activity. The first domain of interest is classroom social norms, particularly those related to participation.

Sociomathematical Norms

Sociomathematical norms are the next social aspect of the classroom microculture framework. In emergent classrooms these norms indicated what was an acceptable solution, a different solution, and an efficient solution. In other words, sociomathematical norms indicated what was valued in mathematics and what was established as mathematical authority (Yackel, Cobb, Wood, Wheatley and Merkel, 1990). Decisions about the validity or status of mathematical processes depended on a range of factors such as the social practices, the classroom power relationships, and norms for resolving argument (Cobb, Wood, Yackel and McNeal, 1992). A feature of the emergent approach is termed 'argumentation' in which mathematical reasoning is based on explanation and justification of mathematical thinking (Cobb, Gravemeijer

et al, 1997). The students participate in this public reasoning, and it is through this process of argumentation that mathematical meanings are negotiated. This study will not examine sociomathematical norms as outlined by the emergent approach because New Zealand mathematics classrooms do not currently focus on argumentation. The ways that mathematical authority are established, challenged or resolved in the classrooms, will be considered simply as one aspect of mathematical negotiation within the classroom context (see section 2.8.1).

Classroom Mathematical Practices

Classroom mathematical practices are the final social aspect of the classroom microculture framework. Research in emergent classrooms traced the establishment and development of classroom mathematical practices over periods of time from several months (Bowers et al, 1999) to one school year (Cobb, Wood and Yackel, 1991). A classroom mathematical practice was a communal mathematical activity that involved a means of symbolising, arguing and validating (Bowers et al, 1999; Cobb, Gravemeijer et al, 1997). Mathematical practices were considered to be established when the practice was 'taken-as-shared'. Taken-as-shared was inferred when the practice no longer needed any explanation or justification by the participants, but could be assumed by the classroom community (Cobb, Gravemeijer et al, 1997). The individual correlate of a classroom mathematical practice was student mathematical activity. Classroom mathematical practices have a particular definition within the emergent approach that may be difficult to use in other classrooms, particularly the reliance on argumentation to establish a practice as 'taken-as-shared'. A key feature of mathematical practices, however, is relevant for this study. The emergent approach emphasised the crucial role of interaction between the participants during the development of a mathematical practice. Interaction involved the participants in the negotiation of mathematical meanings within mathematical practices.

In an explanatory note to distinguish between different uses of the emergent metaphor, Cobb and his colleagues implied that the classroom microculture framework could be used in any classroom as a basis for research about the

emergence of mathematical meaning (Cobb, Gravemeijer et al, 1997, p. 162). There has, however, been no use of this framework for research in 'non-emergent' (traditional) classrooms. Nevertheless, it would seem classroom social norms and classroom mathematical practices are important aspects of a participation perspective for any mathematics classroom. For the purpose of this study, I have adapted the microculture framework (Cobb, Gravemeijer et al, 1997) to include interaction as well as classroom social norms as domains of interest. The domain of interaction encompasses the nature and qualities of classroom interactions when participants jointly develop classroom mathematical practices.

Later emergent studies considered the role of mathematical tools in the process of establishing mathematical practices (Bowers et al, 1999; Cobb, 1995a; Cobb, Stephan, McClain and Gravemeijer, 2001; McClain, Cobb, Gravemeijer and Estes, 1999). The research findings included a tools perspective, as well as a distributed approach to analysing classroom activity. Mathematical tools were talked about and used in the development of classroom mathematical practices. Different tools were used at different times and in different ways by the students in emergent classrooms (Cobb, 1995a). Tools were found to afford or constrain mathematical activity in different lesson contexts. In this way, the tools of the subject matter were incorporated into the analysis of the learning processes in emergent classrooms. As I have indicated earlier, mathematical tools are a particular focus of this study. I propose a third domain of interest, tools, and in particular, the mathematical and technological tools of school mathematics. The three domains, classroom social norms, interaction and tools, are now linked together in a framework for this study.

2.3.3 A Framework for Distributed Mathematical Activity

In this study, I aim to explore participation in mathematical activity in the context of classrooms in the junior secondary schools. While the New Zealand mathematics curriculum states that

> mathematics is most effectively learned through students' active participation in mathematical situations, rather than through passive

> acceptance and repetition of knowledge (Ministry of Education, 1992, p. 18)

there is no further explanation of participation in the curriculum. For students who are in lower or middle band classes, what might be the qualities of participation in classroom activity? How are these normative participation practices established or evolved in classroom situations? When students participate in mathematical activity, what tools do they use and how do they use these tools? What are some of the ways that tools afford or constrain student activity? Junior secondary school students in this country talk a great deal and most classrooms at this level are reasonably chatty places. How does classroom interaction, particularly talk about mathematical activity, influence the negotiation of meanings?

The adapted framework sets out three domains of interest for this study:

(i) Classroom social norms – the normative social practices related to student participation in mathematical activity.

(ii) Tools – tools (particularly mathematical tools), and the ways they afford and constrain mathematical activity.

(iii) Interaction – the interaction processes between teacher and students, and between students, when they communicate about mathematical activity.

In this study, I propose to use these three domains to examine classroom activity. I argue that these three domains address key elements of mathematical learning, namely student participation in mathematical activity, engagement with artefacts of the subject matter, and communication processes in mathematical activity. The three domains contribute to the negotiation of mathematical meaning. This relationship is represented in the diagram below (Figure 2.2).

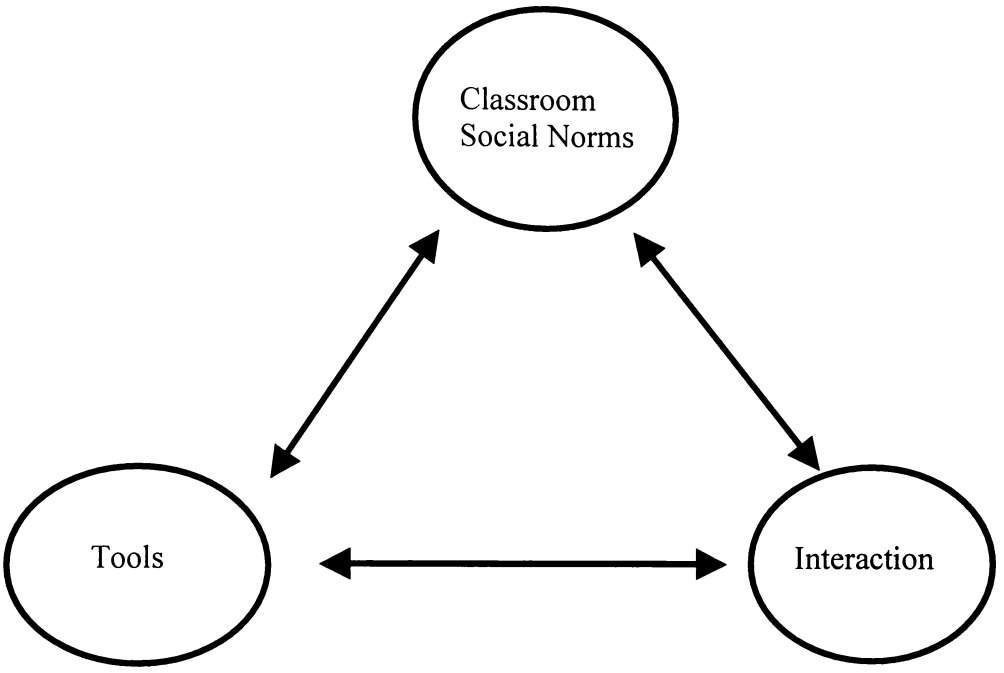

Figure 2.2 Relationship between Domains of Classroom Social Norms, Tools and Interaction.

In the next three sections of this chapter, each of the three domains is outlined in turn. Research findings from emergent classrooms are outlined for each domain and contrasted with research from traditional classrooms. Further classroom research from mathematics and other school subjects is also discussed.

2.4 Classroom Social Norms

2.4.1 Classroom Social Norms

The first domain of interest for this study is classroom social norms. Classroom social norms constitute the expectations and obligations of a social group that become normative over time (Cobb, Gravemeijer et al, 1997). Classroom social norms determine, for example, participation structures in classrooms such as turn taking and gaining access to discussion (Mehan, 1979, 1997; Mousley and Sullivan, 1995). Participation structures organise the opportunities, expectations and obligations for participation within lessons (Cobb, Yackel and Wood, 1992b; Mehan, 1997). Participation opportunities might include lesson events

such as individual tasks, or whole class discussions, providing students with regular opportunities to engage in mathematical activity and to interact with others. Classroom social norms also include expectations and obligations about the roles assumed by teachers and students when they participate in mathematics lessons. Roles indicate the types of contributions, how contributions are valued, and ways of responding to the contributions of others.

Norms determine explicit and implicit social practices in classroom lessons, which shape and frame relationships and status among the classroom participants (Zevenbergen, 2000). Classroom social norms are not necessarily fixed. They can be established, monitored, or re-constituted over time (Cobb, Wood and Yackel, 1991). Classroom social norms related to student participation in mathematical activity are important for this study. The social norms of the emergent approach are now outlined in some detail because this approach addresses the participation of students in classroom mathematical practices. Social norms in traditional mathematics classrooms will then be discussed, followed by research about classroom social norms in New Zealand schools.

2.4.2 Classroom Social Norms for Emergent Classrooms

The emergent approach set up specific classroom social norms based around a particular lesson structure, and the expectations of participants during lesson events. The typical lesson structure was exemplified by the lessons in a second grade classroom (Cobb, Wood, and Yackel, 1991). The teacher began the lesson by spending about five minutes introducing the activities to the children. These activities were presented on activity sheets for pairs of students (sometimes groups of three) who worked on these for the next twenty or so minutes. During this time, the teacher circulated around the room, observing and interacting with the students. The teacher told the class when there was one minute of work time left and the children began to put away any equipment. This signalled the next phase of the lesson, the whole class discussion. The teacher asked for volunteers to explain how they solved the first activity. The teacher chose one volunteer who went to the front of the class, often

accompanied by their partner. The whole class discussion continued for about twenty minutes, then the teacher collected the activity sheets and introduced a whole class activity or some questions for the remaining lesson time (Yackel, Cobb, and Wood, 1991). The two main opportunities for mathematical participation in the lesson were the small group activities and the whole class discussion. Classroom social norms were established for these two phases of the lesson.

Social Norms for Small Group Activity

The social norms or expectations for small group activity (pairs) were that the groups were to "cooperate to complete the activities, and that they reach a consensus" (Cobb, Wood, and Yackel, 1991, p. 168). Co-operation was taken to mean

> explaining personal solutions to partners, listening to and trying to make sense of a partner's explanations and attempting to achieve consensus about an answer, and, ideally, a solution process in situations where a conflict between interpretations or solutions has become apparent (Cobb, Wood, Yackel and Perlwitz, 1992, pp. 486-487).

A consensus was either a mutually agreed solution method or an agreed answer with different solution methods. In addition, students were expected to share materials and to persist in thinking for themselves rather than rush to complete a set number of tasks. These norms afforded sustained activity in small groups without constant monitoring by the teacher.

Social Norms for Whole Class Discussion

The expectation of listening to others and trying to make sense of a peer's explanation was also a norm for whole class discussions. Students who volunteered to share their solution method with the class were expected to explain and justify their solutions. They were expected to explain in such a way that their peers could make sense of their methods. Their peers, in turn, were obliged to listen and to try to make sense of the explanations, and to respond by agreeing, disagreeing or asking questions. Similarly, if the teacher expected the children to honestly explain their methods, then she was obliged to accept their explanations. "Thus the teacher had obligations to the children, just as they did

to her" (Cobb, Wood, and Yackel, 1991, p. 164). These norms constituted expectations and obligations of participants in the emergent classrooms.

The classroom social norms in this emergent classroom set up participation norms that distributed the participation of students and teacher during lesson activity. During the paired activity, the norms of collaboration afforded purpose and opportunity for children to discuss mathematics with each other. Expectations and obligations during this joint activity afforded each child opportunities to explain, listen, question, and respond to one peer. In the whole class discussions, when children shared their solution methods, mathematical thinking was made public and afforded opportunities for peers to respond to the mathematical thinking of others. It also afforded opportunities for students to clarify their own thinking. These emergent classroom social norms afforded access to a greater variety of mathematical resources, that is, the student explanations of solution methods and their mathematical reasoning. This distributed mathematical activity among the social group during classroom interactions.

The Role of the Teacher in Setting Up Classroom Social Norms

In the teaching experiment described here, conducted over a whole year in a second grade classroom, the teacher established the classroom social norms at the beginning of the school year. The teacher, as an authority in the classroom, initiated a class conversation about how the small groups and the whole class would talk about mathematics. They "talked about talking about mathematics" (Cobb, Wood, and Yackel, 1991, p. 164). In the following lessons, the teacher consolidated the classroom social norms by talking about what was expected at crucial times when norms were challenged. For example, instances of children feeling embarrassed with wrong answers, or occasions of put-downs, were times that the teacher intervened, initiated a conversation about expectations, and renegotiated the classroom social norms with the class. By her actions,

> the teacher nurtured the sense of trust that was essential if the children were to talk publicly about their mathematics. She demonstrated that if the children accepted the obligation of expressing their mathematical understandings then she, as an authority, was obliged to protect them. (Cobb, Wood, and Yackel, 1991, p. 166)

All participants were considered to be responsible for the monitoring of these norms. "The students participated in the interactive constitution of the social situation in which their development occurred" (Cobb, Wood, and Yackel, 1993, p. 100). For example, the teacher renegotiated the social norms for small group work when the class returned from two weeks of vacation.

Further Examples of Classroom Social Norms

Other research studies have incorporated the emergent norms into their classrooms. In a year-long study of four Grade 3 children, (Lo, Wheatley and Smith, 1994), the children were "accustomed to talking mathematics with partners and with the whole class" (p. 32). During whole class discussions, the teacher sat among the students and her role was to make sense of students' methods and to facilitate the student discussion. A study in middle high school incorporated the emergent classroom social norms for small group tasks, some of which involved the use of computerised tools (Hershkowitz and Schwarz, 1999). There was an expectation of cooperation among the pairs and of partners reaching consensus as they worked on activities. The emergent perspective gives high status to particular classroom social norms and the authors claim that these contrast with social norms in traditional mathematics classrooms.

2.4.3 Classroom Social Norms in Traditional Classrooms

In all school classroom situations, teachers have authority and power (Zevenbergen, 1995). It is the way the teacher expresses authority that is different in emergent and traditional classrooms. Traditional classroom social norms have been portrayed as dominated by the teacher who has a role of classroom manager (Brown et al, 1993). Norms for whole class discussions require students to respond to the teacher's cues and signals rather than articulate their thinking (Cobb, Gravemeijer et al, 1997; Cobb, Wood, Yackel and McNeal, 1992). Teacher assistance is in the form of re-mediation rather than dialogue and negotiation (Bauersfeld, 1988). Students are silent, passive, and wait for the teacher to tell them how to do the mathematics (Jaworski, 1999). They are expected to listen and their only contributions are responses to questions from the teacher (Askew, 2001). Students are expected to imitate the

demonstrated techniques and their contributions must be channelled through the teacher (Goos, Galbraith and Renshaw, 1999). Norms for class discussions are based around teacher exposition rather than student articulation (Saxe, Gearhart and Seltzer, 1999). Classroom social norms expect the individual learner to be isolated and separated from peers. These norms of traditional classrooms implicate students as passive receivers of the teacher's mathematical knowledge.

2.4.4 Classroom Social Norms in New Zealand Classrooms

While traditional classrooms have been described as dominated by the teacher, some classroom-based research has found that students were not necessarily passive in the generation of social norms. In an ethnographic study of two Fifth Form classes (Year 11- fifteen year olds) in a New Zealand state girls secondary school, Jones (1991) found that classroom norms were qualitatively different for the two classes. The students in the high stream class (5 Simmonds), mostly from middle class Pakeha families, were active in classroom interactions. They questioned the teacher, demanded challenging work and,

> often discussed their work with serious intensity both with their teachers and amongst themselves, inside the classroom and out. Their teachers expected them to express and apply ideas and concepts. They were often asked questions by their teachers which demanded interpretation and thought. (Jones, 1987, p. 2)

Jones identified a norm of individual competition in the ways that these students enthusiastically responded to teacher questions and publicly shared their ideas. It was through these normative social practices that the students in this class learned skills and aptitude for working in middle class, professional work. What was considered to be appropriate classroom activity and valued knowledge was mutually demanded and rewarded by students and by the teacher (Jones, 1991).

Expectations and obligations were also mutually constituted with the low stream class (5 Mason), mostly Pacific Island students from working class families, and their teachers. The class of 5 Mason demanded notes, copied from

the teacher, resisted teacher efforts to engage them in thinking, and spent much of the class time mucking around (Jones, 1991). When teachers asked students to think about the tasks or to work something out for themselves, the students opposed these participation opportunities. Instead they attempted to control the work provided by the teacher.

> For example, they positively reinforced the teacher for giving them copying work to do. When the girls were given copying ... they generally became attentive and copied silently. The quiet 'working' atmosphere thus created was highly rewarding for the teacher ... On the other hand, any attempts by the teacher to encourage discussion or comment about some syllabus topic would be met with disinterest and a lack of co-operation (Jones, 1991, pp. 77-78).

Teacher actions were not only resisted by the students but this class "actively encouraged what they considered to be appropriate teacher behaviour" (Jones, 1991, p.82). Consequently, the classroom activities for 5 Mason were qualitatively different to the activities in the high stream class. Lesson events such as note copying are often cited as examples of prescribed mathematics programmes (see also Boaler, 1997a; Boaler, Wiliam, and Brown, 2000). These judgements are not always appropriate, however, as events become ritualised in particular classroom contexts. An example of a ritualised class interaction was from Human Biology (Jones, 1991), where the teacher asked closed questions demanding short, one word answers, and the class would reply in unison. While this unified chorusing might be considered low-level teaching in Australia (Cooper, Atweh, Baturo, and Smith, 1993), Jones (1991) reported that not only was this chanting "an enjoyable collective activity" (p. 82) but that she herself really enjoyed these interactions, because of the energy and enthusiasm of both teacher and students. In this classroom context, class chorusing was an opportunity for student participation, illustrating social norms based on implicit expectations and obligations for this ritual.

Jones' study of the two different classroom cultures found that the lived classroom experiences were different and social norms were mutually constituted by the students and by their teachers. These norms served to reinforce the students' perceptions of their abilities and their potential prospects as adults. The teaching approaches and what was viewed as valuable knowledge in the low stream class of girls, confirmed students' personal views

of their abilities and their aspirations for semi-skilled working class jobs (Jones, 1991).

Further New Zealand Research

A more recent study in New Zealand schools has been the AIMHI project (Hawk, Cowley, Hill and Sutherland, 2001; Hill and Hawk, 2000) based in nine Decile 1 secondary schools (lowest decile of socio economic ranking). AIMHI was established by the Ministry of Education in response to the low rates of school leaver qualifications for these schools. AIMHI has investigated the practices of exemplary teachers, student views of effective teaching, and the transition from primary to secondary schools. The student study interviewed secondary school students, and found that particular features of the teacher-student relationship were significant for student learning (not only for student enjoyment but engagement with curriculum subjects). When student-teacher relationships were based on fairness, respect and inclusiveness, these aspects indicated reciprocity between students and teacher. Reciprocity is a form of expectations and obligations, particularly of the roles adopted during lessons. Roles adopted by teacher and students were examples of particular classroom social norms found to be important for student learning (Hill and Hawk, 2000). The researchers also identified the teacher-student relationship as critical for student engagement in the learning process in low decile schools. They claimed that students in higher decile schools were more likely to continue to learn even when the relationship with the teacher was not positive and, by implication, even if the social norms were not conducive to social cohesiveness. This finding suggests that normative social practices have different effects in different school and classroom contexts.

Rituals are recurring events within or between lessons. Norms associated with lesson rituals can also have positive or negative impacts. In the AIMHI schools, rituals such as set routines and lesson outlines were found to be important for student learning (Hill and Hawk, 2000). Certain features of these rituals were significant: they were regular and the students understood the expectations of the rituals. This finding was similar to Brown et al (1993), where regular explicitly known rituals afforded student participation in classroom science

activities. Rituals can, however, constrain student activity. In a three year study of ten New Zealand primary school children, Walls (2001) found that certain lesson rituals could have negative social and cognitive effects for some children. The ritual of ten quick questions to begin the lesson was found to be significant. Particular aspects of this starter quiz were problematic for many of the children: the emphasis on speed, the judging of right or wrong answers, the lack of opportunities for discussion or assistance, and the absence of equipment for children. The starter quiz was an individual activity with no assistance available from teacher or peers. According to the classroom micro-culture framework (Cobb, Gravermeijer et al, 1997), social norms have a psychological correlate of an individual student's beliefs about their own role in mathematical activity, and Walls found that seven children indicated that they had lost confidence during the time of the study. The particular norms associated with the quiz and marking were found to constrain children's mathematical identity, particularly for the children who did not experience success in these quiz rituals. Social norms were powerful influences on further individual mathematical activity. Walls' findings have highlighted the need for rituals in mathematics lessons to be socially appropriate and cognitively accessible for all children in the class, particularly for lower ability children (Walls, 2001).

Classroom Social Norms and the New Zealand Mathematics Curriculum

The norms for the starter quiz, marking and review described above (Walls, 2001), run counter to the explicit social norms in the current mathematics curriculum (Ministry of Education, 1992). For example, "opportunities for students to develop communication skills and to participate in collaborative problem-solving situations" (Ministry of Education, 1992, p. 7), are linked to the development of social and cooperative skills. Participation in problem solving teams and "listen[ing] and respond[ing] to the ideas of others" (Ministry of Education, 1992, p. 9), are Achievement Aims for the Mathematical Processes strand. Other expectations similar to the emergent approach are only indirectly mentioned in the mathematics curriculum (Ministry of Education, 1992). The importance of "encouraging students to share ideas" (p. 11) is related to the statement "Learning to communicate about and through mathematics is part of learning to become a mathematical problem

solver and learning to think mathematically" (p.11). Similarly, cooperative group learning tasks are provided as one way to involve girls more actively in mathematics (p.12). Participation, collaboration, and communication are therefore social norms inferred from the New Zealand mathematics curriculum for primary and secondary schools.

2.4.5 Classroom Social Norms and Student Participation

As can be seen from the above discussion, the research literature on classroom social norms is somewhat contradictory. While the emergent approach had specific classroom social norms that set the lesson participation structure, some of these norms also featured in traditional New Zealand classrooms. This suggests that classroom social norms are complex and related to the context of mathematics classrooms (Boaler and Wiliam, 2001). Key aspects of classroom social norms will now be identified for this study.

Rituals

Lesson rituals are recurring actions within the daily lesson structure (Brown et al, 1993), and can provide opportunities for participation in mathematical activity. Rituals can mean routines or organisational procedures, terms commonly used by New Zealand teachers. Rituals may also be the result of a school policy, communicated to students at the beginning of the year. An example of this would be students waiting outside a classroom before a lesson, until the teacher unlocks the classroom door or signals to the students to enter. Other rituals may be inherited as part of a tradition of schooling, such as raising a hand to ask a question, indicating a variety of intentions. A ritual, such as a ten question quiz to start a mathematics lesson, might be explicitly signalled by the teacher, or such an expected part of the mathematics lesson that its omission is signalled by the teacher. In a sixth grade science classroom (Brown et al, 1993), rituals were identified as those relating to participation structures, such as rituals for participating in small groups. These participation rituals were "few and practised repeatedly so that students, and indeed observers, can tell immediately what format the classroom is operating under at any one point of time" (Brown et al, 1993, p. 200). The affordances of these rituals were that

they were easily recognisable by the students, and enabled them to more easily make transitions between activities and to understand the roles expected of them (Brown et al, 1993). Consequently, classroom rituals are closely associated with roles adopted by teacher and students during lesson events.

Roles

The roles adopted by students and teacher influence the nature of their contributions during lesson activity (Collins and Green, 1992). The roles of teacher and student are "constituted through the range of roles members construct" (Tuyay, Floriani, Yeager, Dixon and Green, 1995, p. 1). For example, the classroom social norms of the emergent approach constituted particular roles for the students and for the teacher. During small group activity, student roles were as contributor, as listener and as assistor. During the whole class discussion, there were additional student roles as explainer and as questioner. These roles enabled students to explain their solutions, to listen to the solution methods of others, to ask questions of other students, and to respond to requests for assistance (in various forms). Various teacher roles were also evident during the different phases of the lesson such as commentator and indicator of mathematical significance (Cobb, Wood, Yackel and McNeal, 1992, 1993). Roles can be general such as those related to subject matter: for instance, mathematicians, scientists, social scientists (Tuyay et al, 1995). In the sixth grade science classroom previously mentioned (Brown et al, 1993), student roles were related to a community of scientists, such as "teachers, editors, advisers, and mentors, making comments on one another's work and entering a network of learners with various degrees of expertise in the domain" (p. 204). Providing feedback to one another was also perceived as an important role for every student. Another classroom study identified roles for both students and teacher as generator of knowledge, facilitator of knowledge, and instructor of knowledge. Only the teacher, however, adopted the roles of monitor of knowledge and aligner of knowledge (connecting, signalling) (Collins and Green, 1992).

Student roles in a traditional classroom include some of the above. For example, roles as listener and contributor would be observed in a traditional

mathematics classroom. But features of these roles, such as the nature and frequency of listening or contributing, contrast with an emergent approach. In a traditional classroom, the teacher demonstrates and provides a clear verbal explanation of mathematical procedures (Askew, 2001). Teachers ask questions that request answers rather than explanations or reasoning. Student contributions are expected to reproduce the procedure demonstrated by the teacher (Askew, 2001). Due to qualitatively different student input, the student role of listener is more likely to be listening to the teacher for long periods of time rather than to other students. In a study of six schools in London, lower band classes had more prescriptive teaching approaches than mixed ability classes, including lengthy exposition episodes (Boaler et al, 2000). In traditional classrooms, individualised activity predominated over more shared collaborative approaches to tasks (Bauersfeld, 1988). In order to investigate classroom social norms in New Zealand mathematics classrooms, social norms such as classroom rituals and roles are useful indicators of expectations and obligations of classroom participants.

2.4.6 Summary of Classroom Social Norms

In this study, the classroom social norms domain is one of three domains that contribute to the negotiation of mathematical meaning. The expectations and obligations of classroom social norms influence "both subsequent mathematical activity and discussions in which the teacher and children talked about mathematics" (Cobb, Wood and Yackel, 1993, p. 99). I propose to investigate the classroom social norms through an analysis of rituals and roles. Rituals are defined as recurring actions within and across lessons. The rituals of interest are lesson events that afford student participation in mathematical activity. A focus on two aspects of student roles will be used to indicate participation in negotiation of mathematical meanings. Firstly, the ways students work together either in whole class or small group activity will indicate roles related to collaboration and assistance. Secondly, a focus on the status of student contributions will reveal how students' mathematical resources contribute to classroom interaction. Status indicates the value placed on student contributions by both teacher and peers. For this study, the two indicators of student roles are

ways of working together and the status of contributions. Some classroom social norms influence the organisation of student participation in the classroom. As Klein (2000) pointed out, student engagement in participation is not enough. In a re-interpretation of a norm from the emergent approach (p. 74), she suggested that a particular participation episode may not have been empowering for the student. It is the qualities of participation structures and practices that are important. Classroom social norms can afford opportunities for students to assume roles, particularly roles related to interaction. Classroom social norms that influence student participation also afford opportunities for students to use mathematical tools in activity. This relationship between the three domains is illustrated below in Figure 2.3.

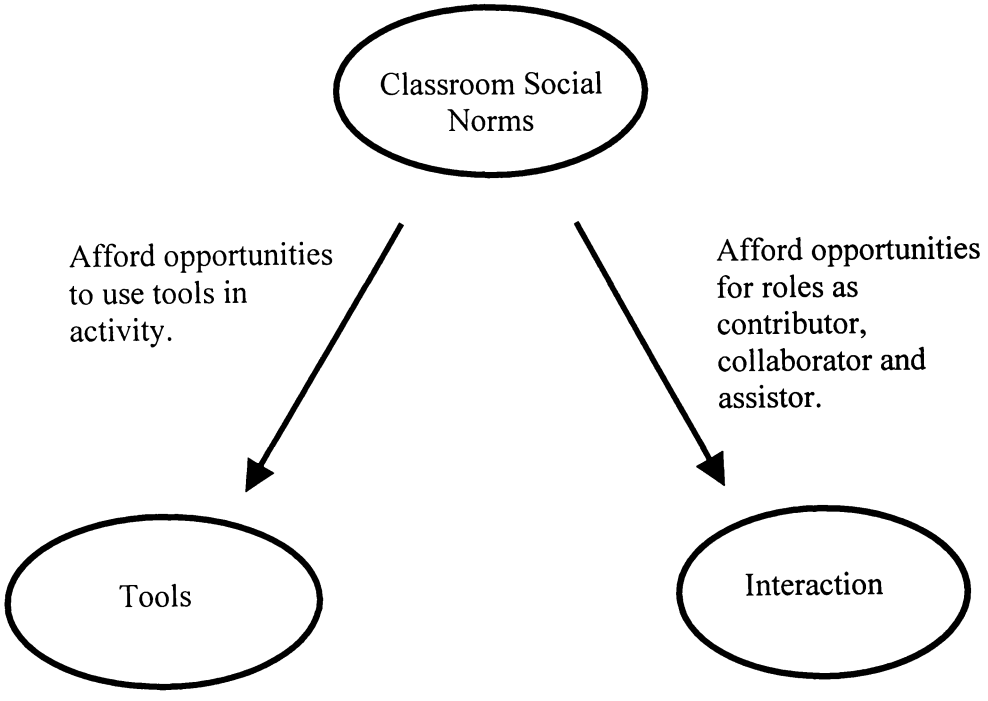

Figure 2.3 The Influence of Participation Social Norms on the Domains of Tools and Interaction.

Classroom mathematical activity involves tools of school mathematics, and the domain of tools is discussed in the next section.

2.5 Tools

When students participate in mathematical activity, they use artefacts of the subject matter, mathematical tools. When students talk about their mathematical thinking, they talk about mathematical tools in the context of an activity. Tools are important constituents of the shared negotiation of mathematical meanings, and tools are the second domain for this study. In this section, general aspects of tools are introduced and examples of school mathematical tools discussed. The roles or functions of tools in mathematical activity are outlined. These roles are related to the affordances and constraints of tools in classroom activity. The special case of the calculator is examined as an example of a technological tool. Finally, the role of tools in distributed classroom activity is discussed.

2.5.1 Introduction to Tools

For this study, I will use a general definition of tools as 'cultural tools' that encompass both physical and mental tools. Physical tools include common educational items, such as rulers and pens, and student textbooks and notebooks (Pea, 1993). Technological tools are considered a special category of physical tools and hand-held calculators are the most significant technological tools in this study. Mental tools will be designated as cognitive tools, in order to distinguish them from physical, including technological, tools (Salomon and Perkins, 1998). This definition of tools has been influenced by sociocultural perspectives of cultural tools which are now discussed in more detail.

The notion of tools as a key aspect of human activity can be traced to Vygotsky (1978). Tools were physical artefacts and considered as an extension of human labour (Vygotsky, 1978). Examples of tools were items such as pencils, hammers and other implements. Vygotsky clearly distinguished between physical tools and mental artefacts which he named as 'sign and symbolic operations' (p. 23) or by the collective term, sign systems. Examples of sign systems were speech and symbolic notation. Vygotsky also identified such actions as tying a knot in a handkerchief, as a sign, a device used as a personal reminder to remember something. While physical tools and sign systems were

different, Vygotsky stated that they were "subsumed under the same category; that of mediated activity" (p. 54). His ideas have been adapted by other writers who have used the collective term, cultural tools, to include both physical and mental artefacts (Cobb, 1995a; Pea, 1993; Salomon and Perkins, 1998).

> Here we use "tools" in a broad sense, including not only physical implements but technical procedures (e.g., the algorithms of arithmetic) and symbolic resources (e.g., those of natural languages and mathematical and musical notation). (Salomon and Perkins, 1998, p. 10)

Another tools perspective has been the use of the term 'inscriptions' to mean both physical objects and symbolic objects, such as maps and diagrams, used to model aspects of the world (Cobb, 2002; Forman and Ansell, 2001, 2002). In addition, tools have been used to describe technologies such as calculators and computers, including some computer software (Abreu, 2000; Pea, 1993; Shaffer and Kaput, 1999).

General Roles of Tools

Vygotsky recognised that the use of tools and sign systems transformed human activity (Vygotsky, 1978). This transformation of activity included the role of tools in replacing human physical labour, as well as in cognitive activity. By transforming cognitive activity, tools could transcend "the limitations of the mind" (Pea, 1987, p. 91). For example, tools mediated activity by substituting for an individual's mental activity, and one use of tools was to "offload" mental effort (Pea, 1993, p. 53).

As well as substituting for mental activity, tools can also amplify human activity. The metaphor of 'cultural amplifier' appears to have first come from Bruner (1966, cited in Pea, 1987, p. 92). Tools provide greater efficiency by increasing precision and by saving time. "Tools are not merely amplifiers of human capabilities, but they lead to the re-organisation and restructuring of activity" (Cobb, Gravemeijer et al, 1997, p. 222). By re-structuring and re-organising activity, tools make further activity possible. Cultural tools have the capacity to transform human activity, and serve particular purposes or functions, such as:

Saving time

Providing greater precision

Offloading mental effort

Reorganising resources

Restructuring activity

I aim to apply these purposes of tools to mathematical activity in classroom lessons.

Tools are not separated from context. Using Wertsch's idea of a tool kit, tools may have different purposes and roles in different contexts. "Some tools are more powerful and efficacious for certain activities or spheres of life, and others are more powerful and efficacious for others" (Wertsch, 1991, p. 102). A tool kit approach encompasses the use of tools nested within other tool use. Wertsch claimed that "tools are acquired in a certain order and therefore *inherently* organized along a continuum from lower to higher or from less powerful to more powerful" (Wertsch, 1991, p. 100, original italics). With an increasing interest in sociocultural perspectives of human activity, tools have been a focus of discussion and research in mathematics education.

2.5.2 Tools and School Mathematics

New Zealand Mathematics Curriculum

Tools are included in several ways in the New Zealand mathematics curriculum. One of the general aims of school mathematics is to provide students "with the mathematical tools, skills, understandings, and attitudes which will help them to cope confidently with the mathematics of everyday life" (Ministry of Education, 1992, p. 8). Tools are not defined but are referred to in the context of practice of mathematical procedures. Fluency of procedures is acknowledged to be important, but teachers are warned that "such routines only become useful tools when students can apply them to realistic problems" (Ministry of Education, 1992, p. 11). In the mathematical processes strand, some of the problem solving strategies are implicit examples of tools being used in the context of solving mathematical problems. Using equipment and other materials involves the use of physical tools to model aspects of the problem situation. Other strategies, such as using a table or drawing a diagram,

involve the use of general cognitive tools in mathematical activity (Ministry of Education, 1992).

Technological tools are explicitly mentioned and referred to as learning tools.

> Calculators, graphics calculators and computers are learning tools which students use to discover and reinforce new ideas. Calculators are powerful tools for helping students to discover numerical facts and patterns, and helping them to make generalisations about, for example, repeated operations. (Ministry of Education, 1992, p. 14)

This marks a shift from previous curriculum documents (Department of Education, 1985, 1987) that emphasised the importance of being proficient with using calculators and computers rather than using these to enhance learning. These are the diverse meanings of mathematical tools implicit in the current curriculum document (Ministry of Education, 1992). Mathematical tools, particularly cognitive tools, will now be discussed.

Cognitive Mathematical Tools

Mathematics is a subject based on cognitive tools and many of these tools are part of school mathematics. In this study, cognitive tools are categorised into three main groups of tools: representations, thinking devices, and analogous models.

Representations

Representations can stand for a physical entity or can be mental versions of the physical object (Greeno, 1991). Mathematical representations include mathematical notation such as numbers and symbols, and other visual representations such as drawings and diagrams. When a person writes the number 28 on a piece of paper, the representation of the number is a more useful resource than retaining a mental representation of twenty-eight things. Representations of mathematical entities are also carriers of information, particularly about the properties of these conceptual entities (Greeno, 1991). For example, the representation of 28 conveys information about the magnitude of the digits, the magnitude of the places, as well as information about the precision of the representation. This information about place value can be used in further mathematical activity.

> [The] physical properties of symbols can be resources for reasoning – in other words, the physical symbols provide affordances for activities of making inferences. The positions of the digits in an integer or the location of a decimal point, a fraction bar, or a ratio colon conveys important information about the relations among numbers or quantities that are denoted by the numerical symbols. They are also crucial cues in the performance of procedures on the symbolic representations that include them. (Greeno, 1991, p. 196)

Mathematical representations afford further activity by providing more information. Representations can serve as labels or markers, reveal features such as reference points or benchmarks, or provide useful cues for further activity.

Mathematical entities can be represented in multiple ways that afford access to further information. When one half is represented in multiple ways as a fraction, or a decimal, or a percentage, different properties of one half are afforded by these representations. Not only do these representations afford stable external records, but representations generated in multiple forms such as an equation, sequence or graph, afford a choice of representations that are more useful for different contexts. For example, writing 28 as 7 x 4 generates a different representation that highlights a particular property of 28 (Greeno, 1991). Multiple representations present mathematical relationships in different ways, and make different resources available for further activity. In these ways, representational tools transform activity by re-organising mathematical resources and re-presenting mathematical information.

Thinking Devices

Cognitive mathematical tools are also thinking devices for further mathematical activity (attributed to Lotman, in Wertsch, 1991). Mathematical tools such as algorithms or algebraic equations are examples of thinking devices used for calculation processes with numbers and quantities (Greeno, 1991). Mental procedures such as algorithms substitute for more complex calculation processes. A multiplication algorithm, for example, is a tool replacing mental computation processes that would otherwise be more complex, take longer and sometimes be less accurate. By offloading complex mental computation, students can focus on other aspects of a mathematical situation. For example,

students can more easily and quickly have access to calculation results as a resource for further activity. In school mathematics, algorithms or algebraic equations can be subsumed into more complex activity, such as problem solving. By using tools as substitutes for mental calculations, students can be freed to consider further aspects of the problem situation, such as interpretation and application of the calculation result.

Tools in the form of multiple representations of numbers can also be used as thinking devices. A calculation strategy involves making decisions about which configurations or representations of numbers or quantities to use, and then using an operation process with these representations. Recomposing or decomposing any number produces a further cognitive tool. For example, knowing that 18 is equivalent to 20 - 2, affords the calculation of 56 + 18 as 56 + 20 – 2. The recomposing of 18 has become an additional tool with a resultant offloading of mental effort compared to other calculation procedures, such as counting or using an algorithm.

In school mathematics, some cognitive tools can be both representations and thinking devices. For example, the number line is a representation of whole number magnitude that affords access to further mathematical tools. If the number line is calibrated in tens, this makes available important information about the decimal number system. The decimal construction of the number system is a property that affords counting in tens or using decades as benchmarks for recomposing or decomposing numbers (Mousley, 1999). The original tool, the number line, provides a visual representation that makes transparent a useful property of the number system, which can then be used as a tool for calculation. Mousley (1999) identified affordances of the number line. Number labels were signifiers of magnitude and sequence, affording information about the number system. The number line was also a thinking device when used to model processes of addition and subtraction with directed numbers. The intervals represented by the number line afforded thinking about the process as a shift along the number line. It is useful for "demonstrating the four operations with positive numbers, and is particularly useful for missing addend problems" (Mousley, 1999, p. 10). Another role of the number line was

that it could be talked about. The terms used, such as plus, minus, positive and negative, were mathematical signifiers, affording further information about integers and the operations. The number line was also a tool that could constrain activity, particularly if students were "thinking about numerals representing points on the number line instead of directed shifts along the line" (Mousley, 1999, p. 10). Similarly, a tool can serve different functions for different mathematical activities. In the case of a physical tool such as a measuring strip (a strip of paper of specified length), the tool was used as a measuring tool in one practice and then used as a thinking device in another practice (Cobb et al, 2001).

Analogous Models

Another type of cognitive tool used in school mathematics is the analogous model. These types of models afford similar features to an object or situation familiar to students. Analogous models can be verbal descriptions or visual representations such as diagrams. For example, an analogous model of an overdraft can be used as a tool for thinking about addition and subtraction of negative integers. Activities related to "directed number becomes meaningfully situated in the practice of school mathematics" (Mousley, 1999, p. 11). The inclusion of stories, metaphors or real life examples are examples of using analogous models as tools that afford illustrations and connections (Boaler, 1993a, 1993b). School mathematics uses analogous models to make links between everyday experiences and mathematical entities, in an effort to gradually formalise learning (Ministry of Education, 1992).

2.5.3 Tools in Classroom Mathematical Activity

Activity can be re-organised and re-structured by tools that "fundamentally change the nature of the task and the requirements of the task" (Goos et al, 1999, p. 38). Classroom mathematical activity that involves the use of physical or cognitive tools engages participants in qualitatively different activity than that without these tools. "The child's appropriation of culturally devised 'tools' comes about through involvement in culturally organised activities in which the tool plays a role" (Newman, Griffin and Cole, 1989, p. 63). The role of the tool

affects the activity and implicitly provides information about the tool (Salomon, 1993). Information about the tool within the context of an activity is significant. Hiebert (1990) emphasised that active student reflection about tools leads to further knowledge of mathematics. For example, reflection on tools such as symbols and procedures can reveal mathematical patterns to the learner. Recognizing patterns is one way of building mathematical relationships

> that reveal the structure and consistency and elegance of the systems. Performing routine procedures provides an entrance into this world of patterns, but only if the goal is to reflect on the procedures rather than to practice them. (Hiebert, 1990, p. 38)

He specifically highlighted the "algorithm's potential to reveal patterns in the system" (p. 38) and that student reflection on the use of the tool helped students to make sense of both the algorithm and the number system. This links with Salomon's (1993) model of the reflexive relationship between tools and the human operator. Salomon suggested that activity occurs in a

> spiral-like fashion whereby individual inputs, through their collaborative activity, affect the nature of the joint distributed system, which, in turn, affects their cognitions such that their subsequent participation is altered (Salomon, 1993, p. 122).

By being used in a particular context, the tool provides feedback about affordances and constraints. The operator's recognition and response to this feedback influences the next interaction with the tool, and so on (Salomon, 1993). Consequently, mathematical activity can lead to learning about tools (Abreu, 2000; Cobb, 1995a; Hershkowitz and Schwarz, 1999).

By involvement in social practices, a person becomes more accustomed to particular tools in different contexts (Kozulin, 1998). "Tools characteristically play a double role: as means to act upon the world and as cognitive scaffolds that facilitate such action" (Salomon and Perkins, 1998, p. 11). This is illustrated by findings from the emergent research projects. The use of tools as a cognitive scaffold was one aspect of the Candy Factory and Candy Shop projects (Bowers et al, 1999; Cobb, Gravemeijer et al, 1997; Cobb, Yackel and Wood, 1992a). In developing and establishing a sequence of classroom mathematical practices, a variety of tools were used for different tasks. Physical tools such as Unifix blocks were initially used to represent various counting

arrangements for the candies (Cobb, Yackel and Wood, 1992a). Drawings and diagrams of rolls and candies were then incorporated as a means of representing the arrangements in hundreds, tens and ones (Cobb, Gravemeijer et al, 1997; Cobb, Yackel and Wood, 1992a). Number lines and empty number lines were used for operations on the quantities of candies (Cobb, Gravemeijer et al, 1997). In one of the projects, a computer platform was introduced as a tool for generating representations of the candies and for subsequent reasoning with these representations (Bowers et al, 1999). The sequence of tools and tasks was carefully planned and structured as part of the instructional design of the research. The tools were assumed to offer both instructional transparency and models of mathematical entities (Cobb, Gravemeijer et al, 1997). Tools such as the Unifix blocks afforded transparency with the place value system. Drawings were tools that were congruent with aspects of some of the tasks, and this was a significant affordance.

> Thus, in line with current theories of distributed cognition, we observe that the presence of the pictures might have profoundly influenced the nature of the children's mathematical activities. ... They could reason with the pictures in this way because the pictured rolls and collection of candies signified numerical composites of 10 for them. (Cobb, Gravemeijer et al, 1997, p. 190)

Within the sequence of instruction, different tools were used as representations and as thinking devices. These tools were scaffolded for increasingly more demanding cognitive tasks and transformed subsequent activity.

Another example of tools as a cognitive scaffold is the idea of 'model of/model for' that has emerged from the Realistic Mathematics Education project (Gravemeijer, 1994). During a study of measurement in a first grade classroom, the children used "particular ways of symbolizing at certain points in the instructional sequence" (Gravemeijer et al, 1998, p. 196). When children first modelled a situation in an informal way, the model (tool) was a *model of* the situation. When the children formalised their reasoning processes, the tool was then used as a *model for* mathematical reasoning. A notable example of this *model of/model for* approach was the use of the empty number line (Beishuizen, 1993; Gravemeijer, 1994) as initially a model of the children's informal arithmetical reasoning that "will subsequently become a model for increasingly

abstract yet personally meaningful arithmetical activity" (Gravemeijer et al, 1998, p. 197). Tools such as rulers, number lines and empty number lines have been identified as useful for both representing quantities and modelling a process. In other words, the tool was used as both a representation and a thinking device. The tools as representations

> created the opportunity for the students to discuss mathematically relevant issues that otherwise would not have become a topic of discussion. In addition to this, the development of mathematical meaning that resulted from such discussions laid the basis for the introduction of more sophisticated ways of symbolizing. (Gravemeijer et al, 1998, p. 208)

When the same tool is used in various ways in different situations, students are more likely to talk about the tool within the contexts of the activity. This affords opportunities for students to describe the use of the tools, and also affords further feedback about the tool. The reflexive relationship of the model of/model for approach is another illustration of the reflexive feedback spiral between tool and student (Salomon, 1993).

Not all activity with tools affords meaningful feedback to students. Research of four second grade children's use of a physical tool, the hundreds board, found that this tool constrained particular activity (Cobb, 1995a). By examining the children's mathematical activity, in particular strategies for counting, Cobb concluded that the hundreds board was not always a carrier of meaning for the children. For one child,

> the numerical meanings she established were interwoven with her use of a cultural tool, place-value notation. The relationship between symbol and conception appears to be reflexive in that the numeral patterns both constrained and enabled her numerical interpretation, and her numerical understanding made the numeral patterns significant when she used the hundreds board to solve tasks (Cobb, 1995a, p. 382).

An important implication was the nested and interrelated analysis of two tools, place value notation and the hundreds board, in the context of the activity. When tools are used in classroom events, they are not free from, or neutral of, cultural values and status (Kozulin, 1998). Knowing about tools is related to what counts as legitimate knowledge in a particular classroom context, as well as to what is valued in a general cultural sense. Abreu observed that an "understanding of the social value of the tool has an impact on the individual",

(Abreu, 2000, p.21). In her research, she found that the same tool, mental arithmetic, had different social markers in two different contexts, home and school. In addition, these social markers had a significant effect on the individual's use of the tool in activity, resulting in qualitatively different activity in the two contexts.

2.5.4 Special Case of Technological Tools

The potential for technological tools to both transform the mathematical experiences for learners and to amplify mathematical thinking has been documented over a long period (Papert, 1980, 1993). In fact, many writers (Groves and Cheeseman, 1995; Kaput, 2001; Noss, 1998; Papert, 1980) have argued that technology has the capacity to not only transform the mathematics curriculum but to challenge teacher expectations and understandings of how students learn mathematics. Technological tools are important for this study and are defined as a special case of a physical tool. Technological tools have also been described as an intelligent technology (Salomon et al, 1991), suggesting that these tools are qualitatively different to other technologies such as pens, whiteboards etc. In mathematics education, technological tools usually refer to calculators and computer software (Arnold, 1998). In New Zealand schools, calculators (simple calculators, scientific calculators and graphical calculators) are more widely used for mathematics than computers. Two research studies have particular relevance for considering calculators as tools. Although based in primary schools, using simple calculators, I argue that the findings are relevant for scientific calculators used in secondary school mathematics lessons.

Early debates about the use of calculators revolved around the issue of the calculator taking over the role of the human brain and by implication, producing less mathematical thinking (McChesney, 1995a; Shuard, 1992). This offloading by the calculator of mental mathematical effort was considered a bad thing for primary school children. In other words, this was the mental corollary of wasted or flabby muscles when a (physical) tool takes over from human effort. The CAN project (Calculator Aware Number curriculum) in England and Wales, introduced simple calculators to six year old children (Ruthven,

1999; Shuard, 1992). The findings of this five year project suggested that children were able to explore more complex tasks using authentic contextual quantities, such as large numbers, and could generate or invent their own cognitive tools such as algorithms for calculation (Ruthven, 1999; Shuard et al, 1991). The calculator was found to be an auxiliary tool that could enhance and amplify children's mathematical activity.

The second significant research project was the Calculators in Primary Mathematics project in Melbourne, Australia (Groves, 1991; Groves and Cheeseman, 1995; Groves, Cheeseman, Allan and Williams, 1992). One of the outcomes of this project was the identification of different roles of the calculator during classroom mathematical activity. The first role identified was that of a recording device or scratchpad, enabling children to generate representations of numbers on the calculator screen. The built-in counting function of the simple calculator generated any counting sequence, providing the calculator with a role as a counting machine. This capacity of the calculator was considered to be different to the capacity of a calculator as a machine for calculating. Another role of the calculator identified in the research was as a number cruncher, a machine capable of operating on a range of numerical data. The final role of the calculator was as a machine to explore in its own right, as an exploratory object. The children in the research classrooms were observed to play and experiment with the functions of the calculator. This in turn developed into significant mathematical activity as a result of independent exploration of the calculator as a technological object. These five roles of a simple calculator are considered relevant for this study: as a recording device, a counting machine, a calculating machine, a number cruncher, and an exploratory object.

The general roles of cultural tools can be applied to these roles of the simple calculator tools. Using the calculator as a recording device offloaded mental effort and saved time, particularly for younger children still coming to grips with writing numerals on paper. The roles of counting machine, calculating machine and number cruncher were examples of the external processing capacity of these kinds of technology (Shaffer and Kaput, 1999). "Viewed in the long tradition of tool making, calculators represent a powerful alternative to

the drudgery and inefficiency of paper-and-pencil arithmetical computation" (Wheatley and Shumway, 1992, p. 2). In addition to offloading mental effort, saving time and increasing precision, the calculator restructured and transformed the activity for the child by producing new quantitative information and making this available to the child. As a machine to explore in its own right, the calculator offered unique opportunities for children to receive rapid feedback about their methods, to facilitate trial and improve methods, and to offer alternative equivalent calculation processes, using different operation keys (Arnold, 1998).

Further research has found that calculators afford classroom mathematical activity (Ruthven, 1999). For example, calculators can offload complex calculations and re-structure tasks so learners can focus on other aspects of mathematics. "Computational tools can free students to reason mathematically without being sidetracked by carrying out time-consuming paper-and-pencil algorithms" (Wheatley and Shumway, 1992, p. 5). When scientific calculators were introduced to Grade 7 and 8 students in the United States (Hirschhorn and Senk, 1992), the students were allowed to use the calculators freely. The students learned to use the calculator more effectively and developed confidence in the use of number and calculation. In addition, "students expressed curiosity about the keys that they did not use in the course, such as those pertaining to logarithms or trigonometric" keys (Hirschhorn and Senk, 1992, p. 85). Students learned more about the capabilities of the calculator, and also about other mathematical tools such as numbers and calculation processes.

Some of the properties of the calculator can constrain mathematical activity, particularly for novice learners of mathematics. Anecdotal evidence suggests that the writing of decimal numbers on the calculator screen can be problematic. For example, many calculators remove the zeroes when numbers such as 2.0 or 5.00 have been entered. When the context is money or some other form of measurement, then the calculator screen does not correspond directly with the real life representations of these kinds of numbers. Human operators of the calculator can make keying in errors, constraining the role as a calculating machine or number cruncher. Technological tools are, however,

powerful learning tools and often under-utilised in school mathematics (McChesney, 1995a; Noss, 1998). This study aims to find out more about the role of calculators in Year 9 and 10 mathematics lessons.

2.5.5 Summary of Tools

To summarise, the cognitive tools used in classroom activity are assumed to be diverse but can be grouped into representations, thinking devices or analogous models. The roles of tools are to offload mental effort, save time, increase precision, re-organise resources and restructure activity. Tools also afford or constrain mathematical activity. For example, by affording mathematical information, this information can be used by students as resources for further activity. Tools can also afford further activity by being objects of communication. In this study, I propose to investigate the role of tools in classroom activity. The tools domain is related to the other domains of social norms and interaction, as shown in the diagram below.

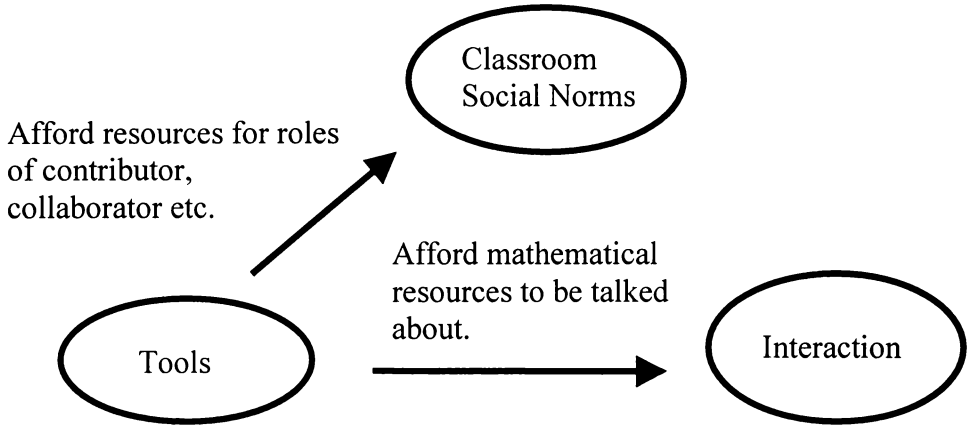

Figure 2.4 The Influence of Tools on the Domains of Social Norms and Interaction.

Cognition is distributed over tools, the human participants and other resources such as talk. Mathematical tools are one kind of resource that mediates classroom mathematical activity. Talk is another resource, arising from interactions between classroom participants. Classroom interactions provide opportunities to describe the tools used and the roles played by the tools.

2.6 Interaction

Interaction is the third domain of interest for this study. Research about classroom interaction processes is now outlined using a variety of classroom contexts. A particular focus is research about the nature of teacher and student contributions to classroom interactions.

2.6.1 Whole Class Interaction in Traditional Classrooms

There have been many research studies of whole class interactions in primary and secondary school classrooms (Edwards and Mercer, 1987; Edwards and Westgate, 1987; Kumpulainen and Wray, 2002). Some studies of verbal communication in school classrooms have focussed on teacher and student questions during whole class discussion (Cazden, 1988; Hogan, Nastasi and Pressley, 1999; Mehan, 1979; Mercer, 1995, 2002a; Mousley and Sullivan, 1995). When questions were analysed, it was found that "teachers, not students, ask questions: right answers are praised, wrong are ignored, with very little probing of why they were given" (Mousley and Marks, 1991, p. 38). Research has found a predominant pattern of questioning that has become known as the initiation-response-evaluation (IRE) sequence (sometimes known as IRF initiation-response-feedback) (Cazden, 1988; Mehan, 1979). This pattern of questioning was characterised by a teacher question (initiation), a student response to the question, followed by teacher evaluation (feedback) of the student response. Within the IRE sequence, the teacher controlled both "the development of the topic (and what counts as relevant to it) and who gets a turn to talk" (Cazden, 1988, p. 30). The student's role was to follow this questioning ritual rather than contribute to the development of the topic.

There are differing views about the value of this type of ritualised interaction pattern in the classroom (Edwards and Mercer, 1987; Edwards and Westgate, 1987; Mercer, 1995, 2002a, 2002b). One interpretation is that it provides a known stable pattern for public classroom interactions, affording opportunities for student contributions, and for teachers to formatively assess student understanding (Mercer, 1995). More often this IRE pattern of questioning has been interpreted as

constraining student contributions (Cazden, 1988; Mehan, 1979). IRE was an indication of a teacher dominated classroom, where teachers controlled conversation in significantly different ways to normal conversations (Mehan, 1979). The student responses were likely to be in the form of atomised skills or facts, indicating an outcome or skill based curriculum (Edwards and Mercer, 1987; Edwards and Westgate, 1987). The teacher was assumed to be simply assessing the validity of the student response in terms of a preconceived correct answer. Cazden noted that in a primary school context, "the children get the teacher to give them the cues they need to find the answer. ... the teacher was providing implicit information about how to answer such questions" (Cazden, 1988, p. 304). This ritualised format was so embedded that students expected the teacher to evaluate each student response. When teachers didn't, or asked the questions again of the same student, the implication was that the answer must have been incorrect (Chazan and Ball, 1999). The IRE pattern of questioning has also been considered a cultural mismatch in New Zealand and other places, because it controlled and inhibited participation in ways that were counter to students' home experience (Bishop and Glynn, 1999; Cazden, 1988).

Questioning patterns are not the only ways that teachers dominate whole class interaction. When one voice dominates, the interaction can be described as univocal (Cobb, 1995b; Knuth and Peressini, 2001; Wertsch, 1991). And in traditional classrooms, the voice that dominates is the teacher's. A univocal approach can be explicit, where teacher contributions to whole class interactions are instructions to be followed. This can occur in mathematics lessons when the teacher leads the class through the steps of a procedure, and the students are expected to imitate these steps (Bauersfeld, 1988; Cobb and Bauersfeld, 1995). The teacher might also ask a series of leading questions that focus on a specific predetermined end. This has been called 'funnelling' and interpreted as an unhelpful process (Bauersfeld, 1988), similar to 'guess what's in the teacher's head' (Mason, 2002). Univocal interactions can also be implicit, for example when teachers invoke a non-standard use of first person plural, "we", when talking about their own preferred calculation methods (Gerofsky, 1999; Rowland, 1999; Senn-Fennell, 1995). Tag and rhetorical questions can also be indicators of an implicit univocal approach as can a cloze

technique where students complete the teacher's sentence (Mason, 2002). Middle class teachers have been found to use questions as directives (Delpit, 1995; Zevenbergen and Lerman, 2001). These interaction patterns are examples of teacher domination of classroom conversations where teachers control student contributions (Hendry, 1996; Mousley and Sullivan, 1995).

2.6.2 Alternative Interpretations of Teacher Contributions

Some interpretations of the IRE questioning sequence have challenged the view that IRE is an educationally low-level interaction. Mehan (1979) claimed that students were active participants in this ritual. Mercer (1995) viewed the teacher responses as not simply evaluative feedback to student responses, but as contributions towards potential scaffolding of student thinking. Scaffolding referred to teachers organising dialogue using an appropriate structure that matched with students' knowledge (Bruner, 1986). The metaphor of scaffolding implied the supporting and sustaining of student contributions. As an interactive process, scaffolding involved a variety of teacher responses depending on the nature and context of each student contribution. Research findings suggest that certain teacher responses to student contributions constituted scaffolding actions. These actions included repeating student responses for emphasis (Edwards and Mercer, 1987), and rephrasing or paraphrasing student responses. Teachers asked students to explain further, or asked other students whether they agreed or disagreed with a contribution (Edwards and Mercer, 1987). A two year study of four secondary school mathematics classrooms in Australia (Goos et al, 1999), analysed ways that mathematical thinking developed through teacher scaffolding of inquiry processes. Teachers called on students to clarify, elaborate, critique and justify their assertions. The teachers also structured student thinking by asking questions that led them through strategic steps. The interweaving of familiar and formal knowledge was found to help students adopt the conventions of mathematical communication.

Teachers also appropriated student contributions by using student ideas and language in the subsequent dialogue. By emphasising some contributions and ignoring others, teachers selected key terms or phrases as signals or cues for the

students (Brown, 2001). These cues, gestures and other signs were used by the teacher to convey meanings valued by the curriculum (Mercer, 2002a). In a study of a Year 6 classroom mathematics programme based on an argumentation approach (Brown, 2001), whole class discussion was viewed as managing the reporting process. During this process, "the teacher may rephrase, paraphrase, and re-present the contributions of particular groups and draw connections between the language used in presentations and that used by mathematicians" (Brown, 2001, p. 108). Edwards and Mercer (1987) identified reconstructive recapitulation as another means of scaffolding. Reconstructive recapitulation involved the following: repeating, paraphrasing and recasting of student contributions in more acceptable form, eliminating errors, changing terminology, and adding information. Teachers contribute further by 'overhearing' student contributions, that is, by filling in around the edges of what students say (Ball, 1997). Overhearing also included teachers returning issues from previous lessons to the present discussion, thereby contributing "historical voices" of the classroom community (Chazan and Ball, 1999, p. 7).

Similarly, scaffolding actions included the idea of calibration (attributed to Bruner) that progressively linked student language with the appropriate conventional terms and language for the subject area (Cooper and McIntyre, 1996). In scaffolded interactions, teachers and students gradually interchanged the use of informal terms known as student register, with more formal terms of the mathematical register (Adler, 1998; Mousley and Marks, 1991). This is linked to code-switching (Adler, 1999), where bilingual students moved between languages or dialects at various stages of the interaction process. In the study of an inquiry approach in four classrooms, students found it helpful when the teacher made explicit reference to mathematical language, conventions and symbolism (Goos et al, 1999). When the teacher linked technical terms to commonsense meanings, and used multiple representations of new terms and concepts, this was found to afford communication in the classroom. In these ways, scaffolding actions of teachers supported and sustained student contributions while at the same time providing messages about the curricular status of contributions.

Teacher funnelling actions have likewise been interpreted in different ways. The emergent approach claimed that, during the funnelling of the dialogue, the roles of the teachers were significantly different to a traditional classroom (Cobb, Boufi, McClain and Whitenack, 1997; Cobb, Wood et al, 1992; Wood, 1995, 2002). In emergent classrooms, the teacher might begin whole class interaction by asking a question in order to encourage genuine dialogue. In a first grade lesson based on 'the monkeys in the trees' problem, the teacher contributed in particular ways that were described as funnelling the dialogue (Cobb, Boufi et al, 1997). By repeating certain words or phrases, and ignoring others, the teacher signalled the status of the mathematical thinking. By this process, the "teacher gave a commentary from the perspective of one who could judge which aspects of the children's activity might be mathematically significant" (Cobb, Boufi et al, 1997, p. 262). The teacher also drew a table on the board to record children's contributions of the possible ways the five monkeys could be arranged in the two trees. This table organised the student contributions, and provided a public and stable record of the student solutions. This cognitive tool was then used to funnel the discussion towards the students' solutions, and afforded the student solutions as the next topic of conversation. Another way to provide focus was when the teacher asked specific questions at particular points in order "to narrow the joint field of attention" (Wood, 1995, p. 219). Consequently, funnelling was a focused kind of scaffolding, where the teacher filtered out 'noise' in the dialogue, providing a more fine-grained focus on particular mathematical meanings.

A further contradictory finding about whole class interaction is from the New Zealand AIMHI study. The use of the word 'we' by the teacher, was interpreted as an indicator of a classroom community rather than of teacher domination. Many teachers

> used 'we' when they were talking about activities the students were doing. In participating, they were modelling a willingness, not only to be part of the class, but also indicating a model of classroom leadership based on power with, rather than over, the students (Hill and Hawk, 2000, p. 4).

Similarly Weade (1995) suggested that the teacher use of 'we' was an indicator of insider knowledge and of the classroom as a distinct community.

Scaffolding and funnelling are useful metaphors for the ways that teachers shape classroom interactions. By responding to student contributions in particular ways, teacher contributions influence the direction of the interaction. Teachers can also shape how the content of the discussion progresses over time (Bauersfeld, 1995). Research from the emergent approach has highlighted the importance of shifts in the discourse to the long-term progress of classroom discussions. Teacher contributions, such as a question or a recording of student responses, can restructure and initiate a shift in the discourse (Cobb, Boufi et al, 1997; Wood, 1995). Examples of teacher-initiated shifts in the discourse were the teacher asking a student to provide a reason, or asking how the class could be sure they had all possible solutions (Wood, 1995). Some aspects of funnelling whole class interaction could also involve shifting the discourse. This occurred in the 'monkeys and the trees' lesson (Cobb, Boufi et al, 1997). After the students had contributed their solutions, the teacher shifted the discourse to considering the student contributions. The teacher's role was then "to guide and, as necessary, initiate shifts in the discourse such that what was previously done in action can become an explicit topic of conversation" (Cobb, Boufi et al, 1997, p. 269). Resources jointly contributed could then be discussed, and further resources contributed. The emergent approach called this process reflective discourse, a significant means of negotiating meaning in communal discussions (Cobb, 1998).

By shifting the discourse, the teacher takes responsibility for shaping the mathematical direction of public interactions. "In seeking to modulate the focus, direction and nature of the discussion productively, teachers must have a repertoire of ways to add, stir, slow, direct the class's work" (Chazan and Ball, 1999, p. 9). This affords opportunities to introduce new ideas, to add further resources such as cognitive tools, to seek clarification at different times, and to adjust or extend the task. In order to do this, teachers draw on their knowledge-in-action (Shulman, 1986) where "teacher moves are selected and invented in response to the situation at hand, to the particulars of the child group or class and to the needs" of the curriculum (Chazan and Ball, 1999, p. 7). Teachers' contextualised knowledge of their students, their pedagogical content knowledge, and their knowledge of school mathematics, all combine in teacher

contributions to whole class interactions. The teacher has responsibility for shaping these interactions and "the teacher is both value-carrier and a value-mediator" (Bishop, 1991, p. 207). The emergent perspective claimed that not just the teacher, but also the students, actively contributed to shifts in the discourse. In this way, whole class discussions were considered to be interactive accomplishments (Cobb, Boufi et al, 1997).

I have shown that research findings about whole class interactions are complex and a little contradictory. Teacher contributions can be viewed as controlling and inflexible. On the other hand, teacher contributions can be viewed as supportive of, and responsive to, student contributions.

2.6.3 Student Contributions

Research about student contributions to whole class interaction is similarly complex. Whole class interactions afford opportunities for students to articulate their ideas, and in the process "they revise their thinking and construct new insights" (Chazan and Ball, 1999, p. 7). It is not always easy for teachers to understand what students mean from their verbal contributions. Students "use nonstandard terms, draw pictures, and make analogies, they get apparently right answers for unsound reasons, and apparently incorrect answers for sound reasons" (Ball, 1997, p. 735). Students often have tacit knowledge rather than descriptive knowledge. A study of a third grade class found that students needed skills in talking about their mathematics. They needed,

> to know both the ideas they intend to communicate and the audience with whom they will communicate. They have to reflect on their problem solving processes and decide how to present their methods based on their anticipation of the audience's expectations. (Lo and Wheatley, 1994, p. 153)

These findings were related to classroom social norms (Yackel, 1993) but also depended on opportunities for students to learn how to communicate mathematically. Skills such as explaining, clarifying, refuting and revising ideas were considered to be important for supporting student mathematical contributions (Lo and Wheatley, 1994). Similarly, Brown (2001) found that the expectation of students to explain their solutions was problematic. The student

contributions were more likely to be descriptions of a procedure rather than explanations of their thinking, and some students resisted discussing their reasoning, preferring instead to articulate their methods. Student contributions can also be constrained when "they get stuck and are not sure what they think; they disagree with one another and with their teacher" (Chazan and Ball, 1999, p. 7). Not all student contributions are necessarily 'public'. In a primary school classroom in New Zealand, Alton-Lee and Nuthall (1992) found ten instances of a student talking to himself during 37 minutes of teacher-directed time. Some students, however, had virtually no instances of talking to themselves.

In a sixth grade mathematics classroom of thirty students, the teacher could not assume that all students had discursive skills for engagement in whole class interactions (Lubienski, 2000). Higher socio-economic status (SES) students were more comfortable with discussion, happy with ideas being debated and were prepared to defend and justify their ideas when challenged. More of the high SES understood the role of the teacher in discussions, whereas low SES students preferred specific direction from the teacher.

> In general, lower-SES students focused more on giving the right answer to a question I asked about a previously assigned problem, whereas higher-SES students were more inclined to discuss a method or an idea. (Lubienski, 2000, p. 396)

A variety of opinions confused the low SES students but high SES students felt they could sort out which ideas were sensible and which were not. The teacher-researcher reported that "lower-SES students also appeared less confident, more often backing down when their contributions were questioned or challenged by me or another classmate" (Lubienski, 2000, p. 396). A social class perspective suggested that middle class students were more used to questions and being guided by the teacher. Working class students were found to be more receptive to being told or shown by the teacher. These findings were similar to the study discussed earlier, of an upper and a lower stream class in a New Zealand secondary school (Jones, 1991).

Responses to Peer Contributions

Peer reactions to student contributions can also be problematic and sometimes result in undesirable consequences (Lampert, Rittenhouse and Crumbaugh,

1996). In a classroom based on the emergent approach, there was a range of outcomes when students did not accept peer contributions. Sometimes this could result in a fruitful whole class discussion, but at other times it could turn into a disruptive argument. "Negative emotions such as anger, confusion, and frustration sometimes occurred during such events" (Lo and Wheatley, 1994, p. 151). This was also a dilemma in a study of classroom discussions with older students (Chazan and Ball, 1999). These studies highlight concerns about the potential for arguments between students during whole class interactions and the possible negative social consequences, irrespective of the ages of the students. Peer responses, however, do not always constrain public interactions. Students have been found to prefer peer models to teacher models because it was easier to reach shared understandings (Cooper and McIntyre, 1996). Students' informal code "has a greater power to communicate their intended meaning" (Zazkis, 2000, p. 39). When peers verbally transformed teacher contributions, more accessible resources were afforded to the discussion. These were resources such as student register, familiar cognitive tools or shared prior experiences.

Student Contributions to Small Group Activity

Interactive processes can take on different qualities in small group activity. Rather than being in the public domain, small group activity is considered as localised semi-public activity (Nuthall, 2000). Much small group activity is independent of the teacher, although the teacher may have fleeting or sustained engagement with the members of the group. Research has investigated different kinds of student talk such as social and task related talk (Barnes, 1999; Edwards and Mercer, 1987; Thomas, 1994). Task related talk has been analysed according to the purpose of the talk and found that "students working together make differing interpretations and yet each assumes that the others share his or her personal interpretation" (Cobb, Yackel and Wood, 1992a, p. 18). In localised activity, students make extended contributions, expand on each other's comments, are willing to express uncertainty, and to offer explanations (Mercer, 2002a). These kinds of contributions afford more opportunities for students to communicate mathematical thinking at different stages of the task.

Student contributions also afford further mathematical resources, often expressed in ways that are compatible for peers.

2.6.4 Summary of Interaction

Interaction is the third domain of interest for this study. Classroom interactions occur during public lesson events, such as exposition or whole class discussion, and also in localised situations during small group, paired or individual activity. There is a substantial body of research about classroom interaction in a range of subjects including mathematics. While the importance of talk in mathematics classrooms is acknowledged (Ministry of Education, 1992), it is the qualities and purposes of talk that are significant. Dialogue is an interactive process, "so that teachers better understand the children's thinking and children gain access to the teacher's mathematical knowledge" (Askew, 2001, p. 114). Language is a social mode of thinking (Bruner, 1986; Gee, 1992; Mercer, 1995; Wells, 1999). Conversational processes such as student description and explanation afford access to more resources. These resources include mathematical tools, student register, and other communication resources. Each classroom context affords and constrains interaction between teacher and student, and between students. The ways that the interaction domain is related to the social norms and tools domains is illustrated in the diagram below.

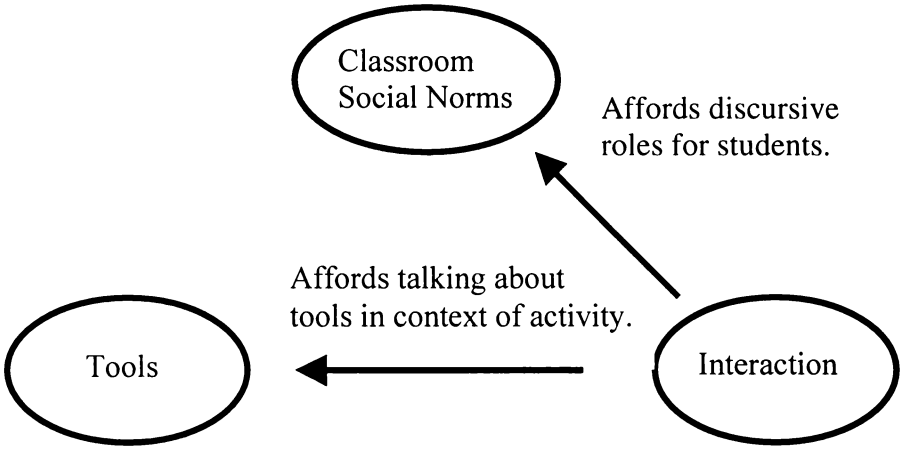

Figure 2.5 The Influence of Interaction on the Domains of Social Norms and Tools.

Interaction processes can both afford and constrain communication. From a teacher's perspective, "managing the differences among ideas in a discussion is one of the crucial challenges for teachers who seek to teach through student exploration and discussion" (Chazan and Ball, 1999, p. 7). Whether in experimental or traditional classrooms, scaffolding, funnelling and shifts in the discourse are mostly initiated by the teacher. By shaping classroom interactions, teachers also convey messages about mathematical values, about what mathematics has status, and implicit assumptions about how students learn (Zevenbergen, 2000). This study will focus on one aspect of school mathematical learning, number sense, and this is discussed in the next section.

2.7 Number Sense

This study has a focus on three domains of the classroom that contribute to learning mathematics: classroom social norms, tools and interaction. In order to provide more focus, the mathematical field is narrowed to number sense and, in particular, to ways students develop number sense in classroom situations. Various descriptions of number sense are now discussed and a key metaphor for number sense is outlined. This environmental metaphor is adapted to encompass a cognitive tools perspective. I argue that the adapted metaphor accounts for key aspects of classroom mathematical negotiation: the roles of teacher and peers, diversity of mathematical achievement, errors, and the impact of technological tools.

2.7.1 Introduction to Number Sense

Number sense has been investigated over the past fifteen years and some of the early international work provided descriptions of number sense. Trafton (1992) suggested that one way of thinking about number sense was to liken it to road sense. A person with good road sense had an integrated mental picture of how various roads were connected, where they led, what they were like, how traffic behaved on them, and how they might be negotiated. In his view,

> A person who possesses number sense might be said to have a well-integrated mental map of a portion of the world of numbers and

> operations, and is able to move flexibly and intuitively throughout the territory (Trafton, 1992, p. 79).

Number sense was being able to make connections and being adaptable to different number situations (McIntosh, 1990; Verschaffel and De Corte, 1996).

Sowder (1992a) provided another description of number sense, expressed as abilities that demonstrated number sense. These included the ability to:

1. compose and decompose numbers; to move flexibly among different representations; to recognise when one representation is more useful than another.
2. recognise the relative magnitude of numbers.
3. recognise the absolute magnitude of numbers.
4. use benchmarks.
5. link numeration, operation and relation symbols in meaningful ways.
6. understand the effects of operations on numbers.
7. perform mental computation through "invented" strategies that take advantage of numerical and operational properties.
8. use numbers flexibly to estimate numerical answers to computations and to recognise when an estimate is appropriate.

(Sowder, 1992a, pp.18-19)

Sowder's number sense description emphasised making connections and being flexible, and acknowledged the importance of patterns and relationships in the number system. In other words, these number sense abilities derived from affordances of the number system such as multiple representations, benchmark numbers and interrelationships between operations. These affordances were relevant to mental calculation, a topic that has recently received more attention in schools (McIntosh, Nohda, Reys and Reys, 1995; Silver, 1994; Sowder, 1988, 1994; Sowder and Kelin, 1993; Trafton, 1994). The ability to 'decompose and recompose' numbers was implicit in mental calculation strategies that relied on representing numbers in multiple ways. Different representations were useful for diverse calculation contexts. The ability to estimate depended on an interconnected knowledge of numbers and operations, and estimation developed alongside other aspects of number learning (Sowder, 1992b). These number sense abilities were portrayed as desirable outcomes by Sowder (1992a), in contrast to curricula based on number knowledge and competency with calculation processes. Student mathematical performance, however, could still be described as deficient in any of the number sense abilities. The number

sense abilities were also general statements with insufficient detail for implementation into school mathematics programmes.

More specific detail was provided by a framework for number sense developed by McIntosh et al (1992). This framework for "examining basic number sense" used three categories: knowledge of and facility with numbers; knowledge of and facility with operations; and applying knowledge of and facility with numbers and operations to computational settings (p. 4). The first two categories were then expanded into specific detail about number knowledge. The third category expanded the focus on active reasoning about numbers and computation. While the extra detail of this framework supported curriculum implementation, it also appeared somewhat prescriptive and similar to existing programmes.

Further international number sense developments included comparative studies (McIntosh et al, 1995; McIntosh, Reys, Reys, Bana and Farrell, 1997), classroom research (Markovits and Sowder, 1994), publication of student materials (McIntosh, Reys and Reys, 1997; McIntosh, Reys, Reys and Hope, 1997), and teacher professional development (Flores, Sowder, Philipp and Schappelle, 1995; McIntosh and Dole, 2000; Sowder, 1995). While this activity was taking place overseas, in New Zealand number sense was relatively slow to catch on as a philosophical approach to the teaching and learning of number (McChesney and Biddulph, 1994).

2.7.2 Number Sense in New Zealand

Number sense appeared for the first time in the current New Zealand mathematics curriculum (Ministry of Education, 1992). It stated in a 'Suggested Learning Experience' at Level 1, that students should be: "developing a number sense by exploring number in the contexts of their own experiences and the world around them; ... by exploring estimation and computation, ..." (p. 33). At Level 5, the curriculum level for the first two years of secondary schools, the curriculum stated that students should be:

> developing their number sense by exploring number in the context of the world around them; and
> developing their number and computation sense by exploring estimation and computation in the context of their everyday lives; (p. 49)

The terms number sense or computation sense, however, were not explained or discussed in the curriculum and in later supplementary teacher materials (Ministry of Education, 1992, 1995, 1996, 1997). The mathematics curriculum also asserted that students needed to become numerate because

> to be able to calculate, estimate, and use measuring instruments, has always been identified as a key outcome for education. Mathematics education aims to contribute to the development of the broad range of numeracy skills. (Ministry of Education, 1992, p. 7)

This linked with the *New Zealand Curriculum Framework* through the eight sets of essential skills, one being numeracy skills (Ministry of Education, 1993).

In recent years the Numeracy projects, and the Numeracy Framework have used a particular definition of numeracy, "To be numerate is to have the ability and inclination to use mathematics effectively – at home, at work and in the community" (Ministry of Education, 2002, p. 6). This definition of numeracy now encompasses mathematics used in everyday life. By being so general, numeracy is now rather meaningless and lacks a focus on patterns and relationships of the number system. Furthermore the Number Framework makes a distinction between strategy and knowledge (Ministry of Education, 2002). This implies an individual view of mathematical learning, where knowledge is transferred from one situation to another. This is contrary to the focus of the preceding descriptions of number sense that emphasise a more integrated approach to mathematical thinking. During the time of this study, I have become increasingly concerned about the prescriptive nature of the Numeracy Framework and Numeracy Projects. Descriptions or definitions of number sense are also inadequate. Instead I sought a means of theorising about distributed activity in the domain of number. In order to find some further direction about number sense, I have returned to an earlier theoretical consideration of number sense.

2.7.3 A Metaphor for Number Sense

Greeno (1991) developed a theoretical basis for number sense by using a metaphor of an environment. Number sense is an "example of knowing in a conceptual domain, the domain of numbers and quantities" (Greeno, 1991, p.170). According to Greeno, the domain of numbers and quantities includes various resources such as number representations and operations. Resources with similar features are grouped together in the locations of the environment. Number sense involves:

> knowing what resources are in the environment
> knowing how to find resources
> knowing which resources are located together
> knowing the properties of resources
> knowing how to use these resources in activity (based on Greeno, 1991, p.170).

By using the metaphor of an environment, Greeno accounted for the complex ways we use number and computation. Number sense results from a person's "ability to find and use the resources" (Greeno, 1991, p. 175). The mathematical resources of the environment include the different kinds of numbers: for example, decimals and fractions, and operations on numbers and on quantities. These resources can be selected or adapted, and then used in activity. Greeno also stated that resources in the environment could be combined together or decomposed into other resources, which can then be used in further activity. This is similar to Sowder's notion that number sense involves the ability to take numbers apart and put them together, and to use different representations in different contexts (Sowder, 1992a). The environment metaphor accounted for the number sense ability to decompose and recompose numbers in meaningful ways for the (quantitative) context.

The environment metaphor has parallels with the three domains of interest for this study. According to Greeno (1991), knowing in a conceptual domain involves participating in conversations. "In a community that uses a conceptual domain, people communicate about the conceptual entities" (Greeno, 1991, p. 183). The domain of social norms includes expectations and obligations about

how and when to contribute, and what kinds of contributions are valued. Communicating about the resources of the domain involves students and teachers describing uses of the resources, or explaining "interesting properties or relations they have heard about or discovered" (Greeno, 1991, p. 183). It also involves asking questions of others and listening to contributions. From Greeno's perspective, communicating about the resources involves the domain of interaction. The conceptual entities and resources relate to the domain of tools. Consequently, the three domains of norms, tools and interaction are encompassed by the environment metaphor. I propose to adapt the metaphor more closely to a tools perspective, and this is discussed in the next section.

2.7.4 The Metaphor Adapted: Number Sense and Tools

When Greeno outlined his metaphor of an environment for number sense, he emphasised knowing about resources in the environment, and knowing how to use these resources. Resources in this sense included talk and other communication resources as well as "objects, materials and tools" (Greeno, 1991, p. 180). If cultural tools, including physical and cognitive tools, are substituted as a more global term for objects, materials and tools, then knowing your way around an environment involves the domain of tools. In other words, number sense is about recognising mathematical tools, knowing the capabilities and effects of the tools, choosing the best tools for the purposes, and knowing how to use the tools. It could also include knowing how to adapt an existing tool for a new situation or purpose, and knowing alternative ways of using a tool in further activity.

Most of the quantitative activity at secondary school involves dealing with representations of quantities rather than with the quantities themselves (Greeno, 1991). In Greeno's terms, mental representations of quantities are important features of a quantitative environment. Representing conceptual entities and recognising conceptual structure in activity situations plays an important role in number sense. He discussed in some detail the importance of constructing and manipulating mental models. One aspect of modelling was using a mental representation of a situation. Knowing in a conceptual domain was "an ability

to construct mental models that conform to the concepts and principles of the domain" (Greeno, 1991, p. 180). Mental representations can be conceptualised as cognitive tools that offload mental effort. For example, different methods for calculating 25 x 48 involved mental models that represented numbers (Greeno, 1991). These mental models constructed mathematical activity rather than being used as aids to illustrate mathematics. Mental models were representations that could be manipulated, for example, combined to form representations of other numbers. According to Greeno (1991), important aspects of number sense were "knowing various ways to make representations of numbers by combining, separating, expanding, and contracting other representations" (p. 186). Mental models can be used as thinking devices, cognitive tools that re-organise and re-restructure mathematical activity. These two roles of mental models, as representations and as thinking devices, are two of the roles of cognitive tools selected for this study.

Although technological tools such as calculators were not explicitly discussed by Greeno, these tools can be incorporated into the metaphor of an environment. Calculators provide additional resources to the environment, in terms of both representations of numbers and of calculations with quantities. Calculators offload mental effort, restructure calculation processes, and provide additional mathematical resources (section 2.5.4). These roles of the calculator generate greater opportunities for students to attend to other processes such as selecting, adapting and manipulating mathematical resources in the environment.

2.7.5 The Metaphor as Distributed Mathematical Activity

The environment metaphor for number sense also has parallels with a distributed perspective for mathematical activity. When proposing his environment metaphor, Greeno contrasted it with an information processing view. Instead of representing concepts and relations within an individual's cognitive structure, Greeno claimed that knowing "involves abilities to find and use the concepts in constructive processes of reasoning" (Greeno, 1991, p. 175). Reasoning with a mental model involves a person placing mental models in the

situation and "manipulating those symbolic objects" (p. 178). Consequently, reasoning is not a solely individual process but is mediated by cognitive tools and by discursive processes. Within a number sense environment, mediated activity involves people, talk, tools and other resources. The metaphor of an environment offers a useful perspective of how activity is distributed over the various elements of mathematical activity.

The properties of tools afford and constrain mathematical activity. According to Greeno (1991), affordances are relational. For example, mental models afford access to other tools such as equivalences, relations and multiple representations. "Finding objects that are the same as other objects or combinations of objects is one of the abilities one would expect of a person who knows the properties of objects in the environment" (Greeno, 1991, p. 193). Knowing the relationships between numbers and recognising tools with similar properties, are important affordances of an environment. By recognising and activating these affordances, a tool can be used in different ways in various task contexts. Number sense is illustrated by flexible calculation, resulting from activating affordances of numbers and operations for the calculation context. Examples of activating affordances are to recognise equivalences, to regroup numbers in mental calculation, and to recognise and use benchmarks. These affordances depend on the task situation, and students activate affordances by "judging and making inferences about quantities with numerical values" (Greeno, 1991, p. 172). Properties of numbers become affordances that can be identified and used when manipulating representations of numbers in different situations.

Mathematical activity is also distributed between people and talk. Activity in an environment involves social interaction (interpersonal), as well as individual cognition (intrapersonal). Knowing about resources in the environment can be extended to knowing about social and discursive resources. Activating social resources can afford and constrain shared mathematical activity in the classroom. In a classroom situation, a number sense environment includes interactions with peers and with the teacher. Knowing when to articulate mathematical ideas to others (or to yourself), knowing when and how to ask for

assistance, and knowing about cues, signifiers and other mathematical language resources, are aspects of participating in a discursive number sense environment. This places number sense as situated and distributed knowing within social settings and practices.

2.7.6 Diversity of Number Sense for Learners

A distributed view of number sense can account for diverse student performance. Differing achievement or attainment for individual students can be viewed as diversity of performance within an environment. Students may not recognise or activate the affordances and constraints of mathematical tools. Pea claimed that experts know or recognise the affordances of a context and that novices do not.

> We know that inscriptional systems often pose vast problems for the learner. Mapping relations between objects in the world and the written number system are problematic for many learners, as in the well documented difficulties of place-value subtraction (Pea, 1993, p. 62).

This can lead to errors, inefficient calculation processes, or misinterpretations of calculation outcomes.

Errors

An error can be considered as either activating a constraint or not recognising an affordance of a mathematical tool in a particular context. Errors can be based on misconceptions, plausible beliefs about numbers but applied to inappropriate situations (Hart, 1981; Johnson, 1989; Maurer, 1987). Errors with decimal numbers have been a topic of much research in both primary and secondary schools (Irwin, 1999; Moloney and Stacey, 1996; Steinle and Stacey, 2001). An example of a decimal misconception is the whole number rule (4.125 is greater than 4.7 because 125 is greater than 7, from Moloney and Stacey, 1996, p. 5). Another misconception relates to the placing of zeros in the decimal number (Irwin, 1999). Decimal misconceptions have been attributed to a variety of factors: application of whole number knowledge to decimal numbers (Steinle and Stacey, 2001), lack of familiarity with particular real life situations such as money, lack of knowledge of place value, or not enough emphasis on multiplicative thinking (Irwin, 1999).

These decimal misconceptions can be re-conceptualised using a distributed perspective in a number sense environment. Rather than using an individual schema approach to account for a misconception (see Stacey and MacGregor, 2002), decimal numbers can be considered as cognitive tools. Using a tools perspective, any representation of a decimal number is a carrier of meaning. As discussed in section 2.5, there are three aspects of meaning carried within the written representation of a decimal. The digit itself carries information about its magnitude. The decimal place of the digits carries information about the relative magnitude of the digits, and the number of decimal places carries information about the overall precision of the number. In other words, a decimal number is a cognitive tool where the place and magnitude of the digits signify important information (Steinbring, 2000). Students also need to recognise the decimal point as a marker that separates the whole number and decimal places. These multiple meanings, carried within a written representation, afford cues for students in their use of decimal numbers as tools (and of other cognitive tools in conjunction with decimal numbers). These signifiers can also be constraints for students. Each signifier of meaning can be mismatched by novices, leading to systematic errors in calculation processes involving decimal numbers. For example, the whole number rule can result from activating the affordance of the magnitude of the digits rather than activating both signifiers together. Being attuned to notice and recognise the information carried in the tool is an aspect of knowing about representations of numbers. Although Greeno did not specifically mention decimals in his number sense examples, he implied that misconceptions are accounted for by attending to the affordances and constraints of mental models. Knowing in a conceptual domain is being attuned to the codes and conventions of the environment. According to Borasi (1996), an error is like getting lost in the city. In order not to get lost, we look for cues, signs of information, and seek out signifiers of resources in the environment. Recognising and activating the affordances of mathematical tools is one aspect of knowing in the domain of number.

Inefficient Calculation Methods

The distributed environment metaphor can account for students using inefficient calculation methods. Some seven to twelve year old students were

found to retain the use of counting strategies rather than using number facts for addition and subtraction (Gray, 1991; Gray, Pinto, Pitta and Tall, 1999; Gray and Pitta, 1996; Gray and Tall, 1994). The less successful students were found to have an over-reliance on counting strategies and inadequate knowledge of number facts. A distributed environment perspective offers alternative interpretations. Students may have believed that counting strategies afforded greater accuracy, and as accuracy was significant and valued in that classroom context, the students activated counting strategies. Some of the signifiers, such as the magnitude of the numbers or the relationship between the two addends, may have cued them towards counting rather than using number facts. The long counting chains were not recognised as a constraint in the context of the school task. Signifiers and cues could also account for the different strategies used by the street vendors in Brazil, based on whether the tasks were presented using different cues: written (school) or verbal (out-of-school) (Nunes et al, 1993).

2.7.7 Classroom Mathematics Lessons

This section discusses the environment metaphor for number sense in the context of classroom mathematics lessons. According to Greeno (1991), knowing means becoming more familiar with the resources in the environment. For this to occur, students need opportunities to participate in mathematical activity, to talk about their activity, and to converse with knowledgeable others, such as teachers or peers.

> The more reciprocal processes of conversation, however, could be an important further educational means by which many more students could reach desirable levels of understanding and capability for quantitative and numerical reasoning. (Greeno, 1991, p. 199)

Coming to know about the environment of numbers is jointly mediated, in which conversations become significant, and teachers and peers play important roles.

The environment metaphor identified the roles of teachers and peers in classroom mathematical activity. A teacher's role was as an expert guide who helped "by indicating what other resources the environment has, where they can be found, what some of the easy routes are, and where interesting sites are that

are worth visiting" (Greeno, 1991, p. 197). These assistance strategies are relative to the resources that the students bring with them to the classroom. As Greeno states,

> An effective guide for learners needs to:
> 1. be sensitive to the information they already have
> 2. connect new information to it
> 3. provide tasks and instructions that can be engaged in productively by beginners
> 4. be aware of potential errors that can result from newcomers' partial knowledge
> 5. help beginners use errors as occasions for learning. (Greeno, 1991, p. 198)

These are examples of teacher scaffolding actions, used to shape classroom experiences. When novices enter a new environment such as a city, we might join an official sightseeing trip. As we become more familiar in the city, there are other means of finding out about the resources of the city. Students at the year 9 and 10 level have already experienced a mathematical sightseeing package at primary school. In order to make more sense of resources in the environment, older students want to know more about contexts, about how to connect together resources, which resources are aligned, the different codes and conventions of resources, and how to use external resources such as guidebooks (textbooks). In a physical environment we rely on others as well as expert guides. Peers are useful travelling companions in classroom lessons. When in difficulties, we may stop a stranger and request assistance. Individual exploration of an environment becomes shared with others such as peers or teachers.

Classroom experiences are opportunities for students to learn more about mathematical resources; in other words, to become more familiar with affordances and constraints of mathematical tools in different contexts. In a later discussion of the environment metaphor, Greeno (1994) emphasised the importance for students of attunement to affordances and constraints. He extended his theory to suggest that there were different kinds of affordances and constraints that related to macro and micro elements of the environment.

> These kinds of knowing involves being attuned to affordances of various kinds of mathematical "materials", including concepts, methods, and representations, and being attuned to constraints about how those

> materials and methods can be combined and how they work. (Greeno, 1994, p. 98)

This raises the issue of what it might mean to become attuned to affordances and constraints. It also highlights the importance for students of lower ability to have opportunities to find out more about the properties of resources in the domain of number. In particular, important properties of mathematical resources are the connections and relations between different tools, cues and signifiers. These properties afford or constrain mathematical activity in different contexts.

A distributed perspective for number sense offers some alternatives to a remedial approach for secondary school students labelled as lower ability. To extend this metaphor to one familiar to thirteen and fourteen year olds, particularly for urban students, is to use a shopping mall as an environment. Knowing your way around the mall is to know what resources are available and where to find them. The use of hand held technology, such as the cell phone, illustrates how a technological tool can provide greater access to mathematical resources at times chosen by the operator. The practice of mathematical skills is similar to staying in one shop in the mall for a longer time in order to figure out the merchandise, how it's organised, and how it compares with another shop. An error or misconception is getting lost in the mall, or going to the wrong shop for a particular item. While the mall provides a great deal of visual information, it is preferable to explore the mall with others.

As has been argued, a metaphor of an environment is useful for conceptualising number sense as situated distributed knowing in the domain of number. The adapted version of this metaphor brings together the three domains of this study and proposes a model of school mathematical activity as distributed over classroom social norms, tools, and interaction. Although Greeno proposed this metaphor in 1991, there has been comparatively little use of it as a research tool in non-experimental classrooms. In this study, I aim to explore how the adapted version of the environment metaphor might account for mathematical activity of lower ability students during mathematical lessons.

2.8 Classroom Negotiation

This final section discusses negotiation of mathematical meanings. This study considers the negotiation of mathematical meaning as distributed across the domains of classroom social norms, tools and interaction. Key aspects of these domains afford and constrain negotiation of mathematical meaning. The final section of the literature review draws together important aspects of negotiation of mathematical meanings, and introduces the research questions of this study.

2.8.1 Negotiation of Meaning

A social view of learning mathematics encompasses the negotiation of mathematical meaning. Bruner (1986) emphasised the importance of interaction and stated that "negotiation is the art of making new meaning" (Bruner, 1986, p. 149). There are various perspectives of negotiation (Wheatley, 1993). One view is the notion of finding common ground (Greeno, 1991; Neyland, 1998).

> To have a successful conversation, the participants construct common ground – that is, a shared body of understanding – and conversational processes are available to check whether the participants' understandings are in fact shared and to repair mismatches that occur" (Greeno, 1991, p. 210).

In order to find common ground, interactive processes are used to check by matching or fitting different meanings (von Glasersfeld, 1990). Due to the complex nature of negotiation, common ground emerges over time (Alro and Skovsmose, 1998; Richards, 1991; Weade, 1995), and is bound by context (Lerman, 1993). Negotiation can also mean reaching a consensus (Mason, 1999; Wheatley, 1991). This can be a complex and lengthy process that involves "give and take, there are questions, answers, doubts, challenges and compromises" (Richards, 1991, p. 19). Negotiation has been acknowledged as an "interactive accomplishment" (Cazden, 1997, p. 304). Negotiation is a joint process between teacher and students (Bruner, 1986, 1990), although research has identified the teacher as the authority in many classroom negotiations (Edwards and Mercer, 1987). Some writers claim that negotiation involves both the individual and the social (Frid and Malone, 1995; Krummheuer, 2000), where the individual "construes his/her own meaning from interactions"

(Lerman, 1999, p. 101). Notions of individual meaning making and intersubjectivity are, however, problematic and contentious (Cobb and Bauersfeld, 1995; Lerman, 1993; Smolka, Del Goes and Pino, 1995; Voigt, 1994).

As an interactive accomplishment, the negotiation of mathematical meanings occurs in discourse activities. These conversations are opportunities to construct meanings for mathematical entities (Sfard, 2000). Talking about mathematical entities, such as models, provides "referents for terms in the conversation" (Greeno, 1991, p. 198). Communicating about mathematical tools and their roles in mathematical activity is one aspect of negotiation. "The mental objects that a person includes in a model provide affordances for mental activities" (Greeno, 1991, p. 204). These affordances are then resources for further activity. Coming to know about tools involves recognising properties of the tools, and activating the affordances of the tools useful for the context. Negotiating meaning is a process of recognising valued positions of mathematical tools, by being receptive to feedback about tools, and attending to the affordances and constraints of tools within particular contexts (Goos, et al, 1999; Greeno, 1991; Van Oers, 2000).

Conversations can be opportunities for resolving uncertainty (Clarke and Helme, 1997), and aspects of interaction can afford or constrain negotiation of mathematical meanings.

> There are some basic elements of the process of establishing a shared understanding, of building an ever-expanding foundation of shared knowledge which will carry the weight of future discourse (and action). These are the offering of new information, reference to past experiences, requests for information, and test or 'checks' on the validity of interpretations offered. (Edwards and Mercer, 1987, p. 6)

Responses to errors and meta-communication about mathematical thinking has been found to afford negotiation (Alro and Skovsmose, 1998). The emergent approach claimed that, to increase the likelihood of understanding each other, discourse needed to be conceptual and involve reasoning as well as description (Cobb, 1998). Reflective discourse (see section 2.6.4) is related to sociomathematical norms for valued, efficient and acceptable solutions. In a

study of two year 5 classrooms in Australia, Frid and Malone (1995) found that students' sources of conviction or perceptions of mathematical authority were significant in their negotiations. This was similar to the finding of Driver et al (1994) of the crucial role of the authority of the teacher in negotiating scientific meanings. Finally, one study of two ninth grade mathematics students found that the learning orientations of individual students afforded or constrained the negotiation processes within a problem-centred learning environment (Wheatley, 1991).

2.8.2 Negotiation is Distributed

Each of the three domains of the study is significant for the negotiation of mathematical meaning. Firstly, classroom social norms set up participation structures for each classroom context and define roles for joint activity. Classroom social norms afford and constrain activity by influencing the qualities of classroom interactions and the use of cultural tools. Secondly, tools are used, described, explained, and appropriated during the joint development of classroom mathematical practices. Mathematical activity can be conceptualised as mathematical tools in action, and interaction provides opportunities to talk about tools within the context of activity. I have argued that negotiation can be considered as distributed over the three domains. Classroom social norms afford participation opportunities, tools afford mathematical resources, and interaction affords discursive resources. The relationships between the three domains are reflexive; that is, the domain of classroom social norms influences the domains of tools and interactions, and vice versa. For example, one of the classroom social norms from the emergent approach was the expectation that students volunteered to explain their mathematical solutions. This particular norm afforded opportunities for students to talk about solution methods involving mathematical tools, and afforded opportunities for peers to engage with and respond to articulated mathematical methods. The domains of tools and interaction were activated by one of the classroom social norms. The ensuing discourse provided opportunities for the negotiation of mathematical meanings. Negotiation was distributed between classroom social norms, mathematical tools, and interactions.

A focus on number sense will provide a focus on an important part of school mathematics for this age group. The environment metaphor for number sense is significant because students are conceptualised as active participants in the environment. The metaphor of an environment can account for diversity of learning in the area of number sense, and for joint mathematical activity. The metaphor also connects with a distributed perspective of situated knowing about tools. Salomon (1993) suggested there was a reciprocating feedback system between tools and human operator that described learning over time. I have adapted the environment metaphor for number sense to be more focussed on the distributed nature of mediated activity with tools. I aim to use this as a means of investigating classroom lessons in lower ability mathematics classes in the first two years of secondary school.

2.8.3 Research Questions

Negotiation of mathematical meaning is distributed over domains of classroom social norms, tools and interaction. Research questions are assigned to each of these domains and are as follows:

Classroom social norms
What classroom social norms were evident in the case study classrooms?
Which of these norms related to student participation in mathematical activity?

These two research questions investigate the significant norms related to student participation in mathematical activity. The main focus is on social norms, associated with rituals and student roles, that indicate participation. These norms will be described in terms of expectations and obligations of classroom participants. Another focus will be how each social norm affords or constrains mathematical activity.

Tools
What cognitive tools were evident in the case study classrooms?
How were these tools, and technological tools, used in the classroom interactions?

The role of tools in mathematical activity highlights the importance for students of knowing about mathematical tools, both physical and cognitive tools. For some time, researchers have been encouraged to focus on what tools might mean for children as they engage in mathematical activity (Cobb, 1995a; Rogoff, 1990). We need to "understand how their negotiation and use of symbolic means supports their mathematical development" (Cobb, 1995a, p. 383). Recent research about tools has been mostly in experimental classrooms or about the use of computerised tools. Most New Zealand mathematics classrooms do not have regular access to computerised tools, and much of the technology research is with graphical calculators at senior level. There is little research in this area from typical school classrooms at Years 9 and 10. This study aims to find out which mathematical tools are used and how these tools afford or constrain classroom mathematical activity.

Interaction

What was the nature of student and teacher contributions during classroom interactions?

This question considers each contribution in terms of the relationship with others' contributions. In order to provide more focus, particular aspects of whole class interaction will be investigated. These aspects are scaffolding actions by teachers, the development of classroom mathematical practices, and shifts in the discourse.

Number sense negotiation

How were mathematical meanings negotiated in the case study classrooms?
How was number sense distributed during lesson activity?

The final two questions bring together the three previous questions with a particular focus on number sense in classroom activity. By considering negotiation as distributed over social norms, tools and interaction, significant aspects of negotiation afford or constrain classroom negotiation.

To summarise, the research questions for this study are:

1. What classroom social norms were evident in the case study classrooms? Which of these norms related to student participation in mathematical activity?

2. What cognitive tools were evident in the case study classrooms? How were these tools, and technological tools, used in classroom activity?

3. What was the nature of student and teacher contributions during classroom interactions?

4. How were mathematical meanings negotiated in the case study classrooms? How was number sense distributed during lesson activity?

2.9 Chapter Summary

This chapter has set out the literature for this study. By drawing on the social, situated and distributed literature, I have proposed three significant domains for mathematics classrooms. The domains are classroom social norms, tools and interaction. Research from experimental and traditional classrooms has been discussed for each of the domains, including some contradictory research findings. For each domain, I have argued for key elements of affordances and constraints on mathematical activity. Rituals and roles are important elements of the domain of classroom social norms. In the tools domain, cognitive tools were discussed as representations, thinking devices or analogous models in the context of classroom mathematical activity. In addition, calculators were identified as technological tools that contribute diverse resources to both joint and individual activity. Interactive processes such as responses to others' contributions, shifts in the discourse, and different registers, were discussed as important aspects of the interaction domain. These aspects shape not only participation of students in classroom mathematical activity, but also the negotiation of mathematical meanings.

Chapter 3
Methodology and Methods

3.1 Introduction

This chapter sets out the methodology and methods for the study. Naturalistic and interpretive methodologies are discussed as appropriate approaches for studies of activity in classroom lessons. Case study and ethnographic approaches are briefly discussed as useful for investigations of interactive contexts. Issues related to trustworthiness of interpretive research are also highlighted. Data collection methods relevant to classroom research are discussed, including tools for data collection. The research design is outlined in chronological order of the research decisions made during the time of the study. The research sites and methods chosen for the study are described, and issues related to the complexities of research in school classrooms are discussed. Ethical issues related to research in classrooms are discussed and specific issues related to informed consent are addressed for this study. Decisions made during the time of the data collection are also recorded. These decisions address aspects of the research process such as the role of the researcher. The analysis of the large amount of data is a significant aspect of this study and the process of analysis is outlined in some detail. Data was organised into lesson transcripts and then episodes selected for analysis. An analysis framework was developed for the purpose of this study, and was used to analyse lesson episodes. In the final section of this chapter, the analysis categories are defined.

3.2 Methodology

3.2.1 Naturalistic and Interpretive Inquiry

In the past twenty years there has been an increasing recognition of the importance of social context in educational research (Bauersfeld, 1988; Romberg and Carpenter, 1986; White, 1985). Mathematics lessons are an example of a social context for teachers and students, within the natural setting of a classroom.

Naturalistic inquiry studies phenomena in natural settings. Salomon (1991) described this investigation of phenomena in their natural setting as 'systemic' research in contrast to 'analytic' research. Systemic research acknowledges the interconnected nature of phenomena and is a useful perspective to bring to the complex context of classroom lessons (Wiliam, 1998). Research methodologies need to account for "the sensitivity of educational phenomena to changes in contexts [that] suggests that we need a different way of thinking about educational research" (Wiliam, 1998, p. 3). Naturalistic inquiry is suited to the study of phenomena in classroom situations (Cohen, Manion and Morrison, 2000).

Particular features of a classroom situation can be studied. White (1985) identified three features of context that needed to be taken into account in the design and reporting of research: "physical conditions, the people involved, and the social conditions" (p. 96). As discussed in the previous chapter (section 2.2.2), each classroom presents a distinct context within the same school setting. Important contextual features are the physical setting, the social group and social practices.

> There are certain characteristics of structure and purpose shared by all classrooms. Equally, each classroom presents a unique social setting, which facilitates and constrains the actions of all participants and the form of the consequent learning. (Clarke, 1996, p. 4)

Time, as well as place, is important because classroom contexts change over the duration of a school term or a school year. For example, early in the school year "social patterns may be more fluid and formal and informal rules of behaviour less established" (White, 1985, p. 96). Consequently, social practices need to be studied over time. Alton-Lee and Nuthall (1992) examined a sequence of lessons rather than individual lessons, because this was "the naturally occurring self-contained opportunity for pupils to learn specific content in classrooms" (p. 1). The same applies for the study of the development of mathematical practices over time (Bowers et al, 1999; Cobb, Gravemeijer et al, 1997).

Associated with studies of natural settings such as classrooms is a research orientation known as an interpretive approach. An interpretive approach studies the world of human experience through the eyes of the participants (Cohen et al,

2000). The social world can be viewed through the meanings that participants have for their actions. Theory emerges from the interpretation of data. Theory becomes "sets of meanings which yield insights and understandings" about contextualised social activity (Cohen et al, 2000, p. 23). An interpretive approach is useful for generating meanings about activity in classrooms.

3.2.2 Case Study Methods

While naturalistic inquiry accepts the importance of context, case study methods provide a focus on particular situations (Cohen and Manion, 1994; Cohen et al, 2000). A case study "is an examination of a specific phenomenon such as a program, an event, a person, a process, an institution, or a social group" ((Merriam, 1988, p. 9). The combinations of people, setting, resources and purposes can be studied as events unfold over time. In a case study approach, the 'case' is the unit of analysis (Miles and Huberman, 1994). A case is defined by both the size of the unit (school, classroom, group of students, individual), and the period of time. Case studies are characterised by inductive reasoning where "new relationships, concepts, and understandings, emerge from an examination of the data" (Merriam, 1988, p. 11). Due to their particular nature, case studies are descriptive and "rely heavily on inductive reasoning in handling multiple data sources" (Merriam, 1988, p. 16). One possible drawback of a case study approach is whether it serves "the needs of the various audiences (all of them) who have some stake in its use" (Guba and Lincoln, 1989, p. 137).

Case study methodology is important for classroom contexts (Bogdan and Biklen, 1992; Erickson and Shultz, 1992; Merriam, 1988; Voigt, 1995). The search for fine-grained detail, particularly about relationships between various facets of classroom contexts, lends itself to a case study approach. For example, case studies of mathematics learning have been used to study mathematics programmes in two schools (Boaler, 1997a), empowerment and interaction patterns in classrooms (Cooper et al, 1993), and teaching approaches at different class levels (Chazan and Ball, 1999). Groups of students (Barnes, 1999), and individual students (Cobb and Whitenack, 1996), can also be considered as cases within a particular classroom context. In these situations, a case study approach

enables more intensive examination of key features or phenomena (Merriam, 1988). In addition, a relatively small number of cases (between two or four) provide sufficient evidence for the research context. For example, a case study approach was used to study the development of the classroom social norms, such as obligations, for first graders in mathematics (Voigt, 1995).

3.2.3 Links with Ethnography

Ethnography is a methodology that lends itself to studies of classroom social relationships and interactions (LeCompte & Preissle, 1993; Mehan, 1979; Zevenbergen, 1998). Ethnography has been used in recent years to generate theories of situation and subject (Nickson, 2000). Ethnography is characterised by 'thick' description providing rich data for analysis. This was the methodology used in the study of two classes in New Zealand (Jones, 1991) (discussed in sections 2.4.4 and 2.6.2). An ethnographic approach enabled the researcher to include multiple features of school life for students. This richly layered approach was particularly useful for examining similarities and differences between the experiences of the girls in the two classes. While ethnography has not been often used in research of mathematics classes, it is useful for "identifying and understanding social and cultural norms in mathematics classrooms" (Zevenbergen, 1998, p. 20).

3.2.4 Methodology of this Study

This study adopts a multi method pluralist approach (van der Aalsvoort and Harinck, 2000) that combines tools from different approaches (Miles and Huberman, 1994). Naturalistic and interpretive perspectives were adopted because these methodologies best served the orientation of the research questions. The domain of social norms in classroom contexts is suited to a naturalistic approach. Interpretive approaches are suited to studies of negotiated meanings (Cohen et al, 2000). And a view of activity as distributed over people, tools and interactions also suits an interpretive approach (Salomon and Perkins, 1998). A case study approach enables a study of mathematics lessons in different

classroom contexts. Ethnographic methods are useful for collecting data about different aspects of classroom activity (Zevenbergen, 1998).

3.2.5 Trustworthiness of Interpretive Inquiry

In adopting an interpretive methodology, notions of validity and reliability can be replaced by considerations of trustworthiness of the research (Lincoln and Guba, 1985). There are four aspects of trustworthiness: credibility, transferability, dependability and confirmability (Guba and Lincoln, 1989). The credibility of the research depends on issues related to the scope of the study, such as prolonged engagement, persistent observation (depth), and member checking (Guba and Lincoln, 1989). Credibility is addressed by comprehensive and well-linked accounts where areas of uncertainty are identified (Miles and Huberman, 1994). Transferability refers to providing extensive and careful descriptions so that readers can apply "the study to their own situations" (Guba and Lincoln, 1989, p. 242). Techniques such as thick description are commonly used to provide sufficient detail for transferability. Any limiting effects should also be discussed (Guba and Lincoln, 1989; Miles and Huberman, 1994). Dependability also relates to the data but is concerned with the stability of the data over time. Any changes and shifts in constructs need to be tracked and the documentation of the study should reveal the logic of process and decisions about method. Guba and Lincoln (1989) call this the dependability audit. The dependability audit involves documenting and tracking the process and method decisions over the duration of the research. This should also include explicit description of the role of the researcher (Miles and Huberman, 1994). Lastly, confirmability is related to the tracking of the data sources. This ensures that the "logic used to assemble the interpretations into structurally coherent and corroborating wholes is both explicit and implicit in the narrative of a case study" (Guba and Lincoln, 1989. p. 243). To achieve confirmability, methods and processes need to be described explicitly throughout the actual sequence of the research process (Miles and Huberman, 1994).

The four aspects of trustworthiness will be addressed in this study. Credibility and dependability will be addressed through detailed recording of and accounting

for decisions made during the research process. Confirmability will be addressed through careful attention to the methods used to analyse the data. The extensive description of the research process, including analysis, will also address transferability. In the following sections, aspects of the dependability audit will be highlighted. The research methods for this study are now described in detail, beginning with the research design, and including the methods used and the role of the researcher. This is followed by a consideration of ethical issues related to this study.

3.3 Research Design

3.3.1 Data Collection Methods

Data collection involves eliciting data from the research site(s) and can take various forms (Cohen et al, 2000; Kellehear, 1993; Miles and Huberman, 1994). One method is participant observation of the research site. Participant observation is commonly used in case study research because it enables first hand observation of activity in the natural setting. The participant observer can stay on site over a substantial period of time (Cohen et al, 2000), and the degree of participation can vary (LeCompte and Preissle, 1993). Advantages of participant observation are that there is opportunity for immersion in the context with more "accurate explanation and interpretation of events rather than relying on the researcher's own inferences" (Cohen et al, 2000). Other considerations for the participant observer are for the researcher to be both unobtrusive yet to stay on site (Miles and Huberman, 1994). Disadvantages of participant observation are that data can be biased, impressionistic or idiosyncratic (Cohen et al, 2000). To avoid this, participant observers need to check for researcher effects and avoid biases stemming from the researcher effects on the site (Miles and Huberman, 1994, from p. 266).

The roles of participant observer in school contexts have been widely documented (Boaler, 1997a; Brown et al, 1993; Cobb, Gravemeijer et al, 1997; Erickson and Shultz, 1992; Jones, 1991; Mehan, 1979; Mercer, 1995). In school settings, the researcher roles are "looking and listening, watching and asking"

(Loflund and Loflund, 1995, p. 19). As Zevenbergen (1998) points out, a participant observer observing in mathematics classrooms can also assume a role as teacher's assistant.

> Such a role enables the researcher to observe such things as the delivery of content and instructions used by the teacher; the ways in which students work or avoid work; teachers interacting with students; the tactics that students use to negotiate the mathematical tasks set by the teacher; the strategies students use to negotiate meanings with their peers or their teacher; and the types and forms of artifacts (such as books and equipment) present in a classroom and how they are used by the teacher." (Zevenbergen, 1998, p. 21)

In many school classrooms, visitors to lessons are likely to be other teachers or teacher aides, people who are familiar with the lesson format and likely to assist students. A silent observer would be unusual and "a potential distraction, whereas another "teacher" in the room would not impact on the teaching and learning of mathematics in any significant way" (Zevenbergen, 1998, p. 21). The researcher can ask spontaneous questions as they arise and respond to student requests for assistance. This may have benefits for the classroom participants as "teachers are appreciative of any assistance that an involved participant observer can provide" (Zevenbergen, 1998, p. 21).

Another method of data collection is the interview of participants. Interviews are a legitimate means of finding out about beliefs and practices (Bogdan and Biklen, 1992). Interviews bring in different perspectives, can occur at any stage of the research process, and can be carried out away from the site (Coffey and Atkinson, 1996; Cohen et al, 2000). This provides flexibility for the researcher as interviews can be used to collect data during classroom lessons, after the lesson, or at the beginning or end of the data collection time. Interviews can also supplement, clarify or validate the data from other sources such as observation data (Loflund and Loflund, 1995).

3.3.2 Tools for Data Collection

Video, audio, field notes, or a combination of these, are the most common data collection tools used in classroom research. Video provides stable visual data that can be returned to for analysis (Brown and Renshaw, 2000; Clarke and Helme,

1997; Cobb, Gravemeijer et al, 1997, Mehan, 1979). Video provides information about who is speaking, actions such as gestures or writing, and is useful when students speak simultaneously. In a study of Year 5/6 mathematics classroom, video provided data of a student using a textbook net for a square pyramid and copying this from the textbook rather than constructing it (Zevenbergen, 1995). This data was not available by usual ethnographic methods and would have been missed without the video data. Video cameras can, however, be disruptive and imposing for the teacher and the students (Cobb and Whitenack, 1996). While video is capable of close observation of group activity, unless there are multiple cameras, other simultaneous classroom activity can be missed.

Audiotaping is a less intrusive alternative to video cameras and provides a record of the participants' talk. A tape recorder is easy to operate and is particularly valuable for teacher-researchers in their classrooms (Lubienski, 2000). Audiotape has the limitation that it only records what is spoken so misses other aspects of classroom interaction, such as gestures, movements, written information, and facial expressions. Relying on only audio evidence provides limited data for making inferences about student activity. Another drawback of audiotaping is the variable quality of the recording. Ambient noise in busy classrooms can make the identification of speech or of the speakers extremely difficult.

Field notes are another means of recording data about lesson events, classroom features and other contextual information relevant to the study (Brown and Dowling, 1998; Clarke, 1996; Kellehear, 1993; Loflund and Loflund, 1995). Field notes can include both the data collected from the lesson and reconstructed thoughts (Tobin, 1992), and are an important source of data (Boaler, 1997a; Jones, 1991). As a supplementary source of data, field notes provide impressions of significant classroom events as well as provide reference markers for interviews (Clarke, 1996).

3.3.3 Methods Used in this Study

Participant observation was the main method chosen to observe joint mathematical activity as classroom lessons unfolded over time. I also needed data

from teachers in order to corroborate my observations and inferences of lesson events, such as rituals and student roles. A semi-structured interview of each teacher was used to find out teacher perspectives of the classroom context.

Collecting data in classrooms poses logistical and technical challenges. In this study, data collection incorporated a variety of tools. I chose audiotaping for the following reasons. Audiotaping collects data about who contributes to verbal interactions and what they say. Audiotape provides data of verbal interaction, a key aspect of negotiation. Other reasons for choosing audiotape were mostly practical. Tape recorders are easy to transport from classroom to classroom and can be quickly organised when time is limited. Tape recorders allow flexibility of location within classrooms, and are easy to operate during the lesson. Tape recorders are relatively small and unobtrusive and consequently less intimidating for the students and teacher. This was important for classroom situations where I anticipated classes were large and classroom space was at a premium. For this study, the choice of audiotape balanced the need to not disturb large classes of junior secondary students, with the need to collect detailed data of classroom interactions.

Other data collection tools were chosen to supplement the audio data. Field notes were used to provide additional observation data. For example, information about what was written on the whiteboard was needed for more complete data about mathematical tools such as written representations. Additional information about student tasks and student seating arrangements was also required.

3.3.4 Research Sites

Decisions about research sites involve considerations of suitability and access. Purposive sampling (Brown and Dowling, 1998; Cohen and Manion, 1994; Cohen et al, 2000) involves identifying key factors before the selection of research sites. This study has a focus on distributed mathematical activity of lower ability students during classroom lessons. In addition, the focus of inquiry was the first two years of secondary school, Years 9 and 10. The key factors for this study related to the typicality of the class context. The class needed to be

coeducational because coeducation is more common than single–sex education in New Zealand secondary schools (Ministry of Education, 2000). The class size also needed to be typical, around thirty students. Another key factor was that the classroom mathematics programme was typical and not subject to curriculum change such as targeted professional development. Another factor was the means that the school used to allocate students to lower band classes. In my experience, schools that use broad banding or streaming in Years 9 and 10 do so for a 'core' of subjects. Consequently, the same group of students is together for subjects such as English, Science, Social Studies and Mathematics. The final factor related to the social environment of the classroom. Given the research focus on negotiation, it was important to find classrooms where student talk was accepted and not where students worked in silence for large periods of time. In my experience, silent Year 9 and 10 classrooms are atypical in New Zealand. I also anticipated that interactive classrooms would be more accepting of another adult in the room.

One urban co-educational secondary school was approached for the study, on the basis that its mathematics classes met the key factors outlined above. This particular school was also likely to have a greater number of lower ability classes than a smaller school. Case study classrooms from just one school provided a common school context and were a form of purposive sampling (Brown and Dowling, 1998; Cohen and Manion, 1994; Cohen et al, 2000). There were also practical advantages for negotiating the approval of the study and for ease of data collection. My initial request was for a lower ability class at each of Year 9 and 10 levels. This would provide a case study classroom at each of the year levels. A third teacher volunteered her Year 9 middle band classroom as a possible research site. This was both opportunity and snowball sampling (Brown and Dowling, 1998; Cohen and Manion, 1994), and I accepted this offer because it provided a further case study opportunity. In this school, the middle and lower band mathematics programmes were similar, based on the same school scheme with common student assessments. I anticipated that the middle and lower band mathematics programmes were likely to have more similarities than differences. The three case study classrooms were typical of Year 9 and 10 mathematics

classes and met the key factors for the research site. I chose a pseudonym for the school, Lemonwood High School.

3.4 Ethical Considerations

This section sets out the two main ethical considerations for the study. The preparation for informed consent and confidentiality is outlined. The process of negotiation of approval and informed consent is set out in some detail because this was an aspect that changed from the planned procedures.

3.4.1 Ethical Consent

The ethical procedures for this study were approved on the basis of balancing the potential harm for teacher and students with the potential benefit of the research. As this study was a non-intervention study, the teachers continued with their usual mathematics programme and the potential for harm to the students' education was considered to be minimal. The first ethical consideration was the process for informed consent of the participants. Informed consent ensures that participants have adequate and transparent information about the study, in order to agree to participate or not (Brickhouse, 1992). Full information about the purpose and design of the research must be provided so individuals can freely choose to take part (Cohen et al, 2000; Miles and Huberman, 1994).

Research in educational settings such as schools requires additional considerations. Requests for ethical or informed consent can involve people with positions of responsibility within the school as well as the teachers and students in the classrooms. Members of the school community may be involved directly or indirectly in the research, and will need to be approached for approval. Researching in classrooms requires sensitivity to the feelings of the participants. Ethnographic approaches focus on classroom processes and relationships rather than on individuals and their actions. Lower ability students in mathematics must be reassured that the research focus is on classroom processes and not on their individual mathematical achievement. For teachers, it is important to "stress continually to the teachers that they are not the focus of the research, but rather it

is the practices inherent in the classroom that are of interest to the researcher" (Zevenbergen, 1998, p. 36). Not only is this an ethical dimension of the research (Cohen et al, 2000; Kellehear, 1993), but when working in a school context, is an example of an ethic of care (Noddings, 1993).

3.4.2 Informed Consent for this Study

Informed consent requires clear and transparent communication about the research focus and design (Miles and Huberman, 1994). In a school context it is important for all participants, including significant people in the school community, to be asked for their informed consent. The informed consent was slightly different to what I had planned, and this process is now described in some detail. I prepared letters to students, teachers, the Principal and the Board of Trustees providing information about the research and seeking the approval of the Principal and Board of Trustees, and the informed consent of the students, their parents or care-givers, and the teachers. My letters described the focus of the research as mathematical learning in classroom lessons with a particular focus on number and calculation. Information about the data collection methods was also included. All participants including the school were assured of confidentiality.

When I explained the ethical approval procedures to the Principal of Lemonwood High School and provided copies of the letters of consent, he suggested that the letters seeking student and parental consent were not necessary. As Principal, in his role of 'in loco parentis', he was prepared to give consent on behalf of the parents and the students. He also agreed to inform the Board of Trustees himself about the research, and advised me that a letter to the Board would not be necessary. It appeared that the Principal wanted to manage the amount of overall communication between the school and parents/caregivers, and may have considered the letter as adding to this volume of paperwork. He was also assured by the details of the proposed study in his school and was prepared to take responsibility for approval and for consent for the students and their caregivers. Over the time of the data collection, two further letters were provided to the Principal to inform him about the progress of the study.

The Principal did not want the students to receive the written consent form but requested that they receive a letter providing information about the research. Acting on the Principal's advice, I provided the students with a letter that outlined the study, described the data collection methods and my role during the lessons. The letter assured students of anonymity by changing their names in any published work. I also explained that students could withdraw their contributions at any time and these would not be included in any publication. The letters to the students were handed out by the classroom teacher; in one class this was not until after the data collection had started.

The Principal was, however, concerned that the teachers were happy with the arrangements, and that the teachers received the letter seeking their approval and written consent to the research. The letters to the teachers provided further detail. I requested that they continue with their normal mathematics programme and set out that I aimed to be as unobtrusive as possible. The data collection methods were described, including the teacher interview. I explained that the interview transcripts would be returned to them for checking, comment and alteration. The possible timeline of the study was outlined and confidentiality for them and their students was assured. A copy of the interview questions was attached to the letter.

I initially approached the Head of the Mathematics Department of Lemonwood High School who agreed to talk to other teachers in the department and ask for volunteers. I requested classes that were lower band according to the school organisation. Two teachers agreed to participate: one with a Year 9 class of Mathematics and one with Year 10. As outlined earlier, another teacher of a Year 9 middle band class also volunteered to be involved. All three teachers provided written consent. I already knew two of the teachers and I met the third at a meeting of local mathematics teachers. Here I was able to speak with two of the teachers, answer any queries and negotiate possible starting dates.

3.4.3 Confidentiality

The second critical issue for this study was the confidentiality of the research in order to maintain the privacy of the participants (Tobin, 1992). Confidentiality is

difficult to achieve in case study research for various reasons. Participants can "jeopardize their own confidentiality in unpredictable ways by telling friends and colleagues of their participation in research" (Brickhouse, 1992, p. 96). I would add that in a small society such as the New Zealand educational community, confidentiality for teachers and students is essential. In this study, participant confidentiality was likely to be the aspect of most potential harm. All participants in this study were assured of anonymity. The case study teachers and students were given pseudonyms. As the school was labelled Lemonwood High School, the two Year 9 classrooms were labelled Case Studies 1 and 2, Lemonwood 1 and 2, and the Year 10 classroom was Case Study 3, Lemonwood 3. Another means of preventing identification of schools or participants is to minimise reporting of "collateral information" (Brown and Dowling, 1998, p. 65), such as specific data about the setting and social group. This was adhered to in the reporting of school setting and the social group within the case study data.

3.5 Data Collection

This next section records information about the collection of data, and reports decisions made in response to unforeseen events within the classroom contexts. It provides further information about the collection of data and the role of the researcher during the data collection time.

3.5.1 Timeline of Data Collection

Data was collected during the final five weeks of the first school term of 2001. This was sufficient time to collect data for a complete unit of work for each class. Term 1 offered an opportunity for observing the setting up of classroom programmes. In addition, the Year 9 students were in their first term at a new school. I anticipated that this might offer opportunities to observe the establishment and consolidation of classroom social norms. Term 1 is typically a time for teaching number topics and this was the case in Lemonwood High School. Teaching periods were scheduled for 60 minutes and included time for moving between classrooms, resulting in lessons of approximately 50 to 55 minutes. Wednesday lessons were fifty minutes long due to an early school

finishing time on that day. The total number of lessons observed for each class during the data collection time is set out in Table 3.1.

Case Study Classroom	Mathematics Lessons per week	Observed Lessons per week	Total Lessons Observed
Lemonwood 1	4	4	12
Lemonwood 2	3	3	9
Lemonwood 3	4	2	7
Total			28

Table 3.1 Number of Lessons Observed.

Observing sequences of lessons over a long period of time provides rich data (Cobb, Gravemeijer et al, 1997; Goos et al, 1999). My original intention was to observe a sequence of lessons for each class but, due to various organisational factors, this was not always possible. A dependability audit needs to record any gaps in the data (Miles and Huberman, 1994), and the following lessons were not included in the collection of data for a variety of reasons. Firstly, there were two timetable clashes each week between the case study classes. At these times, I prioritised the Year 9 classes because these were involved in number topics. Another issue arose when the class teacher was absent from school for a variety of reasons, and the lesson was taken by a relieving teacher. Some lessons involved the entire lesson time in assessment, such as written tests. In both cases, I did not collect data during these lessons. The number of lessons observed in each case study, however, provided sufficient data. By about the fifth lesson, classroom social practices and interaction patterns were apparent and beginning to have a predictive element to them.

3.5.2 The Data Collected

For every observed lesson, data was collected using a range of tools. Firstly, the lessons were audiotaped using two free-standing tape recorders with small sound grabbers. One sound grabber was positioned to optimise the audio quality of whole class talk. These positions turned out to be either at the front or at the back of the classroom on a surface such as a desk or windowsill. Ambient noise was a

difficulty in some of the lessons, particularly during student activity due to student talk and movement. It was also difficult to closely observe in detail during student activity as all the case study classes were large and I had restricted movement around the classroom.

I decided to use 'focus students' to narrow the field of data collection during student activity. During the ongoing review of the audiotapes for each lesson, some students emerged as focus students due to their tendency to readily discuss their mathematical thinking in both whole class teaching and individual student activity. Focus students were a key source of audio data. After about the third lesson, the second sound grabber was placed near a number of these students in order to record their conversations during mathematical activity. Consequently, the focus students appeared in the transcript data more often that other students. In the Year 10 classroom, many of the focus students moved around the room during student activity and this posed problems for collecting audio data during these times. I also carried a small dictaphone when I moved around the room to observe or talk with individuals or groups of students. I turned off the audio recorder during lesson events that did not involve talk about mathematical activity. Events such as the reading of the daily notices and other administration were considered not directly related to the study.

Field notes, background documentation, and teacher interviews provided further data. Field notes were used to record what was written on the whiteboard or overhead projector screen, any observable gestures or movements, the names of students who contributed to discussions, and other observations of student activity or dialogue unlikely to be clearly recorded on the audiotape. The purpose of the field notes was to provide supplementary and corroborating data to the lesson audiotapes. Records of lesson events, such as whole class or student activity phases of the lesson, were included in the field notes. Field notes were also used to record my interactions with students prior to the lesson or when the microphones were near other groups of students. The nature of the field notes changed during the time of each case study. The field notes after about the third lesson for each case study were more detailed and specific, because I knew the students' names more accurately and I could identify the student speakers. I was

also more familiar with the different aspects of classroom life and more attuned to the language and speech patterns of students and teacher.

A range of background documentation was collected and provided further contextual information. During each lesson a 'student seating plan' was completed as a record of where students were sitting and to note any absent students. When a lesson included an activity from a worksheet or textbook, the relevant pages were later photocopied to provide a record of the tasks presented to the students. In addition there were post lesson notes that recorded lasting impressions, comments from the teacher as I packed up the equipment, or conversations with students after the lesson was completed. These post lesson notes were not written for every lesson because lessons sometimes ended abruptly without any further interaction with teacher or students.
The final source of data for this study were the interviews with individual teachers based around a semi-structured set of interview questions with associated probe questions (see Appendix 1). These interviews were recorded on audio tape and lasted between ten to twenty minutes. Although the teacher interviews were initially planned for the beginning of the data collection, the interviews were delayed until the last week of the term, because it was more convenient for the teachers. Consequently, each of the teacher interviews occurred after the classroom data had been collected. This meant that I could include some questions related to some of my observations in their classrooms.

3.5.3 Classroom Role of the Researcher

It is important to describe the role of the researcher during the data collection (Cohen et al, 2000; Miles and Huberman, 1994). On my first day in each of the classrooms, the teacher introduced me using a formal title according to the protocol of Lemonwood High School. I usually sat at the back of the classroom or wherever there was a spare chair. I did not move around the classrooms assisting students unless specifically requested by students (this only happened in the Year 9 classes). Sometimes students sitting near me leaned across and asked questions or asked for assistance. I responded to these requests, always ceasing if the teacher began to talk to the whole class. I also initiated informal interviews

during student activity by talking with individual or groups of students. The packing up time at the end of lessons was an opportunity for me to informally interview students without being disruptive to the lesson. These were usually brief conversations and related to a student contribution during the lesson. These actions were in keeping with the role of another teacher in the classroom as described by Zevenbergen (1998).

This study was a non-intervention one that aimed to gather data from typical mathematics classrooms. Involvement in classroom based qualitative research means that an observer must be sensitive to the needs of the teacher and students and to be flexible in approach. It is essential to be transparent about the degree of disruption or influence that the study may have had on the classroom lessons. In keeping with avoiding researcher effects (Miles and Huberman, 1994), I stayed on site at times other than the lessons. I waited with the Year 9 students for the teacher to unlock the classrooms, and informally chatted with students standing nearby. During interval and lunchtimes I went to the staffroom, and during the fire alarm I stood at the back of the class lines rather than with the teacher. In the context of Lemonwood High School, these were actions in keeping with a teacher aide or a visiting teacher (Zevenbergen, 1998).

The teacher interviews provided teacher perspectives about their classroom programmes and these perspectives appeared to be congruent with the observed lessons. Some indirect evidence also suggested that there was little alteration to the mathematics programmes in each classroom. For example, in Lemonwood 1, the teacher used existing OHT notes from previous years as well as the planned exercises from both the textbook and homework book. The Lemonwood 2 teacher kept on with the set schedule for the term including the start of an algebra unit and the Lemonwood 3 teacher advised me when not to come to class for lessons that were revision for tests. Interruptions to the lesson sequence were typical of the first term in a New Zealand secondary school. Teacher absences and professional development days were prevalent in other secondary schools for that particular year.

3.5.4 Researching Classroom Lessons

This section briefly records more about issues I faced as a researcher in the case study classrooms. This is intended to provide further detail about my role as researcher in the three classroom contexts.

General	Are you the reliever?"
	Are you our reliever?
JM	No, I'm visiting your maths lessons.
Janine	Oh I hate maths.

This was the first time I had met the students of Lemonwood 1 and I was standing with them outside the classroom, waiting for the teacher to arrive and unlock the door. Janine spontaneously told me (and others standing nearby) how she felt about mathematics [Field Notes]. As it turned out, the only spare chair in the room was at the desk next to Janine. So I sat next to her for Lesson 1, attempting to assume a role of classroom researcher. Janine had other ideas. My field notes record that she asked me questions such as "What are you doing?", "Can you help me?", "I'm hopeless at mathematics", "I don't know my times tables". Janine maximised this opportunity for on-call adult assistance at many times during this lesson. This episode from my first lesson with Lemonwood 1 illustrated the complex, contradictory and compromising aspects of my role as researcher.

Establishing relationships with students was important for this study. In my experience, visitors are less frequent in lower ability classes, except for teacher aides. Quickly learning students' names, greeting them around the school and informally chatting about neutral topics prior to the lesson, seemed to establish my role as a visitor to their classroom. The students accepted my presence in their lessons, some asked for assistance during mathematical activity (Zevenbergen, 1998), and others joked about the tape recorders. The Year 10 students of Lemonwood 3 appeared a little wary at first. I assumed that any previous visitors were probably monitoring individual or class behaviour. I tried to appear unfazed by the social dynamics in the lessons, interested in the student activity and equally involved with the dominant students and the quieter ones. All three

teachers were very welcoming and inclusive. In the current climate of teacher appraisal and surveillance, they may have understandably felt a little nervous on the first day. One teacher provided me with a lesson plan for the first lesson but I reassured her that it was not necessary.

3.6 Analysis

The research orientation influences the analysis of classroom interactions (Cohen et al, 2000; Miles and Huberman, 1994). In any interpretive study, the analysis of the data is critical for advancing our understanding of classroom activity (Cobb, Wood, Yackel and McNeal, 1992). This section outlines the process of analysis including the organisation of the data, the selection of episodes, and the design of the analysis framework. The organisation and analysis of the data reflect the orientation of the researcher. As Wiliam (1998) emphasised, " the definition of what counts as plausible is determined by the discourse within which the validation takes place" (Wiliam, 1998, p. 10-11). Details about the process of data analysis contribute to the confirmability of this study (Guba and Lincoln, 1989). The analysis process is now described.

3.6.1 The Data

The first phase of the analysis process identified meaningful data for this study (Coffey and Atkinson, 1996; Miles and Huberman, 1994). The audiotapes for each lesson were reviewed in conjunction with the field notes and sections of the tape were selected and sent away for transcription. Selection was made by identifying segments of the tapes that contained dialogue about mathematical thinking. These segments represented a mix of whole class teaching events, dialogue among groups of students, dialogue between a teacher and an individual student (sometimes more than one student), and between the researcher and groups of students. The omitted sections of the tapes were typically instances of administration, silent copying of notes by students, some behaviour management discussions, or interruptions from outside the classroom. In some classes, the ambient noise during activity time made transcription difficult and these sections

of tape were left for the researcher to transcribe. The entire teacher interviews were also sent away for transcription.

The returned transcriptions were combined with field notes and other student task details to form a detailed 'lesson transcript' for each lesson. This process identified the student speakers, and annotated information about significant gestures or movements, visually presented information, such as written text and diagrams on the whiteboard or overhead projector, and other relevant observations such as student tasks from textbooks. I used the field notes to set out a record of the main events of each lesson in five-minute intervals, and this was matched with the tape times. The audio tapes were reviewed again and additional detail included.

3.6.2 Transcript Symbols

Lesson transcripts are similar to transcripts of other conversations where sentences are incomplete or ungrammatical (Edwards, 1997; Edwards and Mercer, 1987). Clarke and Helme (1997) also noted that students use pronouns and single word utterances (p. 119), so transcripts are often incoherent as written text. Nevertheless, it is possible to analyse transcripts for patterns of interaction, and to assign significance to the lesson contributions (Weade, 1995).

The conversion of verbal dialogue into written text inevitably compromised some of the qualities of the dialogue. I decided to include transcript symbols to signal key features of the dialogue such as pauses and emphasis on particular words. The selection of the transcript symbols was influenced by the work of Edwards (1997) and Barnes (1999). Simplicity and clarity were also important in the final choice of transcript symbols. I included an additional symbol to indicate a rising inflection. A rising inflection is a well-known speech pattern for New Zealanders and indicates a rising pitch at the end of a phrase. A rising inflection can sometimes indicate a question or a tentative statement but is mostly a feature of the Kiwi vernacular. This symbol was included to add some texture to the transcripts. Commas and full stops were also added.

The symbols used in the lesson transcripts were:

[ ] Data from field notes

Pause in speech, approximately one second

(^) Rising inflection, the preceding phrase has ended with a rising inflection

/ Stress, such as emphasis or raised volume, on the preceding word

(xx) Indecipherable utterances

(to) Best guess of word or phrase

( ) Phrase omitted, usually classroom management or administration. The rationale for omission was to maintain coherency of any mathematical dialogue. In addition, numbers have been written in full, for example, using oh if said as oh to represent zero (Clarke and Helme, 1997), and two point three for the decimal number 2.3.

Repeated listening of some sections of audio tape produced a more complete record of student language and patterns of speech. It was at this third stage of transcript review that decisions were made about the quality of the transcript information so that indecipherable words or phrases were acknowledged in the lesson transcript. The names of all students and the teachers were changed to pseudonyms. The pseudonyms were chosen by the researcher to match the gender and ethnicity of students' and teachers' names (Chazan and Ball, 1999). The teacher interviews were also reviewed and any extracts from this interview data were labelled as (Pseudonym of the teacher, Interview).

Exemplar of Transcript

To illustrate the use of pseudonyms and transcript symbols, a section of lesson transcript is presented on the following page (from first case study Lemonwood 1).

[Question on whiteboard 101.552 round to 1 d.p.]

Teacher	OK then put three dp in. ... Now go to one dp ... [points to the tenths place] by putting a line ... [draws a line between the two 5 digits, 101.5/52] after one decimal place, one number past the decimal point ... Do I change it ... to a six?
Mathew	Yep
Fraser	No
Mathew	Yes
Fraser	Oh yeah
Teacher	Yes, it's five, when its five (^) ...
Fraser	You round it up.

Teacher	You do/ round it up, if it's a bit less than five you don't.
Dean	At primary school we um we didn't round the five up (^).
Teacher	Didn't you? Well that would surprise me because it's the same rule ... it is the same rule for everybody. Um. Five or more Dean.
Timu	Dean (xx)
Teacher	You've got to learn it now, five or more ... you do round up. If I change that to a six, [on whiteboard, right hand 5 changed to a 6] ...now Dean, I noticed you've changed this in here, [points to the hundredths place] (Dean), don't go fiddling with this number, ... it's only/ the one on ... to the left of the line you change ... (1/ L11/ 30/ E7)

A question mark beside or instead of a student's name indicated either a best guess of the student speaker or an unidentified speaker. When three or more students spoke together, this has been labelled as General.

3.6.3 The Lesson Episodes

The identification of episodes has been used in studies of mathematics classrooms to provide focus on particular aspects of activity. Episodes were excerpts of talk that encompassed the topic of discussion (Barnes, 1999). The topic of mathematics discussion could be either from fragments of transcripts or from longer sections of transcript. For example, the emergent studies identified episodes that were between five and twenty minutes long (Bowers et al, 1999). Using videotape as the main method of data collection, the project team identified twenty-six episodes over the time of development of five classroom mathematical practices. Studies of secondary school mathematics classrooms have selected episodes which were named as intervention episodes or events (Barnes, 1999; Clarke and Helme, 1997). In this study, the lesson transcripts were examined for key segments of interaction between classroom participants. The segments were chosen as examples of significant mathematical interaction, and referred to as episodes (see Appendices 2, 3 and 4). An episode encompassed a topic of mathematical interaction and consequently episodes varied in length.

Each lesson generated a different number of episodes and each episode was labelled using the following system:
Case Study Number/ Lesson Number/ Time Interval of Lesson/ Episode Number.

For example, an episode labelled as (1/ L11/ 30/ E7) was an episode from Case Study 1 (Lemonwood 1), Lesson 11, in the interval approximately between 26-30 minutes of the lesson, and Episode 7 for that particular lesson. Labelling the time interval retrospectively enabled episodes to be labelled at 0 time, indicating that this episode occurred before the lesson began. In each case study there were up to five episodes generated entirely by field notes, including post-field notes. These particular episodes were typically from researcher discussion with students before the lesson commenced, or observations and interactions during student activity when the audio tape was not close enough to adequately record the student dialogue.

3.6.4 Analysis of Transcript Data

Transcripts of classroom conversations have been analysed in different ways, depending on the focus of the particular research (Edwards, 1997; Edwards and Westgate, 1987; Kumpalainen and Wray, 2002; Macbeth, 2003; Mehan, 1979; Mercer, 1995). This study takes a distributed view of joint classroom activity. Consequently, important aspects of classroom interactions are who contributes and when, and the nature of the contributions. The first aspect is relatively straightforward to identify from transcript data, but the second aspect relies on making inferences about possible meanings of contributions. This is not an easy or transparent process due to the lack of opportunities to converse with participants at the time and find out their mathematical thinking (Clarke, 1996, p. 3). Consequently, the analysis of classroom interactions demands a high level of inference from the researcher. In addition, the process of inference needs to be articulated as part of the dependability audit (Guba and Lincoln, 1989). In this study, the analysis of lesson transcripts focussed on three aspects of distributed activity: relationships between contributions, mathematical content, and language registers. The analysis process for each aspect is now outlined.

Firstly, contributions were analysed in relation to the joint interaction, and in particular, to previous contributions from participants. Other research has taken a similar approach to classroom interactions. Kumpulainen and Mutanen (2000) conducted a functional analysis of student verbal interaction. The function of a

student contribution was "identified in context of their retrospective and prospective effects on the actual discourse both in content and form" (p. 149). This enabled the analysis of student contributions within the context of the classroom activity. Mercer (1995) analysed teacher contributions in terms of the relationship with the joint dialogue. Teacher contributions were found to be either eliciting knowledge from students, responding to what students say, or describing shared classroom experience. Mercer has also analysed student conversations during small group activity. Talk was analysed in relation to the contributions of others: as exploratory, disputational or cumulative (Mercer, 1995). A similar approach was used in a study of small group mathematical activity (Barnes, 1999). Research findings from two mathematics classrooms (Brown and Renshaw, 2000), identified conversational turns as initiations or responses. Examples of contributions identified as initiations were a request for answer, request for explanation, state-answer, or explain. Other contributions were inferred as a response to previous statements such as restatement, expansion, rephrasing, and evaluation. In addition the researchers noted revoicing, rephrasing, and summarising from both teacher and students (Brown and Renshaw, 2000).

3.6.5 Analysis Framework

I have developed an analysis framework that draws on the work described above of Mercer (1995) and of Brown and Renshaw (2000). The purpose of this framework was to infer how each phrase was related within a jointly constituted interaction such as a discussion, or a question and response sequence (Cobb, Yackel and Wood, 1992b). My analysis of lesson transcripts considered each contribution in turn and considered two types of contributions. The explanatory contributions were typically the beginning of conversations such as when teacher or students initiated a new topic of conversation. Examples of these contributions were descriptions or explanations of mathematical thinking. Explanatory contributions also included follow-up contributions from the same speaker where they expanded, clarified or justified their earlier statement. The second type of contribution encompassed responses made to previous statements or actions. These contributions included some reference to a previous idea such as a

repetition, paraphrase or disagreement. A response contribution might be an answer to a previous question or a completion of another's statement. Responses also included brief indicators of affirmation, validation or agreement. My analysis framework is presented below in Table 3.2.

Explanatory contributions	Response contributions
described	repeated
explained	paraphrased
demonstrated	rephrased
modelled	recorded
summarised	overheard
stated	cued
signalled	re-presented
suggested	acknowledged
asked question	ignored
	selected
	disagreed
interpreted	revisited
expanded	evaluated
clarified	affirmed
justified	validated
	corrected
	responded to question
	completed a sentence

Table 3.2 A Framework for Analysing Transcript Data.

Two further aspects of distributed activity were included in the analysis of transcripts. One aspect was the mathematical content of the contributions. Research has analysed classroom conversations using a content frame rather than an interactional one (Alton-Lee and Nuthall, 1992; Edwards and Westgate, 1987). In a mathematics classroom, twenty categories were used to analyse student contributions to whole class discussions (Lubienski, 2000). The categories encompassed the mathematical content, problem context, social content, reasoning, relating to past learning, visual/tactile references, tone, purpose, correctness, insightfulness, mathematical relevance and difficulty level. By taking a distributed perspective, this study draws on the work of the emergent studies that analysed the roles of tools in the development of classroom mathematical practices (Bowers et al, 1999; Cobb, 1995a; Cobb, Gravemeijer et al, 1997).

Consequently, mathematical content will be analysed in terms of the roles of cognitive mathematical tools in classroom conversations.

The role of language is important for making inferences about the meanings of contributions. Conversational language includes different registers such as informal (student) or formal (mathematical) register. Zazkis (2000) interviewed pre-service primary teachers about learning and understanding of concepts of elementary number theory. She found a range of expressions for divisibility such as "can be divided um by 15 evenly, easily divided into, 3 goes into it um 4 doesn't go into it" (Zazkis, 2000, p. 39). Overall she found " 'goes into' (the) most common informal wording used by student (teachers) to communicate divisibility" (p. 39), illustrating the complexity of analysing student register. The final aspect of transcript analysis was the identification of student and mathematical register within classroom conversations. As Richards (1991) noted, school mathematics is a register distinct to mathematicians' register, so my use of mathematics register in this study implicitly refers to school mathematics.

To summarise, the analysis of the classroom transcripts focused on three aspects. The meaning of each phrase was inferred by relating it to previous utterances. An analysis framework was developed for consistency in this process. Secondly, any reference to mathematical tools was identified in the transcript and the role of tools inferred by the mathematical context of the conversation. Lastly, different registers were noted, in particular student register and mathematical register. In this study, the analysis of classroom transcripts was a complex process because students (and sometimes teachers) made incomplete statements, often reversed terms, and used patterns of both New Zealand culture and teenage speech.

Presentation of Analysis of Transcript Data

Due to the large amount of transcript data generated, 'vignettes' were used as representative exemplars of the transcripts (Miles and Huberman, 1994, p. 81). For each case study, the analysis of the transcript data is presented in two ways. One approach is to reproduce extracts from lesson episodes and then discuss. As well, the transcript and analysis is presented in a table with the following columns.

Speaker	Transcript	Transcript Analysis

This sets out a phrase-by-phrase analysis of each contribution aligned to the relevant section of transcript. The significant points of the transcript and analysis are then briefly discussed.

3.6.6 Member Checking

One crucial aspect of establishing credibility is to check "data, preliminary categories, and interpretations with members of the stakeholding groups" (Guba and Lincoln, 1989, p. 238-9). Member checks can be formal or informal, and with individuals or groups. Member checks serve two main purposes. One is for the researcher to verify constructions. The other is the opportunity for the participants to correct any errors and to offer additional information (Guba and Lincoln, 1989). During the process of data analysis, I used both informal and formal member checks with the teachers but not the students. On a few occasions, I telephoned teachers to check some point of fact, for example, the ages of the students in their class or information about resources used in a particular lesson. The more formal member check occurred approximately seven months after the data collection. I sent each teacher a letter and attached the following: the transcript of their interview, a transcript for one of the lessons and the draft text for 'Background Information' and "Mathematics Lessons Observed' for their particular classroom (see the following section). I asked them to check for any errors in the interview transcript and invited them to make changes. Similarly, I requested that they check the description of the classroom environment (Background Information). In identifying their pseudonym, I also requested that they check that their anonymity and that of their students were preserved in the transcript and the draft text. Each teacher provided me with verbal feedback that approved the pseudonyms and corroborated the information about their classroom.

3.7 Analysis Categories

In naturalistic and interpretive studies, the units of analysis exemplify the orientation of inquiry (Coffey and Atkinson, 1996; Guba and Lincoln, 1989; Miles and Huberman, 1994). Part of the dependability audit is to clarify the units of analysis (Miles and Huberman, 1994), and the identification of lesson episodes has already been outlined. This section now sets out the analysis categories for the study. 'Analysis dimensions' were used to group together related categories. The analysis dimensions and their associated categories are recorded in Table 3.3 below.

Analysis Dimensions	Categories
Background Information	Classroom Setting Social Group Classroom Programme
The Mathematics Lessons Observed	
Classroom Curriculum	Typical Lesson Structure Links between Assessment and Teaching Rituals Roles Teacher Mathematical Values
Tools	Classroom Mathematical Tools Technological Tools Other Mathematical Tools
Classroom Interactions	Teacher Guided Participation Whole Class Negotiation Student Initiated Negotiation Error Initiated Negotiation Repair Interactions Technology Mediated Negotiation

Table 3.3 Analysis Dimensions and Categories

Two analysis dimensions, Background Information and The Mathematics Lessons Observed, provided contextual information, important for a situated social perspective of classroom activity (see section 2.2.3). Three analysis dimensions were organised around the domains of interest. Classroom Curriculum encompassed the analysis of data for the domain of classroom social norms. Similarly, Tools and Classroom Interaction were dimensions for the

analysis of the domains of tools and interaction respectively. Within each analysis dimension, categories were generated for a more fine-grained analysis of the case study data. Some additional categories emerged during the analysis process. Each category is now outlined and examples discussed in order to provide information and clarification of the category definition.

3.7.1 Background Information

Sociocultural studies or those based on ethnographic methods describe the setting and social group in some detail (Bleicher, 1998; Collins and Green, 1992; Jones, 1991; Mehan, 1979; Tuyay et al, 1995; Weade, 1995). These descriptions convey information about the physical and social context such as "the characteristics of the learning environment in such settings and the processes by which learning occurs in the classrooms" (Clarke, 1996, p. 4). The dimension of Background Information was organised around three categories: the Classroom Setting, the Social Group and the Classroom Programme. These categories were used to present a rich description of the context for each of the case study classrooms. The cumulative background data was drawn from the field notes, and scrutinised for stable recurring features of the classroom environment as well as lasting impressions gained over the time of the data collection period.

Information about the setting for secondary school mathematics typically identified key physical resources in classrooms such as seating arrangements and access to resources (Boaler, 1997a). The category Classroom Setting provided information about the physical layout of the classroom as well as the organisation of physical resources used by the participants, such as whiteboards and textbooks. Information about these classroom resources was included because access to resources was potentially related to student participation in mathematical activity. The human participants were described as the Social Group for each class. Relevant information about the social group was the number of students of each gender and their approximate ages. Aspects of classroom organisation were noted, such as where the students typically sat during the lessons, providing detail about proximity to other students. Very brief statements about social relationships among the classroom participants were included to give some sense of the

classroom social atmosphere. Finally, the Classroom Programme recorded detail about the number and timing of mathematics lessons for each week, the personal resources used by the students, such as books and equipment, and other detail such as interruptions to lessons.

3.7.2 The Mathematics Lessons Observed

Information about the mathematical context of the observed lessons can be provided in different ways depending on the research orientation. For example, in experimental classrooms, the research team (which often includes the classroom teacher) may plan the mathematical instruction or learning trajectory (Bowers et al, 1999; McClain and Cobb, 2001). Similarly, teacher-researchers use their own records (Chazan and Ball, 1999; Lampert, 1990). For participant observers, there are generally two choices. The teacher can supply the information, but this may impose implicit expectations for the teacher to cover this material during the observed lessons. Alternatively, the researcher can identify and record the mathematical content as it unfolds over the time of the data collection (Tuyay et al, 1995; Weade, 1995). This was the approach adopted in this study. The information about the inter-lesson sequence, the Mathematics Lessons Observed, was presented as my record of the sequence of mathematical content for each case study. This category also provides a record of the number of episodes selected from the lesson transcripts for each observed lesson.

3.7.3 Classroom Curriculum: Analysis of Social Norms Domain

The dimension of Classroom Curriculum encompassed categories related to classroom social norms. The first two categories provided a brief record of features of the lesson structure and extended the contextual information for each case study. The categories of Rituals, Roles and Teacher Mathematical Values focussed on social norms related to student participation in mathematical activity. Evidence of social norms was inferred from the lesson transcripts and the teacher interviews.

Typical Lesson Structure

The category Typical Lesson Structure described events within lessons, such as whole class activity and individual student tasks. The lesson transcripts and the teacher interview provided information about these lesson events. The typical lesson structure was considered important for revealing assumptions about the value and purposes of particular pedagogical events, because these implicit values may be linked with classroom social norms for each case study (Cobb, Gravemeijer et al, 1997).

Links between Assessment and Teaching

Assessment detail can be recorded as information about the mathematical background of the social group; this is relevant if investigating learning over time, or comparing different teaching approaches (Cobb, Wood, Yackel and Perlwitz, 1992). In this study, I have not sought information about the mathematical background of individual students. Instead I noted the type of assessment activity in a minor category Links between Assessment and Teaching. This provided a record of any formal assessment events, such as written tests, and served as a record of the assessment experienced by students in the case studies. Teacher comments about students' current mathematical understandings were also included.

Rituals

Rituals can be either explicit or implicit. Explicit rituals for small group activity and whole class discussion were set up in emergent classrooms (Cobb, Wood and Yackel, 1991; Yackel, 1995) and in other experimental classrooms (Brown et al, 1993; Herschkowitz and Schwarz, 1999; Lo et al, 1994). In non-experimental classrooms where lesson patterns were already established, rituals were implicit and identified from lesson data (Boaler, 1997a; Hill and Hawk, 2000; Jones, 1991). In this study, Rituals were defined as recurring lesson events observed within or between lessons. Rituals linked to student participation in classroom mathematical activity were identified from lesson transcripts. The participation rituals were described in terms of observed characteristics of these regular events. The teacher interviews were also examined for teacher comments related to rituals.

Roles

'Teacher' and 'student' mask the complexities of classroom roles and relationships and have been found to be inadequate to describe the roles of classroom participants (Collins and Green, 1992). Roles however, can be identified by examining what teachers and students do, and how they respond to actions and interactions in the classroom (Brown et al, 1993; Tuyay et al, 1995). Roles can be constructed across time, and in one-to-one conversations as well as in communal activity (Collins and Green, 1992). One research study identified teacher and student roles, such as generator of knowledge, facilitator of knowledge, and instructor of knowledge. The roles of monitor of knowledge and aligner of knowledge (connecting, signalling), were, however, only adopted by the teacher (Collins and Green, 1992). Elbers and Streefland (2000) identified roles of student collaborator and assistor during four lessons of Grade 8 mathematics. Roles can also be signalled by social cues such as "listen, hands up, listen to XX, look at teacher, look at ...., speak when called on, speak loudly so others can hear" (Weade, 1995, p. 44). Roles are social accomplishments and indicators of social norms of the classroom context.

In this study, roles described ways that classroom participants worked together and the status accorded to their contributions. Roles were inferred from features of shared activity, in both whole class and localised student activity. These features were ways in which students and teacher contributed and the nature of these contributions. Examples of student roles were solution describer, question asker, and answer provider. Ways of working together provided information about collaboration and assistance, between teacher and students, or between peers. The initiation of assistance, the kinds of assistance and resolution of assistance also provided information about roles. Some comments from the teacher interview were also included in this category. In each case study, the category of Roles appears in two places. The first is an Introduction to Roles that briefly describes the roles of each case study. In keeping with a distributed view, I assumed that further information about Roles would emerge in the analysis of classroom interactions. Consequently, at the end of the dimension, Classroom Interaction, the category of Roles continues with a more detailed analysis of the student roles.

Teacher Mathematical Values

While student roles are the major focus of the Roles category, this category of Teacher Mathematical Values focuses on teachers. When Marshall (1992) compared transcripts of class observation and interviews of two Grade 5 teachers, she found links between teacher values and the classroom curriculum. For example, one teacher valued work completion while the other valued thinking and challenge for the students. These teacher values were found to be consistent with their observed classroom programmes (Marshall, 1992). The category Teacher Mathematical Values encompassed the analysis of statements from teachers about the nature of both mathematics and school mathematics. Statements were also identified that related to perceptions about the learning of mathematics and to beliefs about what it meant to be a learner of mathematics. Both lesson transcripts and the teacher interviews were analysed for indicators of Teacher Mathematical Values.

3.7.4 Tools: Analysis of Tools Domain

The Tools domain was analysed using two major categories: Classroom Mathematical Tools and Technological Tools, and a minor category of Other Mathematical Tools.

Classroom Mathematical Tools

Classroom Mathematical Tools refer to the cognitive tools used by the teacher during whole class interactions. These tools were introduced by the teacher as a teaching resource in a lesson event or were assumed as prior knowledge for the students. The focus on the teacher-introduced tools for instruction assumed that these tools had high status and potentially long term currency for classroom mathematical activity. When the use of a classroom tool was identified in the lesson episodes, the preceding and subsequent dialogue was examined for explicit and implicit use of the tool. Any student contributions of cognitive tools were included in the next analysis dimension, Classroom Interaction.

Four groups of Classroom Mathematical Tools emerged from the data analysis: representations, thinking devices, analogous models and pedagogical artefacts.

Representations were written or diagrammatic representations of mathematical entities and were recorded on the whiteboard (Greeno, 1991). Thinking devices were tools used to model mathematical processes, such as calculation processes (Pea, 1993). Thinking devices also included metacognitive tools that assisted awareness of the cognition process rather than the process itself. Analogous models were real life situations used to illustrate mathematical entities (Mousley, 1999). I named the final group of Classroom Mathematical Tools as pedagogical artefacts. Pedagogical artefacts accounted for the diagrams, tables, and other means teachers use to represent connections between mathematical entities. An example of this kind of tool is a chart of the different values of each place in decimal notation. A teacher may use a general tool for a pedagogical purpose, such as the use of a table or other organiser to publicly record contributions from students (Cobb, Boufi et al, 1997). Pedagogical artefacts may be products of school pedagogical and curricular traditions, or devised by individual teachers in the context of their classrooms.

Technological Tools

The second tools category was Technological Tools. Technological tools evident in Lemonwood High School were students' and teachers' hand held scientific calculators. All lesson episodes that involved the use of a student calculator or a statement about the value of using a calculator were identified. Each calculator episode was then analysed in order to identify the role(s) of the calculator in mathematical activity. The analysis used the categories of the roles of the calculator from the Calculators In Primary Mathematics Project (Groves and Cheeseman, 1995), but omitted the role of the counting machine as this role applied to simple calculators rather than scientific ones. The roles of the (scientific) calculator are outlined in Table 3.4.

Category	Role of the Calculator
C1	As a recording device
C2	As a calculating machine
C3	As a number cruncher
C4	As an exploratory object

Table 3.4 Roles of the Calculator (based on Groves and Cheeseman, 1995)

The roles of the calculator in activity were inferred from lesson transcripts. This category also identified the initiation of the use of the calculator (students and/or teacher). The analysis of the mathematical negotiation during these calculator episodes is recorded in a later section, Technology Mediated Negotiation.

Other Mathematical Tools

Cognitive tools that were not already identified as Classroom Mathematical Tools were recorded in a minor category of Other Mathematical Tools. This category was then organised into four groups loosely based on the resources discussed by Greeno (1991): Algorithms, Relations and Operations, Multiple Representations, and Other (eg notation etc). This category emerged from the analysis of joint mathematical activity and accounts for the nested use of tools (Cobb, 1995).

3.7.5 Classroom Interaction: Analysis of Interaction Domain

The final dimension of analysis examined patterns of classroom interaction. Six categories were developed. Whole class teacher-initiated interactions were analysed in two categories, Teacher Guided Participation and Whole Class Negotiation. During the analysis of the transcript data, negotiations initiated by students emerged as important opportunities for student participation in whole class negotiations. These negotiations resulted from the volunteered contributions of students. Three categories of student initiations were identified; Student Initiated Negotiation, Error Initiated Negotiation and Repair Interactions. The final category was Technology Mediated Negotiation for the analysis of mathematical interactions during calculator episodes. In the presentation of these interaction categories, exemplar transcripts are presented in more detail using the analysis framework described in section 3.6.5. The transcript exemplars were also a pragmatic solution to managing the large amount of transcript data. Each category is now outlined in more detail, beginning with the two categories of teacher-initiated interactions.

Teacher Guided Participation

The category Teacher Guided Participation examined teacher-initiated activity that scaffolded whole class mathematical activity. The lesson transcripts were

analysed for indicators of the teacher's explicit and implicit support of student learning, both within and between discrete lessons. This category included such teacher scaffolding actions as cueing, modelling, demonstrating and questioning. The purpose of this category was to provide an overview of the teacher's sequencing or connecting practices that supported negotiation of mathematical meaning.

Whole Class Negotiation

Whole Class Negotiation focused on lesson episodes involving the teacher and students in jointly developing classroom mathematical practices. The analysis examined ways that these new practices were introduced, developed and established. Mathematical contributions from participants were identified and analysed. Examples of these contributions were explanations, demonstrations, use of tools, stories or analogies. Although this section has been titled Whole Class Negotiation, localised peer conversations have been included because these were important aspects of whole class interactions (Nuthall, 2000).

The establishment of a classroom mathematical practice was inferred in the emergent studies when a practice was 'taken-as-shared'; that is, when a practice no longer needed justification (Cobb, Gravemeijer et al, 1997). In this study, where justification and reasoning were not necessarily part of the explicit norms of the classroom, I have adopted an alternative stance. Consequently, a classroom mathematical practice is considered established when there is evidence that students have appropriated the practice. Evidence of student appropriation is either verbal articulation of the practice by students or responses that indicate an acceptance of the practice.

Student Initiated Negotiations

The category Student Initiated Negotiation referred to interactions that were initiated by a student contribution, such as an unanticipated question, an unsolicited method, disagreement or resistance. These student contributions initiated further opportunities for joint negotiation of mathematical meaning. This category included whole class episodes or peer episodes among groups of students that may not necessarily have involved the teacher.

Error Initiated Negotiation

The category Error Initiated Negotiation encompassed interactions initiated by a student mathematical error in a verbal answer, solution method or reasoning. The subsequent interaction was analysed with a particular focus on how the error was resolved. Again, these episodes were possible in both whole class and localised activity.

Repair Interactions

Repair Interactions was the last of the student-initiated categories. The notion of repair was taken from Edwards (1997) and referred to a repair to the communication process. When a misinterpretation or incomplete meaning occurred, a repair interaction may be initiated by teacher or students. The initiator might then attempt to repair the dialogue by either repeating or reforming the original statements. This category could also include abandoning rather than re-negotiating a perceived mismatch of meaning.

Technology Mediated Negotiation

Technology Mediated Negotiation was the final category in the dimension of Classroom Interaction. This category expanded on the previous category of Technological Tools with an analysis of calculator episodes that involved mathematical negotiation. These episodes included discussion of calculator functions, mathematical resources produced by the calculator or other contributions influenced by the different roles of the calculator in classroom activity.

To summarise, the data analysis was organised around key dimensions that clustered together related categories. The analysis dimensions reflected significant aspects of this study, such as:

- The importance of the classroom context: physical, social and in terms of the mathematics curriculum.
- The significance of rituals, roles, and teacher mathematical values in mutually establishing normative social practices in each case study context.
- The roles of cognitive tools in the negotiation of mathematical meanings.

- The development of classroom mathematical practices and the whole class and localised negotiation of mathematical meanings.

3.7.6 Further Analysis

The preceding dimensions and categories for analysis were chosen to generate sufficient depth of detail without losing significant features of each case study. The case study data is reported in Chapter 4. In chapter 5, this case study data is combined for each category and the analysis is extended. Significant features of each category are analysed in terms of affordances and constraints on negotiation of mathematical meaning. This further analysis aims to address any relationships between the domains of distribution for mathematical activity: that is, classroom social norms, tools, and interactions.

3.8 Chapter Summary

This chapter has outlined the major theoretical influences on the methodology of the study. An interpretive approach was adopted in order to investigate joint mathematical activity over a series of lessons. A case study approach enabled three classroom contexts to be studied and defined as cases. Participant observation and interviews of teachers were the methods chosen for this study. The design of the research was outlined and issues of accountability were clarified and discussed. This provided a record of the resolution of issues during the time of the design, organisation and data collection.

Analysis of the data involved a layered approach to the analysis of the transcripts and to categorising the data. Some categories were used to initially organise the data while other categories emerged from the data analysis. The analysis categories were based on aspects of contextual information for each case study, and the analysis of lesson activity relevant to each of the three domains of classroom social norms, tools, and interactions. The presentation of the case study data is organised around these analysis categories. The case study data analysis is now presented in the next chapter.

Chapter 4
Case Studies

4.1 Introduction

This chapter records the analysis of the data for each case study classroom. The school setting is introduced first and provides background detail about the school context for the three classrooms. Lemonwood High School was a large urban co-educational secondary school that catered for Year 9 to Year 13 students, thirteen to seventeen year olds. The contributing schools were a nearby Intermediate school (Years 7 and 8) and a number of full primary schools. Although many students lived relatively close to the school, the enrolment zone included some outlying areas, both rural and urban. Students from these areas travelled on school buses, sometimes arriving late for the first period (lesson) of the day. Lemonwood High School was approximately forty years old with mostly wooden buildings based around the 'Nelson' block design. There were also a number of single storey teaching blocks, both re-locatable and of permanent construction, with playing fields at the front and rear of the school.

Lemonwood High School catered for the full range of curriculum subjects with specialist areas such as special needs, careers, bilingual programmes, counselling and international students. The school behaviour management system was based on consequences for infringements of school rules and a chart of the school code was on the wall of every classroom. Most teachers were allocated teaching rooms and students moved to different rooms each period. The Mathematics Department was based in a two-storeyed building and in nearby relocatable classrooms. Although the department had a resource room, teachers kept many of their teaching resources in their classrooms.

This brief information about the physical setting and the school organisation describes the common school context for the case study classrooms. Each case study is described in turn, beginning with one of the Year 9 classrooms labelled as Lemonwood 1.

4.2 Case Study Lemonwood 1

4.2.1 Background Information

The Classroom Setting

The Lemonwood 1 classroom was on the ground floor of a two-storeyed building. There was a whiteboard and an overhead projector screen on the front wall of the room. The overhead projector was on a trolley and moved into position when needed. A number line was drawn in permanent marker at the top of the whiteboard and on the right hand side of the whiteboard was a section marked off for the current homework expectations for each of this teacher's classes. The desk for the teacher was in the middle front of the room between the class and the whiteboard. The 32 student desks filled most of the remaining floor space and were arranged in rows of two along one side of the room, and groups of three or four desks in the remaining space. During some lessons, groups of four desks were rearranged into two rows in response to a request from the teacher. There was very little space for the teacher or students to move around the room between the groups of desks. There were posters on the classroom walls, some produced by students, and a few more formal, including a poster listing problem-solving strategies. A fire exit was at the back of the room and, during one of the lessons with this class, there was a fire alarm.

The Social Group

Lemonwood 1 was a Year 9 class with 31 students, 13 girls and 18 boys, all aged thirteen years at the time of the study. The students were in their first term at secondary school and had come from a variety of contributing schools. The class was designated by the school as a lower band class based on information from the contributing schools, including information about mathematics performance. The teacher, Mrs. Johnson, was an experienced teacher who had taught at Lemonwood High School for a number of years. She was also the class Form Teacher with responsibilities for administration and pastoral support of these students.

It is commonly assumed that, during the first year of secondary school in New Zealand, student friendships and social groupings are formed. In Lemonwood 1, students chose where they sat in the classroom and, during the time of the study, it was noticeable that some students sat in the same places next to the same students, while others sat in different places among more fluid groupings of students. The teacher recognised the social benefits of the students sitting together and noted that the stability of the arrangement of desks had positive social outcomes.

> ... when I set the desks up for their social groups this year. But every time, you know if I turn it back for a test, they don't like it, ( ) ... They are creatures of habit too and they tend to gravitate to the same ... group ... (Mrs. Johnson, Interview)

The students in Lemonwood 1 seemed to get along well with each other. Although many had formed friendship groups, these did not appear to be exclusive. Two students, one girl and one boy, were socially isolated with apparently few friends but they seemed to regularly interact with nearby students. The second language students were in stable friendship groups and participated in many of the class and school activities. During the time of the study, the class had a basketball team in the Year 9 lunchtime competition and many students, and often Mrs. Johnson, watched these lunchtime games. There was a banner made by some students for this competition and, towards the end of the term, it was placed at the front of the classroom.

Focus Students

The Lemonwood 1 focus students were Dean and Mathew (who usually sat together), Timu, Ryan, Fraser, Janine, Natalie and Katrina. Consequently, these students appear more often in the transcript data than other students in the class.

The Classroom Programme

This class had four mathematics lessons per week: three were scheduled for sixty minutes and one for fifty minutes. The students were expected to bring to class their 'squared' mathematics book and their scientific calculator. On two occasions, the teacher reminded students who had left their calculator at home. Each student also had a homework book of practice exercises, *Alpha Mathematics Homework Book* (Barton, 2000). The copies of the mathematics

textbook for this class, *Alpha Mathematics* (Barton, 1999), were kept on shelves at the back of the room and were handed out to students by the teacher when needed. During the Friday lesson, the teacher marked and checked the homework for the week, monitored the students' homework diaries and sometimes collected the homework books for marking over the weekend. When this occurred, part of the Monday lesson included a review of aspects from the homework. Students who had not done their homework, or had not handed in their homework books, were reminded by the teacher about the relevant school policies. The Monday mathematics lesson was the first of the school day. The next two mathematics lessons were after morning interval and the final lesson for the week was Friday after lunch. Consequently, the classroom was locked and the students were expected to wait outside the room until the teacher signalled for them to enter. Some lessons were interrupted by the school 'runner' or by a request for a student to leave class for an appointment. Lesson 9 was interrupted by a fire alarm and the class returned with ten minutes of the lesson time remaining.

4.2.2 The Mathematics Lessons Observed

The mathematical focus of each lesson is recorded in Table 4.1 and includes the number of episodes identified from each lesson transcript.

Lesson	Week	Day	Main Lesson Focus	Number of Episodes
1	1	Fri	Decimals	2
2	2	Wed	Decimals	5
3	3	Mon	Factors	0
4	3	Wed	Factors and Prime Numbers	6
5	3	Fri	Prime numbers and Multiples	4
6	4	Mon	Factors	3
7	4	Tue	Powers	9
8	4	Wed	Revision and Integers	4
9	4	Fri	Addition and Subtraction of Integers	9
10	5	Mon	Revision and order of operations	9
11	5	Tues	Revision and integers	14
12	5	Wed	Revision	9

Table 4.1 Record of Lemonwood 1 Observed Lessons and Episodes

All Year 9 mathematics classes at Lemonwood High School were working from the same school scheme and in the first term (10 teaching weeks), after a 'unit' on angles, the main teaching focus was Number. Lemonwood 1 worked on number topics until the end of the term. Twelve lessons were observed over a period of 5 weeks. When there were tests or the class teacher was absent, the lessons were not observed (section 3.5.1). Lessons 3 to 12 were observed in sequence.

4.2.3 Classroom Curriculum

Typical Lesson Structure

The Lemonwood 1 lessons were observed to begin with a starter quiz of five to ten tasks. These tasks were marked by the whole class, and were followed by a teaching episode. This usually involved exposition by the teacher, followed by a worked example. Students copied brief notes from the overhead projector or sometimes the whiteboard. Then the teacher either provided practice tasks or specified exercises from the textbook. During the practice examples, the teacher walked around the room, responding to student requests for assistance, choosing certain students to talk with and generally monitoring student progress on the tasks. These tasks would then be marked, usually as a whole class. Finally, homework was set from the homework book and recorded on the whiteboard. The exceptions to this lesson structure were Lesson 3, when the class undertook a structured investigation called The Licorice Factory (Lovitt and Clarke, 1992), and Lesson 8, which included a whole class brainstorm about negative numbers and the compilation of a whiteboard structured diagram of student ideas.

Links between Assessment and Teaching

All Year 9 classes in the school completed a P.A.T. (Progressive Achievement Test) in mathematics (Reid, 1993). Lemonwood 1 sat this test in the first week of data collection. There was an end of term common test about number which this class sat on the last day of the term (the lesson after Lesson 12). In the week leading up to the common test, the teacher indicated what the students needed to know for the test. The starter quiz questions and homework often included practice of whole number calculations. The teacher had perceived gaps in the students' mathematical knowledge.

> ... I suppose (what) I noticed was the huge range of children. ... Some come in with huge gaps in their knowledge of just basic maths. I think at this stage the kids sort of know their basic arithmetic. You've got to go right back to adding, subtracting, multiplying, (Mrs. Johnson, Interview)

Rituals

Rituals were recurring lesson events observed within or between lessons (section 3.7.3). The Lemonwood 1 teacher structured the lessons with a progression of different types of activity.

> ... so I try and keep the lesson structure, like the beginning. And then ... what I really want to teach them, and exercises ... and then a puzzle or something at the end. (Mrs. Johnson, Interview)

Two of these ritualised lesson events were identified as important opportunities for student participation. These were the Starter Ritual and the Marking Ritual, and are now described in more detail.

Starter Ritual

Lessons usually began with a starter quiz of between five to ten revision questions. The questions were a mix of one step calculations or word problems either called out by the teacher, written on the whiteboard or an overhead transparency. Designed to review previous mathematical learning, these tasks could also include the content of the day's lesson. The teacher valued this starter quiz as a useful social practice for both herself and the students.

> ... that's sort of an underlying structure, books out, question number one and then I've got a couple of minutes. ... I'm a creature of habit and routine and I find that settles them down anyway, ...(Mrs. Johnson, Interview)

The Starter Ritual occurred in ten out of the twelve lessons observed in this case study. It provided an opportunity early in each lesson for students to engage in mathematical activity. During this ritual, students talked about the tasks with each other and the teacher was available for assistance. The exception to this was in Lesson 5 when the starter was an oral quiz, followed by a series of flash cards of multiplication facts, and was a silent non-collaborative activity. Tasks from the Starter Ritual often re-emerged later in the lesson as topics of conversation.

Marking Ritual

Another Lemonwood 1 ritual was the regular and systematic marking and review of mathematical activities, including the starter quiz, individual practice tasks and homework. Answers were often called out by the teacher; at other times the teacher would ask students to provide the answers. If the questions had been written on the whiteboard or overhead projector, the teacher might also write the answers as a record for students. The teacher usually set about five in-class practice tasks at any time so that marking and review, like that of the Starter Ritual, was relatively immediate. Most lessons included at least two cycles of practice and review of tasks. During the Marking Ritual, students asked questions, disagreed with the answers or told the teacher they had used incorrect numbers, usually by misreading the question. The students seemed to be concerned about whether their answer was correct or incorrect. Sometimes students' incorrect answers stimulated further discussion. At other times, the teacher asked a student to explain their method. This provided opportunities for students to describe methods and provide explanations that could lead to further discussion. These review episodes were important opportunities for student participation in whole class negotiation of mathematical meanings.

Introduction to Roles

Another aspect of participation was the variety of roles adopted by Lemonwood 1 students during mathematical activity. The roles related to students as contributors were identified as solution describer, mathematical explainer, and question asker and answerer. Some student contributions were quite brief, such as a word or phrase; others were more detailed explanations or descriptions. Student contributions were often volunteered but also provided when called on by the teacher. While some students were more likely than others to volunteer solutions during the public teaching episodes, the teacher often called on quieter students for answers to tasks. In this classroom there was an explicit expectation that the students listened to the contributions of peers, commonly expressed in school parlance as "no talking when other people are talking". The teacher monitored this expectation and intervened in situations when she thought that students were not listening to their peers. (Example from Lesson 9 transcript)

> Listening ... nobody, we don't talk over anyone in the room, ...

Similar teacher instructions were observed in most lessons in response to high noise level or lack of attention to peer statements, indicating that the teacher considered this to be an important part of her role. The teacher often recorded student methods on the whiteboard as class exemplars. In this way, student roles of both contributor and listener were accorded high status in whole class and localised activity.

Students working together involved roles of collaboration with each other and of assistance. The expectation of student collaboration was implicit rather than explicit. There were no instances of an activity or task that had been explicitly organised for a shared group process. Student collaboration was, however, evident in their talk and actions during whole class interactions and during individual tasks. For example, during whole class interactions, students (usually neighbours and more likely to be friends) contributed additional ideas, particularly if it appeared that their friend was having some difficulty. During individual work, students discussed aspects of their solutions, compared answers, asked questions or asked for assistance from their neighbours.

Requests for assistance had slightly different forms depending on whether it was a public event (whole class) or a localised event (student tasks). An explicit appeal for assistance could be made to either the teacher or to other students sitting in the group or nearby. During localised activity, student-teacher interactions were semi-public as peers sitting nearby often listened, and sometimes contributed, to these conversations. During the individual tasks, students raised their hands to request teacher assistance or called out to the teacher. Some students waited until the teacher was near them before requesting assistance. As well as the role of assistance requester, students were also assistance providers, mostly to other students but sometimes to the teacher. Students responded to requests for assistance and contributed a variety of responses, depending on the context. No instances of refusal to provide assistance were observed. Student roles have been briefly outlined at this point to provide information about the social context of Lemonwood 1. Roles are revisited in the Classroom Interaction dimension with specific examples of the roles identified in this case study.

Teacher Mathematical Values

Teacher Mathematical Values were related to two issues: mathematics as a subject, and what it meant to be a learner of mathematics (at this level). The Lemonwood 1 teacher presented mathematics as concise, efficient and comprised of rules. Mathematics as a concise and efficient subject was implicit when the teacher introduced powers as a shorter way of expressing multiplication.

> You can say that was two to the power of nine ... and it's, it's much shorter/, isn't it much shorter to write this as two to the power of nine ...() than ... multiplying it by itself that many times (Lesson 7 transcript).

This was also the case when the teacher introduced square root as more concise because, "I don't want to keep saying what number multiplied by itself gives ... 49" (Lesson 7 transcript). These were explicit and implicit messages about how mathematical ideas are expressed symbolically and verbally. There was frequent use of the word 'rule' by the teacher to describe a mathematical procedure. Mathematics was portrayed as an internationally consistent body of knowledge consisting of authoritative procedures or rules. In Lesson 5, during the copying of notes about rules of rounding, Janine called out to the teacher, "How come they are called rules?" The teacher response revealed an implicit value that rules were important enough to be notes and that these were what they, as students of mathematics, needed to learn. More implicit messages about mathematics as rules arose in later lessons. In Lesson 9, the teacher made a point about the generalised nature of rules in this number topic.

> Now there are certain rules when we add/ and subtract ... integers ... and if you listen carefully, get the rules right ...you can do every single question (Lesson 9 transcript).

This statement implied that the rules of mathematics were important for students to remember. When the teacher provided notes for students, these were often recorded as mathematical rules. The teacher explained the purpose of these notes.

> Well I'm a routine sort of person. ( ) I like to have a good set of notes so I give them a good set of notes, and they think they can remember it. I mean, I've got a son who's hopeless at writing notes and there are bits of paper all over the place and when it really comes to it, he hasn't a clue what's happening because he hasn't got anything he can go back to. So I guess that's sort of the way I learned so I tend to, the way I teach. (Mrs. Johnson, Interview)

The teacher viewed notes as an important reference for the Lemonwood 1 students when they were doing homework or revising for an assessment.

Additional messages from the teacher about what it might mean to be learner of mathematics arose during the Marking Ritual. For example, in Lesson 7, after the marking of some homework, the teacher emphasised that reading was an important aspect of mathematics.

> Right ... Some of it is just being alert and ... reading properly. Maths is reading too. ... Leeanne, will you turn around please.... And some questions, there's a lot of reading and you have to be... quite good at English to be good at some aspects. You need to read the question properly. Natalie. (Lesson 7 transcript)

And as the common test drew nearer, the teacher reiterated important aspects for the students to know and provided suggestions for revision.

Teacher	You can see this subject, this number subject is very, very large, there's lots and lots to do in it. ...And it's, your test, I looked at your test, it's quite long and it's quite, there's a lot in it.
Dean	How long will it take to do it?
Teacher	Well you're doing it on Friday, it's a whole period. Yes, and you need to study for it, you need to go right over your notes and you need to do all these things I'm giving you now/ are in your test. (Lesson 11 transcript)

In other words, studying for a test involved the review of the copied notes and the practice of examples from lessons.

4.2.4 Tools

Two categories of tools, Classroom Mathematical Tools and Technological Tools, were observed to play an important role in the mathematical activity in Lemonwood 1. These tools and their roles in classroom activity are now described.

Classroom Mathematical Tools

Classroom Mathematical Tools encompassed the cognitive tools used by the teacher in whole class interactions. The roles of the Classroom Mathematical Tools were categorised as representations, thinking devices, or analogous models.

Eight Classroom Mathematical Tools were identified in Lemonwood 1. These are set out in Table 4.2 below.

Classroom Mathematical Tool	Mathematical Use of Tool
Number Line	Model representing relative magnitude of integers Thinking device for addition and subtraction
Empty Number Line	Model of relative magnitude of decimal numbers For demonstrating the recomposition of 7.9
Chocolate Bar Drawing	Analogous model of relative magnitude of decimals
Rounding Dash	Marker to distinguish designated place for rounding
Factor Tree	Diagram for modelling succesive division by prime numbers Record of prime factors
Lift in a Building	Real life example of a number line Analogous model of subtraction process
Overdraft	Real life example of deficit, negative numbers
Thermometer	Real life example of a number line

Table 4.2 Classroom Mathematical Tools for Lemonwood 1

The Lemonwood 1 teacher used a range of tools in the introduction and development of classroom mathematical practices. The teacher explicitly encouraged the students to incorporate these tools into the 'rules', mathematical practices with high status in this classroom. Tools were also used in response to student questions or errors and some of the real life examples were these supplementary tools. The Number Line is discussed first because it was identified as an important tool for practices related to integers.

Number Line

The main classroom mathematical tool identified in Lemonwood 1 was the Number Line. A vertical number line was implicitly contributed by the teacher in Lesson 4 in response to a student question about subtraction. The teacher drew a diagram on the whiteboard to represent a multi-storeyed building, where each floor was labelled. In Lesson 8, an implicit vertical number line emerged again when a student volunteered that temperature was a real life situation of integers. It was a hot summer's day and the teacher contrasted the current temperature with

a negative temperature. There was a thermometer hanging on the wall of the classroom and the teacher asked the student sitting nearest it to read the temperature. The teacher then implicitly linked the temperature scale of the thermometer to a vertical number line.

[Teacher pointing towards the thermometer hanging on the wall]

> Twenty four, yeah, and you can see where the zero, you can see where the zero line is and anything below that, ... it starts to get into the negative temperatures, so that's one way you really use the negatives. Where else? (1/ L8/ 20/ E2)

In the next lesson, the teacher was more explicit about how a thermometer scale related to the horizontal number line constructed in permanent marker at the top of the whiteboard.

> Okay, so we've got our positive numbers on the right. ... We've got our negative numbers on the left. ... Now, ... if, using this number line ... ... here, those signs are outside the thermometer, isn't it, which is on its side. (1/ L9/ 25/ E3)

The thermometer now was an analogous model that represented the relative magnitude of positive and negative numbers. The teacher then used the whiteboard Number Line to demonstrate the dynamic process for addition and subtraction. This was formally presented as a rule: that addition was to the right and subtraction was to the left. The first addend became the starting point for the rule provided by the teacher; she suggested that the students put their pen on this starting point, and modelled the required number of 'hops' along the Number Line. During these demonstrations the students participated in the solutions and sometimes chorused responses back to the teacher (Lesson 9 and 10). The continued use of the Number Line by students was evident in the whole class and individual student activity. Some students were observed using the Number Line they had drawn in their books, or using their ruler as a physical model of the Number Line. Other students were observed looking at the whiteboard Number Line and silently counting or nodding (from Lessons 9, 10 and 11). These actions were inferred to have been a mental use of the Number Line during student activity.

Rounding Dash

The Rounding Dash was introduced as a mathematical tool for the mathematical practice for rounding numbers. The Rounding Dash was a line drawn between the digits, after the place that had been designated for rounding.

[On whiteboard 101.552 round to 1 dp]
Teacher () Now go to 1 dp ... by putting a line . ... after one decimal place, one number past the decimal point. [101.5/52]
(1/ L11/ 30/ E7)

The Rounding Dash implicitly cued the next steps of the rounding process. To the right of the dash was the digit to be scrutinised for either rounding up or down. The digit to the left of the dash was then either left the same or changed if rounding up. The Rounding Dash was a tool used as a marker to signify important digits for the rounding process. This offloaded the mental effort needed for remembering the required place for rounding. By acting as a cue for further steps in the rounding process, the Rounding Dash re-structured this mathematical activity for the students.

Some of the Classroom Mathematical Tools will be discussed in more detail in the following section on Classroom Interaction. For example, the Empty Number Line was used by the teacher to resolve a student misconception about decimals (see section 4.2.5). Similarly, the Chocolate Bar Drawing and Overdraft are briefly discussed in the next section. There were no interaction episodes identified with the student activity involving the Factor Tree.

Technological Tools

The technological tool used in the Lemonwood 1 classroom was the hand held scientific calculator. Nine episodes of calculator use were observed. The roles of the calculator were identified from the calculator episodes. Data about who initiated the use of the calculator (students and/or teacher), and the nature of each observed calculator event, are set out in Table 4.3 (Categories in brackets indicate a secondary role).

Lesson	Calculator Event	Initiated by	Category
2	Checking of decimal multiplication	Students	C2 - calculating machine (C1 - recording device)
7	Method for finding powers	Teacher	C4 - exploratory object (C2)
7	Method for finding squares	Teacher	C4 (C2)
7	Method for finding square roots	Teacher	C4 (C2)
7	Student sharing of the multi-line function	Student	C4
10	Checking of order of operations examples, whole numbers and decimal examples	Teacher	C2 (C1)
11	Finding square roots	Teacher	C2 (C4)
11	Finding cube roots	Teacher	C4
12	Operations on integers	Teacher	C4 (C2)

Table 4.3 Categories of Calculator Use for Lemonwood 1

In Lemonwood 1, the main role of the calculator was as an exploratory object (7 episodes). Students experimented with alternative methods of calculations, including shortcut methods. The teacher validated and supported student experimentation in different ways. In Lesson 7, the teacher signalled to the students that they had to learn how to use their own calculator and used driving (a car) as an analogy for expertise with the calculator.

Teacher ... Now let me say this ... Sshhh. You have to learn ... listen ... you have to learn how to drive your own calculator.
Dean Drive it.
Teacher Yeah, well because ...
Tony Mine's only a manual!
Teacher These, these calculators are different. You've all got different kinds. ... So you need to work out ... sshhh ... I don't like talking over you. You need to work out how to do it on your/ calculator. Now sometimes, like if you've got a Casio, it's different from a Sharp and you have put it in, in a different way. ...
(1/ L7/ 35/ E6)

The teacher explicitly included the two brands of calculator used by the students. The different calculator brands had different keys and operating sequence, and this provided both a teaching point and an opportunity for further mathematical exploration. The teacher explained how the different calculators operated, and recorded the different keys on the whiteboard. This is illustrated by the following extract about the addition and subtraction of positive and negative integers.

Teacher Right have a look down/ on the bottom right, there's a plus/minus sign. ... Everyone find that key ...you've either got a key like that or a key like that.
[Teacher writes on whiteboard
+/- (for Sharp)
(-) (for Casio)]
Dean I've got it at the bottom.
Teacher You've got the bottom one
Janine I've got a key at the top
Teacher Right, The people that have got the one like the plus minus ... I will tell you first how to do it.
(1/ L12/ 30/ E8)

The student interjections as they focussed on their calculators were typical of those during exploration episodes. The teacher then suggested trying out an example with a known result and to use this as a guide.

> Now will you look for your square root sign, and ... try it on a nine ... the square root with your calculator ... the square root ... nine ... now if you get error, try putting the nine in first and then doing square root. ... Some of them will be different, ...
> (1/ L7/ 40/ E7)

The teacher encouraged student experimentation by providing support and feedforward for any situations that generated a calculator error message. The teacher walked around the room with her calculator during this episode looking at what the students were doing. By having two brands of calculators and validating the use of each one, the teacher implicitly supported different ways of generating calculations. In the joint exploration of powers, square and cube roots, and operations of integers, the calculator was an exploratory tool. Students found new ways of using the keys and functions, including more efficient ways of processing the information into the calculator. Two of these episodes are outlined in the section Technology Mediated Negotiation (section 4.2.5).

Another consequence of this exploratory role was the impact on localised student conversations. Different calculators provided opportunities for peers to help each other and many were interested in how the other calculator worked. Students discussed for example, whether both brands had a square key. The following extracts are from conversations between neighbours.

Ryan	Is that the squared?
Mathew	No, that one there,
Janine	Square it
Natalie	Right
Janine	Oh Mrs Johnson, Is that the square button?
Natalie	Up the top there.
	(1/ L7/ 35/ E6)

Student activity with calculators initiated localised peer discussion. This typically involved each student trying out the example on their own calculator while saying the process out loud or making other comments. When there was a disputed result or an error message, the students would try again and, if not satisfied, might then lean over to look at their neighbour's calculator screen. This led to further shared negotiation. While individual exploration and experimentation were features of this case study, whole class interactions emerged from individual activity with a calculator. These episodes are discussed in more detail in a later section, Technology Mediated Negotiation (section 4.2.5).

Other Mathematical Tools

This category records cognitive tools not already identified as Classroom Mathemtical Tools. These tools were identified from the Lemonwood 1 lesson transcripts, similar tools grouped together, and are presented in the table on the following page.

Tool	Examples of this Tool
Algorithm	Subtraction algorithm for whole numbers Multiplication algorithm for whole, and decimal numbers Division algorithm for whole numbers
Operations/Relations	Addition and subtraction of integers Multiplication and division of integers Powers Square and cube roots Combinations of operations (BEDMAS)
Multiple Representations	Numbers represented by factors and by prime factors Numbers recomposed, decomposed Fractions and decimals
Other (eg notation)	Comparison of relative magnitude (< and >) Whole numbers and decimals - Place value notation

Table 4.4 Other Mathematical Tools Identified in Lemonwood 1

4.2.5 Classroom Interaction

This dimension encompassed the analysis for the domain of interaction for both teacher initiated and student initiated interactions. The first two categories record the analysis of teacher initiated interactions.

Teacher Guided Participation

Examples of Teacher Guided Participation emerged during the twelve lessons observed in Lemonwood 1 (section 3.7.5). These were scaffolding actions that supported student learning. For example, new classroom mathematical practices were introduced via students' assumed prior knowledge. Within each topic, the sequence of the student tasks set by the teacher revealed increasing complexity. This complexity involved, for example, increasing number of decimal places, or progressing from squares to square roots. The teacher explained that the relationship between student prior learning and the teaching sequence was deliberate.

> ... and I like to make sure that I go back to the last piece ... to make sure they understand and then bring them up to the rest of the others. (Mrs. Johnson, Interview)

Two other aspects of Mrs. Johnson's teaching were identified as supporting and guiding student learning. The first was her acceptance of discussions about

primary school mathematics. On a few occasions, students used a discourse of primary school mathematics to articulate a belief contrary to the accepted mathematics. They began their contribution with phrases such as "At primary school, we got told ...", "Last year we were shown ..." or "I got told ...". These phrases distanced the students from the mathematical idea they were proposing and this may have allowed them to publicly declare a contradictory meaning. The teacher actively participated in discussions about primary school mathematics (see following transcript).

The second type of teacher scaffolding action was the acceptance of student register when students were explaining their mathematical ideas. This is illustrated during a whole class review of the relative magnitude of whole numbers before introducing a new classroom mathematical practice in the following lesson. The transcript and analysis are presented below.

Speaker	Transcript	Transcript Analysis
Teacher	If I said to you six and four, which number is bigger, six or four?	Teacher recorded 6 and 4 on whiteboard, question to whole class about relative magnitude, using familiar numbers
Shelley	Six.	Shelley volunteered that 6 (is the bigger number)
Teacher	Six. Who knows the sign that you put for that?	Teacher repeated Shelley's response, asked the class about their knowledge of the convention/sign for bigger than (implicit).
Scott	The alligator.	Scott volunteered his term for the 'greater than sign' (alligator as student register)
Teacher	The alligator ... is greater than, and I'll give you a note on this in a minute ... is greater than, now ... you'll remember probably at primary school they teach you that the alligator's mouth ...	The teacher appropriated Scott's response, validated the student register for this sign (the alligator) and linked it to mathematical register (is greater than). Teacher signalled the provision of notes, recorded > on the whiteboard. She emphasised primary school experiences, i.e. the analogy of the alligator's mouth as nearest the biggest number. Implicitly asked for student response.

Shelley	It's the bigger one.	Shelley volunteered to complete the teacher's sentence by identifying the bigger number as closer to the alligator's mouth (rather than throat!).
[Janine	Cos he spat it out] [Janine talking loudly to herself]	Janine volunteered her way of remembering this analogy, implicitly that the alligator spat out the largest number (semi-public contribution).
Teacher	It's the bigger number and that's a good way to remember ... ... What sign would I put between those two numbers?	Teacher repeated Shelley's response with student register, and validated this analogy as an effective way to remember the convention of these signs. Wrote 4 and 6 on board and asked class which sign should be written between 4 and 6.
Susan	Goes the other way, the mouth pointing	Susan volunteered response, referred to the > on the whiteboard, that it should be the other way, <, with the mouth pointing (implicitly towards the 6, the bigger number).
Teacher	Turn around and that is called is less than, so that's how you read that sign. ... That one means is greater than ... and that one ... that means what?	Teacher rephrased Susan's answer as turn around, (flip the sign), and stated the name of this sign (mathematical register). Reviewed the meaning of the sign, pointed to the >, and then pointed to the sign in 4 < 6. Asked question to assess the meaning of <.
Travis	Is less than.	Travis volunteered response, using mathematics register.

(1/ L8/ 25/ E3)

Figure 4.1 Signs for Greater and Less Than

The teacher attended to student prior knowledge and supported a discourse about primary school mathematics experiences. This provided a space for the students to declare their methods or reasoning without having to personally justify these. The above transcript encapsulates many of the teacher scaffolding actions evident in whole class negotiations. The Lemonwood 1 teacher was observed to support and guide student participation by:

asking for student responses,

accepting volunteered student contributions and student register,
linking student register to mathematical register,
indicating mathematical status,
validating a student contribution as a good way to remember, and
checking the use of the mathematical sign in another situation.

Whole Class Negotiation

This category examined the joint development of classroom mathematical practices. Two teaching incidents from Lesson 9 illustrate typical whole class negotiation for Lemonwood 1. The lesson focus was the development of classroom mathematical practices for the addition and subtraction of integers.

The Number Line Rule

In Lesson 9, the teacher introduced the notion of using 'rules' for addition and subtraction of integers. The examples she demonstrated had first addends that were positive or negative but involved the addition or subtraction of only a positive integer. The teacher used the Number Line to model the process of both operations; addition was to the right and subtraction was to the left on the Number Line. The demonstration of this rule was followed by the whole class solutions of six one-step addition and subtraction examples of increasing complexity. By the final example, students were readily responding as a chorus to the elicitation-recitation pattern. The associated transcript and analysis are presented below.

[Task -1 –3 = was written on whiteboard, horizontal Number Line drawn permanently at top of the whiteboard]

Speaker	Transcript	Transcript Analysis
Teacher	...That's where I start ... negative one ... left or right?	Indicated starting point on number line of negative one, (the first step of the rule). Asked class for next step, which direction.
General	Left.	Chorused response volunteered by many students, signalled recognition of subtraction as moving to the left of the number line.
Teacher	Left. How many hops?	Teacher repeated students' response. Asked question in terms of hops, the informal student register. Implicitly a hop was counting one number along the number line.

General	Three.	Chorus response for number of hops, suggesting the students interpreted subtract three, as three hops to the left along the number line
Teacher	One, two, three, I end up at ...	Teacher demonstrated this by pointing to the number line as she counted the number of hops. Asked for end (point), by a complete-the-statement question.
General	Four, negative	Chorus response (final) number first, then some of the chorus attached the negative (integer) sign. (Possible student register as there were many instances of the students saying the number first, then assigning it the appropriate integer sign. This was observed for both positive and negative integers). Student register implicitly accepted by teacher, i.e. not corrected.

(1/ L9/ 30/ E4)

Figure 4.2 Demonstration of the Number Line Rule

This transcript illustrates the use of the Number Line to directly model the process of addition and subtraction of integers, labelled as the Number Line Rule. From the joint solution of the final example, the Number Line Rule was assumed established as a classroom practice for adding and subtracting integers. While the transcript is an example of a ritualised question, response and complete-the-sentence sequence, it also indicated that many students had appropriated the practice. Students were observed to use the Number Line Rule in the subsequent individual tasks. During this student activity, the lesson was interrupted by a fire alarm and everyone left the classroom.

The Whoops Rule

After twenty minutes, everyone returned to the classroom and the teacher used the remaining ten minutes of the lesson to introduce a new rule for the addition and subtraction of a negative integer. Firstly, she cued the students to recognise situations for the new rule. This cue was that the second addend was a negative integer. Using the whiteboard Number Line, the teacher demonstrated the modification of the rule, which was a change in Number Line direction.

[On whiteboard; 2 + -1]

Speaker	Transcript	Transcript Analysis
Teacher	... but / when you hit a negative, the rule changes ... and you go in the other direction. Now ... I'll show you here, if we did 2 plus negative 1. This is where we start, there's 2, up here, we go to the right, now if you find a little/ negative sign up there, instead of a positive, ... it means that you don't go to the right, the rule changes and you go in the other direction ... So if you're going right, you go left.	Teacher highlighted the different situation, "when you hit a negative" (the second addend is a negative). This was the cue that the rule changed and the change was, that you "go in the other direction" (simple one step change). Teacher demonstration using the number line, Saying and pointing to the starting number (2) on the Number Line, the + sign indicated go to the right (number line convention), but if the second 'addend' was negative (signal that rule changes), then the direction is opposite (instead of go right, go left).
Mathew	If you go left, you go right	Mathew appropriated the teacher's phrase, signalling he understood by applying it to the reverse situation, making it his own (semipublic comment, sitting at back of classroom).

(1/ L9/ 55/ E5)

Figure 4.3 Introduction of the Whoops Rule

The teacher introduced an idiosyncratic invention 'whoops', informal and familiar to the students, and designed to cue the students to the change in number line direction. This cue, not generally found in teaching resources, appeared to be specific to this classroom. This was how the teacher introduced the first use of 'whoops'.

Speaker	Transcript	Transcript Analysis
Teacher	... So ... what happens is, this is where you start, ... go to the right, whoops, no you go to the left.	Teacher repeated the new rule, pointed to the first addend as the starting point in 2 + -1, go to the right for addition (points to +), pointed to the –1 (signal for the rule to change), the first introduction of whoops, go to the left.

		(NB assumption is that this was a spontaneous use of whoops by the teacher, because she didn't say it during the initial explanation of this second rule)
General	Whoops! (laughter)	Many students responded to whoops, (mostly boys saying whoops).
Teacher	Then you change direction, ... so you, let's look on the number line.	Teacher repeated the example for emphasis, visually demonstrated using the Number Line
General	Whoops! Whoops! (laughter)	Students responded with whoops, imitating sirens, laughing. More students joined in this time.
Teacher	That's where we start. Now we go to the right, and we go whoops, no/, we go one to the left and the answer is one.	Teacher repeated the demonstration using the Number Line, retained whoops as a cue.

(1/ L9/ 55/ E5)

Figure 4.4 Teacher Use of Whoops as a Cue

The cue of whoops was enthusiastically taken up and used by most of the students. There were many instances of students saying "whoops", both in the context of some individual tasks and among groups of students (Lessons 9 and 10). This provided evidence that many students had appropriated the teacher's cue for changing direction along the Number Line. The Whoops Rule, was another example of a jointly negotiated classroom mathematical practice, this time using the Number Line to add or subtract a negative integer. A student contribution provided further evidence of student appropriation of the Whoops Rule.

Paul's Analogy: Hitting a Wall

Paul volunteered that the practice of the Whoops Rule was "like running and hitting a wall, then going the other way" (Paul, Lesson 9). Paul's contribution provided an analogy for the Whoops Rule. Hopping along the number line ("running"), stopping ("hitting a wall"), and changing direction ("going the other way"), were the three steps demonstrated by the teacher. Paul's contribution was an example of a student actively interpreting the teacher demonstration. The teacher responded immediately.

Teacher It is. ... good. Now that's a good analogy. Like hitting the wall ...ah, Susan ... so hitting the wall and going the other way. Think of it as a crash. Whoops, crash, go the other way. (1/ L9/ 60/ E7)

The teacher affirmed Paul's contribution, appropriated Paul's analogy, and named another student who was talking, implicitly asking for attention to a peer's contribution. The teacher repeated the student analogy for emphasis, rephrased it as a crash, and stated a jointly constituted summary of the rule; "whoops, crash, go the other way". This episode is important for a number of reasons. Firstly the social norms of Lemonwood 1 meant that students volunteered ideas to whole class interactions. It also indicated that one student Paul, who did not often contribute to whole class discussions, was actively interpreting the Whoops Rule and had connected this to a real situation. The teacher's re-interpretation as a crash contributed a further analogy for students. One group of students was observed to talk about the meaning of the crash analogy.

Speaker	Transcript	Transcript Analysis
Mathew ?	Whoops, crash, go the other way.	A student repeated the teacher's summary to himself, in a semi-public situation
Fraser	You can run away from the crash	Fraser continued the analogy of the crash for 'going the other way' (Paul), by volunteering an explanation of why you would go the other way after hitting a wall. This involved human action of running away from the crash.
Mathew	Whoops	Mathew repeated whoops, the cue for the second rule

(1/ L9/ 60/ E7)

Figure 4.5 Student Interpretations of Crash Analogy (to be continued)

One student Fraser, accommodated the crash analogy by interpreting running away from the crash as congruent to 'going the other way' (Paul). Another student in the class countered the crash analogy.

Speaker	Transcript	Transcript Analysis
Tony	Mrs Johnson, Generally when you crash you just stop. You don't go the other way.	Tony questioned the teacher about this analogy. A crash means stop, rather than go the other way.

Teacher	Ah, Oh, ... oh ... Who thinks they can tell me the answer to that last one?...	Teacher paused but moved on to ask the class who can share the answer for the last example. (Possibly not heard or a strategic ignore)
Tony	You could go back and pick it up (^)	Tony attempted to resolve his dilemma with the crash analogy and generated his own plausible situation when you would go the other way. (Go back and pick up something that may have dropped from the car was a possible interpretation)

(1/ L9/ 60/ E7 continued)

Figure 4.5 Student Interpretations of Crash Analogy

Tony, however, indicated that he was not accepting all of the teacher analogy because it did not match with his notion of a crash. When the teacher did not respond, (and it could be that she did not hear the student's comment), the same student amended his original objections to the crash analogy by offering a resolution to his dilemma. There is no further information about whether Tony appropriated either analogy as part of the Whoops Rule.

In the following week, when helping a student absent from Lesson 9, the teacher used this jointly constituted analogy of a crash. During individual student work, Katrina had signalled to the teacher that she didn't know what was happening. The teacher probed further to find out more specific detail and Katrina indicated that she felt excluded from the Whoops Rule. The teacher quickly recognised that the student may have been absent, checked with the student and once confirmed, initiated a whole class teaching explanation. One interesting feature of this re-teach episode was the way that the teacher had refined the Lesson 9 teaching demonstrations into three key points. The following is an extract from the lesson transcript.

Teacher Katrina, ... which part don't you understand about, ...where do you get lost?
Katrina I just don't, I don't know about the whoops bit.
Mathew /Dean Whoops

Teacher Maybe you, were you away when we did that? Ah, right that explains it. ... OK for those who were away, maybe put their hand up, there's only about three things that you have to remember. ... ( )... Now this is where you start, it's always the number you start on ..., where you put your finger on the number line. ... When you add you go right but if you come across a negative/ integer, it just means change direction. ... It's like a crash, you change, you've got to change and go the other way. So actually do/ it with your finger on the number ...
(1/ L11/ 45/ E12)

In response to a request for assistance, the teacher had used the jointly constituted analogy of a crash as an important part of the explanation. This was an example of a sustained appropriation of a student idea into her teaching repertoire.

The analysis of Whole Class Negotiation in Lemonwood 1 has been illustrated by the two classroom mathematical practices introduced in Lesson 9. The first classroom mathematical practice, the Number Line Rule, was established by whole class demonstration and solution of examples, followed by individual activity. The second classroom mathematical practice, the Whoops Rule, for the special case of adding or subtracting a negative integer, evolved from the first practice. During the establishment of these two practices, students and teacher negotiated mathematical meaning in the following ways. They used the Number Line as a tool for both whole class and individual thinking, they established an in-house expression "whoops" as a cue for reversing a known Number Line strategy, and they jointly constituted a crash analogy for this practice. An analogy was volunteered by a student, appropriated and adapted by the teacher, discussed, questioned and then accepted by other students. The teacher also used this analogy in a later explanation to an individual student. Continued use of the Whoops Rule (Lessons 10 and 11) was observed over the next two lessons.

Student Initiated Negotiation

This category encompassed interactions initiated by a student contribution such as a question, solution method, suggestion or disagreement. These student contributions were not anticipated by the teacher and opened up opportunities for further negotiation of mathematical meaning. Paul's Analogy in the previous

section was one example of a student initiated negotiation. Other examples are now described.

Dean and the Shortcut Rule

Many students were observed to appropriate the Whoops Rule. As the teaching sequence unfolded, not all students wanted to use this practice. In Lesson 9, one student appeared to have found a shortcut method. [For task: 2 + -1]

Dean	What about if you had two minus one is one?
Teacher	That's after we do this. You're right. But I'll, let's do one thing at a time. Alright. We're only to, going to be adding, we're only adding at the moment (Lesson 9 transcript).

The teacher requested that the student continued with the Whoops Rule for the time being. The teacher response was presumably to scaffold the mathematical practices for addition and subtraction of integers by providing more time for students to use the Whoops Rule. In the final minute of Lesson 9, Dean told the teacher again about a shortcut of the rule for addition and subtraction of integers. The teacher did not seem to hear him but two students responded to his idea. The class had been set an example of -4 - -6. Mathew and Dean were sitting next to each other within a group of four students and Ryan was sitting nearby.

This localised discussion illustrated three students actively engaged in negotiating new mathematical meaning independently of the teacher. By describing his Shortcut Rule, Dean had semi publicly contributed more resources to the discussion. While his statements were probably only heard by a proportion of the class, two students responded. Mathew, sitting beside Dean, articulated his own thinking while still affirming his friend, and then possibly became distracted by all the negative signs present in the particular task. Ryan contributed a more general statement, using different language. This could have indicated a perspective that the signs were labels for the numbers rather than indicating operations. Another perspective could be that Ryan recognised patterns in the examples provided. This brief exchange indicated, however, the diversity of student interpretation of mathematical situations and symbols. These three students were actively engaged in making sense of a mathematical practice and,

in doing so, they generated a new and more efficient method for adding and subtracting negative integers, the Shortcut Rule.

The transcript and analysis are presented below.

Speaker	Transcript	Transcript Analysis
Dean	Mrs Johnson, That's just like going minus four, plus ... then minus four and minus ... plus six... ... basically.	Dean volunteered statement about his strategy, implicitly recognising the change in direction as counting to the right of the number line (plus).
Mathew	Yeah, minus ... minus it would.	Mathew agreed with Dean in a localised peer discussion, inferred from his repetition of minus to imply plus. Mathew echoing Dean.
Ryan	Why do the negative, just do the positive?	Ryan (from some distance) generalised and asked why (use the Whoops Rule for subtracting a negative), when it would be easier to do positive (addition).
Mathew	Is it the first number? Oh no don't worry.	Mathew asked whether it was the first addend (that was negative). He appeared to self-correct.

(1/ L9/ 60/ E9)

Figure 4.6 Peer Discussion of Shortcut Rule

In the following lesson, some students were not using the Whoops Rule or even the Number Line. In the last part of the lesson, the teacher publicly introduced the Shortcut Rule by appealing to the previous examples completed using the Whoops Rule (Lesson 10 transcript).

Teacher So you start finding that you can replace these with ... just one sign, what could I replace that with? [Teacher pointed to the two signs]
Dean Plus

Dean, the student who had recognised the shortcut strategy in Lesson 9, promptly responded. Some other students also appeared to have been using a mental method. The teacher stressed that they could replace two signs with one sign.

Implicitly, this was a focus on a rule of changing the sign rather than considering the operational process. The teacher then repeated the Shortcut Rule for more examples and asked the students to find a pattern for the generalised rule. A number of students responded to the teacher's appeal for a pattern by chorusing the answers to the teacher questions.

Teacher	So can anyone tell? Have a look at these. What we are replacing them with? Can anyone see a pattern? ...What happens when the signs are different?
General	They're minus.
Teacher	They're minus. OK. What happens when the signs are the same?
General	They're plus.
Teacher	Good. Right. OK. Well done.
	(1/ L10/ 50/ E9)

This indicated a whole class sanctioned fading of the Whoops Rule, a rule dependent on a Number Line tool, in favour of a 'change-the-signs' method followed by mental addition or subtraction, the Shortcut Rule. The teacher affirmed this new classroom mathematical practice in the next lesson.

Speaker	Transcript	Transcript Analysis
Teacher	OK so now you are starting to do it without having to go whoops(^), you are starting to get....	Teacher initiated a discussion by referring to whoops, the in-house expression, and acknowledged that the students had developed an alternative.
Mathew et al	Whoops(^),whoops(^)	Many students, mostly boys, chorused the still popular whoops.
Dean	Why do we need to go whoops, I just go wooh	Dean questioned the need for whoops and stated his version of whoops, even though he was using the Shortcut Rule
Ryan	I don't, can't see why you can't sort of ... change it ...	Ryan questioned why you can't use the Shortcut Rule rather than the Whoops Rule.
Teacher	Well you can, ... replace it with a ... Like this would be replaced with one sign ... plus. ... Natalie ... [Natalie with hand up]	Teacher affirmed Ryan's contribution and narrowed the focus to just replacing two signs for one. She referred to an example on the whiteboard, indicated that the two minus signs could be replaced by a plus. Teacher then opened up a space for another student contribution.

Natalie	Can you have?, if you have two minuses, you could change it over and make a plus(^)	Natalie asked the teacher a question about changing two minus signs for a plus sign, question rephrased as statement with rising inflection.
Teacher	Plus, ... Yes you can. Yep	Teacher repeated Natalie's last word and affirmed her suggested method.

(1/ L11/ 40/ E10)

Figure 4.7 Further Discussion of Shortcut Rule

Two lessons after Dean first initiated the Shortcut Rule, his rule was repeated and reinforced by various students and the teacher. A significant aspect of the development of this classroom mathematical practice was the diversity of student language and methods over time. Also significant was the recursive aspect of teacher guided participation, generating a new more efficient tool for adding and subtracting integers. Some of the students, namely Dean and Ryan, appeared to generate the rule themselves and appropriated it into their individual mathematical activity. These students tested out their ideas with peers and with the teacher. One girl appeared to need teacher sanction or reassurance before abandoning the established practice of the Whoops Rule in favour of the Shortcut Rule.

Scott's Wicked Number Pairs

Another example of a student initiated negotiation was when a student volunteered a solution method. This time, the class task was to find the possible lengths of sides that would produce rectangles of area 24 squares. Scott signalled across the room to the teacher that he had an exciting solution method (from Lesson 5 transcript).

Scott Oh I've worked them already out.
(a few moments later)
Scott I worked out a wicked way to do it.
Teacher OK we'll hear it in a minute. ...

The teacher acknowledged Scott but delayed him, presumably to provide more time for other students to progress further through the task. After a few minutes,

she opened up a space for Scott to share his method with the class. The relevant transcript and analysis is outlined below.

Speaker	Transcript	Transcript Analysis
Teacher	Right Scott. Can you tell us your way to do it?	Teacher invited Scott to share his method
Grace	It's really hard to get some of them.	Another student commented about the difficulty of finding some of the factors.
Teacher	Listening. Scott.	Teacher directed class attention to Scott. Repeated invitation to Scott.
Scott	Well, You go one and	Scott's first pair of numbers
Teacher	Listening ... nobody, we don't talk over anyone in the room. OK ... Scott.	Teacher intervention for class attention to a student contribution. Repeated invitation to Scott.
Scott	One and twenty four	Scott repeated his first pair of numbers
Teacher	Good, one at each end (^) [Teacher wrote 1 and 24 at either end of space of whiteboard],	Public visual record of Scott's results, organised to visually suggest a pattern.
Scott	Two, two and twelve ... three and eight	Scott shared next two pairs of numbers
Teacher	Still people talking, Timu sit up, that's bad manners ... sit up to your desk, Luke ... right two times twelve ... and three times [Teacher wrote 2 and 12]	Teacher intervened for more class attention, named individual students. Teacher repeated Scott's ideas, in terms of factors multiplied together and recorded these on the whiteboard
Scott	eight ...	Scott repeated eight, to complete the teacher sentence
Teacher	Three times eight [On whiteboard: 1 2 3 8 12 24]	Teacher rephrased as factors, using mathematical register.
Scott	And then four times six	Scott provided next pair of numbers as factors, mirrored mathematical register of teacher.

(1/ L5/ 45/ E3)

Figure 4.8 Scott's Explanation of his Wicked Number Pairs

It was significant that the teacher intervened a number of times to re-iterate an important classroom social norm and this signalled the high status of student methods in Lemonwood 1. Scott's excitement was probably based on the appeal

of the patterns of pairs of numbers, and that he was using existing tools in a new situation. Scott's method to generate pairs of factors for 24 appeared to be a symmetrical method, and this was reinforced by the way the teacher recorded his contributions on the whiteboard. The teacher also intervened to slow down Scott's description of his method and to record his results, again signalling high status for a student method. By recording Scott's method, the teacher was foregrounding additional resources such as the symmetry in the pairs of factors and an implicit pattern in the factors of 24. The teacher progressively included mathematical register for pairs of numbers represented as factors of 24 (implicit area as a product). In his contribution of the final pair of factors, Scott illustrated an appropriation of mathematical register. To conclude this episode, the teacher repeated Scott's final pair of numbers and summarised Scott's method, ending with "they're all factors of twenty four", before moving on to some textbook tasks. The teacher's summary connected all factor pairs and highlighted the mathematical properties of these numbers, that they were all factors. Scott's Wicked Number Pairs was appropriated by some students as a practice for finding all whole number sides for rectangles of fixed area (field notes Lesson 5).

In Lemonwood 1, the student initiated negotiations were important interactions about mathematical activity. Two examples illustrated students initiating solution methods for classroom mathematical practices: Dean and Ryan's Shortcut Rule and Scott's Wicked Number Pairs. These student contributions were resources that initiated further discussion. The ensuing interactions illustrated important aspects of negotiation of mathematical meaning. By re-articulating and refining their contributions, students progressed from using student to mathematical register. In addition, students asked questions about mathematical meanings and processes, disagreed with or disputed solutions, and contributed new methods in whole class activity. Other students appropriated the original student method, and localised conversations were important for peer negotiation.

Error Initiated Negotiation

In Lemonwood 1, the most episodes of error initiated negotiation occurred in lessons involving decimal numbers. A student error, followed by a peer question

about an associated aspect of mathematical content, often initiated discussion about more complex mathematical situations. This first example arose in a whole class situation that included localised student conversations.

Dean's Rounding Error

In Lesson 11, the teacher demonstrated the Dash Method for Rounding using the Rounding Dash, a tool previously described as a thinking device (section 4.2.4). During a whole class review of a rounding task, Dean's answer revealed an error. The transcript and analysis are presented below.

[On whiteboard 8.792 Round to 2 d.p.]

Speaker	Transcript	Transcript Analysis
Teacher	... Dean what did you get for this one?	The teacher asked Dean to publicly contribute his answer
Dean	Ah eight point eight.	Dean's answer was 8.8, the correct answer for rounding to one d.p.
Teacher	Two dp ... means it actually...	Teacher assumed that Dean had misread the number of d.p. Restated two dp as a cue for his error.
Fraser	Eight point seven, it doesn't change.	Fraser volunteered his answer (possibly a semi-public statement addressed to Dean). 8.7 would be the correct first two digits. Inferred as a cue to Dean that these don't change (8.79 correct answer)
Teacher	Doesn't change, does it. So.... Eight point seven nine is two dp. [wrote 8.79 on whiteboard]	Teacher repeated Fraser's comment. Then stated the correct answer.
Dean	Don't you change the seven?	Dean volunteered a question, implicitly revealed his method which was to change the seven digit. Localised comment but knowing that teacher would hear.
Fraser	No	Fraser responded to Dean, semi-public comment.
Teacher	...No you don't even touch ... you don't even touch that at all. ..	Teacher repeated Fraser's comment and pointed to the first decimal place as the place you don't touch.
Timu	It's two dp	Timu cued to Dean the requirement of the question, rounding to 2 d.p. Semi-public to Dean.

Teacher	Look at two dp ...put your line in, [on whiteboard; 8.79/2] then it's ...what was that, two. You're only, you are looking at this [pointing to 2] and you are changing that. ... [pointing to 9] Don't change anything else.	Teacher repeated Timu's cue and proceeded to demonstrate the Dash Method for Rounding. This marker then provided a visual cue for rounding the digit to the right of this line, which was the digit 2. Teacher re-iterated the place of digit for inspection, repeated not to change anything else.

(1/ L11/ 30/ E8)

Figure 4.9 Dean's Rounding Error

This sequence illustrated an error initiated negotiation which involved Dean, the teacher, and nearby peers in assistance roles. In response to a student error, these peers contributed answers and re-iterated ideas. Dean maintained his method, suggesting that he had not appropriated the practice of the Dash Method for Rounding.

Mathew's Rounding of Nines

This interaction generated further discussion when Dean's neighbour and friend Mathew, asked a question about rounding 8.99. Mathew's question indicated that he recognised these types of numbers to be a special case. The students involved in the following transcript were sitting near each other but not in the same group of desks.

Speaker	Transcript	Transcript Analysis
Mathew	If the number was like eight point nine nine,	Mathew asked about a particular example, 8.99
Teacher	Good	Teacher signalled support
Mathew	and, ...nine nine nine and you needed to round it up (^) would you change the eight to a (^)...	Mathew changed his example to a more complex one for rounding, 8.999. He asked would you change the digit to the left of all the nines.
Teacher	Yes, Good question,	Teacher again signalled support, affirmed, wrote 8.999 on whiteboard, as a valued example for the rest of the class

Mathew	That would, ... Would you change the eight to a ... nine and?	Mathew volunteered his idea about rounding the eight to a nine, framed as a tentative question
Teacher	So if I wanted to do it to two d p or one d p?, ...or even,	Teacher re-directed Mathew to consider how many decimal places to be rounded
Mathew	One d p	Mathew chose one d.p.

Figure 4.10 Mathew Initiating a New Example

In this interaction, Mathew identified an example that highlighted the extended process of rounding up nines. Mathew also changed the number from 8.99 to 8.999, which raised the complexity of the task from his perspective. By pursuing Mathew's question, the teacher signalled an interactional space and provided more time for other students to engage with this example. The episode continued.

Speaker	Transcript	Transcript Analysis
Teacher	Let's say two d p first. ... Put your line there. [on whiteboard; 8.99/9] ... Do I round this number up?	The teacher demonstrated 2 d.p. and used the Dash Method of Rounding. The teacher asked students to decide whether to round the digit up, pointing to the digit to the right of the dash
Timu	Yes.	Timu volunteered his idea, (9 rounds up)
Teacher	You're left, now when you've got a nine... it changes into a ten. ... But it also changes this one to a ... [pointing to the middle 9]	Teacher drew attention to the digit 9, rounding it up to ten, causing a change of the digits, then systematically changing each decimal place at a time, recorded this on the board.
Timu	Ten.	Timu completed the teacher sentence, indicating that he was following the teacher's demonstration.
Teacher	So keep going ...	Teacher supported the student to continue with the method
Timu	Is that a nine?	Timu asked for confirmation of the final answer nine
Teacher	... and we'll change that one to a ... nine.	Teacher completed the rounding to the last digit.
Timu	(I see)	Timu, was thinking aloud (confirms teacher statement)

Figure 4.11 Teacher Demonstration of Rounding of Nines

Timu's statements were mostly localised as Timu was thinking out loud to himself. He frequently thought 'out loud' and this was accepted by other class participants. In this instance, the teacher appeared to attend to some, but not all, of Timu's thinking aloud. Timu's contributions also indicated he understood the process of rounding for this particular example.

Another issue was how to indicate an empty decimal place in order to satisfy the requirements of the task, to 2 d.p. Using zeroes as place-holders appeared to be problematic for some students because some calculators omit zeroes from the right hand place of a decimal.

Speaker	Transcript	Transcript Analysis
Dean	But Mrs Johnson	Dean signalled implicit disagreement
Teacher	You have to keep those zeroes in ... to show that you have actually ... considered them.	Teacher explained the role of zeroes, anticipated a possible confusion about writing in the zeroes.
Dean	If, see on that Number 4, if that figure was a nine then you change it?	Dean drew attention back to the original example. (8.792 round to 2 d.p. and reconfigured it to 8.799)
Teacher	If there was a nine you'd change it to a, ... that would go to zero and this will go to an eight. Good.	Teacher demonstrated using the dash between the two nines, rounding 8.799 up to 8.80 (2 dp). Teacher affirmed

(all extracts 1/ L11/ 30/ E8)

Figure 4.12 Dean and the Rounding of Nines

Dean has used this interactional space to initiate further interaction about his previous error. By generating a counter example (8.799) that would round up, he and the teacher negotiated meanings for the special case of rounding up a nine. It appeared that either these students were not using the Rounding Dash as a tool in the rounding process, or they did not see how to use it to round up nines. The teacher, however, continued to use the Dash Method for Rounding in whole class activity.

Natalie's Error

Another example of an error initiated negotiation began with a student statement prompted by an incorrect answer. After the Marking Ritual for the starter quiz in Lesson 7, Natalie raised her hand with a query about one of the Starter tasks.

Natalie Um, I got told that the more, the point, after the point, the number after the point, the lower, the lower it is.
(1/ L7/ 20/ E3)

Natalie began with an implicit appeal to a primary mathematics discourse (section 4.2.2), and stated her reasoning about the magnitude of decimal numbers. The teacher responded in the affirmative but, after some more questioning, it emerged that the prompt for Natalie's question was an earlier task from the Starter Ritual. This task was: 'Order these decimals, 8.8 7.2 8.1 7.9 8.0'. Natalie believed that 7.9 was smaller than 7.2, because 9 was more than 2. This represented a misconception about decimal place value, based around an interpretation of the term 'more'. Examples such as 0.004 is less than 0.4, might be one way her previous teachers had illustrated decreasing magnitude of decimal numbers but can also lead to statements such as 'the more numbers after the decimal point, the smaller it is'. This would be true for 0.4 and 0.004 but not true for 0.2 and 0.634. Here Natalie's reasoning was assumed to be an adapted version of the above statement, where 'more' was taken to mean a larger digit, leading her to believe 7.9 was less than 7.2. As discussed in section 4.2.4: Tools, decimal numbers can pose difficulties for students when they focus on the digits as discrete signifiers without attending to the relationship with the place of the digit.

In response to Natalie's dilemma, the teacher drew a model of a rectangular chocolate bar, dividing it firstly into tenths, and then dividing one of the tenths into ten smaller pieces. This analogous model provided a visual demonstration of the relative magnitudes of tenths and hundredths. The Chocolate Bar Drawing did not produce any response from Natalie, so the teacher asked again for information from Natalie. Ryan, who was sitting across the aisle from Natalie, volunteered an explanation that included a way of representing 7.9 as a decimal number. The transcript and analysis are presented below.

Speaker	Transcript	Transcript Analysis
Ryan	Seven point nine, ah put that after the first one	Ryan said that 7.9 should be the second in the ordered sequence of smaller to larger (7.2 first)
Teacher	Yes	Teacher affirmed and intervened to slow down Ryan's explanation.
Ryan	Because seven point nine would be like ...(five tenths) and four and one would more than (to the next number)	Ryan gave reasoning based on a benchmark of (seven) and recomposed 7. 9 as five tenths (7.5) and four more (tenths), needing only one more tenth to go to the next whole number (8.0). (Using 8.0 as another benchmark)
Teacher	... Did anyone read it on a ruler? ... and you're looking at the scale on your ruler. ...	Teacher linked with physical tool of a ruler.
	[Teacher drew an empty number line model on the whiteboard, labelled 7.0 and 8.0 at ends]	Teacher provided a visual representation to model Ryan's contribution
	There's seven, there's eight, ...	Teacher signalled 7.0 and 8.0 as benchmarks, cue for considering decimals to 1 dp (responding to Ryan's way of representing 7.9)
	there's seven point five.	Teacher labels a point between 7.0 and 8.0, intermediate benchmark of 7.5.
	so one, two, three, four. Where's seven point one? Seven point two is here, isn't it? ...	Then counting up in tenths, to 7.9, making Ryan's explanation visible and public, Comparing with 7.1, 7.2 (7.2 was first number in the sequence).
General	Yep	Signal to teacher from students , a cue that they understood the model.
Teacher	There. Seven point two .. Where's seven point nine?	Teacher pointed to 7.2, where's 7.9? as a comparison.
Ryan	Just before the eight.	Ryan responded by describing 7.9 to nearest benchmark which is 8.0.
Teacher	Just before the eight. So it's almost eight, it is bigger. So the rule is ... you look at ...seven point two, seven point nine. ... You look at that first decimal place. A smaller number means it's smaller and a larger number means that it's larger. ..	Teacher repeated Ryan's statement, and reviewed the example of 7.2 and 7.9. then reiterated a general rule for numbers to one decimal place.

(1/ L7/ 20/ E3 continued)

Figure 4.13 Negotiation Initiated by Natalie's Error

This joint negotiation was prompted by Natalie's original error in the Starter Ritual and initiated by the opportunity for student contributions during the Marking Ritual. Another student, Ryan, assisted her with an explanation of 7.9. He used an affordance of numbers to recompose 7.9 as 7.5 plus 0.4. Ryan also related this new representation to the nearest whole number, 8.0. The teacher appropriated Ryan's re-composition of 7.9 and used a form of the Empty Number Line to record and expand the steps in his explanation. This illustrated how the jointly constituted negotiation included responses to Natalie's statement, appropriation of a student contribution, and use of a classroom mathematical tool to model an alternative representation of a decimal number. This negotiation of mathematical meaning was distributed across the Empty Number Line, representations of numbers, benchmarks, and student and teacher reasoning.

To sum up, in Lemonwood 1 errors initiated not only further interactions but more complex student questions. In the interview, the teacher reported that she considered student errors or incorrect solutions as opportunities to engage in further mathematical negotiation.

> I try and find out how they, what they've done to get it wrong ( ). What did you do to get that? ... For me to find out how their brains work. And just not going, well that's wrong, and leave it. Find out how they're thinking, why they've got it wrong. ( ) If they're close, give it a try and then try and get it from them too. (Mrs. Johnson, Interview)

During the observed lessons, an error often prompted a request to students for explanations of their solutions or their reasoning, leading to further teacher and student negotiation. The student sometimes articulated an acceptable solution in the process of a second attempt. Sometimes a pause from the teacher, or even a signal that there might be an error, prompted the student to self-correct almost immediately or a peer to offer assistance.

Repair Interactions

The third type of student initiated negotiations, repair interactions, arose from instances when meanings were mismatched, recognised and then repaired. For example, during the whole class discussion about a vertical subtraction algorithm for 73-58, Dean suggested a strategy involving negative numbers. Instead of reciting, "three takeaway eight, can't do", he reformulated the process to 3-8 = -5,

(three takeaway 8 is minus 5). The teacher did not initially understand Dean's meaning and another student, Grace, picked up Dean's idea and tried to explain it to the teacher.

Grace	He's meaning, Mrs Johnson.
Teacher	Sorry, ... say that again Dean.
Dean	Oh, Don't worry
Grace	He said if three was on top of the eight
Teacher	Aahh three take away eight... what happens there?
Luke	Minus five.
	(1/ L4/ 20/ E3)

The teacher repaired the situation by appealing to Dean to repeat his question but he had opted out. Grace intervened again and rephrased Dean's question. The teacher responded and re-framed it as a question. Luke's volunteered response indicated that he had followed Dean's initial question. This was an example of the assistance role of peers in repair episodes in this classroom.

Following this relatively brief repair interaction, the teacher initiated a discussion related to negative numbers. She talked about a multi-storeyed building as an analogous model of a lift (elevator) moving between levels, and drew a whiteboard diagram corresponding to a number line. This analogous model was not accepted by all students. For example, Mathew wanted to dispense with the negative signs, as is the case in multi-storeyed buildings (from 1/ L4/ 20/ E3). This was an example of a constraint when using a lift as an analogous model of a vertical number line. The whole class interaction continued with a teacher contribution of bank overdrafts, another real life situation for negative numbers.

Teacher	You can't just get rid of it ... what happens if you have $30 in the bank and you withdraw $80?
Dean	I did that last week
Rachel	You can't
Tony	I put this number in
Teacher	Let's say you have got an
Rachel	You owe the bank ... the money
Teacher	Yes you owe them, let's say you (have) an overdraft facility. How much would you owe the bank?
Mathew	Fifty dollars
Rachel	Fifty dollars
General	Fifty

Teacher	Right, it goes back fifty, and that's like having minus 50 isn't it? .. Um Tony?
Tony	Um some of my savings account, they got a special thing and I don't have an over draft, so I ... if I have no money, I won't have ... to pay ...
General	Yeah, you will.
	(1/ L4/ 20/ E3 continued)

Students volunteered their own experiences of overdrafts, some of which seemed to contradict a subtractive model for negative numbers, i.e. that either the bank would not allow the student to go into negative (overdraft) or, as a student claimed, his bank account did not include an overdraft. Again this was a constraint of the analogous model of overdrafts. The conversations initiated by a repair interaction included teacher contributions of two analogous models for negative numbers. These discussions have illustrated the complexity of relating student's varied life experiences to contexts being offered (by the teacher) as analogous models.

Technology Mediated Negotiation

In the final interaction category, Technology Mediated Negotiation, the transcripts of calculator episodes were analysed for negotiation of mathematical meaning. In Lemonwood 1, students generated mathematical resources with the calculator and these resources were often topics of further conversation. The following example is typical of the localised peer conversations about these mathematical resources.

Mathew	Thirty, ...whoopsy ... . Oh found it
Dean	No, it's not ...
Mathew	Most people's ....
Dean	Two to the power of
Mathew	Whooooa
Dean	Mrs Johnson, you go, no it's not ...
Mathew	Two to the power of fifteen, it came out three two seven six eight.
Dean	Yeah
	(1/ L7/ 30/ E5)

Student sentences were often half completed and statements were addressed to themselves as much as to each other or the teacher. This particular conversation

was semi-public as the teacher heard the last comment and guided students towards a focus on large numbers.

Teacher	Yeah, it's a big number. All right. When you start with raising something to a power, and that's how you say it, you say you've raised something to the power of ... It increases very quickly. ...
Natalie	Wow, it's three two seven six eight
Teacher	That's it. Three two seven ... now let's, you can see how big these numbers get. ... three two seven. What is that number? ... Anita? Can you say that number in words? ... Okay
Anita	Thirty two thousand ... seven hundred and sixty eight
	(1/ L7/ 30/ E5)

After the teacher had demonstrated the power and square root keys, there was about ten minutes of student exploratory activity. It was during this time that Tom had called out to the teacher while she was near his group. A few moments later, the teacher asked for class attention.

Tom's Multiline Function

Tom was excited about his finding, which he had already shown to his neighbour. Tom then contributed his finding about the multi-line function on his calculator.

Teacher	Dean, Tom has something he wants to share with everyone ... might be... Might be interesting ...
Timu	Oh Mrs Johnson
Teacher	What happens, Tom?
Tom	Well, when you click on, ..(yeah) and you takeaway nine times six equals, and then put the (xx) off again,
Dean	You don't put the off switch
Tom	and press on again, where it's press multi-line, at the very top .. and it goes back to ah you what was your last line.
Mathew?	Oh, did it?
Dean	Yes
Tom	Yes
Grace	Like replay
Timu	I haven't got it
??	all the answers
Teacher	Have a play with your calculators and see what they'll do ...more buttons to learn. Do that at home
Mathew	How does it do that?
Fraser	It's just memory.
	(1/ L7/ 50/E8)

Using the calculator as an exploratory object afforded opportunities to learn more about the capabilities of their calculators, and students were eager to share their discoveries. Tom's Multiline Function recalled the last expression that had been keyed into the calculator and provided a record of the student's processes. Other students in the class were also interested; some likened this new function to familiar experiences such as replay on a video, and some were disappointed because their brand of calculator did not have this particular function. Again the teacher re-iterated the importance of exploring their calculators to find out more about their various capabilities. Although this episode did not involve any negotiation of mathematical meanings, it did involve a valued student contribution. It also generated excitement about independently exploring the scientific calculator and finding a tool for recalling student processes.

In addition to the roles of recording device, exploratory object and calculating machine, students used calculator results as resources to resolve errors or conflicting answers. The following episode is an error initiated negotiation but is included in this category because during the negotiation, students used calculator results as a mathematical authority.

Luke's Calculator Result

In Lesson 2, there was a whole class review of a practice task involving the multiplication of decimal numbers (0.02 x 0.04 in vertical form). This episode was a protracted discussion among the class about how many zeroes there would be in the product. The teacher twice demonstrated the method of adding together the number of decimal places of each factor, and counting back the decimal places in the product. The transcript and analysis are presented below.

Speaker	Transcript	Transcript Analysis
Luke	Is it zero point zero eight? zero point zero zero eight?	Luke stated answer as a question, then changed to 0.008. Assume initial misreading of the calculator screen.
Teacher	So it's point zero zero zero eight. ..	Teacher stated correct answer, 0.0008
Luke	I did it on the calculator and it was zero point zero zero eight	Luke appealed to the authority of the calculator and restated his answer.

Teacher	Is that a zero in front?	Teacher focused on the number of zeroes and asked Luke a question, in the hope that Luke's missing zero might be to the left of the decimal point.
Luke	Yeah, take one out of the middle, it'll be right.	Luke interpreted teacher suggestion as a solution to the mismatch in answer. Remove a zero in the teacher's answer. Perceived by Luke to be incorrect.
Teacher	One, two, three four. [pointing to decimal places in the factors]	Teacher counted the number of decimal places in the factors.
Timu	Then it won't be right	Timu responded to Luke's suggestion, and disagreed with Luke.
Luke	It's only, it's zero point zero zero eight	Luke restated his answer
Teacher	One two three four ... It's got to have four/ decimal places.	Teacher repeated the counting of the decimal places, the classroom mathematical practice for finding the number of decimal places in the answer.
General	counting of places etc	Other students joining in the counting of places
Dean	Look on your calculator	Dean tried on his calculator and suggested that Luke do the same.
Teacher	So zero point zero ... four (decimal places)	Teacher repeated the method
Luke	Zero three zeroes ... yeah,	Luke tried on his calculator again and stated zero, then three zeroes, indicated he now agreed with teacher.
Ryan	You got it right Mrs Johnson	Ryan agreed in public with the teacher
General	Zero, point .. I know how you do it ...	Students joining in conversation

(1/ L2/ 35/ E5)

Figure 4.14 Negotiation about Luke's Calculator Result

In this particular example, Luke's perception of the authority of the calculator precipitated a whole class discussion about the multiplication of decimals. It was possible that Luke made a keying-in error, as students may be more likely to make errors in keying in decimal numbers rather than whole numbers. In the end, however, the teacher and students used the calculator to jointly resolve the final result. One student publicly supported the teacher's result. This transcript is an

example of the relationship between mathematical resources contributed by the calculator and student beliefs about mathematical authority. This particular mathematical conflict was resolved by joint negotiation, peer dissent, further contribution of mathematical resources, and student assistance of the teacher.

Overall in Lemonwood 1, significant calculator mediated activity generated new methods, diagnosed errors, settled disputed calculations, promoted peer assistance, and generated further mathematical activity. All of these episodes were examples of further opportunities for students to participate in both localised and whole class negotiation of mathematical meanings. Calculators also afforded further resources for students to resolve calculation errors. The calculator's speed and ease of use generated student excitement about numbers. Students also used opportunities to find out more about the capabilities of the scientific calculator. For example, when Tom found out about the multi-line function, he first experimented with the calculator, showed a neighbour, and then, with the teacher's support of a space in the lesson, contributed this as a class resource.

Roles

In this category, student roles were inferred from the status of their contributions and their ways of working together. Specific examples of student roles are now identified from the analysis of the Classroom Interaction data. The first role identified was as solution describer. Examples were Dean and Ryan's Shortcut Rule and Scott's Wicked Number Pairs. In both cases, students described their solution methods in whole class interactions and their methods were appropriated, not always immediately, by teacher and students. The second student role was as mathematical explainer. This role was mostly identified in response to a teacher request. For example, Shelley's contribution of the alligator sign for less than/greater than was repeated by the teacher who validated the student register. An example of a volunteered mathematical explanation was when Ryan contributed 7.9 as 7.5 plus 0.4 during the resolution of Natalie's Error. Again the teacher appropriated Ryan's contribution and recorded this on the whiteboard. Volunteered student phrases or ideas were frequently acknowledged by the teacher, repeated or appropriated into the teacher's

repertoire. Consequently, student contributions were inferred to have high status in Lemonwood 1 whole class interactions. Students also volunteered further mathematical resources. For example, Paul was identified as an analogy provider for the Whoops Rule (crash analogy), and Tom was a describer of a calculator function (multiline function). These volunteered contributions provided further mathematical resources to whole class interactions.

The roles of question asker and question answerer were also identified in this case study. Mathew and Dean adopted the role of question asker during a negotiation about the rounding of nines. Their questions initiated further negotiation about the rounding process. Natalie's statement was an implicit question and initiated a negotiation about the magnitude of decimal numbers. There were many examples of question answerer in whole class interactions. Even when the answer was incorrect, such as Luke's Calculator Result, Luke was able to assume an additional role of solution describer during the joint resolution of the multiplication of decimal numbers. The teacher and student responses to these contributions provided further evidence of the high status of student contributions in Lemonwood 1.

Another aspect of Roles related to ways students worked together. In Lemonwood 1, there were many examples of student collaboration in localised student activity. One example was during student activity with the calculator as an exploratory object. In Lesson 11, Timu, Tom, Mathew and Dean used their calculators to find the square root of 5. The calculator generated the square root of five to at least six decimal places, setting up the rounding task.
(The square root of five is 2.236067978 to be rounded to two decimal places)

Speaker	Transcript	Transcript Analysis
Timu	Good	Timu tried out the square root key. Pleased that it worked.
Tom	Can I see.....?	Tom leaned over to look at Timu's calculator
Mathew	Two point two three ...six oh six seven nine	Mathew read his calculator display.

		[On whiteboard was 2.23/606, Teacher recorded the Dash Method for Rounding as a prompt for students.]
Timu	Three point	Assume that Timu meant 3 dash
Mathew	So you've got to go that up	Mathew recognised that the 6 to the right of the dash would mean rounding up
Tom	so do you... you go, go two, ah	Tom thinking out loud, looked at his calculator screen, beginning to read calculator screen.
Dean	Wouldn't be three one? With the one	Dean identified the 3 on the left of the dash. Assume he was thinking about rounding the three up by one
Tom	You go two, press, you go like that	Tom showed Dean how to do the square root on his calculator

(1/ L11/ 30/ E5)

Figure 4.15 Roles during Joint Student Activity

The calculator was a recording device and provided a prompt for the students before rounding the number to two decimal places. The conversation of Timu, Tom, Mathew and Dean, demonstrated assistance, collaboration, and jointly mediated activity. In this interaction, Tom adopted roles of assistance requestor and then assistance provider to Dean who had implicitly requested assistance. Assistance also occurred in whole class situations, for example in the repair episode described earlier when Grace rephrased Dean's original question. In Lemonwood 1, assistance was requested explicitly as in the following interchange between Janine and Timu.

Janine Hey, Timu, what do you do if it's not an even number?
Timu Try, ... three
(1/ L6/ 35/ E1)

The importance of student collaboration was tacitly supported by the teacher who accepted the localised student talk even during whole class interactions. In the teacher interview, Mrs Johnson acknowledged that this represented a change in her beliefs as a teacher and she mentioned the pragmatic aspect of spreading assistance during individual tasks.

> Initially when I first started teaching, you know, I thought, oh, you do that by yourself, no talking. But as I've taught, I've seen how they ... the

> groups help each other. And I can't get right round, so often so they'll peer teach each other too, ...

This was another example of a Teacher Mathematical Value about how students learn mathematics, and was related to the observed regular opportunities for students to adopt roles of collaborator and assistor. Overall, the student roles identified in this case study were related to the high status of student contributions. By adopting these roles, students contributed further mathematical resources to joint activity.

4.2.6 Lemonwood 1 Summary

The Lemonwood 1 case study involved the analysis of data from twelve observed lessons. The typical lesson structure was of a traditional classroom and did not include explicit problem solving during the observed lessons. The thirty-one students and the teacher, however, were actively participating in the mathematics lessons. Some features of the case study were significant for participation in mathematical negotiation. The Starter Ritual and the Marking Ritual were two rituals that provided regular opportunities or spaces for student participation. These participation structures, along with the high status given to student contributions, meant that students assumed various important roles in the development of mathematical ideas. Students were solution describers, answer providers, mathematical explainers, and question askers and answerers. Some students contributed analogies and individual experiences to whole class interactions. Students also collaborated with their peers and asked for, and received, assistance.

Mathematical tools were an important feature of this case study. The Number Line was used as a thinking device in the development of two classroom mathematical practices for integers. A Rounding Dash was a marker tool incorporated into the classroom mathematical practice for rounding numbers. Other tools were analogous models, some of which constrained mathematical negotiation for students. The scientific calculator was the only Technological Tool but assumed different roles in this case study. The calculator role as an exploratory object was important for shared student experimentation and trialling of calculator processes.

The teacher initiated and demonstrated classroom mathematical practices. These practices were called 'rules' and had high status within this classroom. The teacher explicitly highlighted, modelled and revisited mathematical tools and practices. The development of classroom mathematical practices for the addition and subtraction of integers was significant for this case study, because this involved a variety of student contributions. These contributions were often appropriated or adapted by other students and sometimes by the teacher. In Lemonwood 1, there were many examples of negotiation of mathematical meanings, initiated by either a student question or comment, an error in a student generated answer or solution, or a mismatched dialogue, subsequently repaired by teacher or student or both. A discourse of primary school mathematics also provided a basis for students to outline their ideas, seek validation or to pose a question.

4.3 Case Study Lemonwood 2

4.3.1 Background Information

The Classroom Setting

The Lemonwood 2 classroom was in a re-locatable building containing another teaching room and a teacher resource room. There were windows along three sides of the classroom and a whiteboard on the front wall. There was a door at each end of the room and at the start of each lesson, the students entered by the door at the back of the room. The teacher's desk was at the front of the room beside the wall and behind this was a large cupboard, shelves and filing cabinet for the teacher's resources. On the right hand side of the whiteboard was a section marked off for the current homework expectations for each of this teacher's classes and in front of the whiteboard was a large box of textbooks which were used in some lessons. There was an overhead projector on a trolley but this was not used during the time of the data collection. The thirty-five student desks in the room were arranged in rows of two desks on each side of the room with rows of three desks in the middle. There were posters on the classroom walls, some produced by students, and some more formal including a poster listing problem-

solving strategies. Lesson 7 was timetabled into a room on the other side of the school due to the mathematics classroom being used for a school examination.

The Social Group

Lemonwood 2 was a Year 9 class comprising 31 students, 13 girls and 18 boys. Again, these students were in their first term at secondary school and came from a variety of contributing schools. The class was designated a middle band class, based on information from the contributing schools about the students' performance in a range of subjects, including mathematics. The teacher, Mrs. Mills, was an experienced teacher who was also the class Form Teacher with responsibilities for administration and pastoral support.

Students chose where they sat in the classroom. Some students always sat in the same places next to the same students, while others sat in different places, usually within the same group of friends. One student appeared to be socially isolated and always sat alone at the front of the room. Another student was regularly absent or, when at school, often called out of class by the school administration. The students in this class were talkative and very sociable. They were actively interested in school activities and demanded regular access to the daily school notices. In Lessons 2 and 3, there was intense discussion about social events such as the school disco. In Lesson 5, the teacher used one of these events, a rock concert, as a real situation for a mathematics activity. This class was also involved in the Year 9 lunchtime basketball competition with many of the students attending as spectators. In the final two weeks of the term, two students from this class were rostered on a daily basis as the school 'runners'.

Focus Students

The Lemonwood 2 focus students were Melanie and Simone (who usually sat together), Stephanie, Amy, Louise, Jeremy, Alex and Shaun. Consequently, these students appear more often in the transcript data than other students in the class.

The Classroom Programme

This class had three mathematics lessons per week. Two lessons were sixty minutes long and one was fifty minutes. The students brought to class their

'squared' exercise book for mathematics and their scientific calculator. At home the students had a homework book, *Alpha Mathematics Homework Book* (Barton, 1999), plus a textbook for additional homework tasks, *National Curriculum Mathematics Level 5, Book 1,* (Catley and Tipler, 1995). An older textbook, *Complete Mathematics for Third Form Students,* (Catley, de Groot and Tipler, 1989)), was kept in the classroom and distributed for in-class student activity. The class had only three mathematics lessons per week because, for Term 1, the school had added a weekly ICT lesson in place of a mathematics lesson. The Tuesday and Wednesday lessons often started with form class administration. The Tuesday mathematics lesson was directly after lunchtime, the Wednesday lesson immediately before morning interval, and the Friday lesson before lunchtime. Some lessons were interrupted by the school 'runner'. Other interruptions were by senior students with notices for the class and, on one occasion, by another teacher who needed to speak to the class.

4.3.2 The Mathematics Lessons Observed

In Lemonwood 2, nine mathematics lessons were observed over a period of the final four weeks of the first school term. The mathematical content of each lesson is recorded in, with the number of episodes identified from each lesson transcript.

Lesson	Week	Day	Main Lesson Focus	Number of Episodes
1	2	Tues	Addition and Subtraction of Integers	4
2	2	Wed	Addition and Subtraction of Integers	7
3	2	Fri	Decimals	6
4	3	Wed	Rounding of Decimals	5
5	3	Fri	Rounding and Significant Figures	6
6	4	Tues	Multiplication/Division of Decimals	9
7	4	Wed	Relations and Graphs	0
8	4	Fri	Revision of Division	3
9	5	Wed	Relations and Graphs	0

Table 4.5 Record of Lemonwood 2 Observed Lessons and Episodes

The lessons were observed in sequence except for Lesson 9 which occurred after the common test on number in the final week of the term. One school day in the third week was missed because the school was closed for teacher professional development. The topic for the majority of observed lessons was Number. The next unit, Relations and Graphs, was started in the fourth week of observation. There were no episodes identified in Lessons 7 and 9 because there were no significant interactions relating to number.

4.3.3 Classroom Curriculum

Typical Lesson Structure

The mathematics lessons in Lemonwood 2 did not have a recognisable pattern of activity. For example, two lessons began with review questions, another two with a review of homework, while two more started with a teaching episode. In every lesson, however, there was at least one whole class teaching episode, of varying length, and some tasks set for the students to complete individually. Sometimes these tasks were written on the whiteboard, but usually they were from the mathematics textbook. The independent student activity was sustained for about ten minutes, although sometimes the teacher intervened with a quick comment or cue to the whole class. This seemed to reflect the teacher's own views about the importance of student practice, where practice encapsulated explanation and checking of solutions.

> Well, probably explanation and practice, giving them the opportunity to practice both in class and for homework certain skills. And also explaining why things happen ... (Mrs. Mills, Interview)

In two of the seven lessons on number, the teacher drew large graphic organisers on the whiteboard that she expected the students to copy into their books. In Lesson 1, the first diagram was a series of nested squares that visually represented the relationship between different types of numbers. In Lesson 3, which began a subtopic on decimals, the teacher used another graphic organiser to signify the beginning of a new topic. The purpose of the graphic organisers appeared to be to provide visual and written connections for the students. Also in Lesson 3, there was a whole class game of Bingo (or Housie) where the numbers called out were fractions, decimals or percentages. Each student had their own

Bingo card, the teacher called out the numbers, and the game ended when the first student had a complete card and shouted "Bingo!".

Links between Assessment and Teaching

Like Lemonwood 1, this class had sat the Progressive Achievement Test in mathematics prior to the data collection (Reid, 1993). The teacher showed the researcher the class marks and commented that there was only a small range in the marks with few outliers. In the interview at the end of the data collection, the Lemonwood 2 teacher mentioned that the students had started high school with minor gaps in their mathematical knowledge.

> There's often little gaps of understanding or little gaps where they might not have covered a particular skill. Quite a few of them hadn't done any division - just one or two concepts, you know, like the concept of thousandths, hundredths, tenths. Some of them didn't fully understand some of those basic concepts. They had probably been taught them, but just the understanding - that was a new revelation for them. (Mrs. Mills, Interview)

Division was identified as a topic that had exposed these gaps but during the number unit, students had responded to the additional experiences and had gained more understanding of division.

Rituals

Rituals were defined as recurring lesson events observed within and between lessons (section 3.7.3). In Lemonwood 2, the rituals identified as significant were the Activity and Chorus Rituals. There were other rituals identified in Lemonwood 2, but these were not observed to be significant for student participation. For example, in the marking ritual, the teacher called out the answers for both in-class tasks and selected homework tasks, but there was little student participation in this ritual. Only a few students were observed to volunteer answers or queries about the answers.

Activity Ritual

During individual student activity, Lemonwood 2 students often worked together while the teacher circulated around the class. This provided opportunities for students to participate in social and mathematical activity. Most students talked with their neighbours including those sitting in front or behind them. The teacher

supported this shared activity because it gave students "the opportunity to check things out with their partner who they are sitting next to" (Mrs Mills, Interview). During this time, she actively responded to student questions and requests for assistance.

Chorus Ritual

The Chorus Ritual consisted of five to ten students calling out together during whole class teaching. The chorus was in response either to a teacher question or to complete a teacher sentence during a pause. This appeared to be a signalling strategy by students, signalling to the teacher that they were following the steps during whole class exposition and thereby cueing the teacher to continue. The chorused response and reciprocating teacher response indicated active student participation in joint mathematical activity. This chorusing was often a variation of the IRE questioning sequence described in Chapter 2 but, because it occurred on a regular basis, I have included this as a ritual. Some students were observed to participate in the Chorus Ritual more often than others, although it seemed that students took turns by chorusing in some lessons and not in others.

Introduction to Roles

In Lemonwood 2, students were observed to adopt similar roles during whole class interactions and the Activity Ritual. The frequency and distribution of these roles was, however, different during these lesson events. For example, in whole class interactions only a few students were observed to adopt roles of solution describer, mathematical explainer, and question asker. The most common student roles were question answerer or choruser (see Chorus Ritual). Another role during whole class interaction was as whiteboard recorder. These written contributions were affirmed by the teacher, incorporated into subsequent discussion, and indicated the high status of student contributions. Student contributions were also incorporated into the notes recorded on the whiteboard. The following transcript illustrates two of these roles.

[Round 6921 to the nearest 1000]

Teacher	... OK Melanie, you want to come up and do the next one? Rounding back to the nearest thousand. [Melanie goes to the whiteboard and puts a dot under the six]

Teacher	The number in the thousand column ... Good ... so once again you look at this one, Aaron, we look at the number on the right/ the big number so the six changes up to a ? ... A seven ... so what do we have to put as place holders?
General	Zero
Teacher	Zero ...

In this whole class demonstration of rounding, Melanie was a whiteboard recorder and other students were chorusers.

During the Activity Ritual, more students were observed to adopt roles of solution describer and mathematical explainer. These roles were indicators of peer collaboration and assistance during these activities. This was valued by the teacher. "I can see quite a lot of chatting going on in the classroom and they're checking out with each other" (Mrs. Mills, Interview). Students were observed to actively request assistance from the teacher and peers. The most common way of requesting assistance from the teacher was to wait until the teacher was nearby and indicate by a raised hand or a direct request. When students were on the opposite side of the room to the teacher, there were only infrequent requests for assistance. Mostly the students would consult their immediate neighbours first, or wait until the teacher was near before requesting assistance. In most lessons, a number of students left their seats to converse with peers. Seeking assistance from peers was associated with social networking, particularly among many of the girls. When directed by the teacher, the conversations quickly ceased, and the students went back to their assigned tasks.

Teacher Mathematical Values

In Lemonwood 2, Teacher Mathematical Values related to beliefs about using calculators, about being a learner of mathematics and about mathematics as a subject. Teacher statements about calculators revealed implicit values about the importance of student mental processes for both calculations and reasoning. In Lesson 2, as the teacher introduced using a calculator for integer calculations, she said,

> Now we know we could probably do it faster in our head, but I still want you to, you've still got to learn to do it on a calculator as well as learn to do it in your head. So, today we're going to use the calculators.

A short time later, the teacher commented to the class

> Did you know ... your own brain can work it out. So don't forget your brain is very fast, it's more superior than any (calculator) in the world.
> (General students: Oh)
> Your brain can do more/ functions ... than any of the most sophisticated computers.
> (Lesson 2 transcript)

These statements indicated a teacher value that the students' brains were more capable than any technological tool and were often faster. The teacher suggested that the students should not be reliant on the calculator for all computations and should be confident in their own mental calculations. While reviewing calculator methods, the teacher emphasised that the students were using valuable skills that would be important for the (unspecified) future.

> Well done to most of you for working through that. It is a bit tedious but we've actually learnt ah quite a bit today because ... some of those ... little things there about pressing equals before you do the dividing, ... just checking things in our head. They are skills which if you learn now, they'll put you in good stead for later on. (Because) we always need those skills. (2/ L2/ 50/ E7)

In Lesson 5, while the students were working on more rounding examples, the teacher was observed talking with some students about the value of more practice of these rounding tasks. The students needed "a little more practice, remember maths is about understanding and practice" (field notes, Lesson 5). This comment conveyed the message that skills and practice were important, and that implicitly this would help students in their mathematical learning.

The Lemonwood 2 teacher regularly talked about strategies for being an independent and competent learner of mathematics. The teacher's encouragement of student use of the textbook as a resource for learning is discussed in section 4.3.5. In the lessons leading up to the common test, the teacher referred to strategies for revision. For example in Lesson 8, she directed the class to the part of the textbook considered prior knowledge and then linked this with ways to revise for the test.

> So that's a good way to revise. You see that whole page is ...So I'm really teaching you to how to, I'm really teaching you now how to use your textbook as a source of revision/ for the test. (Lesson 8 transcript)

These practices illustrated a Teacher Mathematical Value of using textbooks as a resource for students.

4.3.4 Tools.

Classroom Mathematical Tools

The Lemonwood 2 teacher incorporated five Classroom Mathematical Tools in the seven lessons on number. These are set out in Table 4.6 below.

Classroom Mathematical Tool	Mathematical Use of Tool
Number Line	Model representing relative magnitude of integers Thinking device for addition and subtraction
Rounding Dot	Marker to label the designated place for rounding
Chocolate Bar Drawing	Analogous model of relative magnitude of tenths and hundredths
Odometer	Real life model of increasing numbers
Place Value Grid	Pedagogical artefact for displaying the values of decimal places and the corresponding prefixes

Table 4.6 Classroom Mathematical Tools for Lemonwood 2

In Lemonwood 2, the main Classroom Mathematical Tools were the Number Line and the Rounding Dot, used as both representations and thinking devices in classroom activity. A feature of this case study was the teacher's use of real life situations as analogous models of mathematical relationships and processes. One of these models, the speedometer, offered a dynamic model of the process of rounding up. These tools are now described in more detail.

Number Line

The Number Line was used in Lesson 1, although it appeared that the Number Line had been used more intensively in the preceding lessons about integers. The Number Line was used as a thinking device in a whole class review of integer addition and subtraction tasks. The teacher used the Number Line to model the process for a few examples, and she then directed the class to the integer chapter in the textbook. She explicitly pointed out the diagram of the Number Line in the

text and re-iterated the textbook method of using the Number Line to add and subtract integers. This tool was both a model for representing the relative magnitude of integers and a thinking device for calculations with integers.

Rounding Dot

In Lesson 4, during the introduction to the mathematical practice of rounding, the teacher drew a dot on the whiteboard underneath one of the digits of the number. The dot was a marker that signified the designated place for rounding. The Rounding Dot was a tool used by the teacher to support the introduction of a new mathematical practice, discussed in a later section (4.3.5).

Chocolate Bar Drawing

In Lesson 3, a chocolate bar was drawn on the whiteboard to demonstrate the relationship between tenths and hundredths. The demonstration by the teacher emphasised the division of the chocolate into more pieces, but only visually implied that the pieces were smaller.

Teacher	... so when you think, if you got, it's like having a big bit of chocolate like that and you can divide it into ... into ten bits. OK, so I decide to share it among the class ... and I divided it, I still wouldn't have enough ... bits of chocolate. Then if I wanted to divide each of these into ten ... it would actually be a tenth of a tenth ... [on whiteboard, teacher draws model of chocolate 5x2 squares] And so how many bits of chocolate would I end up with altogether? If I divide each of these little bits of chocolate into [Teacher divides one square into tenths]
Melanie	A hundred (2/ L3/ 15/ E2)

This was followed by a chorused student response to the teacher's sequence of clarification questions. The chorus indicated that the students were familiar with this particular model of tens and hundredths.

Odometer

A second real life situation, a car odometer, but referred to as a speedometer by the teacher, was talked about but not drawn by the teacher. The teacher introduced the idea of a speedometer to the whole class at the end of a whole

class review of some rounding tasks in Lesson 4; "it is a bit like the speedometer on your car and you watch it and it goes up to nine nine nine and then it all changes ..." (Lesson 4 transcript). In the next lesson, the speedometer was used a number of times by the teacher when some of the students were having difficulty with the rounding up of a nine digit. For a number with more than one nine, (such as 0.0899 – round to 3 d.p.), the teacher mentioned the speedometer.

Teacher	It's a bit like if you watch the speedometer on your car ... it gets up to eight
General	Oh yes –
Teacher	and they will all flick over to zeroes
Jerry	But everything's got to nine? ... oh yeah, yeah. I've got it. (2/ L5/ 25/ E1)

Jerry was sitting near the front of the class and his statements could have been either a question to the teacher or thinking out loud. During individual student activity later in the lesson, the teacher mentioned the speedometer when a student commented that he hated the ones with all the nines. The teacher said, from across the other side of the room, "but just remember the speedo on the car, when all the numbers roll over" (Lesson 5 transcript).

Place Value Grid

The last classroom mathematical tool was the Place Value Grid. This grid included rows with the prefix for the specific measurement unit and the value of each place in powers of ten. While the teacher was recording this on the whiteboard, the class chimed in with a chorused recitation of the key facts, indicating that they were familiar with both the conventions and the grid. The Place Value Grid was categorised as a pedagogical artefact used to record multiple representations and to illustrate relationships between each decimal place.

Technological Tools

The technological tool in Lemonwood 2 was the hand held scientific calculator, either a Sharp or a Casio brand. In Lesson 2, the teacher issued some school owned solar powered simple calculators to students who did not have their scientific calculators. Data about who initiated the use of the calculator (students

and/or teacher) and the mathematical purposes of the calculator episodes are recorded below in Table 4.7 (Categories in brackets indicate a secondary role).

Lesson	Calculator Event	Initiated by	Category
2	Calculations with negative numbers	Teacher	C2 – calculating machine (C1)
2	Using the convention of BEDMAS – focus on brackets, exponents and division	Teacher	C2
2	Student sharing of shortcut key	Student	C4 – as an exploratory object
3	Student clarification of question about decimals	Student	C1 – as a recording device
5	Calculating decimal numbers then rounding	Teacher (Student)	C1 (C2)
6	Multiplication of decimals	Teacher	C2
6	Discussion resulting from student results on calculator screen	Student	C4
8	Students calculating and checking for long division	Student	C2

Table 4.7 Categories of Calculator Use for Lemonwood 2

In Lemonwood 2, the main role of the calculator was as a calculating machine during the practice of calculations with negative integers, a combination of operations (BEDMAS), and with long division. The calculator offloaded mental or written calculations, saved time, and provided greater accuracy. In particular, the calculator afforded greater efficiency when computing more complex calculations such as using BEDMAS, or using large numbers in long division. The most overt use of the calculator by the teacher occurred in Lesson 2, when calculations with integers were combined with a review of BEDMAS. The teacher showed the class the key for integers [+/-], and explicitly distinguished this key from the "take-away button". The integer key was described as one "which makes the number negative", as opposed to the operation key because "that's just ordinary take-away" (from 2/ L2/ 20/ E2). The teacher then led the class through a simple example with a known answer, modelling the process verbally and recording the keystrokes on the whiteboard. More complex

examples that included brackets and exponents were set for the class. Some students calculated the expression, 3(2-4) + 1, a number of times with varying results. This stimulated much peer discussion and, as the teacher moved around the room, a few students told her of their findings. The teacher replied to the whole class "Yes. Sometimes you, some calculators are different, and sometimes if you press the buttons in the wrong order ... or um ... your calculator ... sometimes it can calculate in the wrong order and can produce the wrong answers" (2/ L2/ 35/ E4).

The calculator was also an exploratory object and many students were observed independently using their calculator, often when they were supposed to be doing other tasks. Students tried out calculations and then tried again, they worked backwards to find a required result, and they experimented with different keys on the scientific calculator. Students were observed to independently key in instructions to the calculator, to delete and try again, and to make changes as necessary. These independent student-initiated explorations produced two episodes when students explained their findings of a new key or way of using the calculator. These episodes will be discussed in a later section, Technology Mediated Negotiation.

To summarise, the roles of the calculator in Lemonwood 2 were as a recording device, as a calculating machine, and as an exploratory object. The different brands of calculator provided alternative ways of expressing numbers and calculations, and supported a flexible and adaptable approach to mathematical activity. The capacity of the calculator to express numbers in different ways afforded multiple pathways to the same result. In particular, the scientific calculators had keys that offered shortcuts for more complex operations. There were also different screen expressions for large numbers and although initially confusing for students, these representations initiated further mathematical activity. The calculator episodes stimulated localised and whole class discussion between peers, and between teacher and students. Finally there were two examples of students using the calculator as a recording device (Lessons 3 and 5), and again these are discussed in the later section, Technology Mediated Negotiation.

Other Mathematical Tools

This category records cognitive tools not already identified as Classroom Mathematical Tools. Other Mathematical Tools were identified from the lesson transcripts and are presented in the table below.

Tool	Examples of this Tool
Algorithm	Multiplication algorithm for decimal numbers Division algorithm for whole and decimal numbers Long division algorithm for whole numbers
Operations/Relations	Addition and subtraction of integers Multiplication and division of decimal numbers Powers Combinations of operations (BEDMAS)
Multiple Representations	Numbers represented by factors and by prime factors Numbers recomposed, decomposed Fractions, decimals and percentages
Other (eg notation)	Relative magnitude of decimals Place value notation Significant figures

Table 4.8 Other Mathematical Tools Identified in Lemonwood 2

4.3.5 Classroom Interaction

This dimension encompasses the analysis for the domain of interaction. The first two categories focus on teacher initiated interactions, the next three categories on student initiated interactions, and the last on interactions involving technology.

Teacher Guided Participation

Teacher Guided Participation in Lemonwood 2 emphasised making connections. For example, the teacher encouraged the students to use their textbooks as a resource for learning. Lesson 2 began with the teacher guiding the class through various features of their textbook, such as the contents page and the overview notes for the integers chapter. She explicitly linked the more expert use of the textbook to better mathematical learning.

> I want you just so that you um get used to using the textbook, which you've used a wee bit, but I always tell you the page number. I want you to just

> look in the contents of this text book ... OK and see if you can find a chapter - where does integers come in the, in the book? (2/ L2/ 15/ E1).

Rather than the teacher always providing the page number, Mrs Mills demonstrated how students could use the textbook more independently. Finding the contents page and then using it to locate a particular topic, in this case integers, modelled this process. The teacher continued to outline how the chapters were organised in this particular textbook (Catley, de Groot and Tipler, 1989).

> OK. So at the beginning page 232, see that they've got, these little books have a note, they have notes at the beginning and ... we've got integers on a number line and we've got adding integers, you've got the rules. They say for integers use the number line to add integers, you can move from left to right. ... They talk about subtracting integers on the, page 233, ... you subtract a negative integer move to the right on the number line, multiplying integers and dividing. So these little books have a wee note at the beginning ... and then they have an exercise. ...
> (2/ L2/ 15/ E1)

The teacher referred to aspects of integers already familiar to students, such as the Number Line and rules for operations on integers, providing cues for students to identify not only the notes and exercises in the text but also specific mathematical information. This theme emerged again in a later lesson, when the teacher provided guidance about effective revision methods in preparation for their imminent common test on number. She advised the class to try out some exercises for revision and to use the worked examples in the text to re-mediate any errors.

A second feature of Teacher Guided Participation was the use of resources to connect aspects of mathematical learning. Graphic organisers have already been discussed and another connecting practice occurred during the game of Bingo. While reading out the numbers, the teacher provided support to the students by saying the different representations of a number or by providing other clues. She also called out similar numbers together so that the students could become familiar with multiple representations for numbers. This demonstrated that the subject matter of the game, and the process of the game itself, was a worthwhile learning experience and important for clarifying the multiple ways of representing, saying, and reading fractions, decimals and percentages.

In addition, the teacher provided regular 'signposts' during lessons. For example, in order to link two topics she said to the class, "We're leaving integer land and entering decimal land" (Lesson 3, transcript). As the teacher moved around the room during individual mathematical activities, she provided explicit cues relevant to the mathematical topic. This might take the form of repeating a rule, or signalling an important point such as "don't forget to ...". These teacher scaffolding practices were mostly assisting actions, regularly connecting student experiences with the resources of school mathematics.

Whole Class Negotiation

This category examined the development of classroom mathematical practices and one example will be described in some detail. In Lesson 4, the teacher introduced the practice of using the Rounding Dot for the rounding of whole numbers and decimals. She began by talking about a purpose for rounding and used a real life utilitarian purpose related to money. The practice of rounding was then set out by the teacher as a two step process. The first step was to identify the place designated for the rounding. The place was identified by either inspection or by counting from the decimal point, and then signified by a dot under the digit in the designated place. The teacher used a red pen to draw the Rounding Dot on the whiteboard, implicitly cueing an important resource. An extract of the initial teacher demonstration of the method is given below.

> [on whiteboard 4534.239]
> OK we look at a number like that/ ... when we round, we could round back to ... any one of those numbers, couldn't we. And ... what we've got to be aware of is what number we round back to. So when we round, step one ... we round ... is that we ... we identify ... we identify ... the number ...We identify the number we are rounding back to ... We are rounding back ... and the way that we do that is, the way I am going to show you, it is just put a wee dot/ under the number that we are rounding back to/ (2/ L4/ 30/ E2).

The second step involved the actual rounding process where the digit immediately to the right of the designated place (marked by the Rounding Dot), was inspected for rounding. The use of the Rounding Dot was a localised strategy used by the teacher to establish a new Classroom Mathematical Practice labelled the Dot Method for Rounding. The teacher then used the same number, 4534.239, to demonstrate rounding to different whole number and decimal places. During

this whole class teaching episode, students were involved in a number of ways. Some students were called on, and some volunteered, to go to the whiteboard and draw the red Rounding Dot in the appropriate position. Some students asked questions, mostly about the rounding process, and many students were observed talking to their immediate neighbours. This is illustrated by the transcript and analysis presented below.

Kirsty and Decimal Places

[4534.239 written on whiteboard]

Speaker	Transcript	Transcript Analysis
Teacher	If I wanted to round it back to the nearest hundredths/ … what number is in the hundredths column, Kirsty?	Teacher directed a question to Kirsty, the first step of identifying the digit in the required place, hundredths.
Kirsty	Five	Kirsty said the digit in the hundreds place
Teacher	Not, not hundreds, hundredths	Teacher interprets Kirsty's error and repeated the desired place
Kirsty	Nine	Kirsty said the digit in the thousandth place. Kirsty may have assumed that the hundredths place would be a mirror image of the hundreds place
Teacher	That's the tenths, that's the hundredths and that's the thousandths. So what number is in the hundredths?	Teacher pointed to whiteboard and stated the different decimal places. Repeated the original question
Kirsty	Five, oh the three	Kirsty repeated her original number 5, then self-corrected to three (assuming the hundredths rather than the three in the tens place)
Teacher	The three, OK. so we put a dot under the three, look at its neighbour on the right/ and it's a big number, so it makes three change to a (^)	Teacher demonstrated the first step of the practice and directed students to consider the second step. Rising inflection to signal to student to complete the sentence.
General	Four	Chorused response that completed teacher statement

(2/ L4/ 35/ E3)

Figure 4.16 Kirsty's Identification of Decimal Places

Kirsty may have misheard the teacher question as "What number is in the hundreds column?", rather than the hundredths, a common mistake recognised by the teacher. When the teacher repeated the question, the student identified the number in the thousandth column. This illustrated a difficulty for students in the naming of the decimal places because the semantically corresponding places are not symmetrically arranged about the decimal point. The teacher response was to review the names of the decimal places, in keeping with the original focus of the task. This possibly indicated the importance of place value understanding in the process of rounding and that this understanding was assumed to be prior knowledge.

During the remainder of the whole class negotiation the teacher consistently articulated the Dot Method for Rounding.

Teacher	I would like you to copy these numbers down. Put the dot under them.
Melanie	Do we have to put the dot ?
Teacher	You can do it in your head if you're confident, ... (2/ L4/ 35/ E3)

In the process of negotiating the practice, Melanie had questioned whether she had to put the dot and this may have been an example of student resistance to recording the dot in their written records. The teacher responded by offering a choice to the students, to either use the dot or to carry out the entire practice mentally, thereby allowing for a fading of the Dot Method for Rounding for some students. Further examples of rounding decimal numbers provoked student questions and concerns. These concerns related not to the method but to the resulting adjustments of the numbers as a consequence of the rules of rounding. Rounding down simply truncated the number beyond the designated place but rounding up involved a further step, that of increasing the digit by one. There was an additional step if the digit was a nine because the increase of this digit resulted in changes to other digits to the left. This could require the use of zero as a placeholder in the final rounded number. It was in this lesson that the teacher contributed the speedometer as an analogous model representing how the digits change, particularly when a nine was increased by one. During the later practice examples of rounding, students were observed to appropriate and adapt the Dot

Method for Rounding, many of them developing their own idiosyncratic processes. One student told me she was not using a dot because she didn't like the dots. Instead she said that she did the rounding process in her head by thinking about the number. Another student was not using the dot because she had her own method in her head that was working for her. When asked why she thought the teacher had shown the class the Dot Method for Rounding, this student replied that the teacher did this to make it easy for the students.

Alex's Calculator Dot

One student, Alex, was using his calculator a great deal during some of the rounding tasks. When I asked Alex about his strategy, he said he used a dot but that he had drawn it on the calculator, directly underneath the screen. In Lesson 5, during a public review of rounding of decimal numbers, this student had rounded directly from the calculator screen using a dot drawn under the screen. The transcript is produced below.

Teacher	Alex, what number did you get for e)? 0.384 times 1.3... do the calculation first were the instructions..
General	Which one? Are we up to??
Teacher	What's the number, Pardon?
Alex	I didn't write it down
Teacher	So what's the number then?
[Alex read from his calculator screen]	
Melanie	Zero point five oh
Alex	Zero point 4992(^) (2/ L5/ 25/ E2)

Because Alex had not recorded the calculated product, he had to repeat the calculation on the calculator. This implied that he had used the calculator as a recording device for the next step, the rounding process, aided by the drawing of a dot underneath the required decimal place on the screen. Alex had generated his own tool for rounding. An affordance was that this tool provided immediate feedback, but one constraint was that the original number was erased from the screen.

In Lemonwood 2, the introduction and development of the Dot Method for Rounding illustrated some key features of teacher-initiated negotiation. The

teacher presented the method as a scaffolded two step process that included a Rounding Dot as a tool. Whole class interactions included student contributions of answers to teacher questions, chorused responses, and whiteboard contributions. When errors were made, the teacher clarified and repeated her questions. In this middle band class, some students implicitly recognised the rationale for the Rounding Dot as an introductory tool. The Dot Method for Rounding appeared to be appropriated in different ways by students. Some students quickly abandoned the teacher tool for their own methods including Alex who recorded a dot on his calculator.

Student Initiated Negotiation

This category encompassed interactions initiated by a student contribution such as a question, solution method, suggestion or disagreement. In Lemonwood 2, the negotiation that resulted from unanticipated student contributions involved greater teacher-student interaction than in the teacher-initiated episodes. This is illustrated by the example below, initiated by a student question.

Melanie's Question

An example of student-initiated negotiation occurred in the first observed lesson when the teacher was reviewing practice integer tasks. The teacher had used the Number Line to illustrate a rule for adding and subtracting positive and negative integers. Melanie asked a question about which number was first, that is, which addend should be considered first on the number line. The transcript and analysis are presented below and although it appears here as a conversation between the teacher and Melanie, many students were observed to be following the dialogue.

Speaker	Transcript	Transcript Analysis
Melanie	What if it's a positive plus a negative?	Melanie asked about a prior situation (second addend negative) but in a generalised situation.
Teacher	Positive plus a negative. … Well, that's a bit like this one here. …	Teacher repeated student question, pointed to an example on a handout sheet. [3 + -4]
Craig?	You swap it around, …	Craig assisted Melanie, volunteered a swap method, implicitly in order to add a positive if at all possible.

Teacher	Positive plus a negative, you just take the difference between the two numbers and whichever sign is dominant or … Give me an example on the board.	Teacher repeated Melanie's question, explained difference method. Asked Melanie to provide an example. Teacher cued the student that she would demonstrate with an example.
Melanie	Wait. How do you know which one is the first?	Signalled for teacher to wait, then asked about how you would identify the first addend. Implicitly the starting point if using a number line model. (possibly ideas related to commutativity with negative numbers eg does it matter which number is first?).
Teacher	On which one are you stuck?	Teacher asked a probe question to narrow the focus of the student question.
Melanie	Um three plus negative four	Melanie stated the first example in the practice tasks already marked.
Teacher	Three plus	[Teacher began writing this example on the board], saying it out loud.
Melanie	That's negative, you go four	Melanie completed the teacher's sentence by continuing with the second addend. (3 plus –4)
Teacher	That's the same as three minus four.	Teacher rephrased as 3 minus 4, combining the operation and the integer sign.
Melanie	Oh,	Exclamation from Melanie, possibly surprise.
Simone	So, sort of like (three …)	[Simone sitting next to Melanie] Simone trying out her interpretation.
Melanie	How come that's ..?	Melanie indicated she was still not convinced. Asked for explanation.

(2/ L1/ 20/ E1)

Figure 4.17 Melanie's Question about Integers

Following on from this episode, the teacher explained the difference method for the example, three minus four, based on the comparison of magnitude between a positive and negative number. Melanie and Simone continued to discuss this.

Melanie	It's actually like take aways
Simone	Um, yeah
Melanie	Isn't it just like, this way? (referring to 'three minus four').

A few minutes later, Melanie again engaged the teacher in a discussion about a possible shorter, more concise method but in her efforts to generalise the rule, she had ignored the importance of the integer sign for each of the numbers involved. This was resolved during the next Activity Ritual when the teacher moved around the room and attended more closely to individual student questions. This next extract was from a localised conversation between Melanie and the teacher, with Jeremy sitting directly in front of Melanie. [14 + -2 was directly after 3 + -4 on the handout sheet]

Melanie	Um um. OK, say if you go um. OK, positive fourteen plus um plus negative two, is it the same as going fourteen plus two?
Teacher	OK, positive fourteen plus negative two is, it's just like saying fourteen take away two, ... which is twelve.
Jeremy	So add a negative number is just like taking it away.
	(2/ L1/ 20/ E1 continued)

Although Melanie has not resolved her dilemma, Jeremy had volunteered a viable rule for adding a negative number. Melanie's persistence in asking questions indicated a strong adherence to prior knowledge of positive numbers and to the notion that mathematics should make sense. During this interaction, Melanie received a number of differing messages about how to resolve her question. She could fiddle with the signs, it's just like takeaways, or change the signs. Melanie wanted to know about the general case of a positive plus a negative integer. There was no further data about how she resolved this dilemma. Later in the lesson, however, Melanie publicly contributed the correct answer to a problem about a series of withdrawals and deposits of amounts of money.

As this episode illustrated, student questions were accepted in this classroom and an interactional space was created in response to a student contribution. The student initiated negotiations in Lemonwood 2 were slightly different to the teacher initiated negotiations: the teacher contributions were briefer and more focussed, student explanations were included, and peers were more likely to volunteer further resources.

Error Initiated Negotiation

This category analysed whole class and localised conversations prompted by a student error in either an answer or solution method or both. Some of the error initiated negotiations during the development of the practice of rounding have been discussed (Kirsty and Decimal Places, Alex's Calculator Dot). Other instances of errors were identified in activity related to multiplication and division of decimal numbers.

Simone's Error

During a whole class review of decimal multiplication tasks, the teacher asked Simone to contribute her answer.

Teacher OK next one, we go … ... Simone, will you do the next one please. [15 x 0.2 on whiteboard]
Simone OK … Two times fifteen is thirty … and there's, one … space after the … decimal point, so it's point three
Teacher No
Simone Oh no … it's three point zero
(2/ L6/ 25/ E4)

Simone self corrected after her explanation and an indication of an error by the teacher. There is not enough evidence to infer the reasons for Simone to change her answer and there was little further interaction.

Monique and Decimal Magnitude

Another student error initiated more substantial interaction in Lesson 3. The teacher had set the class a discrimination task for diagnosing misconceptions about the relative magnitude of pairs of decimals. The task was: Which one is the biggest? 0.39 or 0.4; 0.09 or 0.1; 0.25 or 0.08. The students had been working on a similar exercise from the textbook and the teacher noticed that many of the students were having difficulties. She had told individual students to focus on the tenths column first and to decide which of the decimal numbers was the largest. The teacher repeated this strategy to the whole class when reviewing the three examples. She also drew a model of a chocolate bar on the whiteboard and said that "you'd much rather have the tenths bit than the little tiny bit" (pointing to the hundredths parts). The transcript and analysis for the task review are presented below. (Which is bigger, 0.09 or 0.1?)

Speaker	Transcript	Transcript Analysis
Teacher	For this one ... you can imagine um for this one, you can imagine that some people, say a little child would rather give me that bit of chocolate or that.	Teacher pointed to the two numbers, appealing to size of chocolate
Monique	Would probably say the nine.	Student acknowledged that she would probably say that 0.09 was larger than 0.1 (assume due to the magnitude of digits or the number of digits). Teacher's appeal to a model of chocolate would not be enough to convince student.
Teacher	But in actual fact this is bigger than the next one, yes so this is actually bigger so ...	Teacher repeated the size of area to model the magnitude of different decimal places.
Monique	It doesn't look it.	Student rejected teacher's model, maintained her own view, focussed on the digits rather than the analogous model.

(2/ L3/ 45/ E6)

Figure 4.18 Monique and Decimal Magnitude

The teacher then extended the situation to another example where the digit in the tenths column was the same but one number was still larger than another. The task was: Which is bigger 0.23 or 0.208? (field notes). The teacher extrapolated the strategy into the hundredths and other decimal places. This appeared to be an example of a practice that had been appropriated in a simpler form but then became confusing for more complex examples. When student errors emerged, the teacher responded by repeating the process and providing further examples for the class. Students responded by trying these new tasks and providing answers to teacher questions.

Two errors were identified during the Activity Ritual and both related to calculation algorithms. The Error Initiated Negotiations occurred when these students appealed for assistance from me.

Louise and Short Division

In Lesson 6, Louise was working on short division tasks when she requested assistance. Louise had been dividing 8 into 71.3, written in a short division format and had written 8.9 as the quotient with a remainder still to be sorted out. Louise asked, "What happens to the remainder when it gets to there?" pointing to the decimal place in the quotient. Louise continued, "if there are no numbers in the places, then how can you write the answer to those places, what do you do?". Louise believed that if there were no numbers, even zeroes, in the hundredths or thousandths places of the dividend then you cannot write the quotient to these places. She wanted the dividend and quotient to have the same number of decimal places. By considering the dividend to have (invisible) zeroes in the places to the right of the last digit, Louise was prepared to carry on the division process for the above example, resulting in a quotient of 8.9125 (2/ L6/ 55/ E9 from field notes). This error was inferred to be similar to the error in the multiplication of decimals, that is, the decimal numbers must have identical decimal places. This was possibly a consequence of perceived symmetry in the addition and subtraction of decimal numbers. It could also relate to student perceptions of the representations of decimal numbers and the inclusion of zeroes.

Shaun's Multiplication Bug

In Lesson 8, during revision of multiplication algorithms, Shaun was sitting near me and asked why his answers were incorrect. His misconception in the common algorithm was in adding the carried numbers to the digits before multiplying the next place. For example in 1234 x 13, Shaun employed the long multiplication algorithm for the first place (3x4 =12 put down the 4 and carry the 1), but then added the carried one to the three (in 1234), resulting in 12 rather than 10 for the second step in this algorithm. Shaun explained his method and said his method was better (than the standard algorithm). He also told me that he liked multiplication but didn't like division (Lesson 8 field notes). By this time the teacher had moved the lesson on to another phase and we ended this interaction. These two examples indicated misconceptions in student algorithms for multiplication and division, identified during the Activity Ritual.

Repair Interactions

Repair interactions were observed between teacher and students usually in relation to discussions about decimal numbers. Many of the repair episodes could have been analysed as error initiated negotiations. They have been included in repair because they were occasions when the teacher changed her language and used a variety of repair models. A typical repair episode occurred in Lesson 6 during a teaching episode about the multiplication of decimal numbers. The teacher had demonstrated a method of multiplying as for whole numbers, followed by counting the decimal places. At least two students wanted to have the same number of decimal places in the product as in each of the factors, presumably in an effort for symmetry and consistency with the algorithm for addition of decimal numbers. In an effort to repair this misconception, the teacher introduced the multiplication of the equivalent fractions as a proof. The transcript and analysis are presented below;

[0.6 x 0.3, written on the whiteboard].

Speaker	Transcript	Transcript Analysis
Teacher	There are, there are two digits behind the decimal point there and so there must be two digits. Now I can prove this to you. This is a proof ... Point six as a fraction, Stephanie, you watching? ... point six as a fraction is really six ..	Based on the number of decimal places in the product. Teacher appealed to the students' prior knowledge of fraction multiplication. Rewrote the decimal factors as fractions.
Craig	Tenths	Elicitation-response sequence for re-writing the factors as fractions
Teacher	Tenths, ... OK so let's write six tenths down. Point three as a fraction is really ..	Teacher publicly recorded six tenths, conversion of a decimal to fractions, asked a question using complete the sentence
Craig	three tenths	Craig volunteered, possibly responding on behalf of the class
Teacher	three tenths. Okay. Now if we multiply those together, when we multiply fractions, you probably hadn't done this at intermediate yet	Teacher repeated Craig's response, completed her own sentence, then demonstrated the multiplication of fractions, reassured students they would not have multiplied fractions in the previous year, possible means of finding out information.

Melanie?	Yes we have	Melanie cued to teacher, collective 'we' although students came from different schools
Teacher	Oh you have, that's good, when you multiply fractions together, you multiply across the top and across the...	Teacher acknowledged cue, began review of the algorithm for multiplying fractions
General	Bottom	Student chorus for the steps of the process, suggesting general understanding
Teacher	bottom, so we know three sixes are ...	Teacher completed own sentence, cued in the next step
General	eighteen	Chorus, completed the sentence with the product
Teacher	eighteen, ... ten tens are a ...	Teacher repeated student response and then demonstrated the multiplication of the denominator
General	hundred	chorus, completed the sentence as 10 x 10 = 100
Teacher	hundred, ... and so eighteen hundredths is really ...	Teacher repeated student response and then reformulated for the fraction as a whole, asking for its equivalence using complete-the-sentence
Melanie	Nine fifties	Melanie responded with an equivalent fraction

(2/ L6/ 15/ E1)

Figure 4.19 Using Fractions to Repair Decimal Multiplication

The teacher used what she described as a mathematical proof in this demonstration and explanation. The proof used another representation of decimal numbers (fractions) to repair difficulties with the decimal places of the multiplication product. The teacher ended by stating that "decimals are just like fractions in disguise". Some students contributed a chorus that indicated student participation and signalled to the teacher to keep going. Immediately after this interaction the students carried on working with more decimal multiplication tasks.

Amy, Erin and Louise and Multiplication of Decimals

Three students, Amy, Erin and Louise, who were friends and often worked together, were observed talking about decimal multiplication. One of them was still diffident about a method, but together they assisted each other with two more

tasks. The two boys sitting in front of them, Shaun and Alex, turned around at various times to volunteer assistance. Amy appeared to have a misconception of the same number of decimal places in the product as each factor, particularly when both factors had an identical number of decimal places. The transcript of their localised conversation, with teacher input from across the room, is presented below in Table 4.20.

Speaker	Transcript	Transcript Analysis
Amy	How do we do that one, Louise?	Task was 1.2 x 0.2 on whiteboard. Amy requested assistance from Louise.
Louise	It's just that we have ...	Louise began to explain.
Teacher	Twelve times two is twenty-four. One, two, one, two, zero point two four (pause)	Teacher explained the method to another group, calculate the whole numbers first, followed by counting the places in each factor (one two) then counting places on the product from the right, (one two), with 0.24 as the result
Erin	No, it's alright	Erin thinking aloud
Amy	I don't get it	Amy signalled she doesn't understand, implicit request for assistance.
Shaun	Zero point two four	Shaun turned around and volunteered the correct answer. (A repeat of teacher answer)
		[Teacher continued to help other group with this example, goes to the board to explain]
Amy	Isn't it two point four?	Amy has her method that has one dp in the answer (the same number of dp as each of the factors). Amy's disagreement with Shaun phrased as a question.

Figure 4.20 Amy, Erin and Louise and Multiplication of Decimals

(to be continued)

Amy may have wanted a pattern in the written form of the numbers, and her method relied on consistency of the number of decimal places for factors and product. Amy was prepared to request assistance and to question the teacher's answer. Shaun volunteered assistance with the answer. The teacher heard Amy's last question and reiterated that the product should have two digits after the decimal point. The localised peer conversation continued:

Speaker	Transcript	Transcript Analysis
Amy	How do you do c)?	Amy tried another eg from the whiteboard [0.4 x 0.05 is example (c)]
Erin	c. Oh I do, work it, that's twenty oh point five	Erin showed Amy her method, 4 x 5 is twenty, then start considering the decimal factors. Potential error, either misread or was using 0.5 (an incorrect number)
Louise	Do you do it like, point zero five and then ...	Louise asked about the next step. Correctly saying the factor 0.05
Amy	... cross out that, the numbers...	Amy had a method of crossing out the decimal places in the factors (and presumably counting back the same number of decimal places)
Alex	Zero point zero two	Alex turned around and said the answer
Erin	Why?	Erin asked for justification, implicit she didn't agree
Alex	Work it out ... and two zeroes	Alex implicitly suggested a method, the two zeroes could refer to different aspects of the decimal answer? (assume 0.02 was the answer)
Shaun	I'll, I'll check it	Shaun used his calculator to check

(2/ L6/ 20/ E2)

Figure 4.20 Amy, Erin and Louise and Multiplication of Decimals

This episode has been included as an illustration of student difficulties with multiplication of decimal numbers. The roles of questioner and assistor have passed between the students at various times. The roles of the two boys were as assistor, question answerer and checker. It also illustrated that the teacher's attempts at repair and peer attempt at assistance, were problematic for this group of girls.

Technology Mediated Negotiation

This final interaction category encompassed the calculator episodes that involved negotiations of mathematical meaning. Many students were observed independently using the calculator as an experimental means of producing certain numbers on the screen or testing out particular function keys to find out what was the end product on the screen. The next episodes describe interactions initiated by the used of the calculator as a recording device.

Stephanie and 1.0

During an Activity Ritual Stephanie was observed trying out various things on her calculator. She suddenly called out across the room to the teacher "Is one point zero the same as one?". The teacher responded with "Yes" while writing on the whiteboard. The student exclaimed in satisfaction and returned to keying in on the calculator. Stephanie then showed her neighbour the calculator screen. After the lesson, Stephanie told me that the dilemma about whether 1.0 was the same as 1, was something she had wanted to find out from primary school. She was pleased that she had "Got it sorted today" (field notes Lesson 3). It seemed that the keying in of decimal numbers on the calculator helped to clarify her question to the teacher. When 1.0, for example, is keyed in, some calculators automatically remove the zero as soon as an operation key is pressed. When zero is not a place holder, then it is often not included in an answer on the screen. In this way the calculator may have played the roles of recording device for the student and recorder of different conventions for writing decimal numbers. Stephanie's question was quickly resolved, and was a significant learning episode for Stephanie, initiated by her use of a calculator to represent numbers. The second example of a student using the calculator as a recording device, Alex and the Rounding Dot, has already been described in an earlier section Whole Class Negotiation.

Different kinds of calculators provoked discussion and debate when they produced different results on the screens. In Lesson 6, when multiplying two decimal numbers, 0.004 x 0.02, on his calculator, Jeff had found that his calculator screen read, 8. –5. The teacher had noticed this as she had moved around the room and, after a short time, included this example in a whole class review of the practice tasks. Mrs Mills signalled this unusual result to the class and after demonstrating the "count the number of decimal places" method for finding the correct answer, Jeff read his calculator screen to the class. This unfamiliar notation became a topic for further discussion and, with the teacher's support, some students contributed their ideas about possible meanings. After listening to student contributions, one of which assumed a whole number resolution, the teacher affirmed a response relating to the number of decimal

places. The rest of the class tried this out on their calculators, initiating further interactions such as this one between Alex and Hamish.

Alex	On my calculator it works.
Hamish	It's lazy.
Teacher	What did you get?
Alex	Zero point zero zero zero zero eight.
	(2/ L6/ 30/ E4)

Hamish and the Shortcut Key

The final example in this category arose during calculation tasks involving order of operations. While the teacher had demonstrated a method based around using brackets, many students were observed exploring other features of their calculators. This exploration produced a student-generated method that was contributed by the student with teacher support, and followed up in conversation with his peers sitting near him. The use of a short cut key (ab/c) produced the same calculation result as the bracket method, and the transcript is recorded below;

[On whiteboard $\frac{-2 \times -3 + 3}{-3}$]

Hamish	You just type that first line in and then ... press that number a b c button on the calculator, and then
Alex?	Yeah, mine does too
Teacher	It works, does it? ... I don't mind if you do it that way, but just be aware of that fact that that big line ... is like a bracket ... and it means that you've got to do everything on that top line ... before you do dividing. That's a trap that a lot of people fall into, and I remember when I was first learning to use calculators I can remember falling into that trap as well.
Hamish?	OK
	(2/ L2/ 40/ E6)

This particular activity concluded with comments from the above students to their peers (or to themselves, such as "Yes, negative 3", "Yeah it's negative 3", "I was right", "I did it myself"). Students were using the calculator as an exploratory object and Hamish had found a shortcut key that was an alternative to the method demonstrated by the teacher. There was a great deal of interest in Hamish's method from other students, mostly boys, and indicates this age group's interest in technology and in finding shortcut methods.

To summarise, the calculator afforded student exploration and independent generation of number representations. In Lemonwood 2, this initiated further classroom interactions. Stephanie initiated an opportunity to quickly resolve a question she had about decimal numbers. This was an interaction between only one student and the teacher but was significant for the student. The other two episodes outlined here resulted in further student participation where Jeff and Hamish contributed their findings to their peers.

Roles

In this category, student roles were inferred from the status of their contributions and their ways of working together. Specific examples of Lemonwood 2 student roles are now identified from the analysis of the Classroom Interaction data. As outlined in section 4.3.3, the most common student roles during whole class interaction were as question answerer and choruser. Examples of question answerer were Kirsty and Decimal Places, Simone's Error, and Monique and Decimal Magnitude. In the repair interaction, Craig answered implicit questions by completing the teacher sentences about multiplying fractions. There were some examples of students as mathematical explainer. One example was Hamish's volunteered contribution of the Shortcut Calculator Key, prompted by his exploration of the calculator.

The teacher valued student contributions such as questions during whole class interactions. Melanie sustained her role of question asker through the interaction about adding and subtracting integers. Another example was Emily's role as question asker about calculator processes.

Teacher	Let's check it, check it out with our ... Emily?
Emily	Um, sometimes like when you do negative on your calculator
Teacher	Just a minute, Emily is asking a question. Ssshhh. Can you please just be quiet so, so that we can hear Emily.
Emily	Um OK Sometimes when I do a negative on the calculator and you wait for, do you always get the same answer? (2/ L2/ 35/ E4)

The Activity Ritual provided opportunities for more students to adopt a range of roles. Roles of solution describer, mathematical explainer and answer provider were observed during this time and were associated with collaboration and

assistance. The teacher tacitly supported student collaboration. Students would often consult their neighbours or the students sitting at the desks in front or behind. In the example of Amy, Erin, Louise and the Multiplication of Decimals, the girls asked each other for assistance while the boys volunteered assistance. In this particular interaction, the student questions were not necessarily resolved.

Students gained assistance from the teacher by a variety of means such as raising a hand, calling out a question, or sending a direct message such as, "I don't get it". During the Activity Ritual, whether the teacher was at the whiteboard or helping individual students, there appeared to be an acceptance of students calling out across the room either to their peers or to the teacher. The example of Stephanie calling out across the room about how to write a decimal number was an example of this, as was the teacher's called back response. In Lesson 5, during the marking of some practice examples, the teacher made an error as she was calling out the answers. One or two of the students interjected with the correct answer which was readily accepted by the teacher. In this classroom, assistance was a 'two way street' and students assisted the teacher when necessary.

4.3.6 Lemonwood 2 Summary

The Lemonwood 2 case study involved the analysis of data from seven observed lessons. The lesson structure was varied but interactional spaces were identified, particularly during the Student Activity. The Chorus Ritual was notable for the information cued from student to teacher during whole class interactions. Some students also volunteered to record their mathematics on the whiteboard. The student roles of question asker and answer provider indicated the high status of student contributions. Individual student activity was a part of every observed lesson and this provided sustained opportunities for peer contributions, and for semi-public teacher and student interactions. These localised peer negotiations were important in this classroom and students actively volunteered assistance.

Classroom Mathematical Tools were used by the teacher to demonstrate classroom mathematical practices. For example, the Rounding Dot was introduced by the teacher for the process of rounding. The teacher also

contributed analogous models and multiple representations. Instances of student adaptation, appropriation and rejection of these tools were observed within the sequence of lessons. Similarly, students adapted or appropriated resources generated by the scientific calculator, by using it as an exploratory object and as a recording device.

Whole class interactions in Lemonwood 2 were mostly teacher demonstrations of classroom mathematical practices where students contributed steps in the method, answers, and whiteboard resources. Student questions and errors initiated further joint interaction related to the question or error. The teacher often contributed additional representations or methods as further resources for student activity. In one instance, this was in the form of a proof. It was during the Activity Ritual that more students were engaged in negotiating mathematical meanings. Peers contributed resources such as questions, answers, hints and cues. The teacher also contributed resources, sometimes simultaneously to different groups of students.

4.4 Case Study Lemonwood 3

4.4.1 Background Information

The Classroom Setting

The Lemonwood 3 classroom was a large ground floor room in a two storeyed classroom block. The main door was at the front of the room with a fire exit at the rear. The teacher's desk was at the middle front of the room, and a whiteboard and an overhead projector screen were on the front wall. On both sides of the whiteboard there were shelves and cupboards for teaching resources, including some of the class textbooks. There was an overhead projector on a trolley but this was not used during the time of the data collection. The thirty-three student desks in the room were arranged in rows of two desks on each side of the room with rows of three in the middle. There was another larger desk at the back of the room.

The Social Group

Lemonwood 3 was a Year 10 class comprising 31 students, 15 girls and 16 boys, aged fourteen years at the time of the study. The class was designated by the school as a lower band class and many of the students had been in the same class in the previous year. The teacher, Mr. Larsen, was an experienced teacher who had assigned specific desks to students on the basis of alphabetical order of their family name. A new student to the school was present from Lesson 3 onwards and this caused a minor change to the teacher's seating plan. Two students were regularly absent from school and were present for three of the seven observed lessons with this class. Some students were often late to class, and for the Wednesday lessons; this was due to the arrival times of school buses from outlying areas.

The class was talkative, very sociable and considered by teachers in the school to be difficult to manage. Lesson time was regularly used to resolve issues relating to student behaviour, often following prior incidents from other classes. The teacher used the school system of warning cards to indicate the consequences of the students' actions. This involved about one quarter of the class and included both boys and girls. The class social relationships appeared to operate on different levels. There were four combinations of dominant students. Two of these combinations were pairs of boys; one pair sitting near the front, and the other pair at the back of the classroom. Another student gravitated around the pair of boys sitting at the front of the room. The other dominant combinations were pairs of girls. One pair was regularly absent from school and, when they were in class, they sat in the front row directly in front of the teacher's desk. The final pair of students did not sit together but at any opportunity they moved around the room to talk to each other. These two girls had at least two others who gravitated to them and this group regularly combined with the pair of boys sitting at the back of the room. The dominant students appeared to be the ones who were mostly participating in the school management system. During the lessons, these and other students moved around the room to talk to others, or shouted out across the room. The teacher mentioned two of the dominant students and noted their mathematical thinking.

> I think I notice when I have a student who is a thinker/ ( ) ... ..., yeah, Danny/ would be an example of a thinker, he also (likes) doing other things as well. Donna/ would be a thinker when she's there, but lacking a lot of confidence and yet she can nut it out if she focuses on the work, she's thinking. (Mr Larsen, Interview)

About half the class were lively, sociable and talkative for some lessons but relatively subdued during other lessons. There were some groups of quieter students who talked with those sitting near them. Although this class of students demonstrated volatile, intense, and vocal social relationships among themselves, the teacher seemed to maintain positive relationships with individuals and groups of students.

Focus Students

The Lemonwood 3 focus students were Danny, Finn, Nathan, Shane, Donna, Kara, Kate and Tina. Consequently, these students appear more often in the transcript data than other students in the class.

The Classroom Programme

This class had four mathematics lessons per week: three lessons were sixty minutes long and one was fifty minutes. All Year 10 classes at Lemonwood High School were working from the same school scheme for Mathematics and during the observation period, this class completed a unit on Number and began a Measurement unit. The students were expected to bring to class their 'squared' exercise book for mathematics and their scientific calculator. The students had been issued with a textbook, *Beta Mathematics* (Barton, 1998). Homework was regularly set for the students from their textbook. Some students did not bring their textbooks to each lesson so that they either had to share with others or borrow one from the teacher. The Monday mathematics lesson was directly after lunchtime and the Wednesday lesson was the first in the morning. As with the other case study classrooms, some of the lessons were interrupted by the school 'runner' collecting information about student absences.

4.4.2 The Mathematics Lessons Observed

In Lemonwood 3, seven mathematics lessons were observed. Two mathematics lessons per week were time-tabled at the same time as a lesson for each of Lemonwood 1 and 2 so only two lessons per week were available for observation of this class. The sequence of the observed lessons was also affected by tests or the absence of the class teacher (see section 3.5.1). The final week of the term was revision, using practice worksheets; on the advice of the teacher, I did not observe these lessons. The mathematical content of each lesson is recorded in Table 4.9, with the number of episodes identified from each lesson transcript.

Lesson	Week	Day	Main Lesson Focus	No. of Episodes
1	1	Tues	Estimation	3
2	1	Fri	Addition and Subtraction of Integers	7
3	2	Mon	Multiplication/Division of Integers	3
4	2	Wed	Measurement – units etc	6
5	3	Wed	Measurement – area	6
6	4	Mon	Measurement – circles	4
7	4	Wed	Measurement – circles	6

Table 4.9 Record of Lemonwood 3 Observed Lessons and Episodes

Lessons 1 to 3 were the only designated lessons on number topics. Lessons 4 to 7 about measurement involved activity related to number and have been included in the case study data.

4.4.3 Classroom Curriculum

Typical Lesson Structure

The Lemonwood 3 mathematics lessons were observed to follow a regular structure. Five one-step questions were written by the teacher on the whiteboard, often before the students arrived in the classroom, and it was expected that the students would begin work on these starter questions. The teacher moved around the room checking with students or completing behaviour management administration for individual students during this settling down phase of the lesson. In Lesson 7, Mr Larsen told the class the rationale for the five starter questions.

> Remember that the purposes of that first five questions is to get the brain/ thinking about the maths that's going to happen during the lesson. (3/ L7/ 15/ E2)

The five questions were then reviewed, and the teacher proceeded to a teaching episode involving exposition, demonstration of worked examples and questions directed to the students. The teaching episode was usually followed by exercises set from the textbook. During this student activity, the teacher moved around the classroom, talking with students, responding to requests for assistance, and re-directing any students not engaged in the designated tasks. In Lessons 5 and 6, the textbook activities were replaced by practical measuring tasks, including a measuring activity outside the classroom.

Links between Assessment and Teaching

The teacher was observed to be concerned about the students' levels of engagement with the mathematics and this was apparent in both implicit and explicit ways. The explicit messages were about what would be familiar to the students and the implicit messages suggested how new content related to their existing knowledge. This is illustrated by the following extract from Lesson 2 of a teacher review of addition and subtraction of integers.

[writing on whiteboard]

Teacher ... So listen carefully, I am only/ talking about the signs/ inside/. ... ... ( ) If the signs are the same/, like they're both adding, well you know that, you've been doing that for years. If you/ are/ ... subtracting/ a positive, ... well you've been doing that/ for years, ( ) you've been doing that for years as well too, and that/ is/ subtraction. (3/ L2/ 55/ E6)

In the teacher interview, the teacher mentioned that feedforward of information was important for the students' future learning and also for confidence in themselves.

> (Laugh) ... One of the things I am trying/ to do ... ( ) is to ... to keep this idea of feeding forward, of giving/ them the information about the objectives that are needed to do. ( ) so they actually know/ what they're supposed to be learning/, ( ) ... I think that's, that's sort of a key aspect. The students don't seem to have very much confidence in themselves/ ... as individuals. ... I'm hoping that will give them a wee bit more confidence. (Mr. Larsen, Interview)

Rituals

During the seven observed lessons, a number of rituals were identified but only two were significant for participation. Five lessons began with brief questions or tasks written on the whiteboard, but there was little collaboration between students. Students were observed to either work silently on their own or to continue to socialise with others without engaging in the mathematics activity. Consequently the starter was not identified as a significant participation ritual. In Lemonwood 3, the two rituals significant for student participation were the Activity and the Story Rituals.

Activity Ritual

Collaboration and participation were observed during the student activity time, the Activity Ritual. Individual student activity was based on tasks set by the teacher usually from the textbook. This activity occurred in a social context where neighbouring students assisted each other and the teacher circulated around the class. The Activity Ritual was significant because it was regular and supported the intensive social relationships of this class. It also opened up opportunities for the teacher to interact with individuals and small groups of students. Students' mathematical ideas and methods were included in these localised interactions. Some of these interactions appeared to be re-teach episodes, as a few students were observed to wait until the teacher had acknowledged or interacted with them before attempting any textbook tasks.

Story Ritual

The second participation ritual was the regular teacher contribution of stories, expanded explanations, or background detail about aspects related to the mathematical content of the lesson. Although the teacher was the storyteller, there were contributions from students in the form of questions or comments. These stories were important for socially binding the class together and, during these times, the classroom atmosphere was relaxed and humorous. For these reasons, the Story Ritual was identified as an opportunity for student participation in joint activity. Stories occurred in Lessons 2, 4 and 7.

Two examples of stories from Lesson 4 illustrate how aspects of mathematical content were presented historically, related to real life and, in terms of a mathematical cautionary tale, likely to appeal to this age group and particular class. The teacher used an example of his brother, a builder, to illustrate a real context for choosing appropriate units of measurement. The story began with a focus on imperial units.

Teacher	Who, who, who hears the term/ miles in, in conversations that people have, do you hear people still using it? (^)
General	Yep, yeah
Teacher	Who uses it?
?	Yeah, old people
Finn	The old people.
Lewis	Old people
Teacher	Old people and who else?
Danny	Americans.
	(Lesson 4 transcript).

The continuation of this discussion is included in a later section; Teacher Mathematical Values. Near the end of this lesson, the second story emerged, based on the ill-fated voyager expedition to Mars. An extract of the teacher's story is now presented because it illustrated the teacher's focus on school mathematics, namely, that mathematics is useful.

Teacher	... they do/ have/ the units the scientists/ are using. However, there was a little accident last year ... and it was, it happened/ because/ of a mistake with these, these units. And you may have heard about this/ this/ this spacecraft was sent to take pictures of, of Mars and it was going to go around the back of Mars. Now when you're flying and a spacecraft is going around, around a planet, and it's going fast/ enough/ ... as far as the, enough away it's not going to get drawn into it. [teacher draws diagram on whiteboard of orbit around planet] And so they/ worked out, that while it stayed, um I think the thing was fifty metres, a hundred and fifty metres or maybe five thousand metres. As long as it stayed five thousand metres away, ...
Danny	Five k
Teacher	Five k's. So five thousand metres, as long as it stayed five thousand metres away/, it would take them (close) to the back of Mars that no one can see, and then shoot off and do something else. But/ another/, another scientist who put it in, saw the five thousand/ and programmed it at five thousand feet (^). Now

	there's approximately three feet to one metre, so instead of going here[pointing to the orbit diagram on whiteboard], it was a third of the height. So it crashed. So this one person made this silly/ mistake. I mean he must have felt pretty silly/
Kara?	Was he fired?
Teacher	Oh, I don't know, I haven't heard what's happened to him, but I don't think he'll get the chance to do it again anyway.
Tina?	Did anyone die?
Teacher	No, it was an unmanned spacecraft, but/ it was about one point eight billion dollars worth of project … went down. Why(^)? because someone got their units wrong (^). So please/, don't get your units wrong when we do ours.
Danny?	Or we will crash!!
	(3/ L4/ 60/ E6)

Students appeared attentive during this quite long story from the teacher. Some students interjected; one about a mathematical idea, the others related to the drama of the story.

Introduction to Roles

Student roles related to participation in mathematical activity are briefly outlined in this section. In Lemonwood 3, some students were observed to adopt roles of solution describer, mathematical explainer, question asker and answerer in whole class interactions. These student roles were often in response to teacher questions or requests for contributions, but also included other volunteered responses such as humorous or social comments. Students were expected to listen to their peers and there was a great deal of effort put in by the teacher for relative quiet during peer contributions. This was important for such a socially volatile class. Confident students volunteered contributions and were not distracted or intimidated by the more dominant students. On the other hand, some students never contributed to the whole class discussions, contributing only to localised interactions during the Activity Ritual.

In Lemonwood 3, the students were sitting in alphabetical order specified by the teacher, and many students were not sitting next to their friends. During the Activity Ritual, some students requested assistance from their immediate neighbours, but others moved near their friends to seek assistance. This was another opportunity for further socialising and the dominant students were more likely to be moving around the room. The teacher also moved around the room

usually stopping near groups of students rather than focusing on individuals. Teacher assistance was gained by either a raised hand, or by calling (sometimes shouting) out the teacher's name. Sometimes student's mathematical questions or queries generated localised conversations between the teacher and groups of students sitting nearby. Specific examples of the student participation roles are described in more detail at the end of the Classroom Interaction section, 4.4.5.

Teacher Mathematical Values

Teacher Mathematical Values were inferred from explicit and implicit messages from the teacher about the nature of mathematics, and what it means to be a learner of mathematics. A utilitarian view of mathematics was evident in the Story Ritual and other examples contributed by the teacher. Mathematics was useful in the workplace and in other real life situations. The continuation of a previous Story Ritual about measurement units is presented below.

> What we need to do is look at ... what/ you use, where. And there lies a little bit of a problem. Some of the things we do at, at school/, ... may not happen somewhere else. Now, for example, I've got an older brother who is a builder, and I'll, we spend a lot of time/ with rulers with centimetres on. Now when he's building a house/, he works in two units – metres/ and millimetres. Never/ uses centimetres. And he's talking about something, he'll say, "oh, it's a hundred mil" or ... "it's a thousand mil". He'll never use centimetres, ... but we/, centimetres, ah, millimetres are probably too small for us to use all the time, ... but it's quite important for him to use it (^) ... because he doesn't want to build a building/ with gaps in the wall. If he does that/, he's not going to get another job. ... So/, he's got to do it reasonably accurately, so millimetres are quite important to work to. Centimetres/ wouldn't be a good use, a thing for him to use. So when you're looking at things, how/ they're used, ... the units/ are important. (^). (3/ L4/ 15/ E2)

The teacher stressed the importance of measurement units for accuracy in real situations such as building, and also acknowledged that school mathematics was not necessarily the same as the mathematics used in practical situations.

In the interview, the teacher outlined some of his beliefs about how the Lemonwood 3 students learn mathematics.

> One of the things that I think happens is, that, I don't think many students think/ when they do maths. They/ like the idea of getting some small piece of maths explained to them and they'll go ahead and they'll do/ it. And they'll check the answers or something, and go to the next, next bit and so they really don't, I think I say they don't, they don't turn their brains on,

> and it's all sort of working on a reflex action. And they, they do/ it, they walk out and there's, it's almost a passive sort of thing. They don't get any enjoyment out of it because they haven't actually understood what they were doing.
> (Mr. Larsen, Interview)

The teacher emphasised that it was important for the students to generate their methods of solution or explanations. These methods did not have to be the same as his and this was made explicit to the students (Lessons 1, 4 and 5). When students did contribute different solution methods, the teacher response was immediate and enthusiastic.

> So/ this is where/ maths is going to get a little bit more complicated and I think this is where it gets exciting. ... Well it's exciting for me. (Laughter, and more laughter)
> Not the right word to use, you reckon (^). I'll tell you what/ ... ... (more laughter) For the maths teacher, what makes it exciting/ is/ when you/ come up with your own ways of doing/ things. And my/ job/ is to make sure that your way works all the time. Because if you come up with your own way of doing something, then you'll remember it, but if you are learning and struggling to follow my/ method ... then, you are a lot more likely to forget it. So that's where it gets exciting for me when I see, Now I expect that answer because I think like this. So that's why, when you said that, I just thought that's fantastic. You've come up with a different/ way from what I was thinking. And that's, that's neat (^). And, and do you know why? Because that means I know/ you're thinking. Alright, and that's why that's fantastic. Okay.
> (3/ L7/ 15/ E5)

This extract includes explicit messages about mathematics as a subject, about the students' relationship with mathematics, and also about the roles of teacher and student. Not only did the teacher talk about his excitement at student-generated methods, but he mentioned that student alternative methods indicated to him that the students were thinking. This was an implicit example of the high status he gave to students' own thinking and mathematical methods, and corroborated his statements in the teacher interview. The teacher also demonstrated these values by his assistance interactions with the students during the Activity Ritual. He asked questions such as "what would you do first?", "how would you work it out?", and accepted variations of student methods, followed by feedback and suggestions. In these ways, the Lemonwood 3 teacher supported students to actively generate their own methods.

4.4.4 Tools

Classroom Mathematical Tools

In Lemonwood 3, seven Classroom Mathematical Tools were identified. These are set out in Table 4.10 below.

Classroom Mathematical Tool	Mathematical Use of Tool
Ruler	Physical model of number line Thinking device for addition and subtraction
Number Line	Model representing relative magnitude of integers Thinking device for addition and subtraction
2x2 Array	Grid to show relationship between signs of factors and products, pedagogical artefact
Language Example	Analogous example for addition and subtraction of negative integers
Overdraft	Real life example of a deficit
Thermometer	Analogous model of a number line
Think Boxes	Visual representation of an intermediate strategy for considering component areas of a composite shape

Table 4.10 Classroom Mathematical Tools for Lemonwood 3

Six of these tools related to activity with integers. In Lesson 2, the teacher demonstrated the Ruler and the Thermometer. Although these were variations of the Number Line, the tools were considered as distinct and have been included in this category of Classroom Mathematical Tools.

Number Line

The horizontal Number Line drawn on the whiteboard was used in whole class episodes. The Number Line was both a representation of the relative magnitude of integers and a thinking device, for modelling the process of addition and subtraction. The Number Line provided a starting point for any operation, supported any counting, and culminated at the end point, the answer in these one-step addition and subtraction tasks. Students were also observed to use The Number Line as a thinking device during textbook tasks.

The Ruler

The students' 30 cm perspex ruler was used as a model of a Number Line to be used in the addition and subtraction of integers. As a physical model, the Ruler represented only the positive half of the number line. When counting to the left on a number line, past zero towards the negative numbers, the teacher suggested that the students rotate (or flip around) the Ruler so that it would be facing them with the numbers on the reverse side. From the student's viewpoint, the digits were now reversed and could be interpreted as representing negative numbers. Students then counted back down the (reversed) number line. The following extract is Mr Larsen's explanation for the example, 3 + -4.

Teacher	Now so how do you show (^) that the answer is a negative (x)? Take your ruler and point to three, point to three on the ruler. ... OK Now/, what does/ take four off something mean? ...
(General	Minus, takeaway)
Teacher	It means that it goes, go backwards. ( ) So from three, go back four: one, two, three. Oh I can't go any further, turn the ruler around and go one more. ... OK, So look at what I did again, three then go back four. one, two, three ... I'm going in that direction on my ruler, if I turn my ruler around/ I can keep going. Now I'm on one/, but it's inside out, I am looking at the back, so it's negative one. (3/ L2/ 15/ E1)

The teacher then demonstrated the use of the Ruler for an example of +7 – 15. Later in this lesson, when the students were working on questions from the textbook, Mr Larsen was observed demonstrating the Ruler to two students who had requested assistance.

The Ruler is often used as a physical number line in primary school and was familiar to many students. As a portable physical model for a Number Line, there were possibly some kinaesthetic benefits for students. By flipping around the Ruler, there was a physical signal that numbers were different, implying that negative numbers were now in focus. By being a perspex ruler, the digits were visible and, although reversed, students could infer that any inverted numeral represented a negative number. This tool, therefore, re-structured the process of subtracting integers and was used as a thinking device for individual students. Because the Ruler was calibrated in millimetres, implicitly there were positive

and negative decimal numbers, not just integers embedded within this model. Some possible constraints of this tool were that students could not use opaque rulers, students needed to remember to flip the ruler, and to interpret the reversed number labels as negative.

Analogous Models

Associated with the use of tools for integers, the teacher used two real life situations in whole class discussion. The textbook had a diagram of a thermometer drawn vertically as an example of comparing integers (Barton, 1998, p. 13). In Lesson 2, when the teacher asked the class to decide which sign, < or >, to place between the numbers –3 and –4, a response from one student indicated the use of a Number Line model to think about the relative magnitude of two negative numbers. The teacher affirmed the use of a Number Line and then mentioned two real life situations.

Teacher How come it's that/ way here/ and that way there?
[Teacher pointing to 4 > 3 and –4 < -3 on w/b]
Finn? Negative 3 is nearer zero, eh.
Teacher Because you're working out negatives. Good. Negative 3 is nearer zero, good. Think about your number line. This number is negative 4, is further down the number line so it is/ smaller/ in that negative sense. ... ... Now that gets a wee bit hard/ um to look at and ... (one) again take, if I think of temperature, I think of my bank balance, if I've got, um, if I'm in overdraft of 3 dollars ... that's not as bad as um an overdraft of 4 dollars so that one there's ah, more debt.
(3/ L2/ 20/ E2)

The real life situations of temperature (implicitly a Thermometer), and of Overdrafts, were used as analogies for the comparison of negative numbers. Another analogy for integers was the use of a Language Example. The Lemonwood 3 teacher explained;

Teacher Now I am going to tell you a couple of little things that might help you remember which is which, but subtracting/ a negative/ is the same as adding. Now what it is, is a negative of a negative, ... and if you think in English, and a classic example of this. Although you don't mean this, ... if someone says "What are you doing?" a lot of people answer "I ain't doing nothing". ... Now, literally/, they've just said they are doing something. "You ain't

> doing nothing" is two negatives. Two negatives in a sentence make it a positive. So when you say "Ain't doing nothing" you probably say, "Well, yes, I just threw that rock through a window". Ah, "Well, you ain't doing nothing" means you did do something. But I mean ... OK, so you're actually saying you did something when you (meant to) say you did not. (Lesson 2 transcript)

Mr Larsen then connected this to integers and suggested to the students that they needed to be careful with the signs of integers, and that it was easier to use one sign to represent two adjacent signs. This language analogy was used as another tool for the process of subtracting a negative. By demonstrating and discussing multiple tools for calculation processes with integers, there was greater opportunity for students to appropriate one of these tools for their own activity. This illustrated the Lemonwood 3 teacher's view of the importance for students of finding a meaningful method for themselves.

Think Boxes

The final type of Classroom Mathematical Tool was a meta-cognitive tool, the Think Boxes. In Lesson 5, the teacher demonstrated Think Boxes during examples to find the area of composite shapes. On the whiteboard, the teacher drew 'clouds' around two ways of thinking about how an L-shaped area might be composed. One cloud represented a putting-together strategy of adding two smaller rectangles. The other cloud expressed a taking-a-bite-from strategy where a larger rectangle had a smaller rectangle removed. The whiteboard drawings of the two Think Boxes were each enclosed by a cloud to represent a different way of thinking.

Teacher And so this is/ really/ if you think about it, now I will put these think boxes in. It's eight by ten minus three by four (^). And please/ I have already sort of ... suggested/ and maybe (not as) strongly as I should've/, that when you write the word area/, I don't really mind about the think boxes ... but/ and this will ( ) gets hard with the cuts in it, but if you could show/ your working by this first statement. You go eight times ten, minus three times four, or/ you could have written eighty minus twelve. I know straight away then/ that you have recognised that there's a thing/ taken out of there.
(3/ L5/40/ E5)

These acted as reminders for students to use an analytic strategy to consider shape, before beginning any calculation of area. The teacher also mentioned Think Boxes to the students at this stage, stressing that the students could choose whether or not to use these tools. He added that Think Boxes would be useful when finding the areas of more difficult composite shapes and suggested that students use them in the following tasks. In the teacher interview, when asked about the use of the Think Boxes, the teacher indicated that he had used them before with this class.

A feature of this case study was the way the teacher consistently articulated, validated and supported the use of a range of tools for the calculation of integers. Tools were valued as useful thinking devices for appropriation by students in different contexts. Although not specifically used for number activity, the inclusion of Think Boxes illustrated the use of tools in whole class and individual activity. The use of the Think Boxes supported a multi-tool approach for a higher order task, such as finding the area of a composite shape. This meta-cognitive tool was used in conjunction with other tools, such as drawings and labels, followed by multiplication for finding areas of rectangles, and then combining areas with operations such as addition or subtraction. By using tools in this nested sequence, activity was re-structured and mental effort was offloaded at each stage of the process.

Technological Tools

The technological tool observed in Lemonwood 3 was the scientific calculator. One student had a sophisticated calculator/mini-computer, and was the subject of attention from three boys sitting near him. They alternately leaned over to see what he was doing or borrowed the calculator to have a try out for themselves (from field notes). The main role of the calculator was as a calculating machine. The use of the calculator was subsumed into individual student activity and there were few instances of whole class calculator use in this case study. Lesson 6 had the most observed use of the calculator, involving calculations based on measurements taken by groups of students. The group measurements had been recorded on the whiteboard for the whole class and the calculator was used to find the circumference divided by the diameter for each of the measured circular

objects. This led to the introduction of a ratio that emerged as pi, that had a special key on the scientific calculator. In the final lesson, the students used the calculator-generated value of pi in their calculations of areas of circles. There were five episodes of calculator use in Lemonwood 3 and these are recorded below in Table 4.11.

Lesson	Calculator Event	Initiated by	Category
1	Calculating previously estimated examples	Teacher	C2 (calculating machine)
6	Dividing circumference measurements by diameter measurements	Teacher	C2
6	Finding average of ratios (circumference/diameter)	Teacher	C2
6	Investigating pi on calculator	Teacher	C4 (as an exploratory object)
7	Using the pi key to calculate area of circles	Teacher Student	C2

Table 4.11 Categories of Calculator Use for Lemonwood 3

In Lemonwood 3, the technological tool afforded more efficient and precise calculations with decimal numbers and with the student-generated measurements of circumference and diameter. By offloading mental effort, the calculator provided more rapid generation of empirical values for pi. The students could pay greater attentions to the results, and focus on the mathematical meaning of a constant, and then link this constant value to the special function key for pi. Only a proportion of the class ever had their scientific calculators with them and this influenced both social and mathematical activity in Lessons 6 and 7. Some students missed out on opportunities to autonomously generate calculations and to experiment with a new function key, pi. Consequently, calculators were an item of exchange among the students, and it was also common for students to ask for and receive their neighbour's calculator results. In Lesson 7, some students were observed to say to the teacher that they could not do the area of circle problems because they did not have their calculators. Access to calculators became a means for some students to opt out of participation in individual activity.

Other Mathematical Tools

This category records the cognitive tools not already identified as Classroom Mathematical Tools. Other Mathematical Tools were identified from the Lemonwood 3 transcripts and are presented in the table below.

Tool	Examples of this Tool
Algorithm	Algorithm for area of a circle
Operations/Relations	Addition and subtraction of integers Multiplication of integers Combinations of operations (BEDMAS) Squares (area) Addition, subtraction and multiplication of fractions (area)
Multiple Representations	Numbers recomposed, decomposed Representations of fractions Decimals and measurements
Other (eg notation)	Comparison of relative magnitude (< and >) Pi as a constant Diagrams of fractions

Table 4.12 Other Mathematical Tools Identified in Lemonwood 3

4.4.5 Classroom Interaction

This dimension encompassed the analysis for the domain of interaction, for both teacher initiated interactions and student initiated interactions.

Teacher Guided Participation

Examples of Teacher Guided Participation were identified during the seven lessons observed in Lemonwood 3. This involved the following scaffolding actions that supported student learning: referring to student prior knowledge, stories of real life situations, clarifying the language of mathematics and of instruction, demonstrating tools for assisting with student solutions, and including strategic methods for solving problems. For example, in Lesson 2 during a review of the example 8 + -4, the teacher referred to the students' prior knowledge, modelled the steps in the process, and demonstrated a shortcut method. He also mentioned using tools such as the Ruler, temperature or money, and suggested that the students find something that they could connect with. Later in the same

lesson, the teacher rephrased the method for subtracting a negative integer and used an analogy based on the Language Example identified earlier as an analogous model (section 4.4.4). The teacher had presented multiple ways of thinking about the same mathematical content and had guided the participation of students in integer activity. In addition, the Lemonwood 3 teacher was an interlocutor in some student contributions. He recorded student solutions for the whole class and asked clarification questions, enabling greater access to student mathematical resources. Similarly in the measurement lessons, the teacher included scaffolding actions for the students (Lessons 4 to 7). He demonstrated how to restructure a task into smaller steps and how to identify the first steps of a problem; he also included a metacognitive tool, Think Boxes, to assist the students with particular area tasks. In these ways the teacher guided student participation by sequencing the use of different mathematical tools.

Mr. Larsen was concerned about the students' apparent 'unlearning' – a term he used in class with the students in Lesson 5.

> And what's scary to me is the amount of like unlearning that's happened over the last two years. I don't think it is unusual/ I think it's quite natural … um, and, and what I'm trying to ask you to do with these, right, is, these are good questions to get you ahead. I have said it before think/ about/ what/ you're doing/. Now a number of you are still, when you're doing this ... you get side-tracked onto something else and then as soon as you come to it, you … you 'Oh, how do I do this?' and you stop thinking about it. So/ … um I want, I'll go through a couple of others, cos I think there's lots of good ways of doing these questions.
> (Lesson 5 transcript)

His statement referred to his observations that students' knowledge seemed to slip during the first two years of secondary school, and that one of his roles as a teacher was to help the students with strategies, both generic and specific to the mathematics content, to assist them in their learning. These ideas are related to the Teacher Mathematical Value of encouraging student to find their own methods that make sense to them (section 4.4.2). In Lemonwood 3, the range of observed teacher scaffolding actions were based around re-structuring activity, and making connections between mathematics and real life situations.

Whole Class Negotiation

In Lemonwood 3, most of the classroom mathematical practices related to number were observed to be revision of previously learned material (Lessons 1 to 3 transcripts). Some new practices were introduced and developed, such as the Multiplication of Integers Rule. In Lesson 3, when multiplying a series of negative numbers, the teacher introduced the method in the following way. "There has to be a negative to even it up ... on the answer. So if there's an odd number of negative signs in the question (^), then the answer must be negative/" (3/ L3/ 10/ E1).

Shane's Reasoning and Finn's Explanation

After writing the example 1 x –1 x –1 x 1 x –2 x 3 –1 x 1 = 6, on the whiteboard, the teacher asked students to decide whether the six should be positive or negative. By providing the answer without the sign, he had narrowed the joint field of attention to a focus on the signs of the factors. These signs were then message carriers providing information to students about the sign of the final product. The transcript and analysis are presented below.

Speaker	Transcript	Transcript Analysis
Teacher	Should the answer be positive or negative? ... at the end of this (^) ... , just looking at it.	Teacher question with a narrow focus (deciding on the sign of the product). Chosen factors that students could mentally multiply as well as provided the answer.
Shane	Positive.	Shane volunteered the sign of the final product
Teacher	Why is it positive? You are quite right, but why is it positive?	Teacher asked for reasoning, evaluated the student response and affirmed, repeated question.
Shane	Cos the negative, there's four	Shane continued and identified the number of negative factors, and by implication, an even number of negatives, student register.
(?)	Some of the negatives ...	Other students volunteered contributions, Possible peer assistance
(?)	There's minus four	Assume student focus on negative factors, four negative factors
Teacher	OK, that's, that's not quite what I was saying.	The teacher signalled a partly correct idea.

Finn	Cancel each other out	Finn then volunteered an idea, that a pair of negative factors cancels each other out
Teacher	Cancel it out, OK, Thanks, ( ). On off, on off - that's not a bad way of thinking of it. And what I said before/ was one, two, three - there is four subtractions.	Teacher repeated and appropriated Finn's idea, acknowledged it, reformulated using metaphor of a switch and evaluated this as a way of thinking about the rule. The teacher reviewed the example again, counting the number of negative signs, circling the negative signs as he counted. Multiple terms of negative, minus, subtractions.

(3/ L3/ 10/ E1)

Figure 4.21 Shane's Reasoning and Finn's Explanation

Shane volunteered an answer, the teacher asked Shane for a justification, and Shane's explanation identified an even number of negative factors. Other students assisted by adding their contributions, all related to the negative factors. Finn then volunteered an analogy based on a negative factor cancelling out a positive factor. The teacher appropriated Finn's contribution, and then reformulated it to another analogy, an on/off switch. This analogy corresponded to the pattern for numbers of positive and negative factors. An odd number of negative factors was inferred as 'on' (negative) and an even number of negative factors as 'off' (positive). This brief transcript illustrated some important features of whole class interactions in Lemonwood 3. Student contributions were often brief phrases and usually volunteered. Even Shane's explanation was brief, consisting of two incomplete phrases. Consequently, the teacher inferred meaning from the student contributions, and at the same time appropriated student ideas and register.

Tina's, Danny's and Nathan's Methods

In Lesson 7, the class completed whiteboard tasks using an existing classroom mathematical practice, the Area of the Whole Circle Method. This practice involved using a formula for finding the area of a circle, and using the calculator to generate the value for pi and to carry out the calculations. The teacher then drew a diagram on the whiteboard of three-quarters of a circle with radius of 3. The following transcript and analysis illustrates how the teacher and class jointly

developed the next classroom mathematical practice, the Area of Part Circles Method.

Speaker	Transcript	Transcript Analysis
Teacher	… How would we work out that? … … …	Asked class how to work out the area, requesting a strategy from student
Tina	Do the whole thing and minus the …	Tina volunteer a method of area of whole circle and removed (one quarter)
Teacher	Okay. Now let's develop that. Do the whole thing and take away a quarter, (^) okay. So I'm going to say/ before I do this, area/ is equal to a whole circle minus one quarter (^) .. of a circle. I'll just write the idea down … that's one way of doing it. It's great, so you just work that out. It's pi three squared minus one quarter (^) of pi three squared. … … That's OK. And/ I'm going to write down the steps twenty eight point two seven (28.27) minus … … …	Teacher interrupted Tina to slow down for one step at a time. Repeated Tina's first phrase, appropriated and completed Tina's method, used another term for minus, validated. More slowing down, Repeated full strategy using mathematical register of the student Recorded Tina's method on whiteboard, implicitly high status for student method, Affirmed and validated, Cued in the next step of mathematical calculation. Stated the formula for the whole circle (mathematical register) and included student method [teacher recorded the calculator steps on whiteboard, using known area of whole circle]
	(Long pause)	[Teacher writing on whiteboard]
	OK, Now/ did anyone see/ … another/ way of doing that one? …	Teacher asked for more student methods
Danny	Half plus a quarter.	Danny volunteered a building up strategy
Teacher	What's half plus a quarter? Danny?	Teacher accepted student contribution and asked Danny to consider the combination of these two fractions

Danny	Three quarters (^)	Contributed the combined fraction
Teacher	So you can do this one or/ ... you do one half/ of a circle plus one quarter/ of a circle or/ ... take the area is equal to (^) three quarters/ of a circle. You get the whole circle ...	Teacher validated alternative methods, paraphrased Danny's strategy, and reformulated to three quarters, began to explain next part of process
Nathan	Get the whole circle and divide it by four and times it by three.	Nathan volunteered a method to get three quarters from a whole circle
Teacher	Which is three-quarters. That's right.	Teacher (implicitly) appropriated this strategy and stated the final fraction. Validated strategy.

(3/ L7/ 20/ E4)

Figure 4.22 Tina's, Danny's and Nathan's Methods

This whole class interaction involved three solution methods volunteered by different students. Each method was based on a fraction model of the circle. The drawing of the part circle may have cued students to familiar diagrams of fractions (the ubiquitous fraction pies). It was also possible that the previous use of the Think Boxes in Lesson 5 had influenced some of the student contributions. Although used for areas of composite shapes based around rectangles, the Think Boxes had been stable visual records of a take-a-bite-from method (Tina's Method), and a put-together method (Danny's Method). The negotiation of the Area of Part Circles method involved teacher appropriation of student contributions. The teacher repeated, expanded and recorded the student ideas and register, indicating contributions of high status in this classroom. Immediately following Nathan's contribution, the teacher told the class about mathematics being exciting, as described in section 4.4.2.

As illustrated above, classroom mathematical practices in Lemonwood 3 were jointly developed by students and teacher. Although the teacher played a major role in structuring examples and shaping the interaction process, this may have been a consequence of the challenging social relationships within the class. Although many student contributions were brief and sometimes disjointed,

participants were able to infer meaning from these contributions. In whole class negotiation, students contributed solution methods, reasoning and other ideas, and were actively supported and encouraged by the teacher. He, in turn, validated and appropriated some of these contributions as resources in further interaction.

Student Initiated Negotiation

This category encompassed interactions initiated by an unanticipated student contribution. In Lemonwood 3, these student contributions were usually questions in whole class interactions, or statements, questions or other signals during the Activity Ritual. Student contributions could initiate substantial negotiations as illustrated by the following example. In Lesson 3, the teacher reviewed the Multiplication of Integers Rule. This occurred about fifteen minutes after Finn's Explanation of cancelling out the signs of the factors. During the second review example, (-2 x 6 x –3 =), Donna asked a question about the sign of the product.

Donna's Conjecture

The transcript and analysis are presented below.

Speaker	Transcript	Transcript Analysis
Donna	Oh, So is it always positive if there's more than one negative?.	Donna volunteered a question about a possible pattern for the sign of the product, her implicit conjecture has expanded previous experience of multiplying only two negative numbers, i.e. one negative factor has a negative product, two negative factors have a positive product.
Teacher	No/, but that's a good start/. If I put that/ in there, … it's going to be/ … negative, because, there are three/ negatives.	The teacher said no but also gave a supportive statement, [added a negative sign to the six on the whiteboard] to show the multiplication of three negative numbers. Stated the new situation, product was negative because three negative factors.
Donna	Oh, yes	Donna signalled to teacher that she was following the teacher's example so far
Teacher	If there's an odd/ number then that's the rule, for two …	Teacher generalised for an odd number of negative factors, implicitly the product was negative. Began to explain for two (factors)

Donna	Oh, so if, like a number, there's like … then no negative, then that's obviously positive. Like … like there's … two/ negatives, yeah, there … so it must be positive (^).	Donna interrupted and explained her thinking, based on the first example provided above, still focused on two negative numbers. (possibly still had her conjecture for more than one negative factor)
Teacher	OK, Well, let's see if your rule works.	The teacher signalled an opportunity to test out Donna's method, opened up a space for further discussion.

(3/ L3/ 35/ E3)

Figure 4.23 Donna's Conjecture

At this point, other students started to add in their ideas both in whole class and in localised conversations. Donna, who had initiated this discussion, tracked the teacher's new example and, in a process of scaffolded conversation with the teacher, articulated her version of the rule for multiplying a series of negative numbers. The transcript and analysis continues below.

Speaker	Transcript	Transcript Analysis
Danny	What's the, if it's, four … called?	Four factors, or four negatives, Danny talking to himself (loudly).
General	Localised talk among peers	Time for more students to consider and try out their methods
Teacher	Four times … Is that going to be positive or negative?	Teacher wrote up a new example on the whiteboard, with four factors, (may have heard Danny's question or followed on from three factors), narrowed the focus to the sign of the product, asked question.
Donna	Yeah, you go	Donna volunteered
Finn	Negative	Finn called out the sign for the answer, interrupted Donna
Danny	Is it (^) …	Danny began to ask a public question, or to challenge Finn's answer.
Kara and Ngahia	Is that negative? Yeah.	Kara and Ngahia talking to each other, localised conversation

Teacher	Now, because that one has got one, two, three/ negatives, ... there has to be, and the rule I said before, there has to be/ an even/ number of negatives in the whole problem.	Teacher demonstrated the method with previous example, pointed to each of the three negative factors while counting them. He restated the rule in terms of even numbers of negative factors.
Donna	Ooohhh	Surprise - possibly at the focus on even numbers of negative factors, leading to a positive product.
Teacher	The other way of thinking of it, it's odd, ... so it's negative	Teacher rephrased in terms of odd, implicitly number of negatives, as in the eg on the whiteboard (three factors).
Donna	So it's (^)	Donna began to repeat her method.
Teacher	What's there/ has to be negative	Teacher restated for the previous example to; -2 x –6 x -3
Donna	So if there are two (^), so ... there's, it's got to be an even number of ... negatives to be positive	Donna rephrased her method in terms of an even number (of negative factors) to produce a positive product.
Teacher	Yes	Teacher affirmed Donna's statement.
Donna	So, yeah... so if an odd number of negatives	Donna continued with a corollary of the rule, this time for odd number of negative factors.
Teacher	OK	Teacher affirmed Donna's incomplete statement, assumed that she had appropriated the practice.

(3/ L3/ 35/ E3 continued)

Figure 4.24 Resolution of Donna's Conjecture

Donna initiated this interaction with a question. Her implicit conjecture was that one negative factor generated a negative product while two or more negative factors generated a positive product. Her conjecture was possibly based on the signs of the factors as message carriers, rather than the number of negative factors. The teacher encouraged Donna to keep thinking and provided a counter example for her conjecture. Both Donna and the teacher sustained the interactional space. Donna twice contributed her conjecture and articulated her changing ideas. The teacher supported Donna's contributions by stating, "Well,

let's see if your rule works". This initiated further negotiation and created more time for other students to be involved. While a number of peers participated in both localised and whole class discussion, Donna and the teacher maintained an extended dialogue about her conjecture.

Error Initiated Negotiation

Error Initiated Negotiations were prompted by a student error in a method, answer or explanation. In Lemonwood 3, much of the error initiated negotiation occurred during individual student activity. Negotiations involved the teacher and student, and nearby students often listened in, turned around, and contributed to the discussion. The localised negotiations were similar to the following whole class example. In Lesson 7, when using a formula to find the area of a circle, some students were doubling the radius rather than squaring it. During a review of some practice examples, the following dialogue occurred.

Teacher	Can someone tell/ me the process of doing this one. The one example … just tell me what I should write down. Tina, hand up first.
Tina	Um … the half, half, the um diameter, which is like five (^) … then … um
Teacher	Okay and/ so … and that's twenty-five times is seventy-five ... it's a bit more than seventy-five.
Danny	Just doing pi.
Teacher	No/, because five squared is twenty-five.
Shane	It's not two times (^).
Teacher	You have to be very careful with that.
Shane	Yeah, It's too much (^)
	(3/ L7/ 20/ E4)

Previous tasks had provided the diameter length rather than the radius. This may have caused confusion between doubling and squaring of the radius. When the radius is five, then the diameter is ten and deceptively easy to multiply. It was another student who recognised Danny's error and volunteered two contributions.

Repair Interactions

Repair interactions were also more likely to occur during localised student activity, and most of these were teacher repairs of a task instruction. Teacher and students then jointly reformulated the task and reviewed mathematical tools

useful for the task. An example of this kind of interaction occurred in Lesson 2, during a textbook task ,"Write down the opposite of each of these integers" (Barton, 1998, p. 13). A number of students were unsure about what the question required. The teacher repeatedly used the Ruler and a Number Line as models to explain the meaning of the term "opposite" in this context. Another repair situation occurred during textbook tasks to find areas of shapes. The teacher turned a student's textbook ninety degrees to show a group of students that this shape was familiar. These types of running repair situations were not observed when the teacher generated tasks on the whiteboard.

Technology Mediated Negotiation

In Lemonwood 3, there were no observed episodes of negotiation mediated by technological tools.

Roles

In this category, student roles were inferred from the status of their contributions and their ways of working together. Specific examples of student roles are now identified from the analysis of the Classroom Interaction data. The first role identified was as solution describer. Examples of this role were Tina's, Danny's and Nathan's Methods for finding the areas of three-quarters of a circle. These students contributed multiple ways of representing three-quarters and contributed further mathematical resources to a whole class interaction. The three solution methods were explicitly accorded high status by the teacher. Other student contributions were implicitly accorded high status when student register and other resources were included in further interaction by both teacher and peers. The role of mathematical explainer was observed usually in response to a teacher request rather than volunteered. An example was Shane's Reasoning during the negotiation of the Multiplication Rule for Integers. Associated with this was a volunteered role of provider of analogies and other mathematical resources, such as Finn's Explanation. In whole class interactions, some students adopted the role of question asker. An example was Donna's Conjecture when Donna, one of the dominant students, asked a question of the teacher. This initiated an interaction based on Donna continuing her role of question asker and also adopting another role of inquirer. During this process, Donna contributed mathematical reasoning

as further resources. This also illustrated that some students adopted multiple roles in this classroom, and that roles could change during the process of joint negotiation of mathematical meaning.

In Lemonwood 3, the roles of solution describer, mathematical explainer, question asker and answerer were observed more frequently in the Activity Ritual than in whole class interactions. Consequently the regular Activity Ritual provided the major opportunity for students to adopt the above roles, often subsumed within roles of collaborator and assistor. Collaborator and assistor roles were often indistinguishable during student activity. There were examples of explicit collaboration during measurement activities and in most lessons, implicit collaboration included sharing steps of solution methods. There was also a great deal of borrowing and sharing of physical resources: rulers, paper, pens, and calculators were regularly exchanged between students. Mathematical resources were also exchanged such as calculator results and answers to tasks. Sometimes these were in response to peer requests for assistance or implicit signals such as frustration. At other times these were inferred to be an expectation of joint activity among peers. This illustrated that, in Lemonwood 3, collaboration and assistance roles were influenced by the complex social relationships among the students.

4.4.6 Lemonwood 3 Summary

The Lemonwood 3 case study analysed data from seven observed lessons. In this classroom situation, the complex social relationships between students influenced many aspects of the lessons. For example, the observed rituals and roles demonstrated negotiated norms capable of supporting the intense social demands of the students and joint mathematical activity. The interactional spaces during the Activity Ritual were particularly important for localised negotiations between teacher and students. The teacher assumed a major role in monitoring the social norms and explicitly maintained high expectations for the students. He also contributed a range of mathematical resources such as stories, real life contexts, analogies, mathematical and metacognitive tools. Whole class negotiations involved the use of a range of tools, and explicitly supported choice for students

in the appropriation of these tools. These negotiations also involved student contributions of solution methods, explanations, student register, and questions.

An extract from the teacher interview illustrates many aspects of his approach to teaching and his views of student mathematical learning.

> Well what, ... my usual way now/ is ask them how they've done it, and to explain it to them. And how/ did you get that answer, rather than say it was wrong, And if I would use an analogy of that, um ... when I was talking to my daughter who was all of four. And I had a sausage in front of me, and I said 'how many cuts do I have to make in this to make it into four pieces?'. ... She's sitting up opposite me and she says ... two/ ..., and I said how did you get two and I was looking at her and she (was) looking down to her toast and says, 'one this way and one (at right angles) this way'. Now you could actually do that with sausage as well but what I want to point, I'm not saying she was particularly clever/, what I'm saying is she was thinking toast/, I was thinking sausages/. So an answer that a student has, you know, is it actually wrong? You don't know until you know the process they've gone through (Mr Larsen, Interview).

Lastly, an unforeseen incident in Lesson 4 illustrated the teacher's multiple roles and seems a fitting way to end the description of the case study data. On a hot summer's day with open windows and doors, a fantail flew into the classroom. A fantail is a New Zealand native bird named for its distinctive tail when flying and known for its relative tameness around people and other animals. A fantail flying indoors can be considered to be a sign of bad luck by some Mäori, and this was mentioned by a few students. The teacher reassured the students, supported a discussion among some students about both bad luck and good luck perceptions of the incident, and then re-directed the students to other aspects of the lesson. Two students guided the fantail out of the classroom and out of the building.

4.5 Chapter Summary

4.5.1 Overview of Case Study Contexts

This section briefly summarises aspects of the analysis dimension, Background Information. The classroom settings were typical of New Zealand urban state secondary school classrooms in terms of size and resources such as furniture and

equipment for students and teachers. The number of students in each class is set out in Table 4. 13 below.

Case Study	Year Level	Age (Yrs)	Number Of Boys	Number Of Girls	Total Number of Students
Lemonwood 1	9	13	18	13	31
Lemonwood 2	9	13	18	13	31
Lemonwood 3	10	14	16	15	31

Table 4.13 Class Size of Case Study Classrooms

The class sizes were typical for Years 9 and 10 mathematics classes. The number of boys in the Lemonwood 1 and 2 classes was higher than for girls; in my experience, there often is a greater proportion of boys in the Years 9 and 10 lower and middle banded classes in state co-educational secondary schools.

The organisation of mathematics programmes in the case study classrooms was found to be typical of mathematics classrooms I have visited around New Zealand. The number of mathematics lessons per week, timetabled at varied times of day, is common practice for Years 9 and 10. The resources used were also typical, with the student textbooks being among the most common in current use. The inclusion of homework books for mathematics has increased in recent years and the two Year 9 classes were using a popular published homework book. The students' own resources, such as a squared exercise book and a scientific calculator, are familiar to most Year 9 and 10 students. The incidence of common tests, and the inclusion of revision for these, are also considered common practice for secondary school mathematics at this level.

Other non-mathematical features were typical of daily secondary school life. The use of lesson time for administration and the interruption of lessons by a teacher or a student 'runner' would be regular occurrences, as would the general class participation in social events related to the school. The fire alarm, a fantail flying into the classroom, and the disruptions to weekly timetables, exemplified the texture of contemporary life in New Zealand schools. Last but not least, the dynamic interactive relationships between teachers and students, and between

peers, were indicative of many New Zealand classrooms I have visited. These background details indicated that the three case study classrooms were typical of large, busy classes for this age group, and therefore useful contexts for this study.

4.5.2 Case Studies Summary

The data from each of the three case studies has been organised, described and analysed in this chapter. The case study data was organised into three analysis dimensions that corresponded to the domains of Classroom Social Norms, Tools, and Interaction. Within each domain, common data categories were used for all case studies. The categories of Rituals and Roles contributed to the analysis of the social norms domain for each case study. Rituals and roles were identified that related to student participation in mathematical activity. Tools were identified and organised into three categories: Classroom Mathematical, Technological, and Other Mathematical, Tools. Tools were then analysed in terms of the roles in mathematical activity. The Interaction categories incorporated exemplars of analysed lesson transcripts. Negotiations initiated by the teacher were examined in two categories. This provided information about how classroom mathematical practices were jointly developed in each classroom. The next three categories encompassed examples of unanticipated student contributions, errors or repair interactions. These student-initiated episodes were analysed for evidence of joint negotiation of mathematical meanings. The final interaction category examined negotiations during activity mediated by the scientific calculator. In Chapter 5, the case study data will be combined for each of the main categories and then further analysed.

Chapter 5
Results

5.1 Chapter Introduction

This chapter extends the analysis of the case study data. Data from each category are examined across the three case studies in order to find common features as well as diverse ones (see section 3.7.6). This further analysis examines in more detail the affordances and constraints on mathematical activity. The category data examined are important for the three domains of this study: social norms, tools, and interaction. In the domain of social norms, the categories of Rituals and Roles are further examined and linked to Teacher Mathematical Values. The categories, Classroom Mathematical Tools, Other Mathematical Tools, and Technological Tools are analysed further in the domain of tools. The Technological Tools category includes a further analysis of calculator episodes and encompasses some aspects of the category, Technology Mediated Negotiation. In the interaction domain, the categories of Teacher Guided Participation and Whole Class Negotiation are further examined. The focus of analysis is how teacher and student contributions afford or constrain whole class mathematical negotiation. The data from the student initiated categories, Student Initiated Negotiation, Error Initiated Negotiation, and Repair Interactions, are also further analysed, particularly the ensuing negotiation of mathematical meaning. The final category, Technology Mediated Negotiation, is included as another example of joint mathematical negotiation.

5.2 Classroom Social Norms

In this section, the classroom social norms are examined in terms of two key categories: Rituals and Roles. As defined in Chapter 2, Rituals were recurring actions by the participants. The rituals of interest in this study were those that provided opportunities for student participation in shared mathematical activity. Roles were defined as ways of working together and the status of contributions to joint activity. Rituals and Roles are now examined for affordances or constraints

on joint mathematical activity. Data from the category Teacher Mathematical Values are also included to provide corroborating evidence of the significant rituals and roles.

5.2.1 Rituals

The following rituals were identified as significant for student participation in mathematical activity. These participation rituals are recorded in Table 5.1 below.

Case Study	Ritual	Features of Ritual
Lemonwood 1	Starter Ritual	Quiz questions, students assisting each other
	Marking Ritual	Regular shared review of starter questions, practice and homework tasks
Lemonwood 2	Activity Ritual	Localised social and mathematical interaction with teacher and/or peers during student tasks
	Chorus Ritual	Student chorus during whole class exposition and review
Lemonwood 3	Activity Ritual	Localised social and mathematical interaction with teacher and/or peers during student tasks
	Story Ritual	Teacher stories about mathematics and people

Table 5.1 Participation Rituals for Mathematical Activity

The Marking, Activity and Chorus Rituals will be outlined as exemplars of these significant participation rituals. Features of these rituals will be identified that afforded and constrained negotiation of mathematical meaning.

The Marking Ritual in Lemonwood 1 incorporated student contributions such as answers, questions and descriptions of solution methods. These contributions often initiated further explanation, discussion, negotiation and resolution by the classroom participants. Consequently, the Lemonwood 1 Marking Ritual afforded an 'interactional space' for students and teacher. This regular and ritualised space afforded opportunities for negotiation of mathematical meanings because

students actively contributed their thinking, and the teacher guided and scaffolded the subsequent interaction. One constraint of this Marking Ritual was the potential for time-consuming public interactions that involved only a minority of the class.

In the Chorus Ritual in Lemonwood 2, students and teacher mutually participated in a discourse pattern that had implicit signals and cues for all participants. The teacher contributions were either complete-the-sentence phrases or incorporated a rising inflection to indicate a question. Student contributions were brief, specific and volunteered. The student responses were relatively rapid so that the joint interaction kept pace with the speed of the teacher's whiteboard writing. The affordances of this ritual were the public feedback from students to teacher, the opportunity for students to signal their mathematical understandings, and to influence the pace of whole class exposition and review episodes. Different students contributed to the chorus in different lessons, and the Chorus Ritual may have been a 'safe' option for less confident students. A constraint of the Chorus Ritual was that the student feedback usually related to procedural mathematics and was relatively rapid. Another constraint was that there was not necessarily a student consensus because only a minority of the class was actively involved in the chorus at any time. Some of the silent students may have been in agreement, and some students may not, with either the mathematical ideas or the pace of the interaction.

In Lemonwood 2 and 3, the Activity Rituals provided localised 'interactional' spaces for mathematical negotiation. The Activity Rituals afforded opportunities for joint activity between groups of peers, and between students and the teacher. Students recognised that this afforded an opportunity to request assistance and access further mathematical resources from peers or the teacher. Students' willingness to engage with peers during this time contributed to a sense of community in the classroom. These interactional spaces also afforded regular and sustained opportunities for students to socialise with their peers. While this socialising could constrain mathematical activity, many students used this as a necessary precursor to engaging in mathematical activity. Another constraint of

the Activity Rituals was that access to the teacher was limited to a few students at a time.

Rituals and Teacher Mathematical Values

Teacher Mathematical Values were inferred by teacher statements drawn from the lesson transcripts and from the teacher interviews. Some of the teacher mathematical values were related to the rituals identified in the case study classrooms. The Teacher Mathematical Values related to participation in classroom mathematical activity are recorded in Table 5.2 below.

Case Study	Teacher Mathematical Values
Lemonwood 1	Routine and practice are important Notes are important as a record of mathematical rules Revision involves reviewing notes and practice
Lemonwood 2	Skills and practice are important Textbooks are useful resources for revision
Lemonwood 3	Students are not expected to imitate teacher's methods Students can use own methods, and are more likely to remember own methods School mathematics needs to be linked to real situations

Table 5.2 Teacher Mathematical Values about Learning Mathematics

The value of routines was linked to rituals such as the Starter Ritual (Lemonwood 1), and associated with opportunities for shared mathematical activity. In the teacher interview, Mrs Johnson talked about the importance of a stable lesson structure, particularly at the beginning of the lessons (the Starter Ritual). The importance of practice for student learning implicitly related to marking rituals that afforded rapid feedback to students (Lemonwood 1 and 2). The Lemonwood 3 teacher told the students in Lesson 7 that the purpose of the starter ritual was to get the brain thinking (section 4.4.3). Again during Lesson 7, Mr Larsen explicitly told the class that students' alternative methods indicated that they were thinking. His enthusiastic response to Tina, Danny and Nathan's Methods

indicated the value he placed on student generated methods (sections 4.4.3 and 4.4.5). This was an example of an opportunity for students to publicly share a range of methods, each one supported and affirmed by the teacher. This illustrated how teacher mathematical values were important influences on ritualised opportunities for students to actively contribute to mathematical activity.

In addition, there were ritualised actions such as students raising their hands (Lemonwood 1 and 2), students calling out during whole class interactions (all case studies), and students going to the board to record their ideas (Lemonwood 2). Although rituals were mutually constituted actions, some contradictory rituals co-existed in the same classroom context. For example, a ritual of 'hands up' existed side-by side with a ritual of students 'calling out' (Lemonwood 1 and 2). 'Hands up' enabled students, including quieter ones, to volunteer contributions, and afforded some order to classroom contributions. There was relatively less background noise enabling students to listen to peers' contributions. 'Calling out' afforded immediate student responses to the dynamic exchanges of whole class interactions, but could constrain contributions from quieter students. Some of these additional rituals were considered to be indicators of more sustained activity involving student contributions, and have been subsumed into the next section, Roles.

5.2.2 Roles

Roles comprise another key category of the social norms domain. As outlined in section 2.4.5, roles relate to how students can contribute, when, and the nature of contributions to whole class interactions. In this section, significant roles for participation in mathematical activity are outlined. Firstly, the roles relating to student contributions are identified for all case study classrooms. The status of particular contributions is also outlined and other participation roles are then identified. The student roles are further examined in terms of affordances and constraints on participation in mathematical activity.

In all three classrooms, a significant student role was as a valued contributor to whole class mathematical interactions. Some of these contributor roles were evident in all three classrooms; others were evident in only one of the classrooms. In contributing to the whole class mathematical activity, students adopted the range of roles set out in Table 5.3 below. Not every student was observed to adopt each of these roles, while some students contributed more frequently than others.

Case Study	Student Roles Identified
All Case Studies	Contributor to whole class discussion solution describer answer provider mathematical explainer question answerer question asker
Lemonwood 1	Provider of models, analogies, real situations Describer of new calculator functions
Lemonwood 2	Whiteboard recorder Choruser
Lemonwood 3	Contributor of solution methods different to the teacher's Provider of analogies

Table 5.3 Contributor Roles of Students to Whole Class Interactions

In all three classrooms, student contributions included describing personal solution methods as well as providing answers to tasks. An example of the role of solution describer was Scott's Wicked Number Pairs (section 4.2.5). In Lemonwood 3, the teacher explicitly encouraged students to find their own methods and Tina, Danny and Nathan's Methods were examples of a whole class sharing of these methods (section 4.4.5). When students adopted the role of mathematical explainer, this included providing extra information that could involve mathematical reasoning. During the interaction prompted by Natalie's Error, Ryan contributed an explanation of a recomposition of 7.9 (section 4.2.5). Hamish and the Shortcut Key (section 4.3.5), and Shane's Reasoning (section 4.4.5), were also examples of mathematical explaining and reasoning. In addition, students' contributions included responding to teacher questions, and asking questions of the teacher. Mathew's Rounding of Nines (section 4.2.5), Melanie's

Question (section 4.2.5), and Donna's Conjecture (section 4.4.5), were examples of the range of questions asked by students.

Some student contribution roles were identified in only one of the case studies. In Lemonwood 1 individual students shared stories of real life mathematical contexts (overdrafts) and also contributed mathematical models or analogies (Paul's 'hitting a wall' analogy for the Whoops Rule, section 4.2.5). In Lemonwood 2, students as contributors recorded mathematical resources on the whiteboard. Also in this classroom, and to a lesser extent in Lemonwood 1, students contributed chorused responses to teacher questions and other prompts.

There were a number of important affordances associated with these roles. The roles of solution describer and mathematical explainer afforded opportunities for the speakers to articulate their mathematical thinking. These students were able to talk about mathematical entities and to verbalise written and mental processes. For example, Scott's explanation of his Wicked Number Pairs illustrated the process he used to generate pairs of sides of a rectangle (section 4.2.5). Student contributions also afforded further mathematical resources for public consideration and discussion. Tina, Danny and Nathan's Methods contributed multiple methods for finding the area of a part circle (section 4.4.5). Finn's Explanation (section 4.4.5), and Paul's Analogy contributed resources for thinking about mathematical processes. Student descriptions of solutions, tools and mathematical reasoning, afforded resources that were often more accessible to peers. Ryan's contribution of 7.9 recomposed as 7.5 plus 0.4 afforded a resource of a tool to represent a decimal number. This then became a topic of further discussion when the teacher appropriated this within a visual tool, the Empty Number Line. The role of question asker was also important as it afforded student concerns to be raised in whole class interaction, and again afforded further mathematical negotiation. An example of this was Mathew's Rounding of Nines (section 4.2.5), Melanie's Question (section 4.3.5) and Donna's Conjecture (section 4.4.5). By contributing to whole class interactions, student contributors afforded opportunities for teacher feedback about their mathematical thinking. Student contributions such as chorusing also afforded information to the teacher

about the pace and student agreement with the demonstration of steps in a solution.

The other significant student roles were those of collaborator in tasks, a listener to peers, and an assistor. When peers collaborated in activities such as textbook tasks, they discussed the task, listened to each other and talked about aspects related to the particular task. An example was when Amy, Erin and Louise collaborated on multiplication of decimals tasks, and were also assisted by Shaun and Alex (section 4.3.5). When students requested assistance, there was an expectation that their peer or the teacher would assist. The form of the peer assistance included asking each other questions, responding to peer questions, explaining their generated methods, looking at their peers' written methods and/or answers, or showing each other their calculator screens. A brief example was Janine and Timu's assistance dialogue about finding prime factors (section 4.2.5). The student assistance practices afforded further mathematical resources, accessed at opportune times and in student register. Resources included hints, cues, partial information, the answer, or contributions of mathematical tools. Collaborative student activity afforded access to more mathematical resources than were available in individual activity. When assistance occurred during whole class interactions, it was often a localised conversation between peers. Another type of assistance was the repair negotiations such as in Lemonwood 1, when Grace rephrased a student contribution misunderstood by the teacher (section 4.2.5), and when Ryan publicly explained a model for a decimal number in order to assist another student Natalie (Natalie's Error, section 4.2.5). Here Ryan was both an assistor and a mathematical explainer, indicating that students could assume more than one role at a time.

Student contributions were accorded high status by the teacher and by peers in two ways. Teachers and students listened and responded to others' contributions, and teachers actively monitored and promoted opportunities for student contributions to whole class mathematical interactions. It was common in each case study classroom for the teacher to intervene and to explicitly remind students to listen to each other. Lesson time was regularly taken to redirect students, and to insist on attention to student contributions. This signalled to

students that the teacher valued their contributions, and that listening to their peer's contributions was useful for them as learners. Consequently, students implicitly valued peer contributions, as indicated by students listening to each other and responding to student questions or ideas. Another indicator of the high status of student contributions was when peers repaired communication mismatches in order to maintain the status of student contributions.

5.2.3 Overview of Classroom Social Norms

This section has provided further analysis of the Rituals and Roles categories in the domain of classroom social norms. Significant participation rituals were identified and analysed in terms of affordances and constraints on student participation in mathematical activity. Various students assumed different roles related to being a contributor, a collaborator and assistor in mathematical activity. These roles were particularly related to student contributions of resources such as mathematical tools and solution processes. Roles afforded contribution of student resources to shared interactions, making available such things as student register, student descriptions and ideas. Student contributions also afforded opportunities for students to reason out loud and to verbalise their thinking. These contributions, both whole class and localised, were affordances that other students could access, and potentially appropriate, adapt or reject in individual activity. In addition, student collaboration and assistance roles were valued in all case study classrooms, and occurred in localised and whole class activity.

5.3 Tools

The second domain in this study was tools. Tools and their roles in mathematical activity were identified in the previous chapter (sections 4.2.4, 4.3.4 and 4.4.4). Three categories of tools were used to organise tools identified in the case study classrooms: Classroom Mathematical Tools, Other Mathematical Tools, and Technological Tools. The tools data are now further analysed to identify how tools afforded and constrained mathematical activity.

5.3.1 Classroom Mathematical Tools

In this study, Classroom Mathematical Tools were defined as the cognitive tools explicitly used by the teacher in whole class teaching interactions (section 3.6.4). The observed Classroom Mathematical Tools are now combined across the case studies in four groupings: representations, thinking devices, analogous models, and pedagogical artefacts. Significant tools are analysed in more detail as exemplars of these groupings.

Representations

Representations are written or diagrammatic representations of mathematical entities (Greeno, 1991). The representations identified in the case study classrooms are recorded below in Table 5.4.

Representation	Case Study	Mathematical Topic
Number Line	Lemonwood 1 Lemonwood 2 Lemonwood 3	Integers Integers Integers
Empty Number Line	Lemonwood 1	Decimals
Chocolate Bar Drawing	Lemonwood 1 Lemonwood 2	Decimals Decimals
Factor Tree	Lemonwood 1	Prime factors
Think Boxes	Lemonwood 3	Area of Composite Shapes

Table 5.4 Classroom Mathematical Tools that were Representations

The affordances of these representations were that they were visual, stable, and presented key mathematical information. For example, the Number Line provided a visual representation of the relative magnitude of integers and afforded implicit information about the relationships between positive and negative integers. By making important mathematical features visually available to students, the Number Line offloaded the mental effort needed to compare integers. Secondly, as a representational model of the number system, the Number Line re-presented mathematical information such as zero, multiples of ten and so on. This information afforded benchmarks or reference points for further mathematical activity. Lastly, the information about the number system was publicly available

as a resource for discussion. The comparison of positive and negative integers was a topic of whole class negotiation in two of the case studies.

The Empty Number Line was also a representation and used once in the context of resolving Natalie's Error, a student dilemma about the magnitude of decimal numbers (section 4.2.5). The teacher used the Empty Number Line to model Ryan's contribution of 7.9 as 7.5 and 0.4. The midpoint of 7.5 was a benchmark, a reference point for adding on 0.4. The empty and uncalibrated nature of the Empty Number Line enabled the teacher to narrow the field of attention to a suitable number range and to a meaningful division of the empty space. The flexibility in the range and scale of the Empty Number Line afforded choice of reference points.

The division of the Chocolate Bar Drawing into tenths and hundredths, on the other hand, was a less transparent tool. It provided a dynamic visual model of decreasing size of decimals, using a popular analogy for fractions. The inclusion of this particular tool in two classrooms did not promote further negotiation (sections 4.2.5 and 4.3.5). Consequently, this two dimensional fractional model was inferred to constrain further activity. One reason was possibly because the model was an area rather than a quantitative representation. Another constraint could be that chocolate bars are not commonly arranged in a single row of ten or as two rows of five pieces. Note that while the chocolate bar was an analogous model, it is included as a representation because the teacher talked about and used the drawing of the model. Think Boxes, introduced by the Lemonwood 3 teacher, were the final representational tool observed in this study. Think Boxes provided external representations of different ways to recompose or decompose known shapes to make the required composite shape. The Think Boxes afforded a stable visual record for students to refer back to at any stage of their individual activity (section 4.4.4).

Thinking Devices

The second grouping of classroom mathematical tools were thinking devices and these are recorded in Table 5.5 below.

Thinking Device	Case Study	Mathematical Topic
Number Line	Lemonwood 1 Lemonwood 2 Lemonwood 3	Integers Integers Integers
Ruler	Lemonwood 3	Integers
Empty Number Line	Lemonwood 1	Decimals
Rounding Dash	Lemonwood 1	Rounding
Rounding Dot	Lemonwood 2	Rounding
Think Boxes	Lemonwood 3	Area of Composite Shapes

Table 5.5 Classroom Mathematical Tools that were Thinking Devices

The thinking devices afforded and constrained mathematical activity in different ways. The Number Line was a thinking device for addition and subtraction processes (see for example, the Number Line Rule, section 4.2.5). Firstly, the labelled integers were used as reference points for the calculation process, such as the starting and end points. The presence of a starting point afforded a secure foothold for the rest of the process. A simple calculation with integers was then restructured into a sequence of defined steps that considered the first addend, then the operation sign, followed by counting the last addend in a series of 'hops', finally reaching the end point. The number at the end of the counting process provided the result of the addition or subtraction process. These affordances of the Number Line offloaded mental effort because students only needed to attend to one step at a time. The sequence of steps needed for successful use of the Number Line tool was the same as the order of the symbolic expressions, that is, congruent with considering each term of the symbolic expression at a time, from left to right. By affording this as a scaffold for the students, the Number Line was a tool that provided a systematic model for the conventional mathematical notation. The Number Line also afforded potentially different ways to carry out a calculation process (sections 4.3.4 and 4.4.4). Addition and subtraction could be modelled by counting in ones, or by recomposing the second addend for speedier skip counting. The zero mid-point afforded a useful benchmark for this recomposition. In Lemonwood 1, some students used a Number Line drawn in their books, or a ruler, or a mental image of a number line as a thinking device (section 4.2.4). In this way, the number line was appropriated as a thinking device by students during individual tasks related to addition and subtraction of integers.

Other significant thinking devices were used for rounding whole or decimal numbers. The Rounding Dash and Rounding Dot tools were introduced by the Year 9 teachers who modelled the use of these tools to mark the designated place for rounding. The Rounding Dash was a marker that separated the places (section 4.2.4, the Dash Method for Rounding). To the left of the dash was the place designated for rounding, to the right was the place for scrutiny for any possible rounding. The Rounding Dot in Lemonwood 2 identified the designated place, and implicitly the place to the right of the dot needed to be investigated for rounding (sections 4.3.4 and 4.3.5). These markers afforded identification of the place required for rounding, and afforded a cue to the digit to the right of the marker. In each case, by narrowing the students' field of attention to only two digits, the marker offloaded mental effort, and restructured the activity for the students.

The final significant example of a thinking device was the Think Boxes, used by the Lemonwood 3 teacher to transform a calculation activity for the area of composite shapes. The information recorded in the Think Boxes afforded resources such as diagrams, labels and simpler shapes recognisable to the students. This afforded student thinking about the problem in diverse ways, and illustrated the role of a metacognitive tool in restructuring the activity.

Analogous Models

The third grouping of Classroom Mathematical Tools encompassed the analogous models used by teachers (section 2.5.2). The range of analogous models observed is recorded below in Table 5.6.

Analogous Model	Case Study	Mathematical Topic
Lift in a Building *	Lemonwood 1	Integers
Thermometer	Lemonwood 1 Lemonwood 3	Integers Integers
Overdraft	Lemonwood 1 Lemonwood 3	Integers Integers
Language Example	Lemonwood 3	Integers
Speedometer	Lemonwood 2	Rounding

Table 5.6 Classroom Mathematical Tools that were Analogous Models

* This was not included in representations, because the teacher's main purpose was to use the analogy to illustrate the subtraction of integers, rather than the diagram (section 4.2.5).

The analogous models related to familiar real life situations that modelled mathematical entities. These real life situations were discussed and sometimes recorded as drawings on the whiteboard. As can be seen from the table, most of the analogous models related to integer topics. The thermometer and diagram of a lift, for example, were used to illustrate real life situations of a number line. In Lemonwood 1, the thermometer afforded a visual congruency with a number line, and the lift afforded a visual resource during a whole class Repair Interaction. There were, however, constraints associated with these analogous models for integers. Thermometers measure temperature and for students in Lemonwood 1 on a hot summer's day, temperature was a sensory measure rather than a quantitative one (section 4.2.4). Multi-storeyed buildings use descriptive labels for their underground floors rather than negative numbers (section 4.2.5). The example of overdrafts, as discussed in Lemonwood 1 and 3, afforded a real life example familiar to the students. Overdrafts are, however, a transaction in the form of a debt to be paid back, rather than an abstract quantity. This created confusion for at least one student in Lemonwood 1 and initiated a joint Repair Interaction (section 4.2.5).

The speedometer was an analogous model introduced by the teacher in Lemonwood 2 to assist with the rounding process. The speedometer was familiar to students who may have spent time in cars, or seen significant numbers change on the odometer (speedometer). This analogous model afforded a mental image that modelled the changing of a nine digit to a zero with the neighbouring digit increasing by one. This also afforded a dynamic rather than static model for the more complex case of rounding up digits that were nines.

To summarise, teachers used analogous models to include real life examples in classroom lessons. The analogous models afforded familiar examples that corresponded to important mathematical ideas or processes. The models could constrain activity if key features were absent or did not match the mathematical entities.

Pedagogical Artefacts

The final grouping of classroom mathematical tools observed across the case study classrooms consisted of pedagogical artefacts used by the teacher to organise information. These tools were the Place Value Grid and 2x2 Arrays. The main role of these tools was to display information in ways that implied patterns or other relationships. This group of tools is not analysed further because these artefacts were not involved in any episodes of negotiation of mathematical meaning.

5.3.2 Other Mathematical Tools

The second category of cognitive tools, Other Mathematical Tools, recorded mathematical tools not already identified as Classroom Mathematical Tools. These are recorded in Table 5.7 below.

Tool	Mathematical Tools common to the Year 9 Case Studies *	Mathematical Tools unique to One Case Study Classroom
Algorithms	Multiplication algorithm for whole and decimal numbers Division algorithm for whole numbers	Subtraction algorithm for whole numbers (L1) Long division algorithm for whole numbers (L2) Algorithm for area of a circle (L3)
Operations/ Relations	Addition and subtraction of integers Multiplication and division of integers Powers Combinations of operations (BEDMAS)	Multiplication and division of decimal numbers (L2) Square and cube roots (L1) Squares (area) (L3) Addition, subtraction and multiplication of fractions (L3)
Multiple Representa-tions	Numbers represented by factors and by prime factors Numbers recomposed, decomposed Fractions and decimals	Percentages (L2) Representations of fractions(L3) Decimals and measurements (L3)
Other (eg notation)	Comparison of relative magnitude for whole numbers and decimals Place value notation	Significant figures and approximation (L2) Pi as a constant (L3) Diagrams of fractions (L3)

Table 5.7 Other Mathematical Tools Identified in the Case Study Classrooms

* The two Year 9 case study tools have been combined together because these classes had the greatest proportion of number lessons.

This category was organised into four groups of tools: Algorithms, Relations and Operations, Multiple Representations, and Other, such as notations. These groups have parallels with the groupings used in Classroom Mathematical Tools. Multiple Representations and Other were parallel to representations, and Algorithms and Relations and Operations to thinking devices. Other Mathematical Tools reflect the particular topics of the lessons observed and the focus on revision of prior knowledge and development of new mathematical practices. Other Mathematical Tools were used in whole class or individual activity, and were mostly nested within other cognitive tools (section 3.6.4). For example, the use of the Empty Number Line, a representational tool and thinking device, included nested tools such as place value notation and representations of decimals (Natalie's Error, section 4.2.5). When students described a solution method, or explained mathematical reasoning, they talked about these cognitive tools in the context of mathematical activity. Other Mathematical Tools were important resources for classroom interaction, and affordances and constraints of some of these tools will be discussed in the further analysis of the Interaction domain, particularly in the category Error Initiated Negotiation.

5.3.3 Technological Tools

The only technological tool in the three case study classrooms was the scientific calculator. A cross case analysis of the calculator role categories is now presented and significant aspects of calculator use are outlined.

Roles of the Scientific Calculator

In the previous chapter, calculator activity was analysed according to roles as a recording device, as a calculating machine, and as an exploratory object (section 3.6.4). The two Year 9 classrooms had more episodes of calculator generated activity than the Year 10 classroom. This was due to two factors: the greater numbers of observed lessons and more number related lessons. The introduction of the scientific calculator at Year 9 is another possible influence on the amount of calculator use. Students initiated the recording device episodes and the

exploratory object episodes observed in Lemonwood 2. In contrast, the teachers initiated most of the eleven calculating machine episodes. In each of the case studies, there was one whole class calculating episode initiated by a student. The high number of calculating episodes reflected the mathematical content of the lessons, and often followed on from a demonstration by the teacher of a new calculator function. The calculator as an exploratory object was identified in a relatively high number of episodes in Lemonwood 1. All but one were initiated by the teacher who explicitly introduced the power, square root, and other functions of the scientific calculator. The number and nature of the calculator episodes for each case study are recorded in Table 5.8 below.

Role of Calculator	Case Study	Number of Calculator Episodes	Number of secondary Episodes	Number of Discursive Episodes
As a recording device	Lemonwood 1	0	2	0
	Lemonwood 2	2 (2S)	1	2
	Lemonwood 3	0	0	0
	Total	2	3	2
As a calculating machine	Lemonwood 1	3 (1S)	4	2
	Lemonwood 2	4 (1S)	1	2
	Lemonwood 3	4 (1S)	0	1
	Total	11	5	5
As an exploratory object	Lemonwood 1	6 (1S)	1	2
	Lemonwood 2	2 (2S)	0	2
	Lemonwood 3	1	0	1
	Total	9	1	5

Table 5.8 Roles of the Calculator across all Case Study Classrooms

Note:

(i) (1S) refers to the number of student initiated episodes for each case study, others were teacher initiated.

(ii) The relatively low number of student initiated episodes can be accounted for by the limitations of the data collection. During student activity, audiotape data was collected from only focus students and supplemented by field notes.

(iii) There were no recorded episodes for the calculator role as a number cruncher. This was not surprising given the mathematical topics of the case studies.

Different brands of calculators promoted discussion and sharing of alternative methods, affording opportunities for further investigation (section 4.2.5). Associated with this was the significance of the calculator as an exploratory object. The students became involved in a series of reciprocating interactions with the calculator. For example, after trialling a known mathematical result, students tried a new function key, and, using the feedback provided by the calculator, they evaluated their methods. A student could retain, refine or try a new method. This process was repeated for a number of trials and the student independently generated authentic and reliable methods for using the scientific calculator. This was a different process to the appropriation of another's method. The rapid feedback provided by the calculator illustrated the importance of the ease of using the calculator. It also illustrated the reciprocal relationship between the technological tool and student operator.

Further Analysis of a Technological Tool

The affordances of the scientific calculator are now grouped according to particular features of the technological tool. These affordances illustrate the significant role of the technological tool in contributing more mathematical resources to classroom activity. The additional resources were speedily and accurately contributed. Any keying-in errors were temporary and changeable. The calculator served two seemingly contradictory purposes. On the one hand, the calculator afforded a private and easily erased calculating means for individual students, supporting student exploration and experimentation. An example was Dean and Mathew exploring 2 to the power of 15 (section 4.2.4). On the other hand, the calculator had many common processes and a screen accessible to neighbouring peers. This afforded a resource of their neighbours' calculator screens and further opportunities for interaction with peers. Examples were the assistance dialogues between Janine and Natalie, and between Ryan and Mathew (both from section 4.2.4).

The grouped affordances are listed in the table below with examples of episodes that illustrate particular affordances.

Speed, precision and accessibility of the calculator;	
Affordance	Examples
More efficient calculations, faster and usually more precise	Mathew and Dean calculating 2 to the power of 15
Rapid feedback provided to operator	Timu, Tom and Mathew and the calculation of the square root of 5
Independent, autonomous, and sometimes private, use by students	Tom's Multiline Function Alex and the Calculator Dot Stephanie and 1.0
Functions of the calculator;	
Affordance	Examples
Alternative methods of calculation	For operations with negative integers For powers (section 4.2.5)
Shortcut keys for more ease of calculation	Hamish and the Shortcut Key
A private calculating tool for individual students	Luke's Decimal Result
Student roles of experimentor and generator of mathematics	Tom's Multiline Function Hamish and the Shortcut Key
Different brands of calculator;	
Affordance	Examples
Alternative sequences for keying in calculation processes	Hamish and the Shortcut Key
Multiple representations of numbers on calculator screen (eg large numbers)	Jeff and Decimal Numbers Stephanie and 1.0 Hamish and Alex (section 4.3.5)

Table 5.9 Affordances of Scientific Calculator and Examples of Mediated Activity

The calculator was also accorded mathematical authority when a calculator result was used to settle disputes, correct errors, or to check answers. An example was Luke's (incorrect) Calculator Result in Lemonwood 1 (section 4.2.5). Luke relied on the authority of his calculator result to validate his answer. This illustrated a constraint of the calculator as Luke initially keyed in the incorrect numbers. Luke's incorrect result prompted him to question the public solution, initiating a whole class discussion about how to multiply together two decimal numbers. The calculator was, however, used to settle this dispute as other students used their

calculators to quickly generate resources that were then contributed to the discussion.

5.3.4 Overview of Tools

The identified roles of tools have possible links with Teacher Mathematical Values. Different teachers highlighted the roles and importance of tools in different ways. The Lemonwood 2 teacher, for example, emphasised tools as a means of connecting aspects of mathematics, for example in the fractions and decimal activity (section 4.2.5). The Lemonwood 3 teacher emphasised the use of mathematics (implicitly measurement tools) in both school mathematics and real life activity (section 4.4.3). The Year 9 teachers also initiated, supported and validated the roles of the technological tool, by demonstrating how to use the functions of the calculator and implicitly indicating a high status for calculator processes. The Lemonwood 1 teacher explicitly encouraged students to learn how to "drive" their own calculator. She devoted lesson time to demonstrate various functions of the different brands of calculators, and provided individual assistance as students experimented with new function keys (sections 4.2.4 and 4.2.5). Teachers did not always encourage student use of the calculator. In the first observed lesson in Lemonwood 3, the teacher asked students not to use their calculators for estimation tasks. On one occasion, the Lemonwood 2 teacher explicitly promoted student brains as faster than the calculator.

Data from the three categories in the domain of tools have been analysed further for significant roles of tools in mathematical activity. Tools that were representations and thinking devices were significant for mathematical activity. Tools that represented information, such as benchmarks, reference points, cues and other nested tools, afforded further mathematical resources for activity. These tools afforded important mathematical information in stable and accessible ways. Tools that could be used in diverse ways afforded stability yet flexibility for students in number and calculation activities. Tools that narrowed the focus of attention afforded specific mathematical activity such as resolving student errors. Analogous models, however, were found to both afford and constrain activity for students. A technological tool, the scientific calculator, afforded independent and

shared exploratory mathematical activity. Specific capabilities of the calculator afforded this exploratory role and some students generated their own calculator methods. This further analysis has also highlighted that tools can afford or constrain interaction between classroom participants, the third domain of this study.

5.4 Classroom Interaction

In this section, data from the interaction categories are combined across the three case studies. Teacher scaffolding actions were identified in the first category, Teacher Guided Participation. The second category, Whole Class Negotiation, focused on the teacher-initiated development of classroom mathematical practices. Episodes initiated by student contributions were examined in the next three categories: Student Initiated Negotiation, Error and Repair Interactions. Interactions during activity with scientific calculators were analysed in the final category, Technology Mediated Negotiation. These Negotiation categories are now analysed further in terms of affordances and constraints on negotiation of mathematical meaning.

5.4.1 Teacher Guided Participation

The case study analysis of the Teacher Guided Participation category identified a range of teacher practices that supported and guided student learning. The significant teacher scaffolding practices are now highlighted and the teacher roles within these scaffolding practices are further analysed in terms of affordances and constraints. The most frequently observed teacher scaffolding practices were:

- Sequencing of mathematical content from prior knowledge, then from less complex to more complex
- Demonstrating, explaining and reviewing mathematical practices
- Breaking mathematical practices into a series of steps
- Scaffolding student contributions, e.g. student solution methods
- Accepting student register and progressively introducing mathematical register

- Using an extensive range of cognitive tools such as representations, metaphors, and analogous models
- Indicating mathematical status of student contributions

Within these scaffolding practices, certain teacher actions served to shape student contributions. Examples of these were the rephrasing of student contributions, recording information on the whiteboard (Natalie's Error, section 4.2.5), or asking for further student explanation or reasoning (Donna's Conjecture, section 4.4.5). This afforded more time for the student to repeat and refine their contributions. Students were not necessarily required to contribute all of their ideas at once. The teacher often intervened as an interlocutor, repeating or recording one step at a time, slowing down the student contribution, and providing more time for a student to express the next step or explanation. An example was Scott's explanation of his Wicked Number Pairs (section 4.2.5). The shaping of student responses served to generate a jointly scaffolded contribution that supported students in refining their thinking. It also afforded greater opportunity for peers to listen and to make sense of the whole class mathematical interaction.

Teachers adopted a variety of roles during whole class and student activity. As described by the scaffolding practices above, teacher roles in all three case studies were mathematical explainer, provider of models and analogies, question asker, recorder of student contributions, and answerer of student questions. Individual teachers assumed additional roles. For example, the Lemonwood 1 teacher assumed the role of note provider and calculator demonstrator (sections 4.2.3 and 4.2.5). A significant role for the Lemonwood 2 teacher was as a mathematical connector. She provided diagrams and notes that explicitly connected discrete aspects of the number topics, and during the Bingo game, she publicly provided clues that connected the numbers to more familiar representations (section 4.3.3). The Lemonwood 3 teacher assumed the role of storyteller (section 4.4.3) and metaphor provider (section 4.4.4). He connected mathematics to real contexts, introduced metaphors that were familiar to the students, and demonstrated a metacognitive strategy. As discussed in the previous

chapter, some of these roles were linked to individual Teacher Mathematical Values (sections 4.2.3, 4.3.3 and 4.4.3).

5.4.2 Whole Class Negotiation

This category encompasses the introduction, development and establishment of new classroom mathematical practices (section 3.6.5). In Chapter 4, the development of at least one classroom mathematical practice was outlined for each case study. These particular classroom mathematical practices are listed in Table 5.10 below.

Case Study	Classroom Mathematical Practices
Lemonwood 1	The Number Line Rule The Whoops Rule
Lemonwood 2	The Dot Method for Rounding
Lemonwood 3	The Multiplication of Integers Rule The Area of Part Circles Method

Table 5.10 Classroom Mathematical Practices Analysed in Chapter 4.

These were not the only classroom mathematical practices analysed in the case studies but were described within this category as exemplars of whole class negotiation. Within the development of classroom mathematical practices, contributions of teachers and students were identified as significant when the contribution was inferred to relate to any of the three domains: classroom social norms, tools or interaction. The significant teacher and student contributions are now further analysed in terms of affordances and constraints on mathematical activity.

Teacher Contributions

Firstly teachers assumed responsibility for introducing the classroom mathematical practices. The introductions were often based around student prior knowledge, such as what was familiar and what was important. This afforded implicit and explicit invitations to participate in the classroom mathematical practices. (Examples from Lemonwood 3 were outlined in sections 4.4.3 and

4.4.5). Secondly, teachers contributed new cognitive tools or used a familiar tool in a new way. For example, the Lemonwood 2 teacher introduced and demonstrated the Marker Dot for the Dot Method of Rounding (section 4.3.5). This afforded a demonstration of a new tool for possible student appropriation or adaptation. The Lemonwood 1 teacher demonstrated multiple uses of a familiar tool, the Number Line. This implicitly valued the Number Line as a tool for two classroom mathematical practices, the Number Line Rule and the Whoops Rule (section 4.2.5).

A significant contribution by the Lemonwood 1 teacher was the word 'whoops'. Whoops served as a signifier that cued students to change direction when using the Number Line for adding and subtracting integers. This cue was established by the teacher, and quickly and enthusiastically appropriated by students. 'Whoops' became an in-house expression, so much so that an absent student Katrina noticed her exclusion from this mutually constructed signifier, and requested information from the teacher. Students and teachers kept using this 'whoops' signifier until replaced by a new classroom mathematical practice, the Shortcut Rule.

Student Contributions

In this category of Whole Class Negotiation, significant student contributions were analogies, solution methods, and student register. In Lemonwood 1, Paul contributed an analogy of 'hitting a wall'. The teacher rephrased Paul's Analogy into a 'crash', and after some localised discussion the crash analogy was adopted by other students. Paul's Analogy had been rephrased by the teacher, discussed by other students, one of whom questioned the meaning, and self-resolved. The teacher demonstrated a long-term appropriation of this re-phrased analogy as she continued to use it two lessons later with Katrina who had been absent (section 4.2.5). A similar joint construction in whole class negotiation occurred in Lemonwood 3. Finn's contribution was a 'canceling out' analogy, and reformulated by the teacher into an on/off analogy for the multiplication of integers. Although it was in response to a teacher contribution about a balance of positive and negative factors, Finn's Explanation was validated by the teacher and temporarily appropriated in his explanation. This was soon followed by the

teacher focus on a more efficient method for generating the sign of the product (section 4.4.5).

Students also contributed solution methods in the development of a classroom mathematical practice. In Lemonwood 3, Tina, Danny and Nathan contributed descriptions of different methods for the area of a part circle. These contributions afforded a range of validated methods, explained in accessible ways for peers. The final significant feature of student contributions was the informal language and student register that characterised the examples above and many of the other student contributions to whole class interaction. Student register afforded greater access to mathematical ideas, resources and processes.

Although teacher and student contributions have been analysed separately in this section, whole class negotiation was jointly constituted. Significant aspects of whole class negotiation were the use of tools in the establishment of classroom mathematical practices, the reciprocating nature of student and teacher appropriation of meanings, and the progression from student register to mathematical register. Student contributions afforded opportunities for matching individual ideas with other public contributions. Teachers and peers appropriated some student contributions. Examples were student and teacher responses to Paul's Analogy (section 4.2.5), and Finn's Explanation (section 4.4.5). Teacher appropriation of student contributions included the accepting, amending, reformulating, or ignoring aspects of the contribution. Some students did the same when appropriating teacher or peer contributions (Alex's Calculator Dot, section 4.3.5). In this way, students and teacher actively adapted and appropriated cognitive tools as a means of establishing a new mathematical practice.

In whole class negotiations, all three teachers shaped the timing and structure of much of the interaction. The initial mathematical practice introduced by the teacher could become a new mutually constituted practice. These jointly constituted whole class negotiations afforded more mathematical resources, and opportunities for further discussion. An unanticipated student contribution such as a suggestion, a question, an error or a mismatch of meaning, provided a

resource for further negotiation. The student initiated categories of serendipitous negotiations are now examined in more detail.

5.4.3 Student Initiated Negotiation

The category Student Initiated Negotiation encompassed interactions initiated by a student contribution, such as an unanticipated question, an unsolicited method, a disagreement, or resistance (section 3.6.5). The table below records the student initiated negotiations that were analysed and discussed in the previous chapter.

Case Study	Example of Student Initiated Negotiation
Lemonwood 1	Dean's (and Ryan's) Shortcut Rule Scott's Wicked Number Pairs to find factors of 24 Tom's Multi-line Function for calculator *
Lemonwood 2	Melanie's Question about adding a negative integer
Lemonwood 3	Donna's Conjecture about multiplication of integers (initiated by a question)

Table 5.11 Examples of Student Initiated Negotiation

* This example was included in the category of Technology Mediated Negotiation, but is included here as an example of a student initiated method.

Student initiated contributions afforded a shift in the original discourse. A shift in the discourse was a re-structuring of the focus of conversation, either to a new focus or a return to a previous one. Student questions were found to initiate shifts in the discourse. In the two examples above, both contributed by girls, their questions initiated a sequence of interactions involving the student, the teacher and some peers. Melanie (Lemonwood 2) sustained a series of "What if" questions as an implicit disagreement or resistance to the practice of using the Difference Method for adding and subtracting integers (section 4.3.5). Donna sustained a nested dialogue with the Lemonwood 3 teacher, within a whole class discussion involving other students. Donna's sequence of questions resulted in a relatively lengthy interaction about her conjecture (section 4.4.5). In these two

instances, student questions afforded opportunities for refining, supporting and contributing to whole class negotiation about specific aspects of mathematical meaning. For the examples set out in the previous table, the student contributions shifted the discourse in the following ways.

Student Initiated Negotiation	Resultant Shift in the Discourse
Dean and Ryan's Shortcut Rule	Firstly delayed by teacher but, when repeated, provoked a localised conversation among peers. Teacher used this to develop a new practice, the Shortcut Rule.
Scott's Wicked Number Pairs	Consideration of lengths of sides changed to ordered combinations of factors for 24, as tools for the dimensions of a rectangle of area 24.
Tom's Multi-line Function	Introduced an unknown capability of the calculator as a general student resource.
Melanie's Question	Melanie asked to provide another example for discussion, an opportunity to return to a whole class review of the practice for adding and subtracting negative numbers (the Difference Method).
Donna's Conjecture	Prompted a whole class review of the practice for multiplying integers

Table 5.12 Resultant Shifts in the Discourse from Student Initiated Negotiations

On further analysis, the significant aspects of this category were the role of peers and the roles of mathematical tools in the subsequent negotiation. Often peers would be the first to know about student generated methods. Tom, for example, initially showed another student his calculator finding before sharing it with the rest of the class (section 4.2.5). Although Dean originally told the teacher of his shortcut method, he next discussed this with his neighbours (and friends) before contributing to a later whole class negotiation (section 4.2.5). These localised peer discussions afforded opportunities for the student to explain and refine their ideas before contributing to a whole class discussion. It could be that for student generated methods, students sought out the mathematical authority of their peers before contributing it to the scrutiny of the whole class. Student contributions involving the use of mathematical tools could also result in shifts in the

discourse. Scott's method afforded a new tool, the factors, in finding the lengths of the sides of a rectangle. Tom contributed a capability of the calculator to recall previous student processes that afforded offloading the need to remember prior calculator processes (both from section 4.2.5).

Unanticipated student contributions were found to open up interactional spaces in all three classrooms. These topics or issues may have been anticipated by the teacher but individual students initiated the timing and often contributed to further interactions. The resultant interactions involved contributions of further mathematical resources from both teacher and students. Consequently, these negotiations were distributed over more participants, more mathematical tools and discursive resources.

5.4.4 Error Initiated Negotiation

Error Initiated Negotiation involved interactions prompted by a student mathematical error in a verbal answer, solution method or reasoning (section 3.6.5). As discussed in Chapter 2, an error can be considered as either activating a constraint or not recognising an affordance of a mathematical tool in a particular context (section 2.7.6). An error was considered significant if it initiated an episode of joint negotiation of mathematical meaning. There were many other examples of errors within the lesson data, but these were either student perturbations in the process of establishing a new classroom mathematical practice, and relatively quickly resolved, or instances when students self corrected. An example of this was in Lemonwood 2 when Simone immediately corrected her answer for the product of two decimal factors (section 4.3.5).

The significant errors from the case studies are now further analysed in terms of constraints associated with mathematical tools. The resolution of these significant errors are analysed in terms of the mathematical and discursive resources evident in the ensuing interactions. The significant errors identified in the previous chapter are recorded in Table 5.13 below.

Case Study	Lesson	Student	Error
Lemonwood 1	7	Natalie	For decimal numbers, the greater the digit in the decimal place, the smaller the number
	11	Dean	Rounding to a specified decimal place means rounding to significant figures
	11	Mathew *	When rounding a series of nine digits, zero is not necessary as a place holder
Lemonwood 2	3	Monique	Decimals numbers are bigger if they have more digits or if the digits are greater
	6	Amy	The number of decimal places in the product should be the same as in the factors
	6	Louise **	In the short division algorithm for decimal numbers, you can't have a quotient with more decimal places that the dividend.
	8	Shaun **	In the multiplication algorithm for whole numbers, you add the carried number before multiplying the next digit.
Lemonwood 3	7	Danny	To find the area of a circle, use the diameter rather than the radius.

Table 5.13 Errors Analysed and Discussed in Case Studies

* This example was also a Student Initiated Negotiation but was included in the Error Initiated category because it followed on from Dean's first error of rounding up a nine digit.

** These were errors during localised student activity and not part of whole class mathematical interaction.

Six of the eight errors related to mathematical activity with decimal numbers. These errors were further analysed using the tools perspective of place value notation described in section 2.7.6. For example, Natalie's error in Lemonwood 1, ("the more, the point, after the point, the number after the point, the lower it is") mismatched two signifiers of decimal numbers. The number of decimal places (away from the decimal point) that signified relative magnitude had been mismatched with the signifier of the magnitude of the digit itself. For Natalie, the magnitude of nine in the 7.9 was a larger number (after the decimal point) that had miscued her to the signifier of the number of decimal places. In this way the mismatch of different meanings carried in the tool had presented her with an

incorrect judgement that 7.9 was smaller than 7.2 (section 4.2.5). Similarly, Monique's error (in Lemonwood 2) that 0.09 was larger than 0.1, can be analysed as using only one of the signifiers in the decimal; that is, the meaning carried in the digits. By not recognising the additional meaning carried in the decimal places of the digits, Monique's error was an example of an incomplete use of the tool as a carrier of meanings (section 4.3.5).

Louise's difficulties with the remainders in the division of decimal numbers resulted from another mismatch of meanings in a representation of a decimal number (section 4.3.5). By inferring that the number of decimal places of the dividend was significant, Louise was reluctant to generate the quotient to more decimal places than the dividend. Associated with this was an observation that many students wrote the same number of decimal places for all the numbers in both multiplication and division examples (Lemonwood 2, section 4.3.5). For example, Amy's error was inferred as based on notions of consistency and pattern. 1.2 x 0.2 = 2.4 was consistent with an addition algorithm for decimal numbers. This misconception may be compounded by examples such as 1.2 x 0.5 = 0.6, that appear to have consistency of decimal places. When factors do not have the same number of decimal places (as in 0.4 x 0.05), then this error posed even more difficulties for some students (all examples from section 4.3.5). The numbers in the expression were possibly interpreted as static representations rather than an expression of a calculation process. Amy attended to the cues represented by the number of decimal places rather than the cue carried in the multiplication sign.

Lastly, the rounding errors were also associated with constraints of decimal numbers (section 4.2.5). I have assumed that Dean used 2 dp as a cue either for beginning the rounding process at two decimal places, or for the number of significant figures in the answer. In this example, Dean did not use the Dash Method for Rounding as a practice for the rounding process. Mathew's interest in rounding a series of nine digits illustrated how increasing a digit by one can set off a cascading process of changing the digits to the left. Associated with this was the role of a zero as a 'place holder'. In the rounding examples, a zero carried

meaning about both the magnitude of the place and the precision of the rounding process.

The student errors during whole class interactions were found to be resolved in different ways, mostly initiated by a teacher action. The predominant response was a sequence of questions initiated by the teacher to find out more about the student's method. This afforded an opportunity for the student to explain their method and/or reasoning. A jointly constituted solution of a new example often followed, where the student might have a leading role in identifying steps in the process. This afforded another public opportunity for talking about and demonstrating a solution method. For example, the Lemonwood 1 teacher responded to the rounding errors by going back to the classroom mathematical practice, the Dash Method for Rounding (section 4.2.5). The review of the practice involved more students in the negotiation. The Lemonwood 2 teacher responded in a similar way to a rounding error (section 4.3.5). Another teacher response to a student error was to explain the error, provide reasoning, and invite students to try another example. A further teacher scaffolding action provided a systematic approach that looked at one decimal place at a time, or simplified the solution process into a series of steps. This narrowed the field of attention for students and afforded opportunities for students to focus on specific aspects of a solution process.

Mathematical tools were used by teacher and students as resources in the negotiated resolution of a student error. The Lemonwood 1 teacher contributed representational tools such as the Empty Number Line, and drawings of analogous models such as the chocolate bar and the lift in a building (section 4.2.5). The Lemonwood 2 teacher initiated a proof in response to errors in the multiplication of two decimal numbers. Equivalent representations of fractions were used as tools to demonstrate the multiplication of decimal numbers (section 4.3.5). Not all tools were affordances for resolving a student error. In one instance, the tool of a visual representation of a chocolate bar did not draw any response from the student (Natalie's Error, section 4.2.5). A possible reason was that it represented visually decreasing area as a model for decreasing magnitude of decimal numbers. The original student error was related to the digits as carriers

of meaning about the magnitude of a decimal number, rather than the values of different decimal places. A significant example of a student contribution to an error resolution was Ryan's re-composition of 7.9 (section 4.2.5 and discussed in 5.3.1).

5.4.5 Repair Interactions

This category encompassed lesson episodes in which teacher or students repaired mismatches of mathematical meanings. There were very few repair episodes in whole class negotiations and these occurred in Lemonwood 1 and 2. In the third case study, repair situations were observed during localised conversations between the teacher and students.

Only two episodes from Lemonwood 1 were outlined in the previous chapter and significant features will be briefly reported. The repair episodes were both initiated by students; the first by Grace, on behalf of a peer, Dean, and the second by Tony who volunteered his experience of overdrafts (section 4.2.5). The teacher responded to a repair statement by asking the student (Dean) to repeat his question, or asking for further information. In the first episode, Grace sustained the repair sequence by rephrasing Dean's original statement. The second repair episode incorporated additional student contributions about their experiences of banks and overdrafts. The teacher played a significant role in this interaction in order to clarify both the real situation of bank overdrafts and the coherence of the analogous model. These two repair episodes were initiated by students, mutually repaired by students and teacher, and afforded further opportunities for mathematical negotiation.

5.4.6 Technology Mediated Negotiation

Some of the calculator episodes from the case studies have been subsumed into an earlier section (section 5.3.2). This final Negotiation category expanded on the category of Technological Tools by analysing calculator episodes involving mathematical negotiation (section 3.6.5). The ensuing interactions are now further analysed in terms of affordances and constraints on negotiation of

mathematical meaning. The scientific calculator negotiations of Lemonwood 1 and 2 are set out in Table 5. 14.

Case Study	Examples of Calculator Negotiations
Lemonwood 1	Tom's Multiline Function Luke's Calculator Result
Lemonwood 2	Stephanie and 1.0 Jeff's Calculator Screen Hamish and the Shortcut Key

Table 5.14 Technology Mediated Negotiations

Tom and Hamish used the scientific calculator as an exploratory object within a designated task. They each produced a new calculator process and volunteered an explanation to the teacher. In both instances, the teachers opened up an interactional space for the students to contribute further resources to their peers. These calculator processes were then taken up in localised peer conversations (sections 4.2.5 and 4.3.5). Stephanie's exploration of 1.0 was only tenuously related to the designated task and her interaction with the teacher was brief and focussed. This was, however, an important episode for Stephanie because the immediate teacher feedback afforded access to specific mathematical information (section 4.3.5). Luke and Jeff's episodes were initiated by a computation result on their calculator screens. Luke's result of 0.008 contradicted the teacher's result and this initiated a substantial negotiation and resolution of the calculation result (section 4.2.5). This was also an error-initiated episode where the resolution of an error was mediated by the technology. Jeff's Calculator Screen contrasted with familiar ways of representing decimal numbers. When Jeff brought this to the attention of the teacher, she raised this as a topic for conversation, thereby opening up an interactional space in the lesson. Consequently, decimals expressed in standard form became a topic of further conversation (section 4.3.5).

Finally, calculator negotiations occurred within social practices that supported different phases of the negotiation; that is, the localised peer and the whole class phases. During student activity, students talked with their neighbours, showed

each other their calculator screens, and sometimes talked to themselves while they were using the calculator. Peer assistance and collaboration were supported and students participated in joint activity with another tool, the calculator. In whole class activity, teachers implicitly and explicitly supported the use of the calculator as a valued resource that the students needed to know about. Finally, students appeared to enjoy using their calculators even when they were supposed to be doing other things. In all three classrooms, there were many instances of student surprise and excitement at the results on their calculator screens.

5.4.7 Overview of Classroom Interaction

The further analysis of the interaction categories has identified aspects of the case study interaction patterns that afforded and constrained mathematical activity. Teacher contributions afforded further resources such as mathematical tools and, in particular, visual representations of tools that became shared resources. Student contributions, such as analogies, stories, explanations and solutions methods, were also important affordances for peers. By expressing these in student register and by volunteering contributions at opportune times, student contributions provided further resources that were accessible to peers. Students and teachers appropriated, rephrased, adapted, or rejected others' contributions during whole class and localised mathematical activity. Student contributions that were unanticipated, such as a volunteered method or an error, were significant in all three classrooms. These student-initiated negotiations afforded further opportunities to talk about mathematical activity and to focus on specific mathematical aspects. The role of localised peer interaction was important in whole class negotiations of classroom mathematical practices. Peer interaction supported student contributions of solution methods and calculator functions, and the resolution of some student errors. Finally, a tools-based analysis of the student errors has highlighted the important roles of signifiers and cues. In their search for meanings for mathematical symbols and expressions, students were inferred to have paid attention to some signifiers and not others, leading to mathematical errors.

5.5 Chapter Summary

This chapter has expanded the data analysis by examining combined case study data for the major categories. While not wanting to lose any of the contextual detail of each case study, this further analysis has provided a focus on significant aspects of the three domains. For example, in the domain of social norms, rituals that afforded spaces for student-teacher interactions were significant. These spaces were often activated by students. Adopting particular roles afforded opportunities for students to contribute their mathematical thinking. Student contributions such as solution methods, questions and student register were mathematical resources that afforded joint negotiation of mathematical meaning.

In the tools domain, the significant mathematical tools were further analysed in terms of how the tools afforded or constrained mathematical activity. Significant tools afforded more mathematical resources by, for example, re-representing mathematical information, benchmarks or cues. In addition, social practices linked to opportunities to talk about tools in the context of mathematical activity afforded these tools as shared resources for mathematical activity. In the category of technological tools, particular features of the scientific calculator were identified as affording mathematical activity for students. Negotiations mediated by the scientific calculator, a technological tool, also afforded further mathematical resources to be rapidly contributed to mathematical activity.

Interactions where mathematical tools were contributed, appropriated, or adapted were significant aspects of the domain of interaction. Teacher supporting actions for joint negotiation were important for supporting volunteered student contributions such as solution methods and analogies. Student contributions afforded significant resources for peers during the development of classroom mathematical practices. Students and teachers actively appropriated, adapted or rejected other's contributions in both whole class and localised mathematical activity. Student contributions, including questions and errors, initiated opportunities for further whole class interaction. Findings about significant features of classroom activity that afford and constrain student mathematical activity are now discussed in the final chapter.

Chapter 6
Discussion and Summary

6.1 Chapter Introduction

This chapter returns to the research questions for this study and sets out the findings. The first research focus relates to classroom social norms that afforded student participation in mathematical activity. The significant classroom social norms are discussed and then related to research about social norms in emergent and traditional classrooms. The second research focus explores cognitive tools and the ways these tools, and technological tools, are used in classroom activity. Tools are identified and the roles of tools in mathematical activity are examined. The findings about tools are then discussed in relation to the literature. The next research focus examines the nature of student and teacher contributions during classroom interactions. In particular, classroom interactions during the development of classroom mathematical practices are discussed in relation to the research literature about classroom interaction processes. Findings about student initiated negotiations are also discussed. The final research focus examines the negotiation of mathematical meanings in the case study classrooms. This brings together the three domains of social norms, tools and interaction. A distributed view of number sense is then discussed in relation to the findings of this study. The concluding section of this chapter discusses the implications of the findings and outlines aspects for further research.

6.2 The Social Norms Domain

The first research focus addressed the significant classroom social norms related to student participation in mathematical activity. Classroom social norms are the expectations and obligations of the social group that become normative over time (Cobb, Gravemeijer et al, 1997; Cobb, Wood and Yackel, 1991; Yackel et al, 1991). The significant classroom social norms were found to be related to ritualised student participation in mathematical activity and the status accorded to student contributions. These findings are now discussed.

6.2.1 Ritualised Participation in Discursive Mathematical Activity

Rituals that afforded student participation were found to provide 'interactional spaces' (Weade, 1992), opportunities within lessons for students to engage with, and talk about, mathematics. Key rituals were the Starter, Marking and Activity Rituals. During the Starter Ritual in Lemonwood 1 (section 4.2.3), the students talked with peers and shared resources. Varying kinds of collaboration or assistance were expected and the teacher was obliged to allow sufficient time for this ritual. The Marking Ritual was based on expectations of volunteered student contributions of answers, solution methods or explanations (section 4.2.3). Peers were obliged to listen to and make sense of these contributions. The teacher frequently asked for an indication of the number of correct answers, and some students cheerfully identified their reasons for incorrect answers. The teacher response was usually to calmly make a suggestion for next time, an implicit message that errors could be fixed, and that the number of errors did not indicate mathematical ability.

The Activity Rituals of Lemonwood 2 and 3, and to a lesser extent in Lemonwood 1, provided interactional spaces for students to collaborate and assist in mathematical activity. The teacher also contributed assistance to individuals or groups of students. During the Activity Ritual, students regularly asked each other for assistance, borrowed materials and equipment, shared ideas, strategies or answers for tasks, and generally discussed, debated or disagreed with each other. Examples of student collaboration during the Activity Ritual were Amy, Louise and Erin and Multiplication of Decimals (section 4.3.5) and Mathew and Dean during calculator activity (section 4.2.5). The Chorus Ritual of Lemonwood 2 was a regular opportunity for some of the class to collectively contribute important signalling information to the teacher, and for the teacher to then adjust the pace of the lesson (section 4.3.3). As in Jones' research (1991), this chorus ritual was enjoyed by the students and provided some student choice in participation.

These ritualised interactional spaces were regular and implicitly understood by the social group, a finding similar to research in science classrooms (Brown et al,

1993). The Starter and Marking Rituals of Lemonwood 1 contrasted with other research that found some starter rituals were damaging to primary children's mathematical identity (Walls, 2001). Walls found that some students lost confidence in their mathematics when the starters were conducted under intense time pressure and did not include material or social assistance for individual students. The features of the Starter Ritual were, however, qualitatively different to those from Wall's research. These qualities were the expectations of collaborative activity and access to further resources. The interactional space of the Marking Ritual, activated by student contributions, also distinguished it from marking rituals in other classrooms (Walls, 2001).

The collaboration and assistance norms were related to classroom practices that afforded students' localised talk. For example, in two of the classrooms, students could sit where they wished so that social relationships were supported (sections 4.2.1 and 4.3.1). Desks were in rows or groups but close together so that localised talk was semi-public rather than public. Student groupings were flexible in order to accommodate the fluid social relationships of this age group. In the teacher interviews, the Lemonwood 1 and 2 teachers mentioned the importance of students working together (sections 4.2.1 and 4.3.1). The regular Activity Ritual in Lemonwood 3 was a significant opportunity for a socially volatile group of students to collaborate with and assist each other. Some students moved around the room or otherwise communicated across the room during this time. During lessons, each teacher implicitly supported student collaboration by intervening only to reduce excessive noise level during student activity. Consequently, the norms of student collaboration and assistance appeared to be implicit in each classroom. This was similar to the collaboration and assistance norms of some classrooms (Elbers and Streefland, 2000; Frid and Malone, 1995), but in contrast to the explicit student activity norms set out in emergent classrooms (Cobb, Wood and Yackel, 1991; Herschkowitz and Schwarz, 1999).

6.2.2 Student Contributions were Accorded High Status

Student contributions were accorded high status in all three classrooms. Student contributions during whole class activity were varied and included solution

methods, explanations, reasoning, answers and questions. The high status of student contributions afforded opportunities for students to assume different roles. The significant roles were solution provider, mathematical explainer, and question asker. Examples of solution provider were Scott's Wicked Number Pairs (section 4.2.5) and Tina's, Danny's and Nathan's Methods (section 4.4.5). Examples of the role of mathematical explainer were Ryan's alternative representation of 7.9 (section 4.2.5) and Shane's Reasoning (section 4.4.5). Melanie's Question (section 4.3.5) and Donna's Conjecture (section 4.4.5) illustrated the role of question asker. Two further roles were associated with contributing additional resources. Paul's Analogy (section 4.2.5) and Finn's Explanation (section 4.4.5) were examples of students adopting a role of analogy provider. Tom's Multiline Function and Hamish's Shortcut Key were examples of a student role of explainer of calculator functions.

Student contributions were accorded immediate status because teachers and peers acknowledged and responded to the contributions. An exception was Dean's and Ryan's Shortcut Rule which was initially delayed by the teacher (section 4.2.5). These two boys did not abandon their method but contributed it again at a later opportunity, when the teacher acknowledged and valued their contribution. On the one hand, this could be an example of the teacher funnelling student contributions. On the other hand, it could illustrate the sustained influence of a significant social norm, that student contributions were accorded high status.

The significant student roles of solution provider and mathematical explainer were similar to roles assumed by students in emergent classrooms (Cobb, Wood and Yackel, 1991; Lo et al, 1994; Yackel et al, 1991). The role of question asker was slightly different to the emergent approach, because in this study students mostly asked questions of the teacher rather than of other students in whole class interactions. The role of question asker of peers was confined to the Activity Ritual. At other times, students contributed discrete steps in a procedure or a single answer to a task. Roles of question answerer and provider of procedures, including choruser, were similar to those identified with traditional classrooms (Askew, 2001; Cobb, Wood, Yackel and McNeal, 1992; Jones, 1991).

6.2.3 Synthesis of the Social Norms Domain

Significant social norms for participation in mathematical activity were related to 'interactional spaces', as described by Weade (1992). The interactional spaces were regular and could be activated by students. The norms associated with these spaces were implicitly understood by the participants, and monitored in explicit and implicit ways by the teacher and students. These social norms afforded student contributions of mathematical resources. For example, expectations of voluntary student contributions were important, as were associated obligations that student contributions would be listened, and responded, to in some depth. A significant finding was that traditional lesson events, such as starters, student tasks and marking rituals, provided interactional spaces. These interactional spaces afforded opportunities for students to contribute social resources related to mathematical activity. Some rituals also afforded collaboration and assistance between peers, which in turn provided further opportunities for volunteered student contributions.

A significant feature of the ritualised interactional spaces was the expectation that students take on a range of active roles related to participation in mathematical activity. Significant roles were as a contributor of mathematical resources such as answers, solution methods, and mathematical explanations. These contributions were accorded high status and used by teacher and peers as resources to shape the ensuing interactions. Furthermore, student contributions were diverse in content and timing, and thus not confined to a narrow teacher prescription. Students also adopted collaborator and assistor roles in whole class and localised activity. Depending on the context of the activity, students adopted a range of roles or more than one role at a time. By expecting students to adopt these active roles, mathematical activity was distributed over greater resources than those of an individual student. Consequently, the classroom social norms influenced the other domains for this study, tools and interactions.

6.3 The Tools Domain

This section discusses the significant findings in the domain of tools. The different types of cognitive tools were identified and categorised in the previous chapter (section 5.3.1). The second part of this research focus investigated how the cognitive tools and technological tools were used in the classroom interactions. The roles of tools are now examined in terms of affordances and constraints on mathematical activity.

6.3.1 Tools in Classroom Activity

Classroom activity involved cognitive mathematical tools 'in action'. Cognitive tools were important in transforming activity in the three classrooms. Tools were found to restructure classroom mathematical activity by the general affordances recorded in Table 6.1.

Affordances of Tools	Example of Tools
Providing stable visual representations of important mathematical information or features	Number Line Think Boxes
Providing information such as labels, benchmarks and other reference points	Number Line Empty Number Line
Promoting student choice eg of representations, of reference points, of multiple processes	Number Line Empty Number Line Think Boxes
Narrowing attention or cueing to key mathematical information	Empty Number Line Marker Dot and Dash
Signalling further action	Marker Dot and Dash Think Boxes

Table 6.1 Significant Affordances of Classroom Mathematical Tools

The first affordance related to the stable and visual features of a tool, usually a representational tool, such as the Number Line and Think Boxes. For example,

the Lemonwood 1 teacher used the Number Line to demonstrate the mathematical practices for the addition and subtraction of integers (section 4.2.5). Similarly, the Lemonwood 3 teacher used Think Boxes to represent different ways of thinking about area activities (section 4.4.5). These stable and visual representational tools were significant resources for joint discursive activity, and resources for later individual activity.

The affordance of providing further mathematical information was significant. A tool such as the Number Line made mathematical information available to students. This information included the relative magnitude of numbers and benchmark numbers such as zero or ten. In the classroom mathematical practice of the Number Line Rule (section 4.2.5), students had access to this information as a resource for further mental calculations (also in sections 4.3.5 and 4.4.5). The Empty Number Line afforded dynamic reference points when the teacher drew the additional benchmark of 7.5 in response to Ryan's re-composition of 7.9 (section 4.2.5). This tool also illustrated the next affordance of promoting a choice of number representations or reference points. Tools afforded students with access to multiple ways of representing numbers and operational processes (sections 4.2.5 and 4.4.5). Student choice of representations was tentatively linked to later activity in Lemonwood 3, when Tina, Danny and Nathan's Methods for Area of Part Circle were similar processes to those represented by the Think Boxes (section 4.4.5).

The final two affordances of tools were related to scaffolded mathematical processes. Cues and other signifiers minimised distractions and revealed information to students in recognisable ways. The Marker Dot and Dash were examples of tools that narrowed the field of attention to the designated place for rounding (sections 4.2.5 and 4.3.5). In addition, these tools signalled the next step in the rounding process by cueing the students to the digit to be assessed for rounding. In these ways, the Marker tools afforded information for further use as a thinking device for the rounding process. Not only were these tools able to be used for whole and decimal numbers, but students were able to adapt and eventually abandon these tools in favour of other methods (see Alex's Calculator Dot and section 4.3.5). When students made errors in rounding tasks, the Marker

tools were invoked again by the teacher in negotiations initiated by student errors and questions (see Dean's Rounding Error and Mathew's Rounding of Nines, both from section 4.2.5).

These affordances of tools were related to research findings about tools. In the resolution of Natalie's Error, the teacher used the Empty Number Line (Beishuizen, 1993), as a 'model for' recomposing the decimal number, followed by another use as a 'model of', when comparing the magnitude of 7.2 and 7.9, the focus of Natalie's original error. This tool was an example of a 'model of/model for' approach (Gravemeijer, 1994; Gravemeijer et al, 1998), and then back to 'model of' again. A more useful analysis process for this study, however, was to identify the roles of tools as representations, thinking devices or analogous models (Greeno, 1991; Pea, 1993; Wertsch, 1991). For example, the Number Line was a representation that afforded information such as the magnitude and sequence of positive and negative integers. When the Number Line was used as a thinking device, it afforded thinking about addition and subtraction with directed numbers (Mousley, 1999). The use of the Number Line as both a representation and a thinking device was similar to the two important roles of representations identified by Greeno (1991). These were the roles of representing conceptual entities and for constructing and manipulating mental models. The affordances of tools were also related to the notion of cognitive scaffolds in classroom activity (Salomon and Perkins, 1998). Not all tools, however, afforded mathematical activity. While analogous models are resources that connect mathematical entities with real life situations (Ministry of Education, 1992), features embedded in analogous models were found to constrain mathematical meaning for some students. For example, in contrast to the findings of Mousley (1999), the analogous model of an overdraft appeared to be confusing for some students of Lemonwood 1.

In all case studies, student contributions of cognitive tools were related to the classroom social norms. The high status of student contributions meant that student tools had equivalent status to teacher's tools, exemplified by the appropriation of student mathematical tools by the Lemonwood 1 and 3 teachers (section 5.4.2). Students also appropriated and adapted the teacher's tools, such

as in Lemonwood 2, when students appropriated, adapted or rejected the Marker Dot tool for rounding (section 4.3.5). The Lemonwood 3 teacher explicitly encouraged the students to use a diversity of tools, and to adapt, as well as appropriate, teacher tools (section 4.4.3). These findings suggest that the domains of social norms and tools were reflexively related (Cobb, Gravemeijer et al, 1997). When teachers appropriated student tools, they implicitly valued these contributions, which encouraged further student contributions of mathematical tools.

6.3.2 Technological Tools

The adapted calculator roles from the Calculator in Primary Mathematics project were useful for identifying roles of the scientific calculator in mathematical activity at Years 9 and 10 (Groves and Cheeseman, 1995). The calculator role of a recording device or scratchpad was still important for these secondary school students. The role of the calculator as a recording device afforded an opportunity for Stephanie in Lemonwood 2 to try out different ways of recording 1.0 (section 4.3.5). This activity prompted her to ask a key question of the teacher, and afforded an opportunity to clarify the equivalence of different number representations. As a calculating machine, the speed of response afforded a rapid re-trial of calculator processes for students. In Lemonwood 1 in particular, these features afforded autonomous use of the calculator as an exploratory object (section 4.2.5). Calculator processes were changed quickly, errors were transitory, and results were repeatable.

Calculators were particularly significant for restructuring activity by making further resources available to students, such as multiple ways of representing numbers (section 5.3.3). These new representations promoted further student interaction (sections 4.2.5 and 4.3.5). Different brands of calculator also afforded alternative calculation processes for students (section 4.2.5). In turn, these processes afforded information about patterns and properties of the number system (Greeno, 1991). By using the calculator to generate powers and decimal calculations, localised student activity was distributed between the calculator tool and the mental representations of the calculation results (sections 4.2.5 and

4.3.5). Information provided by the calculator was interpreted by the student operator and further processes initiated. The calculator-student relationship was found to be generative and illustrated the reciprocating tool-operator feedback model of Salomon (1993).

Another finding was that calculator results afforded mathematical conversations (sections 4.2.5 and 4.3.5). In some cases, these conversations were about specific aspects of the domain of number, such as ways to represent numbers and reasons for these representations. The classroom social norms of collaboration and assistance influenced the student activity with the scientific calculator. Students showed each other their screens and, although their conversations appeared secondary to the calculator activity, there were examples of students finishing each other's sentences in paired dialogue. By efficiently producing results of calculations, students were freed to talk about and use these as resources for further mathematical activity (Wheatley and Shumway, 1992).

6.3.3 Synthesis of the Tools Domain

The significant findings in the domain of tools were how tools afforded classroom mathematical activity. Important affordances of mathematical tools were a tool's capacity to make further mathematical information available to the students. When tools re-presented or re-organised mathematical information, further mathematical resources were available to students. This information needed to be readily accessible to the student, that is, recognisable and adaptable for different contexts. Information such as conceptual structure, benchmarks, cues and other signifiers afforded resources for further mathematical activity for students. These affordances of tools were significant for critical incidents such as the resolution of student errors or repair situations.

Some of the significant affordances were identified in the special case of a tool, the scientific calculator. In the roles of recording device and calculating machine, the calculator rapidly and efficiently presented mathematical information to students. This information afforded a resource for further mathematical activity. Different brands of calculator and a range of calculator processes afforded a

diversity of resources including shortcut methods for students. These affordances of the calculator were significant in its role as an exploratory object, where students experimented with calculator processes. Although the calculator did not afford stable or shared visual representations of resources, students compensated by talking about the processes and showing each other their calculator screens. In this way, the technological tool was similar to cognitive tools in affording discursive resources for classroom discussion.

6.4 The Interaction Domain

The third research focus explored the interaction domain; in particular, the nature of student and teacher contributions during classroom interactions. The nature of teacher contributions related to the whole class negotiation of classroom mathematical practices, and to the teacher responses to unanticipated contributions from the students. Student contributions are discussed in more detail in the second section, using examples from the case study classrooms. Student contributions within the constraints of teacher-initiated whole class interactions are discussed, followed by the significant student contributions that initiated shifts in the discourse. Examples of significant student initiated negotiations are examined as examples of serendipitous mathematical interactions. Finally, the relationship between student contributions and the classroom social norms is briefly discussed.

6.4.1 Teacher Contributions

There were a number of findings about the nature of teacher contributions. These findings relate to teacher scaffolding actions that supported student contributions to the classroom dialogue. The IRE pattern of question and response was observed in each classroom and all were initiated by the teacher (sections 4.2.5, 4.3.5 and 4.4.5). These IRE sequences were usually in particular phases of the lesson, such as a review or re-visit of a procedure, where each teacher question related to a step of a procedure. Often questions were asked in the form of completing the teacher's sentence (section 4.2.5). The student responses were usually volunteered and sometimes the student responses were chorused to

indicate a collective response (sections 4.3.2 and 4.3.5). IRE was found to be a familiar and recognisable pattern for whole class interaction and similar to those found in other classrooms (Cazden, 1988; Edwards and Mercer, 1987; Mason, 2002; Mehan, 1979). It was not, however, a dominant pattern of teacher-student interaction in each classroom, as IRE was often interwoven with extended teacher support of student contributions.

Teacher support of student contributions was a significant finding. A common scaffolding action by teachers was to act in the role of an interlocutor during a whole class student explanation (sections 4.2.5 and 4.4.5). Slowing down the contribution into smaller steps, rephrasing, or repeating for emphasis served to provide more time for peers to engage with the mathematical ideas. Scott's Wicked Number Pairs (section 4.2.5), Melanie's Question (section 4.3.5), and Donna's Conjecture (section 4.4.5) were examples of scaffolding using teacher interlocution. The teacher sometimes added in further resources such as signifiers and labels, affording more mathematical tools and also providing time for students to articulate their ideas (see Mathew's Rounding of Nines, section 4.2.5). This implicitly indicated the high status of a student contribution, and afforded greater opportunity and accessibility for other students to engage with peer ideas. These teacher scaffolding actions were similar to findings in many classrooms, not just mathematics classrooms (Edwards and Mercer, 1987; Mercer, 1995, 2002a, 2002b).

Another significant teacher role was as a 'translator' and connector between student register and mathematical register. This was often implicit and subtle, progressively introducing the more formal mathematical terms without hindering the interactive progress (Cooper and McIntyre, 1996). Examples of this shift towards more mathematical register were Scott's Wicked Number Pairs (section 4.2.5), Melanie's Question (section 4.3.5), and Donna's Conjecture (section 4.4.5). In addition, the teacher implicitly scaffolded student register by accepting and fostering in-house language. For example, the development of the Whoops Rule included a signifier that cued students to further mathematical activity. This signifier had a public currency in at least three lessons in Lemonwood 1. Teachers also appropriated and adapted student resources such as analogies,

which were then contributed to later interactions (sections 4.2.5 and 4.4.5). When whole class interactions included student contributions, teachers accepted and supported brief phrases, incomplete ideas and tentative contributions. Teacher shaping actions were significant interactive processes that afforded focus for students, and afforded further mathematical resources. In other words, while the teacher usually initiated a mathematical conversation, the conversation became a joint reciprocating interaction rather than an expositional process (Edwards and Westgate, 1987; Mercer, 1995, 2002a). These jointly constituted interactions were similar to the whole class situations described by Chazan and Ball (1999).

6.4.2 Student Contributions

The findings indicated that students were not passive but active participants in the shaping of whole class interactions. Students contributed mathematical resources, such as solution methods, questions, stories, and analogies, to joint mathematical interactions. Student contributions also included student register as an accessible resource for peers (Zazkis, 2000). 'Talking to yourself' was considered a legitimate contribution in one classroom (Timu, during Mathew's Rounding of Nines, section 4.2.5). Peers and teachers sought out, engaged with, appropriated and supported student contributions, both in whole class and localised activity. This was similar to the findings of Chazan and Ball (1999).

A significant finding was the importance of the unanticipated serendipitous contributions from students. A useful metaphor is a sporting one, where 'first phase' play begins with a 'set' movement such as teacher exposition. An unanticipated event such as an 'intercept', (an unexpected student contribution), can lead to a shift in 'play'. Consequently, 'second phase play' emerged from these shifts in the discourse. The second phase interaction included whole class discussion of conflicting ideas and involved diverse ways of resolving mathematical dilemmas, such as student errors or conceptual conflicts. The 'second phase' aspects of the whole class interactions were the Student Initiated, Error Initiated Negotiation and Repair Interaction episodes. These initiations shifted the discourse into unanticipated (by the teacher) areas of mathematical inquiry. The student initiated interactions provided opportunities for joint

negotiation around specific or critical areas such as student use of tools or errors. Examples of further negotiations were Natalie's Error (section 4.2.5), Mathew's Rounding of Nines (section 4.2.5), Melanie's Question (section 4.3.5), and Donna's Conjecture (section 4.4.5). These episodes were significant for the more fine-grained negotiation of mathematical meaning. The nature of the teacher-student interactions changed both in purpose, mathematical focus and language. These negotiations could be viewed as critical mathematical incidents where mathematical ideas, and sometimes values, were either discussed, disputed, reviewed, or revisited. Further examples of student initiated negotiations were mediated by prior student calculator activity. Tom's Multiline Function and Hamish and the Shortcut Key initiated sustained peer discussions (sections 4.2.5 and 4.3.5 respectively). Jeff's Calculator Screen initiated further discussion about different representations of decimal numbers (section 4.3.5).

Another finding was that not all dialogue was controlled by the teacher and, in all classes, the students were observed to actively shift the discourse while the teacher accepted and responded to these shifts. Students shaped the process of interaction by either sustaining conversations or resisting efforts by others to shift the discourse. For example, Donna sustained a nested conversation with the Lemonwood 3 teacher (section 4.4.5). Mathew resisted a possible teacher move by continuing a further conversation about rounding (section 4.2.5). Students also assisted each other, by adding in ideas, rephrasing others' mathematical contributions, completing peer sentences, interjecting, or drawing the teacher's attention to something. Mathew's Rounding of Nines was an example of collaboration. Grace's Repair and Ryan's contribution to Natalie's Error were examples of assistance. In these ways, student contributions afforded more discursive resources and a greater variety of mathematical resources. These resources then afforded appropriation or adaptation by peers.

Another significant finding of this study was the discourse of primary school mathematics in Lemonwood 1. This discourse provided an implicit interactional space for student contributions, activated by explicit and implicit cues. For example, Natalie's Error negotiation began with Natalie saying "I got told ...", before proceeding with her reasoning about the magnitude of decimals (section

4.2.5). The primary mathematics discourse involved students calling on the past, as a safe or neutral means of revoicing their prior knowledges and experiences without having to necessarily own these mathematical ideas or to justify them. It was also a means of cueing the teacher to recognise their past mathematical experiences and to validate their reasoning. As reported in section 4.2.5, this primary school discourse was mutually constituted and actively supported by the teacher.

6.4.3 Synthesis of the Interaction Domain

Significant findings of the interaction domain were based around the nature of the teacher and the student contributions, and the jointly constructed nature of classroom interactions. Teacher initiated interactions were found to include both IRE patterns of traditional classrooms and the more reciprocating patterns of emergent classrooms. Teachers shaped interactions by actively contributing resources such as tools and mathematical register. Teachers scaffolded student contributions of mathematical resources. Teachers signalled what was valued in the classroom context, by repeating or rephrasing student contributions, or appropriating and adapting volunteered student contributions such as analogies. Teachers also responded to student initiations such as explicit questions, errors, and other implicit signals. In these ways, teacher contributions shaped whole class and localised interactions.

Students were also active in shaping classroom interactions. Student contributions of mathematical resources included mathematical solutions, explanations and analogies. Students used informal language, incomplete sentences and informal indicators of understanding. Students explicitly activated interactional spaces by asking questions or repairing peer contributions. Student errors were implicit means of activating interactional spaces. By activating an interactional space, second phase interactions were jointly initiated. The second phase interactions were particularly significant because these enabled students to shape the negotiation in specific aspects of mathematics. The primary mathematics discourse was an example of a safe indicator for assistance that also activated interactional spaces in the context of Lemonwood 1.

These significant findings of the interaction domain were related to the classroom social norms, particularly the features of interactional spaces in each classroom. Student and teacher contributions were social resources with high status and worthy of response. Contributions included mathematical tools indicating that the tools and interaction domains were related. The interrelationship between the three domains of this study is now discussed.

6.5 Negotiation of Mathematical Meaning

This section draws together the three domains of social norms, tools and interactions. Findings from the three domains are now discussed in terms of the final research focus, the negotiation of mathematical meaning. Specific negotiation examples are further discussed by outlining micro actions and subsequent affordances or constraints on negotiation. These examples illustrate how negotiation was distributed across three domains of classroom learning. Negotiations in the domain of number sense are then discussed in terms of the adapted environment metaphor.

6.5.1 Negotiation as Distributed

Negotiation has been described as constructing common ground (Greeno, 1991; Neyland, 1998) or resolving uncertainty (Clarke and Helme, 1997). When Clarke and Helme (1997) described negotiation of mathematical meaning as the resolution of uncertainty, their focus was on peer contributions during student activity. This study has found many examples of resolution of uncertainty in whole class interactions. In Lemonwood 1, Mathew's Rounding of Nines was a student initiated negotiation to resolve uncertainties associated with the special case of rounding nines (section 4.2.5). In Lemonwood 2, Melanie's Question was another student contribution that initiated a resolution of uncertainty about the processes for adding and subtracting integers (section 4.3.5). And in Lemonwood 3, Donna's Conjecture initiated a nested negotiation between teacher and student in order to resolve uncertainty about the multiplication of integers (section 4.4.5). There were also examples of implicit resolutions of uncertainty. Natalie's Error

in Lemonwood 1 was initially phrased as an explanation of her reasoning. This activated a jointly constructed interaction to resolve uncertainty about the magnitude of decimal numbers. The Lemonwood 1 primary mathematics discourse was another implicit resolution of uncertainty (section 4.2.3). These examples of resolution of uncertainty were all initiated by student questions or errors. The student contributions initiated subsequent 'second phase' negotiations about specific aspects of mathematical uncertainty.

In all of the above examples, classroom social norms provided opportunities for students to both initiate and participate in mathematical negotiation. By activating interactional spaces, students asked questions that initiated significant negotiations. This was not only during whole class interactions. In Lemonwood 2, important interactional spaces occurred during the Activity Ritual. It was during this Activity Ritual that Stephanie resolved her uncertainty about 1 and 1.0 in a very brief interaction between student and teacher. Louise and Short Division involved the student and researcher in resolving uncertainty about remainders. The resolution of uncertainty about the multiplication of decimals also occurred during the Activity Ritual (Amy, Erin and Louise). The subsequent discussion was a protracted negotiation between these three students, with additional peer assistance from Shaun and Alex (all from section 4.3.5).

Negotiation involves "the offering of new information, reference to past experiences, requests for information, and test or 'checks' on the validity of interpretations offered" (Edwards and Mercer, 1987, p. 6). The negotiation episodes illustrated these aspects. Both teachers and students offered new information to interactions. This new information was usually in the form of mathematical tools that were then used as resources for further interaction. References to past experiences included the discourse of primary school mathematics in Lemonwood 1, the use of the fraction proof in Lemonwood 2 (section 4.3.5), and the regular signalling ("you've been doing this for years") and inclusion of stories by the Lemonwood 3 teacher (sections 4.4.3 and 4.4.5). Direct requests for information, such as direct or implicit questions, often initiated negotiations. Validity checks on mathematical meanings were also offered. When Mathew asked about a particular case of rounding, he had

implicitly initiated a validity check for the rounding of nines. In Donna's Conjecture, the teacher opened up an interactional space for her to check her interpretation for the multiplication of integers. In both cases, the whole class nature of these checks on validity ensured a jointly negotiated check rather than an individual justification. These more fine-grained aspects of resolving uncertainty were also related to particular discursive practices, such as checking and repairing (Greeno, 1991), and matching or fitting (von Glasersfeld, 1990).

6.5.2 Examples of Distributed Negotiation

A significant finding was the interrelationship between the three domains of this study: classroom social norms, tools, and interaction. Two examples of joint negotiation of mathematical meaning are now discussed as illustrations of these relationships. The first example is a whole class negotiation of a classroom mathematical practice, the Whoops Rule, (section 4.2.5). Specific discursive actions are set out sequentially and each action is associated with an affordance or constraint. The sequence of actions and associated affordances are recorded below in Table 6.2.

Discursive Action	Affordance/Constraint
Teacher demonstrates the Number Line Rule	Affords Number Line as thinking device and representational tool.
Teacher spontaneously uses 'whoops' in the demonstration of the next classroom mathematical practice, the Whoops Rule	Whoops as a signifier, affords cue to next step. Signifier appropriated by students (value indicated by widespread student use).
Paul contributes analogy of hitting a wall	Analogy as a tool affords an analogous model for the operation on integers (student contribution valued).
Teacher appropriates student model, adapted to a crash analogy	Crash analogy affords another tool as an analogous model.
Further joint discussion of the analogy (jointly constructed)	Opportunities for student appropriation, adaptation or rejection of the analogy. Analogy constrains at least one student.
Peer discussion, self resolved by one student	Shared meaning among peers.
Teacher uses analogy with Katrina.	Affords a pedagogical tool for teacher

Table 6.2 Negotiation of the Whoops Rule (Lemonwood 1)

The first action of the teacher was the contribution of a whole class demonstration of the Number Line Rule. This action within the interaction domain afforded use of a significant tool, the Number Line. Consequently, the joint activity was distributed over the interaction and the tools domain. The Number Line tool afforded further interaction, as it was then used as a resource in the demonstration of another mathematical practice, the Whoops Rule. During this demonstration, mediated by the tool as a thinking device, the teacher contributed a social resource, "whoops". This resource cued students to an important step in the classroom mathematical practice and was enthusiastically appropriated by many students as a signifier. Mathematical activity was again distributed over the interaction and tools domain. The next action brings in the social norms domain when Paul volunteered his analogy of hitting a wall. Two significant classroom social norms were invoked by the classroom participants. The first was that a student was a contributor and, secondly, that this volunteered contribution was accorded high status. By jointly invoking these social norms, Paul's Analogy afforded a resource for the subsequent negotiation. This resource was a tool of an analogous model for the Whoops Rule. The tool (analogy), and signifier ("whoops"), afforded resources for further whole class interaction. The norms of peer assistance afforded a localised peer conversation about the teacher's adapted crash analogy. The ensuing negotiation about the viability of the analogy was distributed over the norms, tools and interaction domains. This example illustrates how the classroom participants activated the three domains at different times during the joint negotiation of mathematical meaning. This negotiation also illustrated the classroom participants finding common ground (Greeno, 1991; Neyland, 1998). Part of the negotiation focussed on establishing common ground for the meaning of the initial analogy (hitting a wall), and the adapted analogy (a crash). Finding common ground appeared to depend on whether the shared meaning of the analogy was a useful mental model for the process of changing direction along the Number Line.

The second example is also from Lemonwood 1 but involved a negotiation initiated by a student error. The negotiation of Natalie's Error is outlined in Table 6.3 below.

Discursive Action	Affordance/Constraint
Marking Ritual reviews the solution to the 'Order the decimals' task.	Public visual record of the order of decimals, uncertainty for Natalie in the order of magnitude of 7.2 and 7.9
Natalie invokes primary school discourse, implicit question in the form of Natalie's reasoning	Student statement accorded high status, Natalie's reasoning affords resources to teachers and peers
Teacher draws a chocolate bar on whiteboard	Diagram affords visual representation of relationship between decimal places (constrains further interaction)
Ryan contributes explanation	Affords tool of 7.9 recomposed as 7.5 plus 0.4 (another representation of 7.9) Affords a resource for teacher appropriation
Teacher draws Empty Number Line, with benchmarks of 7.0 and 8.0, midpoint of 7.5	Empty Number Line affords a visual re-presentation of 7.9 recomposed as 7.5 and 0.4
Ryan continues verbal explanation of recomposing of 7.9	Affords visual representation of 7.9 as close to 8 and 7.2 as close to 7 (Empty Number Line affords benchmarks of 7.0 and 8.0)

Table 6.3 Negotiation of Natalie's Error (Lemonwood 1)

This particular negotiation has been discussed previously as an example of multiple tool use that transformed the activity of comparing decimal numbers. When the negotiation is considered as distributed over three domains, the interrelationship between domains becomes apparent. Prior to this negotiation, the significant social norm of the Starter Ritual afforded a ritualised opportunity for Natalie to participate in mathematical activity. Natalie's performance in the task initiated an uncertainty about the magnitude of decimals (from the Starter Ritual). Natalie's performance also activated an implicit norm that in Lemonwood 1 it was acceptable to get it wrong (the answer to a task), and that students were expected to ask others for assistance. This norm involved the obligation that the teacher or peers would assist the student. The significant social norm of the Marking Ritual afforded an interactional space in Lemonwood 1. Natalie activated this interactional space by phrasing an implicit question in the form of reasoning within the primary mathematics discourse ("I got told ..."). By invoking a number of the significant social norms of the Lemonwood 1 context, Natalie has initiated this particular negotiation.

Natalie's reasoning about the magnitude of decimal numbers afforded resources in the interaction domain. Teacher and peers could respond to her contribution. The teacher responded with a representation of a Chocolate Bar Drawing as a model of decimal relationships. The use of this tool was inferred as a constraint because there were no responses from any of the students. Instead, Ryan invoked significant classroom social norms: the norms of student assistance, of a volunteered contribution, and of high status accorded to a student contribution. Ryan's contribution of 7.9 recomposed into 7.5 and 0.4 afforded a cognitive tool as a resource. This contribution was distributed over both the interaction and tools domains. In addition, the tool activated further resources from the teacher. The teacher contributed the Empty Number Line as a means of representing Ryan's contribution in a more accessible way for students. The teacher talked about how the tool of the Empty Number Line afforded other tools such as the benchmarks of 7.0, 8.0 and 7.5. Joint mathematical activity was again distributed between the interaction and tools domains. These multiple tools were used to visually represent the magnitude of two decimal numbers, and supported the resolution of Natalie's original error. The error-initiated 'second phase' episode was another example of a negotiation distributed over the domains of classroom social norms, tools and interactions. This example also illustrates that activating a resource in the domain of tools does not necessarily afford further negotiation. When the Chocolate Bar Drawing was contributed, for example, there was no further interaction about this tool. It was not until another classroom participant contributed further resources that mathematical negotiation continued.

6.5.3 Number Sense Negotiations

The final research question related to number sense as distributed activity. This section discusses significant findings in terms of Greeno's (1991) environment metaphor for number sense (section 2.7.3). The significant findings were the roles of participants, the importance of social interaction, the range of student-initiated negotiations, and the role of mathematical resources in mathematical activity. These findings are also related to the adapted metaphor that encompassed a distributed perspective.

The findings of the significant roles of students and teachers were closely related to the environment metaphor. In the metaphor, the teacher was conceptualised as a knowledgeable person who guides students around the domain of number. All three teachers were found to adopt the role of a guide. Firstly, teachers provided invitations and encouragement for students. They contributed explicit and implicit signals to students about what was familiar, easy, useful and valued. Many of these examples have been discussed in Teacher Mathematical Values (sections 4.2.3, 4.3.3 and 4.4.3). As guides, teachers responded to questions from students and often sustained prolonged interactions in response to these questions. Mathew's Rounding of Nines, Melanie's Question and Donna's Conjecture (sections 4.2.5, 4.3.5 and 4.4.5 respectively), were examples of teachers guiding the discourse about specific aspects of mathematics. In this way, teachers enabled more time to be spent at a particular site in the environment. Similarly to guided situations, when responding to questions from one individual, some of the class were willing participants, while others chose not to participate. Students were also active in finding their way around the environment. They asked questions, assisted their peers, and sometimes their teachers. Students signalled information to teachers by repeating signifiers (whoops), chorusing, and activating interactional spaces. The significant social norms of this study were implicated in number sense activity, that is, in jointly navigating the environment of the domain of number.

Whole class negotiations usually began with direct instruction from the teacher. The purpose of the direct instruction episodes was inferred to 'fore-ground' opportunities for the students to focus on specific mathematical content. More substantial student contributions were then included, as a result of, for example, the classroom social norms of Lemonwood 1 (section 4.2.3), or the explicit invitation of the Lemonwood 3 teacher (section 4.4.3). These student contributions were explanations, reasoning, descriptions, analogies, and solution methods. By engaging in conversations about conceptual entities, classroom participants constructed meanings for "terms in the conversation" (Greeno, 1991, p. 198). Within the environment metaphor, these contributions included descriptions of interesting properties of numbers and asking questions of others (section 2.7.4). Consequently, the social resources of whole class and localised

negotiations were important resources within the number sense environment (Greeno, 1991).

Student Initiated Negotiations were significant opportunities for students to focus on resolving particular dilemmas. These unanticipated 'second phase' negotiations provided rich opportunities for finding out more about mathematical resources. Within the metaphor, this was similar to stopping at one place in the environment and finding out more about mathematical resources. By exploring resources such as representational models within the context of an activity, students engaged with how numbers were represented and the properties of numbers in various situations. These more focussed negotiations provided opportunities for students to explore how particular resources afforded or constrained mathematical activity.

Greeno emphasised the importance of representations and the relationship with mental models used in mathematical activity (Greeno, 1991). Examples of this relationship were the use of the Number Line, the Rounding Dot and Dash, and the Think Boxes (section 5.3). While teachers contributed the visual representations, there was evidence that the students appropriated or adapted these representations as mental models for use in activity (sections 4.2.5 and 4.3.5). As in Greeno's explanation of the environment metaphor, these models afforded access to further models. The notion of mathematical resources such as representations and models is now expanded to encompass the domain of tools and the adapted metaphor described in sections 2.7.4 and 2.7.5.

The adapted environment metaphor for number sense emphasised recognition of mathematical tools, knowing the capabilities and effects of the tools, choosing the best tools for the purposes, and knowing how to use the tools. The findings of the significant roles of tools, particularly mathematical cognitive tools, appear to support this adapted metaphor. Tools were used, described, explained and appropriated by classroom participants. For example, activity with cognitive tools included recognising cues and signifiers in the tools, adapting tools and trialling these within further activity. Students appeared to be responsive to feedback about tool use (Salomon, 1993). Students also appeared to be attuned to

affordances and constraints of tools (Greeno, 1991, 1994). Examples of students' attunement to tools were identified in the range of classroom activities, such as in developing a new classroom mathematical practice, in resolving a student error, and in using a technological tool (Greeno, 1994; Salomon, 1993).

6.5.4 Distributed Activity and Mathematical Authority

The previous section discussed negotiation of student uncertainties in relation to activity in a number sense environment. This section discusses how the resolutions of uncertainties were related to sources of conviction (Frid and Malone, 1995) and inferences of mathematical authority (Clarke and Helme, 1997). Issues of mathematical authority are briefly discussed and linked to previous brief descriptions of sociomathematical norms of the emergent approach (see section 2.3.2).

Access to, and use of, a range of mathematical resources, such as tools, affords mathematical activity to be distributed. Findings from this study suggest that student access to mathematical resources was significant. When students recognised, activated and used mathematical tools within activity, issues of mathematical authority and sources of conviction were implicitly invoked. Students made judgements about valued, acceptable and efficient mathematical solutions within the classroom context. There were four sources of conviction found to establish mathematical authority. These were joint authority of teacher and students, authority of peers, of a counter-example, and of technology.

Examples of joint mathematical authority were in the negotiation to resolve Natalie's Error and the negotiation of Mathew's Rounding of Nines (both section 4.2.5). In these examples, teacher and students responded with contributions of resources. Resources were visual representations and discursive resources that constituted mental models for the processes involved. Within a distributed view of activity, mathematical authority can be reconceptualised as distributed over the resources of tools and interaction. An example of peer authority was Amy, Erin and Louise and Multiplication of Decimals during the Activity Ritual (section 4.3.5). Implicit peer authority occurred within other negotiations such as the

student resolution of Paul's Analogy (section 4.2.5), and the discussion related to Tony's overdraft (section 4.2.4). While all of these episodes had some teacher input, peers contributed strategic resources. The mathematical authority of a counter-example occurred during the negotiation of Donna's Conjecture (section 4.4.5). Although the teacher contributed the counter-example, it was Donna's resolution of this counter-example that appeared to shift her thinking about the multiplication of integers. The counter-example was a significant resource within the negotiation process, and has been accorded as a source of conviction for Donna.

The final mathematical authority was a technological authority and, in all classrooms, this technology was a scientific calculator. Calculator results provided mathematical authority in both Lemonwood 1 (section 4.2.5), and Lemonwood 2 (section 4.3.5). The authority of the calculator was sometimes an unstable authority, particularly in the role as a calculating machine that produced a calculation result. In Luke's Calculator Result (section 4.2.5), an incorrectly keyed-in calculation produced a result that Luke then used to challenge the teacher's calculation. The dilemma was jointly resolved during further negotiation, shared solution attempts, and a final, possibly fruitful for more than just this one student, resolution of the dilemma.

Mathematical authority was more clearly identified during student initiated negotiations. These negotiations were found to be opportunities for negotiating the sociomathematical norms within each classroom context. In other words, these 'second phase' episodes provided more sharply defined opportunities for students to negotiate socio-mathematical norms. For example, the authority used to establish a solution was highlighted during the resolution of a student error. While teacher authority was perceived to be the dominant feature of traditional mathematics classrooms (Bauersfeld, 1988), there were few instances of teacher authority being used to resolve student dilemmas. The various means of constituting mathematical authority were related to those identified by Clarke and Helme (1997). These sources of conviction were technology, the teacher, and peers (Driver et al, 1994; Frid and Malone, 1995). Significant negotiations implicitly included examples of acceptable solutions, efficient solutions, and

what counted as justification for a solution (Cobb, Gravemeijer et al, 1997). This contrasted with the emergent approach that explicitly focussed on particular sociomathematical norms (Bowers et al, 1999; Cobb, Gravemeijer et al, 1997). The role of prior experience was also identified by Driver et al (1994) as a source of conviction. In Lemonwood 1, however, the discourse of primary mathematics was a means of signalling uncertainty and activating an interactional space. This indicated that prior school mathematics experience might be a transitory form of mathematical authority for this group of students.

6.6 Limitations of the Methodology

There were some limitations to this study in both research design and in analysis. The limitations in research design and the impact on the data are discussed. The limitations in analysis included limitations in the units of analysis and in the research focus of lower ability students.

6.6.1 Limitations in Research Design

The limitations in research design related to technical as well as human limitations. Firstly, problems with the quality of audiotaping as a precise form of data collection have been acknowledged in Chapter 3. The ambient noise on the audiotapes prevented the accurate transcription of some dialogue. This was more likely to occur during individual student activity and in some lessons there were student activities with significant gaps in the lesson transcript. Field notes were used to supplement the data for these instances and, although the record of the dialogue was noted, it was limited by the speed and accuracy of the researcher's writing. The most successful audiotaping of student activity occurred when focus students were identified and a tape recorder placed nearby. This enabled sufficient audio quality of the localised conversation and the researcher could sit in an appropriate place, unobtrusive yet within sufficient visibility to write supplementary field notes. These particular episodes tended to be after about Lesson 3 or 4 of each case study as time was needed to identify focus students and to establish a rapport with groups of students. As has been noted, in Lemonwood 3 some of the focus students moved around the classroom during the

Activity Ritual and their conversations were often not recorded. Although a response to a technical difficulty, the data from focus students may give undue emphasis to some students. For example, Dean and Ryan were attributed with the Shortcut Rule because the microphones were near these focus students. Consequently, there is a lack of data about whether other students had generated this method for the addition and subtraction of integers. Another limitation of the data was the research decision not to collect copies of the students' written work. This decision was made so that there was minimum disruption to students. The written record of the students' mathematical processes in individual activity was not included in the data. Consequently, data was limited about individual student appropriation of classroom mathematical practices.

6.6.2 Limitations in Analysis

One limitation in analysis related to the identification of suitable category units. Within the analysis dimension Classroom Interaction, there emerged some difficulties in the categorisation of episodes. For example, some episodes could have been both an error and a repair initiated negotiation because both features were identified in the discourse. Brief examples of student initiations were also embedded in substantial negotiations. I decided to allocate episodes according to the focus of the ensuing negotiation, rather than the initiation. Another limitation related to any inferences about the mathematical activity of lower ability students. An area of interest for this study was the mathematical negotiation in lower ability classes. Two of the case study classes were lower ability (Lemonwood 1 and 3), while Lemonwood 2 was a middle ability class. By adopting a situated view of learning, each classroom context was taken as unique rather than indicative of either lower or middle ability students. Consequently, findings are not associated specifically with the classroom activity of lower ability students. This issue is discussed further in the next section, Implications.

Overall, while the difficulties of using audiotape in busy classrooms have already been acknowledged, transcript data has provided useful information about classroom mathematical discourse. Large projects with multiple researchers, such as the emergent research classrooms, have generated a great deal of valuable data

over long periods of time. High technology data collection such as video cameras and radio microphones are useful for providing more data of classroom events, but are the province of large scale research teams or centrally funded research projects. Within the acknowledged limitations, this study was a useful example of how an unobtrusive small-scale research endeavour may complement larger projects.

6.7 Implications

The implications of this study relate to two key aspects of school mathematics: social practices in mathematics classrooms, and the relationship between mathematical tools and number sense. These two aspects are now discussed in terms of implications for teachers and other educators. As in any research, findings related to specific questions raise further issues for inquiry. Some further issues will be briefly discussed and areas of interest highlighted for further research.

6.7.1 Implications: Social Practices in Mathematics Classrooms.

This study has found that particular features of classroom social norms were significant affordances of student participation in mathematical activity. The classroom microculture framework of the emergent approach (Cobb, Gravemaijer et al, 1997) was useful for the analysis of social norms. An analysis of classroom social norms appears to offer an alternative perspective of ritualised lesson events. In each classroom context, different norms were associated with the same lesson event. Rather than describing lesson events as traditional or progressive, whole class or group-oriented, a focus on the features of the associated social norms offers more insights into student participation in mathematical activity. For example, the affordances and constraints of a starter quiz can be described within specific classroom contexts (see Walls, 2001). Similarly, students sitting in groups may not necessarily afford collaborative activity (Edwards and Mercer, 1987); collaborative practices are related to the norms associated with lesson events rather than the organisation of students. The classroom events identified in the study of two different school mathematics cultures (Boaler, 1997a), could

also be re-conceptualised in terms of how classroom social norms afforded or constrained student participation in mathematical activity.

A focus on the features of classroom social norms, that is, the expectations and obligations associated with lesson events, presents an alternative conceptualisation of the social practices of secondary school mathematics classrooms. In contrast to a focus on teachers controlling or managing classes, teachers and students jointly negotiate and monitor social norms in each classroom context. Teachers need to make these norms more explicit for students, such as the Lemonwood 3 teacher explanation of the purpose of the starter quiz. Explicit communication may serve to highlight learning goals rather than performance goals for students (Carr, 2001, see p. 30). This may enable teachers, in the context of each classroom situation, to more closely align academic norms with social norms (Alton-Lee, 2003).

In more recent reports of the emergent approach, some of the original classroom social norms were adapted. The adapted norms associated with student activity were that "students often work[ed] individually" and talked "with peers of their choosing as the need arises" (Cobb, 1998, p. 48). This contrasted with an original norm of student pairs reaching a consensus during the activity phase of the lesson. Another change was that whole class discussion did not always follow the small group activity. Whole class discussion occurred only when mathematically significant issues arose during the activity. The teacher then selected certain students to explain methods rather than calling for volunteers (Cobb, 1998). Consequently, the mathematical content of whole class interaction was selectively funnelled by the teacher. These new classroom social norms suggest a more focussed approach to the mathematical negotiation in emergent classrooms. This may also reflect that the later emergent projects have been with older students (Bowers et al, 1999; Cobb and McClain, 2002). The findings of this study, in particular those related to the Activity Rituals and Whole Class Negotiations, concur with these adapted classroom social norms of the emergent projects. Whole class interaction was found to be more nuanced and scaffolded than previously portrayed in traditional classrooms (Boaler and Greeno, 2000). For example, whole class interaction provided opportunities for mutual

appropriation of meanings and tools among classroom participants. Access to the mathematical resources of the teacher was also important for resolving student uncertainty about critical aspects of mathematics. Social norms that provide interactional spaces afford opportunities for students to "express their uncertainties and reveal their confusions, and to request information and explanation from others who are more knowledgeable" (Mercer, 2002a, p. 145). This suggests that teachers could review their mathematics programmes, identify when and how interactional spaces might occur within their classroom contexts, and negotiate norms associated with interactional spaces.

6.7.2 Implications: Mathematical Tools and Number Sense

The findings in the domain of tools were related to student access to mathematical resources such as tools. Access to mathematical tools arose from opportunities to engage with tools. These opportunities arose in unexpected ways with lower ability students of this age group. For example, the process of rounding decimal numbers provided further opportunities for students to engage with the signifiers of place value notation, including the critical role of zero. When viewed from a tools perspective, the place value notation for decimal numbers revealed mathematical information. The representations of numbers revealed implicit clues about the number, information that students then used in further activity. One implication is that lower ability students in Years 9 and 10 want to know about the codes and conventions of representations of numbers. In other words, students want to know about what (representations of) numbers mean and what they can do (be used for) (Greeno, 1991). This supports Pea's (1993) view that the conventions and uses of tools are usually obtuse to the novice. The roles of tools are revealed in "relation to activities of persons in the world" (Pea, 1993, p. 62).

Identifying tools as representations, thinking devices or analogous models appeared to be more useful than the 'model of/model for' approach (Gravemeijer, 1994). This study also highlighted the significance of cognitive tools that were both representations and thinking devices, for example the Number Line and the Empty Number Line. These were examples of stable cognitive tools with

affordances that meant they were used and adapted for a wide variety of activity contexts. Identifying further tools with these qualities may assist lower ability students in secondary school.

Another implication is how students become more attuned to the affordances and constraints of tools in mathematical activity. Activities that require the students to engage with the meanings and signifiers of tools, such as those of place value notation, are valuable opportunities for students to clarify mathematical meanings. For example, Natalie's Error emerged from an 'ordering the decimals' task which required her to consider representations of decimal numbers. In an informal example, Stephanie wanted to know about the meanings of different representations for one (that 1 is the same as 1.0). In terms of Greeno's (1991) metaphor of number sense, this was an example of wanting to find your way around the environment. Knowing when a particular representation is useful provided access to participation in further mathematical activity. An additional discussion about how and why 1.0 rather than 1 is useful may have provided an opportunity for Stephanie to recognise the affordances of these (and other) representations of one in different contexts. Current school tasks that involve these representations may not provide sufficient or transparent feedback about the representations (tools). Stephanie was in Lemonwood 2, the middle band class, which suggests that wanting to know about number representations is not restricted to lower band students in the first year of secondary school.

The environment metaphor for number sense was significant for understanding activity mediated by cognitive mathematical tools. When tools are considered as resources within the domain of number sense, knowing your way around an environment means knowing what resources are available, where to find them, knowing how resources are related, and recognising possible resources for use in the environment (Greeno, 1991). Uses of representations and operations and relations are also important for coming to know about how the number system works, including tools nested within others. A tools perspective may also be useful for helping adults with mathematics, particularly when 're-learning' the subject in post-school study. By considering the roles of tools in activity, the affordances and constraints of these tools in different situations could be the

focus of post-activity conversations. For example, this may be a fruitful approach for assisting beginning primary school teachers to develop both subject matter knowledge and pedagogical content knowledge.

6.7.3 Further Implications

The importance of regular student access to discursive mathematical resources has implications for the current situation in New Zealand schools. Firstly, students who are regularly absent, or constantly change schools, have fewer opportunities to become part of a classroom community. These students miss out on the establishment, negotiation and consolidation of classroom social norms. Due to the relationship of these norms with engagement in mathematical learning, this is likely to adversely affect student performance. According to Gordon (1994), students who constantly change schools are more likely to be in lower socioeconomic schools. Students from lower socioeconomic groups were one group identified as significantly underachieving in mathematics at the Year 8 level (Flockton and Crooks, 2002).

Secondly, the interactional spaces for participation in mathematical negotiation depend on both opportunity and ability to talk about mathematics. Students from a non-English speaking background (NESB) may too easily become silent spectators of the classroom activity rather than active participants in the negotiation of mathematical meanings. Mathematics is often perceived as largely symbolic and independent of context, and consequently assumed to be more accessible to NESB students than some other subjects. NESB students may experience a different and probably lesser engagement with the classroom mathematical discourse during lessons. Edwards' (2003) research in a Year 13 mathematics classroom found that the NESB students were unable to participate in conversational English and the humourous discourse of the classroom. The teacher's "use of humour and her efforts to promote verbal interactions in class added to, rather than diminished, the language problems for her NESB students" (Edwards, 2003, p. 15). Edwards also found that norms of collaboration during class time were not adopted by the NESB students. While the findings were for older and high achieving students, access to discursive mathematical resources

may be particularly crucial for NESB students of lower ability in Years 9 and 10 mathematics. In classrooms with large numbers of NESB students, the teacher may abandon interactive teaching approaches for more formal expositional ones, further limiting access to discursive mathematical resources. Rather than focus on NESB students' lack of English skills, it may be more useful to bring a distributed perspective to student access to discursive resources.

When Lerman (2000) challenged the emergent approach to consider aspects of gender, culture and socioeconomic status in the analysis of classroom learning, he highlighted an important research focus. Although this study was not specifically oriented to gender, class or ethnicity, some of the data suggests that consideration of these dimensions may have added more to the findings. For example, some aspects of classroom negotiation processes suggest gendered discourse patterns for volunteering contributions and sustaining questions. Girls requested public assistance from the teacher through questions or tentative statements such as conjectures. Boys requested public teacher assistance by making statements or asking about examples. In the Year 9 classrooms, it was the boys who contributed the student generated calculator methods. This does not necessarily suggest that girls did not explore mathematics on their calculators, but that they may have been more reticent about publicly contributing a new finding related to technology. Information about the ethnicity and class backgrounds of the students was not collected, so any statements can only be tentative. The social norms of collaboration and assistance, as well as an acceptance of students thinking out loud, were important for the case study students. These practices may support groups of students who are currently under-performing in mathematics assessments and qualifications (Clark, 1999). By encouraging norms that support social relationships, classroom practices may afford better outcomes for all students.

A further influence of the emergent projects has been the interest in an argumentation approach (Brown and Renshaw, 2000; McClain et al, 1999; Voigt, 1994). An argumentation approach is when students publicly justify their mathematical reasoning. By justifying their reasoning and by listening to the justifications of others, students were assumed to come to know mathematics.

This approach assumes that students have the discursive skills needed for argumentation. Lubienski (2000) found that while high socioeconomic students were comfortable with justifying their ideas, the low socioeconomic students in her class valued different discursive skills. I suggest that conversational processes in New Zealand do not include argumentation and for this particular age group, argumentation is too threatening for lower ability students. While argumentation is appealing because it includes discursive resources and seems more closely aligned to the activity of mathematicians, my concern is that we may be setting up another valued position of school mathematics that excludes groups of students. The primary mathematics discourse identified in the lower ability Year 9 classroom afforded individual students a neutral and safe means of contributing mathematical resources without needing to justify or defend their reasoning. The findings of this study suggest that student reasoning involved diverse cultural signifiers, including discursive tools appropriate to the classroom context. This leads me to question the appropriateness of an argumentation approach in a New Zealand context.

The findings of this study also challenge some of the key tenets of the New Zealand Numeracy Projects and the Number Framework. Firstly, the lesson structure of the Numeracy Project is based on a group teaching arrangement for a large proportion of each lesson. This means that a student has access to the teacher's mathematical resources during only one short teaching episode every two lessons. For the rest of the time, a student is engaged in individual activity, which may or may not be collaborative. The group teaching arrangement stands in direct contrast to this study's findings, which highlighted the importance of access to teacher mathematical resources during classroom activity. While group teaching has been the historical practice of teaching reading in primary schools, there is little evidence of improved outcomes for mathematics (Education Review Office, 2002). The hierarchical nature of the Number Framework also stands in contrast to the findings of this study. This study found that developing number sense was a nested and recursive process for the Year 9 and 10 students. Students did not move through stages of the Number Framework. Instead, classroom activity afforded opportunities for students to return to familiar mathematical

entities, and negotiate further mathematical meanings. This illustrated how students were actively engaged in the number sense environment.

6.7.4 Issues for Further Research

The nature of collaboration and assistance practices during mathematical activity is an area for further research. While there has been research about joint student activity (Barnes, 1999), there has been little attention to student assistance and collaboration practices. In this study, interactional spaces provided greater student access to mathematical resources. Greater access to mathematical resources was afforded by peer collaboration and assistance, eavesdropping on localised conversations, and even talking to yourself during activity. A focus on the microroles of student assistor may be useful for this age group. It seems likely that student assistance is an implicit mirroring of the teacher role of assistor, based on years of school mathematics. The teacher, however, has access to more mathematical resources and greater knowledge of affordances or constraints of tools in different contexts. While student assistors have access to fewer mathematical resources, peer assistance appears to be significant and worthy of further investigation. Development of assistance practices may also be linked to greater distributed resources during independent activity.

This study found that collaborative activity with a technological tool, the scientific calculator, afforded opportunities for joint mathematical exploration. The scientific calculator is under-researched compared to the simple or graphical calculator. It is, however, the technology at hand in most New Zealand Year 9 and 10 mathematics classrooms. The scientific calculator is a relatively simple and inexpensive piece of technology that is often used only as a calculating device. As an exploratory object and a generator of further mathematical tools, the scientific calculator may have further implications for the number-related curriculum in Years 9 and 10. Opportunities to engage with the calculator, in shared social activity, could be a focus of further research.

The roles of cognitive mathematical tools also provide issues for further research (Sfard, 2002). In particular, the reciprocating system between operator and tool

has received little attention in mathematics (Salomon, 1993). Higher ability students may be able to independently continue with a spiral of feedback between individual and tool. This may account for why a problem solving approach is more often used with high ability students. For lower ability students, access to further resources may be more critical for this feedback process. An investigation of joint activity may provide more information about tool feedback to students within the context of number activity. We need to know more about how lower ability students recognise and activate affordances of tools and about how students use and adapt tools in activity. Some recent research has provided in depth analyses of lesson interactions with inscriptional tools (Cobb, 2002; Forman and Ansell, 2002; Sfard and McClain, 2002). These lessons were with secondary school students and involved resources from computerized tools. A similar analytical approach might be useful for lessons that involve number sense activity in typical classrooms.

We need to know more about the school experiences of students, such as Janine in Lemonwood 1. How do students come to know about the affordances and constraints of mathematical tools in different activity contexts? A study of Janine's activity within the distributed resources of classroom lessons would be a valuable focus of further study. The case study of Lemonwood 3 also raises an issue for further research. It seems to me that there are many Year 9 and 10 classes in New Zealand with similarly volatile social relationships among the students. The mathematics curriculum and the available textbook constrained participation in mathematical activity for these students. An alternative curriculum that focused on student participation in activity, could be a focus of an experimental approach to contextualised classroom mathematics programmes. This might also provide information about meaningful classroom experiences for students of this age group.

A final issue for further investigation relates to the recent work on student mathematical identity and agency (Boaler, 2000; Boaler and Greeno, 2000). A focus on identity and agency places the individual learner as the unit of analysis and may be restrictive. Student mathematical identities have been found to be situated and fluid (Elbers and Streefland, 2000). Notions of agency and identity

may contribute to a richer understanding of how students of lower ability participate in mathematical activity. It might be possible to bring a distributed perspective to the notion of agency as a feature of students' relationships with social and material resources within a distributed system.

6.8 Summary

A distributed perspective has highlighted the interrelationship between various resources of joint classroom activity. The typical lesson structure of all case studies conformed to what was defined as 'traditional'. Every lesson contained at least one whole class teaching event and at least one episode of individualised student tasks set by the teacher, usually from a textbook. Students activated interactional spaces associated with these lesson events, affording participation in activity. An important feature of student participation for lower ability students was access to further mathematical resources. The affordances and constraints of resources were activated by students and teachers according to the adapted 'environment' metaphor for number sense. The negotiation of mathematical meaning was therefore distributed over more resources, such as tools and discursive resources.

Appendices

Appendix 1

Interview Questions for Teachers

1. Which maths topics have you covered so far this year?
   Which topics are planned for the rest of Term 1?

3. What would a typical maths lesson be like?
   Describe a typical maths lesson with this class.

3. What kinds of routines or expectations do you have with this class?
   (on a day-to-day basis).

4. What are the difficulties that you see some of the students in this class have with maths?

5. What are some of the things you have found useful for this class of students?

Follow-up Questions

6. What is the expectation in your classroom for students working together on mathematical activities?

7. In whole class discussion, how do the students know what is mathematically valid? And how do students know when they may need to revise their ideas?
   (what do you do when the students give a 'wrong' answer/response?)

Organisational Questions:

8. What is the approximate 'ability' level of this class? (within broadbanding)

9. What are the ages of the students in the class?

Appendix 2

Record of Lemonwood 1 Episodes

Lesson 1	Episode 1 Episode 2	Student questions to researcher Janine's comments to researcher
Lesson 2	Episode 1 Episode 2 Episode 3 Episode 4 Episode 5	Teacher comment about calculators Student question about convention for writing decimals Teacher explanation of multiplication of decimals Teacher demonstration of worked example Whole class negotiation of multiplication of decimals and Luke's Calculator Error
Lesson 3	No episodes	
Lesson 4	Episode 1 Episode 2 Episode 3 Episode 4 Episode 5 Episode 6	Student comment to researcher about lab rats Student strategies for mental calculations Teacher review of decomposition, Grace's Repair, discussion about overdrafts Marking ritual for subtraction tasks Teacher explanation of multiples Teacher explanation of lowest common multiple
Lesson 5	Episode 1 Episode 2 Episode 3 Episode 4	Teacher demonstration of rectangles with area of 12 Review of other rectangles Scott's Wicked Number Pairs Teacher explanation of divisibility
Lesson 6	Episode 1 Episode 2 Episode 3	Janine and Timu's conversation about prime factors Review of the prime factors of 875 Susan's error with divisible by 3
Lesson 7	Episode 1 Episode 2 Episode 3 Episode 4 Episode 5 Episode 6 Episode 7 Episode 8	Marking the starter examples Review of the place value tasks Natalie's Error Introduction to powers, student misconceptions Teacher demonstration of calculator powers Teacher demonstration of calculator for squares and cubes Teacher demonstration of calculator for square roots Tom's Multiline Function
Lesson 8	Episode 1 Episode 2 Episode 3 Episode 4	Ryan's error in Starter Student contributions of their ideas about negative numbers, including temperature Review of greater than/less than signs Student questions about greater than/less than signs

Lesson 9	Episode 1	Subtraction of integers using lift of a building
	Episode 2	Student error subtraction of integers, context of temperature
	Episode 3	Use of Number Line for subtraction
	Episode 4	Demonstration of the Number Line Rule
	Fire alarm	
	Episode 5	Teacher introduction of whoops
	Episode 6	Teacher demonstration of the Whoops Rule
	Episode 7	Paul's Analogy, student discussion
	Episode 8	Joint review of Whoops Rule
	Episode 9	Janine's question, Dean and Ryan - Shortcut Rule
Lesson 10	Episode 1	Demonstration of BEDMAS
	Episode 2	Review of example from Starter
	Episode 3	Example from Starter, use of calculator, student errors
	Episode 4	Calculator and Janine's error
	Episode 5	Two calculator examples of BEDMAS
	Episode 6	Teacher rounding of decimals
	Episode 7	Teacher review of multiplication by tens, etc
	Episode 8	Teacher review of integers
	Episode 9	Review of practice integer tasks, teacher intro to Shortcut Rule
	Episode 10	Discussion of Shortcut Rule
Lesson 11	Episode 1	Review of word problems from Starter
	Episode 2	Janine, "How come they are called rules"?
	Episode 3	Review of square and square root, discussion about square root of 5
	Episode 4	Teacher assisting Janine with calculator
	Episode 5	Review of cube root, student discussion using calculators
	Episode 6	Review of Dash method for Rounding
	Episode 7	Further examples of rounding
	Episode 8	Mathew and the Rounding of Nines
	Episode 9	Teacher comments about the common test
	Episode 10	Student questions of teacher about Whoops rule
	Episode 11	Review of practice of Shortcut Rule
	Episode 12	Teacher and Katrina discuss Whoops Rule
	Episode 13	Teacher rules for the multiplication of integers
	Episode 14	Mathew's resistance to rules for multiplying and dividing integers
Lesson 12	Episode 1	Clarification of convention for multiples
	Episode 2	Clarification of lowest common multiple
	Episode 3	Difference between rounding and significant figures
	Episode 4	Clarification of estimation cue
	Episode 5	Integer errors from Nick
	Episode 6	Teacher mathematical value about learning from mistakes
	Episode 7	Student questions about methods for division
	Episode 8	Use of calculators for calculations with integers
	Episode 9	Janine's error

Appendix 3

Record of Lemonwood 2 Episodes

Lesson 1	Episode 1	Whole class review of integer rule using number line, Melanie's Question
	Episode 2	Review of integer worksheet
	Episode 3	IRE review of integer context - overdraft
	Episode 4	Teacher overview of different types of numbers
Lesson 2	Episode 1	Teacher guidance for using textbooks
	Episode 2	Teacher demonstration of using calculator for integer calculations
	Episode 3	Teacher comment about calculators and brains
	Episode 4	Student question about calculators
	Episode 5	Calculators, BEDMAS and brackets
	Episode 6	Hamish and the Shortcut Key
	Episode 7	Teacher Mathematical Values
Lesson 3	Episode 1	Stephanie's question about 1 and 1.0
	Episode 2	Teacher drawing chocolate bar as model for decimals
	Episode 3	Teacher decimal chart and review of measurement units
	Episode 4	Bingo game, multiple representations of decimals, fractions and percentages
	Episode 5	Comparison of two decimals, 0.39 and 0.4
	Episode 6	Monique and Decimal Magnitude
Lesson 4	Episode 1	Contexts for rounding
	Episode 2	Teacher introduction of Dot Method for Rounding
	Episode 3	Worked example of Rounding, Kirsty
	Episode 4	Whole class review of rounding examples, first mention of speedometer
	Episode 5	Dot Method of Rounding for specified decimal places
Lesson 5	Episode 1	Review of Dot Method for Rounding, speedometer mentioned again
	Episode 2	Alex's Calculator Dot
	Episode 3	Different student strategies for rounding
	Episode 4	Using money as a context for significant figures zero as a place holder
	Episode 5	Example of concert as mathematical context for rounding
	Episode 6	Convention for approximation, 'round up'

Lesson 6	Episode 1	Review of multiplication of decimals, teacher use of fraction proof
	Episode 2	Student resistance to multiplication of decimals, Amy, Erin and Louise.
	Episode 3	Shaun's solution for multiplication of decimals
	Episode 4	Student solutions and questions, Simone's Error
	Episode 5	Calculator example, student conversations, Jeff's calculator screen
	Episode 6	Teacher demonstration of division of decimal
	Episode 7	More division, localised and public student methods
	Episode 8	Division of decimals, teacher use of fractions to demonstrate
	Episode 9	Student error for division – Louise and Short Division
Lesson 7	No episodes	Lessons on relations and graphs
Lesson 8	Episode 1	Teacher comments re senior school mathematics
	Episode 2	Shaun's bug in multiplication algorithm
	Episode 3	Teacher comments about revision and textbooks
Lesson 9	No episodes	Lessons on relations and graphs

Appendix 4

Record of Lemonwood 3 Episodes

Lesson 1	Episode 1	Teacher affirming student methods
	Episode 2	Student explanations of methods of estimation
	Episode 3	Finn's articulation of BEDMAS
Lesson 2	Episode 1	Teacher using ruler for addition of negative nos.
	Episode 2	Contexts for integers, temperature and overdrafts
	Episode 3	Organisation of tasks and use of textbooks
	Episode 4	Constraints on language of textbook tasks
	Episode 5	Strategy for integers, changing signs
	Episode 6	Teacher guidance and review of integer calculations
	Episode 7	Teacher's language example as analogous model
Lesson 3	Episode 1	Shane's Reasoning and Finn's explanation
	Episode 2	Teacher explanation of mathematics as abstract
	Episode 3	Donna's Conjecture about multiplication of integers
Lesson 4	Episode 1	Teacher review of measurement units
	Episode 2	Teacher story of school maths and real life maths
	Episode 3	Teacher story continued
	Episode 4	Fantail interruption
	Episode 5	Student strategies for shortest distance
	Episode 6	Rounding in context of measurement of distances, Teacher stories – metric and USA, Mars spacecraft
Lesson 5	Episode 1	Teacher discusses strategies for finding area
	Episode 2	Teacher use of Think Boxes
	Episode 3	Teacher validation of multiple student methods, student questions about specific area examples
	Episode 4	Teacher discussion of student unlearning
	Episode 5	More about Think Boxes
	Episode 6	Further strategies for thinking about area of shapes, Teacher suggestions about use of textbook
Lesson 6	Episode 1	Teacher setting up measurement activity
	Episode 2	Teacher demonstration of measurement activity
	Episode 3	Student methods for finding pi from diameter measurements
	Episode 4	Teacher articulation of pi, average and use of calculator
Lesson 7	Episode 1	Teacher analogy for adding fractions
	Episode 2	Teacher rationale for Starter Ritual
	Episode 3	Danny's explanations for using pi to calculate area, further examples of area of circle
	Episode 4	Student strategies for 3/4 of circle, Tina, Danny and Nathan's Methods
	Episode 5	Teacher excitement at different student methods
	Episode 6	Teacher talking about multiple methods

References

Abreu, G. de (2000). Relationships between macro and micro socio-cultural contexts: Implications for the study of interactions in the mathematics classroom. *Educational Studies in Mathematics, 41*(1), 1-29.

Adler, J. (1998). A language of teaching dilemmas: Unlocking the complex multilingual secondary mathematics classroom. *For the Learning of Mathematics, 18*(1), 24-33.

Adler, J. (1999). The dilemma of transparency: Seeing and seeing through talk in the mathematics classroom. *Journal for Research in Mathematics Education, 30*(1), 47-64.

Alangui, W., Autagavaia, J., Barton, B., Kensington-Miller, B., Lane, J., Paterson, J., Poleki, A. & Van Den Heuvel, A. (2002). The Mathematics Enhancement Project: The pilot phase. In B. Barton, K. C. Irwin, M. Pfannkuch & M. O. J. Thomas (Eds.), *Mathematics Education in the South Pacific* (Proceedings of the 25th annual conference of the Mathematics Education Research Group of Australasia, Auckland, pp. 83-90). Sydney: MERGA.

Alro, H. & Skovsmose, O. (1998). That was not the intention! Communication in mathematics education. *For the Learning of Mathematics, 18*(2), 42-51.

Alton-Lee, A. (2003). *Quality teaching for diverse students in schooling: Best evidence synthesis.* Wellington: Ministry of Education.

Alton-Lee, A. & Nuthall, G. (1992). Children's learning in classrooms: Challenges in developing a methodology to explain "opportunity to learn". *Journal of Classroom Interaction, 27*(2), 1-7.

Anderson, J. R., Reder, L. M. & Simon, H. A. (1996). Situated learning and education. *Educational Researcher, 25*(4), 5-11.

Anderson, J. R., Reder, L. M. & Simon, H. A. (1997). Situative versus cognitive perspectives: Form versus substance. *Educational Researcher, 26*(1), 18-21.

Anthony, G. (1994). Learning strategies in the mathematics classroom: What can we learn from stimulated recall interviews. *New Zealand Journal of Educational Studies, 29*(2), 127-140.

Anthony, G. (1996a). Active learning in a constructivist framework. *Educational Studies in Mathematics, 31*, 349-369.

Anthony, G. (1996b). When students fail to use appropriate learning strategies. *Mathematics Education Research Journal, 8*(1), 23-27.

Anthony, G. (1998). It's all right to be wrong. *Australian Mathematics Teacher, 54*(4), 34-37.

Anthony, G. & Knight, G. (1999). Basic facts: The role of memory and understanding. *New Zealand Mathematics Magazine, 36*(3), 28-40.

Arnold, S. (1998). Technology and mathematics education: From calculators to multimedia. In N. Ellerton (Ed.), *Issues in mathematics education: A contemporary perspective* (pp. 173-198). Perth: MASTEC.

Askew, M. (2001). Policy, practices and principles in teaching numeracy: What makes a difference? In P. Gates (Ed.), *Issues in mathematics teaching* (pp. 105-119). London: Routledge Falmer.

Ball, D. B. (1991). Implementing the *Professional Standards for Teaching Mathematics*: What's all this talk about discourse? *Arithmetic Teacher, 39*(3), 44-48.

Ball, D. B. (1993). Halves, pieces, and twoths: Constructing and using representational contexts in teaching fractions. In T. P. Carpenter, E. Fennema & T. A. Romberg (Eds.), *Rational numbers: An integration of research* (pp. 157-195). Hillsdale, NJ: Erlbaum.

Ball, D. L. (1997). From the general to the particular: Knowing our students as learners of mathematics. *The Mathematics Teacher, 90*(9), 732-737.

Ball, D. L. & Bass, H. (2000). Making believe: The collective construction of public mathematical knowledge in the elementary classroom. In D. C. Phillips (Ed.), *Constructivism in education: Opinions and second opinions on controversial issues* (pp. 193-224). Chicago: The National Society of the Study of Education.

Barnes, M. (1999). Cumulative and exploratory talk in a collaborative learning classroom. In J. M. Truran & K. M. Truran (Eds.), *Making the difference* (Proceedings of the 22nd annual conference of the Mathematics Education Research Group of Australasia, Adelaide, pp. 53-59). Sydney: MERGA.

Barton, D. (1998). *Beta mathematics.* Auckland: Longman.

Barton, D. (1999). *Alpha mathematics.* Auckland: Longman.

Barton, D. (2000). *Alpha mathematics homework book.* Auckland: Longman.

Bauersfeld, H. (1988). Interaction, construction, and knowledge: Alternative perspectives for mathematics education. In D. A. Grouws & T. J. Cooney (Eds.), *Perspectives on research on effective mathematics teaching* (pp. 27-46). Reston, VA: National Council of Teachers of Mathematics.

Bauersfeld, H. (1995). "Language games" in the mathematics classroom: Their function and their effects. In P. Cobb & H. Bauersfeld (Eds.), *The emergence of mathematical meaning: Interaction in classroom cultures* (pp. 271-292). Hillsdale, NJ: Lawrence Erlbaum Associates.

Beishuizen, M. (1993). Mental strategies and materials or models for addition and subtraction up to 100 in Dutch Second Grades. *Journal for Research in Mathematics Education, 24*(4), 294-323.

Bishop, A. J. (1991). Mathematical values in the teaching process. In A. J. Bishop, S. Mellin-Olsen & J. van Dormolen (Eds.), *Mathematical knowledge: Its growth through teaching* (pp. 195-214). Dordrecht: Kluwer Academic Publishers.

Bishop, R. & Glynn, T. (1999). *Culture counts: Challenging power relations in education*. Palmerston North: Dunmore Press.

Bleicher, R. B. (1998). Classroom interactions: Using interactional sociolinguistics to make sense of recorded classroom talk. In J. A. Malone, B. Atweh & J. R. Northfield (Eds.), *Research and supervision in mathematics and science education* (pp. 85-104). Mahwah, NJ: Lawrence Erlbaum Associates.

Blithe, T., Clark, M. & Forbes, S. (1993). *The testing of girls in mathematics*. Wellington, New Zealand: Victoria University.

Boaler, J. (1993a). Encouraging the transfer of 'school' mathematics to the 'real world' through the integration of process and context, context and culture. *Educational Studies in Mathematics, 25*(4), 341-373.

Boaler, J. (1993b). The role of contexts in the mathematics classroom: Do they make mathematics more "real"? *For the Learning of Mathematics, 13*(2), 12-17.

Boaler, J. (1997a). *Experiencing school mathematics: Teaching styles, sex and setting*. Buckingham: Open University Press.

Boaler, J. (1997b). Setting, social class and survival of the quickest. *British Educational Research Journal, 23*(5), 575-595.

Boaler, J. (1997c). When even the winners are losers: Evaluating the experiences of 'top set' students. *Journal of Curriculum Studies, 29*(2), 165-182.

Boaler, J. (1999). Participation, knowledge and beliefs: A community perspective on mathematics learning. *Educational Studies in Mathematics, 40*(3), 259-281.

Boaler, J. (2000). Introduction: Intricacies of knowledge, practice and theory. In J. Boaler (Ed.), *Multiple perspectives on mathematics teaching and learning* (pp. 1-18). Westport, CT: Ablex Publishing.

Boaler, J. & Greeno, J. G. (2000). Identity, agency and knowing in mathematics worlds. In J. Boaler (Ed.), *Multiple perspectives on mathematics teaching and learning* (pp. 171-200). Westport, CT: Ablex.

Boaler, J. & Wiliam, D. (2001). 'We've still got to learn!' Students' perspectives on ability grouping and mathematics achievement. In P. Gates (Ed.), *Issues in mathematics teaching* (pp. 77-92). London: Routledge Falmer.

Boaler, J., Wiliam, D. and Brown, M. (2000). Students' experiences of ability grouping: Disaffection, polarisation and the construction of failure. *British Educational Research Journal, 26*(5), 631-648.

Bogdan, R. & Biklen, S. (1992). *Foundations of qualitative research for education* (2nd Ed). Boston: Allyn & Bacon.

Borasi, R. (1996). *Reconceiving mathematics instruction: A focus on errors.* Norwood: Ablex Publishing Corporation.

Bowers, J., Cobb, P. & McClain, K. (1999). The evolution of mathematical practices: A case study. *Cognition and Instruction, 17*(1), 25-64.

Brickhouse, N. W. (1992). Ethics in field-based research: Ethical principles and relational considerations. *Science Education, 76*(1), 93-103.

Brophy, J. (1999). Perspectives of classroom management. In J. H. Freiberg (Ed.), *Beyond behaviourism: Changing the classroom management paradigm* (pp.43-56). Needham Heights, MA: Allyn & Bacon.

Brown, R. (2001). Using mathematics to communicate understanding in a primary classroom. In J. Bobis, B. Perry & M. Mitchelmore (Eds.), *Numeracy and beyond* (Proceedings of the 24th annual conference of the Mathematics Education Research Group of Australasia, Sydney, pp. 106-113). Sydney: MERGA.

Brown, A. & Dowling, P. (1998). *Doing research/reading research: A mode of interrogation for education.* London: Falmer Press.

Brown, R. A. J. & Renshaw, P. D. (2000). Collective argumentation: A sociocultural approach to reframing classroom teaching and learning. In H. Cowie & G. van der Aalsvoort (Eds.), *Social Interaction in learning and instruction: The meaning of discourse for the construction of knowledge* (pp. 52-66). Amsterdam: Pergamon.

Brown, A., Ash, D., Rutherford, M., Nakagawa, K., Gordon, A. & Campione, J. C. (1993). Distributed expertise in the science classroom. In G. Salomon (Ed.), *Distributed cognitions: Psychological and educational considerations* (pp. 188-228). New York: Cambridge University Press.

Bruner, J. S. (1986). *Actual minds, possible worlds.* Cambridge, MA: Harvard University Press.

Bruner, J. S. (1990). *Acts of meaning*. Cambridge, MA: Harvard University Press.

Burns-Wilson, B. & Thomas, M. (1997). Computers in primary mathematics: Strategies for overcoming barriers. In B. Barbour et al (Eds.), *SAMEpapers 1997* (pp. 40-60). Hamilton: CSMTER, University of Waikato.

Carr, M. (2001). *Assessment in early childhood settings: Learning stories*. London: Paul Chapman Publishing.

Catley, K. M. & Tipler, M. J. (1995). *National curriculum mathematics, Level 5, Book 1*. Christchurch: Caxton.

Catley, K. M., de Groot, H. L. & Tipler, M. J. (1989). *Complete mathematics for third form students*. Christchurch: Caxton.

Cazden, C. B. (1988). *Classroom discourse: The language of teaching and learning*. Portsmouth, NH: Heinemann.

Cazden, C. B. (1997). Performance before competence: Assistance to child discourse in the zone of proximal development. In M. Cole, Y. Engeström & O. Vasquez (Eds.), *Mind, culture, and activity: Seminar papers from the Laboratory of Human Cognition* (pp. 303-310). Cambridge: Cambridge University Press.

Chazan, D. & Ball, D. (1999). Beyond being told not to tell. *For the Learning of Mathematics, 19*(2), 2-10.

Clark, M. (1997). Comfort in the mathematics classroom. *The New Zealand Mathematics Magazine, 34*(2), 14-21.

Clark, M. (1999). Maori and Pacific Islands student performance in mathematics. In *Exploring Issues in mathematics education: Proceedings of a research seminar on mathematics education (Year 0-6 students)* (pp. 31-37). Wellington: Ministry of Education.

Clarke, D. (1996). Learning Research. In P. C. Clarkson (Ed.), *Technology in mathematics education* (Proceedings of the 19th annual conference of the Mathematics Education Research Group of Australasia, Melbourne, pp. 1-9). Melbourne: MERGA.

Clarke, D. & Helme, S. (1997). The resolution of uncertainty in mathematics education. In F. Biddulph & K. Carr (Eds.), *People in mathematics education* (Proceedings of the 20th annual conference of the Mathematics Education Research Group of Australasia , Rotorua, New Zealand, pp. 16-123). Waikato: MERGA.

Cobb, P. (1995a). Cultural tools and mathematical learning: A case study. *Journal for Research in Mathematics Education, 26*(4), 362-385.

Cobb, P. (1995b). Mathematical learning and small-group interaction: Four case studies. In P. Cobb & H. Bauersfeld (Eds.), *The emergence of mathematical meaning: Interaction in classroom cultures* (pp. 25-129). Hillsdale, NJ: Lawrence Erlbaum Associates.

Cobb, P. (1998). Theorizing about mathematical conversation and learning from practice. Extract from: Sfard, A., Nesher, P., Streefland, L., Cobb, P. & Mason, J. Learning mathematics through conversation: Is it as good as they say? *For the Learning of Mathematics, 18*(1), 41-51.

Cobb, P. (2000). The importance of a situated view of learning to the design of research and instruction. In J. Boaler (Ed.), *Multiple perspectives on mathematics teaching and learning* (pp. 45-82). Westport, CT: Ablex Publishing.

Cobb, P. (2002). Reasoning with tools and inscriptions. *Journal of the Learning Sciences, 11*(2&3), 187-215.

Cobb, P. & Bauersfeld, H. (1995). Introduction: The coordination of psychological and sociological perspectives in mathematics education. In P. Cobb & H. Bauersfeld (Eds.), *The emergence of mathematical meaning: Interaction in classroom cultures* (pp. 1-15). Hillsdale, NJ: Lawrence Erlbaum Associates.

Cobb, P., Boufi, A., McClain, K. & Whitenack, J. (1997). Reflective discourse and collective reflection. *Journal for Research in Mathematics Education, 28*(3), 258-277.

Cobb, P. & Bowers, J. (1999). Cognition and situated learning perspectives in theory and practice. *Educational Researcher, 28*(2), 4-15.

Cobb, P., Gravemeijer, K., Yackel, E., McClain, K. & Whitenack, J. (1997). Mathematizing and symbolizing: The emergence of chains of signification in one First-Grade classroom. In D. Kirshner & J. A. Whitson (Eds.), *Situated cognition: Social, semiotic, and psychological perspectives* (pp. 151-233). Hillsdale, NJ: Lawrence Erlbaum Associates.

Cobb, P. & McClain, K. (2002). Supporting students' learning of significant mathematical ideas. In G. Wells & G. Claxton (Eds.), *Learning for life in the 21st Century* (pp. 154-166). Oxford: Blackwell.

Cobb, P., Stephan, M., McClain, K. & Gravemeijer, K. (2001). Participating in classroom mathematical practices. *Journal for the Learning Sciences, 10*(1,2), 113-164.

Cobb, P. & Whitenack, J. (1996). A method for conducting longitudinal analyses of classroom videorecordings and transcripts. *Educational Studies in Mathematics, 30*(3), 213-228.

Cobb, P., Wood, T. & Yackel, E. (1991). A constructivist approach to Second Grade mathematics. In E. von Glasersfeld (Ed.), *Radical constructivism in mathematics education* (pp. 157-176). Dordrecht: Kluwer Academic Publishers.

Cobb, P., Wood, T. & Yackel, E. (1993). Discourse, mathematical thinking, and classroom practice. In E. A. Forman, N. Minick & C. A. Stone (Eds.), *Contexts for learning: Sociocultural dynamics in children's development* (pp. 91-119). New York: Oxford University Press.

Cobb, P., Wood, T., Yackel, E. & McNeal, B. (1992). Characteristics of classroom mathematics traditions: An interactional analysis. *American Educational Research Journal, 29*(3), 573-604.

Cobb, P., Wood, T., Yackel, E. & McNeal, E. (1993). Mathematics as procedural instructions and mathematics as meaningful activity. In R. B. Davis & C. A. Maher (Eds.), *Schools, mathematics and the world of reality* (pp. 119-133). Boston: Allyn & Bacon.

Cobb, P., Wood, T., Yackel, E. & Perlwitz, P. (1992). A follow-up assessment of a second-grade problem-centred mathematics project. *Educational Studies in Mathematics, 23*(5), 483-504.

Cobb, P. & Yackel, E. (1998). A constructivist perspective on the culture of the mathematics classroom. In F. Seeger, J. Voigt & U. Waschescio (Ed.), *The culture of the mathematics classroom* (pp. 158-190). Cambridge: Cambridge University Press.

Cobb, P., Yackel, E. & Wood, T. (1992a). A constructivist alternative to the representational view of mind in mathematics education. *Journal for Research in Mathematics Education, 23*(1), 2-33.

Cobb, P., Yackel, E. & Wood, T. (1992b). Interaction and learning in mathematics classroom situations. *Educational Studies in Mathematics, 23*(1), 99-122.

Cobb, P., Yackel, E. & Wood, T. (1995).The teaching experiment classroom. In P. Cobb & H. Bauersfeld (Eds.), *The emergence of mathematical meaning: Interaction in classroom cultures* (pp. 17-23). Hillsdale, NJ: Lawrence Erlbaum Associates.

Coffey, A. & Atkinson, P. (1996). *Making sense of qualitative data: Complementary research strategies.* Thousand Oaks, CA: Sage.

Cohen, L. & Manion, L. (1994). *Research methods in education (4th Ed.).* London: Routledge.

Cohen, L., Manion, L. & Morrison, K. (2000). *Research methods in education (5th Ed.).* London: Routledge Falmer.

Cole, M. & Engeström, Y. (1993). A cultural-historical approach to distributed cognition. In G. Salomon (Ed.), *Distributed cognitions: Psychological and educational considerations* (pp. 1-46). New York: Cambridge University Press.

Collins, E. & Green, J. L. (1992). Learning in classroom settings : Making or breaking a culture. In H. H. Marshall (Ed.), *Redefining student learning: Roots of educational change* (pp. 59-85). Norwood, NJ: Ablex.

Cooper, T., Atweh, B., Baturo, A. & Smith, R. (1993). Higher-intellectual functioning, empowerment and oral communication in mathematics instruction. In M. Stephens, A. Waywood, D. Clarke & J. Izard (Eds.), *Communicating mathematics: Perspectives from classroom practice and current research* (pp. 11-25). Hawthorn: Australian Council for Educational Research.

Cooper, P. & McIntyre, D. (1996). The importance of power-sharing in classroom learning. In M. Hughes (Ed.), *Teaching and learning in changing times* (pp. 88-108). Oxford: Blackwell.

Crooks, T. & Flockton, L. (2002). *Mathematics: Assessment results 2001.* National Education Monitoring Report 23. Dunedin: Education Assessment Research Unit.

Damarin, S. (1999). Social construction and mathematics education: The relevance of theory. In L. Burton (Ed.), *Learning mathematics: From hierarchies to networks* (pp. 141-150). London: Falmer Press.

Delpit, L. (1995). *Other people's children: Cultural conflict in the classroom.* New York: The New Press.

Department of Education (1985). *Mathematics: Junior classes to Standard Four syllabus for schools.* Wellington: Department of Education.

Department of Education (1987). *Mathematics: Forms 1-4 syllabus for schools.* Wellington: Department of Education.

Driver, R., Asoko, H., Leach, R., Mortimer, E. & Scott, P. (1994). Constructing scientific knowledge in the classroom. *Educational Researcher, 23*(7), 5-12.

Education Review Office (2002). *Maths now: Mathematics education for students in Years 1 to 8.* Wellington: Education Review Office.

Edwards, D. (1997). *Discourse and cognition.* London: Sage.

Edwards, F. (2003). Factors affecting the achievement of NESB students in mathematics. *Many Voices, 21*, 11-18.

Edwards, D. & Mercer, N. (1987). *Common knowledge: The development of understanding in the classroom.* London: Methuen/Routledge.

Edwards, A. D. & Westgate, D. P. G. (1987). *Investigating classroom talk.* London: Falmer Press.

Elbers, E. & Streefland, L. (2000). "Shall we be researchers again?" Identity and social interaction in a community of inquiry. In H. Cowie & G. van der Aalsvoort (Eds.), *Social Interaction in learning and instruction: The meaning of discourse for the construction of knowledge* (pp. 35-51). Amsterdam: Pergamon.

Ellerton, N. F. & Clements, M. A. (1992). Some pluses and minuses of radical constructivism in mathematics education. *Mathematics Education Research Journal, 4*(2), 1-22.

EQUALS (1989). *Mathematics teaching: Making changes.* Auckland: EQUALS Network.

Erickson, F. & Shultz, J. (1992). Students' experience of the curriculum. In P. W. Jackson (Ed.), *Handbook of research on curriculum* (pp. 465-485). New York: Macmillan Publishers.

Flockton, L. & Crooks, T. (1998). *Mathematics: Assessment results 1997.* National Education Monitoring Report 9. Dunedin: Education Assessment Research Unit.

Flores, A., Sowder, J. T., Philipp, R. A. & Schappelle, B. P. (1995). Orchestrating, promoting, and enhancing mathematical discourse in the middle grades: A case study. In J. T. Sowder & B. P. Schappelle (Eds.), *Providing a foundation for teaching mathematics in the middle grades* (pp. 275-299). Albany, NY: State University of New York Press.

Forbes, S. D. (1995). Mathematics and New Zealand Maori Girls. In P. Rogers & G. Kaiser (Eds.), *Equity in mathematics education* (pp. 109-116). London: Falmer Press.

Forman, E. A. & Ansell, A. (2001). The multiple voices of a mathematics classroom community. *Educational Studies in Mathematics, 46*(1-3), 115-142.

Forman, E. A. & Ansell, A. (2002). Orchestrating the multiple voices and inscriptions of a mathematics classroom. *Journal of the Learning Sciences, 11*(2/3), 251-267.

Frid, S. & Malone, J. (1995). Negotiation of meaning in mathematics classrooms: A study of two Year 5 classes. *Mathematics Education Research Journal, 7*(2), 132-147.

Garden, R. A. (1996). *Mathematics performance of New Zealand Form 2 and Form 3 students: National results from New Zealand's participation in TIMSS.* Wellington: Ministry of Education.

Garden, R. A. (1997). *Performance assessment in the third international mathematics and science study: New Zealand results*. Wellington: Ministry of Education.

Gee, J. P. (1992). *The social mind: Language, ideology and social practice*. New York: Bergin & Garvey.

Gerofsky, S. (1999). Genre analysis as a way of understanding pedagogy in mathematics education. *For the Learning of Mathematics , 19*(3), 36-46.

Goos, M., Galbraith, P. & Renshaw, P. (1999). Establishing a community of practice in a secondary mathematics classroom. In L. Burton (Ed.), *Learning mathematics: From hierarchies to networks* (pp. 36-61). London: Falmer Press.

Gordon, L. (1994). Some schools are more equal than others: Schools and market choice in Aotearoa, 1994. *Set: Research information for teachers, 2*, 1-6.

Gravemeijer, K. P. E. (1994). *Developing Realistic Mathematics Education.* Utrecht: Freudenthal Institute.

Gravemeijer, K., McClain, K. & Stephan, M. (1998). Supporting students' construction of increasingly sophisticated ways of reasoning through problem solving. In A. Olivier & K. Newstead (Eds.), *Proceedings of the 22nd annual meeting of the International Group for the Psychology of Mathematics Education,* (Vol 1, pp. 194-209). Stellenbosch, Sth Africa: PME.

Gray, E. M. (1991). An analysis of diverging approaches to simple arithmetic: Preference and its consequences. *Educational Studies in Mathematics, 22*(6), 551-574.

Gray, E., Pinto, M., Pitta, D. & Tall, D. (1999). Knowledge construction and diverging thinking in elementary and advanced mathematics. *Educational Studies in Mathematics, 38*(1-3), 111-133.

Gray, E. & Pitta, D. (1996). Number processing: Qualitative differences in thinking and role of imagery. In L. Puig & A. Gutiérrez (Eds.), *Proceedings of the 20th annual meeting of the International Group for the Psychology of Mathematics Education*, (Vol 3, pp. 35-42). Valencia, Spain: PME.

Gray, E. M. & Tall, D. O. (1994). Duality, ambiguity, and flexibility: A "proceptual" view of simple arithmetic. *Journal for Research in Mathematics Education, 25*(2), 116-140.

Greeno, J. G. (1991). Number sense as situated knowing in a conceptual domain. *Journal for Research in Mathematics Education, 22*(3), 170-218.

Greeno, J. G. (1994). Some further observations of the environment/model metaphor. *Journal for Research in Mathematics Education, 25*(1), 94-99.

Greeno, J. G. (1997). On claims that answer the wrong question. *Educational Researcher, 26*(1), 5-17.

Groves, S. (1991, November). *Calculators as an agent for change in the teaching of primary mathematics: The Victoria College Calculator Project.* Paper presented at the annual conference of AARE, Surfer's Paradise, Australia.

Groves, S. & Cheeseman, J. (1995). Beyond expectations: Using calculators with young children. *Set: Research information for teachers, 1,* 1-4.

Groves, S., Cheeseman, J., Allan, A. & Williams, M. (1992). Calculators In Primary Mathematics - The Calculator Projects Reach Grade 3. In M. Horne & M. Supple (Eds.), *Mathematics: Meeting the challenge* (pp. 379-386). Melbourne: Mathematical Association of Victoria.

Guba, E. G. & Lincoln, Y. S. (1989). *Fourth generation evaluation.* Beverley Hills, CA: Sage.

Hart, K. M. (1981). *Children's understanding of mathematics: 11-16.* London: John Murray.

Hawk, K., Cowley, E. T., Hill, J. & Sutherland, S. (2001, December). *Relationships: The critical factor in teaching Maori and Pasifika students.* Paper presented at the annual conference of NZARE, Dunedin, New Zealand.

Hendry, G. D. (1996). Constructivism and educational practice. *Australian Journal of Education, 40*(1), 19-45.

Hershkowitz, R. & Schwarz, B. (1999). The emergent perspective in rich learning environments: Some roles of tools and activities in the construction of sociomathematical norms. *Educational Studies in Mathematics, 39*(1-3), 149-166.

Hiebert, J. (1990). The role of routine procedures in the development of mathematical competence. In T. J. Cooney & C. R. Hirsch (Eds.), *Teaching and learning mathematics in the 1990s* (pp. 31-40). Reston, VA: National Council of Teachers of Mathematics.

Higgins, J. (1995). "I don't know how much to interfere": Independent group work and teacher intervention in the junior school. In J. Neyland (Ed.), *Mathematics Education: A handbook for teachers, Vol 2* (pp. 21-33). Wellington: Wellington College of Education.

Higgins, J. (2002). *An evaluation of the Advanced Numeracy Project 2001.* Wellington: Ministry of Education.

Hill, J. & Hawk, K. (2000, November). *Four conceptual clues to motivating students: Learning from the practice of effective teachers in low decile, multicultural schools.* Paper presented at the annual conference of NZARE, Hamilton, New Zealand.

Hirschhorn, D. B. & Senk, S. (1992). Calculators in the UCSMP curriculum for Grades 7 and 8. In J. Fey (Ed.), *Calculators in mathematics education* (pp. 70-90). Reston, VA: National Council of Teachers of Mathematics.

Hogan, K., Nastasi, B. K. & Pressley, M. (1999). Discourse patterns and collaborative scientific reasoning in peer and teacher-guided discussions. *Cognition and Instruction, 17*(4), 379-432.

Holton, D. A., Anderson, J. & Thomas, B. (1997). *OPE-N plan for teaching mathematical problem solving*. Dunedin: University of Otago.

Irwin, K. C. (1999). Difficulties with decimals and using everyday knowledge to overcome them. *Set: Research information for teachers, 2,* 1-4.

Irwin, K. C. & Niederer, K. (2002). *An evaluation of the Numeracy Exploratory Study (NEST) and the Associated Numeracy Exploratory Studies Assessment (NESTA)*. Wellington: Ministry of Education.

Jaworski, B. (1999). Tensions in teachers' conceptualizations of mathematics and of teaching. In L. Burton (Ed.), *Learning mathematics: From hierarchies to networks* (pp. 153-172). London: Falmer Press.

Johnson, D. C. (1989). *Children's mathematical frameworks 8-13: A study of classroom teaching*. Windsor: NFER-Nelson.

Jones, A. (1987). What really happens in the classroom. *Set: Research information for teachers, 2*, 1-6.

Jones, A. (1991). *"At school I've got a chance"*. Palmerston Nth: Dunmore Press.

Kaput, J. J. (2001). Implications of the shift from isolated, expensive technology to connected, inexpensive, diverse and ubiquitous technologies. *New Zealand Mathematics Magazine*, *38*(3), 1-26.

Kellehear, A. (1993). *The unobtrusive researcher: A guide to methods*. Sydney: Allen & Unwin.

Kirkwood, V., Carr, M., Bell, B., McChesney, J., Osborne, R. & Symmington, D. (1987). *LISP(Energy): The strategy and some preliminary findings*. Working Paper 302. Hamilton: Science Education Research Centre, University of Waikato.

Klein, M. (2000). Is there more to numeracy than meets the eye?: Stories of socialisation and subjectification in school mathematics. In J. Bana & A. Chapman (Eds.), *Mathematics education beyond 2000* (Proceedings of the 23rd annual conference of the Mathematics Education Research Group of Australasia, Fremantle, pp. 72-78). Sydney: MERGA.

Knuth, E. & Peressini, D. (2001). Unpacking the nature of discourse in mathematics classrooms. *Mathematics Teaching in the Middle School, 6*(5), 320-325.

Kohn, A. (1993). *Punished by rewards: The trouble with gold stars, incentive plans, As, praise and other bribes*. New York: Houghton Mifflin.

Kozulin, A. (1998). *Psychological tools: A sociocultural approach to education*. Cambridge, MA: Harvard University press.

Krummheuer, G. (2000). Mathematics learning in narrative classroom cultures: Studies of argumentation in primary mathematics education. *For the Learning of Mathematics, 20*(1), 22-32.

Kumpulainen, K. & Mutanen, M. (2000). Mapping the dynamics of peer group interaction: A method of analysis of socially shared processes. In H. Cowie & G. van der Aalsvoort (Eds.), *Social Interaction in learning and instruction: The meaning of discourse for the construction of knowledge* (pp. 144-160). Amsterdam: Pergamon.

Kumpulainen, K. & Wray, D. (Eds.) (2002). *Classroom interaction and social learning: From theory to practice*. London: Routledge Falmer.

Lampert, M. (1985). How do teachers manage to teach? Perspectives on problems in practice. *Harvard Educational Review, 55*(2), 178-194.

Lampert, M. (1990). Connecting inventions with conventions. In L. P. Steffe & T. Wood (Eds.), *Transforming children's mathematics education* (pp. 253-265). Hillsdale, NJ: Lawrence Erlbaum.

Lampert, M, Rittenhouse, P. & Crumbaugh, C. (1996). Agreeing to disagree: Developing sociable mathematical discourse. In D. R. Olsen and N. Torrance (Eds.), *The handbook of education and human development* (pp. 731-764). Oxford: Blackwell.

Lave, J. (1988). *Cognition in practice: Mind, mathematics and culture in everyday life*. Cambridge: Cambridge University Press.

Lave, J. (1991). Situated learning in communities of practice. In L. B. Resnick, J. M. Levine & S. D. Teasley (Eds.), *Perspectives on socially shared cognition* (pp. 63-82). Washington: American Pyschological Association.

Lave, J., Murtaugh, M. & de la Rocha, O. (1984). The dialectic of arithmetic in grocery shopping. In B. Rogoff & J. Lave (Eds.), *Everyday cognition: Its development in social context* (pp. 67-94). Cambridge, MA: Harvard University Press.

Lave, J. & Wenger, E. (1991). *Situated learning: Legitimate peripheral participation*. Cambridge: Cambridge University Press.

LeCompte, M. D. & Preissle, J. (1993). *Ethnography and qualitative design in educational research* (2nd Ed.). San Diego: Academic Press.

Lerman, S. (1993). The position of the individual in radical constructivism: In search of the subject. In J. A. Malone & P. C. S. Taylor (Eds.), *Constructivist interpretations of teaching and learning mathematics* (pp. 105-112). Perth: Curtin University for Technology.

Lerman, S. (1999). Culturally situated knowledge and the problem of transfer in the learning of mathematics. In L. Burton (Ed.), *Learning mathematics: From hierarchies to networks* (pp. 93-107). London: Falmer Press.

Lerman, S. (2000). The social turn in mathematics education research. In J. Boaler (Ed.), *Multiple perspectives on mathematics teaching and learning* (pp. 19-44). Westport, CT: Ablex Publishing.

Lerman, S. (2001). Cultural discursive psychology: A sociocultural approach to studying the teaching and learning of mathematics. *Educational Studies in Mathematics, 46*(1-3), 87-113.

Lincoln, Y. S. & Guba, E. G. (1985). *Naturalistic inquiry.* Beverley Hills, CA: Sage.

Lo, J. & Wheatley, G. H. (1994). Learning opportunities and negotiating social norms in mathematics class discussion. *Educational Studies in Mathematics, 27*(2), 145-164.

Lo, J., Wheatley, G. H. & Smith, A. C. (1994). The participation, beliefs, and development of arithmetic meaning of a Third-Grade student in mathematics class discussions. *Journal for Research in Mathematics Education*, 25(1), 30-49.

Lofland, J. & Lofland, L. H. (1995). *Analyzing social settings: A guide to qualitative observation and analysis* (3rd Ed.). Belmont, CA: Wadsworth Publishing.

Lovitt, C. & Clarke, D. (1992). *The Mathematics Curriculum and Teaching Programme: Professional development package, Activity Bank Vol 2.* Carlton, Vict: Curriculum Corporation.

Lubienski, S. T. (2000). A clash of social class cultures? Students' experiences in a discussion-intensive Seventh-Grade mathematics classroom. *The Elementary School Journal, 100*(4), 377-403.

Macbeth, D. (2003). Hugh Mehan's *Learning Lessons* Reconsidered: On the differences between the naturalistic and critical analysis of classroom discourse. *American Educational Research Journal, 40*(1), 239-280.

Markovits, Z. & Sowder, J. (1994). Developing number sense: An intervention study in Grade 7. *Journal for Research in Mathematics Education, 25*(1), 4-29.

Marshall, H. H. (1992). Seeing, redefining, and supporting student learning. In H. H. Marshall (Ed.), *Redefining student learning: Roots of educational change* (pp. 1-32). Norwood, NJ: Ablex.

Masingila, J. O. (1995). Carpet laying: An illustration of everyday mathematics. In P. A. House & A. F. Coxford (Eds.), *Connecting mathematics across the curriculum* (pp. 163-169). Reston, VA: NCTM.

Mason, J. (1999). The role of labels in promoting learning from experiences among teachers and students. In L. Burton (Ed.), *Learning mathematics: From hierarchies to networks* (pp 187-208). London: Falmer Press.

Mason, J. (2002). Minding your Qs and Rs: Effective questioning and responding in the mathematics classroom. In L. Haggarty (Ed.), *Aspects of teaching secondary mathematics: Perspectives on practice* (pp. 248-258). Milton Keynes: Open University Press.

Maurer, S. B. (1987). New knowledge about errors and new views about learners: What they mean to educators and more educators would like to know. In A. H. Schoenfeld (Ed.), *Cognitive science and mathematics education* (pp. 165-187). Hillsdale, NJ: Lawrence Erlbaum Associates.

McChesney, J. (1995a). The difficult road ahead for the simple calculator. In J. Neyland (Ed.), *Mathematics education: A handbook for teachers, Vol. 2* (pp. 2-12). Wellington: Wellington College of Education.

McChesney, J. (1995b). Making sense of number sense. In S. Groves & R. Tyler (Eds.), *Contemporary approaches to research in mathematics and science education* (pp. 85-88). Geelong: Deakin University.

McChesney, J. & Biddulph, F. (1994). Number sense. In J. Neyland (Ed.), *Mathematics education: A handbook for teachers, Vol. 1* (pp. 7-17). Wellington: Wellington College of Education.

McChesney, J., Mitchell, D., Hammonds, M. & Weston, P. (1987). *Case study Number 3: City High School.* Working Paper No. 6 Project One In Seven. Hamilton: University of Waikato.

McClain, K. & Cobb, P. (2001). Supporting students' ability to reason about data. *Educational Studies in Mathematics, 45*(1), 103-109.

McClain, K., Cobb, P., Gravemeijer, K. & Estes, B. (1999). Developing mathematical reasoning within the context of measurement. In L. Stiff (Ed.), *Developing mathematical reasoning (1999 Yearbook of NCTM)* (pp. 93-106). Reston, VA: National Council of Teachers of Mathematics.

McIntosh, A. (1990). Becoming numerate: Developing number sense. In S. Willis (Ed.), *Being Numerate: What counts?* (pp. 24-43). Hawthorn, VA: ACER.

McIntosh, A. & Dole, S. (2000). Mental computation, number sense and general mathematics ability: Are they linked? In J. Bana & A. Chapman (Eds.), *Mathematics education beyond 2000* (Proceedings of the 23rd annual conference of the Mathematics Education Research Group of Australasia, Fremantle, pp. 401-408). Sydney: MERGA.

McIntosh, A., Nohda, N., Reys, B. J. & Reys, R. E. (1995). Mental computation performance in Australia, Japan and the United States. *Educational Studies in Mathematics, 29*(3), 237-258.

McIntosh, A., Reys, B. J. & Reys, R. E. (1992). A proposed framework for examining basic number sense. *For the Learning of Mathematics, 12*(3), 2-8.

McIntosh, A., Reys, B. & Reys, R. (1997). *Number Sense: Simple effective number sense experiences: Grades 6-8.* Palo Alto, CA: Dale Seymour Publications.

McIntosh, A. Reys, B., Reys, R., Bana, J. & Farrell, B. (1997). *Number sense in school mathematics: Student performance in four countries.* Perth: MASTEC, Edith Cowan University.

McIntosh, A., Reys, B., Reys, R. & Hope, J. (1997). *Number Sense: Simple effective number sense experiences: Grades 4-6.* Palo Alto, CA: Dale Seymour Publications.

Mehan, H. (1979). *Learning lessons: Social organization in the classroom.* Cambridge, MA: Harvard University Press.

Mehan, H. (1997). Students' interactional competence in the classroom. In M. Cole, Y. Engeström & O. Vasquez (Eds.), *Mind, culture, and activity: Seminar papers from the Laboratory of Human Cognition* (pp. 235-240). Cambridge: Cambridge University Press.

Mercer, N. (1995). *The guided construction of knowledge: Talk amongst teachers and learners.* Clevedon: Multilingual Matters.

Mercer, N. (2002a). Developing dialogues. In G. Wells and G. Claxton (Eds.), *Learning for life in the 21st Century* (pp. 141-153). Oxford: Blackwell.

Mercer, N. (2002b). How is language used as a medium for classroom education? In B. Moon, S. Mayes & S. Hutchinson (Eds.), *Teaching, learning and the curriculum in secondary schools: A reader* (pp. 169-188). London: Open University Press.

Merriam, S. (1988). *Case Study Research in Education: A qualitative approach.* San Francisco: Jossey Bass Publishers.

Miles, M. B. & Huberman, A. M. (1994). *Qualitative data analysis: An expanded source book* (2nd Ed.). Thousand Oaks, CA: Sage.

Millroy, W.L. (1992). An ethnographic study of the mathematical ideas of a group of carpenters. *Journal for Research in Mathematics Education.* Monograph Number 5.

Ministry of Education (1992). *Mathematics in the New Zealand Curriculum.* Wellington: Learning Media.

Ministry of Education (1993). *The New Zealand Curriculum Framework.* Wellington: Learning Media.

Ministry of Education (1995). *Implementing Mathematical Processes.* Wellington: Learning Media.

Ministry of Education (1996). *Development Band Mathematics.* Wellington: Learning Media.

Ministry of Education (1997). *Developing mathematics programmes.* Wellington: Learning Media.

Ministry of Education (2000). *Education statistics of New Zealand for 1999.* Wellington: Ministry of Education.

Ministry of Education (2001). *Assessing knowledge and skills for life: First results from PISA 2000.* Wellington: Ministry of Education.

Ministry of Education (2002). *The Number Framework.* Wellington: Ministry of Education.

Moloney, K. & Stacey, K. (1996). Understanding decimals. *The Australian Mathematics Teacher, 52*(1), 4-8.

Mousley, J. A. (1999). Understanding negative number: You can take 10 from 7! *Australian Mathematics Teacher, 55*(4), 7-13.

Mousley, J. & Marks, G. (1991). *Discourses in Mathematics.* Geelong: Deakin University.

Mousley, J. & Sullivan, P. (1995, July). *Classroom communication: Investigating relationships between language, subjectivity and classroom organisation.* Paper presented at 19th annual meeting of the International Group for the Psychology of Mathematics Education, Recife, Brazil.

Newman, D., Griffin, P. & Cole, M. (1989*). The construction zone: Working for cognitive change in school.* Cambridge: Cambridge University Press.

Neyland, J. (1998). Outcomes-based mathematics education. In N. F. Ellerton (Ed.), *Issues in mathematics education: A contemporary perspective* (pp. 60-75). Perth: MASTEC Edith Cowan University.

Nickson, M. (2000). *Teaching and learning mathematics: A teacher's guide to recent research and its application.* London: Cassell.

Noddings, N. (1993). Constructivism and caring. In R. B. Davis & C. A. Maher (Eds.), *Schools, mathematics and the world of reality* (pp. 35-50). Boston: Allyn & Bacon.

Noss, R. (1998). New numeracies for a technological culture. *For the Learning of Mathematics, 18*(2), 2-12.

Nunes, T., Schliemann, A. E. & Carraher, D. W. (1993). *Street mathematics not school mathematics.* Cambridge: Cambridge University Press.

Nuthall, G. (2000, November). *Understanding what students learn in school.* Paper presented at the annual conference of NZARE, Hamilton, New Zealand.

Papert, S. (1980). *Mindstorms: Children, computers and powerful ideas.* Brighton: Harvester Press.

Papert, S. (1993). *The children's machine: Rethinking school in the age of the computer.* New York: BasicBooks

Pea, R. D. (1987). Cognitive technologies for mathematics education. In A. H. Schoenfeld (Ed.), *Cognitive science and mathematics education* (pp. 89-122). Hillsdale, NJ: Lawrence Erlbaum Associates.

Pea, R. D. (1993). Practices of distributed intelligence and designs for education. In G. Salomon (Ed.), *Distributed cognitions: Psychological and educational considerations* (pp. 47-87). New York: Cambridge University Press.

Perkins, D. N. (1993). Person plus: A distributed view of thinking and learning. In G. Salomon (Ed.), *Distributed cognitions: Psychological and educational considerations* (pp. 88-110). New York: Cambridge University Press.

PrIME (1989). *The Second Year Of CAN.* Cambridge: PrIME.

Putnam, R. T. & Borko, H. (2000). What do new views of knowledge and thinking have to say about research on teacher learning? *Educational Researcher, 29*(1), 4-15.

Reid, N. A. (1993). *Progressive Achievement Test of mathematics.* Wellington: NZCER.

Resnick, L. B. (1991). Shared cognition: Thinking as social practice. In L. Resnick, J. M. Levine & S. D. Teasley (Eds.), *Perspectives on socially shared cognition* (pp. 1-20). Washington, DC: American Psychological Association.

Richards, J. (1991). Mathematical discussions. In E. von Glasersfeld (Ed.), *Radical constructivism in mathematics education* (pp. 13-51). Dordrecht: Kluwer Academic.

Rogoff, B. (1990). *Apprenticeship in thinking: Cognitive development in social context*. New York: Oxford University Press.

Rogoff, B. (1994). Developing understanding of the idea of communities of learners. *Mind, Culture, and Activity, 1*(4), 209-229.

Rogoff, B. (1997). Evaluating development in the process of participation: Theory, methods, and practice building on each other. In E. Amsel & K. A. Renninger (Eds.), *Change and development: Issues of theory, method and application* (pp. 265-285). Mahwah, NJ: Lawrence Erlbaum Associates.

Romberg, T. A. (1993). How one comes to know: Models and theories of the learning of mathematics. In M. Niss (Ed.), *Investigations into assessment in mathematics education* (pp. 97-111). Dordrecht: Kluwer Academic Publishers.

Romberg, T. A. & Carpenter, T. P. (1986). Research on teaching and learning mathematics. In M. Wittrock, (Ed.), *Handbook of research on teaching* (3rd Ed) (pp. 850-873). New York: Macmillan.

Rowland, T. (1999). Pronouns in mathematics talk: Power, vagueness and generalisation. *For the Learning of Mathematics, 19*(2), 19-26.

Ruthven, K. (1999). The pedagogy of calculator use. In I. Thompson (Ed.), *Issues in teaching numeracy in primary schools* (pp. 195-206). Buckingham: Open University Press.

Salomon, G. (1991). Transcending the qualitative-quantitative debate: The analytic and systemic approaches to educational research, *Educational Researcher, 20*(6), 10-18.

Salomon, G. (1993). No distribution without individual's cognition: A dynamic interactional view. In G. Salomon (Ed.), *Distributed cognitions: Psychological and educational considerations* (pp. 111-138). New York: Cambridge University Press.

Salomon, G. & Perkins, D. (1998). Individual and social aspects of learning. In P. D. Pearson & A. Iran-Nejad (Eds.), *Review of Research in Education* (pp. 1-24). Washington: American Educational Research Association.

Salomon, G., Perkins, D. N. & Globerson, T. (1991). Partners in cognition: Extending human intelligence with intelligence technologies. *Educational Researcher, 20*(3), 2-9.

Saxe, G. B., Gearhart, M. & Seltzer, M. (1999). Relations between classroom practices and student learning in the domain of fractions. *Cognition and Instruction, 17*(1), 1-24.

Scribner, S. (1984). Studying working intelligence. In B. Rogoff & J. Lave (Eds.), *Everyday cognition: Its development in social context* (pp. 9-40). Cambridge, MA: Harvard University Press.

Senn-Fennell, C. (1995). Oral and written communication for promoting mathematical understanding. *Journal of Curriculum Studies, 27*(1), 31-54.

Sfard, A. (1998). On two metaphors for learning and the dangers of choosing just one. *Educational Researcher, 27*(2), 4-13.

Sfard, A. (2000). Symbolizing mathematical reality into being – or how mathematical discourse and mathematical objects create each other. In P. Cobb, E. Yackel & K. McClain (Eds.), *Symbolizing and communicating in mathematics classrooms: Perspectives on discourse, tools, and instructional design* (pp. 37-98). Mahwah, NJ: Lawrence Erlbaum Associates.

Sfard, A. (2001). There is more to discourse that meets the ears: Looking at thinking as communicating to learn more about mathematical learning. *Educational Studies in Mathematics, 46*(1-3), 13-57.

Sfard, A. (2002). The interplay of intimations and implementations: Generating new discourse with new symbolic tools. *Journal of the Learning Sciences, 11*(2&3), 319-357.

Sfard, A. & McClain, K. (2002). Analyzing tools: Perspectives on the role of designed artifacts in mathematics learning. *Journal of the Learning Sciences, 11*(2&3), 153-161.

Shaffer, D. W. & Kaput, J. J. (1999). Mathematics and virtual culture: An evolutionary perspective on technology and mathematics education. *Educational Studies in Mathematics, 37*(2), 97-119.

Shuard, H. (1992). CAN: Calculator use in the primary grades in England and Wales. In J. Fey (Ed.), *Calculators in mathematics education* (pp. 33-45). Reston, VA: National Council of Teachers of Mathematics.

Shuard, H., Walsh, A., Goodwin, J. & Worcester, V. (1991). *Calculators, children and mathematics.* Hemel Hempstead: Simon & Schuster.

Shulman, L. S. (1986). Those who understand: Knowledge growth in teaching. *Educational Researcher, 15*, 4-14.

Silver, E. A. (1994). Treating estimations and mental computation as situated mathematical processes. In R. E. Reys & N. Nohda (Eds), *Computational Alternatives for the Twenty-First Century: Cross-Cultural Perspectives from Japan and the United States* (pp. 147-161). Reston, VA: National Council of Teachers of Mathematics.

Skemp, R. R. (1971). *The psychology of learning mathematics.* Middlesex: Penguin.

Smolka, A. L. B., De Goes, M. C. R. & Pino, A. (1995). The constitution of the subject: A persistent question. In J. V. Wertsch, P. del Rio & A. Alvarez (Eds.), *Sociocultural studies of mind* (pp. 165-184). Cambridge: Cambridge University Press.

Sowder, J. T. (1988). Mental computation and number comparison: Their roles in the development of number sense and computational estimation. In J. Hiebert & M. Behr (Eds.), *Number concepts and operations in the middle grades* (pp. 182-197). Reston, VA: National Council of Teachers of Mathematics.

Sowder, J. (1992a). Teaching computation in ways that promote number sense. In C. J. Irons (Ed.), *Challenging children to think when they compute* (pp. 14-27). Brisbane: Queensland University of Technology.

Sowder, J. T. (1992b). Estimation and related topics. In D. Grouws (Ed.), *Handbook for Research on Mathematics Teaching and Learning* (pp. 371-389). New York: Macmillan.

Sowder, J. T. (1994). Cognitive and metacognitive processes in mental computation and computational estimation. In R. E. Reys & N. Nohda (Eds), *Computational Alternatives for the Twenty-First Century: Cross-Cultural Perspectives from Japan and the United States* (pp. 139-146). Reston, VA: National Council of Teachers of Mathematics.

Sowder, J. T. (1995). Instructing for rational number sense. In J. T. Sowder & B. P. Schappelle (Eds.), *Providing a foundation for teaching mathematics in the middle grades* (pp. 15-30). Albany, NY: State University of New York Press.

Sowder, J. T. & Kelin, J. (1993). Number sense and related topics. In D. T. Owens (Ed.), *Research Ideas for the Classroom: Middle Grade Mathematics* (pp. 41-57). New York: Macmillan.

Stacey, K. (1992). Mathematical problem solving in groups: Are two heads better than one? *Journal of Mathematical Behavior, 11*, 261-275.

Stacey, K. & MacGregor, M. (2002). As though the thinking is out there on the table. In D. Tall & M. Thomas (Eds.), *Intelligence, learning and understanding in mathematics: A tribute to Richard Skemp* (pp. 219-233). Flaxton, Qld: Post Pressed.

Steinbring, H. (2000). Mathematics AS language: Epistemological particularities of mathematical signs in social communication. In B. Barton (Ed.), *Pre-conference publication of Working Group 9* (pp. 81-91). Auckland: University of Auckland.

Steinle, V. & Stacey, K. (2001). Visible and invisible zeros: Sources of confusion in decimal notation. In J. Bobis, B. Perry & M. Mitchelmore (Eds.), *Numeracy and beyond* (Proceedings of the 24th annual conference of the Mathematics Education Research Group of Australasia, Sydney, pp. 434-441). Sydney: MERGA.

Thomas, G. (1994). Peer discussion and learning in mathematics. In J. Neyland (Ed.), *Mathematics education: A handbook for teachers, Vol. 1* (pp. 319-329). Wellington: Wellington College of Education.

Thomas, G. & Ward, J. (2001). *An Evaluation of the Count Me In Too Pilot Project.* Wellington: Ministry of Education.

Thomas, M. O. J. (2002). Versatile learning of mathematics. In D. Tall & M. Thomas (Eds.), *Intelligence, learning and understanding in mathematics: A tribute to Richard Skemp* (pp. 179-204). Flaxton, Qld: Post Pressed.

Thompson, P. W. (2000). Radical constructivism: Reflections and directions. In L. P. Steffe and P. W. Thompson (Eds.), *Radical constructivism in action: Building on the pioneering work of Ernst von Glasersfeld* (pp. 291-315). Dordrecht: Kluwer Academic Press.

Tobin, K. (1992). Ethical concerns and research in science classrooms: Resolved and unresolved dilemmas. *Science Education, 76*(1), 105-117.

Trafton, P. R. (1992). Using number sense to develop mental computation and computational estimation. In C. J. Irons (Ed.), *Challenging children to think when they compute* (pp. 78-92). Brisbane: Queensland University of Technology.

Trafton, P. R. (1994). Computational estimation: Curriculum and development efforts and instructional issues. In R. E. Reys & N. Nohda (Eds.), *Computational Alternatives for the Twenty-First Century: Cross-Cultural Perspectives from Japan and the United States* (pp. 76-86). Reston, VA: National Council of Teachers of Mathematics.

Tuyay, S., Floriani, A., Yeager, B., Dixon, C. & Green, J. (1995). Constructing an integrated, inquiry-oriented approach in classrooms: A cross case analysis of social, literate and academic practices. *Journal of Classroom Interaction, 30*(2), 1-15.

Ueno, N. (1998). Doing mathematics as situated practice. In C. Donlan (Ed.), *The development of mathematical skills* (pp. 111-128). Hove: Psychology Press.

van der Aalsvoort, G. M. & Harinck, F. J. H. (2000). Studying social interaction in instruction and learning: Methodological approaches and problems. In H. Cowie & G. van der Aalsvoort (Eds.), *Social Interaction in learning and instruction: The meaning of discourse for the construction of knowledge* (pp. 5-20). Amsterdam: Pergamon.

Van Oers, B. (2000). The appropriation of mathematical symbols: A psychosemiotic approach to mathematics learning. In P. Cobb, E. Yackel & K. McClain (Eds.), *Symbolizing and communicating in mathematics classrooms: Perspectives on discourse, tools, and instructional design* (pp. 133-176). Mahwah, NJ: Lawrence Erlbaum Associates.

Verschaffel, L. & De Corte, E. (1996). Number and arithmetic. In A. J. Bishop, K. Clements, C. Keitel, J. Kilpatrick & C. Laborde (Eds.), *International handbook of mathematics education* (pp. 99-138). Dordrecht: Kluwer Academic Publishers.

Voigt, J. (1994). Negotiation of mathematical meaning and learning mathematics. *Educational Studies in Mathematics, 26*(2-3), 274-298.

Voigt, J. (1995). Thematic patterns of interaction and sociomathematical norms. In P. Cobb & H. Bauersfeld (Eds.), *The emergence of mathematical meaning: Interaction in classroom cultures* (pp. 163-201). Hillsdale, NJ: Lawrence Erlbaum Associates.

von Glasersfeld, E. (1990). An exposition of constructivism: Why some like it radical. In R. B. Davis, C. A. Maher & N. Noddings (Eds.), *Constructivist views on the teaching and learning of mathematics* (pp. 19-30). Reston, VA: NCTM.

Vygotsky, L. S. (1978). *Mind in society: The development of higher psychological processes*. Cambridge, MA: Harvard University Press.

Walls, F. (2001). Children talk about their mathematics lives. In J. Bobis, B. Perry & M. Mitchelmore (Eds.), *Numeracy and beyond* (Proceedings of the 24th annual conference of the Mathematics Education Research Group of Australasia, Sydney, pp. 499-506). Sydney: MERGA.

Weade, G. (1992). Locating learning in the times and spaces of teaching. In H. H. Marshall (Ed.), *Redefining student learning: Roots of educational change* (pp. 87-118). Norwood, NJ: Ablex.

Weade, G. (1995). When content becomes process: Talking knowledge into meaning in a Fourth Grade mathematics lesson. *Journal of Classroom Interaction, 30*(2), 41-51.

Wells, G. (1999). *Dialogic inquiry: Toward a sociocultural practice and theory of education*. Cambridge: Cambridge University Press.

Wertsch, J. V. (1991). *Voices of the mind: A sociocultural approach to mediated action*. Cambridge, MA: Harvard University Press.

Wheatley, G. H. (1991). Constructivist perspectives on science and mathematics learning. *Science Education, 75*(1), 9-21.

Wheatley, G. H. (1993). The role of negotiation in mathematics learning. In K. Tobin (Ed.), *The practice of constructivism in science education* (pp. 122-134). Washington: American Association for the Advancement of Science Press.

Wheatley, G. H. & Shumway, R. (1992). The potential for calculators to transform elementary school mathematics. In J. Fey (Ed.), *Calculators in mathematics education* (pp. 1-8). Reston, VA: National Council of Teachers of Mathematics.

White, R. (1985). The importance of context in educational research. *Research in Science Education, 15*, 92-102.

Wiliam, D. (1998). A framework for thinking about research in mathematics and science education. In J. A. Malone, B. Atweh & J. R. Northfield (Eds.), *Research and supervision in mathematics and science education* (pp. 1-18). Mahwah, NJ: Lawrence Erlbaum Associates.

Wood, T. (1995). An emerging practice of teaching. In P. Cobb & H. Bauersfeld (Eds.), *The emergence of mathematical meaning: Interaction in classroom cultures* (pp. 203-227). Hillsdale, NJ: Lawrence Erlbaum Associates.

Wood, T. (2002). What does it mean to teach mathematics differently? In B. Barton, K. C. Irwin, M. Pfannkuch & M. O. J. Thomas (Eds.), *Mathematics Education in the South Pacific* (Proceedings of the 25th Annual Conference of the Mathematics Education Research Group of Australasia, Auckland, pp. 61-67). Sydney: MERGA.

Yackel, E. (1993). Children's talk in mathematics class as a function of context. In I. Hirabayashi, N. Nohda, K. Shigematsu & F.-L. Lin (Eds.), *Proceedings of the 17th annual meeting of the International Group for the Psychology of Mathematics Education*. (Vol 2, pp. 199-205). Tsubuka, Japan: PME.

Yackel, E. (1995). Children's talk in inquiry mathematics classrooms. In P. Cobb & H. Bauersfeld (Eds.), *The emergence of mathematical meaning: Interaction in classroom cultures* (pp. 131-161). Hillsdale, NJ: Lawrence Erlbaum Associates.

Yackel, E., Cobb, P. & Wood, T. (1991). Small group interactions as a source of learning opportunities in second-grade mathematics. *Journal for Research in Mathematics Education, 22*(5), 390-408.

Yackel, E., Cobb, P., Wood, T., Wheatley, G. & Merkel, G. (1990). The importance of social interaction in children's construction of mathematical knowledge. In T. Cooney (Ed.), *1990 Yearbook of the National Council of Teachers of Mathematics* (pp. 12-21). Reston, VA: National Council of Teachers of Mathematics.

Young-Loveridge, J. (2000). How children's understanding of the number system varies as a function of ethnicity and socio-economic status. In J. Bana & A. Chapman (Eds.), *Mathematics education beyond 2000* (Proceedings of the 23rd annual conference of the Mathematics Education Research Group of Australasia, Fremantle, pp. 672-679). Sydney: MERGA.

Zazkis, R. (2000). Using code-switching as a tool for learning mathematical language. *For the Learning of Mathematics, 20*(3), 38-43.

Zevenbergen, R. (1995). Students' mathematics culture. In B. Atweh & S. Flavel (Eds.), *Galtha* (Proceedings of the 18th annual conference of the Mathematics Education Research Group of Darwin, pp. 557-563). Darwin: MERGA.

Zevenbergen, R. (1998). Ethnography in the classroom. In J. A. Malone, B. Atweh & J. R. Northfield (Eds.), *Research and supervision in mathematics and science education* (pp. 19-38). Mahwah, NJ: Lawrence Erlbaum Associates.

Zevenbergen, R. (2000). "Cracking the code" of mathematics classrooms: School success as a function of linguistic, social and cultural background. In J. Boaler (Ed.), *Multiple perspectives on mathematics teaching and learning* (pp. 201-223). Westport, CT: Ablex Publishing.

Zevenbergen, R. & Lerman, S. (2001). Communicative competence in school mathematics: On being able to do school mathematics. In J. Bobis, B. Perry & M. Mitchelmore (Eds.), *Numeracy and beyond* (Proceedings of the 24th annual conference of the Mathematics Education Research Group of Australasia, Sydney, pp. 571-578). Sydney: MERGA.

THE UNIVERSITY OF WAIKATO
EDUCATION LIBRARY