The Derivation and Application of a Theoretically and Economically Consistent Version of the Nelson and Siegel Class of Yield Curve Models

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Leo Krippner

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To my parents, James and Teresa Krippner

Abstract

A popular class of yield curve models is based on the Nelson and Siegel (1987) (hereafter NS) approach of "fitting" yield curve data with simple functions of maturity. However, NS models are not theoretically consistent and they also lack an economic foundation, which limits their wider application in finance and economics. This thesis derives an intertemporally-consistent and arbitrage-free version of the NS model, and provides an explicit macroeconomic foundation for that augmented NS (ANS) model. To illustrate the general applicability of the ANS model, it is then applied to four distinct topics spanning finance and economics, each of which are active areas of research in their own right: i.e (1) forecasting the yield curve; (2) investigating relationships between the yield curve and the macroeconomy; (3) fixed interest portfolio management; and (4) investigating the uncovered interest parity hypothesis (UIPH). In each application, the ANS model allows the formal derivation of a parsimonious theoretical framework that captures the essence of the topic under investigation and is readily applicable in practice. Respectively: (1) the intertemporal consistency embedded in the ANS model results in a vector-autoregressive equation that projects the future yield curve from the current yield curve, and forecasts from that model outperform the random-walk benchmark; (2) the economic foundation for the ANS model leads to a single-equation relationship between the current shape of the yield curve and the magnitude and timing of future output growth, and empirical estimations confirm that the theoretical relationship holds in practice; (3) the ANS model provides a theoretically-consistent framework for quantifying risk and returns in fixed interest portfolios, and portfolios optimised ex-ante using that framework outperform a passive benchmark; and (4) the ANS model allows interest rates to be decomposed into a component related to economic fundamentals in the underlying economy, and a component related to cyclical influences. Empirical tests based on the fundamental interest rate components do not reject the UIPH, while the UIPH is rejected based on the cyclical interest rate components. This provides empirical support for suggestions in the theoretical literature that interest rate and exchange rate dynamics associated with cyclical interlinkages between the economy and financial markets under rational expectations may contribute materially to the UIPH puzzle.

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The list of abbreviations and symbols is arranged chapter by chapter, with each approximately in the order of their introduction.

Chapter 2

NS	Nelson and Siegel
ANS	augmented Nelson and Siegel
HJM	Heath-Jarrow-Morton
BGM	Brace-Gatarek-Musiela
AR1, VAR1	first-order autoregression, and vector-autoregression
FRED	Federal Reserve Economic Database
FF, TB3, GS	FRED US federal funds rate, 3-month Treasury bill rate, and Treasury bond yield for the given maturity
FF/GS10	10-year US Treasury bond yield less the federal funds rate
RMSE	root mean squared error
bp(s)	basis point(s)
t	time
m	time to maturity ($m \geq 0$ and $m = T - t$)
T	time of maturity ($T \geq t$ and $T = t + m$)
$f(t, m)$	instantaneous forward rate curve at time t as a function of time to maturity m; $f(t, m) = f_{\mathrm{BGM}}(t, m)$
$f(t, T)$	instantaneous forward rate curve at time t as a function of time of maturity T; $f(t, T) = f_{\mathrm{HJM}}(t, T) = f_{\mathrm{HJM}}(t, t + m) = f_{\mathrm{BGM}}(t, m) = f(t, m)$
$R(t, m)$	continuously-compounding zero-coupon interest rate at time t as a function of time to maturity m
n	index number from 1 to 3 for the NS and ANS coefficients and modes

Chapter 2 (continued)

$\beta_1(t)$	Level coefficient at time t for the NS and ANS model
$\beta_2(t)$	Slope coefficient at time t for the NS and ANS model
$\beta_3(t)$	Bow coefficient at time t for the NS and ANS model
ϕ	a positive constant parameter that governs the rate of exponential decay in the non-Level modes
$g_n(\phi, m)$	mode n for the forward rate mode in the NS model or the expected path of short rate mode in the ANS model, as a function of the parameter ϕ and time to maturity m
$g_1(\phi, m)$	Level mode for the forward rate (in NS model) or expected path of short rate (in ANS model)
$g_2(\phi, m)$	Slope mode for the forward rate (in NS model) or expected path of short rate mode (in ANS model)
$g_3(\phi, m)$	Bow mode for the forward rate (in NS model) or expected path of short rate mode (in ANS model)
$s_n(\phi, m)$	interest rate mode n as a function of the parameter ϕ and time to maturity m
$s_1(\phi, m)$	Level interest rate mode
$s_2(\phi, m)$	Slope interest rate mode
$s_3(\phi, m)$	Bow interest rate mode
E_t	expectations operator conditional on information available at time t
$r(t+m)$	actual path of the short rate as a function of time $t+m$; $r(t+m) = r(T)$
$E_t[r(t+m)]$	expected path, as at time t, of the short rate as a function of future time $t+m$
$\lambda_n(t)$	time-varying coefficients applied to $g_n(\phi, m)$ to represent the expected path of the short rate in the ANS model
σ_n	constant volatility parameters (i.e annualised standard deviations) applied to Wiener increments
$dW_n(t)$	Wiener increments under the physical measure (the source of stochastic changes in the ANS model coefficients)
ρ_n	constant parameters representing the market prices of risk associated with stochastic changes in the ANS coefficients
N	number of independent stochastic processes in the HJM framework
$\alpha_n(\cdot, \cdot)$	deterministic component n for the HJM framework as function of variables in context
$\sigma_n(\cdot, \cdot)$	volatility function n for the HJM framework as function of variables in context
$\phi_n(v)$	market price of risk for component n
$h_n(\phi, m)$	time-invariant functions of maturity representing the effect that a unit of volatility in the coefficient has on the shape of the forward rate curve

Chapter 2 (continued)

$u_n(\phi, m)$	time-invariant functions of maturity representing the effect that a unit of volatility in the coefficient has on the shape of the interest rate curve
γ_n	constant parameters representing the HJM calculation of the effect of the market prices and quantities of risk
$\beta_n(t)$	ANS coefficient n as the sum of γ_n and $\lambda_n(t)$
k	index number from 1 to K for the fixed interest securities that define the yield curve
K	total number of fixed interest securities that define the yield curve
j	index number from 1 to $J[k]$ for cashflows of security k
$J[k]$	number of cashflows for fixed interest security k
ε_k	estimated price residual for security k
a_{kj}	cashflow j for security k
m_{kj}	time to maturity of cashflow j for security k
$R(t, m_{kj})$	interest rate at time t for time to maturity m_{kj}
w_k	weighting on the price residual for the fixed interest security k when estimating the ANS model
i	index number from 1 to T to denote the individual changes in the ANS coefficients
I	number of ANS coefficient observations used to estimate the variances σ_n^2
F	number of observations of ANS coefficients per year, which is used to annualise the variances σ_n^2
τ	an arbitrary finite increment of time ($\tau \geq 0$)
$E_{t+\tau}[r(t+\tau+m)]$	expected path, as at time $t+\tau$, of the short rate as a function of future time $t+\tau+m$
$E_t[r(t+\tau+m)]$	expected path, as at time t, of the short rate as a function of future time $t+\tau+m$
$f(t+\tau, m)$	forward rate curve at time $t+\tau$ as a function of time to maturity m
$f(t, \tau+m)$	forward rate curve at time t as a function of time to maturity $\tau+m$
$\mathbf{g}(\phi, m)$	3-vector of functions $g_n(\phi, m)$
$\boldsymbol{\lambda}(t)$	3-vector of coefficients $\lambda_n(t)$
$\boldsymbol{\lambda}(t+\tau)$	3-vector of $\lambda_n(t)$ after an increment of time τ from t
$\boldsymbol{\gamma}(t)$	3-vector of parameters γ_n
$\boldsymbol{\beta}(t)$	3-vector of ANS coefficients $\beta_n(t)$
$\boldsymbol{\beta}(t+\tau)$	3-vector of $\beta_n(t)$ after an increment of time τ from t
$E_t[\boldsymbol{\beta}(t+\tau)]$	3-vector of the expected value, as at time t, of $\boldsymbol{\beta}(t+\tau)$
$\boldsymbol{\mu}(\phi, \tau)$	3-vector accounting for the effect of term premia within the expression for the evolution of the ANS coefficients

Chapter 2 (continued)

$\mathbf{\Phi}(\phi, \tau)$	3×3 matrix of the VAR1 coefficient matrix as a function of ϕ and τ
$\boldsymbol{\delta}(t + \tau)$	3-vector representing unanticipated movements in $\boldsymbol{\lambda}(t)$ and $\boldsymbol{\beta}(t)$ from time t to $t + \tau$

Chapter 3

BE	Berardi and Esposito
ABE	augmented Berardi and Esposito
GDP	gross domestic product
IGD	inflation rate for the GDP deflator
CBO	Congressional Budget Office
PCE	personal consumption expenditure deflator
PCEX	personal consumption expenditure deflator excluding food and energy
VAR	vector autoregression
j	index number from 1 to J for the real factors of production, and from $J + 1$ to $2J$ for their respective deflators
J	the number of real factors of production in the economy, and the number of associated inflation state variables
$s_j(t)$	real instantaneous growth on returns to the factors of production in the economy at time t (for $j = 1$ to J), and inflation state variables at time t (for $j = J + 1$ to $2J$)
κ_j	positive constant mean-reversion parameter for the state variable j
$\theta_j(t)$	steady-state (i.e long-run) values of $s_j(t)$
$\sigma_{0,j}$	positive constant standard deviation parameters for the steady-state variables j
$\sigma_{1,j}$	positive constant standard deviation parameters for the state variables j
$dz_{0,j}(t)$	independent Wiener variable under the physical (i.e non-risk-neutral) measure representing the source of stochastic changes to steady-state variable j
$dz_{1,j}(t)$	independent Wiener variable under the physical (i.e non-risk-neutral) measure for state variable j
$\pi_j(t)$	instantaneous rate of inflation for the factor of production j
$\sigma_{j,p}$	positive constant parameter representing the standard deviation of instantaneous changes in the deflator j
$\theta_{j,\pi}(t)$	steady-state rate of inflation for the factor of production j

Chapter 3 (continued)

$dY(t)$	instantaneous output growth
$dP(t)$	instantaneous economy-wide inflation
$dY^*(t)$	instantaneous steady-state output growth
$dP^*(t)$	instantaneous steady-state economy-wide inflation
κ_π	mean-reversion parameter for the inflation state variable in the BE model
$E_t[s_j(t+m)]$	expected value, as at time t, of $s_j(t+m)$ as a function of future time $t+m$
$\rho_{0,j}$	market price of risk for innovations in the steady-state variable j
$\rho_{1,j}$	market price of risk for innovations in the state variable j
$B_j(m)$	Vasicek functional form
$E_t[dY(t+m)]$	expected value, as at time t, of $dY(t)$ as a function of future time $t+m$
$E_t[dP(t+m)]$	expected value, as at time t, of $dP(t)$ as a function of future time $t+m$
$E_t[dY^*(t+m)]$	expected value, as at time t, of $dY^*(t)$ as a function of future time $t+m$
$E_t[dP^*(t+m)]$	expected value, as at time t, of $dP^*(t)$ as a function of future time $t+m$
$\text{central}(\kappa_j)$	central measure of κ_j
Δ_j	relative difference between κ_j and $\text{central}(\kappa_j)$
$\beta_{1,t}$	alternative notation for ANS Level coefficient at time t
α^*	constant parameter in the cointegrating relationship between $\beta_{1,t}$ and steady-state nominal GDP
ε_t^*	residual in the cointegrating relationship between $\beta_{1,t}$ and steady-state nominal GDP
ΔY_t^*	annualised steady-state output growth at time t
ΔP_t^*	annualised steady-state inflation at time t
$E_t[dX(t+m)]$	expected instantaneous change, as at time t, in nominal output growth relative to steady-state nominal output growth as a function of future time $t+m$
T_1	beginning of discrete forward time interval $t+T_1$ to $t+T_2$
T_2	end of discrete forward time interval $t+T_1$ to $t+T_2$
$E_t[\Delta X_{t+T_1,t+T_2}]$	average value of $E_t[dX(t+m)]$ over the discrete forward time interval $t+T_1$ to $t+T_2$
$q_n(T_1,T_2)$	average value of $g_n(\phi,m)$ over the discrete forward time interval $t+T_1$ to $t+T_2$
$\Delta X_{t+T_1,t+T_2}$	ex-post realised value of $E_t[\Delta X_{t+T_1,t+T_2}]$
α_{0,T_1,T_2}	constant coefficient in the relationship between $E_t[\Delta X_{t+T_1,t+T_2}]$ and the shape of the yield curve
α_{1,T_1,T_2}	coefficient relating $E_t[\Delta X_{t+T_1,t+T_2}]$ to the time-varying component of the shape of the yield curve

Chapter 3 (continued)

ε_{t,T_1,T_2}	regression residuals for the estimated ANS yield curve versus output growth relationship
D_t	step-dummy variable representing a structural break in term premia around the late 1970s / early 1980s
α_{2,T_1,T_2}	coefficient for the step-dummy variable D_t

Chapter 4

IYC	initial yield curve
YCE	yield curve exposure
MV	market value (or market price)
PV	present value
FOYCE	first-order yield curve exposure
SOYCE	second-order yield curve exposure
BPV_k	basis point value for security k
IYC	initial yield curve
LIBOR	London interbank offered rate
SRT	simulated real time
I/S	in-sample
P/S	pre-sample
M/A	mean-adjustment
$P_k(t)$	MV of fixed interest security k at time t
$p(m)$	PV of a unit cashflow as a function of time to maturity m
$\mathbf{s}(\phi, m)$	3-vector of functions $s_n(\phi, m)$
$\mathbf{v}$	3-vector of ANS coefficient variances σ_n^2
$\mathbf{u}(\phi, m)$	3-vector of functions $u_n(\phi, m)$
$\boldsymbol{\beta}$	$\boldsymbol{\beta}(t)$ with dependency on t omitted for notational brevity
$\mathbf{s}$	$\mathbf{s}(\phi, m)$ with dependencies on ϕ and m omitted for notational brevity
Q or $Q(m)$	time-invariant component of the interest rate curve
$p(\boldsymbol{\beta}, m)$	present value of a unit cashflow as a function of the ANS coefficients $\boldsymbol{\beta}$ and time to maturity m
$\boldsymbol{\delta}$	3-vector $\boldsymbol{\delta}(t+\tau)$, with t and τ omitted for notational brevity, representing an unanticipated yield curve shift over an increment of time from t to $t+\tau$
$p(\boldsymbol{\beta}+\boldsymbol{\delta}, m-\tau)$	PV of a unit cashflow following a yield curve shift $\boldsymbol{\delta}$ over an increment of time from t to $t+\tau$
$\boldsymbol{\lambda}_k$	3-vector of FOYCE components for security k
$\boldsymbol{\Omega}_k$	3×3-matrix of SOYCE components for security k
$P_{k,t}$	alternative notation for $P_k(t)$
$\Delta P_{k,t+\tau}$	change in the MV of security k over over an increment of time from t to $t+\tau$

Chapter 4 (continued)

$P_{k,t}(\boldsymbol{\beta})$	PV of security k at time t based on the ANS model coefficients $\boldsymbol{\beta}(t)$
$P_{k,t+\tau}(\boldsymbol{\beta})$	PV of security k at time $t+\tau$ based on the ANS model coefficients $\boldsymbol{\beta}(t+\tau)$
$\Delta\varepsilon_{k,t+\tau}$	change in the ANS price residual for security k
$E_t[\Delta\varepsilon_{k,t+\tau}]$	expected change, as at time t, in the ANS price residual for security k over the increment of time $t+\tau$
A_k	face-value of security k
$\eta_{k,t}$	ANS yield residual for security k at time t
$\eta_{k,t+\tau}$	ANS yield residual for security k at time $t+\tau$
π_k	hypothesised mean-adjustment for yield residual $\eta_{k,t}$
θ	hypothesised first-order autoregression coefficient for mean-reversion in yield residual $\eta_{k,t}$
$\upsilon_{k,t+\tau}$	hypothesised idiosyncratic component of mean-reversion in yield residual $\eta_{k,t}$
$E_t\left[\Delta\eta_{k,t+\tau}\right]$	expected change, as at time t, in the ANS yield residual for security k over the increment of time $t+\tau$
$A_{0,k,t}$	face-value of security k in the benchmark portfolio
$A_{1,k,t}$	face-value of security k in the optimised portfolio
$A_{1,k,\min}$	minimum face-value of security k in the optimised portfolio
$A_{1,k,\max}$	maximum face-value of security k in the optimised portfolio
$\alpha_{k,t}$	"potential yield enhancement" of a unit of security k at time t;
$\boldsymbol{\alpha}_t$	K-vector of potential yield enhancement values $\alpha_{k,t}$
$\mathbf{A}_{0,t}$	K-vector of face-values for the benchmark portfolio at time t
$\mathbf{A}_{1,t}$	K-vector of face-values for the alternative portfolio at time t
$\boldsymbol{\Lambda}_{k,t}$	4-vector of the quantities $[P_k, \boldsymbol{\lambda}_{k,1}, \boldsymbol{\lambda}_{k,2}, \boldsymbol{\lambda}_{k,3}]_t'$ at time t
$\boldsymbol{\Lambda}_t$	$4 \times K$ matrix of the quantities $[\boldsymbol{\Lambda}_1, \ldots, \boldsymbol{\Lambda}_k, , \ldots, \boldsymbol{\Lambda}_K]_t$ at time t
$\sigma[\cdot]$	standard deviation of the bracketed quantity
$\mathrm{var}[\cdot]$	variance of the bracketed quantity
$\pi_k(t)$	simulated real-time mean-adjustment for yield residual $\eta_{k,t}$
$S(t,x)$	swap rate quoted at time t for maturity x-calendar-years

Chapter 5

CA	denotes a Canadian quantity
US	denotes a United States quantity
CAD	Canadian dollar

Chapter 5 (continued)

USD	United States dollar
RNVA	risk-neutral and volatility-adjusted
e_t	natural logarithm of the nominal exchange rate between the CAD and the USD at time t (defined as the number of USDs per CAD, so a rise in e_t is an appreciation of the CAD against the USD)
$e_{t,m}$	natural logarithm of the forward CAD/USD exchange rate at time t for settlement at $t+m$ years
$\Delta e_{t,m}$	ex-post change in e_t over a time-step m years, lagged m years
$E_t\left[e_{t+m}\right]$	expected value, as at time t, of e_{t+m}
$R_{t,m}^{\mathrm{US}}$	annualised continously-compounding zero-coupon interest rates for the US at time t for maturity $t+m$ years
$R_{t,m}^{\mathrm{CA}}$	annualised continously-compounding zero-coupon interest rates for Canada at time t for maturity $t+m$ years
$R_{t,m}^{\mathrm{US}}\left(\mathrm{ANS}\right)$	ANS-estimated US zero-coupon interest rate at time t for time to maturity m
$R_{t,m}^{\mathrm{CA}}\left(\mathrm{ANS}\right)$	ANS-estimated Canadian zero-coupon interest rate at time t for time to maturity m
$\varepsilon_{t,m}^{\mathrm{US}}\left(\mathrm{ANS}\right)$	ANS-estimated yield residual for the US zero-coupon interest rate at time t for time to maturity m
$\varepsilon_{t,m}^{\mathrm{CA}}\left(\mathrm{ANS}\right)$	ANS-estimated yield residual for the Canadian zero-coupon interest rate at time t for time to maturity m
$\beta_n^{\mathrm{US}}\left(t\right)$	ANS coefficient n for the US interest rate curve at time t
$s_n^{\mathrm{US}}(m)$	ANS interest rate mode n for the US interest rate curve
$\beta_n^{\mathrm{CA}}\left(t\right)$	ANS coefficient n for the Canadian interest rate curve at time t
$s_n^{\mathrm{CA}}(m)$	ANS interest rate mode n for the Canadian interest rate curve
a_m	constant coefficient in the UIPH regression
b_m	coefficient for the interest rate differential in the UIPH regression
w_m	coefficient for the Level component of the interest rate differential in the UIPH regression
x_m	coefficient for the non-Level component of the interest rate differential in the UIPH regression
y_m	coefficient for the Slope plus Bow components of the interest rate differential in the UIPH regression
z_m	coefficient for the residual component of the interest rate differential in the UIPH regression
$v_{t,m}$	regression residuals for the UIPH regression
$v_{t,m}^{*}$	regression residuals for the UIPH regression when the Stock and Watson (1993) method is used

Chapter 1

Introduction

The purpose of this introductory chapter is to provide the objective and motivation for the thesis, and then to provide an overview of the thesis and its main results. To keep both sections brief, references are limited to standard texts and seminal work; details on the related literature and other relevant information are contained within the individual chapters.

1.1 Objective and motivation

The objective of this thesis is to derive a theoretically and economically consistent version of the Nelson and Siegel (1987) class of yield curve models, and then to apply that derived model formally across a variety of finance and economic topics that have a connection to interest rates and the yield curve.

To begin the motivation for this objective from its broadest context, even a casual reference to any finance or economics textbook will readily illustrate the fundamental role that interest rates play in financial markets and the economy; e.g as a financial investment in their own right, as a basis for pricing assets by discounting expected cashflows, as an influence on consumption, investment, and savings decisions, and as a tool for the operation and transmission of monetary policy. It is not surprising then that the study of interest rates has commanded the ongoing attention of practitioners over at least the last five millennia. This is summarised, for example, in Homer (1963) and James and Webber (2000) chapter 2.

The study of interest rates has naturally led researchers and academics to represent them with a variety of models, which differ widely depending on the aspect of interest rates that is emphasised and the intended application of the models. It is useful to classify models of interest rates into three broad categories, which are discussed in turn below, i.e: (1) static yield curve (or term structure) models that emphasise relationships between interest rates of different maturities at a given point in time; (2) dynamic yield curve models that emphasise the relationships between interest rates at different points in time; and (3) macroeconomic models of interest rates that emphasise the dynamic relationships between interest rates and other macroeconomic variables.

Static yield curve models include the pioneering hand-drawn yield curves of Durand (1942), the polynomial spline models originally proposed in McCulloch (1971), and the common practice in financial markets, as formalised in Fama and Bliss (1987), of "bootstrapping" implicit zero-coupon interest rates from observed coupon-bearing interest rate securities. Hull (2000) pp.90-92 and James and Webber (2000) chapter 15 provide overviews of those and other related models of the yield curve. The applications of such interest rate models include representing the shape of the yield curve to imply expectations of future monetary policy, determining potential pricing anomalies between fixed interest securities of different maturities, and providing estimates of zero-coupon yields to use in subsequent analysis requiring such data (e.g discounting projected cashflows into present values). However, in emphasising simple relationships by maturity and the fit to observed yield curve data at a given point in time, static yield curve models overlook the dynamics of interest rates over time. Hence, they are not intertemporally consistent (i.e interest rates and yield curves at different points in time cannot be mapped to each other via an underlying stochastic time-series process), and without the explicit consideration of the effect of volatility on expected returns, static yield curve models may admit arbitrage on themselves. Indeed, related to the latter point, a distinct disadvantage of polynomial splines is that they diverge to plus or minus infinity beyond the longest-maturity point of observed yield curve data. This is obviously not reasonable asymptotic behaviour, and prices based on such interest rates would clearly be subject to arbitrage opportunities. In summary then, the absence of intertemporal consistency and arbitrage-free construction preclude the application of static yield curve models in a time-series context or where no-arbitrage consistency is required.

Dynamic yield curve models include the equilibrium and no-arbitrage interest rate factor models used for pricing derivatives that depend on interest rates, as summarised in Hull (2000) chapters 20-22. Equilibrium models, such as the seminal models of Vasicek (1977) and Cox, Ingersoll and Ross (1985*b*), specify a stochastic differential equation for the evolution of the short rate which is then solved to obtain interest rates as a function of maturity. Equilibrium models are therefore intertemporally consistent by construction, but they can have a poor fit to observed yield curve data at a given point in time. Alternatively, no-arbitrage models of the yield curve, such as the original example developed in Ho and Lee (1986), may be estimated to fit an observation of the yield curve precisely while also allowing for stochastic dynamics. However, Backus, Foresi and Zin (1998) and Brandt and Yaron (2002) note that such models are often applied in an intertemporally-inconsistent manner, essentially because parameters within the model that are assumed to be constant over time are recalibrated at each point in time without regard to historical data.

Other models that implicitly emphasise the stochastic properties of interest rates are those concerned with measuring, monitoring and managing interest rate risk; i.e potential changes in the market value of fixed interest securities or portfolios due to unanticipated changes in interest rates or the shape of the yield curve. These models include the concept of "duration", as originally proposed in Macauley (1938) and discussed in Hull (2000) pp. 108-114, that quantifies the financial exposure to level shifts in the yield curve. Similarly, principal component models of the yield curve, as originally proposed in Litterman and Sheinkman (1991) and summarised in James and Webber (2000) chapter 16, quantify the financial exposure to non-parallel shifts in the yield curve. Standard statistical time-series models, which may be as simple as calculating the standard deviation of historical interest rate or principle component changes, are then used to gauge the distributions of potential movements in interest rates or the yield curve. Applying those distributions to the financial exposures calculated from duration or principal components models then generates the distributions of potential changes in the market value of fixed interest securities or portfolios.

Macroeconomic models of interest rates typically take the form of standard IS-LM-AS (i.e investment-savings, liquidity-money, and aggregate supply) models; e.g see Gordon (1990), Walsh (1998) chapter 5, and Mankiw (2001) chapters 31-33 for an economic perspective, and James and Webber (2000) chapter 11 for a finance perspective. These models typically use a representative single-maturity interest rate to represent the dynamic relationships between interest rates and macroeconomic variables, and are usually specified in discrete time. A representative long-maturity interest rate can also be added in conjunction with the Fisher (1907) relationship between long-maturity interest rates and long-term expected inflation; e.g see James and Webber (2000) pp. 273-274. However, the interest rates within such models are not specified as a continuous function of maturity, and typically no attention is given to intertemporal and/or no-arbitrage consistency. Hence, macroeconomic models of interest rates cannot be applied as static or dynamic yield curve models.

The wide range of models of interest rates discussed above raises the question of whether it might be possible to specify a standard "one size fits all" model of interest rates and the yield curve, i.e a model that is equally consistent by maturity, across time, and in conjunction with macroeconomic data, rather than having to switch between customised models designed for particular applications. Preferably, the model should also meet the criteria that it is tractable and easy to compute, which is mentioned as a key consideration in James and Webber (2000) chapters 3 and 15, and the frequently-referenced article of Bliss (1996). Such a standard model could then be applied quite generally across topics in finance and economics, and may be particularly useful where those fields intersect.

One potential candidate for a standard model of the yield curve is the Nelson and Siegel (1987) (hereafter NS) class of yield curve models. Models of this class are already very popular with practitioners, researchers, and academics in finance and economics, as will be discussed in more detail in chapter 2. For the purposes of this introductory chapter, NS models readily meet the criteria of tractability (being straightforward to estimate by "fitting" to market-quoted yield curve data), and they are also effective static yield curve models (given they provide sensible and intuitive output in the form of an implied forward rate curve as a parsimonious, stable, and asymptotically-bounded function of maturity with a typically good fit to market-quoted data). In addition, NS models have already been applied empirically with some success as dynamic models of the yield curve in a time-series context, and to investigate relationships between the yield curve and macroeconomic data.

That said, the latter two applications push NS models beyond their theoretical limits, which essentially reflects the two shortcomings of NS models. Specifically, the first shortcoming is that NS models are not intertemporally consistent, as discussed in James and Webber (2000) pp. 447-448. The second shortcoming is that NS models lack an economic foundation, which means that any relationships with macroeconomic data are constrained to be statistically-based rather than intrinsic within a self-consistent theoretical system.

The context outlined above therefore motivates the initial objective of this thesis: i.e a theoretically-consistent version of the NS model with a rigorous economic foundation should serve as a standard model of the yield curve that can be applied equally as a static yield curve model, a dynamic yield curve model, and a macroeconomic model of interest rates. The motivation for the subsequent objective of applying the model to a range of topics across finance and economics is then to provide a practical test of its general applicability.

1.2 Overview of the thesis

With reference to the previous section, the central questions asked in this thesis are the following:

1. Is it possible to augment the NS class of yield curve models to obtain an intertemporally-consistent and arbitrage-free model of the yield curve, while retaining the tractability of NS models?

2. Is it possible to provide an explicit economic foundation for the augmented NS (hereafter ANS) model of the yield curve?

3. Is the ANS model generally applicable across the fields of finance and economics? Specifically, how does the ANS model fare when applied to the following topics: (1) forecasting the yield curve; (2) establishing relationships between the yield curve, output, and inflation; (3) managing and optimising fixed interest portfolios; and (4) providing a new perspective on the uncovered interest parity hypothesis (i.e the proposition jthat exchange rates should appreciate/depreciate at a pace that offsets the interest rate discount/premium available between the underlying currencies)? Each topic is an active area of research in its own right, and so the performance of the ANS

model can be compared with investigations in the existing literature.

To address question 1, chapter 2 develops the ANS model of the yield curve by applying the Heath, Jarrow and Morton (1992) (hereafter HJM) framework to the functions of maturity used to represent the forward rate curve in the NS class of yield curve models. Using the HJM framework ensures that, by construction, the ANS model will be theoretically consistent across both time and maturity. The ANS model also retains the tractability of NS models; i.e yield curve data at a given point in time are still summarised by estimating just three coefficients (i.e the Level, Slope, and Bow coefficients), and the additional parameters representing the market prices and quantities of risk may be estimated consistently from the time series of yield curve data and the Level, Slope, and Bow coefficients. The intertemporal consistency of the ANS model is illustrated by explicitly deriving the vector-autoregressive time-series model for the ANS model coefficients. This leads immediately to a theoretical framework for forecasting the yield curve; i.e the first application noted in question 3. Given the forecasting framework is based on a simple theoretical time-series projection, it is considerably more parsimonious and easy to apply than previous approaches in the literature that require the estimation of non-parsimonious time-series models. However, the empirical results are similar, and forecasts from the ANS framework also outperform the random-walk benchmark.[1]

To address question 2, chapter 3 first specifies a generic multifactor version of the standard continuous-time general-equilibrium-economy model from Cox, Ingersoll and Ross (1985*a*), and then derives its associated forward rate curve. Comparing the latter to the ANS model of the yield curve then provides the basis for explicitly linking the ANS model coefficients and parameters to the current and expected state variables of the

[1]The material in chapter 2 is based on the working paper Krippner (2005*b*) and the published article Krippner (2006*a*). Related material in appendix A is based on the working papers Krippner (2003*a*) and Krippner (2003*b*).

underlying economy. It is then shown that the aggregated state variables of the general-equilibrium model correspond to the macroeconomic variables of expected inflation and output growth. This leads immediately to a theoretical framework for interpreting and investigating relationships between the yield curve, output, and inflation; i.e the second application in question 3. Specifically, the ANS model is used to derive theoretical single-equation relationships between the yield curve, output, and inflation that are considerably more parsimonious and easy to apply than previous approaches in the literature, such as vector-autoregressive models. The empirical results are consistent with the predictions of the ANS framework; i.e the estimated long-maturity level of the yield curve given by the Level coefficient in the ANS model is cointegrated with steady-state inflation plus steady-state output growth, and the shape of the yield curve given by the Slope and Bow coefficients in the ANS model corresponds to the profile (i.e the timing and magnitude) of future output growth relative to its steady-state value.[2]

To address fixed interest portfolio risk and optimisation, i.e the third application in question 3, chapter 4 first shows how the stochastic dynamics within the ANS model can be used to represent unanticipated shifts in the level and shape of the yield curve. It then proceeds with a second-order Taylor expansion around the vector of ANS coefficients to derive the financial exposure of fixed interest securities and portfolios to those unanticipated shifts in the yield curve. This provides a framework for measuring the ex-ante financial exposure of portfolios to unanticipated movements in the yield curve, and for attributing returns to yield curve movements ex-post. The empirical application to ex-post attribution shows that nearly all of the variability in portfolio returns is due to first-order yield curve exposures (i.e "duration" effects) from stochastic shifts in the level and shape of the yield

[2] The material in chapter 3 is based on the working papers Krippner (2005*c*) and Krippner (2005*d*). The former was a joint recipient of the February 2005 New Zealand Econometric Study Group best student presentation award and the 2005 A. R. Bergstrom Prize in Econometrics, and the latter was the revised version of that submission.

curve. Second-order yield curve exposures (i.e "convexity" effects) and other contributions are immaterial. Chapter 4 also shows how a simple measure of "relative value" (i.e deviations of the actual yields of fixed interest securities from the yields implied by the ANS model) may present a potential opportunity for generating excess returns. Combining the ex-ante yield curve exposure and the relative value aspects of the ANS model produces a parsimonious framework for portfolio optimisation that can be applied using standard linear programming. The empirical application shows that portfolios optimised ex-ante using this framework significantly outperform an evenly-weighted benchmark over time. This provides support for the idea that the concept of relative value used within financial markets is a quantifiable concept, and maximising that quantity potentially enhances portfolio returns.[3]

To provide a new perspective on the uncovered interest parity hypothesis (UIPH), i.e the fourth application in question 3, chapter 5 uses the economic foundation of the ANS model to decompose the interest rate data used in the standard UIPH regressions into components that reflect expectations of the cyclical and fundamental components of the underlying economy. The empirical analysis then finds that the UIPH is not rejected based on the fundamental components of interest rates, but is soundly rejected based on the cyclical components. These results provide empirical support for suggestions in the existing theoretical literature that interest rate and exchange rate dynamics associated with cyclical interlinkages between the economy and financial markets under rational expectations may contribute materially to the UIPH puzzle.[4]

Chapter 6 very briefly concludes the thesis, and then spends some time discussing potential extensions and refinements of the ANS model and ideas for further applications. Appendix A provides details on how the ANS model could be extended arbitrarily, and how the assumptions underlying the derivation of the ANS model or its extensions could be

[3]The material in chapter 4 is based on the working paper Krippner (2005*a*).
[4]The material in chapter 5 is based on the working paper Krippner (2006*b*).

relaxed, which may prove useful for some particular applications. Appendix B shows that the nature of the results from the generic general-equilibrium economy models in chapter 3 are still obtained outside the special case assumed in that chapter for mathematical convenience. Appendix C provides details on how the ANS portfolio framework can be used for the active management of fixed interest portfolio management.

Chapter 2

A theoretically-consistent version of the NS class of yield curve models

2.1 Introduction

The objective of this chapter is to derive a theoretically-consistent version of the class of yield curve models originally proposed in Nelson and Siegel (1987) (hereafter NS). With reference to chapter 1, it is motivated by the fact that NS models are not intertemporally-consistent or arbitrage-free, and so cannot be applied as dynamic models of the yield curve. The immediate illustrative application of the derived augmented NS (hereafter ANS) model of the yield curve is then to develop a framework for forecasting the yield curve.

The chapter proceeds as follows: section 2.2 introduces the NS class of yield curve models from the existing literature, and also discusses their lack of intertemporal consistency. Section 2.3 specifies, derives, and discusses the ANS model, and section 2.4 explicitly demonstrates the intertemporal consistency of the ANS model by deriving the stochastic time-series process for the ANS model coefficients. Section 2.5 applies the ANS model in tandem with the derived time-series process to obtain out-of-sample forecasts of the United States yield curve over the period 1954 to 2004. Section 2.6 briefly compares the ANS forecasting framework and its empirical results to the existing literature, and section 2.7 summarises and concludes.

2.2 A review of the NS class of yield curve models

The NS class of yield curve models originated with the proposal in NS to represent the forward rate curve with a linear combination of a constant and exponential-polynomial functions. The NS approach has subsequently been extended and respecified in Svensson (1994), Hunt (1995), Bliss (1997), Mansi and Phillips (2001), and Diebold and Li (2006). NS models of the yield curve may be represented by the following forward rate specification:

$$f(t,m) = \sum_{n=1}^{3} \beta_n(t) \cdot g_n(\phi,m) \tag{2.1}$$

where $f(t,m)$ is the (instantaneous) forward rate curve at time t as a function of time to maturity m ($m \geq 0$, so the time of maturity $t+m$ is a future point in time); and $\beta_n(t)$ are the three coefficients at time t that are associated with the three time-invariant functions of maturity $g_n(\phi,m)$. The latter are defined as a constant and then the first two orthonormalised Laguerre polynomials (as detailed in appendix A), i.e:

$$g_1(\phi,m) = 1 \tag{2.2a}$$

$$g_2(\phi,m) = -\exp(-\phi m) \tag{2.2b}$$

$$g_3(\phi,m) = -\exp(-\phi m)(-2\phi m + 1) \tag{2.2c}$$

where ϕ is a positive constant parameter that governs the rate of exponential decay. Figure 2.1 illustrates those three forward rate functions, which are hereafter named the Level,

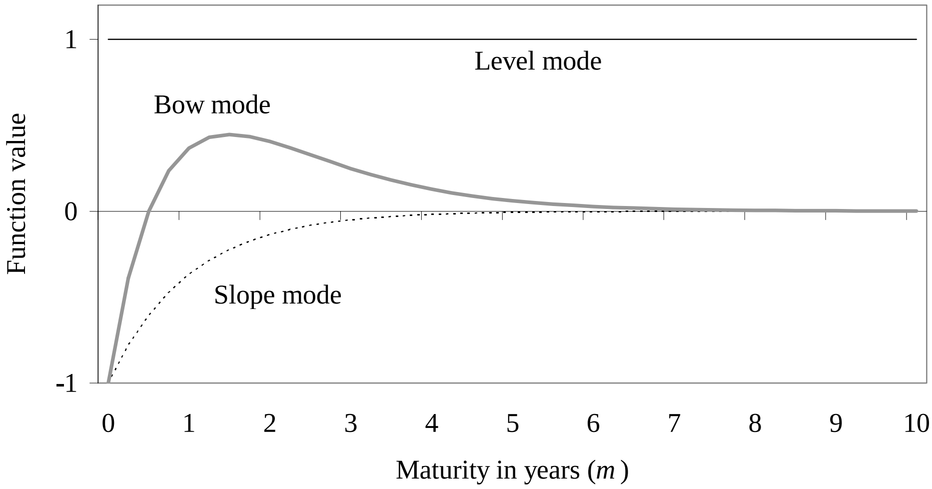

Figure 2.1: The functions $g_1(\phi, m)$, $g_2(\phi, m)$, and $g_3(\phi, m)$ (i.e the Level, Slope, and Bow forward rate modes) for the NS model. $\phi = 1$ for this illustration.

Slope, and Bow modes based on their intuitive shapes. Correspondingly, $\beta_1(t)$, $\beta_2(t)$, and $\beta_3(t)$ are also referred to as the Level, Slope, and Bow coefficients.

The specification in equation 2.1 is linearly equivalent to the models of Nelson and Siegel (1987), Hunt (1995), and Diebold and Li (2006). The models of Svensson (1994), Bliss (1997), and Mansi and Phillips (2001) are analogous to the specification in equation 2.1, but the exponential terms in those models are allowed to have different decay rates, thus giving them additional flexibility by maturity. Alternatively, appendix A shows that additional flexibility could be added as required by arbitrarily extending the NS model via the systematic addition of higher-order exponential-polynomials from the sequence of orthonormalised Laguerre polynomials.

The (continuously-compounding zero-coupon) interest rate curve associated with the NS forward rate curve has a similar functional form to equation 2.1 except the $\beta_n(t)$ coefficients correspond to interest rate functions, i.e: $R(t,m) = \sum_{n=1}^{3} \beta_n(t) \cdot s_n(\phi, m)$, where $s_n(\phi, m) = \frac{1}{m} \int_0^m g_n(\phi, m)\, dm$. The $s_n(\phi, m)$ functions, or interest rate modes, are

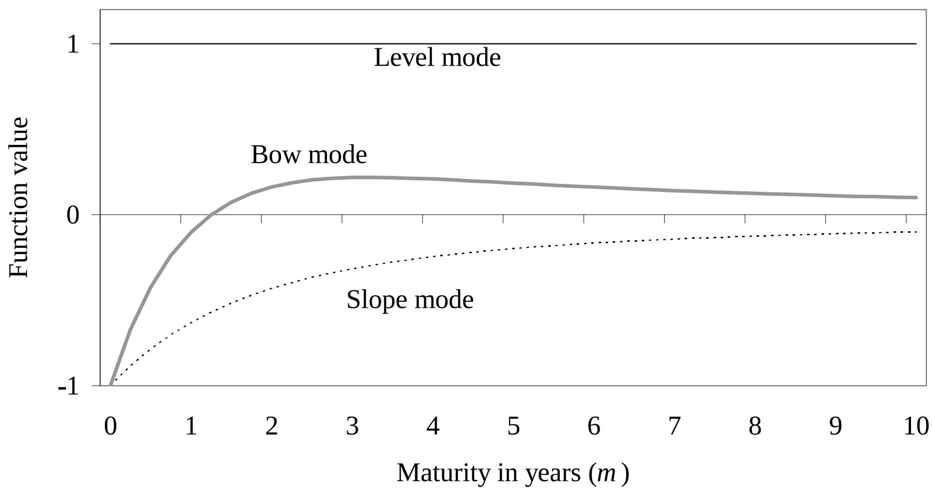

Figure 2.2: The functions $s_1(\phi, m)$, $s_2(\phi, m)$, and $s_3(\phi, m)$ (i.e the Level, Slope, and Bow interest rate modes) for the NS model. $\phi = 1$ for this illustration.

defined below and illustrated in figure 2.2.

$$s_1(\phi, m) = 1 \tag{2.3a}$$

$$s_2(\phi, m) = \frac{1}{\phi m}\left[\exp(-\phi m) - 1\right] \tag{2.3b}$$

$$s_3(\phi, m) = -\frac{1}{\phi m}\left[2\phi m \exp(-\phi m) + \exp(-\phi m) - 1\right] \tag{2.3c}$$

The simple functional form for the interest rate curve facilitates the estimation of the $\beta_n(t)$ coefficients directly from “fitting” cross-sectional observations of market-quoted yield curve data, which is one factor underlying the popularity of NS models. Another factor is that NS models provide sensible and intuitive output in the form of an implied forward and interest rate rate curve that is a parsimonious, continuous, and smooth function of maturity with well-behaved asymptotic properties. In addition, as detailed in Dahlquist and Svensson (1996), Bliss (1997), Seppala and Viertio (1996), Fergusson and Raymar (1998), Subramanian (2001), Ioannides (2003), the empirical results from NS models when applied to yield curve data are typically comparable or superior to more complex and/or customised models of the yield curve.

The popularity of NS models is evident from their frequent use by practitioners, researchers, and academics in a wide variety of markets and applications in finance and economics that require routine yield curve analysis. A comprehensive, but not necessarily exhaustive, list of examples by application includes: (1) forecasting the yield curve as in Fabozzi, Martellini and Priaulet (2005) and Diebold and Li (2006); (2) analysing relative values of fixed interest securities as in Kacala (1993) and Ioannides (2003); (3) deriving monetary policy expectations as in Söderlind and Svensson (1997), Monetary Authority of Singapore (1999), Bank for International Settlements (1999), and Bank for International Settlements (2005);[1] (4) measuring and managing fixed interest portfolio risk as in Barrett, Gosnell and Heuson (1995), Willner (1996), and Diebold, Ji and Li (2005); (5) investigating relationships between the yield curve and macroeconomic time-series data as in Diebold, Rudebusch and Aruoba (2005); (6) investigating interest rate swap spreads as in Brooks and Yong Yan (1999), Fang and Muljono (2003), and Jankowitsch and Pilcher (2004); and (7) providing estimates of zero-coupon yields for subsequent empirical analysis as in Diaz and Skinner (2001), Soto (2001), Schmidt and Kalemanova (2002), and Steeley (2004).

When the NS model specified in equation 2.1 is applied to yield curves observed at different points in time, the natural temptation is to treat it as a dynamic model of the yield curve; i.e as if the NS coefficients were state variables with stochastic dynamics to allow for unanticipated changes to the shape of the yield curve as time evolves. Even if that assumption is not made explicitly, it is implicit when NS coefficients are subjected to time-series analysis, as with many of the applications already noted above.

However, Björk and Christensen (1999), Filipović (1999), and Filipović (2000) have shown explicitly that NS models cannot be intertemporally consistent; i.e the yield curves specified by an NS model at different points in time cannot be mapped to each

[1] The latter two articles contain sub-articles and further references regarding ten central banks (of twelve surveyed) that use NS models.

other via an underlying stochastic time-series process. Bayraktar, Chen and Poor (2005) further extends those results to rule out intertemporal consistency even allowing for jump-diffusions. These conclusions are not surprising, because NS models were originally proposed as static models of the yield curve, which as noted in the introductory chapter emphasise simple relationships by maturity and overlook the dynamics of interest rates over time. The lack of intertemporal consistency in NS models means it is not strictly valid to use them in applications involving a time-series context or where no-arbitrage consistency is required. Fortunately, the following section shows that only a subtle adjustment to the NS approach is required to create a complete dynamic model of the yield curve that is theoretically consistent across both time and maturity (i.e intertemporally consistent and arbitrage-free).

2.3 The ANS model of the forward rate curve

The derivation of the ANS model in this section proceeds in four parts. Section 2.3.1 specifies and discusses the essential assumptions, definitions, and notation involved in constructing the ANS model of the forward rate curve. Section 2.3.2 derives the ANS model, and section 2.3.3 discusses the intuition behind the functional form of the ANS model from an economic and financial perspective. Section 2.3.4 discusses the empirical estimation of the ANS model, including the calculation of the ANS interest rate curve from the ANS forward rate curve.

2.3.1 The assumptions underlying the ANS model of the forward rate curve

The derivation of the ANS model of the forward rate curve is based on the HJM framework, which will be detailed in the following section. The essential intuition at this stage is that, at each point in time, the HJM framework under the physical measure specifies the required relationship between: (1) the forward rate curve; (2) the expected path of the short rate; (3) the volatility structure that dictates how the entire forward rate curve and the expected path of the short rate can potentially change due to stochastic factors; and (4) the market prices of risk. Defining functional forms for the latter three components therefore defines the functional form for the forward rate curve.

The three assumptions below define the functional forms required to derive the ANS model, and also provide a brief discussion of the practical interpretation of the assumptions. Note that the assumptions and the subsequent derivation of the ANS model uses the time and time-to-maturity notation already defined for the NS model in section 2.2; the following section explicitly relates that notation to the time and time-of-maturity used in the HJM framework.

Assumption 1: At time t and as a function of future time $t+m$ ($m \geq 0$), the expected path of the (instantaneous) short rate $E_t\left[r(t+m)\right]$ under the physical measure is defined as:

$$E_t\left[r(t+m)\right] = \sum_{n=1}^{3} \lambda_n\left(t\right) \cdot g_n\left(\phi, m\right) \tag{2.4}$$

where E_t is the expectations operator conditional upon information available as at time t; $\lambda_n\left(t\right)$ are time-varying coefficients, and $g_n\left(\phi, m\right)$ are the modes defined in section 2.2. Equation 2.4 provides the link to the NS model by representing $E_t\left[r(t+m)\right]$ with the modes defined in section 2.2. As time evolves, the market continuously incorporates unpredictable new information relevant to the assessment of $E_t\left[r(t+m)\right]$, and so it will be subjected to unanticipated changes, as defined in the the next assumption.

Assumption 2: Instantaneous stochastic changes to the forward rate curve are defined as:

$$\sum_{n=1}^{3} \sigma_n \cdot g_n(\phi, m) \cdot dW_n\left(t\right) \tag{2.5}$$

where σ_n are constant volatility parameters (i.e standard deviations), and $dW_n(t)$ are Wiener increments under the physical measure. Equation 2.5 represents the impact of unpredictable new information as instantaneous stochastic changes to $f(t,m)$, which will simultaneously be reflected in $E_t[r(t+m)]$. Hence, $E_t[r(t+dt+m)]$ will evolve as $\sum_{n=1}^{3}[\lambda_n(t) + \sigma_n \cdot dW_n(t)] \cdot g_n(\phi, m)$, and so the coefficients $\lambda_n(t)$ will change with a stochastic component $\sigma_n \cdot dW_n(t)$ as time evolves. Equation 2.5 implies that potential stochastic changes to each $\lambda_n(t)$ coefficient, i.e $\sigma_n \cdot dW_n(t)$, are homoskedastic and independent over time (i.e the innovation variance-covariance matrix is constant and diagonal). This assumption and the assumption below regarding the market prices of risk result in the most tractable, parsimonious, and intuitive ANS model. However, appendix A discusses how the ANS model assumptions could be relaxed or generalised if particular applications required that extra flexibility (subject, of course, to the typical trade-off against model tractability, parsimony, and intuition).

Assumption 3: The market prices of risk associated with each mode, i.e ρ_n, are constants. An explicit allowance for the market prices of risk is required because, as with the NS model, the ANS model will inevitably be estimated directly from data observed in a non-risk-neutral environment (i.e under the physical measure). That is, practical yield curve data embeds term premia that compensate investors for bearing risks (i.e potential stochastic changes in the capital value of fixed interest securities) relative to the risk-free investment of a rolling investment in the short rate. In the ANS model, the sources of risk are the stochastic elements $\sigma_n \cdot dW_n(t)$, and the market prices of those risks are assumed to be the constants ρ_n.

2.3.2 The derivation of the ANS model of the forward rate curve

The derivation of the ANS model of the forward rate curve proceeds in four parts: (1) outlining the relevant details of the HJM framework; (2) calculating the effect that volatility in the ANS model coefficients have on the shape of the forward rate curve; (3) calculating the effect that the market prices of risk in the ANS model have on the shape of the forward rate curve; and (4) combining the results together to obtain the ANS model of the forward rate curve.

The HJM framework

The essence of the HJM framework is that it specifies the evolution of the entire forward rate curve via a stochastic process. That is, equation 4 from the HJM framework specifies the dynamics of the forward rate curve under the physical measure as:

$$f(t,T) - f(0,T) = \sum_{n=1}^{N} \int_0^t \alpha_n(v,T)\, dv + \sum_{n=1}^{N} \int_0^t \sigma_n(v,T)\, dW_n(v) \tag{2.6}$$

where $f(t,T)$ is the forward rate curve at time t as a function of time-of-maturity T ($T \geq t$); $f(0,T)$ is the initial forward rate curve at time $t = 0$; N is the number of independent stochastic processes that impart instantaneous random changes to the forward rate curve; $\alpha_n(v,T)$ is the drift or deterministic component for the process n;[2] $\sigma_n(v,T)$ is the volatility function for the process n; $dW_n(v)$ are independent Wiener variables under the physical measure; and v is a dummy integration variable. Equation 18 from HJM specifies that for the avoidance of arbitrage the drift term $\alpha_n(v,T)$ must be restricted to the following form:

$$\alpha_n(v,T) = \sigma_n(v,T)\left[-\phi_n(v) + \int_v^T \sigma_n(v,y)\, dy\right] \tag{2.7}$$

where $\phi_n(v)$ is the market price of risk for process n, and y is a dummy integration variable.

[2]The drift is expressed in this thesis the sum of components $\alpha_n(v,T)$ rather than just the total $\alpha(v,T)$ as expressed in HJM. Hence, the HJM expression $\alpha(v,T) = \sum_{n=1}^{N} \alpha_n(v,T)$, and the HJM expression $\int_0^t \alpha(v,T)\, dv = \int_0^t \sum_{n=1}^{N} \alpha_n(v,T)\, dv = \sum_{n=1}^{N} \int_0^t \alpha_n(v,T)\, dv$.

The relationship between the short rate and the forward rate in the HJM framework is specified in equation 5 from HJM, i.e:

$$r(t) = f(0,t) + \sum_{n=1}^{N} \int_0^t \alpha_n(v,t)\,dv + \sum_{n=1}^{N} \int_0^t \sigma_n(v,t)\,dW_n(v) \tag{2.8}$$

where $r(t)$ is the (instantaneous) short rate at time t. Note that this expression shows that the stochastic dynamics of $r(t)$ are identical to those of $f(0,t)$, given both are driven by the same stochastic processes. Regarding the deterministic component for $r(t)$, equation 25 from HJM specifies the evaluation of the drift integral as:

$$\int_0^t \alpha_n(v,t)\,dv = -\int_0^t \sigma_n(v,t)\,\phi_n(v)\,dv + \int_0^t \sigma_n(v,t)\left[\int_v^t \sigma_n(v,y)\,dy\right]dv \tag{2.9}$$

and substituting that result into equation 2.8 gives:

$$\begin{aligned} r(t) &= f(0,t) + \sum_{n=1}^{N} \int_0^t \sigma_n(v,t)\left[\int_v^t \sigma_n(v,y)\,dy\right]dv \\ &\quad - \sum_{n=1}^{N} \int_0^t \sigma_n(v,t)\,\phi_n(v)\,dv + \sum_{n=1}^{N} \int_0^t \sigma_n(v,t)\,dW_n(v) \end{aligned} \tag{2.10}$$

Equation 2.10 can equally be expressed with the time-of maturity index T, i.e:

$$\begin{aligned} r(T) &= f(0,T) + \sum_{n=1}^{N} \int_0^T \sigma_n(v,T)\left[\int_v^T \sigma_n(v,y)\,dy\right]dv \\ &\quad - \sum_{n=1}^{N} \int_0^T \sigma_n(v,T)\,\phi_n(v)\,dv + \sum_{n=1}^{N} \int_0^T \sigma_n(v,T)\,dW_n(v) \end{aligned} \tag{2.11}$$

In this expression, the term $f(0,T)$ may be obtained by re-ordering equation 2.6 and substituting $\alpha_n(v,T)$ from equation 2.7, i.e:

$$f(0,T) = f(t,T) - \sum_{n=1}^{N} \int_0^t \alpha_n(v,T)\,dv - \sum_{n=1}^{N} \int_0^t \sigma_n(v,T)\,dW_n(v) \tag{2.12a}$$

$$\begin{aligned} &= f(t,T) - \sum_{n=1}^{N} \int_0^t \sigma_n(v,T)\left[\int_v^T \sigma_n(v,y)\,dy\right]dv \\ &\quad + \sum_{n=1}^{N} \int_0^T \sigma_n(v,T)\,\phi_n(v)\,dv - \sum_{n=1}^{N} \int_0^t \sigma_n(v,T)\,dW_n(v) \end{aligned} \tag{2.12b}$$

Substituting this result for $f(0,T)$ into equation 2.11 then gives:

$$\begin{aligned} r(T) &= f(t,T) + \sum_{n=1}^{N} \int_0^T \sigma_n(v,T) \left[\int_v^T \sigma_n(v,y)\, dy \right] dv \\ &\quad - \sum_{n=1}^{N} \int_0^t \sigma_n(v,T) \left[\int_v^T \sigma_n(v,y)\, dy \right] dv \\ &\quad - \sum_{n=1}^{N} \int_0^T \sigma_n(v,T)\, \phi_n(v)\, dv + \sum_{n=1}^{N} \int_0^t \sigma_n(v,T)\, \phi_n(v)\, dv \\ &\quad + \sum_{n=1}^{N} \int_0^T \sigma_n(v,T)\, dW_n(v) - \sum_{n=1}^{N} \int_0^t \sigma_n(v,T)\, dW_n(v) \end{aligned} \tag{2.13}$$

where identical integrals with different limits of integration have been matched for transparency. These integrals may be combined to give:

$$\begin{aligned} r(T) &= f(t,T) + \sum_{n=1}^{N} \int_t^T \sigma_n(v,T) \left[\int_v^T \sigma_n(v,y)\, dy \right] dv \\ &\quad - \sum_{n=1}^{N} \int_t^T \sigma_n(v,T)\, \phi_n(v)\, dv + \sum_{n=1}^{N} \int_t^T \sigma_n(v,T)\, dW_n(v) \end{aligned} \tag{2.14}$$

For the ANS model, assumption 2 in section 2.3.1 specifies that volatility functions are functions of time-to-maturity $m = T - t$ rather than time itself, and assumption 3 specifies that the market prices of risk are constant. Under these assumptions, the integrals over time and time-of-maturity may readily be transformed into integrals over time-to-maturity via integration by substitution. Specifically, the integral for the market price of risk term $\int_t^T \sigma_n(v,T)\, \phi_n(v)\, dv$ will be of the form $\int_t^T \sigma_n(v-t, T-t)\, \rho_n dv$, and with the substitution of $m = T - t$ and $s = v - t$ (hence $ds = dv$, and the limits of integration become $s(T) = T - t = m$, and $s(t) = t - t = 0$), $\int_t^T \sigma_n(v-t, T-t)\, \rho_n dv = \int_0^m \sigma_n(s,m)\, \rho_n ds$. Similarly, the integral for the stochastic term $\int_t^T \sigma_n(v,T)\, dW_n(v)$ is of the form $\int_t^T \sigma_n(v-t, T-t)\, dW_n(v-t) = \int_0^m \sigma_n(s,m)\, dW_n(s)$. The inner integral for the volatility expression $\int_v^T \sigma_n(v,y)\, dy$ will be of the form $\int_v^T \sigma_n(v-t, y-t)\, dy$, and with the substitution of $u = y - t$ (hence $du = dy$, and the limits of integration become $u(T) = T - t = m$, and $u(v) = v - t = s$), $\int_v^T \sigma_n(v,y)\, dy = \int_s^m \sigma_n(s,u)\, du$. The outer integral for the volatility expression may then be transformed as for the market price of risk term above, i.e $\int_t^T \sigma_n(v,T)\left[\int_s^m \sigma_n(v,u)\,du\right]dv$ is of the form $\int_t^T \sigma_n(v-t,T-t)\left[\int_s^m \sigma_n(v,u)\,du\right]dv = \int_0^m \sigma_n(s,m)\left[\int_s^m \sigma_n(s,u)\,du\right]ds$. Substituting these results into equation 2.14 and substituting $t+m$ for T in $r(T)$ and $f(t,T)$ then gives the expression:

$$\begin{aligned} r(t+m) &= f(t,t+m) + \sum_{n=1}^{N}\int_0^m \sigma_n(s,m)\left[\int_s^m \sigma_n(s,u)\,du\right]ds \\ &\quad -\sum_{n=1}^{N}\int_0^m \sigma_n(s,m)\,\rho_n ds + \sum_{n=1}^{N}\int_0^m \sigma_n(s,m)\,dW_n(s) \end{aligned} \tag{2.15}$$

Finally, note that the HJM time and time-of-maturity notation for $f(t,T)$ or $f(t,t+m)$ for the forward rate curve may be equivalently expressed using the time and time-to-maturity notation used for market models, as discussed in James and Webber (2000) p. 208 and originally introduced in Brace, Gatarek and Musiela (1997) (hereafter BGM). Specifically, $f(t,T) = f(t,t+m) = f_{\mathrm{HJM}}(t,t+m) = f_{\mathrm{BGM}}(t,m) = f(t,m)$.

In summary, the HJM framework with the ANS model assumptions defines the relationship between the forward rate curve and the short rate as:

$$\begin{aligned} r(t+m) &= f(t,m) + \sum_{n=1}^{N}\int_0^m \sigma_n(s,m)\left[\int_s^m \sigma_n(s,u)\,du\right]ds \\ &\quad -\sum_{n=1}^{N}\int_0^m \sigma_n(s,m)\,\rho_n ds + \sum_{n=1}^{N}\int_0^m \sigma_n(s,m)\,dW_n(s) \end{aligned} \tag{2.16}$$

where $r(t+m)$ is the short rate at time $t+m$; $f(t,m)$ is the forward rate curve at time t, as a function of time to maturity m ($m \geq 0$); N is the number of independent stochastic processes that impart instantaneous random changes to the forward rate curve; $\sigma_n(s,m)$ is the volatility function for the process n; ρ_n is the market price of risk for the process n; $dW_n(s)$ are independent Wiener variables under the physical measure; and u and s are dummy integration variables.

Applying the expectations operator as at time t to equation 2.16 and re-arranging provides a relationship that will hold at any point in time, i.e:

$$\begin{aligned} f(t,m) &= E_t\left[r(t+m)\right] - \sum_{n=1}^{N} \int_0^m \sigma_n(s,m) \left[\int_s^m \sigma_n(s,u)\,du\right] ds \\ &\quad + \sum_{n=1}^{N} \int_0^m \sigma_n(s,m)\,\rho_n ds \end{aligned} \tag{2.17}$$

where $E_t\left[r(t+m)\right]$ is the expected value, conditional upon information available at time t, of the short rate at time $t+m$ (hereafter this expression in words is abbreviated to "the expected path of the short rate"); and the expectation of the stochastic term in equation 2.16 is zero (see Ross (1997) pp. 541-542).

From the perspective of the ANS model, the functional form for $E_t\left[r(t+m)\right]$ has already been specified in equation 2.4, and it remains to calculate the HJM integral terms using the definitions and assumptions noted in section 2.3.1.[3]

The volatility structure in the ANS model

The HJM integral terms $\int_0^m \sigma_n(s,m)\left[\int_s^m \sigma_n(s,u)\,du\right]$ for the ANS model are calculated from the volatility functions $\sigma_n \cdot g_n(\phi, m)$. The results for the first and second modes are already reported in HJM eq. 39; i.e $\sigma_1^2 \cdot \frac{1}{2}m^2$ and $\sigma_2^2 \cdot \frac{1}{2\phi^2}\left[1-\exp(-\phi m)\right]^2$.[4] The integral term for the third volatility function may be calculated by a routine application of the HJM approach to $\sigma_3 \cdot g_3(\phi, m)$. Following HJM, $\int_s^m \sigma_n(s,u)\,du$ is first calculated as: $\sigma_3 \int_s^m \exp\left[-\phi(u-s)\right]\left[-2\phi(u-s)+1\right] du = \frac{\sigma_3}{\phi}\exp(-\phi m)\exp(\phi s)(1+2\phi m-2s\phi) - \frac{\sigma_3}{\phi}$.

[3]Regarding the six HJM conditions, regularity in the money-market account (C.2), and regularity in bond prices (C.3) are assured because $g_n(\phi, m)$ are smooth, bounded, analytical functions; the existence of market prices of risk (C.4) is as explicitly defined; the uniqueness of the equivalent martingale measure (C.5) is assured because no $g_n(\phi, m)$ function is a linear combination of the other modes (and so its first integral cannot be either); and the existence of the forward rate process (C.1) and the common equivalent martingale measures (C.6) are shown in section 2.4.

[4]HJM uses $\lambda/2$ instead of ϕ, but substituting 2ϕ for λ and simplifying gives the result noted here.

The expression $\int_0^m \sigma_n(s,m)\left[\int_s^m \sigma_n(s,u)\,du\right] ds$ is then:

$$\begin{aligned} & \sigma_3^2 \int_0^m \exp\left[-\phi(m-s)\right]\left([-2\phi(m-s)+1\right] \\ & \times \left[\frac{1}{\phi}\exp(-\phi m)\exp(\phi s)(1+2\phi m-2\phi s)-\frac{1}{\phi}\right] ds \qquad (2.18a) \\ = \; & \sigma_3^2 \cdot \frac{1}{2\phi^2}\left[\exp(-\phi m)+2\phi m\exp(-\phi m)-1\right]^2 \qquad (2.18b) \end{aligned}$$

The market prices of risk in the ANS model

The HJM integral terms $\int_0^m \sigma_n(s,m)\,\rho_n ds$ for the ANS model are also calculated from the volatility functions $\sigma_n \cdot g_n(\phi, m)$. For the first mode of the ANS model $\int_0^m \sigma_1 \cdot g_1(\phi,s)\cdot\rho_1 ds = \sigma_1\rho_1 m$. For the second and third modes, the resultant integrals may be re-written in terms of the $g_n(\phi,m)$ modes. Respectively, $\int_0^m \sigma_2 \cdot g_2(\phi,m)\cdot\rho_2 ds = \frac{\sigma_2\rho_2}{\phi}\cdot\exp(-\phi m) - \frac{\sigma_2\rho_2}{\phi} = -\frac{\sigma_2\rho_2}{\phi}\cdot g_2(\phi,m) - \frac{\sigma_2\rho_2}{\phi}\cdot g_1(\phi,m)$, and $\int_0^m \sigma_3\cdot g_3(\phi,m)\cdot\rho_3 ds = \frac{\sigma_3\rho_3}{\phi}\cdot\exp(-\phi m)(-2m\phi-1) + \frac{\sigma_3\rho_3}{\phi} = \frac{\sigma_3\rho_3}{\phi}\cdot\exp(-\phi m)(-2m\phi+1) - \frac{2\sigma_3\rho_3}{\phi}\cdot\exp(-\phi m) + \frac{\sigma_3\rho_3}{\phi} = \frac{\sigma_3\rho_3}{\phi}\cdot g_3(\phi,m) - \frac{2\sigma_3\rho_3}{\phi}\cdot g_2(\phi,m) + \frac{\sigma_3\rho_3}{\phi}\cdot g_1(\phi,m)$.

Hence, for the ANS model $\sum_{n=1}^3 \int_0^m \sigma_n(s,m)\,\rho_n ds = \sum_{n=1}^3 \gamma_n \cdot g_n(\phi,m)$, where $\gamma_1 = \frac{1}{\phi}(-\sigma_2\rho_2 + \sigma_3\rho_3)$, $\gamma_2 = \frac{1}{\phi}(-\sigma_2\rho_2 - 2\sigma_3\rho_3)$, and $\gamma_3 = \frac{1}{\phi}\sigma_3\rho_3$.

The ANS model forward rate curve

Substituting the results from the previous two sub-sections into equation 2.17 gives the result that the ANS forward rate curve $f(t,m)$, at time t as a function of time to maturity m and under the physical measure, will have the following functional form:

$$f(t,m) = \sigma_1\rho_1 m + \sum_{n=1}^{3}\beta_n(t)\cdot g_n(\phi,m) - \sum_{n=1}^{3}\sigma_n^2\cdot h_n(\phi,m) \qquad (2.19)$$

where $\beta_n(t) = \gamma_n + \lambda_n(t)$, with $\gamma_1 = \frac{1}{\phi}(-\sigma_2\rho_2 + \sigma_3\rho_3)$, $\gamma_2 = \frac{1}{\phi}(-\sigma_2\rho_2 - 2\sigma_3\rho_3)$, $\gamma_3 = \frac{1}{\phi}\sigma_3\rho_3$ (all constants), and $h_1(\phi,m) = \frac{1}{2}m^2$, $h_2(\phi,m) = \frac{1}{2\phi^2}\left[1-\exp(-\phi m)\right]^2$, $h_3(\phi,m) = \frac{1}{2\phi^2}\left[1-\exp(-\phi m)-2\phi m\exp(-\phi m)\right]^2$ (all time-invariant functions of maturity).

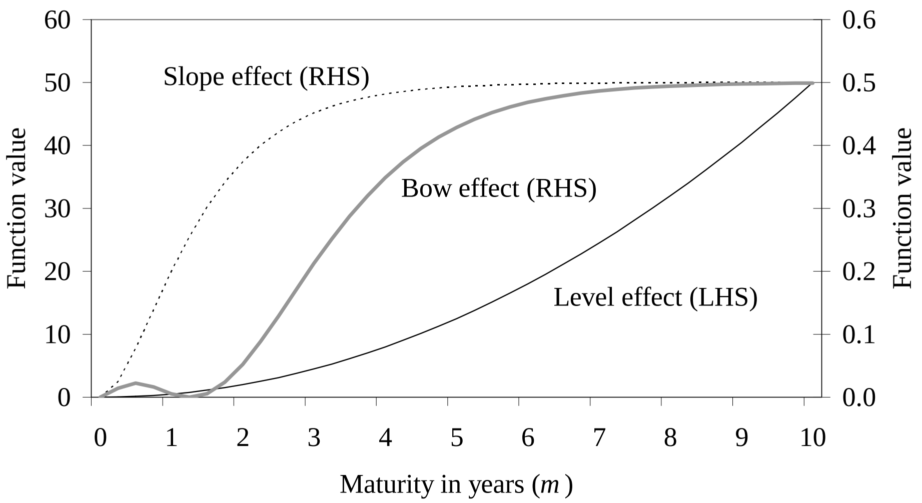

Figure 2.3: The functions $h_1(\phi, m)$, $h_2(\phi, m)$, and $h_3(\phi, m)$ (i.e Level, Slope, and Bow volatility effects) for the ANS forward rate curve. $\phi = 1$ for this illustration.

2.3.3 Discussion of the ANS model of the forward rate curve

It is evident that the ANS model retains a functional form by maturity similar to the NS model, but with two series of augmentations. The first series is $\sum_{n=1}^{3} \sigma_n^2 \cdot h_n(\phi, m)$, which arises from the volatilities of the ANS model coefficients $\lambda_n(t)$ as reflected directly in the volatilities of the coefficients $\beta_n(t)$. The $h_n(\phi, m)$ functions are illustrated in figure 2.3, and may be interpreted as the effects on the shape of the forward rate curve per unit of variance in the stochastic component of each $\beta_n(t)$ coefficient.[5]

The second series of augmentations is related to the market prices of risk applied to the volatilities of the ANS model coefficients. Volatility in the first coefficient of the ANS model leads to an augmentation of non-NS form, i.e $\sigma_1 \rho_1 m$. The remaining augmentations are expressible as $\sum_{n=1}^{3} \gamma_n \cdot g_n(\phi, m)$, and are therefore subsumed directly into $\sum_{n=1}^{3} \beta_n(t) \cdot g_n(\phi, m)$. Together, $\sigma_1 \rho_1 m + \sum_{n=1}^{N} \gamma_n \cdot g_n(\phi, m)$ has the intuitive interpretation as a risk premium function, i.e a time-invariant function of maturity that drives a wedge between

[5] $\sigma_n^2 \cdot h_n(\phi, m)$ are the drift terms in the HJM framework, and may also be seen as "manifold expansions" (i.e the addition of appropriate functions of maturity) analogous to those suggested by Björk and Christensen (1999) pp. 338-339 to make the NS model consistent with the Hull and White (1990) model.

the forward rate curve that would prevail in a stochastic but risk-neutral environment, and the forward rate curve that prevails in practice because risk comes at a price.

Note that the ANS model nests the NS model as a special case where all the volatilities and market prices of risk are set to zero, i.e a deterministic and risk-neutral model of the yield curve. This clearly exposes the theoretical shortcomings of the NS model, because practical yield curve data cannot be expected to accord with these assumptions. Alternatively, this shows again that NS models are only valid for use as static models of the yield curve, and that they should not be applied in a time-series context.

2.3.4 The estimation of the ANS model from market-quoted interest rate data

From a practical perspective, deriving the ANS model of the forward rate curve as a simple function of time to maturity makes its estimation process very similar to the NS model. That is, the interest rate curve is obtained by integrating the forward rate curve, and the ANS model coefficients $\beta_n(t)$ at each point in time may still be estimated directly from the observation of market-quoted yield curve data and the cashflows of the securities that define the yield curve at that point in time. The first part of this section details the aspects that need to be considered in the estimation process for the ANS model, and the second part illustrates the empirical application of the estimation process.

The ANS model of the interest rate curve and its connection to yield curve data

The securities that define the yield curve are typically coupon-bearing, and so the estimation of the ANS model based on those securities requires an allowance for multiple cashflows, each with a different zero-coupon discount rate corresponding to the timing of the cashflow, i.e:[6]

$$\text{Minimise} \quad : \quad \sum_{k=1}^{K} (w_k \cdot \varepsilon_k)^2 \tag{2.20a}$$

$$\text{where} \quad : \quad \varepsilon_k = \sum_{j=1}^{J[k]} a_{kj} \cdot \exp\left[-m_{kj} \cdot R(t, m_{kj})\right] \tag{2.20b}$$

$$\text{and} \quad : \quad R(t,m) = \frac{\sigma_1 \rho_1 m}{2} + \sum_{n=1}^{3} \beta_n(t) \cdot s_n(\phi, m) - \sum_{n=1}^{3} \sigma_n^2 \cdot u_n(\phi, m) \tag{2.20c}$$

where K is the number of fixed interest securities used to define the yield curve; w_k is a weighting factor, which is set to the inverse of the "basis point value" (i.e the price change of the security for a yield change of a single basis point) to obtain a minimisation of yield residuals; $J[k]$ is the number of cashflows for security k; a_{kj} is the magnitude of the cashflow j for security k (defined to be negative for the settlement price, and positive for all cashflows beyond settlement); m_{kj} is the maturity of the cashflow j of security k; and $R(t, m_{kj})$ is the zero-coupon interest rate for maturity m_{kj}.

The zero-coupon interest rates in equation 2.20c are $R(t,m) = \frac{1}{m}\int_0^m f(m)dm$, so $s_n(\phi,m) = \frac{1}{m}\int_0^m g_n(\phi,m)dm$ and $u_n(\phi,m) = \frac{1}{m}\int_0^m h_n(\phi,m)dm$. The functions $s_n(\phi,m)$ are those from equation 2.3. The functions $u_n(\phi,m)$ are defined below and illustrated in figure 2.4.

$$u_1(\phi,m) = \frac{1}{6}m^2 \tag{2.21a}$$

$$u_2(\phi,m) = \frac{1}{4\phi^3 m}\left[2\phi m - 3 + 4\exp(-\phi m) - \exp(-2\phi m)\right] \tag{2.21b}$$

$$u_3(\phi,m) = \frac{1}{4\phi^3 m}\left[\begin{array}{c} 2\phi m - 7 + (12 + 8\phi m)\exp(-\phi m) \\ + \left(-5 - 8\phi m - 4\phi^2 m^2\right)\exp(-2\phi m) \end{array}\right] \tag{2.21c}$$

Given the parameters ρ_1, ϕ, σ_1, σ_2, and σ_3 in equation 2.20c, the system of equations 2.20 is readily optimised using the Newton-Raphson technique to obtain $\beta_n(t)$.

[6] This is the most widely used approach for estimating the coefficients of NS models directly from market-quoted data, and is outlined in the articles in Bank for International Settlements (2005). Zero-coupon interest rate data could also be used by specifying just two cashflows for each security. However, that zero-coupon data would originally be derived from market-quoted coupon-bearing data anyway, and so the direct estimation method is more efficient.

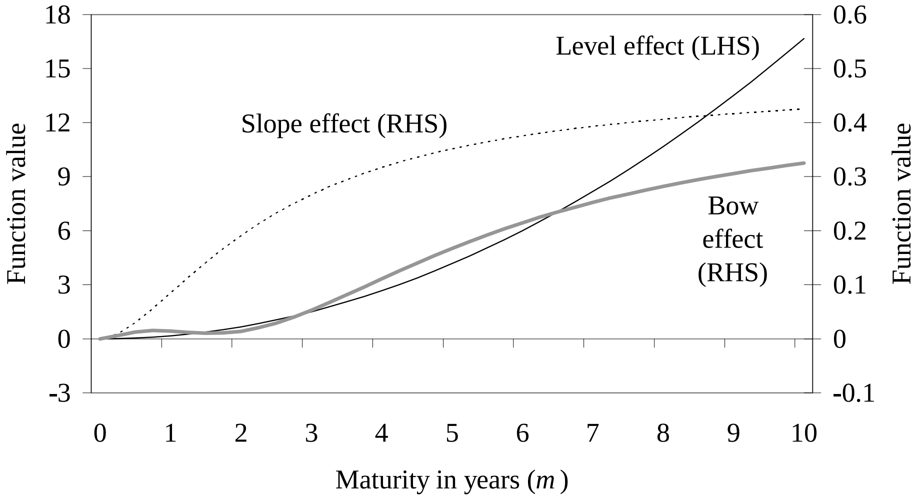

Figure 2.4: The functions $u_1(\phi, m)$, $u_2(\phi, m)$, and $u_3(\phi, m)$ (i.e Level, Slope, and Bow volatility effects) for the ANS interest rate curve. $\phi = 1$ for this illustration.

Of course, the parameters ρ_1, ϕ, σ_1, σ_2, and σ_3 must also be estimated themselves. To ensure consistency across time, they must be estimated based on the available (or the appropriate) historical data, not just the current observation of the yield curve. This is also analogous to typical estimations of the NS model in practice, where a single value of ϕ is selected to provide the "best fit" (e.g to minimise total squared or absolute residuals) over the historical data available, rather than being allowed to change independently for each yield curve observation.

In principle, the estimation of the time-varying ANS model coefficients in conjunction with the constant ANS model parameters across time could be undertaken using any joint estimation process (e.g maximum likelihood with panel data). However, a more convenient method that is also asymptotically efficient is a simple grid search on the unrestricted parameters ρ_1 and ϕ, while σ_1, σ_2, and σ_3 are calculated to be internally consistent. Specifically: (1) select a combination of ρ_1 and ϕ from the pre-defined allowable values in the grid; (2) set $\sigma_n^2 = 0$ (the parameters for equation 2.20c are now all defined); (3) for each historical yield curve observation (which defines the cashflows in equation 2.20b), estimate the $\beta_n(t)$ coefficients in the system 2.20 using the Newton-Raphson technique; (4) using these initial estimates of the $\beta_n(t)$ coefficients, calculate estimates of the σ_1, σ_2, and σ_3 using the usual definition of annualised variance noted in Hull (2000) pp. 368-369; i.e $\sigma_n^2 = \frac{F}{I}\sum_{i=1}^{I}[\Delta\beta_n(i)]^2$, where I is the number of data points, and F is the number of observations per year (e.g 12 for monthly data) required to annualise the variance calculation; (5) re-estimate the $\beta_n(t)$ coefficients using the estimates of the σ_1, σ_2, and σ_3;[7] and (6) record the total sum of squared yield residuals from the actual and estimated yields over the historical estimation period against the combination of ρ_1 and ϕ. Steps 1 to 6 are repeated for all combinations of ρ_1 and ϕ, and the combination associated with the minimum total sum of squared yield residuals gives the estimated parameter values for ρ_1 and ϕ (or the grid search can be refined as required). Note that the estimation of the NS model would equate to a grid search over ϕ using steps 1, 3, and 6 with ρ_1, σ_1, σ_2, and σ_3 set to zero.

The empirical application of the ANS model to market-quoted yield curve data

Anticipating the detailed discussion of the data in the empirical application in section 2.5, figure 2.5 illustrates the estimation of the ANS model by "fitting" United States (US) yield curve data observed for the month of February 2004. Each observation of yield curve data gives an associated estimate of the ANS Level, Slope, and Bow coefficients. Hence, any time series of yield curve observations may be processed into the corresponding three time series of ANS Level, Slope, and Bow coefficients, i.e $\beta_1(t)$, $\beta_2(t)$, and $\beta_3(t)$. Figure 2.6 illustrates the time series of two of the seven yields used to define the yield curve at each point in time, and figure 2.7 plots the three time series of estimated ANS coefficients obtained using the full sample of yield curve data.

[7] Steps 4 and 5 create a two-step process that could be iterated to convergence, but the volatility estimates obtained from the initial estimation of the $\beta_n(t)$ coefficients were immaterially different from subsequent estimates. Also, the results of section 2.4 show that, technically, estimates of $\delta_n(i)$ should be used to calculate the variances for the Slope and Bow coefficients, rather than $\Delta\beta_n(i)$. However, the impact in practice was again immaterial.

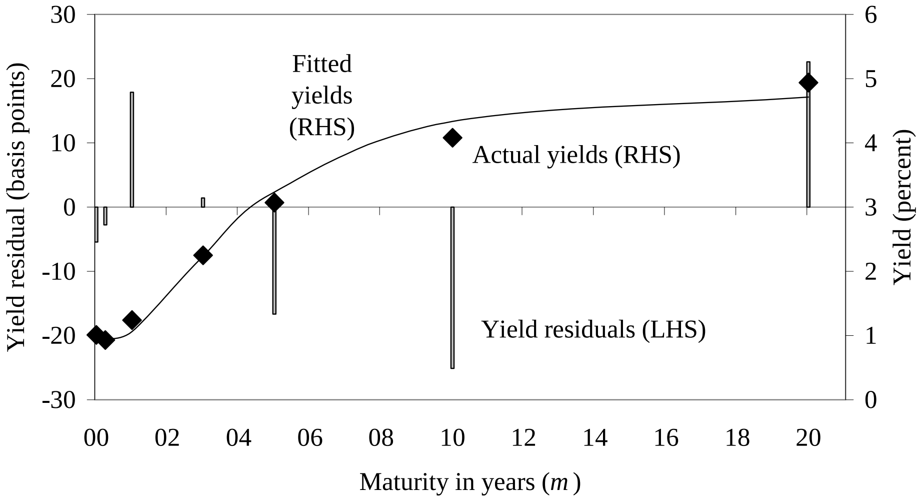

Figure 2.5: US yield curve data for the month of February 2004, and the "fitted" yields based on the estimated ANS model. The estimated Level, Slope, and Bow coefficients are, respectively, β_1 (Feb-04) $= 5.46\%$, β_2 (Feb-04) $= 7.80\%$, and β_3 (Feb-04) $= -3.50\%$. The ANS parameters estimated over the entire sample are $\phi = 1.09$, $\rho_1 = 2.57\%$, $\sigma_1 = 0.79\%$, $\sigma_2 = 2.31\%$, and $\sigma_3 = 1.78\%$.

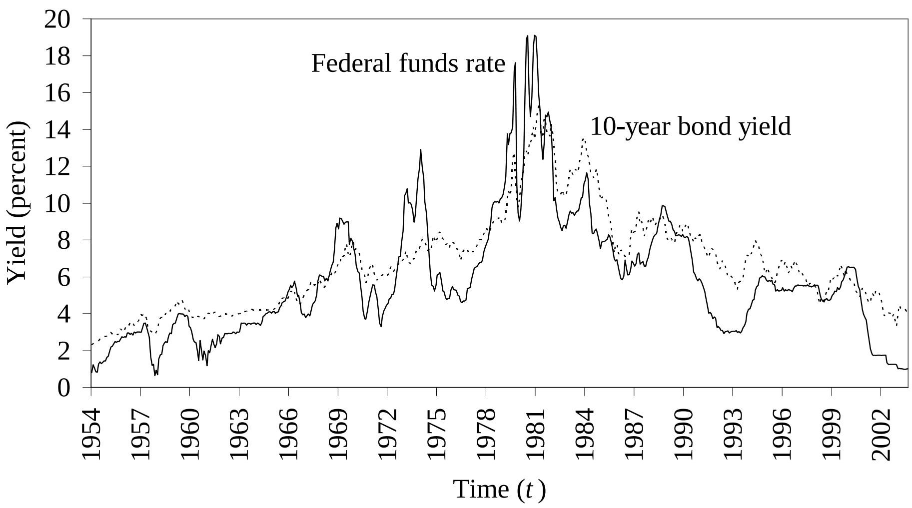

Figure 2.6: The time series of two of seven US interest rates on the yield curve that are used to estimate the time series of ANS Level, Slope, and Bow coefficients plotted in figure 2.7.

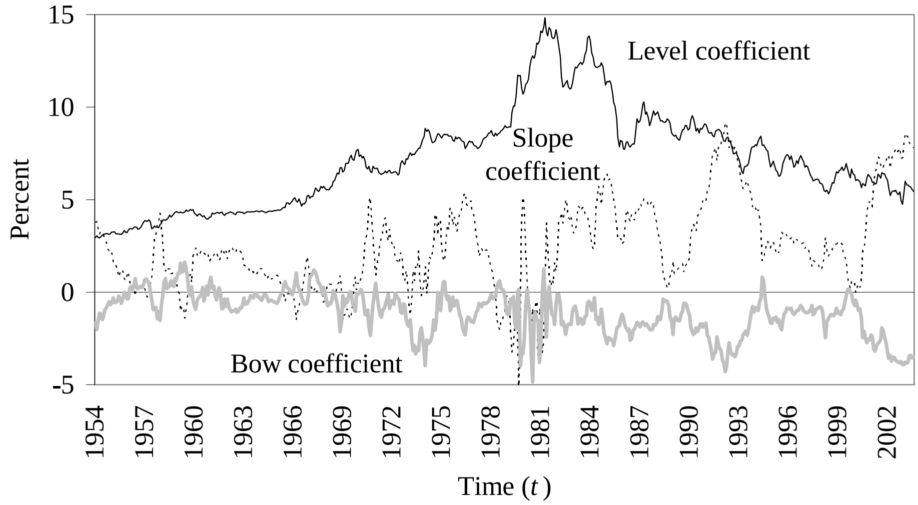

Figure 2.7: The time series of the estimated ANS Level, Slope, and Bow coefficients, i.e $\beta_1(t)$, $\beta_2(t)$, and $\beta_3(t)$. The ANS coefficients at each point in time are estimated using the seven points of yield curve data observed at that point in time, as in the illustration for February 2004 in figure 2.5. The ANS parameters estimated over the entire sample are $\phi = 1.09$, $\rho_1 = 2.57\%$, $\sigma_1 = 0.79\%$, $\sigma_2 = 2.31\%$, and $\sigma_3 = 1.78\%$.

2.4 The intertemporal consistency of the ANS model

While the intertemporal consistency of the generic ANS model is implicit by its construction via the HJM framework, that property can also be illustrated explicitly by deriving the stochastic time-series process for the ANS model coefficients. Section 2.4.1 undertakes that derivation, and section 2.4.2 discusses the intuition behind the derived stochastic time-series process from an economic and financial perspective.

2.4.1 The derivation of the intertemporal relationship for the ANS model coefficients

This derivation of the intertemporal relationship for the ANS model coefficients proceeds in two parts: (1) deriving the intertemporal relationship between expected paths of the short rate as time evolves within the HJM framework; and (2) substituting the expected paths of the short rate as defined within the ANS model into the result from the

HJM framework.

The expected path of the short rate within the HJM framework

Within equation 2.17, define the deterministic component of the HJM framework as $\alpha_n(s,m) = \sigma_n(s,m)\left[-\rho_n + \int_0^m \sigma_n(s,u)\,du\right]$, so that $f(t,m) = E_t[r(t+m)] - \sum_{n=1}^N \int_0^m \alpha_n(s,m)\,ds$. Given a finite time-increment τ, $f(t,\tau+m) = E_t[r(t+\tau+m)] - \sum_{n=1}^N \int_0^{\tau+m} \alpha_n(s,m)\,ds$, and $f(t+\tau,m) = E_{t+\tau}[r(t+\tau+m)] - \sum_{n=1}^N \int_\tau^{\tau+m} \alpha_n(s,m)\,ds$. Substituting these expressions into equation 4 from HJM, i.e $f(t+\tau,m) = f(t,\tau+m) + \sum_{n=1}^N \int_0^\tau \alpha_n(s,m)\,ds + \sum_{n=1}^N \int_t^{t+\tau} \sigma_n(s,m)\,dW_n(s)$, gives the equality $E_{t+\tau}[r(t+\tau+m)] - \sum_{n=1}^N \int_\tau^{\tau+m} \alpha_n(s,m)\,ds = E_t[r(t+\tau+m)] - \sum_{n=1}^N \int_0^{\tau+m} \alpha_n(s,m)\,ds + \sum_{n=1}^N \int_0^\tau \alpha_n(s,m)\,ds + \sum_{n=1}^N \int_t^{t+\tau} \sigma_n(s,m)\,dW_n(s)$.[8] The right-hand side of this equality contains two identical integrals with different upper limits of integration, and combined into a single integral with a new lower limit of integration, i.e $-\sum_{n=1}^N \int_0^{\tau+m} \alpha_n(s,m)\,ds + \sum_{n=1}^N \int_0^\tau \alpha_n(s,m)\,ds = -\sum_{n=1}^N \int_\tau^{\tau+m} \alpha_n(s,m)\,ds$, the latter integral identically cancels with the same term on the left-hand side of the equality, leaving the result:

$$E_{t+\tau}[r(t+\tau+m)] = E_t[r(t+\tau+m)] + \sum_{n=1}^N \int_t^{t+\tau} \sigma_n(s,m)\,dW_n(s) \tag{2.22}$$

This is intuitive; the expected path of the short rate would be realised but for the impact of unpredictable new information represented by the summation of stochastic integrals. The integrals will not necessarily have closed form solutions but $E_t\left[\int_t^{t+\tau} \sigma_n(s,m)\,dW_n(s)\right] = 0$ (see Ross (1997) pp. 541-542).

The expected path of the short rate within the ANS model

It is convenient at this stage to introduce vector notation for the three coefficients in equation 2.4; i.e $\boldsymbol{\lambda}(t) = \{\lambda_1(t), \lambda_2(t), \lambda_3(t)\}'$, which is a 3-vector containing

[8]Note that m on the left-hand side of the equality and $\tau+m$ on the right-hand side of the equality refer to the same future point in time, which is denoted by T (the time of maturity) in HJM.

the three time-varying components of the ANS model coefficients at time t. The expected path of the short rate at times $t+\tau$ and t from equation 2.22 may then be expressed respectively as $E_{t+\tau}\left[r\left(t+\tau+m\right)\right] = \left[\boldsymbol{\lambda}\left(t+\tau\right)\right]' \mathbf{g}\left(\phi, m\right)$, and $E_t\left[r\left(t+\tau+m\right)\right] = \left[\boldsymbol{\lambda}\left(t\right)\right]' \mathbf{g}\left(\phi, \tau+m\right)$, where $\mathbf{g}(\phi, m) = \left\{g_1\left(\phi, m\right), g_2\left(\phi, m\right), g_3\left(\phi, m\right)\right\}'$, and $\mathbf{g}(\phi, \tau+m) = \left\{g_1(\phi, \tau+m), g_2\left(\phi, \tau+m\right), g_3\left(\phi, \tau+m\right)\right\}'$. Each stochastic term $\int_t^{t+\tau} \sigma_n\left(s, m\right) dW_n\left(s\right)$ may be written as $\delta_n\left(t+\tau\right) \cdot g_n\left(\phi, m\right)$ where $\delta_n\left(t+\tau\right)$ has a normal distribution with mean zero and standard deviation σ_n; i.e $\delta_n\left(t+\tau\right) \sim N\left(0, \sigma_n\left[t+\tau\right] - t\right) = N\left(0, \sigma_n \tau\right)$. Hence, $\sum_{n=1}^{N} \int_t^{t+\tau} \sigma_n\left(s, m\right) dW_n\left(s\right)$ may be written in vector form as $\left[\boldsymbol{\delta}\left(t+\tau\right)\right]' \mathbf{g}\left(\phi, m\right)$, where $\boldsymbol{\delta}\left(t+\tau\right) = \left\{\delta_1\left(t+\tau\right), \delta_2\left(t+\tau\right), \delta_3\left(t+\tau\right)\right\}'$. Substituting these expressions into equation 2.22 gives the equality $\left[\boldsymbol{\lambda}\left(t+\tau\right)\right]' \mathbf{g}\left(\phi, m\right) = \left[\boldsymbol{\lambda}\left(t\right)\right]' \mathbf{g}\left(\phi, \tau+m\right) + \left[\boldsymbol{\delta}\left(t+\tau\right)\right]' \mathbf{g}\left(\phi, m\right)$. It may be verified by direct matrix multiplication and simplification that $\mathbf{g}\left(\phi, \tau+m\right) = \left[\boldsymbol{\Phi}\left(\phi, \tau\right)\right]' \mathbf{g}\left(\phi, m\right)$, where $\left[\boldsymbol{\Phi}\left(\phi, \tau\right)\right]'$ is the transpose of the time-invariant 3×3 matrix as a function of ϕ and τ, i.e:

$$\boldsymbol{\Phi}\left(\phi, \tau\right) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \exp\left(-\phi\tau\right) & -2\phi\tau \exp\left(-\phi\tau\right) \\ 0 & 0 & \exp\left(-\phi\tau\right) \end{bmatrix} \tag{2.23}$$

This enables the equality for the expected path of the short rate at times $t+\tau$ and t to be written as $\left[\boldsymbol{\lambda}\left(t+\tau\right)\right]' \mathbf{g}\left(\phi, m\right) = \left[\boldsymbol{\lambda}\left(t\right)\right]' \left[\boldsymbol{\Phi}\left(\phi, \tau\right)\right]' \mathbf{g}\left(\phi, m\right) + \left[\boldsymbol{\delta}\left(t+\tau\right)\right]' \mathbf{g}\left(\phi, m\right)$. Adding the time-invariant risk premium function $\boldsymbol{\gamma}' \mathbf{g}\left(\phi, m\right)$, where $\boldsymbol{\gamma} = \left\{\gamma_1, \gamma_2, \gamma_3\right\}'$, to both sides of the equality then gives $\left[\boldsymbol{\beta}\left(t+\tau\right)\right]' \mathbf{g}\left(\phi, m\right) = \boldsymbol{\gamma}' \mathbf{g}\left(\phi, m\right) + \left[\boldsymbol{\lambda}\left(t\right)\right]' \left[\boldsymbol{\Phi}\left(\phi, \tau\right)\right]' \mathbf{g}\left(\phi, m\right) + \left[\boldsymbol{\delta}\left(t+\tau\right)\right]' \mathbf{g}\left(\phi, m\right)$, where $\boldsymbol{\beta}\left(t+\tau\right) = \boldsymbol{\gamma} + \boldsymbol{\lambda}\left(t+\tau\right) = \left\{\beta_1\left(t+\tau\right), \beta_2\left(t+\tau\right), \beta_3\left(t+\tau\right)\right\}'$.

Factoring out the common term $\mathbf{g}(\phi, m)$, and taking the transpose then gives the result $\boldsymbol{\beta}\left(t+\tau\right) = \boldsymbol{\gamma} + \boldsymbol{\Phi}\left(\phi, \tau\right) \boldsymbol{\lambda}\left(t\right) + \boldsymbol{\delta}\left(t+\tau\right)$. The latter may be rewritten as $\boldsymbol{\beta}\left(t+\tau\right) = \left[\mathbf{I} - \boldsymbol{\Phi}\left(\phi, \tau\right)\right] \boldsymbol{\gamma} + \boldsymbol{\Phi}\left(\phi, \tau\right)\left[\boldsymbol{\gamma} + \boldsymbol{\lambda}\left(t\right)\right] + \boldsymbol{\delta}\left(t+\tau\right)$, where $\mathbf{I}$ is the 3×3 identity matrix, hence giving:

$$\boldsymbol{\beta}(t+\tau) = \boldsymbol{\mu}(\boldsymbol{\phi},\boldsymbol{\tau}) + \boldsymbol{\Phi}(\phi,\tau)\,\boldsymbol{\beta}(t) + \boldsymbol{\delta}(t+\tau) \tag{2.24}$$

where $\boldsymbol{\beta}(t) = \boldsymbol{\gamma} + \boldsymbol{\lambda}(t) = \{\beta_1(t), \beta_2(t), \beta_3(t)\}'$, and $\boldsymbol{\mu}(\boldsymbol{\phi},\boldsymbol{\tau}) = [\mathbf{I} - \boldsymbol{\Phi}(\phi,\tau)]\,\boldsymbol{\gamma}$, which is a time-invariant 3-vector as a function of ϕ and τ. Note that $\boldsymbol{\Phi}(\phi,\tau)$ and $\boldsymbol{\mu}(\boldsymbol{\phi},\boldsymbol{\tau})$ will both be constant for a given value of the constant parameter ϕ and a given time-step τ.

2.4.2 Discussion of the intertemporal relationship for the ANS model coefficients

The intuition underlying the derivation of equation 2.24 is the expectations hypothesis of the yield curve; i.e after allowing for term premia, the maturity τ rate from the forward rate curve at time t implies an expectation of the short rate at time $t+\tau$. Within the HJM framework, the entire forward rate curve at time t defines an expectation of the path of the short rate from time t, which in turn defines an expected path of the short rate at time $t+\tau$ (as shown in the first part of section 2.4.1). In the ANS model, the current and future expected paths of the short rate are represented (to within the constants γ_1, γ_2, and γ_3) by the three coefficients $\beta_1(t)$, $\beta_2(t)$, and $\beta_3(t)$ applied respectively to the time-invariant modes $g_1(\phi, m)$, $g_2(\phi, m)$, and $g_3(\phi, m)$ in section 2.3, and so the expectations hypothesis within the HJM framework condenses into a stochastic time-series processes for the ANS model coefficients in vector form (as shown in the second part of section 2.4.1).

Equation 2.24 is a standard first-order vector autoregressive (VAR1) process in discrete time, and its interpretation is very intuitive from a financial perspective. Firstly, $\boldsymbol{\mu}(\phi,\tau)$ arises directly from the time-invariant risk premium function in the ANS model. Secondly, applying the expectations operator as at time t to equation 2.24 gives the result that $E_t[\boldsymbol{\beta}(t+\tau)] = \boldsymbol{\mu} + \boldsymbol{\Phi}(\phi,\tau)\,\boldsymbol{\beta}(t)$, where $E_t[\boldsymbol{\beta}(t+\tau)]$ is the expected value of $\boldsymbol{\beta}(t+\tau)$ as at time t. Hence, $\boldsymbol{\beta}(t)$ not only summarises the current shape of the yield curve, but in conjunction with the parameters $\boldsymbol{\mu}$ and $\boldsymbol{\Phi}(\phi, \tau)$ it completely summarises the expected evolution of the yield curve. In other words, the coefficients of the ANS model are valid state variables for the yield curve. Thirdly, $\boldsymbol{\delta}(t+\tau)$ is the ex-post realised forecast error, i.e $\boldsymbol{\beta}(t+\tau) - [\boldsymbol{\mu} + \boldsymbol{\Phi}(\phi, \tau)\boldsymbol{\beta}(t)]$, which represents the fact that $\boldsymbol{\beta}(t+\tau)$ will inevitably differ from $E_t[\boldsymbol{\beta}(t+\tau)]$ due to the impact of unpredictable new information that arrives between time t and $t+\tau$.

As time evolves, the current $\boldsymbol{\beta}(t)$ will always reflect the up-to-date expectations embedded in the yield curve, and the change in $\boldsymbol{\beta}(t)$ between any two points in time can be decomposed into an ex-ante expected change and an accumulation of ex-post errors (i.e the deterministic and stochastic components, respectively, of the evolution of $\boldsymbol{\beta}(t)$).

2.5 Forecasting US interest rates and the yield curve with the ANS model

The empirical application of the ANS model in this chapter is to forecasting the yield curve, i.e predicting individual yields and the spreads between yields of different maturities on the future yield curve. Section 2.5.1 describes the data and its context, and sections 2.5.2 and 2.5.3 respectively apply the ANS forecasting framework (i.e the VAR1 process derived in section 2.4 for the ANS model coefficients) without and with term premia to undertake the forecasts.

2.5.1 Description of the data

The interest rate data used are monthly interest rates for constant maturities obtained from the online Federal Reserve Bank of St. Louis economic database (FRED). Specifically, the series used are the federal funds rate (FF), the 3-month Treasury bill rate (TB3), and the yields-to-maturity of the 1-year, 3-year, 5-year, 10-year, and 20-year or

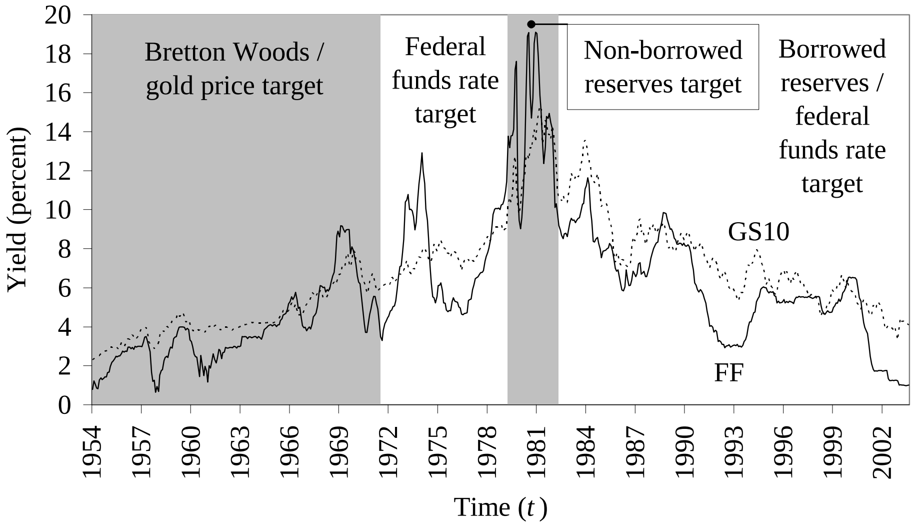

Figure 2.8: The time-series data for the federal funds rate (FF) and the 10-year government bond yield (GS10). The shading indicates the four different monetary policy regimes that prevailed over the sample.

30-year constant maturity bonds (GS1, GS3, GS5, GS10, and GS20 or GS30 respectively). Note that GS20 data is unavailable from January 1987 to September 1993, and so GS30 data is used during this period (with a 30-year maturity in the estimation of the ANS model).

The sample period is July 1954 (the first month FF data is available) to February 2004 (the last month available at the time of the analysis), giving 593 monthly observations of the yield curve. Figure 2.8 illustrates the FF and GS10 data, the longest and shortest maturity rates available for the entire data period. The FF/GS10 spread measure used in the forecast performance analysis is the difference between these two rates.

The sample period spans four distinct monetary policy regimes, as specified in Walsh (1998), and these are used for sub-sample analysis. The regimes are identified in figure 2.8 and are the Bretton Woods / gold price target (start-of-sample to December 1971), the federal funds rate target (January 1972 to September 1979), the non-borrowed reserves target (October 1979 to October 1982), and the borrowed reserves / federal funds rate target (November 1982 to end-of-sample).

The method used to estimate the ANS model coefficients and parameters has already been detailed in section 2.3.4. The cashflows for the bond yield data are assumed to correspond to a par bond for the specified maturity (i.e a settlement price of -1, a principal of 1 at maturity, and semi-annual coupons between settlement and maturity [inclusive] equal to half the yield). This assumption is necessary because the data does not specify the precise maturity and the coupons of the underlying bonds. However, the approximation will be close because new US benchmark bonds are regularly issued at approximately par. Applying the ANS model to the full sample of 593 monthly yield curve observations provides the time series of Level, Slope, and Bow coefficients that have already been illustrated in figure 2.7.

2.5.2 Forecasting interest rates and the yield curve assuming no term premia

As an initial gauge of the importance of term premia with respect to forecasting the yield curve, the first application uses the ANS forecasting framework with no term premia. This is obtained by setting ρ_1 to zero in equation 2.19 and $\boldsymbol{\mu}(\phi, \tau) = \{0, 0, 0\}'$ in equation 2.24.

The forecasting is undertaken out-of-sample using recursive estimation of the ANS framework parameters. Specifically, the first three years of data (July 1954 to June 1957) are used to determine the initial estimates of the parameters ϕ, σ_1, σ_2, and σ_3, and then the following steps are used to obtain yield curve forecasts from the July 1957 yield curve data: (1) $\boldsymbol{\beta}(\text{Jul-57})$ is estimated using the July 1957 yield curve data and the initial estimates of ϕ, and σ_1, σ_2, and σ_3; (2) $\boldsymbol{\beta}(\text{Jul-57})$ is used to obtain the forecasts of $\boldsymbol{\beta}(\text{Jul-57} + \tau)$ for the horizons of 3 months, 6 months, 1 year, 1.5 years, 2 years, and 3 years (i.e using equation 2.24 with $\boldsymbol{\mu}(\phi, \tau) = \{0, 0, 0\}'$ and $\tau = 0.25$, 0.5, 1, 1.5, 2, and 3 respectively); (3) the forecasts of $\boldsymbol{\beta}(\text{Jul-57} + \tau)$ are used in equation 2.19 with the initial estimates of ϕ, σ_1,

σ_2, and σ_3 to obtain forecasts of the forward rate curve and hence the interest rate curve at times Jul-57+τ; (4) the forecast rates or yields-to-maturity for FF, TB3, GS1, GS3, GS5, GS10 at times Jul-57+τ are reconstructed using the forecast interest rate curve;[9] (5) the forecast FF/GS10 spread is calculated as the GS10 forecast less the FF forecast; and (6) the estimates of ϕ, σ_1, σ_2, and σ_3 are updated using all of the historical data up to the current month. This is the most naive method of recursive estimation, and avoids any hint of data mining by using a favourable moving window. Note that the recursively-estimated values of ϕ ranged from 0.83 to 1.33.

These six steps are repeated for each subsequent observation of the yield curve from August 1954 to February 2004, producing six series of forecast yields and the series of forecast FF/GS10 spreads for each of the six forecasting horizons τ. The corresponding forecast errors (in basis points, or bps, where 1 bp = 0.01 percentage points) are calculated as the actual data at time $t + \tau$ less the corresponding forecasts made at time t for horizon τ.

Table 2.1 contains the root-mean-squared forecast errors (RMSEs) for the ANS framework forecasts. Only the results for FF, GS10, and FF/GS10 are shown; the results for intermediate maturities fall between these sets of results. The RMSEs broadly show an increase by horizon, as expected because the yield curve will be subjected to more new information from the time of forecast. The magnitudes of the RMSEs in each regime also broadly follow the changing interest rate volatilities visually apparent in figure 2.6 (also as expected, because higher interest rate volatility will tend to result in larger forecast errors).

Table 2.2 contains the RMSEs for the ANS framework forecasts less the RMSEs for the random walk forecasts, which is the typical naive benchmark used to assess fore-

[9]For the bonds, this reconstruction obtains the semi-annual coupon rate that corresponds to a par bond using the forecast interest rate curve to provide the discount factors. While this process is more complex than simply using zero-coupon yield data for the entire excercise, it is worthwhile because it avoids any model-induced bias in the forecast error analysis (i.e the forecast yields are compared directly to the original yield curve data rather than to model-generated or pre-processed data).

Table 2.1: RMSEs from forecasting the yield curve using the ANS model

Forecast horizon (years)	Yield or spread forecast	Monetary policy regime				
		Full sample	Bretton-Woods / gold price target	Federal funds rate target	Non-borrowed reserves target	Borrowed reserves / federal funds rate target
0.25	FF	122	68	148	367	56
	GS10	61	31	35	139	66
	FF/GS10	106	55	139	279	73
0.5	FF	169	108	216	446	90
	GS10	88	45	46	150	98
	FF/GS10	133	81	190	323	92
1	FF	220	148	272	557	154
	GS10	130	63	64	240	150
	FF/GS10	152	107	223	353	110
1.5	FF	259	157	296	505	221
	GS10	160	75	87	284	186
	FF/GS10	158	115	229	254	124
2	FF	285	151	278	362	274
	GS10	178	83	96	299	207
	FF/GS10	161	113	207	141	136
3	FF	316	155	169	n/a	338
	GS10	206	105	95	n/a	235
	FF/GS10	170	114	168	n/a	153

Note: Root-mean-squared forecast errors (RMSEs) for the ANS framework, in bps by horizon and monetary policy regime. The small size of the non-borrowed reserves sub-sample leaves insufficient results to calculate the 3 year RMSE.

casting performance. A negative entry (non-shaded) indicates an outperformance of the ANS framework forecasts over the random walk forecasts. The statistical significance of each entry is estimated using the Diebold and Mariano (1995) method.[10] Over the whole sample, the ANS framework forecasts for FF and FF/GS10 outperform those of the random walk, and the magnitude and significance of the outperformances tend to rise by forecast horizon. However, the ANS framework forecasts for GS10 consistently underperform the random walk forecasts over the full sample for all horizons.

The sub-sample results offers some insight into the GS10 results; i.e the general outperformance of the ANS framework during the Bretton Woods, federal funds rate target, and non-borrowed reserves regimes is more than offset by the significant underperformance during the borrowed reserves target regime. The FF forecasts in the borrowed reserves target regime also move from an outperformance for shorter horizons, to an increasing underperformance for longer horizons. Further investigation confirmed that even when the ϕ, σ_1, σ_2, and σ_3 parameters were re-estimated using just the data from borrowed reserves regime alone, the ANS framework with no allowance for term premia maintained a strong bias to over-forecast yields during the borrowed reserves regime for all of the horizons investigated.

The sub-sample results suggest that term premia may have become relatively more important in the borrowed reserves regime. This is also consistent with evidence of structural changes in the yield curve from around the late-1970s/early-1980s that have previously been noted in the literature (and which will be discussed in more detail in section 3.4.1 of the following chapter).

[10] Following Diebold and Mariano (1995), the bandwidth is set to the forecast horizon in months less one to allow for serially-correlated forecast errors that arise due to the frequency of the data being greater than the forecast horizons.

Table 2.2: ANS model forecast RMSEs less random walk forecast RMSEs

Forecast horizon (years)	Yield or spread forecast	Monetary policy regime				
		Full sample	Bretton-Woods / gold price target	Federal funds rate target	Non-borrowed reserves target	Borrowed reserves / federal funds rate target
0.25	FF	1	5	24	-24	-3
	GS10	4 **	-2 **	0	2	9 ***
	FF/GS10	1	2	28	-48	12
0.5	FF	3	3	12	-3	-8
	GS10	6 **	-3 *	-2	-1	12 **
	FF/GS10	-2	-1	7	-29	-1
1	FF	-8	-9	-36	24	-5
	GS10	10 *	-4 **	-7	-1	21 **
	FF/GS10	-32 *	-12	-60	-56	-31 *
1.5	FF	-24	-29	-118 ***	80	15
	GS10	13 *	-5 *	-12	2	31 ***
	FF/GS10	-59 **	-22	-146 **	-100	-40 ***
2	FF	-36	-37	-165 ***	122	29
	GS10	15 *	-4	-15	-3	39 ***
	FF/GS10	-75 **	-25	-191 ***	-151	-40 ***
3	FF	-44	-9	-225 ***	n/a	47 *
	GS10	18	-4 *	-15	n/a	53 ***
	FF/GS10	-87 **	19	-200 ***	n/a	-49 ***

Note: Root-mean-squared forecast errors (RMSEs) for the ANS framework forecasts less the RMSEs for the random-walk forecasts, in bps by horizon and monetary policy regime. A negative entry (non-shaded) indicates ANS framework outperformance relative to the random walk, and ***, **, * respectively represent 1, 5, and 10 percent two-tailed levels of significance using the Diebold and Mariano (1995) method. The small size of the non-borrowed reserves sub-sample means that statistical significance cannot be ascertained, so no indications are given.

2.5.3 Forecasting interest rates and the yield curve allowing for term premia

Given the evidence for term premia noted in the previous section, the forecasting exercise is repeated for the borrowed reserves regime using the ANS framework with an allowance for term premia. This requires estimates of the parameters ρ_1 in equation 2.19 and $\boldsymbol{\mu}(\phi, \tau)$ in equation 2.24, in addition to estimates of ϕ, σ_1, σ_2, and σ_3.

The increase in parameters makes the estimation and forecasting process more complex, and so a single estimation is undertaken for all parameters over an appropriate period of history rather than using recursive estimation. Specifically, the period October 1986 to January 1994 (88 months) is chosen as the parameter estimation period because it spans the first full monetary policy cycle (i.e a trough-to-trough cycle in the federal funds rate, and a similar cycle in long–maturity yields) following the late-1970s/early-1980s structural change noted in the previous section. The point estimates of the parameters are $\phi = 0.80$, $\rho_1 = 1.62$ percentage points, and σ_1, σ_2, and σ_3 are 0.84, 1.49, and 1.17 percentage points respectively. The point estimates of $\boldsymbol{\mu}(\phi, \tau)$ for each horizon are contained in table 2.3, and are estimated using the mean realised forecast errors over the parameter estimation period; i.e $\boldsymbol{\mu}(\phi, \tau) = \frac{1}{88-12\tau} \sum_{t=\text{Oct-86}+12\tau}^{\text{Jan-94}} [\boldsymbol{\beta}(t+\tau) - \boldsymbol{\Phi}(\phi, \tau)\boldsymbol{\beta}(t)]$, where $88 - 12\tau$ is the number of realised forecast errors available for the given horizon. To illustrate the practical effect of these estimates of $\boldsymbol{\mu}(\phi, \tau)$, figure 2.9 plots $-[\boldsymbol{\mu}(\boldsymbol{\phi}, \boldsymbol{\tau})]' \mathbf{g}(\phi, m)$ for the one-year horizon. This shows that the yield forecasts for the one-year horizon would overstate realised yields by a material margin if the term premia in the yield curve were not allowed for (e.g the forecast 2-year zero-coupon yield would overstate the realised 2-year zero-coupon yield by more than 100 basis points).

Using these estimated parameters, the out-of-sample forecasting exercise proceeds as outlined in section 2.5.2 (but without the parameter updating step) from February 1994

Table 2.3: ANS forecasting framework term premium vector estimates

$\mu(\phi,\tau)$	Forecast horizon (years)					
component	0.25	0.5	1	1.5	2	3
$\mu(1)$	-0.04	-0.09	-0.23	-0.34	-0.45	-0.58
$\mu(2)$	0.29	0.64	1.50	2.33	3.04	4.66
$\mu(3)$	-0.34	-0.62	-1.11	-1.44	-1.67	-2.05

Note: Estimates of the three components of the vector $\boldsymbol{\mu}(\phi, \tau)$ from the period October 1986 to January 1994, in percentage points by forecast horizon. These are used for the out-of-sample forecasting from February 1994 to February 2004 allowing for term premia.

Figure 2.9: The estimated effect of term premia when the ANS model is used to forecast interest rates over a one-year horizon. The function is $-\{-0.23\%, 1.50\%, -1.11\%\}' \mathbf{g}(\phi, m)$.

to February 2004. The resulting RMSEs from this process less the RMSEs from the random-walk forecasts over the same period are contained in table 2.4. Negative entries (non-shaded) again indicate an outperformance of the ANS framework over the random walk forecasts, and the Diebold and Mariano (1995) method provides the indicated levels of statistical significance.

Table 2.4: ANS ex-TP forecast RMSEs less random walk forecast RMSEs

Yield or spread	Forecast horizon (years)					
	0.25	0.5	1	1.5	2	3
FF	-15 **	-25	-33	-26	-32	-65
TB3	-7	-16	-21	-20	-31	-71
GS1	-1	-3	-6	-11	-27	-66
GS3	0	-1	-4	-10	-25	-64
GS5	-1	-3	-8	-14	-26	-62
GS10	5	1	-6	-11 *	-22 ***	-53 ***
GS20	5	4	1	-4	-9	-29 **
FF/GS10	-5	-26 *	-53 *	-40	-25	-32

Note: Root-mean-squared forecast errors (RMSEs) for the ANS framework forecasts allowing for term premia less the RMSEs for the random walk forecasts, in bps by horizon for the period February 1994 to February 2004. A negative entry (non-shaded) indicates ANS framework outperformance relative to the random walk, and ***, **, * respectively represent 1, 5, and 10 percent two-tailed levels of significance using the Diebold and Mariano (1995) method.

The main point to note from table 2.4 is that the ANS framework forecasts now outperform the random walk forecasts for all maturities over all horizons, except for some minor underperformance for long-maturity yields over short horizons. In addition, the magnitude and significance of the outperformances tend to rise by forecast horizon, like the previous results for the other regimes, although the smaller sample size results in less instances of statistical significance than in table 2.2.

Overall, these results suggest that term premia became more material in the US yield curve after the late-1970s/early-1980s, and that the ANS framework provides a straightforward method of allowing for those premia when forecasting the US yield curve.

2.6 Comparison of the ANS forecasting framework with the existing literature

Forecasting yields from yield curve models is an ongoing area of research in the existing literature, and so this section briefly compares the merits of the ANS forecasting framework against previous approaches. Firstly, Duffee (2002) finds that when used for forecasting yields, affine models of the yield curve (i.e where the yield curve is a direct function of stochastically evolving state variables, such as in the generic model of Dai and Singleton (2000)) underperform the random-walk benchmark. Conversely, the empirical results in section 2.5 show that the ANS model significantly outperforms the random-walk benchmark so long as an appropriate period is used to estimate the effect of term premia. Hence, by implication, the ANS forecasting framework outperforms the forecasts based on affine models of the yield curve.

Duffee (2002) proposes an "essentially affine" model of the yield curve, where the yield curve is effectively a function of underlying state variables augmented with an allowance for time-varying risk premia. The model is relatively complex, with the number of parameters ranging from 21 to 27 depending on the various restrictions assumed. After estimating the model using four decades of yield curve data, forecasts from the essentially affine model offer a 7.7 bp RMSE improvement over the random-walk benchmark for the 10-year maturity on a one-year horizon over the period 1995 to 1998. Conversely, the ANS model uses just three coefficients and five freely-estimated parameters (i.e ρ_1, ϕ, and the three components of $\boldsymbol{\mu}(\phi, \tau)$), and after using a single interest rate cycle over seven years to estimate an appropriate adjustment for the effect of term premia, the ANS model outperforms the random-walk benchmark by 9.9 bps for the same maturity, horizon, and period. Hence, the ANS forecast framework produces comparable empirical results with a theoretical model that is less complex and easier to estimate than that of Duffee (2002).

Diebold and Li (2006) forecasts yields using the NS model in conjunction with autoregressive processes for the NS model coefficients; i.e independent univariate first-order autoregressive (AR1) models for the NS coefficients individually, and an unrestricted VAR1 model for the combined NS coefficients. Based on recursive estimation beginning from 1985 to 1994, table 6 of Diebold and Li (2006) notes the RMSEs for the 10-year maturity on a one-year horizon over the period 1994 to 2000 are 98.1 bps for the AR1 forecasts and 127.9 bps for the unrestricted VAR1. For the corresponding maturity, horizon, and period, the ANS framework produces a RMSE of 91.4 bps. Hence, the ANS forecasting framework produces comparable empirical results to the Diebold and Li (2006) AR1 method and superior results to the Diebold and Li (2006) unrestricted VAR1 method. Of course, the ANS also has the advantage of being derived from a theoretically-consistent model. Conversely, the Diebold and Li (2006) forecasting framework is based on the NS model, which is not intertemporally-consistent, and that framework also imposes the AR1 or VAR1 processes without regard for theoretical consistency.

2.7 Conclusion

This chapter has derived the ANS model; an intertemporally-consistent and arbitrage-free version of the popular class of yield curve models originally introduced in Nelson and Siegel (1987).

The ANS model is a complete dynamic model of the yield curve that is theoretically consistent across both time and maturity. However, the functional form of the ANS model turns out to be only a subtle augmentation to that of NS models, and so the ANS model retains the properties by maturity that have made NS models so popular.

Applying the ANS model to yield curve forecasting gives forecasts that outperforms the random-walk benchmark, and the results are also comparable to more complex and less-parsimonious forecasting frameworks from the existing literature.

Chapter 3

An economic foundation for the ANS model

3.1 Introduction

The objective of this chapter is to develop an economic foundation for the ANS model of the yield curve. With reference to chapter 1, the motivation is to allow the ANS model to be applied in a theoretically-consistent manner within an economic context. This should prove especially useful in the growing field of "macro-finance", where macroeconomics and finance overlap. The immediate illustrative application of the economic foundation of the ANS model is to develop a theoretically-consistent, yet easy-to-apply framework for interpreting and investigating relationships between the yield curve, output, and inflation.

The chapter proceeds as follows: section 3.2 develops the economic foundation for the ANS model of the yield curve, and section 3.3 uses that foundation to derive econometric relationships between the yield curve, inflation, and output growth that are analogous to those used in the existing literature. Section 3.4 uses US data to estimate the econometric relationships derived in section 3.3, and section 3.5 compares the ANS economic framework to the existing macro-finance literature. Section 3.6 summarises and concludes.

3.2 Developing an economic foundation for the ANS model of the yield curve

This section develops a rigorous economic foundation for the ANS model of the yield curve. Section 3.2.1 specifies a generic continuous-time model of the economy, section 3.2.2 derives its associated forward rate curve, and section 3.2.3 explicitly relates the state variables of the derived forward rate curve to the coefficients of the ANS model.

3.2.1 The ABE model of the economy

The model developed in this section is based on the generic multifactor model of Berardi and Esposito (1999) (hereafter BE), which is itself a version of the standard continuous-time general-equilibrium-economy model proposed by Cox et al. (1985*a*). However, the augmented BE (hereafter ABE) model developed here differs from the BE model in several respects, which are discussed at the end of this section in light of the macroeconomic quantities subsequently defined for the ABE model.

The ABE economy is based on J real factors of production (e.g capital, labour, etc., potentially by industry sector), each with its own associated deflator/inflation factor. The dynamics of the ABE economy are represented by $2J$ stochastic differential processes analogous to the Vasicek (1977) specification, i.e:

$$ds_j(t) = -\kappa_j \left[s_j(t) - \theta_j(t)\right] dt + \sigma_{1,j} dz_{1,j}(t) \tag{3.1}$$

where $s_j(t)$ for $j = 1$ to J are the real state variables representing instantaneous growth on returns to the factors of production in the economy at time t; κ_j are positive constant mean-reversion parameters; $\theta_j(t)$ are the steady-state (i.e long-run) values of $s_j(t)$ which vary stochastically over time as $d\theta_j(t) = \sigma_{0,j} dz_{0,j}(t)$; $\sigma_{0,j}$ and $\sigma_{1,j}$ are positive constant standard deviation parameters with $\sigma_{0,j} \ll \sigma_{1,j}$; and $dz_{0,j}(t)$ are $dz_{1,j}(t)$ are independent Wiener variables under the physical (i.e non-risk neutral) measure. For $j = J+1$ to $2J$, $s_j(t)$ are the inflation state variables. BE shows that these have the form $s_j(t) = \pi_j(t) - \sigma^2_{j,p}$, where $\pi_j(t)$ is the instantaneous rate of inflation for the factor of production j, and $\sigma^2_{j,p}$ are positive constant parameters representing the variances of instantaneous changes in the deflator j. Similarly, for $j = J+1$ to $2J$, $\theta_j(t) = \theta_{j,\pi}(t) - \sigma^2_{j,p}$, where $\theta_{j,\pi}(t)$ is steady-state rate of inflation for the factor of production j. Otherwise, the remaining parameters for the inflation state variables are analogous to those of the real state variables. Following BE, this chapter also assumes for mathematical and expositional convenience that all state variables $s_j(t)$ and their associated steady-state variables $\theta_j(t)$ are orthogonal (i.e the innovations $dz_{0,j}(t)$ and $dz_{1,j}(t)$ are uncorrelated), and that the mean-reversion parameters κ_j are the elements of a diagonal mean-reversion coefficient matrix. Even in the completely general case, section B.2 of appendix B shows that an orthogonal representation of the economy can be constructed from the original $2J$ state variables and $2J$ steady-state variables, and that a non-diagonal mean-reversion coefficient matrix will give a solution for the expected path of the short rate (as detailed in the following section) that is a summation of constants and exponential decay terms.

For later use, macroeconomic quantities may be defined from the state variables of the ABE model. Firstly, define real instantaneous output growth as $dY(t) = \sum_{j=1}^{J} s_j(t)$. That is, the sum of instantaneous growth on returns to the factors of production in the economy equals instantaneous output growth. Secondly, define real instantaneous steady-state (i.e potential) output growth at time t as $dY^*(t) = \sum_{j=1}^{J} \theta_j(t)$. That is, if the returns to the factors of production are all growing at their steady-state values, then output must be growing at its steady-state value. Thirdly, define an economy-wide inflation state variable as the sum of all inflation state variables, i.e $dP(t) = \sum_{j=J+1}^{2J} s_j(t)$, and finally define an economy-wide steady-state inflation variable as the sum of all steady-state inflation variables, i.e $dP^*(t) = \sum_{j=J+1}^{2J} \theta_j(t)$. Nominal output growth is therefore $dP(t) + dY(t)$, and steady-state nominal output growth is $dP^*(t) + dY^*(t)$.

Defining these macroeconomic quantities highlights several key differences between the ABE model and the BE model. That is, time-varying $\theta_j(t)$ values in the ABE model allow steady-state output growth and inflation to vary over time, and Gaussian innovations allow output growth and inflation to take on negative values, which are properties consistent with realised historical macroeconomic data.[1] Appendix B also shows that the ABE model could readily be modified to allow for arbitrary empirical covariances between inflation, output growth, and their steady-state values (which might arise, for example, through IS-LM-AS relationships and monetary policy reaction functions, etc.). Conversely, the BE model assumption of constant $\theta_j(t)$ values would result in constant steady-state output growth. The BE model can also be specified with Cox et al. (1985*b*) dynamics (i.e innovations of $\sqrt{s_j(t)} \cdot dz_{1,j}(t)$ in equation 3.1), but that would prohibit output growth and inflation from becoming negative. The BE model also requires additional assumptions for the single inflation state variable it uses. Specifically, BE assumes that innovations in inflation are independent of innovations in the original real state variables, and that mean-reversion in inflation is low (to allow for inflation persistence and consistency with the Fisher hypothesis).[2]

[1] From a financial perspective, Gaussian innovations imply that interest rates have a non-zero probability of becoming negative. This can safely be ignored in practice (as is often done when Vasicek (1977) models are used) unless interest rates are already materially close to zero. Two formal treatments in the latter case are to model interest rates as options (given that cash is an alternative to a zero interest rate investment), as in Black (1995), or to impose a reflecting boundary at zero, as in Goldstein and Keirstead (1997). These are both beyond the scope of this chapter, but the principles are raised again in section 6.2.2 of the concluding chapter.

[2] Incidentally, these assumptions result in the BE model with Gaussian dynamics being a special case of the ABE model, and therefore allows the BE model to be related directly to the ANS model as in section 3.2.3. That is, setting $\theta_j(t) = \theta_j$ and $\sigma_{0,j} = 0$ in equation 3.10 recovers the BE expression for $f(t, m)$.

Finally, note that the assumption of random walks for the steady-state variables in the ABE model implies that both potential output growth and inflation expectations will follow random walks. While this is suitable in practice (for example, the empirical results later in this chapter cannot reject the unit root hypothesis for potential output growth and inflation), it does have the undesirable theoretical property that potential output growth and inflation expectations can adopt arbitrarily low or high values. At the cost of two extra parameters per ABE steady-state variable (i.e a long-run constant and a low mean-reversion parameter) it would be possible to specify weakly mean-reverting processes for the steady-state variables. However, that change would also need to be matched by a change to a weakly mean-reverting process for β_1 in the ANS model, as will become apparent in section 3.2.3 and equation 3.13.

3.2.2 The derivation of the forward rate curve for the ABE model

This section derives the ABE forward rate curve using the HJM framework outlined in the first part of section 2.3.2 from chapter 2, and the stochastic differential processes defined in equation 3.1. The derivation proceeds in four parts: (1) calculating the expected path of the short rate $E_t\left[r\left(t+m\right)\right]$ for the ABE model; (2) defining and calculating the effects that the market prices and quantities of risk in the ABE model coefficients have on the shape of the forward rate curve; (3) combining the results together to obtain the ABE model of the forward rate curve; and (4) defining $E_t\left[r\left(t+m\right)\right]$ and the forward rate curve in terms of the ABE macroeconomic quantities introduced in section 3.2.1.

Then, by setting $\kappa_\pi = 0$ and noting that $\lim_{\kappa_\pi \to 0} B_1\left(m\right) = m$ and $\lim_{\kappa_\pi \to 0} \left[B_1\left(m\right)\right]^2 = m^2$ (by L'Hôpital's rule), the inflation component of $E_t\left[r\left(t+m\right)\right]$ and $f\left(t,m\right)$ in the BE model may be related precisely to the Level component of $E_t\left[r\left(t+m\right)\right]$ and $f\left(t,m\right)$ in the ANS model.

The ABE expected path of the short rate

Following BE and Cox et al. (1985b), the nominal short rate at any given time is the summation of state variables $s_j(t)$, i.e $r(t) = \sum_{j=1}^{2J} s_j(t)$. This equality holds at all points in time, and so $E_t[r(t+m)] = \sum_{j=1}^{2J} E_t[s_j(t+m)]$, where E_t is the expectations operator at time t; m ($m \geq 0$) denotes a horizon from time t (so $t+m$ represents a future point in time); and $E_t[s_j(t+m)]$ are the expected values of the state variables j, all as at time t as a function of horizon m.

Heuristically, the quantities $E_t[s_j(t+m)]$ may be calculated by applying the expectations operation E_t to equation 3.1 and noting that $E_t[\theta_j(t+m)] = \theta_j(t)$; hence $E_t[ds_j(t+m)] = -\kappa_j\{E_t[ds_j(t+m)] - \theta_j(t)\}\,dm$. This ordinary differential equation in m has the solution $E_t[s_j(t+m)] = \theta_j(t) + A_j \cdot \exp(-\kappa_j m)$. The boundary condition at $m = 0$ is $s_j(t) = \theta_j(t) + A_j$, so $A_j = s_j(t) - \theta_j(t)$, and therefore $E_t[s_j(t+m)] = \theta_j(t) + [s_j(t) - \theta_j(t)] \cdot \exp(-\kappa_j m)$. Summing this result across all $2J$ state variables gives the ABE expected path of the short rate, i.e:

$$E_t[r(t+m)] = \sum_{j=1}^{2J} \theta_j(t) + \sum_{j=1}^{2J} [s_j(t) - \theta_j(t)] \cdot \exp(-\kappa_j m) \tag{3.2}$$

More formally, the calculation of $E_t[s_j(t+m)]$ needs to simultaneously account for the stochastic impacts from innovations $dz_{0,j}(t)$ in the steady-state variables and the innovations $dz_{1,j}(t)$ in the state variables. This calculation also highlights how those stochastic components effect innovations to the expected path of the short rate. Hence, with reference to equation 3.1, take a a point u where $t < u < t+m$ so that $ds_j(u) = -\kappa_j[s_j(u) - \theta_j(u)]\,du + \sigma_{1,j}dz_{1,j}(u)$. This may be re-arranged as $ds_j(u) + \kappa_j s_j(u)\,du = \kappa_j\theta_j(u)\,du + \sigma_{1,j}dz_{1,j}(u)$ and expressed as $d[s_j(u) \cdot \exp(\kappa_j u)] = \kappa_j\theta_j(u) \cdot \exp(\kappa_j u)\,du + \sigma_{1,j} \cdot \exp(\kappa_j u)\,dz_{1,j}(u)$. Integrating from t to $t+m$ and taking the result for the lower limit of integration from the left-hand to the right-hand side gives the result:

$$\begin{aligned} s_j(t+m)\cdot\exp(\kappa_j[t+m]) &= s_j(t)\cdot\exp(\kappa_j t)+\kappa_j\int_t^{t+m}\theta_j(u)\cdot\exp(\kappa_j u)\,du \\ &\quad+\int_t^{t+m}\sigma_{1,j}\cdot\exp(\kappa_j u)\,dz_{1,j}(u) \end{aligned} \tag{3.3}$$

Regarding the evaluation of the second term on the right-hand side, note that $d\theta_j(v)=\sigma_{0,j}dz_{0,j}(v)$. Integrating from t to u and taking the result for the lower limit of integration from the left-hand to the right-hand side gives the result:

$$\theta_j(u)=\theta_j(t)+\sigma_{0,j}\int_t^u dz_{0,j}(v) \tag{3.4}$$

and therefore:

$$\begin{aligned} &\kappa_j\int_t^{t+m}\theta_j(u)\cdot\exp(\kappa_j u)\,du \\ =\ &\kappa_j\int_t^{t+m}\left[\theta_j(t)+\sigma_{0,j}\int_t^u dz_{0,j}(v)\right]\cdot\exp(\kappa_j u)\,du && (3.5)\\ =\ &\kappa_j\theta_j(t)\int_t^{t+m}\exp(\kappa_j u)\,du+\kappa_j\sigma_{0,j}\int_t^{t+m}\exp(\kappa_j u)\left(\int_t^u dz_{0,j}(v)\right)du && (3.6) \end{aligned}$$

The first integral is $\theta_j(t)\left[\exp(\kappa_j u)\right]_t^{t+m}=\theta_j(t)\left[\exp(\kappa_j[t+m])-\exp(\kappa_j t)\right]$, and the second integral is $\kappa_j\sigma_{0,j}\int_t^{t+m}\left(\int_v^{t+m}\exp(\kappa_j u)\,du\right)dz_{0,j}(v)$, where the stochastic Fubini theorem has been used to reverse the sequence of integration. Evaluating the latter integral then results in $\sigma_{0,j}\int_t^{t+m}\left[\exp(\kappa_j[t+m])-\exp(\kappa_j v)\right]dz_{0,j}(v)$, and so:

$$\begin{aligned} &\kappa_j\int_t^{t+m}\theta_j(u)\cdot\exp(\kappa_j u)\,du \\ =\ &\theta_j(t)\left[\exp(\kappa_j[t+m])-\exp(\kappa_j t)\right] \\ &+\sigma_{0,j}\int_t^{t+m}\left[\exp(\kappa_j[t+m])-\exp(\kappa_j v)\right]dz_{0,j}(v) \end{aligned} \tag{3.7}$$

Substituting this result into equation 3.3 gives:

$$\begin{aligned} s_j(t+m)\cdot\exp(\kappa_j[t+m]) &= s_j(t)\cdot\exp(\kappa_j t)+\theta_j(t)\cdot\left[\exp(\kappa_j[t+m])-\exp(\kappa_j t)\right] \\ &\quad+\sigma_{0,j}\int_t^{t+m}\left[\exp(\kappa_j[t+m])-\exp(\kappa_j v)\right]dz_{0,j}(v) \\ &\quad+\sigma_{1,j}\int_t^{t+m}\exp(\kappa_j u)\,dz_{1,j}(u) \end{aligned} \tag{3.8}$$

Taking $s_j(t) \cdot \exp(\kappa_j t)$ to the right-hand side and factoring out $\exp(\kappa_j [t+m])$ across the entire equation gives:

$$\begin{aligned} s_j(t+m) &= s_j(t) \cdot \exp(-\kappa_j m) + \theta_j(t) \cdot [1 - \exp(-\kappa_j m)] \\ &\quad + \sigma_{0,j} \int_t^{t+m} dz_{0,j}(v) \\ &\quad - \sigma_{0,j} \int_t^{t+m} \exp(-\kappa_j [t+m-v]) \, dz_{0,j}(v) \\ &\quad + \sigma_{1,j} \int_t^{t+m} \exp(-\kappa_j [t+m-u]) \, dz_{1,j}(u) \end{aligned} \tag{3.9}$$

where the individual stochastic terms have been deliberately separated for transparency.

Applying the expectations operator as at time t then gives $E_t[s_j(t+m)] = \theta_j(t) + [s_j(t) - \theta_j(t)] \cdot \exp(-\kappa_j m)$, and summing over all j reproduces equation 3.2. Regarding dynamics, the second and third lines of equation 3.9 respectively show that an innovation $dz_{0,j}(t)$ will simultaneously result in a parallel shift of $\sigma_{0,j} \cdot dz_{0,j}(t)$ and a non-parallel shift of $-\sigma_{0,j} \cdot \exp(-\kappa_j m) \cdot dz_{0,j}(t)$ to $E_t[s_j(t+m)]$ as a function of maturity m. Similarly, the fourth line of equation 3.9 shows that an innovation $dz_{1,j}(t)$ will result in a non-parallel shift of $-\sigma_{1,j} \cdot \exp(-\kappa_j m) \cdot dz_{1,j}(t)$ to $E_t[s_j(t+m)]$.

The effect of volatility and market prices of risk in the ABE model

The HJM volatility functions for each component of the ABE model are defined by the stochastic innovations $dz_{0,j}(t)$ and $dz_{1,j}(t)$ for each factor j applied to the components of $E_t[r(t+m)]$ associated with that factor, as will be detailed below. The market prices of risk associated with the innovations $dz_{0,j}(t)$ and $dz_{1,j}(t)$ are assumed respectively to be the constants $\rho_{0,j}$ and $\rho_{1,j}$.

An innovation $dz_{0,j}(t)$ will result in a parallel shift of $\sigma_{0,j} \cdot dz_{0,j}(t)$ to $E_t[r(t+m)]$ and $f(t,m)$ simultaneously. Therefore, the volatility function is $\sigma_n(v,m) = \sigma_{0,j}$ for any j, making the second integral in equation 2.17 $\int_0^m \sigma_{0,j}\rho_{0,j} dv = \sigma_{0,j}\rho_{0,j} \cdot [v]_0^m = \sigma_{0,j}\rho_{0,j} \cdot m$, and the first integral in equation 2.17 $\int_0^m \sigma_{0,j} \cdot \left\{\int_v^m \sigma_{0,j} du\right\} dv = \int_0^m \sigma_{0,j} \cdot \{\sigma_{0,j} \cdot [u]_v^m\} dv = \int_0^m \sigma_{0,j}^2 \cdot [m-v]\, dv = \sigma_{0,j}^2 \cdot \left[mv - \frac{v^2}{2}\right]_0^m = \frac{1}{2}\sigma_{0,j}^2 \cdot m^2$.

An innovation $dz_{1,j}(t)$ will result in a non-parallel shift of $\sigma_{1,j} \cdot \exp(-\kappa_j m) \cdot dz_{1,j}(t)$ (i.e an exponential decay function by horizon/maturity) to $E_t[r(t+m)]$ and $f(t,m)$ simultaneously. Therefore, the volatility function is $\sigma_n(v,m) = \sigma_{1,j} \cdot \exp(-\kappa_j m)$ for any j, making the second integral in equation 2.17 $\int_0^m \sigma_{1,j} \cdot \exp(-\kappa_j v) \cdot \rho_{1,j} dv = \sigma_{1,j}\rho_{1,j} \cdot \left[-\frac{1}{\kappa_j}\exp(-\kappa_j v)\right]_0^m = \sigma_{1,j}\rho_{1,j} \cdot B_j(m)$ where $B_j(m) = \frac{1}{\kappa_j}[1-\exp(-\kappa_j m)]$. Note that $B_j(m)$ is the typical Vasicek (1977) functional form that arises from mean reverting stochastic process with Gaussian innovations. The first integral in equation 2.17 is calculated in two steps, i.e: $\int_v^m \sigma_n(v,u)\, du = \int_s^m \sigma_{1,j} \cdot \exp(-\kappa_j[u-v])\, du = \sigma_{1,j} \cdot \left[-\frac{1}{\kappa_j}\exp(-\kappa_j[u-v])\right]_v^m$ $= \frac{\sigma_{1,j}}{\kappa_j} \cdot [1-\exp(-\kappa_j[m-v])]$. Then $\int_0^m \sigma_n(v,m)\left\{\int_v^m \sigma_n(v,u)\, du\right\} dv$ is calculated as: $\int_0^m \sigma_{1,j} \cdot \exp(-\kappa_j[m-v]) \cdot \frac{\sigma_{1,j}}{\kappa_j} \cdot [1-\exp(-\kappa_j[m-v])]dv = \frac{\sigma_{1,j}^2}{\kappa_j} \cdot \int_0^m [\exp(-\kappa_j[m-v]) - \exp(-2\kappa_j[m-v])]dv = \frac{\sigma_{1,j}^2}{\kappa_j^2} \cdot [\exp(-\kappa_j[m-v]) - \frac{1}{2}\exp(-2\kappa_j[m-v])]_0^m = \frac{\sigma_{1,j}^2}{\kappa_j^2} \cdot [1 - \exp(-\kappa_j m) - \frac{1}{2} + \frac{1}{2}\exp(-2\kappa_j m)] = \frac{\sigma_{1,j}^2}{2\kappa_j^2} \cdot [1-\exp(-\kappa_j m) + \exp(-2\kappa_j m)] = \frac{\sigma_{1,j}^2}{2\kappa_j^2} \cdot [1 - \exp(-\kappa_j m)]^2 = -\frac{1}{2}\sigma_{1,j}^2 \cdot [B_j(m)]^2$

An innovation $dz_{0,j}(t)$ will also result in a non-parallel shift of $-\sigma_{0,j} \cdot \exp(-\kappa_j m) \cdot dz_{0,j}(t)$ to $E_t[r(t+m)]$ and $f(t,m)$ simultaneously, in addition to the parallel shift already noted earlier. The integrals for these non-parallel components follow those for $dz_{1,j}(t)$ above, giving the results $-\sigma_{0,j}\rho_{0,j} \cdot B_j(m)$ and $\frac{1}{2}\sigma_{1,j}^2 \cdot [B_j(m)]^2$.

The ABE forward rate curve

Substituting $E_t\left[r\left(t+m\right)\right]$ from the first sub-section and the calculations from the second sub-section into equation 2.17 gives the ABE forward rate curve, i.e:

$$\begin{aligned} f\left(t,m\right) &= \sum_{j=1}^{2J}\theta_j\left(t\right)+m\cdot\sum_{j=1}^{2J}\sigma_{0,j}\rho_{0,j}-m^2\cdot\sum_{j=1}^{2J}\frac{1}{2}\sigma_{0,j}^2 \\ &\quad +\sum_{j=1}^{2J}\left[s_j\left(t\right)-\theta_j\left(t\right)\right]\cdot\exp\left(-\kappa_j m\right)+\sum_{j=1}^{2J}\left[\sigma_{1,j}\rho_{1,j}-\sigma_{0,j}\rho_{0,j}\right]\cdot B_j\left(m\right) \\ &\quad -\sum_{j=1}^{2J}\frac{1}{2}\left[\sigma_{1,j}^2-\sigma_{0,j}^2\right]\cdot\left[B_j\left(m\right)\right]^2 \end{aligned} \tag{3.10}$$

Note that the market prices of risk arise in $f\left(t,m\right)$ because the ABE model is specified under the physical measure where, relative to the risk-free rolling investment in the short rate, investors will demand compensation for the risk associated with owning fixed interest securities (i.e unanticipated changes in market value imparted by the innovations $dz_{0,j}\left(t\right)$ and $dz_{1,j}\left(t\right)$ in $E_t\left[r\left(t+m\right)\right]$ and $f\left(t,m\right)$ as time evolves).

The ABE expected path of the short rate and forward rate in terms of economy-wide inflation and output growth

The ABE expected path of the short rate and the forward rate curve may also be defined in terms of the current and expected values of the macroeconomic quantities from section 3.2.1. Firstly $\sum_{j=1}^{2J}\theta_j\left(t\right)=dP^*\left(t\right)+dY^*\left(t\right)$, which shows the long-run expected path of the short rate is the current steady-state inflation plus output growth, as one would intuitively expect. The long-maturity forward rate (and interest rate) is steady-state inflation plus output growth modified by the effect of the market prices and quantities of risk in the steady-state variables.

Secondly, $\sum_{j=1}^{2J}\left[s_j\left(t\right)-\theta_j\left(t\right)\right]\cdot\exp\left(-\kappa_j m\right)$ may be expressed as the expected values of expected inflation and output growth relative to their expected steady-state levels. That is, denote $dP\left(t+m\right)$, $dY\left(t+m\right)$, $dP^*\left(t+m\right)$, and $dY^*\left(t+m\right)$ respectively as the economy-wide inflation state variable, real output growth, the economy-wide steady-state inflation variable, and steady-state output growth, all as a function of future time $t+m$. Then, as at time t, $E_t[dP\,(t+m)+dY\,(t+m)-dP^*\,(t+m)-dY^*\,(t+m)]$ may be expressed as $E_t\,[dP\,(t+m)+dY\,(t+m)]-E_t[dP^*\,(t+m)+dY^*\,(t+m)]$. The evaluation of the first term has already been undertaken in equation 3.2, and the second term may be evaluated directly using the macroeconomic quantities from section 3.2.1 and the result from equation 3.4. That is:

$$
\begin{aligned}
E_t[dP^*\,(t+m)+dY^*\,(t+m)] &= E_t\left[\sum_{j=1}^{2J}\theta_j\,(t+m)\right] && (3.11a)\\
&= E_t\left[\theta_j\,(t)+\sigma_{0,j}\int_t^{t+m} dz_{0,j}\,(v)\right] && (3.11b)\\
&= \theta_j\,(t) && (3.11c)
\end{aligned}
$$

and therefore:

$$
\begin{aligned}
& E_t[dP\,(t+m)+dY\,(t+m)\,dP^*\,(t+m)-dY^*\,(t+m)] && (3.12a)\\
= & \sum_{j=1}^{2J}\theta_j\,(t)+\sum_{j=1}^{2J}[s_j\,(t)-\theta_j\,(t)]\cdot\exp\left(-\kappa_j m\right)-\sum_{j=1}^{2J}\theta_j\,(t) && (3.12b)\\
= & \sum_{j=1}^{2J}[s_j\,(t)-\theta_j\,(t)]\cdot\exp\left(-\kappa_j m\right) && (3.12c)
\end{aligned}
$$

3.2.3 The correspondence between the ABE model and the ANS model

Using the generic ABE model of the economy and the forward rate curve from section 3.2.2, it is now possible to provide an economic foundation for the ANS model coefficients and parameters from chapter 2. The section proceeds in three parts: (1) showing the correspondence between the ABE steady-state variables and the Level component of the ANS model; (2) showing the correspondence between the ABE non-steady-state variables and the non-Level component of the ANS model; and (3) discussing some additional considerations regarding the correspondence between the ABE model and the ANS model.

The ABE steady-state variables and the ANS Level coefficient

Comparing the ANS expected path of the short rate from equation 2.4 to the ABE expected path of the short rate from equation 3.2 gives the equality $\lambda_1(t) = \sum_{j=1}^{2J} \theta_j(t) = dP^*(t) + dY^*(t)$. That is, $\lambda_1(t) \cdot g_1(\phi, m) = \lambda_1(t) = dP^*(t) + dY^*(t)$, and substituting $\lambda_1(t) = \beta_1(t) - \gamma_1$ gives:

$$\beta_1(t) = \gamma_1 + dP^*(t) + dY^*(t) \tag{3.13}$$

Hence, the Level coefficient from the ANS model at time t is composed of a constant term premium component γ_1, and the economy-wide steady-state inflation variable plus steady-state output growth at time t.

Because both the ANS and ABE models are specified with Gaussian dynamics, the innovations in the ANS Level coefficient correspond precisely to the ABE steady-state innovations, i.e $\sigma_1 dW_n(t) = \sum_{j=1}^{2J} \sigma_{0,j} dz_{0,j}(t)$. In addition, the component of the ANS forward rate curve associated with the Level coefficient and its dynamics corresponds precisely to the component of the ABE forward rate curve associated with the steady-state components and their dynamics. Specifically, the Level component of the ANS forward rate equation 2.19 is $\sigma_1 \rho_1 m + \beta_1(t) - \sigma_1^2 \cdot \frac{1}{2} m^2$, which corresponds to the steady-state component of the ABE forward rate equation (the first line of equation 3.10) with $\sigma_1 \rho_1 = \sum_{j=1}^{2J} \sigma_{0,j} \rho_{0,j}$ and $\sigma_1^2 = \sum_{j=1}^{2J} \sigma_{0,j}^2$. This correspondence between forward curves ensures that when yield curve data observed at time t are "fitted" using the ANS model, the Level coefficient $\beta_1(t)$ will be a consistent estimate, to within the term premium γ_1, of the sum of the steady-state components of the ABE model at time t.

The ABE non-steady-state components and the non-Level ANS components

The "remainder" of the yield curve as estimated by the ANS model (i.e the Slope and Bow components, and the residuals from the yield curve estimation) will correspond to the non-steady-state components of the ABE model, i.e $\sum_{j=1}^{2J}\left[s_j\left(t\right)-\theta_j\left(t\right)\right]\cdot\exp\left(-\kappa_j m\right)$. It can be shown that Slope and Bow components alone are a precise first-order approximation to the expected nominal output growth relative to steady state nominal output growth.

Specifically, define ϕ as a central measure (e.g the median) of the values of κ_j for $j = 1$ to $2J$, i.e $\phi = \text{central}(\kappa_j)$, where ϕ will be a positive constant because all κ_j are positive constants. Each κ_j may then be expressed as a relative deviation from ϕ, i.e $\kappa_j = \phi\left(1+\Delta_j\right)$, and the non-steady-state component of equation 3.2 may therefore be written as $\sum_{j=1}^{2J}\left[s_j\left(t\right)-\theta_j\left(t\right)\right]\cdot\exp\left(-\kappa_j m\right) = \exp\left(-\phi m\right)\cdot\sum_{j=1}^{2J}\left[s_j\left(t\right)-\theta_j\left(t\right)\right]\cdot\exp\left(-\Delta_j\phi m\right)$. Now write each $\exp\left(-\Delta_j\phi m\right)$ as a first-order Taylor expansion around $\Delta_j = 0$; i.e substituting $\exp\left(-\Delta_j\phi m\right) \simeq 1-\Delta_j\phi m$ and expanding gives: $\sum_{j=1}^{2J}\left[s_j\left(t\right)-\theta_j\left(t\right)\right]\cdot\exp\left(-\kappa_j m\right) \simeq \exp\left(-\phi m\right)\cdot\sum_{j=1}^{2J}\left[s_j\left(t\right)-\theta_j\left(t\right)\right]-\exp\left(-\phi m\right)\cdot\sum_{j=1}^{2J}\left[s_j\left(t\right)-\theta_j\left(t\right)\right]\Delta_j\cdot\phi m$. The right-hand side may be rearranged as $-\exp\left(-\phi m\right)\cdot-\sum_{j=1}^{2J}\left[s_j\left(t\right)-\theta_j\left(t\right)\right]\cdot\left(1-\frac{1}{2}\Delta_j\right)-\exp\left(-\phi m\right)\cdot -\sum_{j=1}^{2J}\left[s_j\left(t\right)-\theta_j\left(t\right)\right]\Delta_j\cdot\frac{1}{2}\left(-2\phi m+1\right) = \lambda_2\left(t\right)\cdot g_2\left(\phi,m\right)+\lambda_3\left(t\right)\cdot g_3\left(\phi,m\right)$, where $\lambda_2\left(t\right) = -\sum_{j=1}^{2J}\left[s_j\left(t\right)-\theta_j\left(t\right)\right]\cdot\left(1+\frac{1}{2}\Delta_j\right)$ and $\lambda_3\left(t\right) = -\sum_{j=1}^{2J}\left[s_j\left(t\right)-\theta_j\left(t\right)\right]\frac{1}{2}\Delta_j$.

Hence, $\sum_{n=2}^{3}\lambda_n\left(t\right)\cdot g_n\left(\phi,m\right) \simeq -\sum_{j=1}^{2J}\left[s_j\left(t\right)-\theta_j\left(t\right)\right]\cdot\exp\left(-\kappa_j m\right)$, which upon the substitution of $\beta_n\left(t\right)-\gamma_n = \lambda_n\left(t\right)$ gives:

$$\sum_{n=2}^{3}\left[\beta_n\left(t\right)-\gamma_n\right]\cdot g_n(\phi,m) \simeq -\sum_{j=1}^{2J}\left[s_j\left(t\right)-\theta_j\left(t\right)\right]\cdot\exp\left(-\kappa_j m\right) \tag{3.14}$$

Using the result from equation 3.12c, equation 3.14 becomes:

$$\begin{aligned}\sum_{n=2}^{3}\left[\beta_n\left(t\right)-\gamma_n\right]\cdot g_n(\phi,m) \simeq\ & -E_t[dP\left(t+m\right)+dY\left(t+m\right) \\ & -dP^*\left(t+m\right)-dY^*\left(t+m\right)]\end{aligned} \tag{3.15}$$

This shows that ignoring the yield residuals after estimating the ANS model is equivalent to ignoring the second-order and higher terms from the Taylor expansion of the ABE model.

The stochastic components of the summation $-\sum_{j=1}^{2J}\left[s_j\left(t\right)-\theta_j\left(t\right)\right]\cdot\exp\left(-\kappa_j m\right)$ are $-\sum_{j=1}^{2J}\left[\sigma_{1,j}dz_{1,j}\left(t\right)-\sigma_{0,j}dz_{0,j}\left(t\right)\right]\cdot\exp\left(-\kappa_j m\right)$. Following the first-order Taylor expansion approach outlined above, the latter expression may be expressed as $\sum_{n=2}^{3}\sigma_n \cdot g_n(\phi,m)\cdot dW_n(t)$. This shows that the combination of the innovations in the Slope and Bow components of the ANS model is a precise first-order approximation to innovations in the non-steady-state component of the ABE model.

The analysis above has established that the first-order approximation to the non-steady-state components of the ABE expected path of the short rate has the ANS form $\sum_{n=2}^{3}\lambda_n(t)\cdot g_n(\phi,m)$ with innovations $\sum_{n=2}^{3}\sigma_n \cdot g_n(\phi,m)\cdot dW_n(t)$. Using these functional forms, section 2.3.2 has shown that the market price of risk and volatility integral terms for the ANS forward rate curve are respectively $\sum_{n=2}^{3}\gamma_n \cdot g_n(\phi,m)$ and $\sum_{n=2}^{3}\sigma_n^2 \cdot h_n(\phi,m)$. From section3.2.2, the market price of risk and volatility integral terms for the ABE forward rate curve are respectively $\sum_{j=1}^{2J}\left[\sigma_{1,j}\rho_{1,j}-\sigma_{0,j}\rho_{0,j}\right]\cdot B_j(m)$ and $-\sum_{j=1}^{2J}\frac{1}{2}\left[\sigma_{1,j}^2-\sigma_{0,j}^2\right]\cdot [B_j(m)]^2$, and so $\sum_{n=2}^{3}\gamma_n \cdot g_n(\phi,m) \simeq \sum_{j=1}^{2J}\left[\sigma_{1,j}\rho_{1,j}-\sigma_{0,j}\rho_{0,j}\right]\cdot B_j(m)$ and $\sum_{n=2}^{3}\sigma_n^2 \cdot h_n(\phi,m) \simeq -\sum_{j=1}^{2J}\frac{1}{2}\left[\sigma_{1,j}^2-\sigma_{0,j}^2\right]\cdot [B_j(m)]^2$. Hence, the non-Level component of the ANS forward rate curve approximates the non-steady-state components of the ABE forward rate curve as:

$$\begin{aligned}
& \sum_{n=2}^{3}\left[\beta_n(t)-\gamma_n\right]\cdot g_n(\phi,m)+\sum_{n=2}^{3}\gamma_n\cdot g_n(\phi,m)+\sum_{n=2}^{3}\sigma_n^2\cdot h_n(\phi,m) \\
\simeq\ & \sum_{j=1}^{2J}\left[s_j(t)-\theta_j(t)\right]\cdot \exp\left(-\kappa_j m\right)+\sum_{j=1}^{2J}\left[\sigma_{1,j}\rho_{1,j}-\sigma_{0,j}\rho_{0,j}\right]\cdot B_j(m) \\
& -\sum_{j=1}^{2J}\frac{1}{2}\left[\sigma_{1,j}^2-\sigma_{0,j}^2\right]\cdot [B_j(m)]^2 \qquad (3.16)
\end{aligned}$$

Discussion of the ANS model with respect to the ABE model

The previous two sub-sections show that the ANS forward rate model using just three state variables and seven parameters is a parsimonious and practically amenable representation of the generic ABE model. Note that the latter remains a theoretical construct only, given that a "complete" specification based on multiple factors of production by multiple industry sectors would quickly inflate to a practically untenable number of variables and parameters. For example, an approximation based on just two industries and the three typical factors of production (i.e capital, labour, and total factor productivity) would contain 12 state variables, 12 steady-state variables, and 60 parameters. Even a minimal approximation based on a single industry with a single factor of production would require two state variables, two steady-state variables, and ten parameters (and this is after the implicit orthogonalisation of the original state variables noted in the assumptions of section 3.2.1; allowing for covariances between the original state variables would require further parametrisation).

The approximation in equation 3.15 can also be seen as a reduction in dimensionality that is commonly undertaken using latent factor models for the yield curve (as will be noted in section 3.5), but with an underlying theoretical structure. Specifically, the two time-varying coefficients $\beta_2(t)$ and $\beta_3(t)$ applied to the factors $g_2(\phi, m)$ and $g_3(\phi, m)$ are being used to represent the expected evolution of $2J$ state variables relative to their steady-state values, i.e $\sum_{j=1}^{J}[s_j(t) - \theta_j(t)] \cdot \exp(-\kappa_j m)$, each with a different mean-reversion parameter κ_j.

Appendix A shows how the ANS model can be extended arbitrarily by adding higher-order exponential-polynomial functions. Section A.5 shows those extensions are equivalent to adding terms in the Taylor expansion of the non-steady state components of the ABE model around ϕ. In this sense, the ANS model or its extensions are a natural approximation to the ABE model. Conversely, while any other series of functions could be used to approximate the ABE model (such as the simple polynomials of McCulloch (1971) or the Chebyshev polynomials in Pham (1998)), these would be "unnatural" approximations in the sense that the addition of each higher-order function would not directly correspond to an extra term in the Taylor expansion of the ABE model.

3.3 Econometric relationships for the ANS model coefficients, inflation and output growth

This section derives single-equation econometric relationships between the yield curve, output, and inflation using the ANS model and its explicit economic foundation defined in the previous section. The derivations essentially require the conversion of the continuous-time relationships in section 3.2.3 into discrete-time relationships. Hence, section 3.3.1 derives the discrete-time relationships for the ANS Level component, and section 3.3.2 derives discrete-time relationships for the non-Level ANS components.

3.3.1 The ANS Level coefficient, steady-state output growth and inflation

The elements of equation 3.13 are all contemporaneous, and so its discrete-time version may simply be written as:

$$\beta_{1,t} - \Delta P_t^* - \Delta Y_t^* = \alpha^* + \varepsilon_t^* \tag{3.17}$$

where $\beta_{1,t}$ is the estimated Level coefficient, ΔP_t^* is annualised steady-state inflation, and ΔY_t^* is annualised steady-state output growth, all at time t. The constant parameter α^* captures the term premium component γ_1 for the ANS model (see section 2.3), and the parameters $\sigma_{j,p}^2$ (see section 3.2.1). Equation 3.17 is therefore a (1,-1) cointegrating relationship between $\beta_{1,t}$ and $\Delta P_t^* + \Delta Y_t^*$, and ε_t^* represents a time series of estimated residuals that should be stationary. Note that given all of the data in equation 3.17 are Gaussian processes, ordinary least squares (OLS) estimation and standard unit root tests are applicable.

3.3.2 The ANS non-Level coefficients, non-steady-state output growth and inflation

Equation 3.15 is an intertemporal relationship, so its discrete-time version requires appropriate notation for time, the future horizon, and the width of the future horizon. Hence, denote a forward interval relative to time t by $t + T_1, t + T_2$ where T_1 ($T_1 \geq 0$) and T_2 ($T_2 > T_1$) are both constants. The evaluation of discrete quantities from functions defined in continuous time $t+m$ over the forward interval $t+T_1$ and $t+T_2$ may be obtained by integration with respect to m. Stylistically, this may be represented as:

$$[t] - - - - - - - - - - - - [t + T_1] - - - - - - - - - - - - [t + T_2]$$

$$[t] - [t + m]$$

where $t + T_1$ and $t + T_2$ define the start and finish of the forward interval, and the width of the forward interval is $T_2 - T_1$.

For notational convenience, then define the expected change in instantaneous nominal output growth relative to steady-state nominal output growth as $E_t\left[dX\left(t+m\right)\right] = E_t\left[dP\left(t+m\right) + dY\left(t+m\right) - dP^*\left(t+m\right) - dY^*\left(t+m\right)\right]$. The discrete-time measure of $E_t\left[dX\left(t+m\right)\right]$ over the forward horizon $t + T_1, t + T_2$, hereafter denoted $E_t\left[\Delta X_{t+T_1,t+T_2}\right]$, is then the annualised average of $E_t\left[dX\left(t+m\right)\right]$ over the forward interval.For example, $E_t\left[\Delta X_{t,t+1}\right]$ is expected nominal output growth relative to steady-state nominal output growth from t to one year from t, and $E_t\left[\Delta X_{t+0.5,t+0.75}\right]$ is expected annualised nominal output growth relative to steady-state nominal output growth over the period t plus two quarters to t plus three quarters. The corresponding averages of the ANS model terms from equation 3.15, i.e $-\sum_{n=2}^{3}\left[\beta_n\left(t\right) - \gamma_n\right] \cdot g_n(\phi, m)$, may be calculated via integration by maturity over the forward interval, i.e:

$$\begin{aligned}\frac{-1}{T_2-T_1}\int_{t+T_1}^{t+T_2}\left[\sum_{n=2}^{3}\left[\beta_n(t)-\gamma_n\right]\cdot g_n(\phi,m)\right]dm &= -\sum_{n=2}^{3}\gamma_n\cdot q_n(T_1,T_2)\\ &\quad+\sum_{n=2}^{3}\beta_n(t)\cdot q_n(T_1,T_2) \quad (3.18)\end{aligned}$$

where $q_n(T_1,T_2)=\frac{-1}{T_2-T_1}\int_{T_1}^{T_2}g_n(\phi,m)dm$. The two calculations required are:

$$q_2(T_1,T_2) = \frac{-1}{\phi(T_2-T_1)}\left[\exp(-\phi T_2)-\exp(-\phi T_1)\right] \quad (3.19a)$$

$$q_3(T_1,T_2) = \frac{-1}{\phi(T_2-T_1)}\left[\begin{array}{c}\exp(-\phi T_2)(-2\phi T_2-1)\\ -\exp(-\phi T_1)(-2\phi T_1-1)\end{array}\right] \quad (3.19b)$$

and table 3.3 contains the values of $q_2(T_1,T_2)$ and $q_3(T_1,T_2)$ (denoted as $q_n(T_1,T_2)$ for Slope and Bow) that correspond to the forward horizons tested in the empirical work. As an example, take the ANS Slope and Bow coefficients estimated for June 2004; i.e $\beta_2(\text{Jun-2004}) = 6.80\%$ and $\beta_3(\text{Jun-2004}) = -2.09\%$. Using these in conjunction with the values of $q_2(0,1)$ and $q_3(0,1)$ from the third-to-bottom line of table 3.3 implies that, as at June 2004 and excluding any allowance for term premia, expected nominal output growth relative to steady-state nominal output growth from June 2004 to June 2005 was $6.80\%\times 0.61 + -2.09\%\times 0.07 = 4.03\%$. Similarly, using the values of $q_2(0.5,0.75)$ and $q_3(0.5,0.75)$ from the third line of table 3.3, expected annualised nominal output growth relative to steady-state nominal output growth from December 2004 to March 2005 was $6.80\%\times 0.51 + -2.09\%\times -0.17 = 3.84\%$. Regarding the term premium estimates, because γ_2 and γ_3 are constant parameters in the ANS model, the quantity $-\sum_{n=2}^{N}\gamma_n\cdot q_n(T_1,T_2)$ for any given forward horizon will be a constant. Denoting the latter as α_{0,T_1,T_2} gives the discrete-time single-equation relationship for an arbitrary forward horizon as:

$$E_t\left[\Delta X_{t+T_1,t+T_2}\right]=\alpha_{0,T_1,T_2}+\alpha_{1,T_1,T_2}\cdot\sum_{n=2}^{3}\beta_n(t)\cdot q_n(T_1,T_2)+\varepsilon_{t,T_1,T_2} \quad (3.20)$$

Note that the parameter α_{1,T_1,T_2} should equal 1 for each forward horizon, because the intertemporal profile of $E_t\left[\Delta X_{t+T_1,t+T_2}\right]$ is already embedded in $q_n(T_1,T_2)$. As standard in the existing literature (e.g see Estrella, Rodrigues and Schich (2003)), the estimation of equation 3.20 proceeds by substituting realised $\Delta X_{t+T_1,t+T_2}$ data for $E_t\left[\Delta X_{t+T_1,t+T_2}\right]$. All of the data in equation 3.20 are Gaussian processes, and so OLS estimation is applicable, as is the use of the Newey-West technique to correct the estimated standard errors for the effect of moving-average serial correlation induced in ε_{t,T_1,T_2} whenever the forward horizon exceeds the frequency of the data (this technique is standard in the existing literature; e.g see Estrella et al. (2003)).

3.4 The empirical application to US data

This section tests the predictions of the ANS model framework empirically using US data. To make the results directly comparable to the existing literature and to allow for the longest period of estimation, the empirical analysis is undertaken in-sample only using standard published data. Section 3.4.1 outlines that data, section 3.4.2 discusses the results from estimating the predicted yield curve/inflation relationships, and 3.4.3 discuss the results from estimating the predicted yield curve/output growth relationships.

3.4.1 Description of the yield curve, inflation, and output data

The interest rate data used in the empirical application are as detailed in section 2.5.1, except the sample period is from July 1954 to June 2005 (the last month available at the time of the analysis). This gives 612 monthly observations of yield curve data, and figure 2.7 has already illustrated the monthly time series of ANS Level, Slope, and Bow coefficients obtained from such data.[3] Taking the last month of each quarter from September 1954 to June 2005 provides the relevant quarterly data (a sample size of 204 data points) for the

[3] The model was re-estimated by the procedure in section 2.3.4, but the ANS parameters were almost identical to those obtained up to February 2004. Specifically, the one difference to two decimal places was $\phi = 1.09$, compared to $\phi = 1.07$ obtained previously.

application in this chapter. Note that the ANS coefficients are already on an annualised basis, given that they are estimated from yield curve data expressed on an annualised basis.

The analysis also requires data for steady-state inflation, steady-state output growth, and nominal output growth relative to its steady-state value. These data are not measured directly, and so proxies are necessarily required. The primary proxy for economy-wide steady-state inflation is prevailing inflation in the GDP deflator (hereafter IGD). This choice makes nominal output growth relative to its steady-state value equal to real output growth relative to its steady-state value, i.e $\Delta X_{t+T_1,t+T_2} = \Delta Y_{t+T_1,t+T_2} - \Delta Y^*_{t+T_1,t+T_2}$, for which there is standard data available. Specifically, for the analysis in this chapter, the proxy for real output growth is real GDP growth, and the proxy for steady-state output growth is Congressional Budget Office potential GDP growth (hereafter CBO ΔY^*_t).[4] Prevailing inflation in personal consumption expenditure deflators, including and excluding food and energy (hereafter PCE and PCEX respectively), are also used as alternative proxies for steady-state inflation when testing for the cointegration implied by equation 3.17.[5] All index levels for the series mentioned are available from the FRED on a quarterly basis, and the inflation and growth data are calculated as annualised changes in the logarithm of those levels.

Equations 3.17 and 3.20 are estimated using both annualised quarterly data, and annual data at a quarterly frequency; the former for maximum precision when investigating the intertemporal relationships, and the latter for comparability to the existing literature. The existing literature also forecasts GDP growth, rather than GDP growth relative to

[4]The CBO measure was selected because the data are independently-calculated and are readily available/verifiable from the FRED. The CBO methodology is statistically-based with economic foundations; see Congressional Budget Office (2001) for calculation details. Of course, as noted in Congressional Budget Office (2004), there are many different methods for calculating potential output, including purely statistical techniques (e.g centred moving averages, bandpass filters, HodrickPrescott filter, and Kalman filter, vector autoregressive models (VARs), and structural VARs).

[5]Surveyed long-term, or even short-term, CPI inflation expectations would arguably make superior proxies for steady-state inflation. However, the availability of that data is limited; e.g 10-year inflation expectations from the Philidelphia Federal Reserve website are only available from 1991, and Michigan year-ahead inflation expectations from the FRED are only available from 1978.

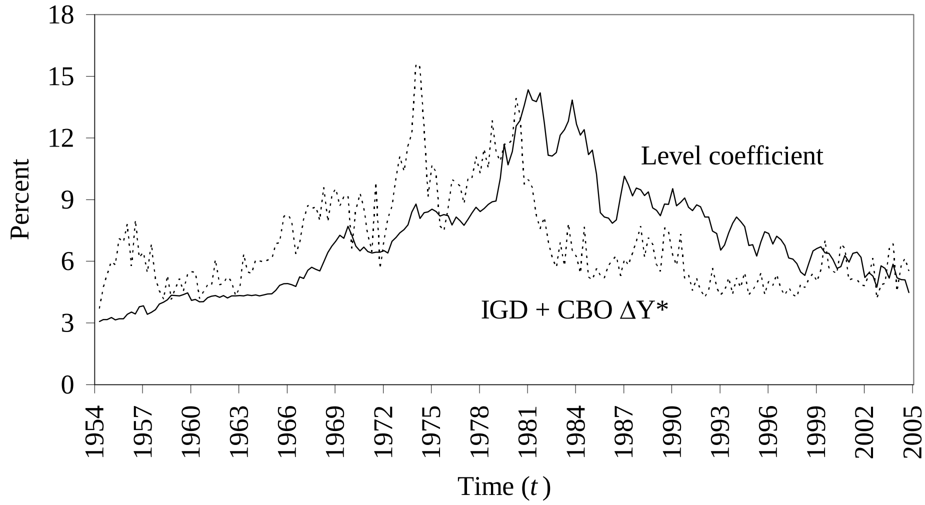

Figure 3.1: The time series of the ANS Level coefficient (i.e $\beta_1(t)$), and annualised quarterly inflation in the GDP deflator (IGD) plus annualised quarterly growth in Congressional Budget Office potential GDP (CBO ΔY^*). Note the apparent structural change between the two series from around the late-1970s/early-1980s.

potential GDP growth. Using a constant estimate of potential output growth (i.e $\Delta Y_t^* = 3.31\%$, which is the average of annualised quarterly GDP growth over the entire sample) instead of the time-varying CBO ΔY_t^* obtains $\Delta X_{t+T_1,t+T_2}$ data equivalent to GDP growth to within a constant.

To allow a visual inspection of some of the relationships to be estimated, figure 3.1 plots the time series of the ANS Level coefficient and the annualised quarterly IGD plus annualised quarterly CBO ΔY_t^* data, and figure 3.2 plots the difference between those series. Figure 3.3 illustrates the annual GDP growth, the annual CBO ΔY_t^*, and the constant $\Delta Y_t^* = 3.31\%$ data that are used to calculate the annual $\Delta X_{t+T_1,t+T_2}$ data subsequently used in the estimation of equation 3.20. Figure 3.4 plots $\Delta X_{t,t+1}$ based on CBO ΔY_t^*, and the corresponding predicted values of $\Delta X_{t,t+1}$ using the ANS Slope and Bow coefficients that prevailed at time t.

Figures 3.1, 3.2, and 3.4 all show prima facie evidence of structural breaks in the yield curve/inflation and yield curve/output growth relationships from around the late

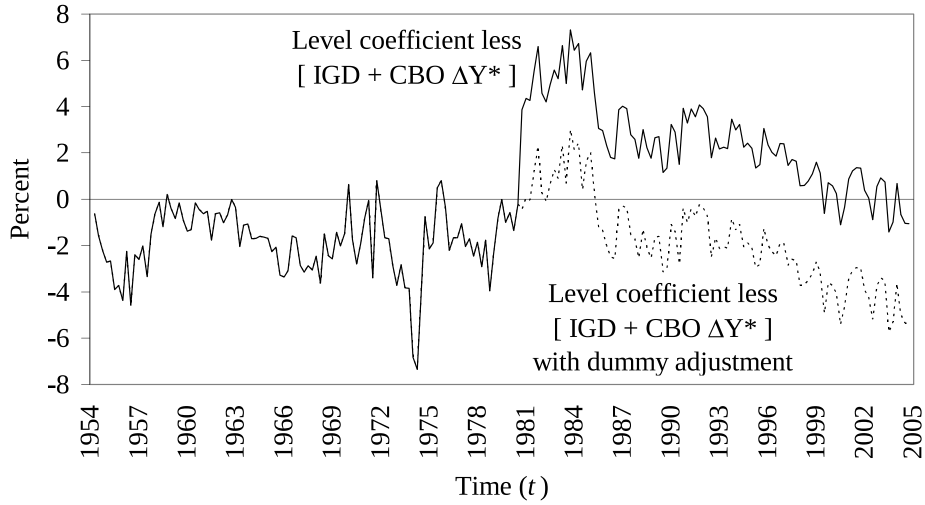

Figure 3.2: The time series of the ANS Level coefficient (i.e $\beta_1(t)$) less annualised quarterly inflation in the GDP deflator (IGD) and annualised quarterly growth in Congressional Budget Office potential GDP (CBO ΔY^*). The alternative series allows for structural change in the difference between the two series from 1982:Q1.

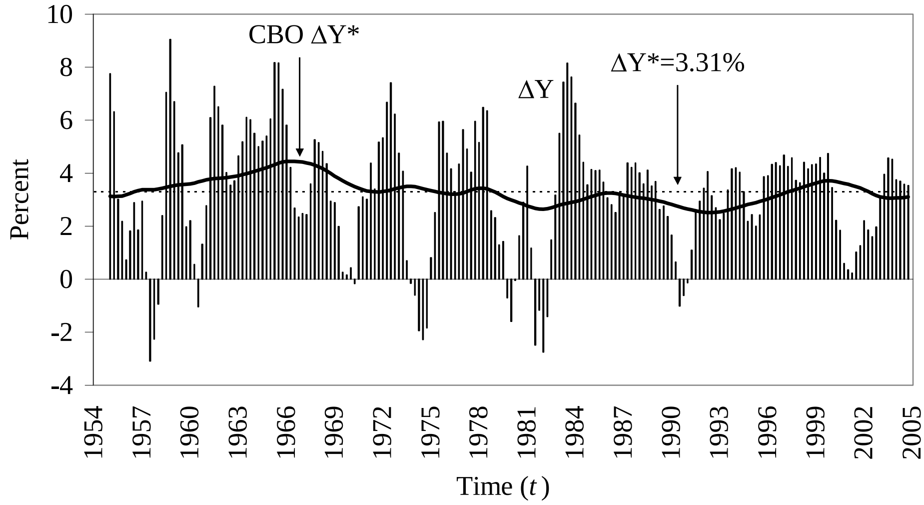

Figure 3.3: Annual output growth data used to construct the $\Delta X_{t+T_1,t+T_2}$ data used for the estimation of equation 3.20. ΔY is the annual growth in GDP, CBO ΔY^* is annual growth in Congressional Budget Office potential GDP, and $\Delta Y^* = 3.31\%$ is the estimate of constant potential growth. The difference between ΔY and the CBO ΔY^* is plotted in figure 3.4.

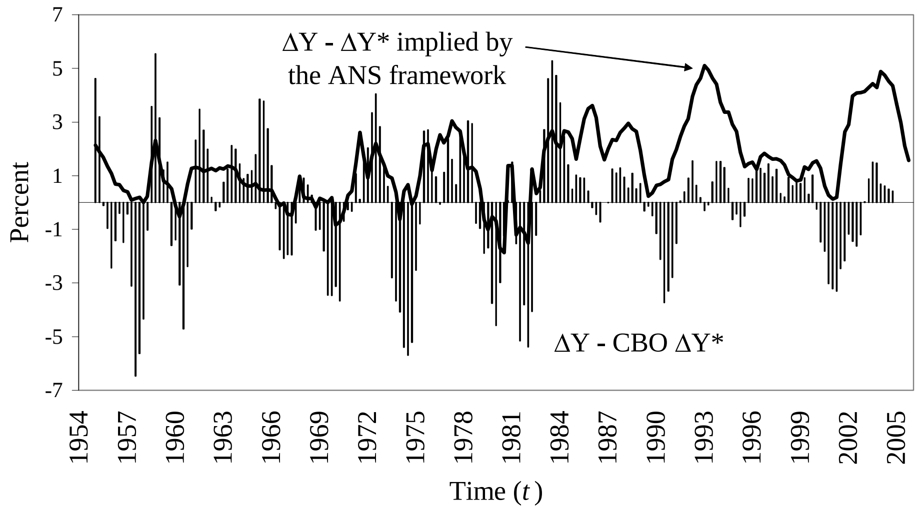

Figure 3.4: The time series of $\Delta Y_{t,t+1} - \Delta Y^*_{t,t+1} (= \Delta X_{t,t+1})$ based on the ΔY and CBO ΔY^* data plotted in figure 3.3, and $E_t\left[\Delta Y_{t,t+1} - \Delta Y^*_{t,t+1}\right]$ $(= E_t\left[\Delta X_{t,t+1}\right])$ as implied by the ANS framework. Note the apparent structural change in the relationship from around the late-1970s/early-1980s.

1970s/early 1980s. This implies a change in term premia embedded in the yield curve, which was also apparent in the yield curve forecasting application of section 2.5. That change in term premia is not surprising given the context of substantial change in the US economic and financial environment during the late-1970s to the mid-1980s. For example, a significant economic change was the Federal Reserve's Volcker-led disinflation from October 1979, and the subsequent maintenance of low inflation. Significant financial changes were progressive market deregulation, including eliminating interest rate restrictions and rationalising reserve requirements, and an increasing role for securitisation. Prior empirical work that will be discussed in section 3.5 also mentions these reasons when documenting structural breaks between 1979:Q4 to 1984:Q1.[6]

The analysis for the full sample therefore proceeds with the inclusion of a step

[6]For example, based on statistical tests for unknown breakpoints, Estrella et al. (2003) identifies structural breaks in October 1979 and October 1982 when the yield curve is used as an indicator of future inflation, and in September 1983 when the yield curve is used as an indicator of future output. Using a similar technique, Aïssa and Jouini (2003) documents a break in the inflation process in June 1982, and Lai (2004) suggests a break between mid-1980 and early-1981. Jardet (2004) documents a break in the yield curve/output relationship in March 1984.

dummy term $\alpha_{2,T_1,T_2} \cdot D_t$ in equations 3.17 and 3.20, where $D_t = 0$ up to the period immediately before the breakpoint, $D_t = 1$ from the given breakpoint, and α_{2,T_1,T_2} is a constant parameter to be estimated. The breakpoints for the analysis in this chapter are 1982:Q1 for inflation and 1984:Q1 for output, which were selected to be consistent with the prior empirical work noted above. The analysis is also undertaken for the two sub-samples pre-1979:Q4 and post-1984:Q1, which excludes all of the breakpoints referenced in prior empirical work.

3.4.2 The ANS Level coefficient and inflation

Tables 3.1 and 3.2 contain the test results for cointegration, as implied by equation 3.17, between the ANS Level coefficient and the annualised quarterly and annual measures of inflation plus CBO potential GDP growth.[7] Note that the critical statistics for the various tests vary. Firstly, the critical statistics for the unit root tests are taken from Hamilton (1994) table B.6 Case 2 (which are based on the standard unit root process with an estimated constant). Those critical statistics are also appropriate when testing for cointegration with imposed/restricted vectors. The critical statistics when testing for cointegration allowing for an estimated step dummy variable are not standard, and so were obtained by the simulation of a unit root process with an appropriate break in the mean.[8]

Both sets of results are moderately supportive of the hypothesis of cointegration over the whole sample and the two sub-samples. Specifically, the test statistics in the top half of the table typically do not reject the unit root hypothesis,[9] but the Level coefficient

[7]For consistency, all quarterly results use one lag for the augmented Dickey-Fuller tests and a window of one for the Phillips-Perron tests, and all annual results use four lags and a window of four (to allow for the expected MA(3) serial correlation plus one). The results using optimal lag and window selection were similar, but implausibly long lag lengths were occasionally selected.

[8]I thank John McDermott for supplying the Gauss code and the summary critical statistics based on a simulation size of 10,000.

[9]The quarterly measures of inflation often produce materially negative test statistics, but MacKinnon (1996) p. 615 notes that test statistics on the annual measures of inflation are more reliable (essentially because unobservable measurement errors in inflation data over short intervals tend to bias the unit root test statistics downward, but that bias fades over longer intervals). Of course, any downward bias in the

less the measures of inflation with or without potential growth added typically do reject the unit root hypothesis. Consistent with the prior discussion on structural change, the cointegration results over the whole sample are stronger when the step dummy variable is included. Interestingly, the results are also better when CBO ΔY_t^* is ignored (which is equivalent to replacing CBO ΔY_t^* with the constant $\Delta Y_t^* = 3.31\%$), and/or measures of consumption inflation are used as proxies for steady-state inflation. These latter results suggest that long-maturity yields may be more responsive to movements in consumption inflation measures, rather than economy-wide inflation and/or variations in steady-state output growth.

That said, any conclusions must remain tentative given what is essentially modest and variable evidence for cointegration in the results. The modesty of the results might arise because current inflation and potential output growth are not good proxies for their steady-state counterparts, but an alternative explanation is that the term premia related to steady-state inflation and/or steady-state growth might vary more than the step dummy variable allows for. Indeed, if the combination of those term premia over time are represented by the time series $\beta_{1,t} - \Delta P_t^* - \Delta Y_t^*$, then figure 3.2 shows four distinct levels: i.e a low (but variable) level up to the late-1970s/early-1980s; a peak level from the early-1980s to 1986; a moderate level from 1986 to 1998; and a return to a relatively low level (i.e consistent with pre-1979 levels) from 1998. It would be intriguing to formally test for structural breaks in the time series $\beta_{1,t} - \Delta P_t^* - \Delta Y_t^*$, and to see how those breaks correspond to changes in the economic and financial environments that prevailed at the time. That investigation is beyond the scope of this chapter, but is discussed in section 6.2.2 of the concluding chapter.[10]

unit root tests for inflation will also translate into the cointegration tests, but the latter test statistics are typically of a larger magnitude than the unit root test statistics on inflation itself.

[10]Buraschi and Jiltsov (2005) provide evidence for a time-varying risk premium on inflation using an arbitrage-free structural model of the macroeconomy and yield curve. However, the more parimonious ANS framework should prove more amenable to investigating such phenomena.

Table 3.1: Quarterly cointegration tests for the ANS Level coefficient

Period	Full sample		Up to 1979:Q3		From 1984:Q1	
Unit root tests	ADF	PP	ADF	PP	ADF	PP
Level coefficient (β_1)	-1.7	-1.6	-0.2	-0.2	-2.4	-3.0 **
Pot. GDP growth (ΔY^*)	-2.8 *	-0.9	-1.9	-0.8	-2.7 *	-0.8
Dummy (D)	-1.0	-1.0	-	-	-	-
Δ GDP deflator (IGD)	-2.7 *	-3.2 **	-1.6	-2.2	-3.6 ***	-5.0 ***
Δ PCE deflator (PCE)	-2.8 *	-3.5 ***	-1.2	-1.9	-3.4 **	-4.7 ***
Δ PCEX deflator (PCEX)	-2.1	-2.6	-1.4	-1.7	-2.1	-2.9 **
IGD + ΔY^*	-2.7 *	-3.3 **	-1.8	-2.4	-3.9 ***	-5.3 ***
PCE + ΔY^*	-2.9 *	-3.6 ***	-1.5	-2.2	-3.6 ***	-5.0 ***
PCEX + ΔY^*	-2.2	-2.6 *	-1.7	-1.9	-2.3	-3.2 **
Cointegration tests	ADF	PP	ADF	PP	ADF	PP
β_1 - IGD	-2.7 *	-3.1 **	-3.6 ***	-4.5 ***	-2.8 *	-3.2 **
β_1 - PCE	-2.9 *	-3.5 **	-3.1 **	-4.4 ***	-3.2 **	-3.7 ***
β_1 - PCEX	-2.9 *	-3.2 **	-3.7 ***	-4.1 ***	-3.7 ***	-3.8 ***
β_1 - [IGD + ΔY^*]	-2.5	-2.8 *	-3.8 ***	-4.9 ***	-2.7 *	-3.0 **
β_1 - [PCE + ΔY^*]	-2.6 *	-3.2 **	-3.4 **	-4.7 ***	-3.1 **	-3.5 **
β_1 - [PCEX + ΔY^*]	-2.5	-2.8 *	-3.6 ***	-4.0 ***	-3.5 **	-3.6 ***
β_1 - IGD - D	-3.8 **	-4.7 ***	-	-	-	-
β_1 - PCE - D	-3.9 **	-5.1 ***	-	-	-	-
β_1 - PCEX - D	-4.0 ***	-4.9 ***	-	-	-	-
β_1 - [IGD + ΔY^*] - D	-3.6 **	-4.5 ***	-	-	-	-
β_1 - [PCE + ΔY^*] - D	-3.8 **	-4.9 ***	-	-	-	-
β_1 - [PCEX + ΔY^*] - D	-3.7 **	-4.5 ***	-	-	-	-

Note: Tests for cointegration between the ANS Level coefficient, and annualised quarterly measures of inflation with and without annualised quarterly growth in CBO potential GDP growth and with and without an estimated step dummy variable. ADF is augmented Dickey-Fuller, and PP is Phillips-Perron. ***, **, * respectively represent 1, 5, and 10 percent levels of significance. As mentioned in section 3.4.1, potential GDP growth and inflation measures are changes in the logarithm of the level.

Table 3.2: Annual cointegration tests for the ANS Level coefficient

Period	Full sample		Up to 1979:Q3		From 1984:Q1	
Unit root tests	ADF	PP	ADF	PP	ADF	PP
Level coefficient (β_1)	-1.7	-1.7	-0.3	-0.3	-0.8	-2.6 *
Pot. GDP growth (ΔY^*)	-2.0	-1.3	-1.0	-1.3	-1.9	-1.1
Dummy (D)	-0.9	-0.9	-	-	-	-
Δ GDP deflator (IGD)	-1.7	-1.9	-0.5	-1.0	-1.3	-2.1
Δ PCE deflator (PCE)	-2.0	-2.2	-0.6	-1.1	-1.3	-1.9
Δ PCEX deflator (PCEX)	-1.5	-1.7	-0.9	-1.1	-0.9	-1.3
IGD + ΔY^*	-1.7	-2.0	-0.8	-1.3	-1.5	-2.2
PCE + ΔY^*	-2.1	-2.3	-0.9	-1.4	-1.5	-2.0
PCEX + ΔY^*	-1.5	-1.8	-1.3	-1.4	-1.1	-1.5
Cointegration tests	ADF	PP	ADF	PP	ADF	PP
β_1 - IGD	-2.1	-2.1	-2.7 *	-2.5	-1.0	-2.4
β_1 - PCE	-2.4	-2.4	-2.8 *	-2.7 *	-0.9	-3.0 **
β_1 - PCEX	-2.2	-2.5	-2.8 *	-2.9 *	-2.0	-3.7 ***
β_1 - [IGD + ΔY^*]	-2.0	-1.9	-3.1 **	-2.9 *	-0.9	-2.3
β_1 - [PCE + ΔY^*]	-2.3	-2.2	-1.4	-1.8	-1.4	-2.3
β_1 - [PCEX + ΔY^*]	-2.2	-2.2	-1.4	-1.7	-1.5	-2.7 *
β_1 - IGD - D	-2.7	-2.7	-	-	-	-
β_1 - PCE - D	-3.1 *	-3.1 *	-	-	-	-
β_1 - PCEX - D	-3.3 *	-3.4 **	-	-	-	-
β_1 - [IGD + ΔY^*] - D	-2.6	-2.7	-	-	-	-
β_1 - [PCE + ΔY^*] - D	-3.1 *	-3.0	-	-	-	-
β_1 - [PCEX + ΔY^*] - D	-3.3 *	-3.0	-	-	-	-

Note: Tests for cointegration between the ANS Level coefficient, and annual measures of inflation with and without annual growth in potential GDP growth and with and without an estimated step dummy variable. ADF is augmented Dickey-Fuller, and PP is Phillips-Perron. ***, **, * respectively represent 1, 5, and 10 percent levels of significance. As mentioned in section 3.4.1, potential GDP growth and inflation measures are changes in the logarithm of the level.

3.4.3 The ANS Slope and Bow coefficients and output growth

Table 3.3 contains the results from estimating equation 3.20 over the full sample, using the dummy variable with the 1984:Q1 breakpoint and $\Delta X_{t+T_1,t+T_2}$ based on CBO ΔY_t^*.[11] The results are supportive of the hypothesis that the ANS framework provides a gauge of the profile (i.e the timing and magnitude) of the future changes in output growth relative to potential output growth. That is, the estimates of α_{1,T_1,T_2} are highly significant and positive for forward horizons up to one year, become insignificant while remaining positive through the second year (although the estimate is significant on an annual basis), and become insignificant and negative for most forward horizons over two years. In addition, the estimates α_{1,T_1,T_2} are insignificantly different from the theoretical value of 1, except for the marginal rejection of that hypothesis for the 2.25 to 2.5 year, and the 2.5 to 2.75 year horizons.

The results are also consistent with a term premium existing before the structural break, and becoming larger after the structural break. That is, both the estimates of α_{0,T_1,T_2} and the dummy variable parameter α_{2,T_1,T_2} are highly significant for short horizons, and remain consistently signed (with occasional exceptions) but insignificant after that. Note that the negative value of both coefficients equates to positive term premia; i.e the yield curve would persistently over-forecast $\Delta X_{t+T_1,t+T_2}$, so a negative adjustment is required to remove that persistent bias.

Table 3.4 contains the results for equation 3.20 estimated over each sub-sample, and this provides an interesting insight into the results for the entire sample. That is, up to 1979:Q3 the shape of the yield curve was best at predicting $\Delta X_{t+T_1,t+T_2}$ over short forward horizons, although it tended to under-predict those changes. Conversely, beyond 1984:Q1

[11] The Newey-West window used in each estimation is the number of quarters to T_2 less 1. This allows for the induced serial correlation expected in theory for both the annualised quarterly and the annual data. For example, both $\Delta X_{t,t+1}$ and $\Delta X_{t+0.75,t+1}$ data will induce moving-average serial correlation of order 3 into ε_{t,T_1,T_2}.

Table 3.3: Full-sample ANS model predictions of real output growth

Horizon T_1, T_2	$q_n(T_1,T_2)$ Slope	Bow	R^2 in %	Constant coefficient	ANS coefficient	Dummy coefficient	ANS coeff. less 1
0 - 0.25	0.88	0.65	10.0	-0.64 *	0.76 ***	-1.16 **	-0.24
0.25 - 0.5	0.67	0.14	15.7	-0.95 **	1.10 ***	-1.81 ***	0.10
0.5 - 0.75	0.51	-0.17	8.5	-0.83 *	0.89 ***	-1.34 **	-0.11
0.75 - 1	0.39	-0.34	6.3	-0.78	0.85 ***	-1.13	-0.15
1 - 1.25	0.30	-0.42	5.7	-0.76	0.91 ***	-1.05	-0.09
1.25 - 1.5	0.23	-0.44	1.1	-0.42	0.46	-0.39	-0.54
1.5 - 1.75	0.18	-0.43	1.0	-0.34	0.50	-0.45	-0.50
1.75 - 2	0.13	-0.40	1.3	-0.38	0.67 *	-0.51	-0.33
2 - 2.25	0.10	-0.36	0.5	-0.23	0.49	-0.34	-0.51
2.25 - 2.5	0.08	-0.32	0.0	-0.04	-0.18	0.14	-1.18 *
2.5 - 2.75	0.06	-0.28	0.0	-0.06	-0.11	0.07	-1.11 *
2.75 - 3	0.05	-0.24	0.8	-0.29	1.13	-0.38	0.13
0 - 1	0.61	0.07	25.4	-0.82 **	0.93 ***	-1.40 ***	-0.07
1 - 2	0.21	-0.43	4.9	-0.48	0.66 ***	-0.59	-0.34
2 - 3	0.07	-0.30	0.2	-0.15	0.28	-0.12	-0.72

Note: Full-sample estimates of equation 3.20 using a step dummy variable and $\Delta X_{t+T_1,t+T_2}$ based on CBO potential output growth. ***, **, * respectively represent 1, 5, and 10 percent two-tailed levels of significance.

the shape of the yield curve was best at predicting $\Delta X_{t+T_1,t+T_2}$ over medium forward horizons; while remaining useful for short horizons, it tended to over-predict $\Delta X_{t+T_1,t+T_2}$. The combination of these sub-sample results evidently offset to give estimates of the coefficients α_{1,T_1,T_2} that are close to unity over the full sample.

Tables 3.5 and 3.6 contain the results for estimating equation 3.20 using $\Delta X_{t+T_1,t+T_2}$ based on $\Delta Y_t^* = 3.31\%$. These estimations are directly analogous to the regressions of GDP growth on lagged yield curve spreads from the existing literature that will be discussed in section 3.5.

Table 3.4: Sub-sample ANS model predictions of real output growth

	Up to 1979:Q3				From 1984:Q1			
Horizon T_1, T_2	R^2 in %	Const. coeff.	ANS coeff.	ANS cf. less 1	R^2 in %	Const. coeff.	ANS coeff.	ANS cf. less 1
0 - 0.25	17.8	-1.31 ***	1.53 ***	0.53	4.2	-0.61	0.27 *	-0.73 ***
0.25 - 0.5	17.2	-1.57 ***	1.82 ***	0.82 **	5.1	-0.77	0.31 *	-0.69 ***
0.5 - 0.75	9.2	-1.28 **	1.47 ***	0.47	8.0	-1.01	0.41 **	-0.59 ***
0.75 - 1	13.1	-1.55 ***	1.93 ***	0.93 *	6.2	-0.91	0.40 *	-0.60 **
1 - 1.25	4.7	-0.96	1.26 **	0.26	5.9	-0.90	0.44 *	-0.56 **
1.25 - 1.5	0.8	-0.45	0.58	-0.42	4.3	-0.78	0.44	-0.56 *
1.5 - 1.75	0.2	-0.22	0.31	-0.69	9.4	-1.16	0.75 **	-0.25
1.75 - 2	0.3	-0.26	0.47	-0.53	7.4	-1.05	0.81 *	-0.19
2 - 2.25	0.0	-0.09	0.17	-0.83	6.7	-1.02	0.94 **	-0.06
2.25 - 2.5	1.9	0.36	-1.58	-2.58	1.3	-0.48	0.51	-0.49
2.5 - 2.75	0.7	0.19	-1.16	-2.16 *	0.2	-0.24	0.27	-0.73
2.75 - 3	0.3	-0.16	0.91	-0.09	1.1	-0.43	0.70	-0.30
0 - 1	38.0	-1.56 ***	1.80 ***	0.80 **	12.1	-0.80	0.33 *	-0.67 ***
1 - 2	2.7	-0.43	0.68 *	-0.32	15.2	-1.00	0.61 **	-0.39
2 - 3	0.2	0.07	-0.37	-1.37	3.8	-0.56	0.66	-0.34

Note: Sub-sample estimates of equation 3.20 using $\Delta X_{t+T_1,t+T_2}$ based on CBO potential output growth. ***, **, * respectively represent 1, 5, and 10 percent two-tailed levels of significance.

Table 3.5: Full-sample ANS model with constant potential output growth

Horizon T_1, T_2	$q_n(T_1,T_2)$ Slope	Bow	R^2 in %	Constant coefficient	ANS coefficient	Dummy coefficient	ANS coeff. less 1
0 - 0.25	0.88	0.65	8.7	-0.41	0.71 ***	-1.49 ***	-0.29 *
0.25 - 0.5	0.67	0.14	13.5	-0.70	1.03 ***	-2.09 ***	0.03
0.5 - 0.75	0.51	-0.17	6.8	-0.57	0.80 ***	-1.59 **	-0.20
0.75 - 1	0.39	-0.34	4.8	-0.51	0.75 ***	-1.37 *	-0.25
1 - 1.25	0.30	-0.42	4.3	-0.50	0.79 ***	-1.29 *	-0.21
1.25 - 1.5	0.23	-0.44	0.6	-0.16	0.32	-0.63	-0.68 **
1.5 - 1.75	0.18	-0.43	0.7	-0.07	0.34	-0.70	-0.66 **
1.75 - 2	0.13	-0.40	0.9	-0.11	0.48	-0.76	-0.52
2 - 2.25	0.10	-0.36	0.5	0.04	0.27	-0.60	-0.73 *
2.25 - 2.5	0.08	-0.32	0.5	0.22	-0.43	-0.13	-1.43 **
2.5 - 2.75	0.06	-0.28	0.4	0.20	-0.40	-0.21	-1.40 **
2.75 - 3	0.05	-0.24	0.7	-0.04	0.79	-0.68	-0.21
0 - 1	0.61	0.07	20.5	-0.57	0.86 ***	-1.68 ***	-0.14
1 - 2	0.21	-0.43	3.1	-0.21	0.50 **	-0.84	-0.50 **
2 - 3	0.07	-0.30	0.7	0.11	0.00	-0.40	-1.00 *

Note: Full-sample estimates of equation 3.20 using a step dummy variable and $\Delta X_{t+T_1,t+T_2}$ based on constant potential output growth (i.e $\Delta Y_t^* = 3.31\%$). ***, **, * respectively represent 1, 5, and 10 percent two-tailed levels of significance.

Table 3.6: Sub-sample ANS model with constant potential output growth

	Up to 1979:Q3				From 1984:Q1			
Horizon T_1, T_2	R^2 in %	Const. coeff.	ANS coeff.	ANS cf. less 1	R^2 in %	Const. coeff.	ANS coeff.	ANS cf. less 1
0 - 0.25	15.3	-0.90 *	1.43 ***	0.43	1.7	-0.60	0.17	-0.83 ***
0.25 - 0.5	14.3	-1.13 *	1.67 ***	0.67 *	2.3	-0.71	0.21	-0.79 ***
0.5 - 0.75	7.0	-0.81	1.30 ***	0.30	4.1	-0.91	0.30	-0.70 ***
0.75 - 1	10.3	-1.08 *	1.74 ***	0.74	2.7	-0.77	0.27	-0.73 ***
1 - 1.25	3.2	-0.49	1.05	0.05	2.8	-0.79	0.31	-0.69 ***
1.25 - 1.5	0.3	0.02	0.35	-0.65	1.9	-0.68	0.30	-0.70 **
1.5 - 1.75	0.0	0.24	0.07	-0.93	4.5	-0.96	0.54	-0.46
1.75 - 2	0.1	0.20	0.20	-0.80	2.6	-0.77	0.49	-0.51
2 - 2.25	0.0	0.37	-0.13	-1.13	2.4	-0.74	0.57	-0.43
2.25 - 2.5	2.7	0.80	-1.90	-2.90 *	0.1	-0.29	0.13	-0.87
2.5 - 2.75	1.2	0.63	-1.51	-2.51 **	0.5	0.07	-0.39	-1.39 **
2.75 - 3	0.1	0.26	0.55	-0.45	0.2	-0.34	0.30	-0.70
0 - 1	29.3	-1.07 *	1.62 ***	0.62 *	5.7	-0.77	0.23	-0.77 ***
1 - 2	1.1	0.04	0.45	-0.55	7.6	-0.90	0.44	-0.56 *
2 - 3	0.7	0.51	-0.67	-1.67	0.8	-0.45	0.31	-0.69

Note: Sub-sample estimates of equation 3.20 using $\Delta X_{t+T_1,t+T_2}$ based on constant potential output growth (i.e $\Delta Y_t^* = 3.31\%$). ***, **, * respectively represent 1, 5, and 10 percent two-tailed levels of significance.

3.5 Comparison of the ANS economic framework with the existing macro-finance literature

The field of macro-finance, or using financial market data in conjunction with macroeconomic data, is a growing field of research within the existing literature. This section briefly summarises that literature and then compares the merits of the ANS economic framework against previous approaches.

Statistical relationships between the yield curve, output, and inflation have already been well-established in the existing literature. That is, single-equation OLS regressions typically show a strong relationship between the current slope of the yield curve (often measured as the 10-year less the 90-day interest rate) and future output growth, a moderate relationship between the current slope of the yield curve and future inflation, and a cointegrating relationship between term interest rates and inflation. A comprehensive, but not necessarily exhaustive, list of examples reporting these results based on single-equation OLS estimation include: (1) on the yield curve/output relationship see Bernanke (1990), Chen (1991) Estrella and Hardouvelis (1991), Davis and Henry (1994), Hardouvelis (1994), Plosser and Rouwenhorst (1994), Haubrich and Dombrosky (1996), Dueker (1997), Bernard and Gerlach (1998), Estrella and Mishkin (1998), Filardo (1999), Atta-Mensah and Tkacz (2001), Hamilton and Kim (2002), Estrella et al. (2003), Bordo and Haubrich (2004), Nakaota (2005), and Paya, Matthews and Peel (2005); (2) on the yield curve/inflation relationship see Mishkin (1990*a*), Mishkin (1990*b*), Mishkin (1991), Jorian and Mishkin (1991), Abken (1993), Schich (1999), and Schich (2000); and (3) on both the yield curve/output and yield curve/inflation relationships see Gamber (1996), Davis and Fagan (1997), Estrella and Mishkin (1997), Kozicki (1997), Berk (1998), Stock and Watson (2001), and Diebold, Rudebusch and Aruoba (2005); and (4) on interest rate/inflation cointegration see Fahmy and Kandi (2003) and Lai (2004). Berk (1998) also provides a useful survey of the literature on yield curve, output, and inflation relationships.

However, as noted in Estrella (2003) pp. 1-4, the various justifications advanced for these empirical relationships are generally informal or heuristic: e.g real business cycles, countercyclical monetary policy, and life-cycle consumption to justify yield curve/output relationships; and the Fisher hypothesis with assumed constant or stationary real interest rates to justify yield curve/inflation relationships and interest rate/inflation cointegration. More formal macroeconomic foundations include Rendu de Lint and Stolin (2003) and Estrella (2005). Specifically, Rendu de Lint and Stolin (2003) derives a yield curve/output relationship via an intertemporal production and endowment economy, and Estrella (2005) derives yield curve, output, and inflation relationships via a standard dynamic model of the macroeconomy incorporating a short-maturity and a long-maturity interest rate. However, the yield curves in those models are not constructed to be arbitrage-free.

An alternative approach in the existing literature to investigating yield curve, output, and inflation relationships is the use of VAR models containing selected macroeconomic and yield curve data, such as in Bernard and Gerlach (1998), Ang and Piazzesi (2003), Jardet (2004), Diebold, Rudebusch and Aruoba (2005), and Wu (2005*b*). VAR models allow some theoretical structure to be imposed via parameter restrictions, including arbitrage-free constructions in Ang and Piazzesi (2003) and Wu (2005*b*). That said, the models noted still have an implicit atheoretical element given that VAR dynamics are simply imposed, as is the order of the autoregression and the parameter restrictions. Also, a practical issue associated with VAR models is that they can be challenging to estimate and interpret due to their lack of parsimony, even after simplifying restrictions are imposed to prevent overfitting and avoid parameter instability. For example, the Ang and Piazzesi (2003) model requires the estimation of 18 parameters (via the multistep application of maximum likelihood) for the ten variables it uses (i.e three price series, four indicators of real activity, and interest rates for five maturities). Similarly, the VAR model of Diebold, Rudebusch and Aruoba (2005) requires the estimation of 66 parameters (via the application of the Kalman filter and maximum likelihood) for the 20 variables it uses (i.e a price variable, a real activity variable, a monetary policy variable, and interest rates for 17 maturities linked using the NS model as a latent factor model of the yield curve).

By comparison with the single-equation OLS approaches, the ANS economic framework developed in this chapter allows the derivation of parsimonious, discrete-time, single-equation econometric relationships between the yield curve, output, and inflation that are analogous to those in the existing literature. However, the relationships derived from the ANS economic framework have the theoretical underpinning from the ABE model of the economy, which also provides theoretical parameter values to compare against the empirically estimated values. The ANS economic framework also helps explain the results of single-equation OLS estimations in the existing literature. That is, the literature finds that the explanatory power of the OLS regressions for future output growth on the slope of the yield curve is highest for short forward horizons and fades quickly past forward horizons of one year (e.g see Hamilton and Kim (2002) table 2). Equation 3.20 shows this is to be expected, given that the $\sum_{n=2}^{3} \beta_n (t) \cdot q_n (T_1, T_2)$ "signal" decreases (due to the falling magnitudes of $q_n (T_1, T_2)$) while the ε_{t,T_1,T_2} "noise" increases (due to the aggregation of more expectational surprises) as the forward horizon lengthens. Secondly, the ANS framework results show better explanatory power using CBO ΔY_t^* rather than $\Delta Y_t^* = 3.31\%$, suggesting that it is worthwhile allowing for time-varying potential output growth when using the yield curve as an indicator for future output growth.

By comparison to the VAR approaches, the single-equation econometric relationships between the yield curve, output, and inflation derived from the ANS economic framework are far easier to estimate, and their parsimony mitigates the overfitting and parameter instability that can arise in the empirical application of VAR models. The ANS model can also be viewed as a factor model of the yield curve, analogous to the interpretation in Diebold, Rudebusch and Aruoba (2005), but the ANS model has the theoretical economic underpinnings that NS models lack (and also the consistency across time and maturity as discussed in section 2.2 of chapter 2).

3.6 Conclusion

This chapter has developed a theoretically-consistent and easy-to-apply framework for interpreting and investigating relationships between the yield curve, output, and inflation. The empirical results based on US data are consistent with the predictions of the ANS framework; i.e the estimated long-maturity level of the yield curve given by the Level coefficient in the ANS model is cointegrated with steady-state inflation plus steady-state output growth, and the shape of the yield curve given by the Slope and Bow coefficients in the ANS model corresponds to the profile (i.e the timing and magnitude) of future output growth relative to its steady-state value. The estimation techniques used within the ANS economic framework are routine, so its practical application should be well-suited to researchers and market practitioners.

Chapter 4

Applying the ANS model to fixed interest portfolio management

4.1 Introduction

This chapter uses the ANS model to develop a framework applicable to fixed interest portfolio management. With reference to chapter 1, the broad motivation is to further illustrate how the ANS model of the yield curve can be formally applied in a financial setting. More specifically, the literature review in section 4.2 below indicates that the application of yield curve models to fixed interest portfolio management is at present separated into the distinct topics of measuring portfolio risk, or identifying potential opportunities to enhance portfolio returns through security selection. As a standard model of the yield curve, the ANS model should be applicable to both of those aspects simultaneously, and representing risk and potential return within one model should therefore allow a framework for optimising fixed interest portfolios.

Following the literature review, the remainder of the chapter proceeds as follows: section 4.3 outlines the intuition of the elements of the ANS model relevant to developing the risk and return framework; section 4.4 formally develops the theoretical risk and return framework based on the ANS model, and shows how the combination of risk and return leads to a formal portfolio optimisation framework; and section 4.5 contains the empirical application to swaps data, including ex-post return attribution and simulated real-time ex-ante portfolio optimisation. Section 4.6 summarises and concludes.

4.2 A review of the existing literature on the use of yield curve models for fixed interest portfolio management

The existing literature on using yield curve models as a basis for fixed interest portfolio management essentially falls into two distinct categories, which are discussed in turn below, i.e: (1) using analytical yield-curve-based frameworks to measure and/or immunise interest rate risk and attribute returns; and (2) using yield or price residuals from yield curve estimation to identify "relative value" (i.e potential excess returns) from the universe of securities that define the yield curve.

The measurement and immunisation of interest rate risk in fixed interest portfolios has been an active and ongoing area of theoretical and empirical research for many decades. One stream of this literature is the development of "duration" and "convexity" measures, i.e analytical first-order and second-order approximations of the change in portfolio market-value for a given yield curve change. For example, Macauley (1938) and Fisher and Weill (1971) developed the traditional duration measures for parallel changes to the yield curve, while Elton and Gruber (1995) pp. 540-541, and Hull (2000) pp. 112-113 note the convexity measures for parallel yield curve changes. More recently, duration measures have been developed for non-parallel changes to the yield curve, e.g see Chambers, Carleton and McEnally (1988), Reitano (1990), Reitano (1991), Reitano (1992), Reitano (1996), and Mann and Ramanlal (1997). Other analytical approaches to the measurement and immunisation of interest rate risk are the generalised M-vector approach of Nawalkha, Soto and Zhang (2003), "gap" management (e.g see Hull (2000) pp. 113-114), key rate durations (Ho 1992), and generic value-at-risk analysis (e.g see Golub and Tilman (2000) chapter 5). However, these are not explicitly based on an analytical model of the entire yield curve and/or its potential movements.

Duration measures have also been extended to multiple dimensions. For example, Willner (1996) and Diebold, Ji and Li (2005) use NS models of the yield curve to define duration measures with three components. These measures simultaneously represent the risks associated with three potential ways that the yield curve may change within the NS model, i.e a level/shift/parallel change, a slope/twist/curve change, and a bow/barbell/butterfly/curvature change, to use some of the intuitive names familiar to fixed interest portfolio managers. A conceptually similar approach is based on principal components analysis, which uses historical data to empirically estimate the three dominant ways that the yields of different maturities along the yield curve may change relative to each other; e.g see Litterman and Sheinkman (1991), Barber and Copper (1996), Hull (2000) pp. 357-361, and Kopprasch (2004).

The concept of estimating or "fitting" the yield curve with smooth analytical functions and using the resulting yield or price residuals (i.e actual less estimated yield or price) as indications of relative value is used widely by financial market participants, e.g see Brown and Giurda (2003), HSBC Bank (2003) and Malik, Barry and Xiao (2003). Several financial market participants use NS models to identify over-valued and under-valued bonds in a wide range of sovereign bond markets, e.g see Kacala (1993) and HSBC Bank (2001). In the literature, Sercu and Wu (1997) applies the Vasicek (1977), Cox et al. (1985*b*), and polynomial spline models of the yield curve to Belgian government bond data, and finds a significant relationship between the resulting bond price residuals and future excess returns. Ioannides (2003) applies the Sercu and Wu (1997) approach to the UK government bond market models, and obtains the best excess returns from an out-of-sample trading strategy based on the price residuals from NS models. In related work not explicitly based on yield curve modelling, Ronn (1987) exploits the "mis-pricing" of fixed interest securities to significantly enhance returns on portfolios of US Treasury securities while meeting future cashflow obligations, and Cornell and Shapiro (1989) provides a case study of an apparent pricing anomaly in the US Treasury market.

The ANS portfolio framework developed in this chapter makes several extensions relative to the existing literature: (1) the multi-dimensional analytical risk measures from the ANS model are developed to second-order (whereas previous work has focussed on first-order dynamics); (2) the ANS portfolio framework explicitly accounts for expected returns due to the passage of time and unexpected returns due to unanticipated yield curve shifts, rather than using the typical assumption of instantaneous yield curve shifts; and (3) using the ANS model as a basis for both expected and unexpected returns allows those elements to be formally combined into a fixed interest portfolio optimisation framework. A fourth extension that is implicit in the development and application of the ANS portfolio framework is that the potential yield curve shifts have an explicit macroeconomic interpretation via the ANS economic framework from chapter 2. This aspect is not discussed further in this chapter, but is revisited in section 6.2.2 of the concluding chapter.

4.3 The ANS model, relative value, and yield curve shifts

This section outlines the intuition of elements of the ANS model that are later used to formally develop the risk and return framework in section 4.4. Section 4.3.1 introduces a convenient vector notation for the ANS model, section 4.3.2 shows how the ANS model may be used to measure relative value at a given point in time, and section 4.3.3 discusses how the ANS model may be used to represent yield curve shifts across time.

4.3.1 Vector notation for the ANS model of the interest rate curve

As detailed in chapter 2, the ANS model represents the shape of the interest rate curve using three time-varying coefficients that are applied to three time-invariant modes. For the ANS portfolio framework developed in this chapter, it is convenient to use a vector notation for the interest rate curve analogous to that of the forward rate curve introduced in section 2.4.1. That is, the interest rate $R(t, m)$ at time t and as a function of time to maturity m in the ANS model may be expressed as:

$$R(t,m) = [\boldsymbol{\beta}(t)]' \mathbf{s}(\phi, m) + \frac{1}{2}\sigma_1\theta_1 m - \mathbf{v}'\mathbf{u}(\phi, m) \tag{4.1}$$

where $\boldsymbol{\beta}(t)$ is the 3-vector of the coefficients $\beta_n(t)$ noted in section 2.4.1; $\mathbf{s}(\phi, m) = \{s_1(\phi, m), s_2(\phi, m), s_3(\phi, m)\}'$ is a time-invariant 3-vector function of time to maturity m containing the three interest rate modes $s_n(\phi, m)$; $\mathbf{v} = \left\{\sigma_1^2, \sigma_2^2, \sigma_3^2\right\}'$ is a constant 3-vector of variance coefficients σ_n^2; and $\mathbf{u}(\phi, m)$ is a time-invariant 3-vector function of time to maturity m. Another advantage of using the vector notation is that it incorporates arbitrary extensions of the ANS model to more than three modes, as detailed in appendix A.

4.3.2 Representing relative value in the ANS model

Section 2.3.4 shows that the estimation of the ANS model from the securities used to represent the yield curve at any point in time results in the differences between the actual yields (or prices) of the securities used to represent the yield curve. Anticipating the empirical application and detailed discussion of the data in section 4.5, figure 4.1 illustrates the application of the ANS model to US interest rate swap data. That is, applying the ANS model to an observation of the US swaps yield curve (i.e 16 market-quoted mid-

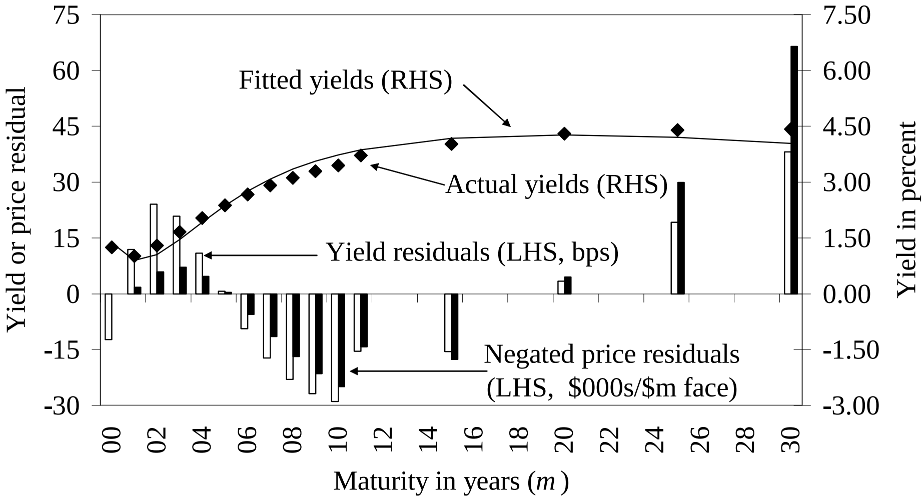

Figure 4.1: The actual and estimated US swaps curve on Monday 16 June 2003, and the associated yield residuals and negated price residuals. The estimated coefficients and parameters are $\boldsymbol{\beta}\,(\text{16-Jun-2003}) = (6.16, 9.04, -4.27)\,\%$, $\phi = 0.62$, $\theta_1 = 0.88\%$, and $\mathbf{v} = \left(1.03^2, 1.65^2, 1.59^2\right)\%^2$.

yields for securities with maturities ranging from overnight to 30-years, all observed at the close-of-market on Monday 16 June 2003) results in the estimated ANS coefficients $\boldsymbol{\beta}\,(\text{16-Jun-03}) = (6.16, 9.04, -4.27)\,\%$. This coefficient vector in tandem with the other ANS model parameters noted in figure 4.1 defines the underlying zero-coupon yield curve that prevailed on that day, which may then be used to reconstruct the fitted market prices and market yields using the cashflows of each security. Those fitted price and yields do not correspond perfectly to the market-quoted prices and yields of the securities that compose the yield curve, and so the estimation also produces 16 price and yield residuals. Table 4.1 in section 4.4.1 contains a detailed numerical example of the calculation of the fitted price, the price residual, and the yield residual for the 2-year swap.

The differences of the prices or yields of fixed interest securities quoted in the market relative to the estimates of the prices or yields from the ANS model may be used as quantitative estimates of "relative value" for each security. That is, actual yields above the ANS-estimated yields offer the prospect for higher running yields and/or capital gains if the yields revert back to "fair value" (i.e the yields derived using the ANS model), compared to alternative securities with actual yields equal to the ANS estimated yields. Similarly, actual yields below the ANS-estimated yields offer the prospect for lower running yields and/or capital losses if the yields revert back to fair value.

In general, section 2.3.4 shows that the ANS estimation of a yield curve defined by K securities at time t will generate K relationships $P_k(t) = P_k[\boldsymbol{\beta}(t)] + \varepsilon_k(t)$, where $P_k(t)$ is the market price (or market value, MV) of security k, $P_k[\boldsymbol{\beta}(t)]$ is the fitted price of security k (determined by the cashflows of security k discounted using the yield curve defined by the ANS model), and $\varepsilon_k(t)$ is the price residual of security k. The price residuals may be equivalently expressed as yield residuals, i.e $\eta_k(t) = -\varepsilon_k(t) / \mathrm{BPV}_k(t)$, where $\mathrm{BPV}_k(t)$ is the "basis point value" (i.e the change in the security price for a single bp change in the yield) of security k at the time the yield curve is estimated.[1] Changes to $\eta_k(t)$ or $\varepsilon_k(t)$ therefore represent a potential source of return (and marginal risk) to the portfolio, as will be detailed in section 4.4.2.

4.3.3 Representing yield curve shifts in the ANS model

To illustrate the intuition behind using the ANS model to represent yield curve changes, figure 4.2 illustrates how the shape of the yield curve may be represented by the 3-vector $\boldsymbol{\beta}(t) = (5.00, 2.00, -1.00)\,\%$, comprised of the Level, Slope, and Bow coefficients at time t, applied to the modes in figure 2.2. Figure 4.2 also shows how an instantaneous increase of 50 basis points (bps, where 1 bp = 0.01 percentage points) in the Level coefficient represents a parallel upward shift of the yield curve (i.e the interest rates of all maturities rise by 50 bps), and an instantaneous 75 bps increase in the Slope coefficient represents a "steepening" of the yield curve (i.e the short rate moves down by 75 bps, infinite-maturity rates remain unchanged, and intermediate-maturity rates move down in proportion to the

[1] BPV is sometimes called dV01, or PV01 in the jargon of fixed interest portfolio management.

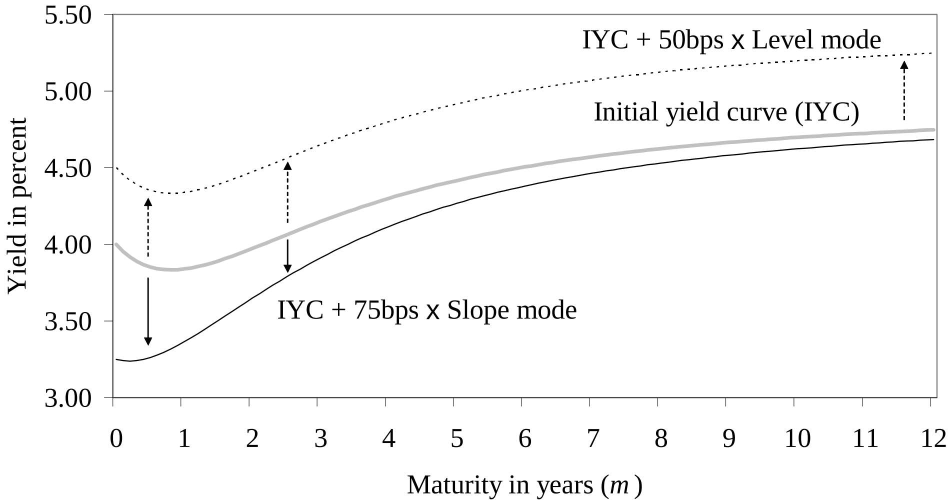

Figure 4.2: Example of an initial yield curve (IYC), and changes to the Level and Slope coefficients. The IYC is $\boldsymbol{\beta}(t) = (5.00, 2.00, -1.00)\,\%$, IYC + 50bps $\times$ Level mode is $\boldsymbol{\beta}(t) = (5.50, 2.00, -1.00)\,\%$, and IYC + 75bps $\times$ Slope mode is $\boldsymbol{\beta}(t) = (5.00, 2.75, -1.00)\,\%$. For this illustration, $\phi = 1$ and all other parameters have been set to zero.

magnitude of the Slope mode by maturity).

Figure 4.3 shows how an instantaneous 75 bp increase in the Bow coefficient represents an "up-bowing" of the yield curve (i.e the short rate moves down by 75 bps, infinite-maturity rates remain unchanged, and intermediate-maturity rates move up or down in proportion to the sign and magnitude of the Bow mode by maturity). Figure 4.3 also contains an example of a simultaneous instantaneous change to the Level, Slope and Bow coefficients, represented by the 3-vector $\boldsymbol{\delta}(t) = (+50, -75, +75)$ bps, which results in a new yield curve shape represented by the 3-vector $\boldsymbol{\beta}(t) = (5.50, 1.25, -0.25)\,\%$. Hence, any actual or potential changes to the shape of the yield curve can be represented by changes to the three ANS model coefficients. As an aside, note that allowing for potential changes in three coefficients is consistent with the suggestion, originally in Litterman and Sheinkman (1991), that three principal components may be used to adequately capture interest rate risks.

In practice, changes to the coefficient vectors $\boldsymbol{\beta}(t)$ will be measured over finite

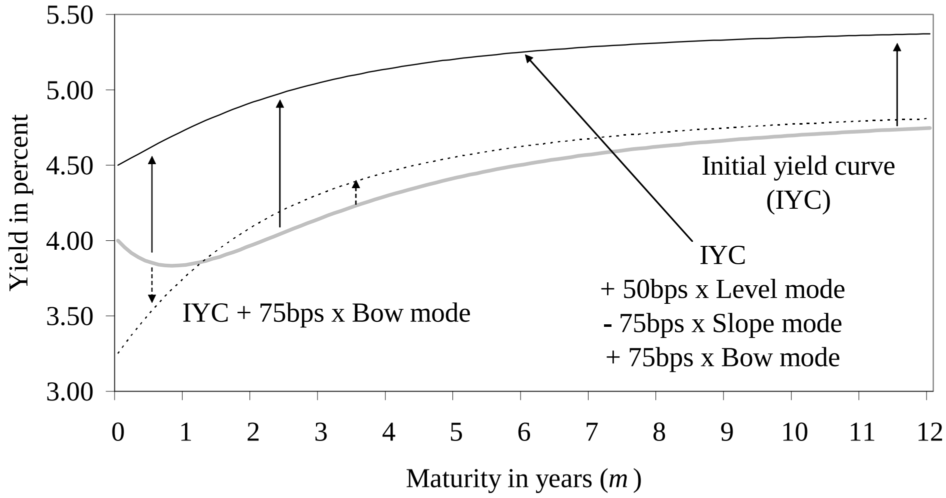

Figure 4.3: Example of an initial yield curve (IYC), a change to the Bow coefficient, and a simultaneous change to all coefficients. The IYC is $\boldsymbol{\beta}(t) = (5.00, 2.00, -1.00)\,\%$, IYC + 75bps $\times$ Bow mode is $\boldsymbol{\beta}(t) = (5.00, 2.00, -0.25)\,\%$, and IYC + the combined Level, Slope, and Bow mode shifts is $\boldsymbol{\beta}(t) = (5.50, 1.25, -0.25)\,\%$. For this illustration, $\phi = 1$ and all other parameters have been set to zero.

periods of time, rather than instantaneously as assumed in the illustrations above. Again anticipating the empirical application in section 4.5, figure 4.4 plots the time series of three of the 16 yields used to define the yield curve at each point in time, and figures 4.5 and 4.6 summarise the corresponding output from the ANS model; i.e respectively, the time series of Level, Slope, and Bow coefficients, and the time series of yield residuals for three of the 16 swaps data series.

Using the finite increment of time τ introduced in section 2.4, the changes to the shape of the yield curve between any two points in time will be $\boldsymbol{\beta}(t+\tau) - \boldsymbol{\beta}(t)$. However, $\boldsymbol{\beta}(t+\tau) - \boldsymbol{\beta}(t)$ will contain both a deterministic (anticipated) component, and a stochastic (unanticipated) component $\boldsymbol{\delta}(t+\tau)$. The stochastic component can be isolated using the result from section 2.4 that $E_t[\boldsymbol{\beta}(t+\tau)] = \boldsymbol{\mu}(\phi,\tau) + \boldsymbol{\Phi}(\phi,\tau)\,\boldsymbol{\beta}(t)$, and then substituting $E_t[\boldsymbol{\beta}(t+\tau)]$ into the definition $\boldsymbol{\delta}(t+\tau) = \boldsymbol{\beta}(t+\tau) - E_t[\boldsymbol{\beta}(t+\tau)]$, giving:

$$\boldsymbol{\delta}(t+\tau) = \boldsymbol{\beta}(t+\tau) - \boldsymbol{\Phi}(\phi,\tau)\,\boldsymbol{\beta}(t) - \boldsymbol{\mu}(\phi,\tau) \tag{4.2}$$

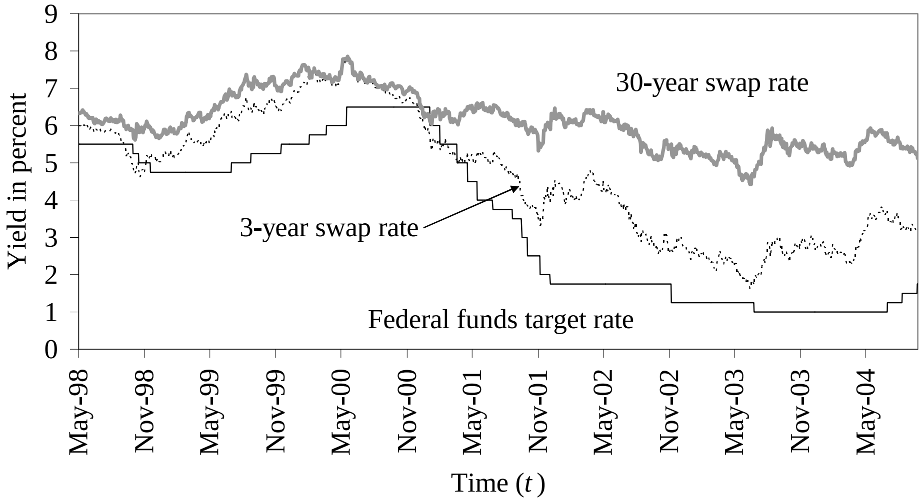

Figure 4.4: The time series of three of the 16 rates used to define the US swaps yield curve over the sample period.

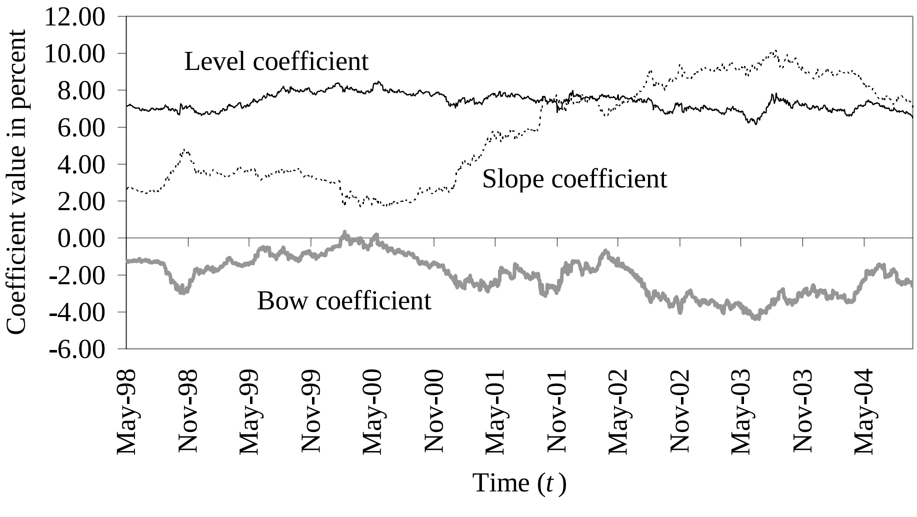

Figure 4.5: The time series of the estimated Level, Slope, and Bow coefficients over the full sample. The estimated parameters are $\phi = 0.62$, $\theta_1 = 0.88\%$, $\sigma_1 = 1.03\%$, and $\mathbf{v} = \left(1.03^2, 1.65^2, 1.59^2\right) \%^2$.

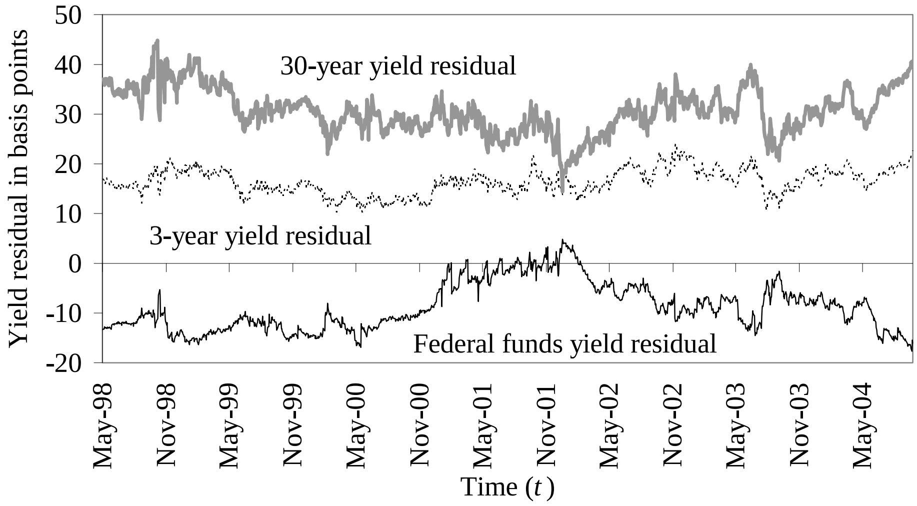

Figure 4.6: The time series of estimated yield residuals for three of the 16 rates used to define the yield curve. The yield residuals are the actual yields in figure 4.4 less the ANS model fitted yields using the estimated ANS coefficients plotted in figure 4.5, and the ANS parameters mentioned in the note to figure 4.5.

As will be detailed in section 4.4.1, $\boldsymbol{\delta}\,(t+\tau)$ delivers unanticipated returns to the portfolio and so represents a source of portfolio risk. Conversely, section 4.4.2 shows that the deterministic component of $\boldsymbol{\beta}\,(t+\tau)-\boldsymbol{\beta}\,(t)$ is a source of interest accrual to the portfolio, i.e an expected return, or "running yield" in the jargon of fixed interest portfolio management.

4.4 The derivation of a framework for fixed interest portfolio risk, relative value, and optimisation

This section formally develops a framework for portfolio risk, relative value, and optimisation based on the ANS model. For clarity and economy of notation, the explicit time notation for $\boldsymbol{\beta}\,(t)$ and $\boldsymbol{\delta}\,(t+\tau)$, and the functional dependence of $\mathbf{s}(\phi, m)$ and $\mathbf{u}(\phi, m)$ on ϕ and m are omitted in this chapter from this point onward. Also, because only $\boldsymbol{\beta}$ is time-varying in the framework developed in this chapter, equation 4.1 may be further abbreviated for convenience to $R(t,m) = \boldsymbol{\beta}'\mathbf{s} + Q$, where $Q = Q\,(m) = \frac{1}{2}\sigma_1\theta_1 m - \mathbf{v}'\mathbf{u}\,(\phi, m)$.[2]

The section proceeds as follows: section 4.4.1 discusses interest rate risk for fixed interest portfolios, section 4.4.2 discusses expected returns for fixed interest portfolios, and section 4.4.3 combines those risk/return elements together to create a framework for fixed interest portfolio optimisation.

4.4.1 Present value and yield curve exposures within the ANS model

The development of a framework for fixed interest portfolio risk, or yield curve exposure (YCE), proceeds in three successive parts: (1) deriving the present value (PV) and YCEs for an individual unit cashflow; (2) representing the PV and YCEs of securities that have multiple cashflows; and (3) representing the PV and YCEs of practical portfolios that contain multiple securities.

The PV and YCEs of a unit cashflow

The PV of single unit cashflow is, by definition, $p\,(m) = \exp\left[-R(m)\cdot m\right]$. Hence, for a given initial value of $\boldsymbol{\beta}$ and Q, the PV according to the ANS model (hereafter abbreviated to PV) may be expressed as $p(\boldsymbol{\beta}, m) = \exp\left[-\left(\boldsymbol{\beta}'\mathbf{s} + Q\right)\cdot m\right]$. After a stochastic disturbance $\boldsymbol{\delta}$ over a time horizon τ, the PV of the unit cashflow will now be $p(\boldsymbol{\beta}+\boldsymbol{\delta}, m-\tau) = \exp\left[-\left(\left[\boldsymbol{\beta}+\boldsymbol{\delta}\right]'\mathbf{s} + Q\right)\cdot(m-\tau)\right]$. This relationship is non-linear, and so the changes in τ and the components of $\boldsymbol{\delta}$ will result in non-proportional changes to the PV. However, the attributions of the change in PV due to τ and the components of $\boldsymbol{\delta}$ may be approximated to the desired degree using a Taylor expansion. Using the notation of Greene (1997), the

[2] The risks from unanticipated changes to the volatility coefficients are not considered in this chapter, although it would be important in a portfolio that contained material interest rate optionality (e.g options on interest rates, or mortgage-backed securities). The complete treatment of the effect of changing volatility would require option valuation within the ANS model framework, which is beyond the scope of this chapter, but is discussed in section 6.2.2 of the concluding chapter. However, as a first-order approximation, an option on a fixed interest security may be included in the framework developed in this chapter by “delta-weighting” the cashflows of the underlying security (i.e applying the probability of option excercise to each cashflow of the security), or equivalently delta-weighting the yield curve exposures of the underlying security.

second-order Taylor expansion of $p(\boldsymbol{\beta}+\boldsymbol{\delta}, m-\tau) = \exp\left[-\left([\boldsymbol{\beta}+\boldsymbol{\delta}]'\,\mathbf{s}+Q\right)\cdot(m-\tau)\right]$ around the column 4-vector $[\beta_1, \beta_2, \beta_3, m]'$ is defined as:

$$\begin{aligned} p(\boldsymbol{\beta}+\boldsymbol{\delta}, m-\tau) &\simeq p(\boldsymbol{\beta}, m) + \left[\frac{\partial p(\boldsymbol{\beta}, m)}{\partial \boldsymbol{\beta}}', \frac{\partial p(\boldsymbol{\beta}, m)}{\partial m}\right] \begin{bmatrix} \boldsymbol{\delta} \\ -\tau \end{bmatrix} \\ &\quad + \frac{1}{2}\left[\boldsymbol{\delta}', -\tau\right] \begin{bmatrix} \frac{\partial^2 p(\boldsymbol{\beta},m)}{\partial\boldsymbol{\beta}\partial\boldsymbol{\beta}'} & \frac{\partial^2 p(\boldsymbol{\beta},m)}{\partial m\partial\boldsymbol{\beta}} \\ \frac{\partial^2 p(\boldsymbol{\beta},m)}{\partial m\partial\boldsymbol{\beta}}' & \frac{\partial p(\boldsymbol{\beta},m)}{\partial m^2} \end{bmatrix} \begin{bmatrix} \boldsymbol{\delta} \\ -\tau \end{bmatrix} \end{aligned} \tag{4.3}$$

where, for notational convenience, $\left[\boldsymbol{\delta}', -\tau\right]$ is the row 4-vector $[\delta_1, \delta_2, \delta_3, -\tau]$ partitioned as the row 3-vector $\boldsymbol{\delta}'$ and the scalar τ,[3] and the first-order and second-order components in equation 4.3 have been partitioned in accordance with this notation. Expanding equation 4.3 using the given partitioned components gives:

$$\begin{aligned} p(\boldsymbol{\beta}+\boldsymbol{\delta}, m-\tau) &\simeq p(\boldsymbol{\beta}, m) + \left[\frac{\partial p(\boldsymbol{\beta}, m)}{\partial \boldsymbol{\beta}}\right]' \boldsymbol{\delta} + \frac{1}{2}\boldsymbol{\delta}' \left[\frac{\partial^2 p(\boldsymbol{\beta}, m)}{\partial\boldsymbol{\beta}\partial\boldsymbol{\beta}'}\right]\boldsymbol{\delta} \\ &\quad - \frac{\partial p(\boldsymbol{\beta}, m)}{\partial m}\cdot\tau + \frac{1}{2}\frac{\partial p(\boldsymbol{\beta}, m)}{\partial m^2}\cdot\tau^2 - \tau\cdot\left[\frac{\partial^2 p(\boldsymbol{\beta}, m)}{\partial m\partial\boldsymbol{\beta}}\right]\boldsymbol{\delta} \end{aligned} \tag{4.4}$$

where the first line of equation 4.4 contains the capital value terms, and the second line contains the interest accrual terms.

The partial derivatives in the first line of equation 4.4 may be calculated directly, i.e:

$$\frac{\partial p(\boldsymbol{\beta}, m)}{\partial\boldsymbol{\beta}} = \frac{\partial \exp\left[-\left(\boldsymbol{\beta}'\mathbf{s}+Q\right)\cdot m\right]}{\partial\boldsymbol{\beta}} \tag{4.5a}$$

$$= \frac{\partial \exp\left[-\left(\boldsymbol{\beta}'\mathbf{s}+Q\right)\cdot m\right]}{\partial\left(\boldsymbol{\beta}'\mathbf{s}+Q\right)}\frac{\partial\left(\boldsymbol{\beta}'\mathbf{s}+Q\right)}{\partial\boldsymbol{\beta}} \tag{4.5b}$$

$$= -m\cdot\exp\left[-\left(\boldsymbol{\beta}'\mathbf{s}+Q\right)\cdot m\right]\mathbf{s} \tag{4.5c}$$

$$= -m\cdot p(\boldsymbol{\beta}, m)\,\mathbf{s} \tag{4.5d}$$

where the second line applies the chain rule of differentiation (in a scalar sense, because $\boldsymbol{\beta}'\mathbf{s}+Q = R(m)$, which is a scalar function of m), and the third line makes the substitution

[3] In full, $[\delta_1, \delta_2, \delta_3, -\tau] = [\beta_1+\delta_1, \beta_2+\delta_2, \beta_3+\delta_3, m-\tau] - [\beta_1, \beta_2, \beta_3, m]$, or $[\boldsymbol{\delta}, -\tau] = [\boldsymbol{\beta}+\boldsymbol{\delta}, m-\tau] - [\boldsymbol{\beta}, m]$.

$\boldsymbol{\beta}'\mathbf{s} + \mathbf{Q} = \mathbf{s}'\boldsymbol{\beta} + Q$ (because both expressions are the scalar function $R(m)$) and applies the result from Greene (1997) p. 51 that $\frac{\partial[\mathbf{s}'\beta]}{\partial\beta} = \mathbf{s}$. Using similar techniques, the second partial derivative may be calculated using the result from equation 4.5, i.e:

$$
\begin{aligned}
\frac{\partial^2 p(\boldsymbol{\beta}, m)}{\partial\boldsymbol{\beta}\partial\boldsymbol{\beta}'} &= \frac{\partial}{\partial\boldsymbol{\beta}}\left[\frac{\partial p(\boldsymbol{\beta}, m)}{\partial\boldsymbol{\beta}'}\right] = \frac{\partial}{\partial\boldsymbol{\beta}}\left[\frac{\partial p(\boldsymbol{\beta}, m)}{\partial\boldsymbol{\beta}}\right]' && \text{(4.6a)}\\
&= \frac{\partial\left\{-m \cdot \exp\left[-\left(\boldsymbol{\beta}'\mathbf{s} + Q\right) \cdot m\right]\mathbf{s}'\right\}}{\partial\boldsymbol{\beta}} && \text{(4.6b)}\\
&= -m \cdot \frac{\partial \exp\left[-\left(\boldsymbol{\beta}'\mathbf{s} + Q\right) \cdot m\right]}{\partial\left(\boldsymbol{\beta}'\mathbf{s} + Q\right)} \frac{\partial\left(\boldsymbol{\beta}'\mathbf{s} + Q\right)}{\partial\boldsymbol{\beta}}\mathbf{s}' && \text{(4.6c)}\\
&= -m \cdot \left\{-m \cdot \exp\left[-\left(\boldsymbol{\beta}'\mathbf{s} + Q\right) \cdot m\right] \cdot \mathbf{s}\mathbf{s}'\right\} && \text{(4.6d)}\\
&= m^2 \cdot \exp\left[-\left(\boldsymbol{\beta}'\mathbf{s} + Q\right) \cdot m\right]\mathbf{s}\mathbf{s}' && \text{(4.6e)}\\
&= m^2 \cdot p(\boldsymbol{\beta}, m)\,\mathbf{s}\mathbf{s}' && \text{(4.6f)}
\end{aligned}
$$

In summary then, the second-order Taylor expansion of $p(\boldsymbol{\beta} + \boldsymbol{\delta}, m - \tau)$ excluding interest accrual terms is:

$$
p(\boldsymbol{\beta} + \boldsymbol{\delta}, m - \tau) \simeq p(\boldsymbol{\beta}, m) - m \cdot p(\boldsymbol{\beta}, m)\,\mathbf{s}'\boldsymbol{\delta} + \boldsymbol{\delta}'\left[\frac{1}{2}m^2 \cdot p(\boldsymbol{\beta}, m)\,\mathbf{s}\mathbf{s}'\right]\boldsymbol{\delta} \qquad (4.7)
$$

where $p(\boldsymbol{\beta}, m)\,\mathbf{s}$ is the first-order yield curve exposure (FOYCE), a column 3-vector; and $m^2 \cdot p(\boldsymbol{\beta}, m)\,\mathbf{s}\mathbf{s}'$ is the second-order yield curve exposure (SOYCE), a 3×3 symmetric matrix. The interpretation of equation 4.7 may be clarified with a simple example: i.e assume an instantaneous parallel shift in the yield curve by Δy. In this case, $\boldsymbol{\delta} = (\Delta y, 0, 0)\,\%$, equation 4.7 becomes $p(\boldsymbol{\beta} + \boldsymbol{\delta}, m) \simeq p(\boldsymbol{\beta}, m) - m \cdot p(\boldsymbol{\beta}, m) \cdot \Delta y + \frac{1}{2}m^2 \cdot p(\boldsymbol{\beta}, m) \cdot \Delta y^2$, and rearranging gives $\frac{\Delta p}{p(\boldsymbol{\beta}, m)} \simeq -m \cdot \Delta y + \frac{1}{2}m^2 \cdot \Delta y^2$, where $\Delta p = p(\boldsymbol{\beta} + \boldsymbol{\delta}, m) - p(\boldsymbol{\beta}, m)$. This is the familiar second-order approximation of the relative price sensitivity of a unit cashflow to a level shift in the yield curve (e.g see Hull (2000) pp. 108-114, and note that a single cashflow has duration m and convexity $\frac{1}{2}m^2$).

To show that the elements of the second line of equation 4.4 represent interest accrual terms, the first term may be derived directly (which is simplified by writing $R(m)$

as the equivalent scalar function of m), i.e:

$$-\frac{\partial p(\boldsymbol{\beta},m)}{\partial m} = -\frac{\partial \exp[-R(m)\cdot m]}{\partial m} \tag{4.8a}$$

$$= -\frac{\partial \exp[-R(m)\cdot m]}{\partial [R(m)\cdot m]}\frac{\partial [R(m)\cdot m]}{\partial m} \tag{4.8b}$$

$$= \exp[-R(m)\cdot m]\cdot f(m) \tag{4.8c}$$

$$= p(\boldsymbol{\beta},m)\cdot f(m) \tag{4.8d}$$

where equation 4.8c uses the result that $\frac{d[R(m)\cdot m]}{dm} = f(m)$, where $f(m)$ is the forward rate as a function of time to maturity m.[4] $p(\boldsymbol{\beta},m)\cdot f(m)\cdot\tau$ therefore represents the interest earned on the PV of the unit cashflow over the horizon τ.

The calculations for the remaining second-order terms of equation 4.4 are omitted for brevity, but in summary $\frac{1}{2}\frac{\partial p(\beta,m)}{\partial m^2}\cdot\tau^2 = p(\boldsymbol{\beta},m)\left\{[f(m)]^2 + \frac{\partial f(m)}{\partial m}\right\}\cdot\frac{1}{2}\tau^2$, which represents "interest on interest" over the horizon τ, and $-\tau\cdot\left[\frac{\partial^2 p(\beta,m)}{\partial m\partial\beta}\right]'\boldsymbol{\delta} = -\tau\cdot p(\boldsymbol{\beta},m)[m\cdot f(m)\,\mathbf{s}+m\mathbf{g}+\mathbf{s}]'\boldsymbol{\delta}$, which represents "interest on changes in PV" over the time-step τ, and $\mathbf{g}(\phi,m) = \frac{\partial[\mathbf{s}(\phi,m)\cdot m]}{\partial m}$.

The PV and YCEs of a fixed interest security

Following the notation of section 2.3.4, a unit face-value of fixed interest security k may be defined as a collection of $J[k]$ cashflows, each of amount a_{kj} occurring at time m_{kj}. The PV of security k will therefore initially be $P_k(\boldsymbol{\beta}) = \sum_{j=1}^{J[k]} a_{kj}\cdot p(\boldsymbol{\beta}+\boldsymbol{\delta},m_{kj})$. Excluding interest accrual terms, the PV to a second-order approximation following a stochastic disturbance $\boldsymbol{\delta}$ is:

$$P_k(\boldsymbol{\beta}+\boldsymbol{\delta},m-\tau) \simeq P_k(\boldsymbol{\beta}) - \boldsymbol{\lambda}_k'\boldsymbol{\delta} + \boldsymbol{\delta}'\boldsymbol{\Omega}_k\,\boldsymbol{\delta} \tag{4.9}$$

where $\boldsymbol{\lambda}_k = \sum_{j=1}^{J} -a_{kj}m_{kj}\cdot p(\boldsymbol{\beta},m_{kj})\,\mathbf{s}$, which represents the FOYCE of security k; and

[4]This result follows in turn from the definition $R(m) = \frac{1}{m}\int_0^m f(x)\,dx$, and the second fundamental theorem of integral calculus noted, for example, in Thomas and Finney (1984) p. 286, i.e $\frac{d}{dm}[R(m)\cdot m] = \frac{d}{dm}\int_0^m f(x)\,dx = f(m)$.

$\mathbf{\Omega}_k = \frac{1}{2} \cdot \sum_{j=1}^{J} a_{kj} m_{kj}^2 \cdot p(\boldsymbol{\beta}, m_{kj}) \, \mathbf{ss}'$, which represents the SOYCE of security k.

Table 4.1 contains a detailed numerical example of the calculation of the YCEs (i.e the FOYCE and SOYCE components) for the fixed cashflows of the 2-year swap on 16 June 2003. Note that the FOYCE components are expressed as the dollar sensitivity for a $million face-value per 1 bp change in the associated coefficient, which is analogous to BPV. For example, the PV of $1 million of the 2-year swap in table 4.1 would decrease (increase) by $200.05 for a 1 bp increase (decrease) in the Level coefficient, and the PV would increase (decrease) by $114.55 for a 1 bp increase (decrease) in the Slope coefficient.

The PV and YCEs of a fixed interest portfolio

A fixed interest portfolio may be defined as a collection of K securities, each with face-value A_k. The PV of the portfolio will therefore initially be $\sum_{k=1}^{K} A_k \cdot P_k(\boldsymbol{\beta})$. Excluding interest accrual terms, the PV to a second-order approximation following a stochastic disturbance $\boldsymbol{\delta}$ is:

$$\begin{aligned} \sum_{k=1}^{K} A_k \cdot P_k(\boldsymbol{\beta} + \boldsymbol{\delta}, m - \tau) \quad \simeq \quad & \sum_{k=1}^{K} A_k \cdot P_k(\boldsymbol{\beta}) \\ & - \left[\sum_{k=1}^{K} A_k \boldsymbol{\lambda}_k \right]' \boldsymbol{\delta} + \boldsymbol{\delta}' \left[\sum_{k=1}^{K} A_k \mathbf{\Omega}_k \right] \boldsymbol{\delta} \end{aligned} \tag{4.10}$$

where $\sum_{k=1}^{K} A_k \boldsymbol{\lambda}_k$ represents the FOYCE of the portfolio, and $\sum_{k=1}^{K} A_k \mathbf{\Omega}_k$ represents the SOYCE of the portfolio. Table 4.2 contains a detailed numerical example of how the YCEs of a portfolio of fixed interest securities are derived from the unit YCEs of the constituent securities as at 16 June 2003. Again, the FOYCE components are expressed as the dollar sensitivity for a $million face-value per 1 bp change in the associated coefficient, so the PV of the portfolio in table 4.2 would decrease (increase) by $144,600 for a 1 bp increase (decrease) in the Level coefficient, and the PV would increase (decrease) by $21,053 for a 1 bp increase (decrease) in the Slope coefficient.

Table 4.1: A 2-year interest rate swap within the ANS framework

Cashflow date	Wed. 18-Jun-03	Thu. 18-Dec-03	Fri. 18-Jun-03	Mon. 20-Dec-04	Fri. 20-Jun-05	
Cashflow maturity (m)	0.01	0.51	1.01	1.52	2.01	
Cashflow magnitude	-1	0.0065	0.0065	0.0065	1.0065	
Level mode value at m	1	1	1	1	1	
Slope mode value at m	-0.9983	-0.8587	-0.7444	-0.6496	-0.5724	
Bow mode value at m	-0.9949	-0.6040	-0.3289	-0.1354	-0.0046	
Total volatility adjustment	0.000	0.002	0.007	0.014	0.020	
Total risk adjustment	0.000	0.000	0.002	0.005	0.007	
$R(t,m)$ in percent	1.39	0.98	0.83	0.86	1.00	
Unit PV	0.9999	0.9950	0.9916	0.9870	0.9802	
Cashflow PV	-0.9999	0.0064	0.0064	0.0064	0.9865	0.0058
Unit MV						0
Unit price residual						-0.0058
Unit yield residual						24.1

Unit λ vector	CF1	CF2	CF3	CF4	CF5	Total
$\lambda(1)$	-0.0055	-0.5043	-0.9998	-1.4954	-1.9737	-2.0005
$\lambda(2)$	0.0055	0.4331	0.7443	0.9714	1.1297	1.1455
$\lambda(3)$	0.0055	0.3046	0.3288	0.2025	0.0091	0.0091

Unit Ω matrix elements	CF1	CF2	CF3	CF4	CF5	Total
$\Omega(1,1)$	0.0000	0.1278	0.5040	1.1328	1.9873	2.0115
$\Omega(1,2)$	0.0000	-0.1097	-0.3752	-0.7358	-1.1374	-1.1527
$\Omega(1,3)$	0.0000	-0.0772	-0.1657	-0.1534	-0.0091	-0.0118
$\Omega(2,2)$	0.0000	0.0942	0.2793	0.4780	0.6510	0.6607
$\Omega(2,3)$	0.0000	0.0663	0.1234	0.0996	0.0052	0.0071
$\Omega(3,3)$	0.0000	0.0466	0.0545	0.0208	0.0000	0.0008

λ vector	Values
Level FOYCE	-200.05
Slope FOYCE	114.55
Bow FOYCE	0.91

Ω matrix	Level	Slope	Bow
Level	201.15	-115.27	-1.18
Slope	-115.27	66.07	0.71
Bow	-1.18	0.71	0.08

Note: An example of the fixed cashflows of the 2-year swap (1.295% quote on Monday 16 June 2003), and the calculation of the relative value and YCEs using the 16 June 2003 ANS coefficients and parameters, i.e $\boldsymbol{\beta}(t) = (6.16, 9.04, -4.27)\,\%$, $\phi = 0.62$, $\theta_1 = 0.88\%$, and $\mathbf{v} = \left(1.03^2, 1.65^2, 1.59^2\right)\%^2$.

Table 4.2: A fixed interest portfolio within the ANS framework

Security k	1	2	3	4		Portfolio potential value
Security name	2-year swap	5-year swap	10-year swap	30-year swap		
Face-value (**A**' vector)	70	10	5	65		
Price residual vector ε'	-0.006	0.000	0.025	-0.066	$\varepsilon'\mathbf{A}$	-4.612
Yield residual vector η'	24.1	0.7	-28.9	38.1	$\eta'\mathbf{A}$	4024

Λ matrix	$\Lambda(1)$	$\Lambda(2)$	$\Lambda(3)$	$\Lambda(4)$	Portfolio $\Lambda\mathbf{A}$ vector
Market-value	0	0	0	0	0
$\lambda(1)$	-200	-475	-828	-1872	-144600
$\lambda(2)$	115	150	150	166	21053
$\lambda(3)$	1	-101	-132	-145	-11061
$\mathbf{\Omega}(1,1)$	201	1168	3895	22484	1506681
$\mathbf{\Omega}(1,2)$	-115	-365	-663	-1508	-113071
$\mathbf{\Omega}(1,3)$	-1	252	624	1475	101435
$\mathbf{\Omega}(2,2)$	66	115	119	130	14840
$\mathbf{\Omega}(2,3)$	1	-78	-106	-117	-8891
$\mathbf{\Omega}(3,3)$	0	55	101	113	8435

Note: An example of an arbitrary portfolio composed of 2, 5, 10, and 30-year swaps as at Monday 16 June 2003. The 16 June 2003 ANS coefficients and parameters are $\boldsymbol{\beta}(t) = (6.16, 9.04, -4.27)\,\%$, $\phi = 0.62$, $\theta_1 = 0.88\%$, and $\mathbf{v} = \left(1.03^2, 1.65^2, 1.59^2\right)\%^2$.

Before moving onto the relative value and expected returns within the ANS portfolio framework, it is worth noting that the YCEs can be expressed in proportional terms (analogous to traditional duration and convexity), aggregated to a single value-at-risk (VaR) measure using variance and covariances of the ANS model coefficients, and expressed relative to a benchmark portfolio. Also, given a view of how the yield curve might change, the portfolio manager can construct the portfolio to take active exposures on all or selected components of the yield curve. These aspects are not central to this chapter, and so are relegated to appendix C.

4.4.2 Relative value and expected returns within the ANS model

Section 4.3.1 introduced the decomposition of the price of a fixed interest security into $P_k(t) = P_k[\boldsymbol{\beta}(t)] + \varepsilon_k(t)$ via the ANS model. Table 4.1 contains a detailed numerical example of the calculation of $P_k[\boldsymbol{\beta}(t)]$ and $\varepsilon_k(t)$ for the 2-year swap. Similarly, table 4.2 contains an example of the calculation of the total price residual for a portfolio of interest rate swaps.

The expected return from a fixed interest security over a time horizon τ is, by definition, $E_t[\Delta P_{k,t+\tau}] = E_t[P_{k,t+\tau}(\boldsymbol{\beta}) - P_{k,t}(\boldsymbol{\beta})] + E_t[\Delta\varepsilon_{k,t+\tau}]$, where E_t is the expectations operator applied at time t, $\Delta P_{k,t+\tau} = P_{k,t+\tau} - P_{k,t}$ is the change in the MV, $[P_{k,t+\tau}(\boldsymbol{\beta}) - P_{k,t}(\boldsymbol{\beta})]$ is the change in the PV, and $\Delta\varepsilon_{k,t+\tau} = \varepsilon_{k,t+\tau} - \varepsilon_{k,t}$ is the change in the price residual. The expected return on a portfolio of K securities with face values $A_{k,t}$ is then the summation:

$$\sum_{k=1}^{K} A_{k,t} \cdot E_t[\Delta P_{k,t+\tau}] = \sum_{k=1}^{K} A_{k,t} \cdot E_t[P_{k,t+\tau}(\boldsymbol{\beta}) - P_{k,t}(\boldsymbol{\beta})] + \sum_{k=1}^{K} A_{k,t} \cdot E_t[\Delta\varepsilon_{k,t+\tau}] \quad (4.11)$$

The first right-hand-side summation of equation 4.11 simply represents the interest accrual on the portfolio, i.e the aggregation of expected returns from each security due to the fully-anticipated passage of time.

The second right-hand-side summation represents a potential source of expected return if any $E_t\left[\Delta\varepsilon_{k,t+\tau}\right] \neq 0$. In general, if $E_t\left[\Delta\varepsilon_{k,t+\tau}\right]$ is different for each security, expected portfolio returns will differ according to the weighting of each security held in the portfolio. In other words, a portfolio that is overweight securities with positive $E_t\left[\Delta\varepsilon_{k,t+\tau}\right]$ would offer excess expected returns relative to a portfolio with lower weights of those securities.

Any predictability of $E_t\left[\Delta\varepsilon_{k,t+\tau}\right]$ may be captured in a time-series process for the yield residual $\eta_k(t) = -\varepsilon_k(t)/\mathrm{BPV}_k(t)$. The representation that is most tractable, as adopted in this chapter, is to assume that each $\eta_{k,t+\tau}$ follows an independent and stationary first-order autoregressive process, or AR1, with identical rates of mean-reversion, i.e:

$$\eta_{k,t+\tau} - \pi_k = \theta\left(\eta_{k,t} - \pi_k\right) + \upsilon_{k,t+\tau} \tag{4.12}$$

where π_k is a "mean-adjustment", i.e a constant that allows for any persistent deviations of $\eta_{k,t+\tau}$ away from zero due to security-specific factors external to the ANS model framework (e.g liquidity premia and/or preferred habitats, as noted in Elton and Gruber (1995), pp. 513-518); θ is the AR1 coefficient that is assumed to be $0 < \theta < 1$;[5] and $\upsilon_{k,t+\tau}$ represents unpredictable stochastic noise, which will be distributed $\upsilon_{k,t+\tau} \sim N\left(0, \sigma_\upsilon^2\right)$ for any security k if the ANS model is estimated by minimising squared yield residuals (as in the procedure outlined in section 2.3.4). An advantage of assuming this time-series process is the high degree of parsimony imparted to the optimisation framework derived in section 4.3.3; in particular, it turns out that an estimate of θ is not required.[6]

[5]The mean reversion implied by this assumption is theoretically justified, because financial arbitrage would preclude the yield of a single security diverging arbitrarily from the other securities that define the yield curve. Or in other words, the time series for the yield residual of a given security cannot be a unit root series, as that would imply the possibility that the yield of the of that security could diverge arbitrarily from the yield curve.

[6]In principle, any stationary time-series process could be assumed for the residuals or estimated from the data (e.g a general vector autoregression), and the resulting expected returns would be used in the optimisation framework developed in section 3.3. However, the complexity of estimation might prove prohibitive in practical applications, and it is well known that improving the in-sample fit of a model is often detrimental to predictability relative to a parsimonious model (see, for example, the dicussion in Diebold and Li (2006) on the shrinkage principle).

Applying the expectations operator to equation 4.12 gives $E_t\left[\eta_{k,t+\tau}\right] - \pi_k = \theta\left(\eta_{k,t} - \pi_k\right)$, which means that $E_t\left[\Delta\eta_{k,t+\tau}\right] = (\theta - 1)\left(\eta_{k,t} - \pi_k\right)$. Hence, a security with positive $\left(\eta_{k,t} - \pi_k\right)$, i.e the yield residual above the typical yield residual, would be expected to contribute positive returns equal to $-(\theta - 1)\left(\eta_{k,t} - \pi_k\right) \cdot \mathrm{BPV}_k(t)$, over the horizon τ, and contribute risk in the order of $\sigma_v \cdot \mathrm{BPV}_k(t)$. Conversely, a security with negative $\left(\eta_{k,t} - \pi_k\right)$ would be expected to contribute negative returns. For later use, it is convenient to define $\alpha_{k,t} = \eta_{k,t} - \mu_k$ as the "potential yield enhancement" of a unit of security k at time t. This is so-named because the MV of security k could potentially be enhanced by $\alpha_{k,t} \cdot \mathrm{BPV}_k(t)$ before further expected changes to $\Delta\eta_{k,t+\tau}$ become zero.

4.4.3 Fixed interest portfolio optimisation within the ANS model

The ANS portfolio framework incorporates both risk (as YCEs) and expected returns, and this section uses those elements to derive a framework for fixed interest portfolio optimisation. This section proceeds in two parts: (1) introducing a convenient vector/matrix notation for the fixed interest portfolio; and (2) formally deriving the optimisation system.

Vector/matrix notation for fixed interest securities and portfolios

To dynamically combine the risks and returns of individual securities into portfolios, it is convenient to re-express the MV and FOYCE components for each security at each point in time in an alternative vector/matrix notation. Specifically, use the following three steps: (1) for each security, "stack" the MV and the three individual components of the FOYCE vector into a column 4-vector $[P_k, \boldsymbol{\lambda}_{k,1}, \boldsymbol{\lambda}_{k,2}, \boldsymbol{\lambda}_{k,3}]_t'$ denoted as $\boldsymbol{\Lambda}_{k,t}$;[7] (2) collect the vectors $\boldsymbol{\Lambda}_{k,t}$ of each security that may exist in the portfolio into a $4 \times K$ matrix

[7] Using the MV anticipates the typical practical constraint that trading be cash-neutral (so that cash injections or withdrawals are not required). The SOYCEs could also be included if required, in which case the six unique individual elements of the SOYCE matrix $\boldsymbol{\Omega}_{k,t}$, i.e $\boldsymbol{\Omega}_{k,11}, \boldsymbol{\Omega}_{k,12}, \boldsymbol{\Omega}_{k,13}, \boldsymbol{\Omega}_{k,22}, \boldsymbol{\Omega}_{k,23}, \boldsymbol{\Omega}_{k,33}$, would also be stacked into $\boldsymbol{\Lambda}_{k,t}$ to capture the second-order effects.

$[\mathbf{\Lambda}_1, \ldots, \mathbf{\Lambda}_k, , \ldots, \mathbf{\Lambda}_K]_t$, denoted as $\mathbf{\Lambda}_t$; and (3) represent the individual face values of the securities in the portfolio as a column K-vector $[A_{0,1}, \ldots, A_{0,k}, \ldots, A_{0,K}]'_t$, denoted as $\mathbf{A}_{0,t}$. The MV and the FOYCE components for the portfolio will now be summarised by the column 4-vector $\mathbf{\Lambda}_t \mathbf{A}_{0,t}$.

Regarding expected returns, collect the potential yield enhancements $\alpha_{k,t}$ for each security that may exist in the portfolio into a column K-vector $[\alpha_1, \ldots, \alpha_k, \ldots, \alpha_K]'_t$, denoted as $\boldsymbol{\alpha}_t$. Table 4.2 in section 4.4.1 contains a detailed numerical example of $\mathbf{\Lambda}_{k,t}$, $\mathbf{\Lambda}_t$, $\mathbf{A}_{0,t}$, $\mathbf{\Lambda}_t \mathbf{A}_{0,t}$, and $\boldsymbol{\alpha}'_t \mathbf{A}_{0,t}$ assuming no mean-adjustment (i.e if $\pi_k = 0$, then $\boldsymbol{\alpha}_t = \boldsymbol{\eta}_t$) for a portfolio as at 16 June 2003.

The optimisation of portfolios of fixed interest securities

The mean/variance approach of Markowitz (1959), as noted in Elton and Gruber (1995), essentially seeks to maximise expected portfolio returns versus the expected standard deviation of those returns while respecting given constraints on individual securities and the overall portfolio. The approach in this chapter is analogous in that it seeks to maximise the expected returns of the fixed interest portfolio while keeping the expected standard deviation unchanged and respecting practical constraints on the face values of securities allowed in the portfolio.

Specifically, using the notation from section 4.4.3, define a benchmark portfolio by the face value vector $\mathbf{A}_{0,t}$, and then propose an alternative portfolio defined by the face value vector $\mathbf{A}_{1,t}$ that has the same expected standard deviation but the maximum expected return. This optimisation problem may be summarised as the system:

$$
\begin{aligned}
\text{Maximise} \quad &: \quad \sum_{k=1}^{K} A_{1,k,t} \cdot -(\theta - 1) \cdot \alpha_{k,t} \cdot \mathrm{BPV}_k(t) \\
&\quad + \sum_{k=1}^{K} A_{1,k,t} \cdot E_t\left[P_{k,t+\tau}(\boldsymbol{\beta}) - P_{k,t}(\boldsymbol{\beta})\right] && \text{(4.13a)} \\
\text{subject to} \quad &: \quad \sum_{k=1}^{K} A_{1,k,t} \cdot P_{k,t} = \sum_{k=1}^{K} A_{0,k,t} \cdot P_{k,t} && \text{(4.13b)} \\
\text{and} \quad &: \quad \sigma\left[\mathbf{A}_{1,t}\right] = \sigma\left[\mathbf{A}_{0,t}\right] && \text{(4.13c)} \\
\text{and} \quad &: \quad A_{1,k,\min} \leq A_{1,k} \leq A_{1,k,\max} && \text{(4.13d)}
\end{aligned}
$$

where $\sigma(\cdot)$ denotes the standard deviation of portfolio returns using $\mathbf{A}_{0,t}$ or $\mathbf{A}_{1,t}$, and $A_{1,k,\min}$ and $A_{1,k,\max}$ are given minimum and maximum constraints on the face values of $A_{1,k,t}$ that may be held in the portfolio at any point in time (e.g $A_{1,k,\min} = 0$ would prohibit negative face values or “short” positions in any security).

The equations in system 4.13 may be simplified substantially using three reasonable assumptions. These assumptions are collected here for convenience, including a brief justification, and their validity will later be discussed in light of the empirical application.

Assumption 1: The total interest accrual $\sum_{k=1}^{K} A_{1,k,t} \cdot E_t\left[P_{k,t+\tau}(\boldsymbol{\beta}) - P_{k,t}(\boldsymbol{\beta})\right]$ will be approximately constant for all feasible portfolios. This follows from the restriction that the universe of feasible alternative portfolios must all have the same portfolio MV, as specified by the equality constraint in equation 4.13b. Hence, given that the only difference between PV and MV are the relatively small price residuals $\varepsilon_k(t)$, the PV of the feasible portfolios will be almost identical, and so the interest accrual returns from any feasible portfolio should therefore be similar.

Assumption 2: Scaling the potential value of each security in the objective function by $1/\mathrm{BPV}_k(t)$ will leave all feasible portfolios with similar contributions to expected portfolio standard deviation from changes to relative value. This follows from the discussion in section 3.2 that the stochastic component on the yield residual $\upsilon_{k,t+\tau}$ in equation 4.12

is distributed as $N\left(0,\sigma_v^2\right)$ for all securities. Hence, the expected standard deviation on the price residual for a unit of security k will be $\sigma_v \cdot \mathrm{BPV}_k\left(t\right)$, and so scaling by $1/\mathrm{BPV}_k\left(t\right)$ will leave the expected contribution to portfolio standard deviation from a unit of security k at σ_v. Note that this scaling effectively "encourages" the optimisation process to add or subtract smaller amounts of higher BPV securities relative to lower BPV securities when maximising relative value, which therefore avoids excessive change to overall portfolio risk due to security selection.

Assumption 3: Feasible portfolios with identical FOYCE components $\sum_{k=1}^{K} A_k \boldsymbol{\lambda}_k$ will have very similar expected portfolio standard deviations. This is because portfolios with $\sum_{k=1}^{K} A_k \boldsymbol{\lambda}_k$ identical will have $\sigma\left\{\left[\sum_{k=1}^{K} A_k \boldsymbol{\lambda}_k\right]' \boldsymbol{\delta}\right\}$ identical (given $\boldsymbol{\delta}$ is independent of the portfolio structure), and $\sigma\left\{\left[\sum_{k=1}^{K} A_k \boldsymbol{\lambda}_k\right]' \boldsymbol{\delta}\right\}$ is the first-order contribution to the standard deviation of the portfolio (which follows from the results derived in section 3.1).

Assumption 1 means the second line of the objective function equation 4.13a may be eliminated, and then the scalar $-\left(\theta-1\right)$ may be eliminated from the first line (being identical for each security). Assumption 2 then scales the remainder of the objective function by $1/\mathrm{BPV}_k\left(t\right)$. The objective function may now be written as: Maximise: $\sum_{k=1}^{K} A_{1,k,t} \cdot \alpha_{k,t}$, or using the notation of section 3.3.1: Maximise: $\boldsymbol{\alpha}_t' \mathbf{A}_{1,t}$.

Regarding the constraints, using the vector notation from section 3.3.1, the MV and variance constraints of 4.13b and c may be replaced by $\boldsymbol{\Lambda}_t \mathbf{A}_{1,t} = \boldsymbol{\Lambda}_t \mathbf{A}_{0,t}$. That is, if the first component of the 4-vector $\boldsymbol{\Lambda}_t \mathbf{A}_{1,t}$ equals that of $\boldsymbol{\Lambda}_t \mathbf{A}_{0,t}$, then the MVs of the two portfolios will be identical, and if the second to fourth components of $\boldsymbol{\Lambda}_t \mathbf{A}_{1,t}$ equal those of $\boldsymbol{\Lambda}_t \mathbf{A}_{0,t}$, then the FOYCE components will be identical.[8]

[8]And if the six unique SOYCE components were also included, the SOYCE components would be identical if the fifth to tenth components of $\boldsymbol{\Lambda}_t \mathbf{A}_{1,t}$ equalled those of $\boldsymbol{\Lambda}_t \mathbf{A}_{0,t}$.

The system represented by equations 4.13a to d therefore reduces to the system:

$$\text{Maximise:} \quad \boldsymbol{\alpha}_t' \mathbf{A}_{1,t} \tag{4.14a}$$

$$\text{subject to:} \quad \boldsymbol{\Lambda}_t \mathbf{A}_{1,t} = \boldsymbol{\Lambda}_t \mathbf{A}_{0,t} \tag{4.14b}$$

$$\text{and:} \quad A_{1,k,\min} \leq A_{1,k,t} \leq A_{1,k,\max} \tag{4.14c}$$

which is a linear programme. Compared to the alternative approach (in principle) of maximising expected returns versus standard deviations defined via variances and covariances, the advantages of the linear programming approach are twofold: (1) the optimisation may now be undertaken using the simplex algorithm, a standard and straightforward method of optimisation;[9] (2) the optimisation problem has ready intuition; i.e the portfolio with the highest potential value and with MV and FOYCE components identical to the initial/benchmark portfolio will offer the highest expected returns for the same risks.

Note that the optimisation system does not allow for transaction costs, which follows the precedent set in the literature by Sercu and Wu (1997) and Ioannides (2003). Of course, transactions costs would likely be an important consideration in practice, and so section 6.2.2 of the concluding chapter and appendix C.3 contain further discussion on that aspect. Without transactions costs, the empirical application of the optimisation framework in this chapter addresses only the issue of whether the concept of relative value used by financial market participants is a valid source of potential excess returns, rather than whether those potential returns could necessarily be exploited in practice.

[9]See, for example, Murty (1983).

4.5 The empirical application of the ANS portfolio framework

This section applies the ANS portfolio framework empirically. Section 4.5.1 describes the data, section 4.5.2 discusses the results from attributing fixed interest portfolio returns ex-post, and section 4.5.3 discusses the results from optimising fixed interest portfolios ex-ante.

4.5.1 Description of the data

The empirical analysis is undertaken using US fixed-for-floating interest rate swaps data. Swaps data are used rather than US Treasury market data for the following reasons: (1) swaps are a new class of security on which to investigate relative value, while the issue of relative pricing in sovereign bond markets has already been addressed previously in Sercu and Wu (1997), Ioannides (2003), and for the US Treasury market in Ronn (1987) and Cornell and Shapiro (1989); (2) swaps data are quoted for standard maturities making the analysis more straightforward than for sovereign bond markets where the investment universe must be continuously adjusted to allow for maturities and new issuance; and (3) swaps are more standardised and homogeneous than government bonds, so there is less chance of unique market-structure factors influencing the results. The latter applies especially to the US Treasury market, where the relative prices of securities are influenced dynamically and materially by on-the-run/off-the-run effects, issuance/buyback effects, liquidity considerations, differences in tax treatment, and differences in the effective underlying funding rates.[10]

The data are obtained from Datastream, and are the daily closing mid-rates for

[10]See Fleming (2003) for a discussion of these aspects in the context of measuring market liquidity. Note that the effective funding rate for each US Treasury security is its associated repurchase rate, and these often differ markedly between bonds due to bonds going "special" (i.e being tightly held by a few market participants) in the physical market.

the federal funds target rate, and the rates for the 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 12, 15, 20, 25, and 30-year fixed-for-floating interest rates swaps. This gives 16 rates in total, and the sample period is from 1 May 1998 (when data for the 20, 25, and 30-year swaps first became available) to 22 September 2004 (the latest data available at the time of the analysis). Note that the federal funds rate is used to provide a contemporaneous representative short-maturity rate for the swaps yield curve. While a bank-risk short-maturity rate would be more ideal (to be consistent with the swaps rates that are also bank-risk), the London interbank offered (LIBOR) rates that are available are fixed in the London morning, which would not be contemporaneous with the swaps rates at the US market close. Using the federal funds rate may make a minor impact on the outright attribution of interest accrual returns, as discussed in section 4.5.2, but will make no impact on the comparative analysis in section 4.5.3.

While the sample period is relatively short in chronological time, there is no reason to suspect it should not be representative; i.e the data spans 1,599 trading days, it captures a full monetary policy cycle (i.e the 1999 to 2000 sequence of federal funds rate hikes, the 2001 to 2003 sequence of cuts, and the 2004 sequence of hikes to-date), and it captures a full trough-peak-trough cycle in long-maturity rates. The sample also includes the financial market stress events of the Asian/Russian/LTCM crisis, the 11 September 2001 World Trade Centre tragedy, the 1999 30-year Treasury buy-back programme and the subsequent 2001 cessation of issuance, and the deflationary scare of 2003 to 2004. Before beginning the empirical analysis, 24 obvious data anomalies occurring over 11 days of the dataset were corrected,[11] and non-trading days were removed from the dataset. Figure 4.4 in section 4.3.3 has already illustrated the time series of three of the 16 data series used in the empirical analysis.

[11] Specifically, one "big figure error" (i.e an incorrect percentage point for one swap rate) and "stale quotes" indicated by daily changes in yields for individual swap maturities that were 10 to 50 bps inconsistent with the daily changes for swaps rates of similar maturities.

Regarding the precise cashflows for the swaps, Hull (2000) pp. 132-133 notes that a fixed-for-floating rate swap agreement is equivalent to a fixed coupon bond funded by a floating rate note liability. A market-quoted swaps rate defines the coupon of a par fixed coupon bond, and the other parameters are defined by agreed market convention; i.e a US swaps rate $S(t,x)$ quoted at date t for maturity x-calendar-years implies notional settlement of the unit face-value (i.e a cashflow of -1) on date $t+2$-working-days, with the first coupon (i.e a cashflow of $+S(t,x)/2$) on date $t+2+6$-calendar-months, subsequent coupons (i.e cashflows of $+S(t,x)/2$) each 6-calendar-months thereafter, and the final coupon payment and notional return of principal (i.e a cashflow of $1+S(t,x)/2$) at the maturity date of $t+2+x$-calendar-years.[12] Figure 4.1 has already shown an example of the fixed cashflows implied by the 2-year swap rate quoted on Monday 16 June 2003. The floating rate leg of the swap is a par floating rate note with notional drawdown of the unit face-value on date $t+2$-working-days, subsequent payments of interest at three-monthly intervals based on the 3-month LIBOR rate, and the notional payback at the maturity date of $t+2+x$-calendar-years. However, these floating cashflows make no contribution to the valuation and the interest rate risk of the swap agreement implied by the market-quoted rate, and may therefore be ignored for the analysis in this chapter.[13]

4.5.2 The ex-post attribution of fixed interest portfolio returns

The investigation of ex-post portfolio returns is undertaken using a benchmark portfolio constructed as follows: (1) the benchmark portfolio is established as at 1 May 1998 with zero cash, a $10 million face-value for each swap maturity (to give a total market value of zero, because the MV of floating leg of the swap equals the MV of the fixed leg); (2)

[12] All subject to the modified following business day convention, as noted in Hull (2000) p. 128.

[13] The floating leg of the swap will only contribute valuation and interest rate risk once the first floating rate is set, and therefore becomes a known cashflow. In the analysis of this chapter, the swaps are effectively terminated (via the exchange of cash equal to the market value of the swap) before the floating leg becomes effective.

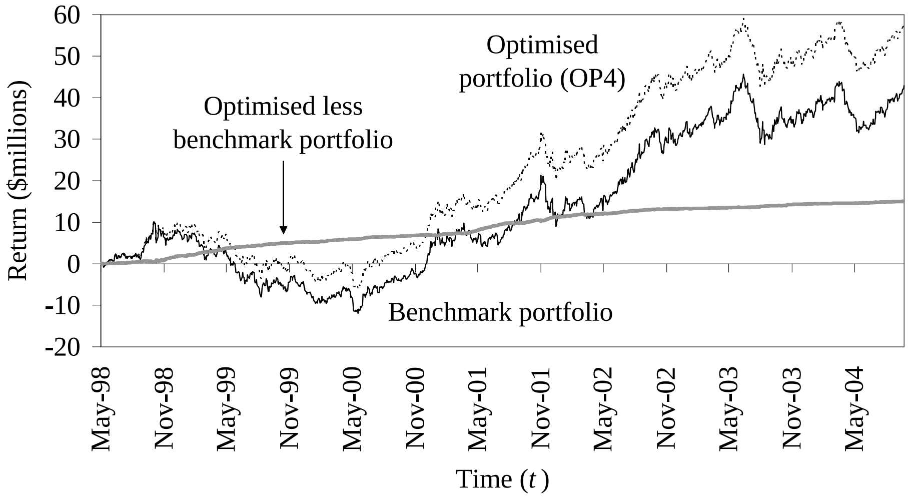

Figure 4.7: Cumulative returns for the benchmark portfolio and the optimised portfolio 4 (OP4). As detailed in section 4.5.3, OP4 uses in-sample estimates of the parameters ϕ, θ_1, $\mathbf{v}$, and in-sample estimates of π_k.

this portfolio is carried over to the following trading day, and the daily return is calculated by revaluing the cashflows of the swaps using the zero-coupon curve "boot-strapped" from the new prevailing yield curve;[14] (3) the face-values in the portfolio are reset to \$10 million; and (4) steps 2 and 3 are repeated for the entire sample. This process gives a time-series of 1,598 independent daily returns for the benchmark portfolio. The cumulative returns for the benchmark portfolio are plotted in figure 4.7.

Attributing ex-post portfolio returns to the YCEs for a given day firstly requires an ex-post estimate of $\boldsymbol{\delta}$ for that day, which is provided by applying equation 4.2 to the time series of $\boldsymbol{\beta}$ illustrated in figure 4.5. Note that an internally-consistent ex-post estimate of $\boldsymbol{\mu}(\phi, \tau)$ for the sample may be estimated as the average of the time series $\boldsymbol{\beta}(t+\tau) - \boldsymbol{\Phi}(\phi, \tau)\boldsymbol{\beta}(t)$ calculated for each day of the sample. This ensures that the average of the realised $\boldsymbol{\delta}$ values will identically equal zero (which is the expected value of $\boldsymbol{\delta}$) over the sample. Secondly, the calculations of the vector $\sum_{k=1}^{K} A_k \boldsymbol{\lambda}_k$ and the matrix $\sum_{k=1}^{K} A_k \boldsymbol{\Omega}_k$

[14] Hull (2000) p. 150 discusses the concepts behind this technique. The analysis in this chapter uses a stepwise-continuous zero-coupon curve based on the linear interpolation of the continuously-compounding interest rates at the maturity of each swap.

for the given day are undertaken using the estimated ANS model for the given day, and the cashflows of each of the securities in the benchmark portfolio on that day. Finally, substituting the values of $\boldsymbol{\delta}$, $\sum_{k=1}^{K} A_k \boldsymbol{\lambda}_k$, and $\sum_{k=1}^{K} A_k \boldsymbol{\Omega}_k$ into equation 4.10 gives the returns for that day that are attributable to the three individual FOYCE components and the six unique SOYCE components. Repeating this over the entire sample gives the time series of attributions to the FOYCE and SOYCE components.

Portfolio returns due to changes in the relative value of the portfolio are calculated directly by comparing the relative value of each security to its relative value on the following day. The final attribution is the interest accrual return, which is estimated as the difference between the actual benchmark portfolio returns less the FOYCE, SOYCE, and relative value returns already attributed above. Note that the variability in the interest accrual returns in the tables that follow mostly reflects the uneven spacing of working days over calendar time; i.e there will be more interest accrual expected over a weekend or holiday than between adjacent weekdays.[15]

The ex-post portfolio attribution results are summarised in table 4.3. This shows that the dispersion of ex-post daily returns (as measured by the standard deviation, minimum, maximum, or the spread between maximum and minimum) are dominated by the FOYCE components. For example, the standard deviation rankings are σ (Level FOYCE) $>$ σ (Slope FOYCE) $>$ σ (Bow FOYCE) $\gg$ σ (Relative value) $>$ σ (Accrual returns) $>$ σ (SOYCEs). Table 4.4 contains the variances and covariances between each of the attribution groups, and it is apparent that the variances and covariances outside of the "FOYCE block" are very small. Specifically, the FOYCE block variance is within 3% of total portfolio variance, and therefore the FOYCE standard deviation would be within 1.5% of the total portfolio standard deviation.

[15] Also, being a "remainder", the interest accrual term will implicitly capture third-order and higher effects ignored in the second-order Taylor approximation of section 4.4, but those should be very small.

Table 4.3: Statistical summary of benchmark portfolio returns

Attribution	Sum	Mean	Std dev.	Min.	Max.	Max. less min.
Actual return	42.852	0.027	0.659	-2.852	3.107	5.959
$\lambda(1)$ FOYCE	18.652	0.012	0.660	-3.629	3.825	7.453
$\lambda(2)$ FOYCE	8.308	0.005	0.213	-0.972	1.354	2.326
$\lambda(3)$ FOYCE	6.301	0.004	0.140	-0.776	0.636	1.412
Total FOYCE	33.261	0.021	0.668	-2.936	3.086	6.021
$\Omega(1,1)$ SOYCE	4.048	0.003	0.006	0.000	0.085	0.085
$\Omega(1,2)$ SOYCE	-0.738	0.000	0.002	-0.036	0.005	0.042
$\Omega(1,3)$ SOYCE	0.203	0.000	0.001	-0.014	0.010	0.025
$\Omega(2,2)$ SOYCE	0.256	0.000	0.000	0.000	0.006	0.006
$\Omega(2,3)$ SOYCE	0.272	0.000	0.000	-0.001	0.005	0.006
$\Omega(3,3)$ SOYCE	0.164	0.000	0.000	0.000	0.003	0.003
Total SOYCE	4.206	0.003	0.005	0.000	0.057	0.057
Relative value	1.516	0.001	0.006	-0.040	0.029	0.069
Interest accrual	3.870	0.002	0.007	-0.032	0.033	0.065

Note: Summary of benchmark portfolio returns ($millions) and ex-post attributions of those returns to the 11 components noted in the text.

Table 4.4: Summary of benchmark portfolio return covariances

	Total FOYCE	Total SOYCE	Relative value	Interest accrual
Total FOYCE	0.4467	-0.0005	-0.0017	-0.0044
Total SOYCE	-0.0005	0.0000	0.0000	0.0000
Relative value	-0.0017	0.0000	0.0000	0.0000
Interest accrual	-0.0044	0.0000	0.0000	0.0000
	Total	0.4336	FOYCE/Total	1.0300

Note: Variances and covariances of the benchmark portfolio attributed returns ("Total FOYCE" and "Total SOYCE" are aggregates of the individual components contained in table 4.3).

These ex-post attribution results offer an important insight into ex-ante portfolio risks; i.e $\boldsymbol{\delta}$ is a random quantity ex-ante, and so $\left[\sum_{k=1}^{K} A_k \boldsymbol{\lambda}_k\right]' \boldsymbol{\delta}$ and $\boldsymbol{\delta}' \left[\sum_{k=1}^{K} A_k \boldsymbol{\Omega}_k\right] \boldsymbol{\delta}$ represent risks to portfolio returns due to unanticipated changes in the Level, Slope, and/or Bow of the yield curve. It is therefore evident that the risks of the portfolio are adequately captured by the FOYCE components. This result accords with assumption 3 in section 4.4.3, and it also suggests that the SOYCE components of the YCEs may be ignored in the practical management of fixed interest portfolios.[16]

Regarding returns, the FOYCE and SOYCE returns simply reflect the aggregation of the changes to the shape of the yield curve that prevailed over the sample period applied to the YCEs of the benchmark portfolios. For example, the attribution to the Level FOYCE component is positive because the portfolio had negative Level FOYCE over a period when the average change to the Level coefficient was negative (i.e the portfolio was "long duration in a falling rate environment" to use the jargon of fixed interest portfolio management). Similarly, the attribution to the Slope FOYCE component is positive because the portfolio had positive Slope FOYCE over a period when the average change to the Slope coefficient was positive (i.e the portfolio had "a curve steepener and the yield curve steepened", again using the jargon of fixed interest portfolio management). The returns attributed to relative value are relatively small, which suggests that the contributions from relative value tend to average out to zero over time in the benchmark portfolio. The interest accrual returns are positive, which is worthy of note. In a risk-neutral environment, this interest accrual component should be identically zero, because the interest accrual from all cashflows should be identical, and a swap is equivalent to a fixed interest asset exactly offset by a floating rate

[16] The latter suggestion is consistent with the results of Soto (2001), where it is found that constraints on "level, slope and curvature of term structure shifts are necessary to guarantee a return close to target", while differences in traditional convexity have little impact over horizons of one and two years. That said, SOYCE effects will aggregate steadily over time (because they are effectively the sums of the squared components of the vector $\boldsymbol{\delta}$), which means they will ultimately make material contributions to portfolio returns over long horizons.

liability (effectively the federal funds rate in this analysis). The positive interest accrual of \$3.870 million (which equates to 39 basis points per annum on the constant face-value of \$150 million over the 6.57 years of the sample period) therefore reflects the risk-averse environment that would typically be expected in financial markets; i.e the interest accrual is implicitly higher on the cashflows of the longer-maturity fixed interest asset than on the floating rate liability due to the higher compensation required for risk. As noted in the previous section, some of these positive returns may be due to the use of the federal funds rate instead of a bank-risk short-maturity rate in the analysis. However, the 3–month rate LIBOR has averaged only 17.8 bps above the federal funds rate over the sample, and since the overnight LIBOR rate was introduced in January 2001, it has averaged only 7.7 bps above the federal funds rate. Hence, the interest accrual component would still be positive even allowing for adjustments of those magnitudes.

4.5.3 Fixed interest portfolio optimisation

The optimisation of fixed interest portfolios relative to the benchmark portfolio is undertaken in five different ways, denoted as optimal portfolio (OP) 1 to 5. OP1 and OP2 are genuine out-of-sample tests undertaken in simulated real-time (SRT);[17] i.e the optimisation at each point in time uses only information that would have been available at that point in time. Specifically, the ANS parameters used in the ex-ante optimisation OP1 and OP2 are the pre-sample (P/S) values previously estimated in section 2.5.3; i.e $\sigma_1 = 0.84\%$, $\rho_1 = 1.62$, $\phi = 0.80$, and $\mathbf{v} = \left(0.84^2, 1.49^2, 1.17^2\right)\%^2$. These were obtained from the monthly data for the government bond curve from October 1986 to January 1994, and so these estimates would obviously have been available if the portfolio optimisation had begun in 1 May 1998. Ideally, it would be more desirable to use ANS parameters

[17]The term "simulated real-time" is adopted from the simulated real-time forecasting of Stock and Watson (2002) in a macroeconomic context.

estimated from swaps yield curve data before 1 May 1998, but that is not possible given that long-maturity swaps data are not available before 1 May 1998.

The difference between OP1 and OP2 is the mean-adjustment (M/A) applied to the yield residuals of each security to obtain $\boldsymbol{\alpha}_t$ for the optimisation process. Specifically, OP1 uses no mean-adjustment, so $\pi_k = 0$; and OP2 uses SRT mean-adjustments, so π_k is set by recursive estimation using the mean of the yield residuals up to the previous working day, i.e $\pi_k(t) = \frac{1}{t-1}\sum_{i=1}^{t-1} \eta_{k,i}$ (the initial value π_k (1-May-98) is set to zero, given that the yield residual from the previous day would not be available).

For later comparison to OP1 and OP2, the optimised portfolios OP3, OP4, and OP5 use varying degrees of in-sample (I/S) information. Specifically, OP3, OP4, and OP5 all use I/S estimates of the ANS parameters, which were calculated as $\phi = 0.62$, $\rho_1 = 0.88\%$, and $\mathbf{v} = \left(1.03^2, 1.65^2, 1.59^2\right)\%^2$ using the full sample of swaps yield curve data and the procedure outlined in section 2.3.4. Regarding the M/As: OP3 uses $\pi_k = 0$; OP4 uses the full-sample estimated means for the yield residuals (i.e $\pi_k = \frac{1}{1598}\sum_{i=\text{1-May-98}}^{\text{22-Sep-04}} \eta_{k,i}$); and OP5 uses the SRT estimates of π_k as for OP2.

The portfolio optimisations relative to benchmark are undertaken as follows: (1) the benchmark portfolio is established as at 1 May 1998 with zero cash, and a \$10 million face-value for each swap maturity; (2) the ANS model is estimated using the parameters already noted above and the yield curve data at time t, and this is used to calculate the yield residuals for each swap security and the FOYCEs for the benchmark portfolio; (3) the M/A parameter π_k is set according to the alternatives discussed for OP1 to OP5 above, and $\boldsymbol{\alpha}_t$ is calculated; (4) the alternative portfolio is optimised using the linear programme in equation 4.14 (i.e with the alternative portfolio MV and FOYCE components equal to those of the benchmark portfolio on that day), and the constraint that the face-values of each swap security are maintained between \$0 and \$20 million, and cash is maintained at zero; (5) this optimised portfolio is carried over to the following trading day and the daily return is calculated by revaluing the cashflows of the swaps using the zero-coupon curve "boot-strapped" from the new prevailing yield curve; and (6) steps 2 to 4 are repeated for the entire sample.

This process gives a time series of 1,598 independent daily returns for the optimised portfolios. Figure 4.7 plots the cumulative returns for OP4, which uses I/S estimates of the ANS parameters and the M/As. It is evident that the returns for OP4 are higher than for the non-optimised benchmark (by $15.049 million over the full sample), and those excess returns accrue steadily over the sample period (hence, the excess performance does not result from one or more fortuitous events). The other optimised portfolios also outperform the benchmark portfolio, which will be discussed later.

To gauge the source of the excess returns for OP4, the returns are attributed ex-post as for the benchmark portfolio. Those results are shown in table 4.5. The attributions to the FOYCE components are identical to the benchmark, which occurs by definition because the optimisation process exactly matches the FOYCE components of the optimised and benchmark portfolios. The attributions to the SOYCE components are very similar to those of the benchmark, indicating that leaving the SOYCE components unconstrained makes an immaterial difference to portfolio returns (which is again consistent with assumption 3 in section 4.4.3).

The largest difference is in the relative value component, which is $14.859 million higher in the optimised portfolio. This accords with the premise of the optimisation framework; i.e the maximisation of relative value in the optimisation process should deliver excess returns over time relative to a non-optimised benchmark portfolio. There is also a slight difference between the optimised and benchmark interest accrual components, but this is several orders of magnitude smaller than the relative value differences. Indeed, the similarity of the interest accrual returns accords with assumption 1 in section 4.4.3 that interest accrual returns do not differ much between feasible portfolios. Specifically, the total interest accrual return of \$4.051 million equates to 41 basis points per annum, compared to the 39 basis points per annum in the benchmark portfolio.

Table 4.5: Statistical summary of optimised portfolio returns

Attribution	Sum	Mean	Std dev.	Min.	Max.	Max. less min.
Actual return	57.902	0.036	0.658	-3.035	3.323	6.357
λ(1) FOYCE	18.652	0.012	0.660	-3.629	3.825	7.453
λ(2) FOYCE	8.308	0.005	0.213	-0.972	1.354	2.326
λ(3) FOYCE	6.301	0.004	0.140	-0.776	0.636	1.412
Total FOYCE	33.261	0.021	0.668	-2.936	3.086	6.021
Ω(1,1) SOYCE	4.058	0.003	0.006	0.000	0.093	0.093
Ω(1,2) SOYCE	-0.738	0.000	0.002	-0.036	0.005	0.042
Ω(1,3) SOYCE	0.203	0.000	0.001	-0.014	0.011	0.025
Ω(2,2) SOYCE	0.256	0.000	0.000	0.000	0.006	0.006
Ω(2,3) SOYCE	0.272	0.000	0.000	-0.001	0.005	0.006
Ω(3,3) SOYCE	0.163	0.000	0.000	0.000	0.003	0.003
Total SOYCE	4.215	0.003	0.005	0.000	0.066	0.066
Relative value	16.375	0.010	0.035	-0.325	0.242	0.567
Interest accrual	4.051	0.003	0.007	-0.042	0.040	0.082

Note: Summary of the returns (\$millions) of optimised portfolio OP4 and ex-post attributions of those returns to the 11 components noted in section 4.2. OP4 uses in-sample estimates of the parameters ϕ, θ_1, $\mathbf{v}$, and in-sample estimates of π_k, as detailed in the text.

Table 4.6: Summary of optimised portfolio return covariances

	Total FOYCE	Total SOYCE	Relative value	Interest accrual
Total FOYCE	0.4467	-0.0006	-0.0027	-0.0043
Total SOYCE	-0.0006	0.0000	0.0000	0.0000
Relative value	-0.0027	0.0000	0.0012	0.0001
Interest accrual	-0.0043	0.0000	0.0001	0.0000
	Total	0.4328	FOYCE/Total	1.0319

Note: Variances and covariances of the optimised portfolio OP4 attributed returns ("Total FOYCE" and "Total SOYCE" are aggregates of the individual components contained in table 4.5).

Regarding the dispersion of attributed returns for OP4, the standard deviations in table 4.5 and the variances and covariances in table 4.6 are typically identical or very similar to those of the benchmark portfolio. An exception is the component related to relative value, where the standard deviation of returns is an order of magnitude larger than for the benchmark portfolio. While this does not accord exactly with assumption 2 in section 4.4.3 (that the variation of returns from relative value should be similar among all feasible portfolios), table 4.6 shows that the greater variance from relative value is offset by greater negative covariance with the FOYCE components, leaving the total variance of OP4 similar to that of the benchmark portfolio. Hence, this indicates that OP4 is not taking on excess risk relative to the benchmark to achieve the excess returns.

Figure 4.8 plots the cumulative returns of the OP2, OP4, and OP5 (i.e the portfolios optimised after allowing for either I/S or SRT M/As) less the cumulative benchmark returns. Each of these series indicate excess returns accruing steadily over the sample period, with similar total excess returns by the end of the sample. Table 4.7 contains the summary annualised statistics for the excess returns. The information ratios (i.e annualised returns divided by the annualised standard deviations) are extremely high, and the corresponding t-statistics underlying the information ratios are extremely significant (i.e well beyond the 1% threshold).

Table 4.7: Summary of optimised less benchmark portfolio returns

Optimised portfolio relative to the benchmark portfolio	Annualised return ($million)	Annualised standard deviation ($million)	Information ratio	t-statistic
OP1 (P/S parameters, no M/A)	0.47	0.65	0.72	1.82 *
OP2 (P/S parameters, SRT M/A)	2.47	0.50	4.90	12.37 ***
OP3 (I/S parameters, no M/A)	0.09	0.69	0.13	0.33
OP4 (I/S parameters, I/S M/A)	2.35	0.55	4.26	10.78 ***
OP5 (I/S parameters, SRT M/A)	2.28	0.51	4.47	11.30 ***

Note: Statistical summary of the returns of the optimised portfolios relative to the returns of the benchmark portfolio. "I/S" is in-sample, "P/S" is pre-sample, "M/A" is mean-adjustment, and "SRT" is simulated real time. Details on the parameter values and M/As are contained in the text.

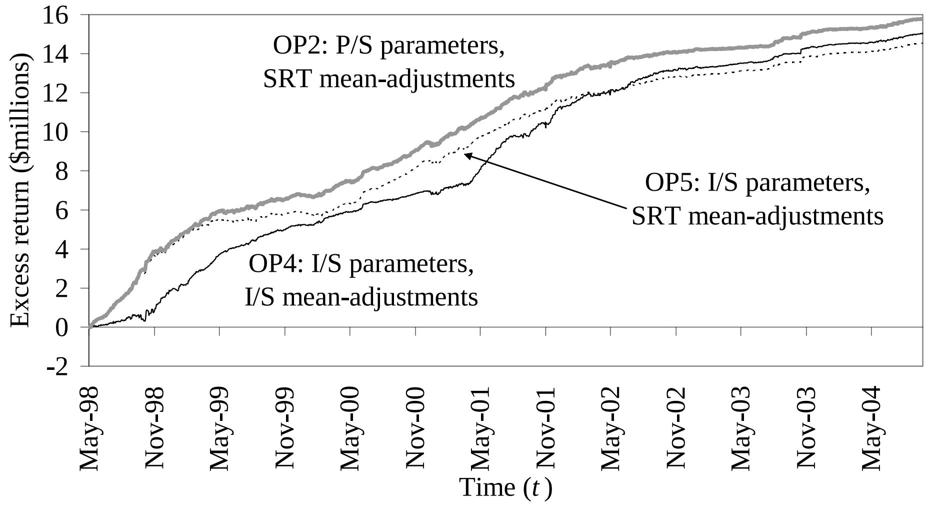

Figure 4.8: Cumulative returns for the optimised portfolios OP2, OP4, and OP5. "I/S" is in-sample, "P/S" is pre-sample, and "SRT" is simulated real time. Details on the parameter values and mean-adjustments are contained in the text.

Figure 4.9 plots the cumulative returns of the OP1 and OP3 (i.e the portfolio optimised without allowing for any M/As, i.e $\pi_k = 0$) less the cumulative benchmark returns. While the cumulative excess returns are still positive, the end-of-sample excess returns are much less than for OP2, OP4, and OP5. Table 4.7 shows that the information ratio for OP1 is moderate (with the t-statistic only significant to the 10% level), and the information ratio for OP3 is small (with an insignificant t-statistic).

Table 4.8 compares the returns for the optimised portfolios to each other. OP4 is the natural benchmark for the performances of the various optimised portfolio, because it uses the "ideal" parameters for the optimisation (i.e I/S parameters for the ANS model and I/S estimates for the M/As). Within the optimised portfolios that use the I/S ANS parameters, line 1 of table 6 (i.e OP3 less OP4) shows that the difference in excess returns is significantly negative using no M/A, but line 2 (i.e OP5 less OP4) shows the difference is insignificant using the SRT M/As. This suggests that optimisation performance deteriorates materially when inappropriate M/As are used, but using consistent estimates provided by the SRT M/As makes little practical impact. As an aside, the deterioration of the

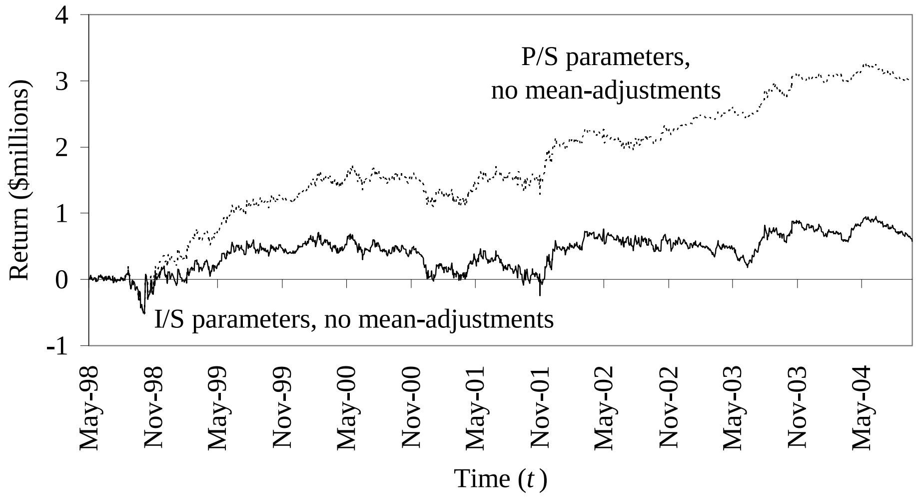

Figure 4.9: Cumulative returns for the optimised portfolios OP1 and OP3. "I/S" is in-sample, and "P/S" is pre-sample. Details on the parameter values are contained in the text.

optimisation results using no M/As tentatively suggests that factors external to the ANS model framework (e.g liquidity premia and/or preferred habitats, as noted in Elton and Gruber (1995) pp. 513-518) may be influencing the shape of the US swaps curve over the sample period, although further research would be required to make any firm conclusions on that aspect.

Comparing the returns of optimised portfolios that use P/S ANS parameters to OP4, line 3 of table 4.8 (i.e OP1 less OP4) again shows the material deterioration of optimisation performance with no M/A, while line 4 of table 4.8 (i.e OP2 less OP4) indicates that performance is not materially affected by using different ANS parameters when a consistent M/A is made. This suggests that the optimisation results are much more sensitive to whether consistent M/As are being made, rather than whether the "ideal" ANS parameters are being used. This is confirmed in line 5 of table 4.8 (i.e OP1 less OP2), where performance materially deteriorates with no M/A even when the same P/S ANS parameters are used.

Finally, lines 6 and 7 in table 6 (i.e OP1 less OP3, and OP2 less OP5) indicate

Table 4.8: Summary of optimised portfolio differences

Specified relative returns between optimised portfolios	Annual-ised return ($million)	Annual-ised standard deviation ($million)	Infor-mation ratio	t-statistic
OP3 (I/S parameters, no M/A) less OP4 (I/S parameters, I/S M/A)	-2.26	0.75	-3.02	-7.63 ***
OP5 (I/S parameters, SRT M/A) less OP4 (I/S parameters, I/S M/A)	-0.08	0.48	-0.16	-0.40
OP1 (P/S parameters, no M/A) less OP4 (I/S parameters, I/S M/A)	-1.89	0.68	-2.76	-6.97 ***
OP2 (P/S parameters, SRT M/A) less OP4 (I/S parameters, I/S M/A)	0.12	0.48	0.24	0.61
OP1 (P/S parameters, no M/A) less OP2 (P/S parameters, SRT M/A)	-2.00	0.87	-2.30	-5.82 ***
OP1 (P/S parameters, no M/A) less OP3 (I/S parameters, no M/A)	0.38	0.21	1.78	4.49 ***
OP2 (P/S parameters, SRT M/A) less OP5 (I/S parameters, SRT M/A)	0.19	0.19	1.02	2.58 ***

Note: Statistical summary of the returns of the optimised portfolios relative to other optimised portfolios. “I/S” is in-sample, “P/S” is pre-sample, “M/A” is mean-adjustment, and “SRT” is simulated real time. Details on the parameter values and mean-adjustments are contained in the text.

that when the method of M/A is the same between the optimisations, the choice of ANS parameters can have a material influence on performance. However, using "non-ideal" ANS parameter estimates is evidently not necessarily detrimental to optimisation performance, because the optimisations using P/S ANS parameter estimates show higher returns than with the "ideal" ANS parameter estimates.

4.6 Conclusion

This chapter uses the ANS model derived in chapter 2 to develop a framework applicable to fixed interest portfolio management. In the empirical application using six years of US interest rate swaps data, the ex-post attribution analysis shows that nearly all of the variability in portfolio returns is due to first-order yield curve exposures (i.e FOYCEs, or "duration" effects) from stochastic shifts in the level and shape of the yield curve; second-order yield curve exposures (i.e SOYCEs, or "convexity" effects) and other contributions are immaterial. Ex-ante, those yield curve changes are unpredictable, and so represent sources of risk to the portfolio.

The second empirical application shows that portfolios optimised ex-ante using the ANS model risk/return framework significantly outperform a naive evenly-weighted benchmark over time. This provides support for the idea that "relative value" (i.e deviations of actual yields from the estimated yields implied by the ANS model) is a quantifiable concept, and maximising that quantity potentially offers a way of enhancing portfolio returns.

Chapter 5

Using the ANS model to investigate the uncovered interest parity hypothesis

5.1 Introduction

This chapter uses the ANS model to investigate the uncovered interest parity hypothesis (UIPH), i.e the proposition that exchange rates should appreciate/depreciate at a pace that offsets the interest rate discount/premium available between the underlying currencies. With reference to chapter 1, the broad motivation is to further illustrate how the ANS model of the yield curve with its economic foundation can be formally applied in an economic setting. More specifically, the literature review in section 5.2 below shows that the empirical failure of the UIPH still presents a puzzle, and the ANS model offers a means of empirically investigating theoretical suggestions that rationally-based interest rate and exchange rate dynamics associated with cyclical influences in the wider economy may contribute to those puzzling results.

Following the literature review, the outline of the remainder of the chapter is as follows: section 5.3 proposes how the ANS model can be used to investigate the UIPH; section 5.4 describes the data and discusses points relevant to the empirical estimation; and section 5.5 presents and discusses the empirical results. Section 5.6 summarises and concludes.

Before proceeding, note that the notation and examples in this chapter refer explicitly to the Canadian and United States data (superscripted CA and US respectively) subsequently used in the empirical application of section 5.4. Of course, the approach applies quite generally to the exchange rates and yield curves of any currency pair, with the potential exception of when the interest rates of one or both of the currencies are close to zero (which would allow a material probability of negative interest rates, as discussed in section 3.2.1 of chapter 3). Also, for notational convenience and clarity, the explicit functional dependence of $s_n^{\mathrm{US}}(\phi^{\mathrm{US}}, m)$ on ϕ^{US} and $s_n^{\mathrm{CA}}(\phi^{\mathrm{CA}}, m)$ on ϕ^{CA} has been omitted from the notation introduced in chapter 2.

5.2 The UIPH and a review of the existing literature

The UIPH and its parallel specification as the forward rate unbiasedness hypothesis (i.e the FRUH, where exchange rates should appreciate/depreciate at a pace that matches the forward exchange premium/discount) both originate from the covered interest parity relationship, which defines the forward exchange rate as:

$$e_{t,m} = e_t + m\left(R_{t,m}^{\mathrm{US}} - R_{t,m}^{\mathrm{CA}}\right) \tag{5.1}$$

where e_t is the natural logarithm of the nominal exchange rate between the Canadian dollar (CAD) and the United States dollar (USD) at time t (defined as the number of USDs per CAD, so a rise in e_t is an appreciation of the CAD against the USD); $e_{t,m}$ is the natural logarithm of the forward CAD/USD exchange rate at time t for settlement at $t+m$ years; and $R_{t,m}^{\mathrm{CA}}$ and $R_{t,m}^{\mathrm{US}}$ are respectively the annualised continuously-compounding zero-coupon interest rates for Canada and the US at time t for maturity $t+m$ years.

Equation 5.1 precludes outright arbitrage opportunities between the forward exchange market and the interest rates of the two currencies. That is, if covered interest parity did not hold, then it would be possible to arbitrage between the forward exchange rates $e_{t,m}$ and the equivalent alternative of directly borrowing and investing at the prevailing exchange rate and interest rates on the underlying currencies.[1] Covered interest parity is well supported empirically, as noted, for example, in Sarno and Taylor (2003) ch. 2.

Assuming that agents are rational, the ex-ante relationship between e_t and $e_{t,m}$ should be $E_t\left[e_{t+m}\right]=e_{t,m}$, where E_t is the expectations operator conditional on information available at time t. Substituting $E_t\left[e_{t+m}\right]$ for $e_{t,m}$ in equation 5.1 and re-arranging then gives the UIPH, i.e:

$$E_t\left[e_{t+m}\right]-e_t=m\left(R_{t,m}^{\mathrm{US}}-R_{t,m}^{\mathrm{CA}}\right) \tag{5.2}$$

which essentially states that the expected change in the exchange rate over the horizon m should equal the prevailing difference in interest rates with a time to maturity of m. Alternatively, re-arranging equation 5.1 to express $m\left(R_{t,m}^{\mathrm{US}}-R_{t,m}^{\mathrm{CA}}\right)$ as the forward exchange premium $e_{t,m}-e_t$, and substituting that into equation 5.2 gives the FRUH specification, i.e $E_t\left[e_{t+m}\right]-e_t=e_{t,m}-e_t$.

The UIPH is typically tested by estimating the following equation using ex-post exchange rate and interest rate data:

$$\Delta e_{t,m}=a_m+b_m\cdot m\left(R_{t,m}^{\mathrm{US}}-R_{t,m}^{\mathrm{CA}}\right)+v_{t,m} \tag{5.3}$$

where $\Delta e_{t,m}$ is $e_{t+m}-e_t$ (i.e the change in e_t from time t to $t+m$) lagged m years; a_m

[1] That is, equivalent under the typical assumptions in the literature that capital markets are unconstrained, returns are not distorted by tax considerations, and transactions costs are negligible.

is an estimated constant which allows for any systematic risk premia; b_m is the estimated slope parameter; and the innovation terms $v_{t,m}$ represent unanticipated differences between expected and realised exchange rates, which should be distributed with mean zero. The estimation of a_m and b_m is typically the primary consideration in empirical tests of the UIPH/FRUH, and this chapter follows that precedent.[2] Hence, if the UIPH holds, then a statistical test on the estimated parameter b_m should not reject the theoretical value of 1, while the estimated parameter a_m may be non-zero to allow for any systematic premia that may arise because the exchange rate and interest rate data are observed in a non-risk-neutral environment. Similarly, the FRUH is typically tested by estimating the equation $\Delta e_{t,m} = a_m + b_m \cdot (e_{t,m} - e_t) + v_{t,m}$ using lagged ex-post exchange rate and forward exchange rate data.

It is well established that the UIPH/FRUH is typically rejected based on the standard regressions noted above. The frequently-referenced surveys summarising those results are Hodrick (1987), Froot and Thaler (1990), and Engle (1996). Recent examples of empirical investigations that reject the UIPH/FRUH are Liu and Maynard (2005), Wu (2005*a*), and Zhou and Kutan (2005). Indeed, rather than yielding the expected coefficient of $b_m = 1$, UIPH/FRUH regressions frequently produce significantly negative estimates of b_m, implying that exchange rates move contrary to the predictions of the UIPH/FRUH.

That said, recent empirical investigations based on longer horizons/maturities, rather than the weekly, monthly, or quarterly data often used, have been more supportive of the UIPH/FRUH. For example, Alexius (2001) generally does not reject the UIPH using 10-year interest rates and exchange rate changes over the corresponding 10 year horizon. Meredith and Chinn (2004) reports similar results using 5- and 10-year interest rates over the corresponding horizons, while rejecting the UIPH based on 3-, 6-, and 12-month matu-

[2] The additional test of whether the information available at time $t - m$ was used efficiently is that $v_{t,m}$ should exhibit no serial correlation beyond the moving-average correlation induced when the horizon m is greater than the frequency of the data, but that aspect is not tested in this chapter.

rities/horizons. Similarly, Razzak (2002) generally does not reject the FRUH on a 1-year horizon, but rejects it for the 1-month horizon.

These mixed empirical results have prompted further bodies of literature, as summarised in the survey of Sarno (2005), on how the UIPH/FRUH might be reconciled with the data. For example, Fama (1984) originally proposed that deviations of the data from the UIPH/FRUH might reflect time-varying risk premia, although subsequent investigations using standard finance/economic models with plausible parameter values have not been able to provide satisfactory sources of those risk premia or the required magnitude of variation.[3] A second class of proposals with some empirical support is that failures of the UIPH/FRUH might reflect departures from the rational expectations assumed in the formulation of the UIPH/FRUH.[4] Another strand of the literature suggests that the puzzling results from the standard regression tests of the UIPH/FRUH might be largely a statistical artifact arising from the time-series properties of the data over finite samples.[5]

The strand of literature most closely connected with the material in this chapter suggests that rationally-based interest rate and exchange rate dynamics associated with cyclical interlinkages between the economy and financial markets may be an important factor contributing to the UIPH/FRUH puzzle. McCullum (1994) originally illustrated this concept by augmenting the UIPH relationship with a simple monetary policy reaction function to represent the smoothing of the path of interest rates by the central bank. Solving the two-equation stochastic system analytically under rational expectations then produces an expected negative slope coefficient for the standard UIPH regression. Meredith and Chinn

[3]See Sarno (2005) pp. 676-678. Wu (2005*a*) is a recent addition showing that time-varying interest rate risk cannot explain the failure of the UIPH.

[4]See Sarno (2005) pp. 678-679. The heterogenous agent model of the exchange rate market in Grauwe and Grimaldi (2006) is a recent theoretical addition to this literature.

[5]For example, Baillie and Bollerslev (2000) shows via simulation that persistent time-varying volatility in the data would lead to a diffuse distribution for the FRUH regression slope coefficient, and Maynard and Phillips (1998) considers the implications of I(0) changes in the exchange rate and I(1) forward exchange premia. Empirically, Sarno (2005) pp. 679-683 discusses that there is better support for the FRUH based on long-run cointegrating relationships between the levels of the exchange rate and the forward exchange rate, and Delcoure, Barkoulas, Baum and Chakraborty (2003) is a recent addition to that literature.

(2004) expands the McCullum (1994) approach into a more realistic macroeconomic model that includes the UIPH, a Taylor rule monetary policy reaction function, a Phillips curve inflation relationship, an investment-savings output equation, and a short-maturity and long-maturity interest rate to represent the yield curve. Applying the standard UIPH regression to artificial interest rate and exchange rate data generated from stochastic simulations of that calibrated model under rational expectations reproduces the typical empirical results discussed above; i.e negative slope coefficients and rejections of the UIPH for short horizons, but slope coefficients near 1 and non-rejections of the UIPH for longer horizons. Lim and Ogaki (2003) obtains a similar pattern of results by simulating a rational-expectations open-economy model that includes an exogenous domestic interest rate process with temporary and persistent innovations.

While these theoretical and simulation results illustrate that deviations of exchange rate and interest rate data from the UIPH can occur without recourse to time-varying risk premia and/or non-rational expectations, that strand of literature currently has no direct empirical support. Indeed, the illustrative model of McCullum (1994) has been rejected in a subsequent empirical investigation by Mark and Wu (1996). The models of Lim and Ogaki (2003) and Meredith and Chinn (2004) could in principle be tested empirically, but the estimation of rational expectations models is practically challenging, and the number of parameters in the Lim and Ogaki (2003) and Meredith and Chinn (2004) models would hinder inference in any case.

However, an alternative approach is to use the ANS model and its foundation within a general-equilibrium/rational expectations economy. That is, as detailed subsequently in the following section, the components of the ANS yield curve model may be used to decompose the interest rates used within tests of the UIPH regression into their rationally-based fundamental and cyclical components. Testing the UIPH using the cyclical components of interest rates may then provide a direct gauge of the contribution that rationally-based cyclical component of interest rates might make to the UIPH puzzle.

5.3 Investigating the UIPH using the ANS model of the yield curve

The first use of the ANS model to investigate the UIPH is simply as a convenient means of generating zero-coupon interest rate data from market-quoted yield curve data that are typically coupon-bearing for maturities of one year and beyond.[6] That is, once estimated from the available yield curve data (as in the example of figure 5.1), the ANS model provides a continuous zero-coupon interest rate function for any maturity over the interval $0 \leq m < \infty$. Tests of the UIPH can then be undertaken using a time series of estimated zero-coupon interest rates for an arbitrary given maturity m, i.e:

$$\Delta e_{t,m} = a_m + b_m \cdot m \left[R^{\text{US}}_{t,m} (\text{ANS}) - R^{\text{CA}}_{t,m} (\text{ANS}) \right] + v_{t,m} \tag{5.4}$$

where $R_{t,m}$ (ANS) is the annualised continuously-compounding zero-coupon interest rate as a function of time to maturity m from the ANS model derived in chapter 2. Note that the use of estimated zero-coupon interest rate data is common practice in the literature.[7] However, given both market-quoted and ANS-estimated zero-coupon interest rates are available for the 3- and 6-month horizons/maturities investigated in this chapter, it is worthwhile undertaking the UIPH tests with both sets of data to ensure that using estimated interest rate data does not materially influence the empirical results.

[6] This chapter works exclusively with the UIPH specification from this point onward, given interest rates may be calculated directly from the ANS yield curve model. (Formulating and testing FRUH relationships would require an additional transformation of the interest rate differentials into forward exchange rate data, and the subsequent empirical estimations for the FRUH would be identical to those for the UIPH in any case.)

[7] For example, the analysis in Soto (2001), Schmidt and Kalemanova (2002), and Fang and Muljono (2003) is based on interest rates estimated using the Nelson and Siegel (1987) approach. The method of "bootstrapping" (e.g see Hull (2000) p. 150), is an alternative method of estimation that precisely replicates the market-quoted yields, but longer maturity yields are subject to distortions due to "errors" in the data (e.g bid-ask bounce or stale quotes from market-quoted data).

As previously mentioned, the more important use of the ANS model is to decompose the interest rates used in the UIPH regression into their rationally-based cyclical and fundamental components. The essential principles underlying that decomposition follow from the derivations in chapter 3; i.e the ANS model is a parsimonious first-order approximation to the ABE model, which is in turn a generic general-equilibrium economy and therefore a rational expectations model. Furthermore, chapter 3 shows that the ANS Level component at any given point in time captures the persistent component of interest rates across all maturities, which in turn reflects the steady-state variables or fundamentals of the underlying economy prevailing at that point in time. Conversely, the non-Level components of the ANS model capture the non-persistent or cyclical component of interest rates, which in turn reflects any dynamic relationships that influence the expected evolution of the state variables of the economy relative to the steady-state variables. More formally, section B.3 in appendix 3 details how the generic structure of the ABE model can readily be extended into a model with two economies, two yield curves, and a bilateral exchange rate.

In effect then, applying the ANS model to the yield curve data of each country allows the decomposition of the interest rate differential between those two countries into rationally-based cyclical and fundamental components, and the UIPH can be directly tested using those components. This has two distinct advantages over the structural economic models proposed by Lim and Ogaki (2003) and Meredith and Chinn (2004). Firstly, the generic specification of the general-equilibrium economy underlying the ANS model avoids the need to explicitly specify and model the myriad of potential relationships that may influence the dynamics of the economy (such as Phillips curve relationships, monetary policy reaction functions, monetary policy credibility effects, exchange rate influences on inflation and/or the real economy, etc.). In other words, the ANS model allows the decomposition of the interest rate data used in the UIPH into its cyclical and fundamental components while remaining agnostic about the precise dynamics that generate those components in each of the underlying economies. The second advantage of the ANS model approach is parsimony, which makes the ANS model very straightforward to apply in practice. That said, one disadvantage of the ANS model approach is that it offers no direct means of decomposing the exchange rate data used to test the UIPH into its cyclical and fundamental components. The latter would enable a more comprehensive series of UIPH tests based on the cyclical and fundamental components of both interest rates and exchange rates. Such tests may be more revealing in cases where much of the cyclicality of the underlying economy is reflected in deviations of the exchange rate from its fundamental value, or as alluded to in the discussion of section 5.4, where monetary authorities deliberately influence the exchange rate away from its fundamentals.[8] This chapter proceeds with just the interest rate decomposition as discussed, but section 6.6.2 of the concluding chapter discusses potential methods for the decomposition of exchange rate data into its cyclical and fundamental components.

The first step in decomposing the interest rates used to test the UIPH is to remove the ANS-model-estimated market prices of risk and volatility components, i.e $\frac{\sigma_1 \theta_1 m}{2} - \sum_{n=1}^{3} \sigma_n^2 \cdot u_n(m)$, from the interest rate data. This gives risk-neutral volatility-adjusted (RNVA) zero-coupon interest rates, and the UIPH tests based on RNVA interest rate differentials may be expressed as:

$$\begin{aligned} \Delta e_{t,m} \quad = \quad & a_m + b_m \cdot m \left[\left\{ \varepsilon_{t,m}^{\mathrm{US}} + \sum_{n=1}^{3} \beta_n^{\mathrm{US}}(t) \cdot s_n^{\mathrm{US}}(m) \right\} \right. \\ & \left. - \left\{ \varepsilon_{t,m}^{\mathrm{CA}} + \sum_{n=1}^{3} \beta_n^{\mathrm{CA}}(t) \cdot s_n^{\mathrm{CA}}(m) \right\} \right] + v_{t,m} \end{aligned} \tag{5.5}$$

where zero-coupon estimates of $\varepsilon_{t,m}^{\mathrm{US}}$ and $\varepsilon_{t,m}^{\mathrm{CA}}$ will only be available for the 3- and 6-month maturities in this chapter (given only those securities are non-coupon-bearing).[9] In the

[8] Mark and Wu (2004) shows that unanticipated exchange rate interventions within a rational expectations framework can produce deviations from the UIPH without recourse to time-varying risk premia and/or non-rational expectations.

[9] The yield-to-maturity of a coupon-bearing security is effectively an internal rate of return on the coupons

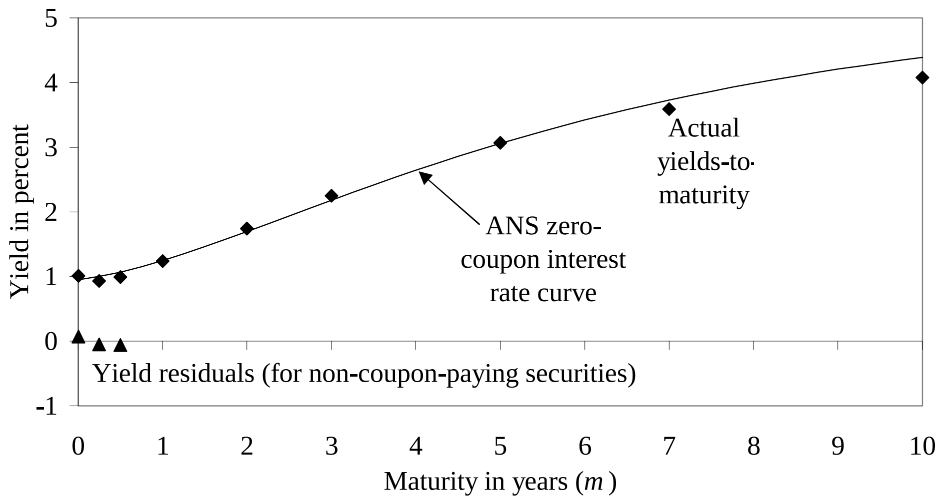

Figure 5.1: The US yield curve for February 2004. The diamond points are the market-quoted yields-to-maturity of the different securities, the triangle points are the estimated yield residuals for the non-coupon-paying securities, and the line is the continuously-compounding zero-coupon interest rate curve for the estimated ANS model with β_1^{US} (Feb-2004) $= 6.47\%$, β_2^{US} (Feb-2004) $= 7.98\%$, β_3^{US} (Feb-2004) $= -2.46\%$.

absence of zero-coupon estimates for the other maturities, $\varepsilon_{t,m}^{\mathrm{US}}$ and $\varepsilon_{t,m}^{\mathrm{CA}}$ are simply set to zero, and the time-varying component of equation 5.5 becomes the RNVA ANS model interest differential, i.e: $\sum_{n=1}^{3} \beta_n^{\mathrm{US}}(t) \cdot s_n^{\mathrm{US}}(m) - \sum_{n=1}^{3} \beta_n^{\mathrm{CA}}(t) \cdot s_n^{\mathrm{CA}}(m)$. The RNVA interest rates for any given maturity may then be decomposed into their Level components, and non-Level components (i.e the ANS Slope plus Bow components, and the yield residuals $\varepsilon_{t,m}$ for the 3- and 6-month securities).

Anticipating the discussion of the data in section 5.4, the following figures give detailed examples of using the ANS model to decompose the interest rate data used in the UIPH tests. Figure 5.1 illustrates the US yield curve data observed for February 2004, the associated yield curve estimated using the ANS model, and the estimated yield residuals for the non-coupon-paying securities.

Figure 5.2 illustrates the RNVA ANS interest rate curve and the Level and Slope

and principle. Similarly, the estimated yield residual for a coupon-bearing security will be on an internal-rate-of-return basis, which is not zero-coupon.

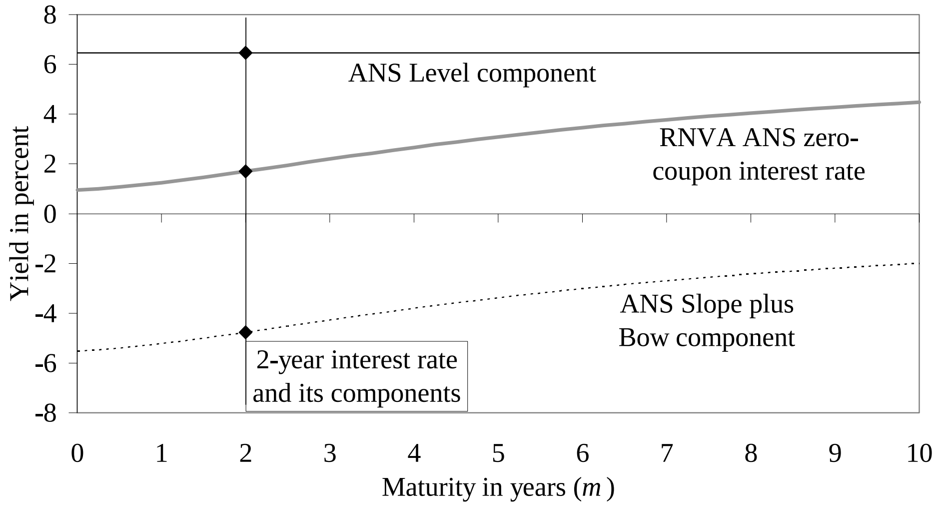

Figure 5.2: The RNVA ANS zero-coupon interest rate curve and its components for the February 2004 US yield curve observation. The RNVA ANS zero-coupon interest rate curve is $\sum_{n=1}^{3} \beta_n^{\mathrm{US}}$ (Feb-2004) $\cdot\, s_n^{\mathrm{US}}(m)$, the ANS Level component is β_1^{US} (Feb-2004) $\cdot\, s_1^{\mathrm{US}}(m)$, and the ANS Slope plus Bow component is β_2^{US} (Feb-2004) $\cdot\, s_2^{\mathrm{US}}(m)+$ β_3^{US} (Feb-2004) $\cdot\, s_3^{\mathrm{US}}(m)$.

plus Bow components of that curve for the February 2004 US yield curve in figure 5.1. The estimated RNVA ANS interest rate curve is $R_{\text{Feb-2004},m}^{\mathrm{US}}(\mathrm{ANS}) = \sum_{n=1}^{3} \beta_n^{\mathrm{US}}$ (Feb-2004) $\cdot\, s_n^{\mathrm{US}}(m)$, the Level component of that curve is β_1^{US} (Feb-2004) $\cdot\, s_1^{\mathrm{US}}(m) = \beta_1^{\mathrm{US}}$ (Feb-2004), and the non-Level component is β_2^{US} (Feb-2004) $\cdot\, s_2^{\mathrm{US}}(m) + \beta_3^{\mathrm{US}}$ (Feb-2004) $\cdot\, s_3^{\mathrm{US}}(m)$. Figure 5.2 also highlights the RNVA ANS interest rate for the 2-year maturity. This has the value $\sum_{n=1}^{3} \beta_n^{\mathrm{US}}$ (Feb-2004) $\cdot\, s_n^{\mathrm{US}}(2)$, with the Level component β_1^{US} (Feb-2004), and the non-Level component β_2^{US} (Feb-2004) $\cdot\, s_2^{\mathrm{US}}(2) + \beta_3^{\mathrm{US}}$ (Feb-2004) $\cdot\, s_3^{\mathrm{US}}(2)$.

Continuing the example, figure 5.3 then illustrates the difference between the estimated RNVA ANS yield curves for the US and Canada as at February 2004, and the difference between the ANS Level and non-Level components of the US and Canadian RNVA ANS yield curve. As highlighted in figure 5.3, the 2-year RNVA interest rate differentials and the Level and non-Level components of those differentials are just the respective function values at $m = 2$.

The example above illustrates how an interest rate differential and its Level and

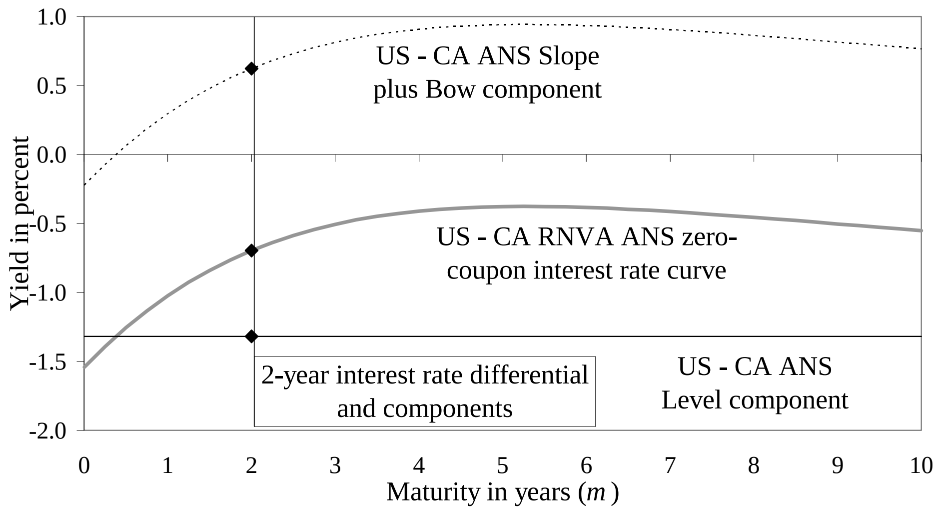

Figure 5.3: The RNVA ANS zero-coupon interest rate differential and its components for the February 2004 US and Canadian yield curves. The RNVA ANS zero-coupon interest rate differential is $\sum_{n=1}^{3} \beta_n^{\mathrm{US}}$ (Feb-2004) $\cdot\, s_n^{\mathrm{US}}(m) - \sum_{n=1}^{3} \beta_n^{\mathrm{CA}}$ (Feb-2004) $\cdot\, s_n^{\mathrm{CA}}(m)$, the Level component is β_1^{US} (Feb-2004) $-\, \beta_1^{\mathrm{CA}}$ (Feb-2004), and the non-Level component is $\sum_{n=2}^{3} \beta_n^{\mathrm{US}}$ (Feb-2004) $\cdot\, s_n^{\mathrm{US}}(m) - \sum_{n=2}^{3} \beta_n^{\mathrm{CA}}$ (Feb-2004) $\cdot\, s_n^{\mathrm{CA}}(m)$.

non-Level components are generated at a single point in time. Repeating the estimation of the ANS model for each observation of the Canadian and US yield curve data over the entire sample period therefore allows the generation of a time series of interest rate differentials of the required maturity, and the generation of the Level and non-Level components of those interest rate differentials. That data can then be used in conjunction with changes in the exchange rate over the horizon corresponding to the interest rate maturity to test the UIPH for that horizon.

The different expressions of interest rate data and the decomposition of those interest rates into components provides many different permutations of UIPH tests, particularly for the 3- and 6-month maturities where zero-coupon estimates of the yield residuals $\varepsilon_{t,m}$ are available. Hence, for the 3- and 6-month horizons, tests of the UIPH are undertaken for: (1) the market-quoted zero-coupon interest rate (equation 5.3); (2) the interest rate from the ANS model (equation 5.4); and (3) the RNVA interest rate (equation 5.5). The additional UIPH tests on the underlying interest rate components are as follows: (4) the

Level component and non-Level components, with the latter separated out as the Slope plus Bow components and the yield residual components, i.e:

$$\begin{aligned}\Delta e_{t,m} &= a_m + w_m \cdot m \left[\beta_1^{\mathrm{US}}(t) - \beta_1^{\mathrm{CA}}(t)\right] + y_m \cdot m \left[\sum_{n=2}^{3} \beta_n^{\mathrm{US}}(t) \cdot s_n^{\mathrm{US}}(m)\right. \\ &\quad \left. - \sum_{n=2}^{3} \beta_n^{\mathrm{CA}}(t) \cdot s_n^{\mathrm{CA}}(m)\right] + z_m \cdot m \left[\varepsilon_{t,m}^{\mathrm{US}} - \varepsilon_{t,m}^{\mathrm{CA}}\right] + v_{t,m} \qquad (5.6)\end{aligned}$$

(5) the Level component and non-Level components, i.e:

$$\begin{aligned}\Delta e_{t,m} &= a_m + w_m \cdot m \left[\beta_1^{\mathrm{US}}(t) - \beta_1^{\mathrm{CA}}(t)\right] + x_m \cdot m \left[\left\{\varepsilon_{t,m}^{\mathrm{US}} + \sum_{n=2}^{3} \beta_n^{\mathrm{US}}(t) \cdot s_n^{\mathrm{US}}(m)\right\}\right. \\ &\quad \left. - \left\{\varepsilon_{t,m}^{\mathrm{CA}} + \sum_{n=2}^{3} \beta_n^{\mathrm{CA}}(t) \cdot s_n^{\mathrm{CA}}(m)\right\}\right] + v_{t,m} \qquad (5.7)\end{aligned}$$

(6) the Level component only, i.e:

$$\Delta e_{t,m} = a_m + w_m \cdot m \left[\beta_1^{\mathrm{US}}(t) - \beta_1^{\mathrm{CA}}(t)\right] + v_{t,m} \qquad (5.8)$$

(7) the non-Level components only, separated out as the Slope plus Bow components and the yield residual components, i.e:

$$\begin{aligned}\Delta e_{t,m} &= a_m + y_m \cdot m \left[\sum_{n=2}^{3} \beta_n^{\mathrm{US}}(t) \cdot s_n^{\mathrm{US}}(m) - \sum_{n=2}^{3} \beta_n^{\mathrm{CA}}(t) \cdot s_n^{\mathrm{CA}}(m)\right] \\ &\quad + z_m \cdot m \left[\varepsilon_{t,m}^{\mathrm{US}} - \varepsilon_{t,m}^{\mathrm{CA}}\right] + v_{t,m} \qquad (5.9)\end{aligned}$$

and (8) the non-Level component of the RNVA ANS model, i.e:

$$\begin{aligned}\Delta e_{t,m} &= a_m + x_m \cdot m \left[\left\{\varepsilon_{t,m}^{\mathrm{US}} + \sum_{n=2}^{3} \beta_n^{\mathrm{US}}(t) \cdot s_n^{\mathrm{US}}(m)\right\}\right. \\ &\quad \left. - \left\{\varepsilon_{t,m}^{\mathrm{CA}} + \sum_{n=2}^{3} \beta_n^{\mathrm{CA}}(t) \cdot s_n^{\mathrm{CA}}(m)\right\}\right] + v_{t,m} \qquad (5.10)\end{aligned}$$

For horizons/maturities of one year and beyond, estimated residuals are not available on a zero-coupon basis. Hence, equations 5.3, 5.6, and 5.9 cannot be estimated, and the estimation of the other equations proceeds with $\varepsilon_{t,m} = 0$.

5.4 The data and empirical estimation

The data used for the analysis in this chapter are the month-end CAD/USD exchange rates, and month-end Canadian and US yield curve data. These data were chosen for the investigation for several reasons. Firstly, the CAD/USD exchange rate is set within a relatively unhindered floating regime and is the only currency pair within the Group of Seven (G7) currencies that has been relatively untainted by major currency market events in recent decades. Regarding the other G7 currencies, Germany, France, and Italy were subjected to alignment of their currencies within the European Monetary System and they subsequently adopted the euro currency in 1999, Japan has been subject to a degree of exchange rate management including occasional large interventions as recently as 2003,[10] and the United Kingdom (UK) was subjected to major foreign exchange speculation and subsequent withdrawal of the UK pound from the European Monetary System in 1992. Secondly, the US and Canada central bank websites readily provide long time series of detailed market-quoted yield curve data (as detailed below), while the data for other currencies is limited. That is, long time-series of market-quoted data generally consist of only two points on the yield curve (e.g a 90-day rate and a 10-year bond yield), while more detailed market-quoted curve data is only available for relatively short periods (e.g Datastream data for Germany only dates from 1996).[11]

The CAD/USD exchange rate data are taken from the online Federal Reserve Economic Database (FRED) available on the Federal Reserve Bank of St. Louis website. The US yield curve data are constant maturity interest rates obtained from the FRED database. The specific series are the federal funds rate, the 3-month and 6-month Treasury bill rates (both zero-coupon securities), and the 1-year, 2-year, 3-year, 5-year, 7-year, 10-

[10] Also, in reference to the discussion in section 2.2, short-term Japanese interest rates have been held at zero almost continuously since the late-1990s, which would invalidate the application of the ANS model.

[11] The Bundesbank and Bank of England websites offer zero-coupon yield curve data obtained using curve-fitting methods applied to market-quoted yield curve data, but not the market-quoted data itself.

year, and 20-year or 30-year constant maturity bond rates (all semi-annual coupon-paying securities).[12] The Canadian yield curve data used in the empirical application are constant maturity interest rates obtained from the Bank of Canada website. The specific series are the Bank of Canada policy rate, the 3-month and 6-month Treasury bill rates (both zero-coupon securities), and the 2-year, 3-year, 5-year, 7-year, 10-year, and 25-year or 30-year constant maturity bond rates (all semi-annual coupon-paying securities).[13] The sample period is chosen as January 1985 to December 2005, giving 252 monthly observations of the yield curve. The start of this period was chosen to be beyond the late-1970s/early-1980s structural change for US yield curve data that was previously discussed in section 3.4.1 of chapter 3, and December 2005 was the last month available at the time of the analysis.

The estimation of the ANS model from a time series of yield curve observations is detailed in section 2.3.4 of chapter 2. An example of the output from that estimation has already been discussed in section 5.3, and the estimated ANS parameters for the US are $\phi = 0.51$, $\rho_1 = 1.48\%$, $\sigma_1 = 0.81\%$, $\sigma_2 = 1.72\%$, and $\sigma_3 = 1.32\%$, while those for Canada are $\phi = 0.44$, $\rho_1 = 1.67\%$, $\sigma_1 = 1.12\%$, $\sigma_2 = 3.01\%$, and $\sigma_3 = 2.25\%$. As an example of the data used for testing the UIPH, figure 5.4 illustrates the time series of the RNVA 1-year interest rate differential and annual changes in the CAD/USD lagged one year (i.e $\Delta e_{t,1}$). Figures 5.5 and 5.6 respectively illustrate the Level and Slope plus Bow components of the RNVA ANS 1-year interest rate and $\Delta e_{t,1}$. For the 3-month, 6-month, and 1-year horizons, figure 5.7 plots the difference between lagged ex-post changes in the exchange rate, and the ex-ante changes predicted by the UIPH.

While economic theory would suggest that the data should be stationary in the long-run (because exchange rates cannot appreciate or depreciate indefinitely, and interest rate differentials should have an upper bound related to relative economic fundamentals),

[12] 20-year data are unavailable from January 1987 to September 1993, and so 30-year data (with a 30-year maturity) are used during this period for the estimation of the ANS model.

[13] 30-year data are available from January 1991, and 25-year data are available before then.

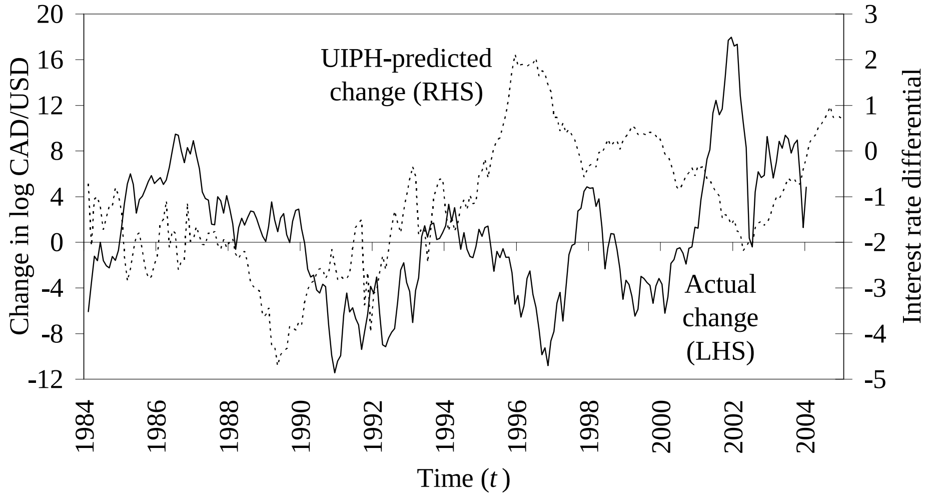

Figure 5.4: UIPH data for the 1-year horizon using the ANS model 1-year interest rate. The series are the lagged annual change in the CAD/USD exchange rate (i.e $\Delta e_{t,1}$) and the US less Canadian interest rate differential from the RNVA ANS model for the 1-year maturity (i.e $m = 1$).

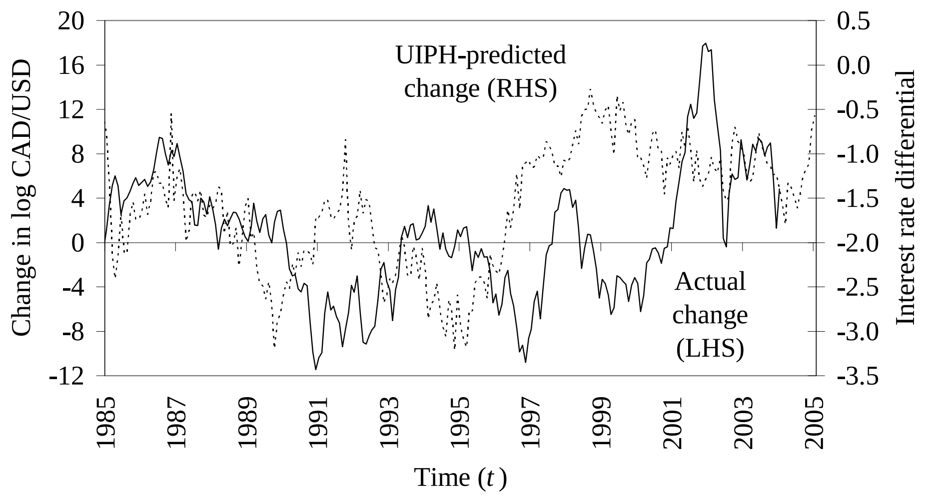

Figure 5.5: UIPH data for the 1-year horizon using the Level component of the 1-year interest rate. The series are the lagged annual change in the CAD/USD exchange rate (i.e $\Delta e_{t,1}$) and the US less Canadian interest rate differential for the Level component of the ANS model for the 1-year maturity (i.e $m = 1$).

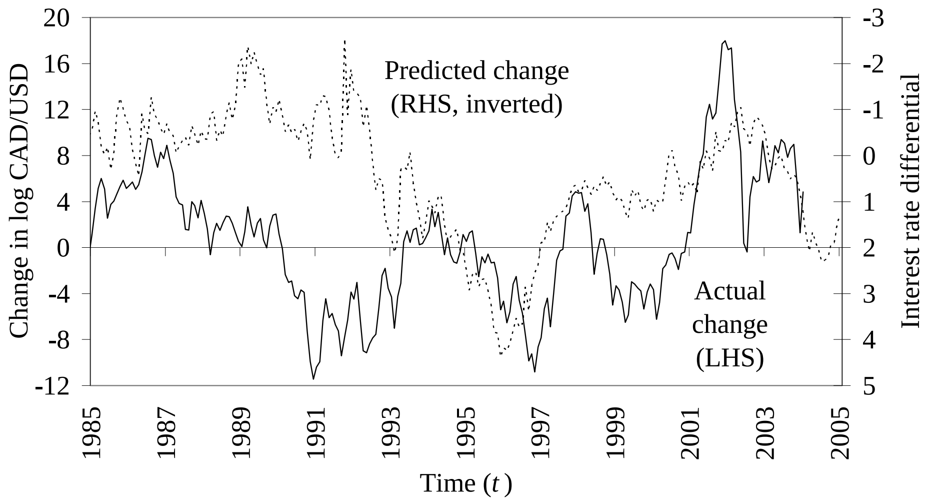

Figure 5.6: UIPH data for the 1-year horizon using the Slope plus Bow component of the 1-year interest rate. The series are the lagged annual change in the CAD/USD exchange rate (i.e $\Delta e_{t,1}$) and the US less Canadian interest rate differential for the Slope plus Bow component of the ANS model for the 1-year maturity (i.e $m = 1$). The latter has been inverted to better illustrate the apparent inverse relationship with $\Delta e_{t,1}$.

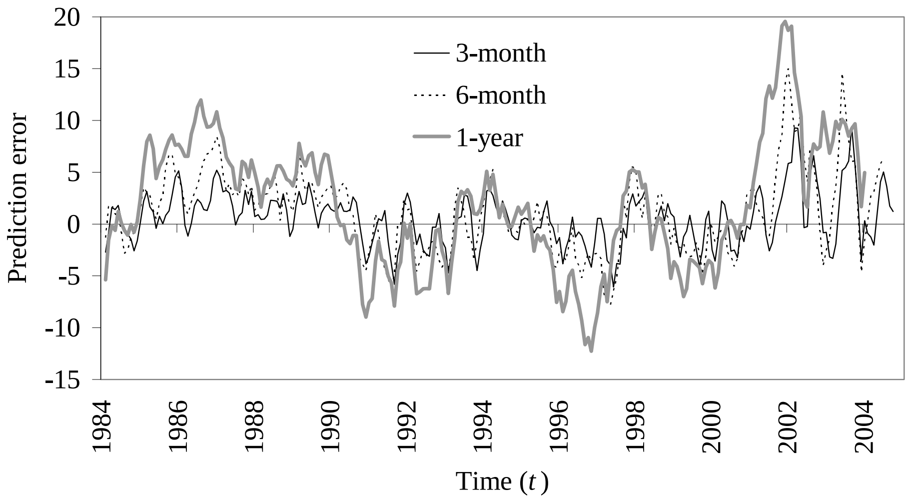

Figure 5.7: UIPH prediction errors, i.e the lagged changes in the CAD/USD exchange rate (i.e $\Delta e_{t,m}$) less the interest rate differential for the 3-month, 6-month, and 1-year horizons.

an inspection of figures 5.5, 5.6, and 5.7 suggests that the data did have relatively high persistence over the sample period. Indeed, the results contained in table 5.1 for the 1-year horizon data suggest that the hypothesis of stationarity is frequently rejected, or alternatively the hypothesis of a unit root frequently cannot be rejected over the sample period. However, the results in table 5.2 indicate that the data is at least cointegrated.

Table 5.1: Unit root and stationarity tests on the 1-year UIPH data

Unit root or stationarity test	Change in LN USD/CAD	ANS interest rate differential	RNVA ANS interest rate differential	Level component of ANS interest rate differential	Non-Level component of ANS interest rate differential
PP fixed window	-2.81 * 12	-1.78 12	-1.78 12	-2.85 * 12	-2.26 12
PP auto window	-2.82 * 11	-1.78 12	-1.78 12	-2.81 * 11	-2.26 12
ADF fixed lags	-1.87 12	-1.86 12	-1.86 12	-1.86 12	-2.03 12
ADF auto lags	-1.87 12	-1.52 7	-1.52 7	-2.60 * 1	-2.41 0
KPSS fixed window	0.28 12	0.95 *** 12	0.95 *** 12	0.53 ** 12	0.66 ** 12
KPSS auto window	0.30 11	0.95 *** 12	0.95 *** 12	0.57 ** 11	0.66 ** 12

Note: PP is Phillips-Perron, ADF is augmented Dickey-Fuller, KPSS is Kwiatkowski-Phillips-Schmidt-Shin, and the window width/number of lags is given below each statistic. *, **, and *** respectively denote a 10, 5, and 1 percent level of significance.

The unit root and cointegration results for the 1-year horizon are typical for the other horizons investigated in this chapter. Hence, the analysis follows the advice in Hamilton (1994) p. 447 and tests the standard UIPH regression assuming both stationary data and cointegrated data to ensure that the results are not sensitive to the persistence in the data over the sample period.[14] The UIPH regression allowing for cointegrated data uses the

[14] This chapter does not consider mixed integration, i.e where exchange rate changes are I(0) and the interest rate differential data is highly persistent or indistinguishable from I(1). While Maynard and Phillips (1998) shows that mixed integration does have the potential to distort critical values in finite samples, Liu and Maynard (2005) finds in practice that the effect is relatively modest compared to the standard FRUH regression (and the FRUH is still rejected).

Table 5.2: Cointegration tests on the 1-year UIPH data

Cointegration tests versus USD/CAD	Change in LN USD/CAD	ANS interest rate differential	RNVA ANS interest rate differential	Level component of ANS interest rate differential	Non-Level component of ANS interest rate differential
PP fixed window	n/a n/a	-2.98 ** 12	-2.98 ** 12	-3.09 ** 12	-2.91 ** 12
PP selected window	n/a n/a	-2.97 ** 11	-2.97 ** 11	-3.10 ** 11	-2.91 ** 11
ADF fixed lags	n/a n/a	-2.39 12	-2.39 12	-2.38 12	-2.32 12
ADF selected lags	n/a n/a	-2.39 12	-2.39 12	-2.38 12	-2.32 12

Note: The cointegration tests are unit root tests on the interest rate differential measures less the change in the exchange rate data for the 1-year horizon UIPH. PP is Phillips-Perron, ADF is augmented Dickey-Fuller, and the window width/number of lags is given below each statistic. ** denotes a 5 percent level of significance.

method of Stock and Watson (1993), which essentially results in the estimated equations being augmented with the leads and lags of changes in the interest rate differential.[15] For example, equation 5.3 becomes:

$$\Delta e_{t,m} = a_m + b_m \cdot m \left(R^{\mathrm{US}}_{t,m} - R^{\mathrm{CA}}_{t,m} \right) + \sum_{\tau=-m}^{m} \Delta \left[m \left(R^{\mathrm{US}}_{t-\tau,m} - R^{\mathrm{CA}}_{t-\tau,m} \right) \right] + v^{*}_{t,m} \tag{5.11}$$

Unfortunately, this augmentation rapidly reduces the degrees of freedom as the horizon being tested increases, and the implications are discussed in the following section in light of the empirical results.

Finally, note that all of the horizons tested are greater than the monthly frequency of the data, and so the order of moving-average serial correlation induced in all of the equations to be estimated will be the horizon in months less 1. Hence, the Newey and West (1987) method with a window of the horizon in months less 1 is used to correct the standard

[15]Following Hamilton (1994) pp. 608-613, the window width for the Stock and Watson (1993) method is determined by the correlation between residuals $v_{t,m}$ in the original regression and leads and lags in the changes in the right-hand side data. Because changes in the exchange rate data are lagged m years, the expected window of correlation is m (a result that was also confirmed empirically), and so the appropriate symmetric window width is $2m$.

errors of the regressions for the expected autocorrelation, and will at the same time correct for any heteroskedasticity that is a typical feature of exchange rate data.[16]

5.5 Results and discussion

Table 5.3 contains the results from estimating equation 5.4, for which data are available for all maturities. Assuming stationary data, the estimates of b_m are negative and significantly different from 1 for horizons up to two years, and are positive and insignificantly different from 1 for horizons from three to five years. This pattern by horizon is consistent with the results in the existing literature, as referenced earlier. The estimates of a_m are insignificantly different from zero for all horizons. This result is common to all of the subsequent estimations in this chapter, and so is not discussed again.[17]

Assuming cointegrated data, the results for the 3- and 6-month horizons confirm the results for the stationary versions of the regressions; i.e the estimates of b_m are negative and significantly different from 1. The remaining results confirm the pattern in the results assuming stationary data, except the estimates of b_m become positive from the 2-year horizon onward. Note that the R^2 statistics show increasing evidence of overfitting, which results from the degrees of freedom dropping rapidly with the increasing horizon.

The remainder of the analysis focusses on the UIPH tests for the 3-month, 6-month, and 1-year horizons, which unambiguously reproduce the typical puzzling result of negative estimates of b_m. Tables 5.4, 5.6, and 5.8 contain the results for the series of UIPH tests on the 3-month, 6-month, and 1-year horizons assuming stationary data, and tables 5.5, 5.7, and 5.9 contain the parallel estimations assuming cointegrated data.

[16]Examples noting heteroskedasticity in exchange rate data are Baillie and Bollerslev (1989) and Huisman, Koedijk, Kool and Palm (2002).

[17]This result probably arises from the use of government-risk interest rates in the analysis. This means that the interest rate differentials should not reflect any differences in default risk, which would be more of a factor when non-government interest rates are used.

Table 5.3: UIPH test results using ANS model interest rates

Estimates assuming stationary data							
	3 months	6 months	1 year	2 years	3 years	4 years	5 years
a_m	-0.51%	-0.42%	-0.50%	-0.17%	0.47%	0.44%	0.38%
s.e	1.13%	1.05%	1.10%	1.20%	1.43%	1.36%	1.03%
P(0)	0.652	0.687	0.652	0.889	0.743	0.746	0.712
b_m	-0.85	-0.81	-0.89	-0.63	0.25	0.71	0.99
s.e	0.39	0.40	0.58	0.87	0.99	0.96	0.70
P(0)	0.028	0.045	0.126	0.472	0.804	0.462	0.157
P(1)	0.000	0.000	0.001	0.063	0.446	0.763	0.992
R^2	0.021	0.031	0.050	0.026	0.004	0.043	0.102
DF	247	244	238	226	214	202	190
Estimates allowing for cointegrated data							
a_m	0.06%	0.06%	0.36%	0.83%	1.92%	3.62%	2.53%
s.e	1.13%	1.20%	1.32%	1.80%	0.95%	0.19%	1.96%
P(0)	0.960	0.961	0.787	0.647	0.045	0.000	0.199
b_m	-0.45	-0.49	-0.24	0.73	3.49	5.84	4.92
s.e	0.42	0.51	0.76	1.35	0.87	0.29	1.63
P(0)	0.280	0.333	0.752	0.591	0.000	0.000	0.003
P(1)	0.001	0.004	0.103	0.840	0.005	0.000	0.017
R^2	0.099	0.079	0.157	0.391	0.716	0.951	0.996
DF	236	224	200	152	104	56	8

Note: P(0) and P(1) are p-values for the respective hypotheses that the parameters equal 0 or 1. The UIPH (i.e $b_m = 1$) is rejected at the 5 percent level of significance for short horizons, but is not rejected for longer horizons. Note that the estimates allowing for cointegrated data show increasing evidence of overfitting beyond the 1-year horizon.

Table 5.4: Stationary UIPH test results for the 3-month horizon

	Actual	ANS	RNVA	L+SB+R	L+SBR	L	SB+R	SBR
a_m	-0.53%	-0.51%	-0.53%	0.73%	0.81%	1.05%	0.71%	0.81%
s.e	1.14%	1.13%	1.14%	2.34%	2.26%	2.29%	1.03%	0.94%
P(0)	0.639	0.652	0.639	0.755	0.719	0.648	0.491	0.391
b_m	-0.85	-0.85	-0.85					
s.e	0.37	0.39	0.37					
P(0)	0.022	0.028	0.022					
P(1)	0.000	0.000	0.000					
w_m				0.01	0.00	0.23		
s.e				1.07	1.07	1.09		
P(0)				0.993	0.999	0.834		
P(1)				0.356	0.349	0.479		
x_m					-0.97			-0.97
s.e					0.40			0.40
P(0)					0.016			0.017
P(1)					0.000			0.000
y_m				-0.85			-0.85	
s.e				0.49			0.50	
P(0)				0.087			0.090	
P(1)				0.000			0.000	
z_m				-3.47			-3.47	
s.e				5.44			5.43	
P(0)				0.524			0.523	
P(1)				0.412			0.411	
R^2	0.023	0.021	0.023	0.028	0.027	0.000	0.028	0.027
DF	247	247	247	245	246	247	246	247

Note: UIPH test results for the 3-month horizon assuming stationary data. P(0) and P(1) are p-values for the respective hypotheses that the parameters equal 0 or 1. L is the ANS Level component, SB is the ANS Slope plus Bow component, R is the yield residual component, and SBR is the Slope plus Bow plus yield residual component. The UIPH (i.e $b_m = 1$) is strongly rejected, which is attributable to the Slope plus Bow component of interest rates (i.e $y_m << 1$).

Table 5.5: Cointegrated UIPH test results for the 3-month horizon

	Actual	ANS	RNVA	L+SB+R	L+SBR	L	SB+R	SBR
a_m	0.00%	0.06%	0.00%	2.80%	2.97%	2.21%	0.48%	0.77%
s.e	1.14%	1.13%	1.14%	2.48%	2.41%	2.41%	1.25%	0.97%
P(0)	0.997	0.960	0.997	0.261	0.220	0.360	0.700	0.424
b_m	-0.47	-0.45	-0.47					
s.e	0.39	0.42	0.39					
P(0)	0.230	0.280	0.230					
P(1)	0.000	0.001	0.000					
w_m				1.49	1.38	0.93		
s.e				1.18	1.19	1.17		
P(0)				0.210	0.248	0.429		
P(1)				0.680	0.749	0.953		
x_m					-0.83			-0.63
s.e					0.44			0.40
P(0)					0.056			0.120
P(1)					0.000			0.000
y_m				-0.51			-0.29	
s.e				0.68			0.68	
P(0)				0.459			0.672	
P(1)				0.028			0.059	
z_m				-8.76			-7.44	
s.e				9.31			9.22	
P(0)				0.348			0.420	
P(1)				0.295			0.361	
R^2	0.101	0.099	0.101	0.142	0.124	0.015	0.094	0.081
DF	236	236	236	220	228	236	228	236

Note: UIPH test results for the 3-month horizon assuming cointegrated data. The notation and results are as for table 5.4.

Note firstly that for the 3- and 6-month horizons, the estimates of b_m are immaterially different whether market-quoted or ANS zero-coupon interest rates are used. Similarly, whenever the yield residuals for the 3- and 6-month maturities are included as a separate explanatory variable, the estimated coefficients z_m are statistically insignificant. Hence, even if market-quoted zero-coupon data were available for horizons from one year and beyond for the UIPH estimations in table 5.3, it is unlikely that the empirical results would be materially different from the results based on the ANS interest rate data. Secondly, note that the results using the RNVA interest rates are immaterially different from the results based on market-quoted or ANS zero-coupon interest rates. Indeed, for any given maturity m, the function $\frac{\sigma_1 \rho_1 m}{2} - \sum_{n=1}^{3} \sigma_n^2 \cdot u_n(m)$ is time invariant, and so the adjustment of the data to RNVA interest rates only affects the estimate of a_m.[18]

The UIPH estimations using the individual components for the RNVA interest rates show that the coefficients w_m for the Level component of interest rates are positive and insignificantly different from 1, and the coefficients y_m for the Slope plus Bow component of interest rates are negative and significantly different from 1. These results suggest that the negative estimates of b_m are due to the influence of the Slope plus Bow component of interest rates, i.e the cyclical component of interest rates when the ANS model is related back to a generic general-equilibrium economy.

The remaining tests of the UIPH use the Level and non-Level components of interest rates independently. Hence, using just the Level component and omitting the non-Level component from the UIPH estimation effectively filters out the cyclical components

[18] As an aside, the immaterial differences in the estimates of a_m after the adjustment to RNVA interest rates confirms the result in Wu (2005*a*) that time-varying interest rate risk alone cannot plausibly explain the deviation of the data from UIPH. More specifically, an inspection of figure 5.7 shows that time-varying term premia would at times have to account for persistent prediction errors of $\pm 2\%$ in the 3-month horizon, and $\pm 5\%$ in the 1-year horizon. Noting that $\frac{\sigma_1 \rho_1}{2} = 0.0094\%$ for Canada and 0.0060% for the US, and assuming volatility σ_1 does not vary by several orders magnitude across time (the data in figures 5.4 to 5.6 do not suggest otherwise), then the market price of risk ρ_1 would have to vary by a minimum of around 1000 times around its long-term average (i.e $10\% \div 0.0094\% = 1066$ for the 1-year horizon, and $4\% \div 0.0094\% \div 4 = 1705$ for the 3-month horizon).

Table 5.6: Stationary UIPH test results for the 6-month horizon

	Actual	ANS	RNVA	L+SB+R	L+SBR	L	SB+R	SBR
a_m	-0.48%	-0.42%	-0.48%	1.64%	1.10%	1.48%	1.36%	0.65%
s.e	1.15%	1.05%	1.15%	2.59%	2.22%	2.26%	1.82%	0.95%
P(0)	0.679	0.687	0.678	0.528	0.621	0.513	0.457	0.491
b_m	-0.72	-0.81	-0.72					
s.e	0.37	0.40	0.37					
P(0)	0.049	0.045	0.049					
P(1)	0.000	0.000	0.000					
w_m				0.19	0.28	0.52		
s.e				0.98	1.02	1.02		
P(0)				0.847	0.781	0.614		
P(1)				0.409	0.483	0.637		
x_m					-0.86			-0.88
s.e					0.37			0.36
P(0)					0.021			0.017
P(1)					0.000			0.000
y_m				-1.09			-1.10	
s.e				0.56			0.56	
P(0)				0.054			0.050	
P(1)				0.000			0.000	
z_m				1.81			1.86	
s.e				4.06			4.08	
P(0)				0.655			0.649	
P(1)				0.841			0.834	
R^2	0.028	0.031	0.028	0.046	0.038	0.002	0.046	0.038
DF	244	244	244	242	243	244	243	244

Note: UIPH test results for the 6-month horizon assuming stationary data. The notation and results are as for table 5.4.

Table 5.7: Cointegrated UIPH test results for the 6-month horizon

	Actual	ANS	RNVA	L+SB+R	L+SBR	L	SB+R	SBR
a_m	-0.03%	0.06%	-0.03%	4.00%	4.08%	3.11%	1.21%	0.72%
s.e	1.30%	1.20%	1.30%	2.90%	2.75%	2.72%	2.50%	1.01%
P(0)	0.981	0.961	0.981	0.170	0.140	0.254	0.630	0.477
b_m	-0.48	-0.49	-0.48					
s.e	0.22	0.16	0.22					
P(0)	0.032	0.003	0.032					
P(1)	0.000	0.000	0.000					
w_m				2.25	2.14	1.50		
s.e				1.51	1.40	1.31		
P(0)				0.138	0.127	0.251		
P(1)				0.410	0.415	0.700		
x_m					-0.96			-0.84
s.e					0.56			0.42
P(0)					0.088			0.048
P(1)					0.001			0.000
y_m				-0.97			-1.10	
s.e				0.86			0.82	
P(0)				0.261			0.182	
P(1)				0.023			0.011	
z_m				-2.60			0.35	
s.e				6.84			6.36	
P(0)				0.704			0.956	
P(1)				0.599			0.919	
R^2	0.070	0.079	0.070	0.164	0.130	0.036	0.108	0.074
DF	224	224	224	196	210	224	210	224

Note: UIPH test results for the 6-month horizon assuming cointegrated data. The notation and results are as for table 5.4.

of the original interest rates before applying the UIPH regression. Similarly, using just the non-Level component and omitting the Level component effectively filters out the steady-state or fundamental components of the original interest rates before applying the UIPH regression.

Table 5.8: Stationary UIPH test results for the 1-year horizon

	ANS	RNVA	L+SB	L	SB
a_m	-0.50%	-0.51%	1.73%	1.98%	0.99%
s.e	1.10%	1.10%	2.17%	2.35%	1.18%
P(0)	0.652	0.647	0.427	0.401	0.403
b_m	-0.89	-0.89			
s.e	0.58	0.58			
P(0)	0.126	0.126			
P(1)	0.001	0.001			
w_m			0.48	0.88	
s.e			1.02	1.02	
P(0)			0.637	0.389	
P(1)			0.613	0.905	
x_m			-1.15		-1.19
s.e			0.58		0.56
P(0)			0.049		0.033
P(1)			0.000		0.000
R^2	0.050	0.050	0.087	0.012	0.084
DF	238	238	237	238	238

Note: UIPH test results for the 1-year horizon assuming stationary data. P(0) and P(1) respectively represent tests that the parameters equals 0 or 1. L is the ANS Level component, SB is the ANS Slope plus Bow component. The UIPH (i.e $b_m = 1$) is strongly rejected, which is attributable to the Slope plus Bow component of interest rates (i.e $x_m << 1$).

The UIPH estimations using the independent Level component of the RNVA interest rates show that the coefficients w_m are positive and insignificantly different from 1. In other words, the UIPH is not rejected when the cyclical component of interest rates is excluded from the UIPH regression.

Conversely, the UIPH estimations using the independent non-Level component of the RNVA interest rates show that the coefficients x_m are all negative and significantly different from 1. In other words, the cyclical component of interest rates appears to be responsible for the negative coefficients obtained in the UIPH regressions, which suggests that the rejection of the UIPH over short horizons may in part be due to rationally-based cyclical dynamics in interest rates.

Table 5.9: Cointegrated UIPH test results for the 1-year horizon

	ANS	RNVA	L+SB	L	SB
a_m	0.36%	0.35%	5.71%	5.28%	1.04%
s.e	1.32%	1.32%	3.76%	3.50%	1.35%
P(0)	0.787	0.789	0.130	0.133	0.440
b_m	-0.24	-0.24			
s.e	0.76	0.76			
P(0)	0.752	0.752			
P(1)	0.103	0.103			
w_m			3.07	2.89	
s.e			2.01	1.78	
P(0)			0.128	0.105	
P(1)			0.305	0.289	
x_m			-0.73		-1.00
s.e			1.02		0.74
P(0)			0.475		0.175
P(1)			0.091		0.007
R^2	0.157	0.157	0.276	0.187	0.128
DF	200	200	174	200	200

Note: UIPH test results for the 1–year horizon assuming cointegrated data. The notation and results are as for table 5.8.

5.6 Conclusion

This chapter applies the ANS model of the yield curve to investigate the UIPH. After decomposing the interest rate data used in the UIPH regressions into components that reflect rationally-based expectations of the cyclical and fundamental components of the underlying economy, it is found that the UIPH is not rejected based on the fundamental components of interest rates, but is soundly rejected based on the cyclical components. These results provide empirical support for suggestions in the theoretical models of McCullum (1994), Lim and Ogaki (2003) and Meredith and Chinn (2004) that rationally-based interest rate and exchange rate dynamics associated with cyclical interlinkages between the economy and financial markets may contribute materially to the UIPH puzzle.

Chapter 6

Conclusion, and ideas for further work

The purpose of this concluding chapter is to provide an overview of the main themes of the thesis, and then to discuss ideas for potential extensions to the work within this thesis. Because section 1.2 of the introductory chapter and the individual chapters themselves have already provided overviews and conclusions, section 6.1 is kept very brief.

6.1 Conclusion

This thesis has developed the ANS model of the yield curve, an intertemporally-consistent and arbitrage-free version of the NS model, and has also given the ANS model a rigorous economic foundation. The ANS model should prove to be a practically useful tool for practitioners, researchers, and academics, and its theoretical and economic consistency should allow it to be applied quite generally to exercises in finance and economics. This is illustrated by the applications of the ANS model in this thesis to four distinct topics spanning finance and economics. In particular, the applications show that the ANS model allows the derivation of theoretical frameworks that capture the essence of the topics under investigation, and the parsimony of those theoretical frameworks means that they are readily applicable in practice.

6.2 Potential extensions of the thesis

The potential extensions of this thesis are grouped into two sections: section 6.2.1 discusses extensions and variations of the ANS model itself that might prove appropriate for particular applications. Section 6.2.2 discusses potential applications of the ANS model, including extensions of the applications in this thesis.

6.2.1 Potential extensions and variations of the ANS model

The first potential extension of the ANS model is to increase the number of components (i.e the number of coefficients and modes) that are used to represent the forward rate and interest rate curve at any point in time. Such extensions would give more flexibility by maturity than the three-component ANS model allows, and would therefore give a better fit to observations of yield curve data. This may be an important aspect in some applications where very close correspondences between market-quoted and model-estimated prices are required or desired (e.g for pricing options). In the extreme, if the number of components in the extended model equals the number of securities that define the yield curve, then the model will provide a precise fit.

As noted in chapter 2, appendix A details how the NS model may be arbitrarily extended using the sequence of orthonormalised Laguerre polynomials (OLPs). Then, analogous to the transformation of the NS model into the ANS model in chapter 2, appendix A also shows that any OLP model can be transformed into an intertemporally-consistent and arbitrage-free model of the yield curve, generically called the volatility-adjusted OLP

(VAO) model. Appendix A.5 also shows that the economic foundation provided for the ANS model in chapter 3 continues to apply to the VAO model of any order. Indeed, the addition of each coefficient and mode corresponds to an additional term in the Taylor expansion of the generic general-equilibrium economy model detailed in chapter 3.

The ANS model and its generalisation to the VAO model in appendix A assumes that the variance-covariance matrix of the innovations is constant and diagonal, and that the market prices of risk are constant. These assumptions could be relaxed, although at the usual trade-off between precision and parsimony. For example, by ease of accommodation: (1) the HJM framework would continue to give analytical functions of maturity for the forward rate curve with deterministic volatilities and/or market prices of risk (examples of such functions are contained in section A.4); (2) a constant non-diagonal innovation variance-covariance matrix could be allowed for by the appropriate rotation of the $g_n(\phi, m)$ modes noted in section 2 (further details are provided in section A.4); (3) stochastic volatilities within the HJM framework could be allowed for using the approach of Valchev (2004); and (4) jump dynamics, which have recently become increasingly popular in the literature (e.g see Duffie, Pan and Singleton (2000)), could be allowed for within the HJM framework using the approach of Bayraktar et al. (2005).

6.2.2 Potential applications of the ANS model and extensions of the applications in this thesis

The ANS model and its generalisation to the VAO model are intertemporally-consistent and arbitrage-free. Hence, following the approach of HJM and Brenner and Jarrow (1993), analytical expressions for pricing options on discount bonds could be derived directly from the ANS or VAO models. That said, the assumption of Gaussian dynamics in the ANS/VAO coefficients allows the non-zero probability of negative interest rates, which might not be acceptable in some applications. Specifying square-root stochastic processes (i.e CIR dynamics) for the ANS/VAO coefficients would preclude negative interest rates, but such dynamics would then be inconsistent with the underlying generic general-equilibrium-economy model, as discussed in section 3.2.1 of chapter 3.

Alternatively, it would be possible to use the theoretical and economic consistency of the ANS/VAO model to price options on interest rates in a way that precludes the probability of negative interest rates and is consistent with a general-equilibrium-economy model. That is, the zero-bound on interest rates means that at any point in time the expected path of the short rate should not adopt negative values for any maturity. In principle, this problem condenses to modelling the coefficients of the ANS/VAO model (e.g by "trees" or by Monte Carlo simulation), and ruling out combinations of those coefficients that would result in negative values of the expected path of the short rate. Although such a framework would not result in an analytical solution, it would be consistent with economic theory; indeed, this approach would effectively treat interest rates as real options within an economic setting, as in the discussion of Black (1995).

The economic foundation for the ANS/VAO model allows it to be used for many applications within the field of macro-finance. One obvious practical application that follows directly from the application in chapter 3 is to use the ANS framework to extract implied market expectations of inflation and output growth directly from the yield curve, and to track changes in those expectations over time (particularly to gauge the response to economic and financial events such as data releases or monetary policy decisions). Central banks could use the information extracted via the ANS economic framework as inputs into its own economic assessments, and its formulation, implementation, and communication of monetary policy with respect to its policy targets. For example, a rise in implied steady-state inflation to above the stated inflation target might add to a case for tightening monetary policy.

That said, time-varying term premia would also need to be considered as an influence on the yield curve. Indeed, the investigation of the ANS Level coefficient versus steady-state nominal output growth in chapter 3 provides some preliminary evidence of structural breaks in term premia. Formal tests such as those of Andrews (1993),[1] Andrews and Ploberger (1994), and Bai and Perron (1998) could be applied to the levels and/or changes in the time series of chapter 3 to identify the timing of term premia changes. The ANS economic framework also offers a convenient theoretical framework for linking volatility in inflation and output growth back to term premia within the yield curve. In principle, this link could potentially be exploited to decompose the term premia and their changes into market-determined prices and quantities of risks to inflation and output growth.

One direct extension of the empirical application in chapter 3 would be to estimate the derived inflation and output growth equations simultaneously. This would lead to greater efficiency in the econometric estimation, and from an economic perspective it may also offer some insights into how "shocks" to output growth and inflation have been interrelated with each other over history. Similarly, correlations between the Level and non-Level components of the yield curve at given points in time may offer insights on how the market perceived the relationship between output growth and inflation at that time. For example, a strong positive covariance between the Level and non-Level coefficients of the ANS model during particular periods might imply heightened market sensitivity that upside surprises on output growth would translate to higher inflation. These relationships would be empirically driven by the data, and could vary over time, rather than being pre-specified and constant as in the models of Ang and Piazzesi (2003) and Estrella (2003).

A straightforward extension of the model in chapter 3 would be to create an

[1] The table in Andrews (2003) contains some corrections to the orginal critical values.

ANS framework for just the real side of the economy (which would simply omit the ABE deflator/inflation state variables, and the associated steady-state variables and parameters). This model would then be applicable to inflation-indexed yield curve data. When applied simultaneously with the nominal ANS framework, the differences between the real and nominal ANS models would imply information about the expected path of inflation, and on the term premia associated with risks to inflation.

The economic foundation of the ANS model in conjunction with its intertemporal and no-arbitrage consistency may allow the yield curve to be used in valuing and hedging some of the macroeconomic derivatives that have been suggested by Shiller (1993) and Shiller (2003), and that have been provided to the market over recent years (e.g see Frankel and O'Neill (2002), Chicago Mercentile Exchange (2005), and Goldman Sachs (2005)).

An obvious application of the ANS portfolio framework developed in chapter 4 would be to optimally hedge bond index swaps. For example, at the onset of a bond index swap, just four physical securities or three derivative securities would be needed to replicate the market value of zero and the Level, Slope, and Bow exposures inherent in the bond index swap. The ANS portfolio framework can also be used in many ways for the management of fixed interest portfolios. For example, appendix C details how value-at risk can be measured within the ANS portfolio framework, in absolute terms, relative to an index, and/or in duration terms. The economic foundation of the ANS model also allows that risk to be interpreted in economic terms, or in reverse, the ANS economic and portfolio frameworks together provide a rigorous foundation for converting non-consensus macroeconomic views into optimal active positions within fixed interest portfolios. For example, a view that consensus inflation expectations are about to rise would best be expressed as an active exposure to an increase in the Level coefficient, while a view that monetary policy is about to tighten without a rise in inflation expectations would best be expressed as an active exposure to a fall in the Slope coefficient (i.e a flattening of the yield curve). Section C.3 in appendix C shows how those exact exposures can be obtained in conjunction with optimal security selection. Section C.4 also discusses how transaction costs could be incorporated into the ANS portfolio framework, although unfortunately this would come at the expense of the straightforward linear programming approach outlined in section 4.5.3 of chapter 4.

An obvious application of the work in chapter 5 would be to investigate the UIPH for other countries, and an extension would be to forecast the exchange rate out-of-sample using the current shape of the yield curve. It was also mentioned in chapter 5 that decomposing the UIPH exchange rate data into cyclical and fundamental components would provide the basis for a more exhaustive series of UIPH tests. Time-series filtering techniques may offer a means of decomposing the exchange rate data. For example, Baxter (1994) uses spectral methods to filter out the noise from high-frequency exchange rate data, leaving data that better reflect exchange rate dynamics at the business-cycle frequency (and using that data improves the correlation between interest rate differentials and expected exchange rate depreciation). However, the wide range of choices of filters (e.g Hodrick-Prescott, Kalman, etc.), and the choices of parameters for those filters would make this time-series filtering approach somewhat subjective.

The broader principle from chapter 5 is that the ANS model should be applicable to the consistent modelling of other financial market variables in connection with interest rates and the economy. For example, a standard method of valuing an equity is to discount the path of expected cashflows from that equity using the interest rates associated with each cashflow, and that principle extends to the aggregation of individual equities within an index. Under the assumption that the aggregate expected cashflows of an equity index are related to the expected evolution of the broad economy, the ANS model of the yield curve with its economic foundation provides a parsimonious framework for modelling those expected cashflows and the interest rates used to discount those cashflows. In other words, the ANS model would provide an internally-consistent framework for investigating the fundamental value of equity indices.

Bibliography

Abken, P. (1993), 'Inflation and the yield curve', *Federal Reserve Bank of Atlanta Economic Review* **78**, 13–31.

Aïssa, M. and Jouini, J. (2003), 'Structural breaks in the US inflation process', *Applied Economics Letters* **10**, 633–636.

Alexius, A. (2001), 'Uncovered interest parity revisited', *Review of International Economics* **9(3)**, 505–517.

Andrews, D. (1993), 'Tests for parameter instability and structural change with unknown change point', *Econometrica* **61(4)**, 821–856.

Andrews, D. (2003), 'Tests for parameter instability and structural change with unknown change point: a corrigendum', *Econometrica* **71(1)**, 395–397.

Andrews, D. and Ploberger, W. (1994), 'Optimal tests when a nuisance parameter is present only under the alternative', *Econometrica* **62(6)**, 1383–1414.

Ang, A. and Piazzesi, M. (2003), 'A no-arbitrage vector autoregression of term structure dynamics with latent variables', *Journal of Monetary Economics* **50**, 745–787.

Anton, H. (1984), *Elementary Linear Algebra, Fourth Edition*, Wiley and Sons.

Atta-Mensah, J. and Tkacz, G. (2001), 'Predicting recessions and booms using financial variables', *Canadian Business Economics* **8(3)**, 30–36.

Backus, D., Foresi, S. and Zin, S. (1998), 'Arbitrage opportunities in arbitrage-free models of bond pricing', *Journal of Business and Economic Statistics* **16**, 13–26.

Bai, J. and Perron, P. (1998), 'Estimating and testing linear models with multiple structural breaks', *Econometrica* **66(1)**, 47–78.

Baillie, R. and Bollerslev, T. (1989), 'The message in daily exchange rates: a conditional variance tale', *Journal of Business and Economic Statistics* **7**, 297–305.

Baillie, R. and Bollerslev, T. (2000), 'The forward premium anomaly is not as bad as you think', *Journal of International Money and Finance* **19**, 471–488.

Bank for International Settlements (1999), 'Zero-coupon yield curves: technical documentation', *BIS Papers* **March**.

Bank for International Settlements (2005), 'Zero-coupon yield curves: technical documentation', *BIS Papers* **25**.

Barber, J. and Copper, M. (1996), 'Immunisation using principal components analysis', *Journal of Portfolio Management* **Fall**, 99–105.

Barrett, W., Gosnell, T. and Heuson, A. (1995), 'Yield curve shifts and the selection of immunization strategies', *Journal of Portfolio Management* **Fall**, 53–64.

Baxter, M. (1994), 'Real exchange rates and interest rate differentials', *Journal of Monetary Economics* **33**, 5–37.

Bayraktar, E., Chen, L. and Poor, V. (2005), 'Consistency problems for jump-diffusion models', *Applied Mathematical Finance* **12(2)**, 101–119.

Berardi, A. and Esposito, M. (1999), 'A base model for multifactor specifications of the term structure', *Economic Notes* **28(2)**, 145–170.

Berk, J. (1998), 'The information content of the yield curve for monetary policy: a survey', *De Economist* **146**, 303–320.

Bernanke, B. (1990), 'On the predictive power of interest rates and interest rate spreads', *Federal Reserve Bank of Boston New England Economic Review* **November/December**, 51–68.

Bernard, H. and Gerlach, S. (1998), 'Does the term structure predict recessions? The international evidence', *International Journal of Finance and Economics* **3**, 195–215.

Björk, T. and Christensen, B. (1999), 'Interest rate dynamics and consistent forward rate curves', *Mathematical Finance* **9(4)**, 323–348.

Black, F. (1995), 'Interest rates as options', *Journal of Finance* **50**, 1371–1376.

Bliss, R. (1996), 'Testing term structure estimation methods', *Federal Reserve Bank of Atlanta Working Paper* **96-12a**.

Bliss, R. (1997), 'Testing term structure estimation methods', *Advances in Futures and Options Research* **9**, 197–231.

Bordo, M. and Haubrich, J. (2004), 'The yield curve, recessions, and the credibility of the monetary regime: long run evidence 1875-1997', *NBER Working Paper Series* **10431**.

Brace, A., Gatarek, D. and Musiela, M. (1997), 'The market model of interest rate dynamics', *Mathematical Finance* **7**, 127–155.

Brandt, M. and Yaron, A. (2002), 'Time-consistent no-arbitrage models of the term structure', *Working Paper, Wharton School, University of Pennsylvania* .

Brenner, R. and Jarrow, R. (1993), 'A simple formula for options on discount bonds', *Advances in Futures and Options Research* **6**, 45–51.

Brooks, R. and Yong Yan, D. (1999), 'London inter-bank offer rate (LIBOR) versus Treasury rate: evidence from the parsimonious term structure model', *Journal of Fixed Income* **2**, 71–83.

Brown, R. and Giurda, F. (2003), 'Euro-govvie relative value - overview', *UBS Investment Research* **3 September**.

Buraschi, A. and Jiltsov, A. (2005), 'Inflation risk premia and the expectations hypothesis', *Journal of Financial Economics* **75**, 429–490.

Chambers, D., Carleton, W. and McEnally, R. (1988), 'Immunizing default-free bond portfolios with a duration vector', *Journal of Financial and Quantitative Analysis* **23(1)**, 89–104.

Chen, N. (1991), 'Financial investment opportunities and the macroeconomy', *Journal of Finance* **46(2)**, 529–554.

Chicago Mercentile Exchange (2005), 'An introduction to CME economic derivatives'.
URL: *http://www.cme.com*

Congressional Budget Office (2001), 'CBO's method for estimating potential output: an update'.
URL: *http://www.cbo.gov*

Congressional Budget Office (2004), 'A summary of alternative methods for estimating potential GDP'.
URL: *http://www.cbo.gov*

Cornell, B. and Shapiro, A. (1989), 'The mispricing of U.S. Treasury bonds: a case study', *Review of Financial Studies* **2**, 297–310.

Courant, R. and Hilbert, D. (1953), *Methods of Mathematical Physics, First English Edition*, Interscience Publishers.

Cox, J., Ingersoll, J. and Ross, S. (1985*a*), 'An intertemporal general equilibrium model of asset prices', *Econometrica* **53**, 363–384.

Cox, J., Ingersoll, J. and Ross, S. (1985*b*), 'A theory of the term structure of interest rates', *Econometrica* **53**, 385–407.

Dahlquist, M. and Svensson, L. (1996), 'Estimating the term structure of interest rates for monetary policy analysis', *Scandinavian Journal of Economics* **98(2)**, 163–183.

Dai, Q. and Singleton, K. (2000), 'Specification analysis of affine term structure models', *Journal of Finance* **55(5)**, 1943–1978.

Davis, E. and Fagan, G. (1997), 'Are financial spreads useful indicators of future inflation and output growth in EU countries?', *Journal of Applied Econometrics* **12(6)**, 701–714.

Davis, E. and Henry, S. (1994), 'The use of financial spreads as indicator variables: evidence for the United Kingdom and Germany', *International Monetary Fund Staff Papers* **41(3)**, 517–525.

Delcoure, N., Barkoulas, J., Baum, C. and Chakraborty, A. (2003), 'The forward rate unbiasedness hypothesis reexamined: evidence from a new test', *Global Finance Journal* **14(1)**, 83–93.

Diaz, A. and Skinner, F. (2001), 'Estimating corporate yield curves', *Journal of Fixed Income* **3**, 95–103.

Diebold, F., Ji, L. and Li, C. (2005), 'A three-factor yield curve model: non-affine structure, systematic risk sources and generalised duration', *Working Paper, University of Pennsylvania* .

Diebold, F. and Li, C. (2006), 'Forecasting the term structure of government bond yields', *Journal of Econometrics* **130(2)**, 337–364.

Diebold, F. and Mariano, R. (1995), 'Comparing predictive accuracy', *Journal of Business and Economic Statistics* **13(3)**, 253–263.

Diebold, F., Rudebusch, G. and Aruoba, S. (2005), 'The macroeconomy and the yield curve: a dynamic latent factor approach', *Journal of Econometrics (forthcoming)* .

Donohue, C. and Yip, K. (2003), 'Optimal portfolio rebalancing with transactions costs', *Journal of Portfolio Management* **Summer**, 49–63.

Dueker, M. (1997), 'Strengthening the case for the yield curve as a predictor of US recessions', *Federal Reserve Bank of St. Louis Review* **79(2)**, 41–51.

Duffee, G. (2002), 'Term premia and interest rate forecasts in affine models', *Journal of Finance* **57(1)**, 405–443.

Duffie, D., Pan, J. and Singleton, K. (2000), 'Transform analysis and asset pricing for affine jump-diffusions', *Econometrica* **68(6)**, 1343–1376.

Durand, D. (1942), 'Basic yields of corporate bonds, 1900-1942', *Technical Paper, National Bureau of Economic Research* **3**.

Elton, E. and Gruber, M. (1995), *Modern Portfolio Theory and Investment Analysis*, Wiley and Sons.

Engle, C. (1996), 'The forward discount anomaly and the risk premium: a survey of recent evidence', *Journal of Empirical Finance* **3**, 123–192.

Estrella, A. (2003), 'Why does the yield curve predict output and inflation?', *Working Paper, Federal Reserve Bank of New York* .

Estrella, A. (2005), 'Why does the yield curve predict output and inflation?', *Economic Journal* **115**, 722–744.

Estrella, A. and Hardouvelis, G. (1991), 'The term structure as a predictor of real economic activity', *Journal of Finance* **46(2)**, 555–576.

Estrella, A. and Mishkin, F. (1997), 'The predictive power of the term structure of interest rates in Europe and the United States: implications for the European Central Bank', *European Economic Review* **41(7)**, 1375–1402.

Estrella, A. and Mishkin, F. (1998), 'Predicting U.S. recessions: financial variables as leading indicators', *Review of Economics and Statistics* **80(1)**, 45–61.

Estrella, A., Rodrigues, A. and Schich, S. (2003), 'How stable is the predictive power of the yield curve? Evidence from Germany and the United States', *Review of Economics and Statistics* **85(3)**, 629–644.

Fabozzi, F., Martellini, L. and Priaulet, P. (2005), 'Predictability in the shape of the term structure of interest rates', *Journal of Fixed Income* **2**, 40–53.

Fahmy, Y. and Kandi, M. (2003), 'The Fisher effect: new evidence and implications', *International Review of Economics and Finance* **12**, 451–465.

Fama, E. (1984), 'Forward and spot exchange rates', *Journal of Monetary Economics* **14**, 319–338.

Fama, E. and Bliss, R. (1987), 'The information in long-maturity forward rates', *American Economic Review* **77**, 680–692.

Fang, V. and Muljono, R. (2003), 'An empirical analysis of the Australian dollar swap spreads', *Pacific-Basin Finance Journal* **11**, 153–173.

Fergusson, R. and Raymar, S. (1998), 'A comparative analysis of several popular term structure estimation models', *Journal of Fixed Income* **1**, 17–32.

Filardo, A. (1999), 'How reliable are recession prediction models?', *Federal Reserve Bank of Kansas City Economic Review* **83**, 35–55.

Filipović, D. (1999), 'A note on the Nelson-Siegel family', *Mathematical Finance* **9(4)**, 349–359.

Filipović, D. (2000), 'Exponential-polynomial families and the term structure of interest rates', *Bernoulli Journal* **6(6)**, 1081–1107.

Fisher, I. (1907), *The Rate of Interest*, Macmillan Publishers.

Fisher, L. and Weill, R. (1971), 'Coping with the risk of interest rate fluctuations', *Journal of Business* **44**, 408–431.

Fleming, M. (2003), 'Measuring Treasury market liquidity', *Federal Reserve Bank of New York Bulletin* **9(3)**, 83–108.

Frankel, O. and O'Neill, J. (2002), 'Economic derivatives: what is their benefit?', *Derivatives Week* **11(38)**.

Froot, K. and Thaler, R. (1990), 'Anomalies: foreign exchange', *Journal of Economic Perspectives* **4**, 179–192.

Gamber, E. (1996), 'The policy content of the yield curve slope', *Review of Financial Economics* **5(2)**, 163–179.

Goldman Sachs (2005), 'Economic derivatives: options on economic statistics'.

URL: *http://www.gs.com*

Goldstein, R. and Keirstead, W. (1997), 'On the term structure of interest rates in the presence of reflecting and absorbing boundaries', *Working Paper Series, Ohio State University* .

Golub, B. and Tilman, L. (2000), *Risk Management Approaches for Fixed Income Markets*, Wiley and Sons.

Gordon, R. (1990), *Macroeconomics, Fifth Edition*, Harper Collins.

Grauwe, P. and Grimaldi, M. (2006), 'Exchange rate puzzles: a tale of switching attractors', *European Economic Review* **50**, 1–33.

Greene, W. (1997), *Econometric Analysis, Third Edition*, Prentice Hall.

Hamilton, J. (1994), *Time Series Analysis*, Princeton University Press.

Hamilton, J. and Kim, D. (2002), 'A re-examination of the predictability of economic activity using the yield spread', *Journal of Money, Credit, and Banking* **34(2)**, 340–360.

Hardouvelis, G. (1994), 'The term structure spread and future changes in long and short rates in G7 countries', *Journal of Monetary Economics* **33**, 255–283.

Haubrich, J. and Dombrosky, A. (1996), 'Predicting real growth using the yield curve', *Federal Reserve Bank of Cleveland Economic Review* **32(1)**, 26–35.

Heath, D., Jarrow, R. and Morton, A. (1992), 'Bond pricing and the term structure of interest rates: a new methodology for contingent claims valuation', *Econometrica* **60(1)**, 77–106.

Ho, T. (1992), 'Key rate durations: measures of interest rate risks', *Journal of Fixed Income* **3**, 29–44.

Ho, T. and Lee, S. (1986), 'Term structure movements and pricing interest rate contingent claims', *Journal of Finance* **41**, 1011–1029.

Hodrick, R. (1987), *The Empirical Evidence on the Efficiency of Forward and Futures Foreign Exchange Markets*, Harwood Academic.

Homer, S. (1963), *A History of Interest Rates: 2000 BC to the Present*, Rutgers University Press.

HSBC Bank (2001), 'Government fair value models', *HSBC Bank Bloomberg site* **18 October**.

HSBC Bank (2003), 'Gilt fair value model', *HSBC Bank Bloomberg site* **24 November**.

Huisman, R., Koedijk, K., Kool, C. and Palm, F. (2002), 'The tail-fatness of FX returns reconsidered', *De Economist* **150(3)**, 299–312.

Hull, J. (2000), *Options, Futures and Other Derivitives, Fourth Edition*, Prentice Hall.

Hull, J. and White, A. (1990), 'Pricing interest rate derivative securities', *Review of Financial Studies* **3(4)**, 573–592.

Hunt, B. (1995), 'Fitting parsimonious yield curve models to Australian coupon bond data', *Working Paper, School of Finance and Economics, University of Technology Sydney* **51**.

Ioannides, M. (2003), 'A comparison of yield curve estimation techniques using UK data', *Journal of Banking and Finance* **27**, 1–26.

James, J. and Webber, N. (2000), *Interest Rate Modelling*, Wiley and Sons.

Jankowitsch, R. and Pilcher, S. (2004), 'Parsimonious estimation of credit spreads', *Journal of Fixed Income* **4**, 49–63.

Jardet, C. (2004), 'Why did the term structure of interest rates lose its predictive power?', *Economic Modelling* **21**, 509–524.

Jorian, P. and Mishkin, F. (1991), 'A multi-country comparison of term structure forecasts at long horizons', *Journal of Financial Economics* **29(1)**, 59–80.

Kacala, V. (1993), 'Zero coupon curves part 3: fair bond yields', *Research Paper, Buttle Wilson* .

Kopprasch, R. (2004), 'A look at a variety of duration measures', *United States Fixed Income Research, Citibank* .

Kozicki, S. (1997), 'Predicting real growth and inflation with the yield spread', *Federal Reserve Bank of Kansas City Economic Review* **82**, 39–57.

Krippner, L. (2003*a*), 'Modelling the yield curve with orthonormalised Laguerre polynomials: a consistent cross-sectional and intertemporal approach', *Working Paper, University of Waikato* **03/02**.

Krippner, L. (2003*b*), 'Modelling the yield curve with orthonormalised Laguerre polynomials: an intertemporally consistent approach with an economic interpretation', *Working Paper, University of Waikato* **03/01**.

Krippner, L. (2005*a*), 'Attributing returns and optimising United States swaps portfolios using an intertemporally-consistent and arbitrage-free model of the yield curve', *Working Paper, University of Waikato* **05/03**.

Krippner, L. (2005*b*), 'An intertemporally-consistent and arbitrage-free version of the Nelson and Siegel class of yield curve models', *Working Paper, University of Waikato* **05/01**.

Krippner, L. (2005*c*), 'Investigating the relationships between the yield curve, output, and inflation using an arbitrage-free version of the Nelson and Siegel class of yield curve models', *Working Paper, University of Waikato* **05/02**.

Krippner, L. (2005*d*), 'A new framework for yield curve, output, and inflation relationships', *Working Paper, University of Waikato* **05/05**.

Krippner, L. (2006*a*), 'A theoretically consistent version of the Nelson and Siegel class of yield curve models', *Applied Mathematical Finance* **13(1)**, 39–59.

Krippner, L. (2006*b*), 'A yield curve perspective on uncovered interest parity', *Working Paper, University of Waikato* **06/10**.

Lai, K. (2004), 'On structural shifts and stationarity of the ex-ante real interest rate', *International Review of Economics and Finance* **13**, 217–228.

Lim, H. and Ogaki, M. (2003), 'A theory of the exchange rate and the term structure of interest rates', *Working Paper, University of Rochester* **504**.

Litterman, R. and Sheinkman, J. (1991), 'Common factors affecting bond returns', *Journal of Fixed Income* **2**, 54–61.

Liu, W. and Maynard, A. (2005), 'Testing forward rate unbiasedness allowing for persistent regressors', *Journal of Empirical Finance* **12**, 613–628.

Macauley, F. (1938), *Some Theoretical Problems Suggested by the Movements of Interest Rates, Bond Yields, and Stock Prices in the United States since 1865*, Columbia University Press.

MacKinnon, J. (1996), 'Numerical distribution functions for unit root and cointegration tests', *Journal of Applied Econometrics* **11(6)**, 601–618.

Malik, H., Barry, J. and Xiao, Q. (2003), 'US Treasury relative value and asset swap report', *JP Morgan internet website* **25 November**.

Mankiw, N. (2001), *Principles of Economics, Second Edition*, Harcourt College Publishers.

Mann, S. and Ramanlal, P. (1997), 'The relative performance of yield curve strategies', *Journal of Portfolio Management* **Summer**, 64–70.

Mansi, S. and Phillips, J. (2001), 'Modelling the term structure from the on-the-run Treasury yield curve', *Journal of Financial Research* **24(4)**, 545–564.

Mark, N. and Wu, Y. (1996), 'Risk, policy rules, and noise: rethinking deviations from uncovered interest parity', *Working Paper, Ohio State University* **14**.

Mark, N. and Wu, Y. (2004), 'Official interventions and occasional violations of uncovered interest parity in the dollar-DM market', *Working Paper, University of Notre Dame* .

Markowitz, H. (1959), *Portfolio Selection: Efficient Diversification of Investments*, Wiley and Sons.

Maynard, A. and Phillips, P. (1998), 'Rethinking an old empirical puzzle: econometric evidence on the forward discount anomaly', *Journal of Applied Econometrics* **16(6)**, 671–708.

McCulloch, J. (1971), 'Measuring the term structure of interest rates', *Journal of Business* **44**, 19–31.

McCullum, B. (1994), 'A reconsideration of the uncovered interest parity relationship', *Journal of Monetary Economics* **23**, 105–132.

Meredith, G. and Chinn, M. (2004), 'Monetary policy and long-horizon uncovered interest parity', *International Monetary Fund Staff Papers* **51(3)**.

Mishkin, F. (1990*a*), 'The information in the longer maturity term structure about inflation', *Quarterly Journal of Economics* **105(3)**, 815–828.

Mishkin, F. (1990*b*), 'What does the term structure tell us about future inflation', *Journal of Monetary Economics* **25**, 72–95.

Mishkin, F. (1991), 'A multi-country study of the information in the shorter maturity term structure about future inflation', *Journal of International Money and Finance* **10**, 2–22.

Monetary Authority of Singapore (1999), 'Extracting market expectations of future interest rates from the yield curve: an application using Singapore interbank and interest rate swap data', *Occasional Paper, Economics Department, Monetary Authority of Singapore* **17**.

Murty, K. (1983), *Linear Programming*, Wiley and Sons.

Nakaota, H. (2005), 'The term structure of interest rates in Japan: the predictability of economic activity', *Japan and the World Economy* **17**, 311–326.

Nawalkha, S., Soto, G. and Zhang, J. (2003), 'Generalised M-vector models for hedging interest rate risk', *Journal of Banking and Finance* **27**, 1581–1604.

Nelson, C. and Siegel, A. (1987), 'Parsimonious modelling of yield curves', *Journal of Business* **October**, 473–489.

Newey, W. and West, K. (1987), 'A simple, positive semi-definite, heteroskedasticity and autocorrelation consistent covariance matrix', *Econometrica* **55**, 703–708.

Paya, I., Matthews, K. and Peel, D. (2005), 'The term spread and real economic activity in the US inter-war periods', *Journal of Macroeconomics* **27(2)**, 331–343.

Pham, T. (1998), 'Estimation of the term structure of interest rates: an international perspective', *Journal of Multinational Financial Management* **8**, 265–283.

Plosser, C. and Rouwenhorst, K. (1994), 'International term structure and real economic growth', *Journal of Monetary Economics* **33**, 133–155.

Rainville, E. and Bedient, P. (1981), *Elementary Differential Equations, Sixth Edition*, Collier Macmillan.

Razzak, W. (2002), 'The forward rate unbiasedness hypothesis revisited', *International Journal of Finance and Economics* **7**, 293–308.

Reitano, R. (1990), 'Non-parallel yield curve shifts and durational leverage', *Journal of Portfolio Management* **Summer**, 62–67.

Reitano, R. (1991), 'Non-parallel yield curve shifts and spread leverage', *Journal of Portfolio Management* **Spring**, 82–87.

Reitano, R. (1992), 'Non-parallel yield curve shifts and immunisation', *Journal of Portfolio Management* **Spring**, 36–43.

Reitano, R. (1996), 'Non-parallel yield curve shifts and stochastic immunisation', *Journal of Portfolio Management* **Winter**, 71–105.

Rendu de Lint, C. and Stolin, D. (2003), 'The predictive power of the yield curve: a theoretical assessment', *Journal of Monetary Economics* **50**, 1603–1622.

Ronn, E. (1987), 'A new linear programming approach to bond portfolio management', *Journal of Financial and Quantitative Analysis* **22(4)**, 439–466.

Ross, S. (1997), *Introduction to Probability Models, Sixth Edition*, Academic Press.

Sarno, L. (2005), 'Viewpoint: towards a solution to the puzzles in exchange rate economics: where do we stand?', *Canadian Journal of Economics* **38(3)**, 673–708.

Sarno, L. and Taylor, M. (2003), *The Economics of Exchange Rates*, Cambridge University Press.

Schich, S. (1999), 'The information content of the German term structure regarding inflation', *Applied Financial Economics* **9**, 385–395.

Schich, S. (2000), 'What the yield curves say about inflation: does it change over time?', *Working Paper, OECD Economics Department* **227**.

Schmidt, B. and Kalemanova, A. (2002), 'Applying a three-factor defaultable term structure model to the pricing of credit default options', *International Review of Financial Analysis* **11**, 139–158.

Seppala, J. and Viertio, P. (1996), 'The term structure of interest rates: estimation and interpretation', *Discussion Paper, Bank of Finland* **19**.

Sercu, P. and Wu, X. (1997), 'The information content in bond model residuals: an empirical study on the Belgian bond market', *Journal of Banking and Finance* **21**, 685–720.

Shiller, R. (1993), *Macro Markets: Creating Institutions for Managing Society's Largest Economic Risks*, Oxford University Press.

Shiller, R. (2003), *The New Financial Order: Risk in the 21st Century*, Princeton University Press.

Söderlind, P. and Svensson, L. (1997), 'New techniques to extract market expectations from financial instruments', *Journal of Monetary Economics* **40**, 383–429.

Soto, G. (2001), 'Immunization derived from a polynomial duration vector in the Spanish bond market', *Journal of Banking and Finance* **25**, 1037–1057.

Steeley, J. (2004), 'Estimating time-varying risk premia in UK long-term government bonds', *Applied Financial Economics* **14**, 367–373.

Stock, J. and Watson, M. (1993), 'A simple estimator of cointegrating vectors in higher order integrated systems', *Econometrica* **61**, 783–820.

Stock, J. and Watson, M. (2001), 'Forecasting output and inflation: the role of asset prices', *Working Paper, National Bureau of Economic Research* **8180**.

Stock, J. and Watson, M. (2002), 'Macroeconomic forecasting using diffusion indexes', *Journal of Business and Economic Statistics* **20(2)**, 147–162.

Subramanian, K. (2001), 'Term structure estimation in illiquid markets', *Journal of Fixed Income* **2**, 77–86.

Svensson, L. (1994), 'Estimating and interpreting forward interest rates: Sweden 1992-4', *Discussion Paper, Centre for Economic Policy Research* **1051**.

Thomas, G. and Finney, R. (1984), *Calculus and Analytic Geometry, Sixth Edition*, Addison Wesley.

Valchev, O. (2004), 'Stochastic volatility Gaussian Heath-Jarrow-Morton models', *Applied Mathematical Finance* **11**, 347–368.

Vasicek, O. (1977), 'An equilibrium characterisation of the term structure', *Journal of Financial Economics* **5**, 177–188.

Walsh, C. (1998), *Monetary Theory and Policy*, MIT Press.

Willner, R. (1996), 'A new tool for portfolio managers: level, slope, and curvature durations', *Journal of Fixed Income* **2**, 48–59.

Wolfram, S. (1996), *The Mathematica Book, Third Edition*, Cambridge University Press.

Wu, S. (2005*a*), 'Interest rate risk and the forward premium anomaly in foreign exchange markets', *Working Paper, University of Kansas* .

Wu, T. (2005*b*), 'Macro factors and the affine term structure of interest rates', *Journal of Money, Credit and Banking (forthcoming)* .

Zhou, S. and Kutan, A. (2005), 'Does the forward premium anomaly depend on the sample period used or the sign of the premium?', *International Review of Economics and Finance* **14**, 17–25.

Appendix A

The orthonormalised Laguerre polynomial model of the yield curve

The primary purpose of this appendix is to show how the NS and ANS models of the yield curve may be extended using the sequence of orthonormalised Laguerre polynomial (OLP) functions. Such extensions might be useful for obtaining a more precise fit to the observed yield curve data than can be obtained using just three coefficients and modes of the NS and ANS models of the yield curve. The appendix also shows how some of the assumptions underlying the ANS model or its extensions could be relaxed if that was required for particular applications.

Section A.1 introduces the OLP functions, and section A.2 shows how the NS model can be arbitrarily extended using those OLPs. Section A.3 then shows how any OLP model of the yield curve can be made intertemporally-consistent and arbitrage-free using the HJM framework to create the volatility-adjusted OLP (VAO) model of the yield curve. Section A.4 discusses how some of the assumptions underlying the parameters in the

ANS/VAO model may be relaxed. Finally, section A.5 shows that the VAO model of any order retains the economic foundation conferred by the ABE model developed in chapter 3.

A.1 Laguerre and orthonormalised Laguerre polynomials

As noted in standard texts, such as Courant and Hilbert (1953) and Rainville and Bedient (1981), Laguerre polynomials are of the form:

$$L_n(x) = \sum_{k=0}^{n} \frac{(-1)^k n! x^k}{(k!)^2 (n-k)!} \tag{A.1}$$

where n and k are integers. Laguerre polynomials do not by themselves form an orthonormal set, but the related set of functions $\exp(-x/2) \cdot L_n(x)$ are orthonormal for the interval $0 \leq x < \infty$. That is:

$$\int_0^\infty e^{-x} L_i(x) L_j(x)\, dx = \delta_{i,j} \tag{A.2}$$

Substituting $x = 2\phi m$ into $\exp(-x/2) \cdot L_n(x)$ and negating, i.e:

$$g_n(\phi, m) = -\exp(-\phi m) \cdot \sum_{k=0}^{n-2} \frac{(-1)^k (n-2)!(2\phi m)^k}{(k!)^2 (n-2-k)!} \tag{A.3}$$

gives a series of orthonormalised Laguerre polynomials (hereafter OLPs) for $n \geq 2$, where ϕ is a fixed positive constant that governs the rate of exponential decay.

A.2 The generic OLP model of the yield curve

The series of OLPs in equation A.3 can be used to arbitrarily extend the NS model of the forward rate curve as follows:

$$f(t, m) = \beta_1(t) + \sum_{n=2}^{N} \beta_n(t) \cdot g_n(\phi, m) \tag{A.4}$$

where $\beta_1(t)$ is the coefficient for a constant function with value 1; β_n are the coefficients of the OLPs; and N is the number of components in the OLP model (i.e the number of coefficients and modes, which is also termed the order of the OLP model hereafter). Note that direct evaluation of equation A.3 at the lower boundary of $m = 0$ gives $g_n(\phi, 0) = -1$, and the limiting value is $\lim_{m\to\infty} g_n(\phi, m) = 0$ due to the dominance of the exponential term over the polynomial terms. In addition, OLPs are continuous, as are their differentials and integrals, which therefore ensures that the functions $g_n(\phi, m)$ will be well-behaved over $0 \leq m < \infty$. These properties will also apply to any linear combination of OLPs, which ensures that equation A.4 will be well-behaved, and will always converge asymptotically to the value of the constant $\beta_1(t)$.

The interest rate curve is defined directly from the forward rate curve as:

$$
\begin{aligned}
R(t,m) &= \frac{1}{m}\int_0^m f(t,m)\,dm && \text{(A.5a)}\\
&= \sum_{n=1}^{N} \beta_n\,(t)\cdot\frac{1}{m}\int_0^m g_n(\phi,m)\,dm && \text{(A.5b)}\\
&= \sum_{n=1}^{N} \beta_n\,(t)\cdot s_n(\phi,m) && \text{(A.5c)}
\end{aligned}
$$

Hence, the coefficients $\beta_n(t)$ for the interest rate modes are the same coefficients as for the forward rate modes. The integrals of $g_n(\phi, m)$ will also be of exponential-polynomial form, and so will be well-behaved with the limiting value being $\lim_{m\to\infty} s_n(\phi, m) = 0$. It can also be confirmed that the interest rate modes are properly defined at the lower boundary of $m = 0$. That is, the first form of L'Hôpital's rule, e.g from Thomas and Finney (1984) pp. 231-235, is: if $y(0) = z(0) = 0$, $y'(m)$ and $z'(m)$ exist, and $z'(m) \neq 0$ then $\lim_{m\to 0}\frac{y(m)}{z(m)} = \frac{y'(0)}{z'(0)}$. Hence, expressing $y(m) = \int_0^m g_n(\phi, m)\,dm$ and $z(m) = m$, $y(0) =$

$z(0) = 0$, and so:

$$\lim_{m \to 0} s_n(\phi, m) = \frac{y'(0)}{z'(0)} \tag{A.6a}$$

$$= \frac{\frac{d\left[\int_0^m g_n(\phi,m)dm\right]}{dm}}{\frac{dm}{dm}} \tag{A.6b}$$

$$= g_n(\phi, m)]_{m=0} \tag{A.6c}$$

$$= -1 \tag{A.6d}$$

Discount factors are defined as $\exp[-m \cdot R(t,m)]$, which will be a well-behaved function over $0 \leq m < \infty$. Direct evaluation at the lower boundary of $m = 0$ gives $\exp[-0 \cdot R(t,m)] = 1$. The limiting value is $\lim_{m \to \infty} \exp[-m \cdot R(t,m)]$, which is equivalent to $\lim_{m \to \infty} \exp[-m \cdot \beta_1(t)] = 0$.

Figure A.1 summarises some prior empirical results from applying the OLP model with N ranging from 1 to 5 to daily New Zealand yield curve data over the period July 1997 to February 2002. This illustrates that the addition of each mode improves the fit of the OLP model to the observed data. That said, the marginal effects of adding further modes beyond $N = 3$ are relatively minor. As an aside, allowing for time-varying values of ϕ (i.e estimating the value of ϕ at each point in time, as is sometimes done in applications of NS models) makes little difference in the fit to the model across the entire time series of yield curve data relative to using a fixed value of ϕ across the entire sample.

A.3 The generic volatility-adjusted OLP (VAO) model of the yield curve

An intertemporally-consistent version of any OLP model can always be derived following the approach from chapter 2, but the generalisation from 3 to N modes requires some additional notation. Hence, begin with the generalisation of the assumptions underlying the ANS model, i.e:

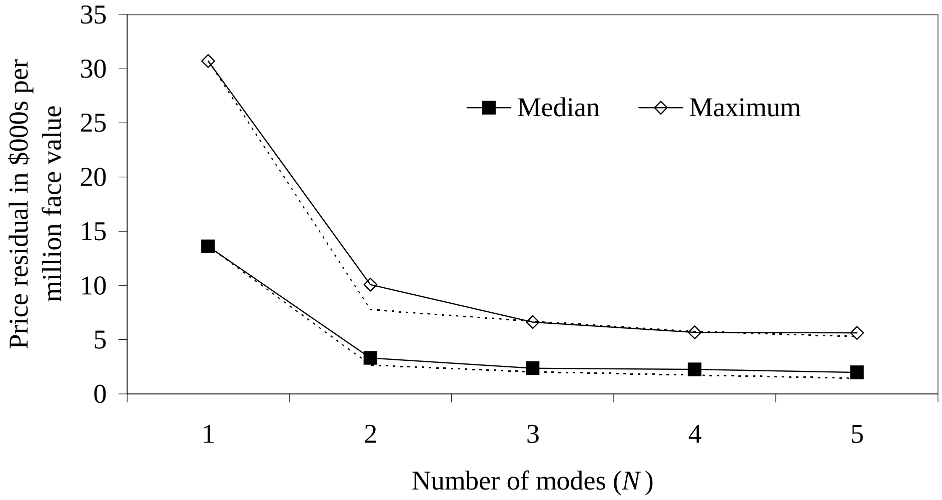

Figure A.1: Summary statistics for fitting the New Zealand yield curve using the OLP model with $\phi = 1$, and the number of modes N ranging from 1 to 5. The maximum absolute price residual for each daily estimation of the yield curve from July 1997 to February 2002 was recorded, thereby giving a time series of maximum absolute price residuals for each value of N. The medians and maximums of those time series of maximum absolute price residuals are plotted. The associated dotted lines allow for time-varying values of ϕ instead of fixing $\phi = 1$ over the sample.

Assumption 1: $E_t\left[r(t+m)\right]$ under the physical measure is:

$$E_t\left[r(t+m)\right] = \sum_{n=1}^{N} \lambda_n(t) \cdot g_n(\phi, m) \tag{A.7}$$

Assumption 2: Instantaneous stochastic changes to the forward rate curve are:

$$\sum_{n=1}^{N} \sigma_n \cdot g_n(\phi, m) \cdot dW_n(t) \tag{A.8}$$

Assumption 3: The market prices of risk for each mode are constants ρ_n.

A.3.1 The volatility structure in the VAO model

To show that the volatility integral term in equation 2.17 may be evaluated for arbitrary N, define a generic exponential-polynomial volatility function as $\sigma(t, t+m) = \sigma \cdot \exp(-\phi m)(\phi m)^a$, where $a\ (\geq 0)$ is an integer.[1] Following the HJM approach, $\int_s^m \sigma_n(s,u)\,du$

[1] In the HJM notation using time t and time of maturity T, this would be written $\sigma(t,T) = \sigma \cdot \exp(-\phi[T-t])(\phi[T-t])^a$, so $T = t+m$.

is calculated as:

$$
\begin{aligned}
& \sigma \cdot \int_s^m \exp\left(-\phi\left[m-u\right]\right)\left(\phi\left[m-u\right]\right)^a du && \text{(A.9a)}\\
= \; & \sigma \cdot \left[-\frac{1}{\phi}\Gamma\left[1+a,\phi\left[m-u\right]\right]\right]_s^m && \text{(A.9b)}\\
= \; & \frac{\sigma}{\phi} \cdot \left(-\Gamma\left[1+a,\phi\left(m-s\right)\right]+\Gamma\left[1+a,0\right]\right) && \text{(A.9c)}
\end{aligned}
$$

where $\Gamma\left[\cdot,\cdot\right]$ is the incomplete Gamma function.[2] Note that $\Gamma\left[1+a,0\right]=a!$, the factorial definition, and these expressions are used interchangeably below. Substituting equation A.9c into $\int_0^m \sigma_n\left(s,m\right)\left[\int_s^m \sigma_n\left(s,u\right)du\right]ds$ gives:

$$
\begin{aligned}
& \frac{\sigma^2}{\phi} \cdot \int_0^m \left[\begin{array}{c} \exp\left(-\phi\left[m-s\right]\right)\left(\phi\left[m-s\right]\right)^a \\ \times\left(-\Gamma\left[1+a,\phi\left(m-s\right)\right]+a!\right)\end{array}\right] ds && \text{(A.10a)}\\
= \; & \frac{\sigma^2}{2\phi^2}\left[2a!\Gamma\left[1+a,\phi\left(m-s\right)\right]-\left(\Gamma\left[1+a,\phi\left(m-s\right)\right]\right)^2\right]_0^m && \text{(A.10b)}\\
= \; & \frac{\sigma^2}{2\phi^2}\left[2\left(a!\right)^2-\left(a!\right)^2-2a!\cdot\Gamma\left[1+a,\phi m\right]+\left(\Gamma\left[1+a,\phi m\right]\right)^2\right] && \text{(A.10c)}\\
= \; & \frac{\sigma^2}{2\phi^2}\left(a!-\Gamma\left[1+a,\phi m\right]\right)^2 && \text{(A.10d)}
\end{aligned}
$$

The functions $h_n(\phi,m)$ for $n>1$ are a summation of exponential-polynomial terms, i.e $\sigma_n\left(m\right)=\sigma_n\cdot g_n(\phi,m)=\sigma_n\cdot\exp\left(-\phi m\right)\cdot\sum_{k=0}^{n-2}\frac{(-1)^k(n-2)!(2\phi m)^k}{(k!)^2(n-2-k)!}$. The latter may be expressed equivalently as $\sigma_n\cdot\sum_{k=0}^{n-2}\frac{(-2)^k(n-2)!}{(k!)^2(n-2-k)!}\exp\left(-\phi m\right)\left(\phi m\right)^k$, and so the volatility integral terms will always be expressible as linear combinations of the generic form in equation A.10d.

A.3.2 The market prices of risk in the VAO model

The market price of risk integral from equation 2.17, i.e $\int_0^m \sigma_n\cdot g_n\left(\phi,s\right)\cdot\rho_n ds$ for $n>1$, may always be expressed as a linear expression of the modes $g_n\left(\phi,m\right),g_{n-1}\left(\phi,m\right),\ldots,$ $g_1\left(\phi,m\right)$. This can be shown by the direct calculation of each integral, i.e: $g_n\left(\phi,s\right)=-\exp\left(-\phi s\right)\cdot\sum_{k=0}^{n-2}\frac{(-1)^k(n-2)!(2\phi s)^k}{(k!)^2(n-2-k)!}$. Write $u\left(s\right)=\sum_{k=0}^{n-2}p_{n,k}\cdot s^k$ so $du\left(s\right)=\sum_{k=0}^{n-2}q_{n,k}\cdot$

[2] That is, $\Gamma\left[1+a,z\right]=\int_z^\infty x^a\exp\left(-x\right)dx$; see Wolfram (1996) p. 740.

$s^{k-1}ds$ where $p_{n,k}$ and $q_{n,k}$ capture all of the associated constants, and write $dv(s) = -\exp(-\phi s)\,ds$ so $v(s) = \frac{1}{\phi}\exp(-\phi s)$. Integration by parts, i.e $\int u(s)\,dv(s) = u(s)\,v(s) - \int v(s)\,du(s)$, will result in the indefinite integral $\exp(-\phi s) \cdot \sum_{k=0}^{n-2} w_{n,k} \cdot s^k - \int \exp(-\phi s) \cdot \sum_{k=0}^{n-2} x_{n,k} \cdot s^{k-1}ds$, where $w_{n,k}$ and $x_{n,k}$ capture all of the associated constants, and the maximum order of the polynomial term in the new integration term has been reduced by 1. Hence, the repeated application of integration by parts will ultimately result in a finite sequence of exponential-polynomial functions, with the maximum order of the polynomial terms $n-2$ and the minimum order 0. This may be evaluated at the limits of integration 0 and m, and the resulting series of exponential-polynomial functions (with a maximum order of $n-2$ and a minimum order of 0) may be re-arranged into an equivalent sequence of OLP functions plus a constant.

A.3.3 The intertemporal consistency of the VAO model

The VAO model of any order will always be intertemporally consistent, with the relationship between the VAO model coefficients given by the stochastic time-series process:

$$\boldsymbol{\beta}(t+\tau) = \boldsymbol{\mu} + \boldsymbol{\Phi}(\phi,\tau)\,\boldsymbol{\beta}(t) + \boldsymbol{\delta}(t+\tau) \tag{A.11}$$

where $\boldsymbol{\beta}(t) = \{\beta_1(t), \beta_2(t), \ldots, \beta_N(t)\}'$ is a column N-vector containing the VAO model coefficients at time t; τ (> 0) is a parameter representing an arbitrary future horizon from time t; $\boldsymbol{\beta}(t+\tau)$ is the N-vector of VAO model coefficients at time $t+\tau$; $\boldsymbol{\mu}$ is a column N-vector of constants; $\boldsymbol{\Phi}(\phi,\tau)$ is a time-invariant $N \times N$ matrix with a top right entry of 1, a block diagonal sub-matrix with entries that are explicit functions of ϕ and τ, and eigenvalues of $\{1, \exp(-\phi m), \ldots, \exp(-\phi m)\}$; and $\boldsymbol{\delta}(t+\tau) = \{\delta_1(t+\tau), \delta_2(t+\tau), \ldots, \delta_N(t+\tau)\}'$ is a column N-vector of independent random variables.

The proof of this proposition follows the principles in section 2.4.1. That is, the first sub-section of 2.4.1 derives the intertemporal relationship for the expected path of the short rates within the HJM framework as $E_{t+\tau}\left[r\left(t+\tau+m\right)\right] = E_t\left[r\left(t+\tau+m\right)\right] + \int_t^{t+\tau}\sigma_n\left(s,m\right)dW_n\left(s\right)$. Following the second sub-section of 2.4.1, the latter expression may then be written within the VAO model as:

$$[\boldsymbol{\lambda}\left(t+\tau\right)]'\,\mathbf{g}\left(\phi,m\right) = [\boldsymbol{\lambda}\left(t\right)]'\,\mathbf{g}\left(\phi,\tau+m\right) + [\boldsymbol{\delta}\left(t+\tau\right)]'\,\mathbf{g}\left(\phi,m\right) \qquad \text{(A.12)}$$

where $\boldsymbol{\lambda}\left(t+\tau\right) = \left\{\lambda_1\left(t+\tau\right),\ldots,\lambda_N\left(t+\tau\right)\right\}'$; $\mathbf{g}\left(\phi,m\right) = \left\{g_1\left(\phi,m\right),\ldots,g_N\left(\phi,m\right)\right\}'$; $\boldsymbol{\lambda}\left(t\right) = \left\{\lambda_1\left(t\right),\ldots,\lambda_N\left(t\right)\right\}'$; and $\mathbf{g}\left(\phi,\tau+m\right) = \left\{g_1\left(\phi,\tau+m\right),\ldots,g_N\left(\phi,\tau+m\right)\right\}'$. Showing that an explicit time-series process of the form in equation A.11 will result for the VAO model of any order then requires re-expressing $\mathbf{g}\left(\phi,\tau+m\right)$ in terms of $\mathbf{g}\left(\phi,m\right)$ to obtain $\boldsymbol{\Phi}\left(\phi,\tau\right)$ for the general case.

Hence, beginning with the definition for the VAO modes for $n \geq 2$ in equation A.3, $g_n\left(\phi,\tau+m\right) = -\exp\left(-\phi\left[\tau+m\right]\right)\cdot\sum_{k=0}^{n-2}\frac{(-1)^k(n-2)!(2\phi[\tau+m])^k}{(k!)^2(n-2-k)!}$. For notational convenience, express this as $g_n\left(\phi,\tau+m\right) = -\exp\left(-\phi\left[\tau+m\right]\right)\cdot\sum_{k=0}^{n-2}p_{n,k}\cdot\left(\tau+m\right)^k$, where $p_{n,k} = \frac{(-1)^k(n-2)!(2\phi)^k}{(k!)^2(n-2-k)!}$. The expression $\left(\tau+m\right)^k$ fully expanded will have the form $m^k + \tau^k + \sum_{j=1}^{k-1}\frac{k!}{(k-j)!j!}\cdot\tau^{k-j}\cdot m^j$, where m^k is the highest power of m, τ^k is the highest power of τ, and the summation captures the cross-terms containing powers of m and τ. For notational convenience, the summation may be expressed as $\sum_{j=1}^{k-1}q_{j,k}\cdot\tau^{k-j}\cdot m^j$, where $q_{j,k} = \frac{k!}{(k-j)!j!}$. Making the additional substitution of $\exp\left(-\phi\tau\right)\cdot\left[-\exp\left(-\phi m\right)\right]$ for $-\exp\left(-\phi\left[\tau+m\right]\right)$, the function $g_n\left(\phi,\tau+m\right)$ written in full is then:

$$\begin{aligned} g_n\left(\phi,\tau+m\right) &= \exp\left(-\phi\tau\right)\cdot\left[-\exp\left(-\phi m\right)\right] \\ &\quad\times\sum_{k=0}^{n-2}p_{n,k}\cdot\left(m^k+\tau^k+\sum_{j=1}^{k-1}q_{j,k}\cdot\tau^{k-j}\cdot m^j\right) \end{aligned} \qquad \text{(A.13)}$$

For the highest order of the given VAO model of the (i.e $n = N$), an inspection of equation A.13 shows that $g_N\left(\phi,\tau+m\right)$ will include the highest-order OLP $g_N\left(\phi,m\right)$, given $\exp\left(-\phi\tau\right)\cdot\left[-\exp\left(-\phi m\right)\right]\cdot\sum_{k=0}^{N-2}p_{N,k}\cdot m^k = \exp\left(-\phi\tau\right)\cdot g_N\left(\phi,m\right)$. And $g_N\left(\phi,\tau+m\right)$ will also include a series of exponential-polynomial functions that may be expressed as a linear combination of the lower-order (i.e $1 < n < N$) OLP functions from $\exp(-\phi\tau) \cdot g_{N-1}(\phi, m)$ to $\exp(-\phi\tau) \cdot g_2(\phi, m)$; i.e the $n = 2$ term is evident from an inspection of equation A.13 given $\exp(-\phi\tau) \cdot [-\exp(-\phi m)] \cdot \sum_{k=0}^{N-2} p_{N,k} \cdot \tau^k = \exp(-\phi\tau) \cdot g_2(\phi, m) \cdot \sum_{k=0}^{N-2} p_{N,k} \cdot \tau^k$. Hence, $g_N(\phi, \tau + m) = \exp(-\phi\tau) \cdot \left[g_N(\phi, m) + \sum_{k=2}^{N-1} Q_{N,k} \cdot g_k(\phi, m)\right]$, where $Q_{N,k}$ are the constant linear coefficients for the lower-order OLP functions, and $Q_{N,k}$ will in turn be composed of the constants $p_{n,k}$ and τ^k.

This linear combination for $g_N(\phi, \tau + m)$ may be written equivalently in vector form as $g_N(\phi, \tau + m) = \exp(-\phi\tau) \cdot [0, Q_{N,2}, \ldots, Q_{N,N-1}, 1]\, \mathbf{g}(\phi, m) = \exp(-\phi\tau) \cdot \mathbf{x}_N \mathbf{g}(\phi, m)$, where $\mathbf{x}_N$ is a row N-vector of constants. The first element of $\mathbf{x}_N$ is 0 (given that $g_N(\phi, \tau + m)$ for $n \geq 2$ does not include a constant, and so the coefficient associated with $g_1(\phi, m) = 1$ must be zero), the elements from 2 to N are the linear coefficients $Q_{N,k}$ associated with $g_2(\phi, m)$ to $g_{N-1}(\phi, m)$, and the element N associated with $g_N(\phi, m)$ is 1.

Repeating the steps above for $n = N - 1$ results in $g_{N-1}(\phi, \tau + m) = \exp(-\phi\tau) \cdot [0, Q_{N-1,2}, \ldots, Q_{N-1,N-2}, 1, 0]\, \mathbf{g}(\phi, m) = \exp(-\phi\tau) \cdot \mathbf{x}_{N-1} \mathbf{g}(\phi, m)$. Note that the final term in the vector $\mathbf{x}_{N-1}$ is 0 because the highest order of m^k in $g_{N-1}(\phi, \tau + m)$ is m^{N-3}, and so $g_{N-1}(\phi, \tau + m)$ cannot contain a non-zero $Q_{N,k} \cdot g_N(\phi, m)$ term. Repeating the steps again for $n = N-2$ gives $\mathbf{x}_{N-2}$, and so on until the term $n = 2$ results in $\mathbf{x}_2 = [0, Q_{2,2}, 0, \ldots, 0]$; i.e $g_2(\phi, \tau + m) = \exp(-\phi\tau) \cdot [0, Q_{2,2}, 0, \ldots, 0]\, \mathbf{g}(\phi, m) = \exp(-\phi\tau) \cdot \mathbf{x}_2 \mathbf{g}(\phi, m)$. Finally, for $n = 1$, note that $g_1(\phi, \tau + m) = g_1(\phi, m) = 1$. Hence, $\mathbf{x}_1 = [1, 0, \ldots, 0]$, i.e $g_1(\phi, \tau + m) = [1, 0, \ldots, 0]\, \mathbf{g}(\phi, m)$.

The vector $\mathbf{g}(\phi, \tau + m)$ may therefore be written as $[\boldsymbol{\Phi}(\phi, \tau)]' \mathbf{g}(\phi, m)$, where $[\boldsymbol{\Phi}(\phi, \tau)]'$ is the $N \times N$ matrix obtained by "stacking" the N row vectors from $\mathbf{x}_1$, $\exp(-\phi\tau) \cdot \mathbf{x}_2$, ..., $\exp(-\phi\tau) \cdot \mathbf{x}_N$ (i.e $[\boldsymbol{\Phi}(\phi, \tau)]'$ is a column N-vector of the N row N-vectors). Note that form of $[\boldsymbol{\Phi}(\phi, \tau)]'$ will be $\begin{bmatrix} 1 & \mathbf{0}' \\ \mathbf{0} & \exp(-\phi\tau) \cdot [\boldsymbol{\Psi}(\phi, \tau)]' \end{bmatrix}$, where $[\boldsymbol{\Psi}(\phi, \tau)]'$ is a lower-diagonal matrix with diagonal elements 1 and the lower off-diagonal elements polynomial functions of $\phi\tau$; and $\mathbf{0}$ is the column $(N-1)$-vector of zeros. Hence, given $[\mathbf{\Phi}(\phi,\tau)]'$ is a lower-diagonal matrix, the eigenvalues of $[\mathbf{\Phi}(\phi,\tau)]'$ and $\mathbf{\Phi}(\phi,\tau)$ are simply the diagonal elements; i.e 1 and $\exp(-\phi\tau)$.

Substituting $\mathbf{g}(\phi,\tau+m) = [\mathbf{\Phi}(\phi,\tau)]'\mathbf{g}(\phi,m)$ into equation A.12 results in the intertemporal relationship for the expected path of the short rate within the VAO model being $[\boldsymbol{\lambda}(t+\tau)]'\mathbf{g}(\phi,m) = [\boldsymbol{\lambda}(t)]'[\mathbf{\Phi}(\phi,\tau)]'\mathbf{g}(\phi,m) + [\boldsymbol{\delta}(t+\tau)]'\mathbf{g}(\phi,m)$. Adding the time-invariant term premium $\boldsymbol{\gamma}'\mathbf{g}(\phi,m)$ to both sides, factoring out the common term $\mathbf{g}(\phi,m)$, and transposing gives $\boldsymbol{\gamma} + \boldsymbol{\lambda}(t+\tau) = \boldsymbol{\gamma} + \mathbf{\Phi}(\phi,\tau)\boldsymbol{\lambda}(t) + \boldsymbol{\delta}(t+\tau)$. The latter may be rewritten as:

$$\boldsymbol{\gamma} + \boldsymbol{\lambda}(t+\tau) = [\mathbf{I} - \mathbf{\Phi}(\phi,\tau)]\boldsymbol{\gamma} + \mathbf{\Phi}(\phi,\tau)[\boldsymbol{\gamma} + \boldsymbol{\lambda}(t)] + \boldsymbol{\delta}(t+\tau) \tag{A.14}$$

where $\mathbf{I}$ is the $N \times N$ identity matrix. Given $\boldsymbol{\beta}(t+\tau) = \boldsymbol{\gamma} + \boldsymbol{\lambda}(t+\tau)$ and $\boldsymbol{\beta}(t) = \boldsymbol{\gamma} + \boldsymbol{\lambda}(t)$, this results in equation A.11, with $\boldsymbol{\mu}(\boldsymbol{\phi},\boldsymbol{\tau}) = [\mathbf{I} - \mathbf{\Phi}(\phi,\tau)]\boldsymbol{\gamma}$, which is a time-invariant N-vector as a function of ϕ and τ.

A.4 Relaxing the VAO model assumptions

The ANS model and its generalisation to the VAO model in this appendix assumes that the variance-covariance matrix of the innovations is constant and diagonal, and that the market prices of risk are constant. However, based on the US data discussed in section 2.5.1 of chapter 2, figures A.2 and A.3 respectively show that the volatilities of the ANS coefficients and the market price of risk for the Level coefficient (i.e ρ_1) have changed over time (seemingly in response to the changes in monetary regimes at the time; e.g there was a sharp rise in volatility following the onset of the Volcker-led disinflation of the late-1970s/early-1980s).

That said, the assumption of constant volatilities and market prices of risk would

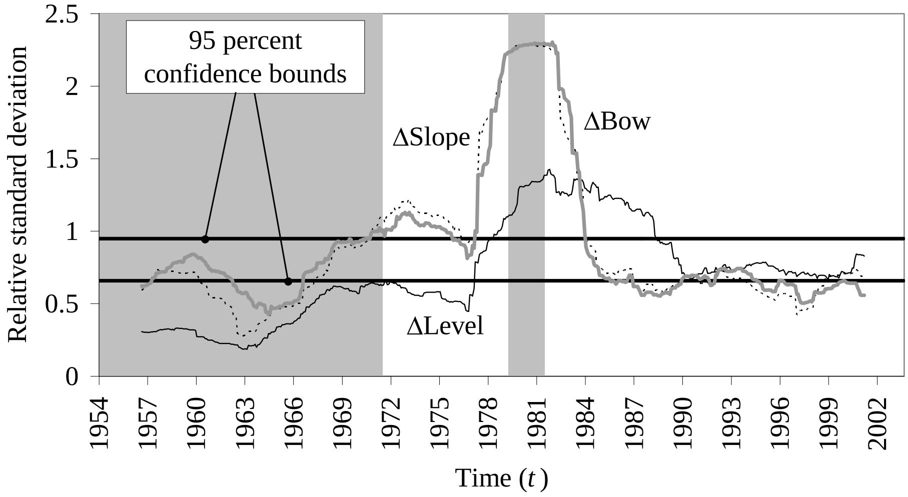

Figure A.2: The standard deviation of changes in each of the ANS coefficients calculated on a centred five-year window divided by the standard deviation for the entire sample. The 95 percent confidence bounds are for the null hypothesis that the five-year window calculations are equal to the standard deviation over the entire sample.

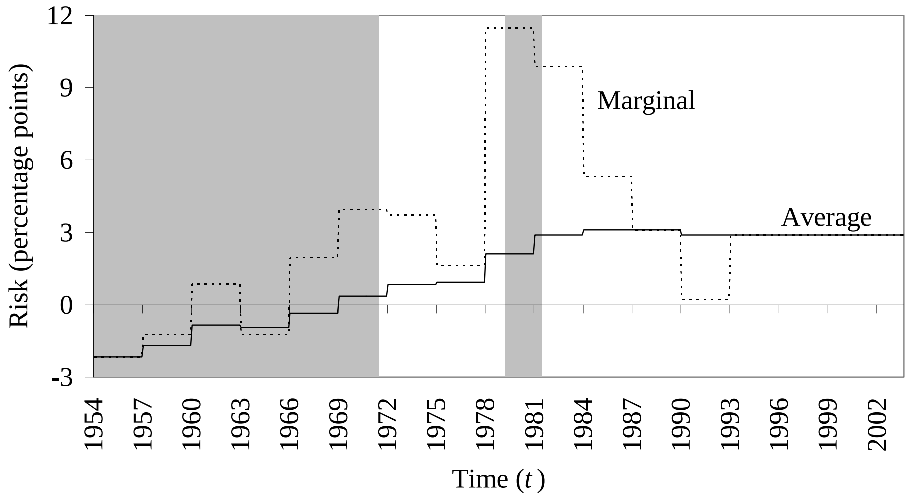

Figure A.3: The average market price of risk is the estimate of ρ_1 from the grid search using all of the data from the start of the sample up to the end of each three-year period. The marginal price of risk is the implied value of ρ_1 over each three-year period that would be required to equate the updated three-year average with the previous three-year average.

remain appropriate so long as at any point in time it is reasonable to assume that the market expects volatility and the market prices of risk to remain constant. However, if that assumption was not reasonable for a particular application, then the HJM framework would continue to give analytical functions of maturity for the forward rate curve with deterministic volatilities and market prices of risk. For example, if current volatilities were materially different from their expected long-run values, then that could be modelled using a simple exponential decay over time, i.e:

$$E_t\left[\sigma_n\left(t+m\right)\right] = \eta_{\sigma,n} + \left[\sigma_n\left(t\right) - \eta_{\sigma,n}\right] \cdot \exp\left(-\chi_{\sigma,n} m\right) \tag{A.15}$$

where $E_t\left[\sigma_n\left(t+m\right)\right]$ is the expected path of volatility as a function of future time $t+m$; $\sigma_n\left(t\right)$ is current volatility; $\eta_{\sigma,n}$ is the long-run level of volatility; and $\chi_{\sigma,n}$ is the exponential decay that governs the rate at which the prevailing level of volatility reverts back to the long-run level. Similarly, if the current market prices of risk were materially different from their expected long-run values, then that could be modelled using a simple exponential decay over time, i.e:

$$E_t\left[\rho_n\left(t+m\right)\right] = \eta_{\rho,n} + \left[\rho_n\left(t\right) - \eta_{\rho,n}\right] \cdot \exp\left(-\chi_{\rho,n} m\right) \tag{A.16}$$

where $E_t\left[\rho_n\left(t+m\right)\right]$ is the expected path for the market price of risk as a function of future time $t+m$; $\rho_n\left(t\right)$ is the current market price of risk; $\eta_{\rho,n}$ is the long-run level of the market price of risk; and $\chi_{\rho,n}$ is the exponential decay that governs the rate at which the prevailing market price of risk reverts back to the long-run level.

Also based on the US data discussed in section 2.5.1, figure A.4 shows that the correlations between innovations in the ANS coefficients have varied over time, and the correlation between the Slope and Bow coefficient innovations has always been significantly negative.

While this time-variance and negative correlation is not strictly consistent with

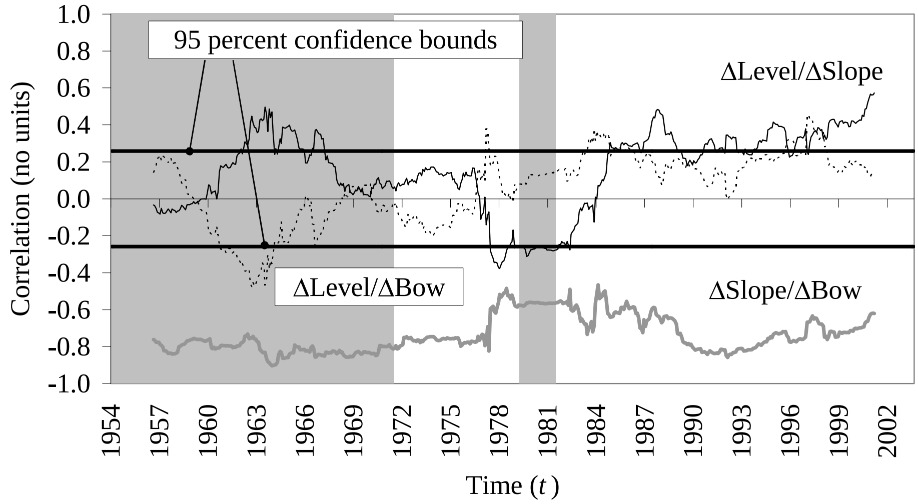

Figure A.4: The correlations of changes in each of the coefficients calculated on a centred five years window. The 95 percent confidence bounds are for the null hypothesis that the five year window calculations are zero.

the assumptions underlying the ANS model, any practical impact from correcting for those aspects would be negligible. Firstly, figure 2.4 of chapter 2 shows that the effect of volatility in the Slope and Bow coefficients on the interest rate curve is several orders of magnitude less than for volatility in the Level coefficient. Hence, the practical effect of any covariance between the Slope and Bow coefficients on the shape of the forward or interest rate curve would also be very small. Secondly, figure A.4 shows that the correlation between innovations in the Level and Slope coefficients, and between innovations in the Level and Bow coefficients, has typically been statistically insignificant.

However, if complete orthogonality in coefficient innovations was required for a particular application, this could be obtained by the appropriate rotation of the $g_n(\phi, m)$ modes noted in section A.1. Specifically, the initial estimation of the ANS/VAO model with the original set of N modes $\mathbf{g}(\phi, m) = \{g_1(\phi, m), g_2(\phi, m), \ldots, g_N(\phi, m)\}'$ would produce a time-series vector of innovations $\boldsymbol{\delta}(t + \tau)$. These innovations by component and across time may be expressed as:

$$\mathbf{D} = \begin{bmatrix} \delta_{1,1} & \delta_{1,2} & \cdots & \delta_{1,T} \\ \delta_{2,1} & \delta_{2,2} & \cdots & \delta_{2,T} \\ \vdots & \vdots & \vdots & \vdots \\ \delta_{N,1} & \delta_{N,2} & \cdots & \delta_{N,T} \end{bmatrix}_{N \times T} \tag{A.17}$$

where $\mathbf{D}$ is an $N \times T$ matrix containing the time series of innovations for the Level coefficients $\delta_{1,1}, \ldots, \delta_{1,T}$, the time series of innovations for the Slope coefficients $\delta_{2,1}, \ldots, \delta_{2,T}$, and so on up to the time series of innovations for the coefficient N, i.e $\delta_{N,1}, \ldots, \delta_{N,T}$. The sample covariance matrix of innovations is then $\frac{1}{T} \cdot \mathbf{DD'}$.

If $\mathbf{DD'}$ is not diagonal, then the Gramm-Schmidt process (e.g see Anton (1984) pp. 192-194) may be used to construct an orthogonal basis. This orthogonal basis would then have the property that $\frac{1}{T} [\mathbf{PD}] [\mathbf{PD}]' = \frac{1}{T}\mathbf{PDD'P'} = \mathbf{V}$, where $\mathbf{P}$ is an $N \times N$ matrix with values of 1 down the leading diagonal, and constant coefficients a_{ij} in the lower diagonal, i.e:

$$\mathbf{P} = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ a_{21} & 1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ a_{N1} & a_{N2} & \cdots & 1 \end{bmatrix}_{N \times N} \tag{A.18}$$

and where $\mathbf{V}$ is the diagonal matrix:

$$\mathbf{V} = \begin{bmatrix} \sigma_1^2 & 0 & \cdots & 0 \\ 0 & \sigma_2^2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & 0 & \cdots & \sigma_N^2 \end{bmatrix} \tag{A.19}$$

The original modes $\mathbf{g}(\phi, m)$ may then be adjusted using $\mathbf{P}$. That is, the complete set of new modes would be $\mathbf{Pg}(\phi, m)$, and the individual modes would be of the form:

$$[\mathbf{Pg}(\phi, m)]_n = g_n(\phi, m) + \sum_{i=1}^{n-1} a_{ni} \cdot g_i(\phi, m) \tag{A.20}$$

However, changing from the original modes would then require new calculations of the effects of volatility and the market prices. Hence the respective HJM integrals $\int_0^m \sigma_n(s, m) \left[\int_s^m \sigma_n(s, u)\, du\right] ds$ and $\int_0^m \sigma_n(s, m)\, \rho_n ds$ would need to be calculated using the functions $[\mathbf{Pg}(\phi, m)]_n$. The resulting new ANS/VAO model would then be estimated, giving a new matrix $\mathbf{D}$, and the process could be iterated to convergence.

The potential extensions of the ANS/VAO model noted above are, of course, subject to the typical trade-off of model precision against tractability, parsimony, and intuition. For example, modelling volatility and the market prices of risk as the time-varying quantities noted above would add two additional state variables and two parameters per ANS/VAO coefficient relative to the ANS/VAO model that assumed constant volatilities and market prices of risk. Accounting for covariance between innovations in the ANS coefficients would add three additional parameters.

A.5 The economic foundation of the VAO model

To show the correspondence between the OLP functional form in equation A.7 that is used to represent the expected path of the short rate within the VAO model, and equation 3.2 in section 3.2.1 from chapter 2, first define ϕ as a central measure of the values of κ_j for $j = 1$ to $2J$, i.e $\phi = \text{central}(\kappa_j)$ (which is a constant, because κ_j are constants). Hence, $\kappa_j = \phi(1 + \Delta_j)$ with $-1 < \Delta_j < 1$,[3] and equation 3.2 may be written as:

$$\sum_{j=2}^{J} E_t[s_j(t+m)] = \sum_{j=2}^{J} \theta_j(t) + \exp(-\phi m) \cdot \sum_{j=2}^{J} [s_j(t) - \theta_j(t)] \cdot \exp(-\Delta_j \phi m) \tag{A.21}$$

[3] This restriction on Δ_j is always possible by construction; in the extreme case, ϕ could be defined as $\max(\kappa_j)$, and then $-1 < \Delta_j \leq 0 < 1$ (because the lower bound for each κ_j is zero).

Now write each exponential term containing Δ_j as a Taylor expansion around $\Delta_j = 0$ to order $N-2$; i.e $\sum_{j=2}^{J} E_t\left[s_j\left(t+m\right)\right]$ may be approximated to arbitrary precision as:[4]

$$\begin{aligned}
& \sum_{j=2}^{J} \theta_j\left(t\right) + \exp\left(-\phi m\right) \cdot \sum_{j=2}^{J} \left[s_j\left(t\right) - \theta_j\left(t\right)\right] \left[\sum_{n=2}^{N} \frac{\left(-\Delta_j \phi m\right)^{(n-2)}}{(n-2)!}\right] && \text{(A.22a)} \\
= & \sum_{j=2}^{J} \theta_j\left(t\right) + \exp\left(-\phi m\right) \cdot \sum_{n=2}^{N} \omega_n\left(t\right) \cdot \left(\phi m\right)^{(n-2)} && \text{(A.22b)} \\
= & \sum_{j=2}^{J} \theta_j\left(t\right) - \sum_{n=2}^{N} \lambda_n\left(t\right) \cdot - \exp\left(-\phi m\right) \sum_{k=0}^{n-2} \frac{\left(-1\right)^k \left(n-2\right)! (2\phi m)^k}{\left(k!\right)^2 \left(n-2-k\right)!} && \text{(A.22c)}
\end{aligned}$$

where the coefficients $\omega_n\left(t\right)$ in equation A.22b are the collections of the coefficients on powers of $\left(\phi m\right)^{(n-2)}$ from the full expansion of the double summation in equation A.22a, and equation A.22c is a rearrangement of the summation of exponential-polynomials into an equivalent summation of OLP functions. Equation A.22c is the generic OLP form noted in equation 3.2, which shows that the order of the VAO model can be increased to obtain an arbitrary approximation to the ABE model. That is, $N-2$ represents the order of the Taylor expansion.

Note that empirical significance of higher-order modes in the VAO model should indicate the relative distribution of Δ_j, i.e the magnitudes of the mean-reversion coefficients for the real state variables κ_j relative to central(κ_j). If higher-order modes in the VAO model quickly become empirically insignificant, this would suggest that the magnitudes of κ_j are generally similar, and/or that factors of production with κ_j materially different from the average make up a relatively small proportion of the economy. The empirical success of three-mode OLP models in many different markets suggests that one or both of these conditions generally hold.

[4]The residual term $\sum_{n=N+1}^{\infty} \frac{\left(-\Delta_j \phi m\right)^{(n-2)}}{(n-2)!}$ associated with the Taylor expansion approximation will always converge to a finite value (and that value may be made arbitrarily small) because $\left|\Delta_j\right| < 1$.

Appendix B

Generalising the economic model underlying the ANS/VAO model

The primary purpose of this appendix is to show that the solutions presented in chapter 3 remain valid even with a completely general specification of the economy. That is, an orthogonal representation of the economy can be constructed from the $2J$ state variables and $2J$ steady-state variables, and a non-diagonal mean-reversion coefficient matrix will still give a solution for the expected path of the short rate that is a summation of constants and exponential decay terms. Hence, the ANS model will also be a first-order approximation to the generalised ABE model.

Section B.1 firstly outlines these generalised results for the BE economy (which is essentially the ABE model with constant steady-state variables), and then section B.2 uses the analogous approach to establish the generalised results for the ABE economy. As referenced in chapter 5, section B.3 provides the details for the implicit economic model that underlies the UIPH analysis in chapter 5 (i.e a model with two economies, two yield curves, and a bilateral exchange rate).

B.1 The generalised BE economy

In its most general form, the BE economy may be expressed as the following stochastic vector differential equation:

$$d\mathbf{s}(t) = -\boldsymbol{\kappa}[\mathbf{s}(t) - \boldsymbol{\theta}]\,dt + \boldsymbol{\sigma}d\mathbf{z}(t) \tag{B.1}$$

where $\mathbf{s}(t) = [s_1(t), \ldots, s_{2J}(t)]'$ are the state variables; $\boldsymbol{\theta} = [\theta_1, \ldots, \theta_{2J}]'$ are the steady-state variables; $d\mathbf{z}(t) = [dz_1(t), \ldots, dz_{2J}(t)]'$ are Wiener increments; $\boldsymbol{\kappa}$ is the mean-reversion coefficient matrix $\begin{bmatrix} \kappa_{1,1} & \cdots & \kappa_{1,2J} \\ \vdots & \ddots & \vdots \\ \kappa_{2J,1} & \cdots & \kappa_{2J,2J} \end{bmatrix}$; and $\boldsymbol{\sigma}$ is the standard deviation coefficient matrix $\begin{bmatrix} \sigma_{1,1} & \cdots & \sigma_{1,2J} \\ \vdots & \ddots & \vdots \\ \sigma_{2J,1} & \cdots & \sigma_{2J,2J} \end{bmatrix}$. The outer product of the stochastic terms is $\boldsymbol{\Omega} = \boldsymbol{\sigma}d\mathbf{z}(t)[d\mathbf{z}(t)]'\boldsymbol{\sigma}' = \boldsymbol{\sigma}\mathbf{I}\boldsymbol{\sigma}' = \boldsymbol{\sigma}\boldsymbol{\sigma}'$. The matrix $\boldsymbol{\Omega}$ will in general be non-diagonal (which allows for relationships between innovations in the growth rates of the factors of production and their rates of inflation), but it may be rotated into a diagonal representation. That is, using the notation of Greene (1997) pp. 35-38 for characteristic vectors and values (or eigenvectors and eigenvalues) $\boldsymbol{\Omega} = \mathbf{C}\boldsymbol{\Lambda}\mathbf{C}'$, where $\mathbf{C} = \{\mathbf{c}_1, \ldots, \mathbf{c}_{2J}\}$ is a matrix of order $2J\times\, 2J$ (i.e a $2J$-row vector of $2J$-column eigenvectors); and $\boldsymbol{\Lambda} = \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & \lambda_{2J} \end{bmatrix}$ is a $2J\times\, 2J$ diagonal matrix of eigenvalues. Pre-multiplying equation B.1 by $\mathbf{C}'$ then gives an orthogonal basis, i.e:

$$\mathbf{C}'d\mathbf{s}(t) = -\mathbf{C}'\boldsymbol{\kappa}[\mathbf{s}(t) - \boldsymbol{\theta}]\,dt + \mathbf{C}'\boldsymbol{\sigma}d\mathbf{z}(t) \tag{B.2}$$

which is orthogonal given that $\mathbf{C}'\boldsymbol{\sigma}_1 d\mathbf{z}_1(t)[d\mathbf{z}_1(t)]'\boldsymbol{\sigma}_1'\mathbf{C} = \mathbf{C}'\mathbf{C}\boldsymbol{\Lambda}\mathbf{C}'\mathbf{C} = \mathbf{I}\boldsymbol{\Lambda}\mathbf{I} = \boldsymbol{\Lambda}$.

Taking the expectation of equation B.2 gives:

$$E_t\left[\mathbf{C}'d\mathbf{s}\left(t+m\right)\right]=-\mathbf{C}'\boldsymbol{\kappa}\left\{E_t\left[\mathbf{s}\left(t+m\right)\right]-\boldsymbol{\theta}\right\}dt \tag{B.3}$$

and noting that $\mathbf{CC}'=\mathbf{I}$, the right-hand side of equation B.3 may be re-expressed, giving:

$$E_t\left[\mathbf{C}'d\mathbf{s}\left(t+m\right)\right]=-\mathbf{C}'\boldsymbol{\kappa}\mathbf{C}\left\{E_t\left[\mathbf{C}'\mathbf{s}\left(t+m\right)\right]+\mathbf{C}'\boldsymbol{\theta}\right\}dt \tag{B.4}$$

With the exception of an extraordinary coincidence, the matrix $-\mathbf{C}'\boldsymbol{\kappa}\mathbf{C}$ will not be diagonal, and so the solution will not be as straightforward as solving the scalar differential equation for separate elements of the vector $\mathbf{C}'d\mathbf{s}\left(t+m\right)$. However, Rainville and Bedient (1981) pp. 247-273 shows how to obtain a solution in the general case of $d\mathbf{X}=\mathbf{AX}+\mathbf{B}$ using eigenvectors and eigenvalues. Hence, substitute $\mathbf{PQP}'=-\mathbf{C}'\boldsymbol{\kappa}\mathbf{C}$, where $\mathbf{P}=\{\mathbf{p}_1,\ldots,\mathbf{p}_{2J}\}$ is a matrix of order $2J\times\ 2J$ (i.e a $2J$-row vector of $2J$-column eigenvectors), and $\mathbf{Q}=\begin{bmatrix} q_1 & 0 & \cdots & 0 \\ 0 & q_2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & q_{2J}\end{bmatrix}$ is a $2J\times\ 2J$ diagonal matrix of eigenvalues. Following Rainville and Bedient (1981), the solution is then:

$$E_t\left[\mathbf{C}'\mathbf{s}\left(t+m\right)\right]=\mathbf{C}'\boldsymbol{\theta}+\sum_{j=1}^{2J}w_j\mathbf{p}_j\exp\left(q_jm\right) \tag{B.5}$$

where w_j are constants. The constants w_j may be identified using the boundary condition at $m=0$, i.e:

$$\begin{aligned}\mathbf{C}'\mathbf{s}\left(t\right) &= \mathbf{C}'\boldsymbol{\theta}+\sum_{j=1}^{2J}w_j\mathbf{p}_j \\ &= \mathbf{C}'\boldsymbol{\theta}+\mathbf{Pw}\end{aligned} \tag{B.6}$$

where $\mathbf{w}=\left[w_1,\ldots,w_{2J}\right]'$. Hence, $\mathbf{w}=\mathbf{P}'\mathbf{C}'\left[\mathbf{s}\left(t\right)-\boldsymbol{\theta}\right]$, given the property of eigenvectors that $\mathbf{P}^{-1}=\mathbf{P}'$.

Under the very mild requirement that the eigenvalues q_j are real and negative (as is effectively assumed in the special case of chapter 3), this establishes that the functional form for each state variable will be a constant plus a summation of exponential decay terms.[1] Hence, the expected path of the short rate will be the summation of constants and exponential decay terms.

B.2 The generalised ABE economy

In its most general form, the ABE economy may be expressed as the following stochastic vector differential equation:

$$\begin{bmatrix} d\mathbf{s}(t) \\ d\boldsymbol{\theta}(t) \end{bmatrix} = -\begin{bmatrix} \boldsymbol{\kappa}_1 & \boldsymbol{\kappa}_0 \\ \mathbf{0} & \mathbf{0} \end{bmatrix} \begin{bmatrix} \mathbf{s}(t) - \boldsymbol{\theta}(t) \\ \boldsymbol{\theta}(t) \end{bmatrix} dt + \begin{bmatrix} \boldsymbol{\sigma}_{11} & \boldsymbol{\sigma}_{10} \\ \boldsymbol{\sigma}_{01} & \boldsymbol{\sigma}_{00} \end{bmatrix} \begin{bmatrix} d\mathbf{z}_1(t) \\ d\mathbf{z}_0(t) \end{bmatrix} \tag{B.7}$$

where $\mathbf{s}(t) = [s_1(t), \ldots, s_{2J}(t)]'$ are the state variables; $\boldsymbol{\theta}(t) = [\theta_1(t), \ldots, \theta_{2J}(t)]'$ are the steady-state variables; $d\mathbf{z}_x(t) = [dz_{x,1}(t), \ldots, dz_{x,2J}(t)]'$ for $x = 1$ and 0 are Wiener increments; $\boldsymbol{\kappa}_x = \begin{bmatrix} \kappa_{x,1,1} & \cdots & \kappa_{x,1,2J} \\ \vdots & \ddots & \vdots \\ \kappa_{x,2J,1} & \cdots & \kappa_{x,2J,2J} \end{bmatrix}$ for $x = 1$ and 0 are the mean-reversion sub-matrices; and $\boldsymbol{\sigma}_{xy} = \begin{bmatrix} \sigma_{xy,1,1} & \cdots & \sigma_{xy,1,2J} \\ \vdots & \ddots & \vdots \\ \sigma_{xy,2J,1} & \cdots & \sigma_{xy,2J,2J} \end{bmatrix}$ for the combinations of $xy = 11$, 10, 01, and 00 are the standard deviation coefficient sub-matrices. Note that the $\mathbf{0}$ sub-matrix entries in the deterministic coefficient matrix ensure that the steady-state variables evolve as unit root processes. Other restrictions may also be introduced for compatibility with economic theory (e.g $\boldsymbol{\sigma}_{01} = \mathbf{0}$, so that short-run dynamics cannot influence long-run dynamics), but that does alter the mathematical nature of the exposition given here.

Following the approach used for solving the generalised BE model in section B.1,

[1]With reference to the economic interpretation of chapter 3, the requirement of real and negative eigenvalues implies that nominal GDP growth will be stationary and without regular cycles (i.e without the damped sinusoidal cycles that would result from the presence of any complex eigenvalues). These properties are readily evident from the casual observation of realised historical data.

the outer product of the stochastic terms is then:

$$\boldsymbol{\Omega} = \begin{bmatrix} \boldsymbol{\sigma}_{11} & \boldsymbol{\sigma}_{10} \\ \boldsymbol{\sigma}_{01} & \boldsymbol{\sigma}_{00} \end{bmatrix} \begin{bmatrix} \boldsymbol{\sigma}_{11} & \boldsymbol{\sigma}_{10} \\ \boldsymbol{\sigma}_{01} & \boldsymbol{\sigma}_{00} \end{bmatrix}' \tag{B.8}$$

As with the generalised BE model, the matrix $\boldsymbol{\Omega}$ will in general be non-diagonal, but it may be rotated into a diagonal representation. That is, $\boldsymbol{\Omega} = \mathbf{C}\boldsymbol{\Lambda}\mathbf{C}'$, where $\mathbf{C} = \{\mathbf{c}_1, \ldots, \mathbf{c}_{4J}\}$ is a matrix of order $4J\times\ 4J$ (i.e a $4J$-row vector of $4J$-column eigenvectors), and $\boldsymbol{\Lambda}$ is a $4J\times\ 4J$ diagonal matrix of eigenvalues. Pre-multiplying equation B.7 by $\mathbf{C}'$ then gives an orthogonal basis, i.e:

$$\begin{aligned} \mathbf{C}' \begin{bmatrix} d\mathbf{s}(t) \\ d\boldsymbol{\theta}(t) \end{bmatrix} &= -\mathbf{C}' \begin{bmatrix} \boldsymbol{\kappa}_1 & \boldsymbol{\kappa}_0 \\ \mathbf{0} & \mathbf{0} \end{bmatrix} \begin{bmatrix} \mathbf{s}(t) - \boldsymbol{\theta}(t) \\ \boldsymbol{\theta}(t) \end{bmatrix} dt \\ &\quad + \mathbf{C}' \begin{bmatrix} \boldsymbol{\sigma}_{11} & \boldsymbol{\sigma}_{10} \\ \boldsymbol{\sigma}_{01} & \boldsymbol{\sigma}_{00} \end{bmatrix} \begin{bmatrix} d\mathbf{z}_1(t) \\ d\mathbf{z}_0(t) \end{bmatrix} \end{aligned} \tag{B.9}$$

Applying the expectations operator as as at time t to equation B.9 gives:

$$E_t \left\{ \mathbf{C}' \begin{bmatrix} d\mathbf{s}(t+m) \\ d\boldsymbol{\theta}(t+m) \end{bmatrix} \right\} = -\mathbf{C}' \begin{bmatrix} \boldsymbol{\kappa}_1 & \boldsymbol{\kappa}_0 \\ \mathbf{0} & \mathbf{0} \end{bmatrix} \begin{bmatrix} E_t[\mathbf{s}(t+m)] - \boldsymbol{\theta}(t) \\ \boldsymbol{\theta}(t) \end{bmatrix} \tag{B.10}$$

and equation B.10 may be re-expressed as:

$$E_t \left\{ \mathbf{C}' \begin{bmatrix} d\mathbf{s}(t+m) \\ d\boldsymbol{\theta}(t+m) \end{bmatrix} \right\} = -\mathbf{C}' \begin{bmatrix} \boldsymbol{\kappa}_1 & \boldsymbol{\kappa}_0 \\ \mathbf{0} & \mathbf{0} \end{bmatrix} \mathbf{C} \begin{bmatrix} E_t[\mathbf{C}'\mathbf{s}(t+m)] - \mathbf{C}'\boldsymbol{\theta}(t) \\ \mathbf{C}'\boldsymbol{\theta}(t) \end{bmatrix} \tag{B.11}$$

As with the generalised BE model, the matrix $-\mathbf{C}' \begin{bmatrix} \boldsymbol{\kappa}_1 & \boldsymbol{\kappa}_0 \\ \mathbf{0} & \mathbf{0} \end{bmatrix} \mathbf{C}$ will in general not be diagonal, but the solution is obtained using the eigenvector and eigenvalue approach already outlined in section B.1. Hence, substitute $\mathbf{PQP}' = -\mathbf{C}' \begin{bmatrix} \boldsymbol{\kappa}_1 & \boldsymbol{\kappa}_0 \\ \mathbf{0} & \mathbf{0} \end{bmatrix} \mathbf{C}$, where $\mathbf{P} = \{\mathbf{p}_1, \ldots, \mathbf{p}_{4J}\}$ is a matrix of order $4J\times\ 4J$ (i.e a $4J$-row vector of $4J$-column eigenvectors), and $\mathbf{Q} = \begin{bmatrix} q_1 & 0 & \cdots & 0 \\ 0 & q_2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & q_{4J} \end{bmatrix}$ is a $4J \times 4J$ diagonal matrix of eigenvalues. Note that many of the eigenvalues q_j will be zero, given the unit root processes assumed for the steady-state variables. Following Rainville and Bedient (1981), the solution is then:

$$E_t \left\{ \mathbf{C}' \begin{bmatrix} \mathbf{s}(t+m) \\ \boldsymbol{\theta}(t+m) \end{bmatrix} \right\} = \sum_{j=1}^{4J} w_j \mathbf{p}_j \exp(q_j m) \tag{B.12}$$

where w_j are constants, and when $q_j = 0$, $\exp(q_j m) = \exp(0 \cdot m) = 1$. The constants w_j may be identified using the boundary condition at $m = 0$, i.e:

$$\begin{aligned} \mathbf{C}' \begin{bmatrix} \mathbf{s}(t) \\ \boldsymbol{\theta}(t) \end{bmatrix} &= \sum_{j=1}^{4J} w_j \mathbf{p}_j \\ &= \mathbf{P}\mathbf{w} \end{aligned} \tag{B.13}$$

where $\mathbf{w} = [w_1, \ldots, w_{4J}]'$. Hence, $\mathbf{w} = \mathbf{P}'\mathbf{C}' \begin{bmatrix} \mathbf{s}(t) \\ \boldsymbol{\theta}(t) \end{bmatrix}$. As for the generalised BE model, the functional form for each state variable will be a summation of constants (that result from the zero eigenvalues q_j) plus a summation of exponential decay terms (that result from the terms with non-zero eigenvalues q_j, again under the mild assumption that the latter are real and negative). Hence, the expected path of the short rate will be the summation of constants and exponential decay terms.

B.3 Two generalised ABE economies with an exchange rate

In its most general form, two ABE economies with an bilateral exchange rate may be expressed in the same form as equation B.7, i.e:

$$\begin{bmatrix} d\mathbf{s}_1(t) \\ d\boldsymbol{\theta}_1(t) \\ d\mathbf{s}_2(t) \\ d\boldsymbol{\theta}_2(t) \\ de(t) \\ d\theta_e(t) \end{bmatrix} = -\boldsymbol{\kappa}_{1,2,e} \begin{bmatrix} \mathbf{s}_1(t) - \boldsymbol{\theta}_1(t) \\ \boldsymbol{\theta}_1(t) \\ \mathbf{s}_2(t) - \boldsymbol{\theta}_2(t) \\ \boldsymbol{\theta}_2(t) \\ e(t) \\ \theta_e(t) \end{bmatrix} + \boldsymbol{\sigma}_{1,2,e} d\mathbf{z}_{1,2,e}(t) \tag{B.14}$$

where $\mathbf{s}_1(t)$ and $\mathbf{s}_2(t)$ are respectively the vectors of the state variables for economy 1 and 2; $\boldsymbol{\theta}_1(t)$ and $\boldsymbol{\theta}_2(t)$ are respectively the vectors of the steady-state variables for economy 1 and 2; $e(t)$ and $\theta_e(t)$ are respectively the state variable and steady-state variable for the bilateral exchange rate; $d\mathbf{z}_{1,2,e}(t)$ is the Wiener variable vector for the two economies and the exchange rate; $\boldsymbol{\kappa}_{1,2,e}$ is the mean-reversion coefficient matrix for the two economies and the exchange rate; and $\boldsymbol{\sigma}_{1,2,e}$ is the standard deviation coefficient matrix for the two economies and the exchange rate.

The terms $\boldsymbol{\kappa}_{1,2,e}$ and $\boldsymbol{\sigma}_{1,2,e}$ are, respectively, generalisations of the mean-reversion and standard deviation coefficient matrices in the generalised ABE model of section B.2. These will allow for dynamic dependencies between the state variables and steady-state variables of the two economies and the exchange rate. Suitable zero restrictions will be required to ensure that the steady-state variables evolve as unit root processes, and other restrictions may also be introduced for compatibility with economic theory (e.g so short-run dynamics cannot influence long-run dynamics).

The orthogonalisation of the state variables and the steady-state variables will then follow the processes already outlined for the generalised ABE model in section B.2.

The solution of the expected paths of the short rate in both economies will follow from that orthogonal representation, as for the generalised ABE model in section B.2. The result will again be a functional form for each state variable that is a summation of constants (that result from the zero eigenvalues q_j associated with the unit root processes for the steady-state variables) plus a summation of exponential decay terms (that result from the terms with non-zero eigenvalues q_j, again under the mild assumption that the latter are real and negative). Hence, the expected paths of the short rate in both economies will be the summation of constants and exponential decay terms.

Appendix C

Other aspects of fixed interest portfolio management

The primary purpose of this appendix is to discuss how the ANS portfolio framework developed in chapter 4 can be applied more generally to aspects of fixed interest portfolio management. The appendix also discusses the effect that transaction costs would have on the optimisation framework developed in chapter 4 (which excludes transaction costs, as standard in the literature), and in particular highlights why the additional complexity makes the inclusion of transactions costs beyond the initial scope of investigation in this thesis.

Section C.1 discusses how ex-ante risk can be calculated within the ANS portfolio framework, section C.2 shows how YCEs within the portfolio can be interpreted in terms of duration measures, and section C.3 discusses how portfolios can be optimised in conjunction with active trading. Section C.4 discusses how transaction costs could be introduced into the ANS portfolio framework.

C.1 Measuring ex-ante risk in fixed interest portfolios

As an extension to the FOYCE calculations in section 4.4.1 from chapter 4, the ANS model offers a very straightforward way to calculate value-at-risk (VaR) for fixed interest portfolios if the distribution of potential changes in the ANS coefficients can be calculated. That is, the results in section 3.1 show that the expected variance of the PV of the portfolio to a first-order approximation is $\mathrm{var}\left\{\left[\sum_{k=1}^{K} A_k \boldsymbol{\lambda}_k\right]' \boldsymbol{\delta}\right\}$. This may be expressed equivalently as $\left[\sum_{k=1}^{K} A_k \boldsymbol{\lambda}_k\right]' \mathrm{var}(\boldsymbol{\delta}) \left[\sum_{k=1}^{K} A_k \boldsymbol{\lambda}_k\right]$, where $\sum_{k=1}^{K} A_k \boldsymbol{\lambda}_k$ is a 3-vector containing the FOYCE components for the portfolio, and $\mathrm{var}(\boldsymbol{\delta})$ is a 3×3 variance-covariance matrix for changes in the ANS model coefficients over the required horizon. Under the typical assumptions of a multi-variate normal distribution (e.g see Hull (2000) pp. 345-351), the ANS coefficients on a daily basis give the following calculation of $\mathrm{var}(\boldsymbol{\delta})$ over the full sample period noted in the section 4.5.1:

$$\mathrm{var}\left(\boldsymbol{\delta}\right) = \begin{bmatrix} 38.7 & 28.1 & 8.6 \\ 28.1 & 103.4 & -76.8 \\ 8.6 & -76.8 & 103.4 \end{bmatrix} \mathrm{bp}^2 \tag{C.1}$$

Hence, the standard deviation of changes in the Level, Slope, and Bow coefficients are respectively 6.2, 10.2, and 10.2 bps. It is also evident that changes in the coefficients do not occur independently; i.e over the sample period there is material positive covariance between changes in the Level coefficient and changes in the Slope coefficient, and substantial negative covariance between changes in the Slope coefficient and changes in the Bow coefficient.

For the portfolio in table 4.2, $\sum_{k=1}^{K} A_k \boldsymbol{\lambda}_k$= (-\$144,600, \$21,053, -\$11,061)$'$. Hence, the standard deviation calculation for the daily VaR of this portfolio is:

$$\begin{aligned} \left[\sigma\left(\text{1-day}\right)\right]^2 &= \sqrt{\left[\sum_{k=1}^{K} A_k \boldsymbol{\lambda}_k\right]' \mathrm{var}\left(\boldsymbol{\delta}\right) \left[\sum_{k=1}^{K} A_k \boldsymbol{\lambda}_k\right]} \\ &= \$871,552 \end{aligned}$$

The daily VaR corresponding to a given threshold level of significance x is $\sigma(\tau) \cdot \Phi^{-1}(x)$, where $\Phi^{-1}(x)$ is the inverse normal distribution. A typical threshold level significance is 1%, and $\Phi^{-1}(0.01) = -2.33$. Hence, the 1% daily VaR for the portfolio in table 4.2 is $\$871,552 \times -2.33 = -\$2,027,509$; i.e there is a 1% probability of a loss of \$2,027,509 or more in a single day.

This procedure is analogous to the principal components approach noted in Hull (2000) pp. 357-363. However, the advantages of using the ANS model as a basis for calculating VaR are: (1) it can simultaneously be used to determine the relative values of securities that define the yield curve; (2) it can be applied precisely to any maturity or series of cashflows, rather than "bucketing" securities into maturity bands; and (3) the risk components can be visualised and related back to expectations of inflation and output growth in the underlying economy (and section C.3 will show how the risk components can be targeted precisely and independently from each other). Note also that the ANS framework SOYCE components could be included to extend the VaR calculation to a quadratic approximation, as with the model noted in Hull (2000) pp. 352-355.

Of course, one major critique of the typical VaR calculation is the assumption of multi-variate normal distributions; in practice, the tails of the distributions of financial market variables do not often accord closely to those of the normal distribution. However, VaR calculations independent of the multi-variate normal distributions could still be undertaken conveniently within the ANS framework using alternative methods to generate the distribution of $\boldsymbol{\delta}$. For example, the historical simulation approach noted in Hull (2000) p. 356 could be undertaken using simulations based on the historical values of $\boldsymbol{\delta}$, and then applying those to the FOYCE vector (and the SOYCE matrix in the quadratic approximation) of the current portfolio to build up a distribution of potential changes in portfolio value. Alternatively, samples of $\boldsymbol{\delta}$ could be generated via multi-variate time-series models of the Level, Slope, and Bow coefficients (potentially allowing for generalised time-varying volatility, e.g ARCH or GARCH models), which would again be applied to the FOYCE vector of the current portfolio to build up a distribution of potential changes in portfolio value.

C.2 Level, Slope, and Bow durations

Another extension to the FOYCE calculations in section 4.4.1 is to express them as standardised Level, Slope, and Bow durations analogous to the traditional measure of duration; i.e the percentage change in portfolio value for a 1 percentage point level shift in the yield curve. This is achieved by simply dividing each FOYCE component by the portfolio MV. Hence, $\frac{1}{\mathrm{MV}}\sum_{k=1}^{K} A_k \boldsymbol{\lambda}(1)_k$ is the percentage change in the value of the portfolio for a 1 percentage point change in the Level coefficient (i.e a shift in the yield curve by 1 percentage point for all maturities). The quantities $\frac{1}{\mathrm{MV}}\sum_{k=1}^{K} A_k \boldsymbol{\lambda}(2)_k$ and $\frac{1}{\mathrm{MV}}\sum_{k=1}^{K} A_k \boldsymbol{\lambda}(3)_k$ are the Slope and Bow durations respectively, which have the interpretation of the percentage change in the portfolio MV given a percentage point change in the Slope or Bow coefficients. These are analogous to the partial duration measures of Golub and Tilman (2000) pp. 24-25.

As an example, the portfolio in table 4.2 is composed of \$150 million face-value swaps. This has a MV of zero, but if it were backed by \$150 million of overnight cash (i.e a maturity of $m = 1/365$ years), then the MV would be \$150 million, and the FOYCE components would remain essentially unchanged. The Level duration would then be -\$144600 per bp / \$150 million $\times$ 1 percentage point per 100 bps $=$ -9.64 (no unit); i.e approximately 9.6 years of traditional duration. The Slope duration would be 1.40, and so a 1 percentage point increase in the Slope coefficient (i.e a steepening of the yield curve) would increase the MV of the portfolio by 1.4%. The Bow duration would be -0.74, so a "down-bowing" of mid-maturity bonds relative to short and long-maturity bonds would increase the MV of the portfolio.

The VaR may also be expressed in proportional terms. That is, using the calculation from section A.1, the proportional VaR would be \$2,027,509 / \$150 million = 1.35. Hence, there is a 1% probability of a loss of 1.35% of portfolio value or more in a single day.

The second-order terms in the matrix $\sum_{k=1}^{K} A_k \mathbf{\Omega}_k$ can also be scaled by $\frac{1}{\mathrm{MV}}$ to make the second-order sensitivities of portfolios with different market values comparable. For example, $\frac{1}{\mathrm{MV}} \sum_{k=1}^{K} A_k \mathbf{\Omega}\,(1,1)_k$ is analogous to the traditional measure of convexity, while the remaining diagonal elements would give the Slope and the Bow convexities, and the off-diagonal elements would give the Level-Slope, Level-Bow, and Slope-Bow cross-term convexities. These are analogous to the partial convexity measures of Golub and Tilman (2000) pp. 24-25.

C.3 Portfolio optimisation with active trading

The ANS portfolio framework developed in this chapter is also directly applicable to active yield curve trading; i.e where the portfolio manager deliberately seeks to take on YCEs relative to the initial/benchmark portfolio based on a view of how the yield curve is likely to change. If the view is proven correct, then the PV and hence MV of the portfolio will increase relative to the initial/benchmark portfolio, but a relative loss will occur if the yield moves in the opposite direction.

As background, a minimum of four securities is required to perfectly match any given MV and three FOYCE components. Hence, in principle, any four securities could be transacted to change the MV and three FOYCE components to those desired. Given a set of four securities, the required face-values to transact could be found by straightforward matrix algebra. However, the transaction might not be allowed if it breached any constraints on the amounts of securities allowed in the portfolio, so substantial trial and error on the selection of the four trading securities (from the allowable universe) might be required before an allowable transaction is found.

Conversely, the optimisation framework in this chapter automatically calculates the optimal feasible solution, if a solution exists. The optimisation problem expressed relative to a benchmark or initial portfolio is a trivial variation on the linear programme of equation 4.14, i.e:

$$\text{Maximise:} \quad \boldsymbol{\alpha}_t' \mathbf{A}_{1,t} \tag{C.2a}$$

$$\text{subject to:} \quad \boldsymbol{\Lambda}\mathbf{A}_{1,t} = \boldsymbol{\Lambda}\mathbf{A}_{0,t} + \boldsymbol{\kappa}_t \tag{C.2b}$$

$$\text{and:} \quad A_{1,k,\min} \leq A_{1,k,t} \leq A_{1,k,\max} \tag{C.2c}$$

where $\boldsymbol{\kappa}_t$ represents the desired (or acceptable) differences, at time t, between the MV and FOYCEs of the alternative portfolio and the initial/benchmark portfolio. For example, a pure relative slope/twist/curve trade may be specified by $\boldsymbol{\kappa}_t = (0, 0, \$x_2, 0)'$, and a portfolio constructed with $\boldsymbol{\Lambda}\mathbf{A}_1 = \boldsymbol{\Lambda}\mathbf{A}_0 + \boldsymbol{\kappa}_t$ would return $\$x_2$ relative to the initial/benchmark portfolio for each bp increase in the Slope coefficient. Any changes in the Level or Bow coefficients would deliver zero change relative to the initial/benchmark portfolio. Similarly, $\boldsymbol{\kappa}_t = (0, 0, 0, \$x_3)'$ represents a pure relative bow/barbell/butterfly/curvature trade. Hybrid trades with several distinct exposures to the yield curve could be specified using several non-zero entries in $\boldsymbol{\kappa}$; e.g an upward level shift exposure in tandem with a downward bow exposure would be expressed as $\boldsymbol{\kappa}_t = (0, \$x_1, 0, -\$x_3)'$. If cash were being injected or withdrawn relative to the initial/benchmark portfolio, then the first element of $\boldsymbol{\kappa}_t$ could be set to a non-zero amount to represent the change in the MV of the portfolio relative to the initial/benchmark portfolio.

Note that one possible output of the linear programme of equation C.2 is "infeasible". This would indicate that the desired MV and FOYCEs cannot be obtained simultaneously given the portfolio constraints, and therefore $\boldsymbol{\kappa}_t$ would need to be adjusted (or the constraints relaxed, if possible) to obtain a feasible solution.

For active portfolio exposures, the ANS portfolio framework is highly desirable relative to a "black box" of variances and covariances, for four reasons: (1) the intended exposure/s to yield curve changes may be visualised using the ANS modes; (2) the intended active risks may be precisely specified and constructed, as noted above; (3) the yield curve exposures have a direct link back to risks around the consensus in the underlying economy; and (4) the optimisation framework determines the portfolio with the highest relative value that achieves the desired exposure/s to yield curve changes. This will act to enhance portfolio returns at the margin, independently of whether the active yield curve trading is successful or not.

C.4 Portfolio optimisation including transactions costs

Most transaction costs vary depending on volume, and are incurred regardless of the direction of the transaction.[1] For example, when securities are transacted at the bid or the ask rate, a cost (i.e a loss of MV) of half the bid-ask spread is incurred immediately when the portfolio is subsequently revalued at mid-rates. Brokerage is another variable cost, being proportional to the face values or market values of the traded securities (e.g the cost per unit on futures contracts).

Variable transaction costs enter into the objective function as an absolute value,

[1]Fixed transactions costs (e.g fixed overheads and/or non-volume-dependent settlement charges) could readily be included in the linear programme of equation 4.14 without changing the nature of the optimisation problem.

and so the objective function of the linear programme of equation 4.14 would become:[2]

$$\begin{aligned}
\text{Maximise} \quad &: \quad \sum_{k=1}^{K} \alpha_k A_k - z_k \left|\Delta A_k\right| && \text{(C.3a)}\\
\text{subject to:} \quad & \quad \mathbf{\Lambda}_t \mathbf{A}_{1,t} = \mathbf{\Lambda}_t \mathbf{A}_{0,t} && \text{(C.3b)}\\
\text{and:} \quad & \quad A_{1,k,\min} \leq A_{1,k,t} \leq A_{1,k,\max} && \text{(C.3c)}
\end{aligned}$$

where ΔA_k represents the change to the face value of security k; z_k represents the variable cost of transacting security k, and the absolute value is applied to ensure the transaction cost will always be positive regardless of whether ΔA_k is positive or negative. Unfortunately, the latter makes the optimisation problem non-linear (and highly so because the absolute value function is discontinuous in the first derivative).

Another consideration with transactions costs is that it is not optimal to rebalance back to the target (i.e the benchmark or the desired active) YCEs. Rather, as noted in Donohue and Yip (2003), if the portfolio is within allowable thresholds from the target (which could be defined by risk tolerances), then no transaction is required. If a portfolio is outside allowable thresholds, then the optimal transaction is that which takes the portfolio back to the allowable threshold, not to the benchmark. This is relatively easy to operationalise if there is only a threshold in a single dimension (e.g a bond versus equity allocation in a balanced portfolio), but it becomes increasingly complex as the number of dimensions increases. The ANS model has three dimensions, and would therefore require three thresholds relating to allowable tolerances on Level, Slope, and Bow exposure.

[2]Variable transaction costs cannot be included in the ANS portfolio optimisation framework by simply subtracting the cost of the trade that the optimisation framework recommends, because that would often lead to non-optimal transactions. For example, it would not be optimal to transact a trade with positive relative value if the cost of the transaction outweighs that relative value.