



Angular Dependence of Third-order Law in Anisotropic MHD Turbulence

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Abstract

In solar wind turbulence, the energy transfer/dissipation rate is typically estimated using MHD third-order structure functions calculated using spacecraft observations. However, the inherent anisotropy of solar wind turbulence leads to significant variations in structure functions along different observational directions, thereby affecting the accuracy of energy dissipation rate estimation. An unresolved issue is how to optimise the selection of observation angles under limited directional sampling to improve estimation precision. We conduct a series of MHD turbulence simulations with different mean magnetic field strengths, B_0 . Our analysis of the third-order structure functions reveals that the global energy dissipation rate estimated around a polar angle of $\theta = 60^\circ$ agrees reasonably with the exact one for $0 \leq B_0/b_{\text{rms}} \leq 5$, where b_{rms} denotes the rms magnetic field fluctuation. The speciality of 60° polar angle can be understood by the mean value theorem of integrals, since the spherical integral of the polar-angle component ($\tilde{T}_\theta$) of the divergence of Yaglom flux is zero, and $\tilde{T}_\theta$ changes sign around 60° . Existing theory on the energy flux vector as a function of the polar angle is assessed, and supports the speciality of the 60° polar angle. The angular dependence of the third-order structure functions is further assessed with virtual spacecraft data analysis. The present results can be applied to measure the turbulent dissipation rates of energy in the solar wind, which are of potential importance to other areas in which turbulence takes place, such as laboratory plasmas and astrophysics.

Unified Astronomy Thesaurus concepts: Solar wind (1534); Interplanetary turbulence (830); Magnetohydrodynamical simulations (1966); Magnetohydrodynamics (1964); Space plasmas (1544)

1. Introduction

Magnetohydrodynamic (MHD) turbulence, characterized by high Reynolds numbers, is ubiquitous in natural environments, including the solar wind (J. Coleman & J. Paul 1968; J. Jokipii & J. V. Hollweg 1970; E. Parker 1979; W. Matthaeus & M. Goldstein 1982; C. Tu & E. Marsch 1995; R. Bruno & V. Carbone 2013). The solar wind, a steady flow of charged particles that fills the heliosphere, plays a significant role in space weather and satellite operation. Therefore, the study of solar wind is vital for both scientific inquiry and engineering solutions. At scales larger than the kinetic ones, MHD turbulence is capable of modeling solar wind fluctuations. An important parameter in the study of turbulence is the energy dissipation rate (ε), which also measures the intensity of the cascade process. Understanding the energy cascade in MHD turbulence is crucial to understanding the heating phenomena in the solar wind (W. Matthaeus & M. Goldstein 1982). However, it is not currently possible to directly measure ε (or its cousins) in the solar wind. In space environments, plasmas generally act as (almost) collisionless systems, so that viscosity and resistivity may be difficult to even define, and thus direct measurements of viscous and resistive dissipation rates are typically not feasible (see Y. Yang et al. 2024; S. Adhikari et al. 2025; L. Adhikari et al. 2025).

An alternative approach is to estimate the energy dissipation rates indirectly, based on the MHD third-order law, obtained by extending Kolmogorov theory to MHD turbulence (H. Politano & A. Pouquet 1998a, 1998b). Assuming isotropic MHD turbulence (H. Politano & A. Pouquet 1998a; L. Sorriso-Valvo et al. 2007; B. MacBride et al. 2008; J. Stawarz et al. 2009), one obtains the isotropic third-order law:

$$Y_\ell^\pm(\ell) \stackrel{\text{def}}{=} \langle \delta z_\parallel^\mp (\delta z^\pm)^2 \rangle = -\frac{4}{3} \bar{\varepsilon}^\pm \ell. \quad (1)$$

In a standard notation, $z^\pm = \mathbf{u} \pm \mathbf{b}$ are the Elsässer variables with $\mathbf{u}$ the velocity fluctuation field and $\mathbf{b}$ the fluctuating magnetic field in Alfvén units (i.e., $\mathbf{b}$ is scaled by $\sqrt{4\pi\rho}$ with ρ the uniform mass density). The Elsässer increments are defined as $\delta z^\pm(\mathbf{x}, \ell) = z^\pm(\mathbf{x} + \ell) - z^\pm(\mathbf{x})$ and $\delta z_\parallel^\pm = \delta z^\pm \cdot \ell/\ell$, for general position vector $\mathbf{x}$ and lag ℓ . ℓ vector in the case of solar wind measurements is the “longitudinal” direction aligned with the mean flow. Here, $\bar{\varepsilon}^\pm$ represent the globally averaged dissipation rates of Elsässer energies, $\langle |z^\pm(\mathbf{x})|^2 \rangle/2$.

An important limitation of Equation (1) is that it relies on the isotropic assumption, which is often violated in MHD systems with a mean magnetic field, such as the solar wind. Turbulent systems threaded by a mean magnetic field, B_0 , inherently display anisotropy (e.g., J. Shebalin et al. 1983; S. Oughton et al. 1994; W. Matthaeus et al. 1996; T. S. Horbury et al. 2008; Q. Luo & D. Wu 2010; S. Oughton & W. Matthaeus 2020). This anisotropy introduces complications regarding the measurement of energy transfer, which



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future solar wind missions such as HelioSwarm will seek to address (W. H. Matthaeus et al. 2022; H. E. Spence 2019; A. Retinò et al. 2022; M. F. Marcucci & A. Retinò 2024).

Anisotropic aspects of third-order laws have been investigated in several MHD simulation studies. M. Wan et al. (2009, 2010) derived a spherically integrated form of the third-order moment in anisotropic MHD turbulence in the presence of a velocity shear, and verified it with direct numerical simulation (DNS). Later, Y. Wang et al. (2022) and B. Jiang et al. (2023) studied the integral form numerically under external mean magnetic fields. Regarding the angular dependence of the third-order law, A. Verdini et al. (2015) demonstrated that the isotropic third-order law, when evaluated at a fixed polar angle θ , underestimates the globally averaged dissipation rates for θ below 40° and overestimates them beyond 70° . Thus, for the purposes of estimating dissipation rates, there may be an optimal polar angle between 40° and 70° . In an investigation of the anisotropic energy transfer induced by the mean magnetic field, B. Jiang et al. (2023) found that the isotropic third-order law evaluated at $\theta = 60^\circ$ matches the fully direction-averaged third-order law (see Section 2 for definitions) to better than 5%. Whereas, at other values of θ , the discrepancy can be as high as 45%. An aim of the present work is to confirm and understand this speciality of the 60° polar angle.

In practice, unfortunately, in situ spacecraft observations typically lack full directional coverage, limiting the directional averaging that may be performed (K. T. Osman et al. 2011). Consequently, many of the studies using solar wind data to calculate longitudinal third-order moments have in essence employed Equation (1) at a fixed polar angle. Single-point measurements with clear evidence of linear scaling for the third-order moment were first presented by L. Sorriso-Valvo et al. (2007). They also found that there are some regions without such isotropic third-order law scaling, which they suggested was mainly due to the anisotropy associated with the high-latitude regions, with compressibility and inhomogeneity as secondary factors. Based on the isotropic third-order law, B. MacBride et al. (2008) adopted a 1D + 2D method to estimate the dissipation rate, reducing the estimation error. However, J. Stawarz et al. (2009) argued that statistical convergence can significantly affect third-order scaling, and claimed that it requires about a year of ACE data at 64 s cadence to reduce the fractional error of the dissipation rate estimation below 30%. The majority of previous works have leaned toward the use of the isotropic third-order law, with several exceptions. For example, K. T. Osman et al. (2011) and R. Bandyopadhyay et al. (2018) used the Cluster (C. P. Escoubert et al. 2001) and Magnetospheric Multiscale (MMS) missions (J. L. Burch et al. 2016) data, respectively, with the directional averaging technique to estimate the dissipation rates. More details regarding such observational studies can be found in the review paper (R. Marino & L. Sorriso-Valvo, 2023). F. Pecora et al. (2023b) employed MMS data and the lag polyhedral derivative ensemble method (F. Pecora et al. 2023a) to estimate the divergence of the Yaglom flux in lag space directly, and attain more accurate and robust estimations for the dissipation rate, without making any assumption about isotropy. It is clear that studying the angular dependence of the third-order law is meaningful, especially in attempting to find an optimized angle for employing spacecraft measurements.

The layout of the paper is as follows. Section 2 provides a concise overview of the third-order law. Section 3 presents the

numerical method, covering the simulation configurations and some important characteristics of the (statistically steady) flow and magnetic fields. In Section 4, the examination of the energy flux, including its vector properties and divergence, will be studied to explore the speciality of the 60° polar angle. Existing theoretical relationships between the polar angle and energy flux components will be verified. Further, the effects of the azimuthal average will be assessed. Finally, the angular dependence will be explored with virtual spacecraft data analysis. The conclusions are summarized at the end.

2. Third-order Law

This section outlines three different averaging methods pertinent to the third-order law, as discussed in B. Jiang et al. (2023). Beginning with the von Kármán–Howarth (vKH) equation (T. de Karman & L. Howarth 1938; A. Monin & A. Yaglom 1971; U. Frisch 1995; H. Politano & A. Pouquet 1998b), we employ a general expression for an MHD third-order law. In an inertial range (i.e., the scales for which the nonstationary and dissipative terms in the vKH equation are negligible), the cross-scale energy transfer may be expressed as

$$\nabla_\ell \cdot \mathbf{Y}^\pm = \nabla_\ell \cdot \langle \delta \mathbf{z}^\mp |\delta \mathbf{z}^\pm|^2 \rangle = -4\varepsilon^\pm, \quad (2)$$

where $\mathbf{Y}^\pm(\ell) = \langle \delta \mathbf{z}^\mp |\delta \mathbf{z}^\pm|^2 \rangle$ are third-order structure functions, also called the Elsässer energy or Yaglom flux vectors, and angle brackets $\langle \cdot \rangle$ denote ensemble averaging (assumed equal to averaging over $\mathbf{x}$ for simulation data). As noted earlier, the Elsässer variables are $\mathbf{z}^\pm = \mathbf{u} \pm \mathbf{b}$, with increments of $\delta \mathbf{z}^\pm(\mathbf{x}, \ell) = \mathbf{z}^\pm(\mathbf{x} + \ell) - \mathbf{z}^\pm(\mathbf{x})$. Our simulations will employ hyper-dissipation so that explicit forms for the mean dissipation rates of Elsässer energies are $\varepsilon^\pm = \nu_h \sum_{\mathbf{k}} k^{2h} \langle |\hat{\mathbf{z}}^\pm(\mathbf{k}, t)|^2 \rangle$,

where h denotes the hyperviscosity index, and ν_h denotes the hyperviscosity coefficient, taken equal to the hyper-resistivity. The gradient ∇_ℓ operates in lag space (i.e., wrt the coordinates of the lag vector ℓ). In particular, in spherical coordinates we have

$$\begin{aligned} \nabla_\ell \cdot \mathbf{Y}^\pm &= \frac{1}{\ell^2} \frac{\partial(\ell^2 Y_\ell^\pm)}{\partial \ell} + \frac{1}{\ell \sin \theta} \frac{\partial(\sin \theta Y_\theta^\pm)}{\partial \theta} \\ &+ \frac{1}{\ell \sin \theta} \frac{\partial(Y_\phi^\pm)}{\partial \phi} = T_\ell^\pm + T_\theta^\pm + T_\phi^\pm, \end{aligned} \quad (3)$$

where, for convenience of description, the rightmost terms, $T_\ell^\pm$, $T_\theta^\pm$, and $T_\phi^\pm$, are shorthand for the contributions to this divergence. We take the z -axis to be in the direction of the external mean field, $\mathbf{e}_z$, so that θ is the angle between the lag vector and $\mathbf{B}_0$. Note that because the turbulence is homogeneous and Equation (2) involves (ensemble) averaging, there is no dependence of $\mathbf{Y}^\pm$ on position $\mathbf{x}$, and the $\varepsilon^\pm$ are independent of both position $\mathbf{x}$ and lag ℓ .

Equation (2) can be reformulated in terms of various levels of angle averaging. First, taking a volume integral over a sphere with radius $\ell = |\ell|$ yields

$$\iiint_{|\ell| \leq \ell} \nabla_\ell \cdot \mathbf{Y}^\pm dV = \iiint_{|\ell| \leq \ell} -4\varepsilon^\pm dV = -\frac{16\pi}{3} \varepsilon^\pm \ell^3. \quad (4)$$

Using Gauss's theorem, this can be written as a surface integral,

$$\oint_{|\ell|=\ell} Y_\ell^\pm dS = -\frac{16\pi}{3}\varepsilon^\pm\ell^3, \quad (5)$$

where the longitudinal third-order structure functions $Y_\ell^\pm = \langle \delta z_\ell^\mp |\delta z^\pm|^2 \rangle$ is the projection of the energy flux vectors along ℓ and $\delta z_\ell^\mp = \delta z^\mp \cdot \frac{\ell}{\ell}$. Using spherical coordinates, we may express this in terms of the solid angle average of $Y_\ell^\pm$:

$$\frac{1}{4\pi} \int_0^{2\pi} \int_0^\pi Y_\ell^\pm \sin \theta d\theta d\phi = -\frac{4}{3}\varepsilon^\pm\ell, \quad (6)$$

where θ represents the polar angle (from the $\mathbf{B}_0$ axis) and ϕ the azimuthal angle. Considering that no assumptions about rotational symmetry are made in going from Equations (2) to (6), the physical content of Equation (6) is as general as the derivative form of Equation (2). The full generality of Equation (6) follows from the rigorous theorem given by Q. Nie & S. Tanveer (1999) and restated in more accessible terms by M. A. Taylor et al. (2003) and Y. Wang et al. (2022). However, Equation (6) is simpler in the sense that accurate determination of integration only requires the longitudinal component of the energy flux vectors, $Y_\ell^\pm$, on the spherical surface spanned by the coordinates (θ, ϕ) in the 3D lag space.

The most general form of $Y_\ell^\pm$ should be a function of ℓ , θ , and ϕ ; that is, $Y_\ell^\pm(\ell, \theta, \phi)$. The theory of tensor invariants may be used to obtain general constraints on the functional form for $Y_\ell^\pm$ (e.g., H. P. Robertson 1940; G. K. Batchelor 1970; J. Podesta et al. 2007). Previous studies have typically restricted themselves to the purely isotropic assumption ($Y_\ell^\pm$ independent of θ and ϕ) or treated anisotropic turbulence with azimuthal symmetry ($Y_\ell^\pm$ independent of ϕ) as implemented, for example, by J. Stawarz et al. (2009). Assuming isotropy, Equation (6) reduces to

$$\frac{1}{4\pi} Y_{\ell,\text{iso}}^\pm(\ell) \int_0^{2\pi} \int_0^\pi \sin \theta d\theta d\phi = Y_{\ell,\text{iso}}^\pm(\ell) = -\frac{4}{3}\varepsilon^\pm\ell, \quad (7)$$

which we refer to as the *isotropic (third-order) law*. This is the same as Equation (1).

To better understand the anisotropic energy transfer in the inertial range, we provide here a systematic study of the dependence of the (longitudinal) structure function $Y_\ell^\pm(\ell, \theta, \phi)$ on θ and ϕ for situations with different guide field magnitudes. Since simulation (and spacecraft) data are only available at discretely spaced points, we will form estimates of $Y_\ell^\pm$ using discrete sets of angles. Specifically, to cover the sphere, we select to use lag vectors in 37 directions, uniformly spaced in azimuthal and polar angles. That is, $\theta \in [0^\circ: 15^\circ: 90^\circ]$ and $\phi \in [0^\circ: 60^\circ: 300^\circ]$; the middle value in the square brackets indicates the step size for the angle ($\Delta\theta = 15^\circ$ and $\Delta\phi = 60^\circ$). Note that for $\theta = 0^\circ$, the azimuthal angle does not play a role, since it is formally undefined. Moreover, due to the uniform external mean magnetic field, we may restrict the range of θ to $[0^\circ, 90^\circ]$. Thus, we have $N_i = 7$ polar angles and $N_j = 6$ azimuthal angles (except for $\theta = 0$ when $N_j = 1$). To ensure accuracy when calculating the divergence of the energy flux in Section 4.2 and the fitting of the structure function with varying θ in model verification in Section 4.3, we employ a

finer grid with $N_i = 23$ uniformly spaced polar angles. Also, as the divergence involves factors of $\sin \theta$ in several denominators, we drop the $\theta = 0$ points from these calculations. A 3D Lagrangian interpolation is used to estimate data values located between grid points.

In order of increasing levels of angle averaging, the three estimates for $Y_\ell^\pm$ we consider are as follows:

Method (I). $Y_\ell^\pm$ is evaluated at discrete pairs of angles, θ_i and ϕ_j , but with no averaging over angles:

$$Y_\ell^\pm(\ell, \theta_i, \phi_j). \quad (8)$$

This represents a local radial (or longitudinal) energy transfer, where “local” means at the specific azimuthal and polar angles. The total radial energy transfer at scale ℓ is the sum of the Equation (8) contributions from all azimuthal ϕ_j and polar θ_i directions at the same lag length. Separate estimates are made for each of the 37 directions; that is, $Y_\ell^\pm(\ell, \theta_i, \phi_j)$ is calculated for $\theta_i \in [0^\circ: 15^\circ: 90^\circ]$ and $\phi_j \in [0^\circ: 60^\circ: 300^\circ]$.

Method (II). The azimuthally averaged form of the third-order structure function

$$\begin{aligned} \widetilde{Y}_\ell^\pm(\ell, \theta_i) &= \frac{1}{2\pi} \int_0^{2\pi} Y_\ell^\pm(\ell, \theta_i, \phi) d\phi \\ &\approx \frac{\sum_{j=1}^{N_j} Y_\ell^\pm(\ell, \theta_i, \phi_j)}{N_j} \end{aligned} \quad (9)$$

describes the (polar) anisotropy of local radial energy transfer, where here “local” means at a specific polar angle. The total transfer rate is the sum over all θ_i of the contributions given by Equation (9) at a given ℓ . In practice, this estimate just makes appropriate averages over ϕ_j of the estimates determined using Method I, Equation (8).

Method (III). The “full”⁶ direction-averaged form of the third-order structure function,

$$\begin{aligned} \overline{Y}_\ell^\pm(\ell) &= \frac{1}{4\pi} \int_0^{2\pi} \int_0^\pi Y_\ell^\pm \sin \theta d\theta d\phi \\ &\approx \frac{\sum_{j=1}^{N_j} \sum_{i=1}^{N_i} Y_\ell^\pm(\ell, \theta_i, \phi_j) \sin \theta_i}{N_j \sum_{i=1}^{N_i} \sin \theta_i}. \end{aligned} \quad (10)$$

Recall, $N_i (=7)$ and $N_j (=6, \text{ usually})$ indicate the number of θ_i and ϕ_j angles used, respectively. This direction-averaged version of the (longitudinal) third-order structure function is formulated straight from the vKH equation—without assuming statistical isotropy—and is solely dependent on the lag length ℓ . Assuming that the conditions needed for Equation (6) (and Equation (2)) to hold apply, normalizing $\overline{Y}_\ell^\pm$ with the factor $-4\ell/3$ provides an estimate of the cross-scale energy transfer rate (or energy dissipation rate), ε . Additionally, the range of scales over which this linear scaling with ℓ holds may be used to estimate the range of scales comprising the inertial range.

These three variants of the third-order structure functions enable approximations of the true energy dissipation rates, $\varepsilon^\pm$. For example, Method I, Equation (8), is commonly employed in observational studies involving a single spacecraft. Our focus herein is on the *total* energy dissipation rate, so that for

⁶ Here, “full” means with respect to the discrete θ_i, ϕ_j grid, rather than over the continuum of angles.

Table 1
Numerical Configuration Parameters and Steady-state Values for Some Key Quantities

Cases	B_0	h	Grids	ε_v	ε_b	b_{rms}	$\text{Re}_{\lambda,v}$	$\text{Re}_{\lambda,b}$	$k_{\text{max}}\eta_{k,v}$	θ_v (deg)	Averaging Period (T_e)
A	0	2	512^3	0.67	1.24	0.92	846	255	1.73	55	[15:30]
B	2	2	512^3	0.77	1.12	1.14	951	473	1.71	72	[8:68]
C	5	2	512^3	0.83	1.02	1.08	2415	435	1.70	83	[200:275]
D	5	1	512^3	0.93	0.84	1.04	289	164	1.68	81	[8:13]
E	5	1	1024^3	0.81	0.80	1.02	467	166	1.71	83	[9:18]

Note. The external mean magnetic field strength is B_0 . h denotes the hyperviscosity index, and ε_v and ε_b represent the kinetic and magnetic dissipation rates, respectively. $\text{Re}_{\lambda,v}$ and $\text{Re}_{\lambda,b}$ are the kinetic and magnetic Taylor Reynolds numbers; $k_{\text{max}}\eta_{k,v}$ illustrates the grid resolution, where $\eta_{k,v}$ and k_{max} are respectively the Kolmogorov length scale in the velocity field and the maximum resolved wavenumber (a third of the total grid in one direction); T_e refers to the large-eddy turnover time during the statistically steady period for the $B_0 = 0$ case. b_{rms} denotes the rms magnetic field fluctuation in the stationary period. θ_v describes the anisotropy angle for the velocity field as introduced in J. Shebalin et al. (1983).

each of the three forms $Y_\ell^\pm$, $\widetilde{Y}_\ell^\pm$, and $\overline{Y}_\ell^\pm$, we average their positive and negative components. These are then normalized by $-\frac{4}{3}\varepsilon\ell$, where $\varepsilon = (\varepsilon^+ + \varepsilon^-)/2 = \varepsilon_b + \varepsilon_v$ signifies the total energy dissipation rate. This yields, for example, $-3(Y_\ell^+ + \widetilde{Y}_\ell^-)/(8\varepsilon\ell)$, which should be unity for inertial range scales. As our simulations have low cross helicity, we expect statistical equality of the positive and negative components so that averaging them should not make much difference to the ε estimates. The situation in the inner heliosphere solar wind is different, with significant cross helicity levels present there.

3. Numerical Configurations

We solve the 3D incompressible MHD equations numerically, in Fourier space, using a pseudo-spectral method, incorporating the two-thirds rule for dealiasing (S. A. Orszag 1971, 1972; D. Gottlieb & S. A. Orszag 1977; S. A. Orszag & C.-M. Tang 1979; C. Canuto et al. 2007). Our simulations are conducted in a cubic domain $[0, 2\pi]^3$, with periodic boundary conditions and the second-order Adams–Bashforth method for time integration. An external force $\mathbf{f}_v$ is included in the momentum equation (Y. Yang et al. 2021). This forcing is exerted solely on the first two wavenumber shells ($k = 1, 2$) to support development of a wide inertial range; here, $k = |\mathbf{k}|$.

Some key observables for the simulations are listed in Table 1. Since the inertial range in the hyperviscous setup is longer than the standard Laplacian one, which is helpful to test the model from J. Podesta et al. (2007), we chose to present the hyperviscous results (Cases A, B, and C) in the main text, and Cases D and E as references to verify the effects of Re and hyperviscosity in the Appendix. These variables have been averaged over time; see B. Jiang et al. (2023) for further details. An external mean magnetic field, $\mathbf{B}_0 = B_0\mathbf{e}_z$, is imposed along the z -direction. The magnetic fluctuation is $b_{\text{rms}} \approx 1$ as shown in Table 1; thus, $0 \leq B_0/b_{\text{rms}} \leq 5$. The anisotropic effects on the coherent structures can be illustrated directly using the current intensity magnitude, $J = |\nabla \times \mathbf{b}|$; see Figure 1. With increasing B_0 , the structures tend to be more elongated along the mean magnetic field direction ($\mathbf{e}_z$), while the dynamics in the perpendicular plane remain statistically isotropic. Furthermore, the turbulent length scale for the parallel ($\mathbf{B}_0$) direction is obviously larger than that for the perpendicular direction, consistent with established results (e.g., D. C. Robinson & M. G. Rusbridge 1971; S. J. Zweben et al. 1979; J. M. Weygand et al. 2009; M. E. Ruiz et al. 2011).

We note that the values of the cascade rates, ε_v and ε_b , are strongly influenced by the energy injection from the forcing term. The kinetic dissipation rate, ε_v , and the magnetic dissipation rate, ε_b , are calculated as $\varepsilon_v = \nu_h \sum_k k^{2h} \langle |\hat{\mathbf{u}}(\mathbf{k}, t)|^2 \rangle$, $\varepsilon_b = \eta_h \sum_k k^{2h} \langle |\hat{\mathbf{b}}(\mathbf{k}, t)|^2 \rangle$, where $\hat{\mathbf{u}}$ and $\hat{\mathbf{b}}$ denote the velocity and magnetic vectors in the Fourier space. For the present study, we focus on a steady-state analysis, for which the average energy transfer/cascade rates are equal to the dissipation rates, allowing us to use the same symbol(s), $\varepsilon^\pm$, for these conceptually distinct quantities.

4. Results

In this section, the angular dependence of the third-order law will be assessed, using the three methods described in Section 2, and the speciality of the 60° polar angle will be highlighted. An analysis of the divergence of the energy flux vector provides a partial explanation for why $\theta \approx 60^\circ$ has elevated relevance. This will be followed by a discussion of how hints from this analysis provide insight into the modeling of the energy flux vector in axisymmetric MHD turbulence with external mean magnetic fields. Associated with this, the dependence of the third-order law on azimuthal angle will also be explored. Finally, the above findings regarding angular dependence will be verified using virtual spacecraft measurements.

4.1. Third-order Law along the 60° Polar Angle

We begin by demonstrating the speciality of the third-order law along $\theta = 60^\circ$ empirically. Figure 2 displays normalized longitudinal third-order moments after azimuthal averaging, $\widetilde{Y}_\ell(\ell, \theta_i)$, and also after “full” directional averaging, $\overline{Y}_\ell(\ell)$. From Figure 2(a), we see that the estimated energy dissipation rate peaks at large scales for parallel angles and at progressively smaller scales for more perpendicular angles. We also see that the maximum dissipation rate in the perpendicular plane ($\theta = 90^\circ$) is larger than that in the parallel plane ($\theta = 0^\circ$), consistent with other results (e.g., B. MacBride et al. 2008; A. Verdini et al. 2015). Figure 2(b) reveals the intriguing result that the (fully) direction-averaged profiles are well approximated by the azimuthally averaged profile for a specific polar angle, namely, $\theta = 60^\circ$, i.e., $\widetilde{Y}_\ell(\theta \approx 60^\circ) \approx \overline{Y}_\ell$. However, the quality of this approximation decreases with increasing B_0 . Specifically, the maximum deviation on the plateau is approximately 1% when $B_0 = 2$ and increases to

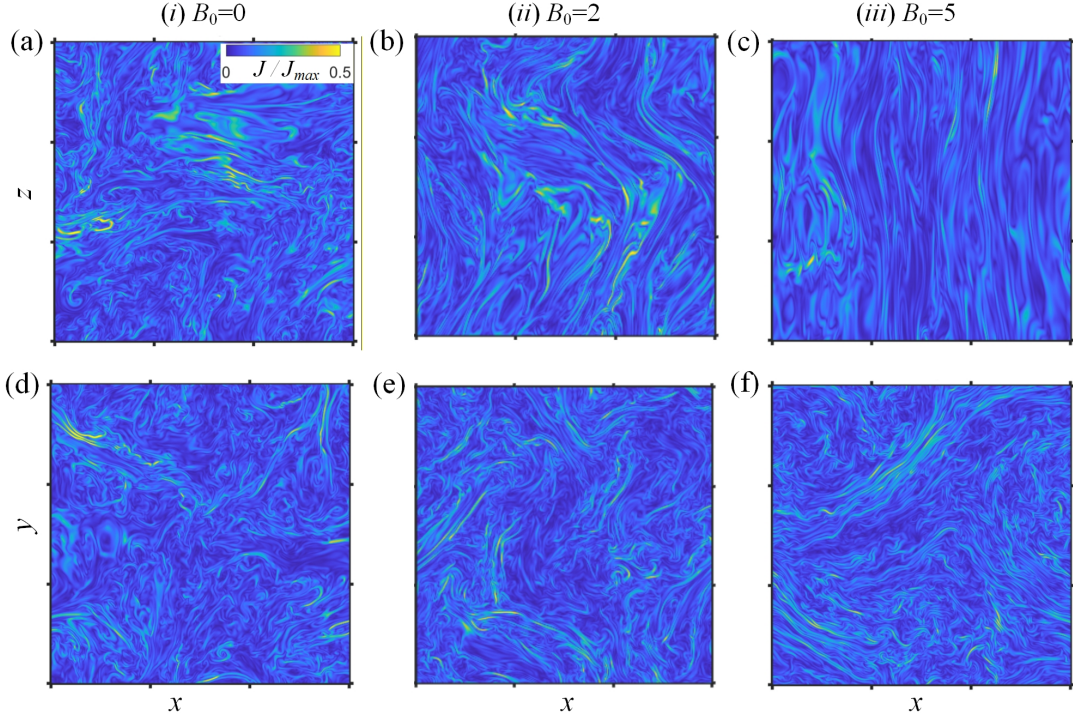


Figure 1. Current intensity ($J = |\nabla \times \mathbf{b}|$) in domain cross sections that are (a), (b), (c) parallel-perpendicular (z - x) and (d), (e), (f) strictly perpendicular (x - y) with respect to the mean magnetic field, for $B_0 = 0$ (panels (a), (d)), 2 (b), (e), and 5 (c), (f). Note the increasing alignment of structures with the mean field direction seen in the top row.

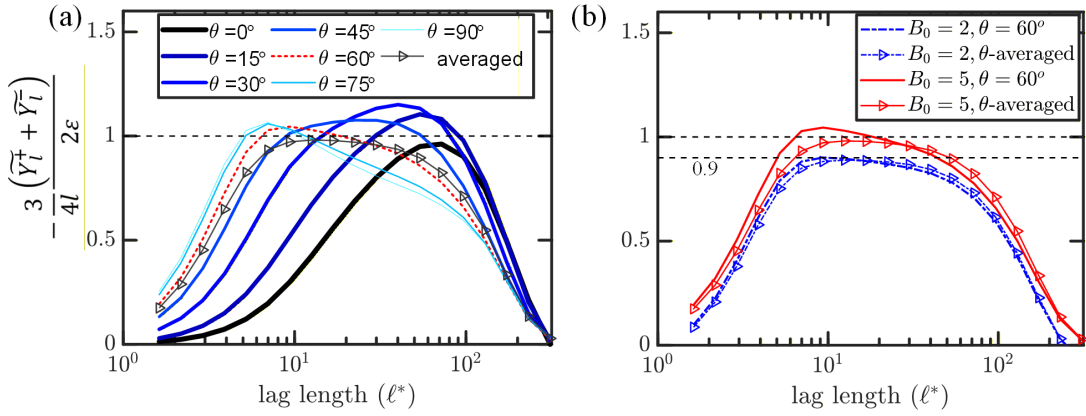


Figure 2. Normalized longitudinal third-order structure functions, $Y_\ell = (Y_\ell^+ + Y_\ell^-)/2$, for various types of angle averaging. (a) Azimuthal average using Method II, $\tilde{Y}_\ell$ in Equation (9), for the $B_0 = 5$ simulation at the indicated values of θ . The solid line with triangles represents the average of these and is equivalent to the “full” direction-averaged profile obtained via Method III, Equation (10): $-3(\overline{Y}_\ell^+ + \overline{Y}_\ell^-)/(8\varepsilon\ell)$. (b) Direction-averaged (Equation (10), lines with triangle symbols) and azimuthally averaged at fixed $\theta = 60^\circ$ (Equation (9), lines without triangle symbols) for the $B_0 = 2$ and 5 simulations. Recall that θ is the angle between the lag vector and the external mean magnetic field, $B_0\mathbf{e}_z$. Time averaging has been employed in all cases; see Table 1. In panel (b), the results for $B_0 = 2$ are shifted by -0.1 to avoid visual clutter with the $B_0 = 5$ results.

approximately 8% when $B_0 = 5$. Standard Laplacian dissipation and hyper-dissipation simulations behave similarly in this regard (see Figure 10 in the Appendix). A detailed explanation of the $\theta = 60^\circ$ polar angle will be provided in Sections 4.2 and 4.3.

4.2. Divergence of the Energy Flux Vector

To directly illustrate the angular dependence of the energy transfer, Figure 3 displays the divergence of the total energy flux vector, normalized by an appropriate multiple of the dissipation rate and azimuthally averaged; see Equation (3).

Starting with the $B_0 = 5$ simulations and comparing Figure 3(f) with Figure 2(a)—which employs $\tilde{Y}_\ell$ rather than the divergence of $\tilde{\mathbf{Y}}$ —we see that the main differences occur in the range of $\theta \approx [15^\circ, 45^\circ]$, in which Method II, Equation (9), overestimates the true dissipation rate (albeit this observation may change for larger Reynolds number simulations with wider inertial ranges). For $B_0 = 0$, Figure 3(d) reveals that there is a range over which all the curves collapse, indicating the expected angular independence, or isotropy. This range gets shorter with increasing B_0 , however. To investigate these trends with B_0 , in the Appendix, we compare the results of this

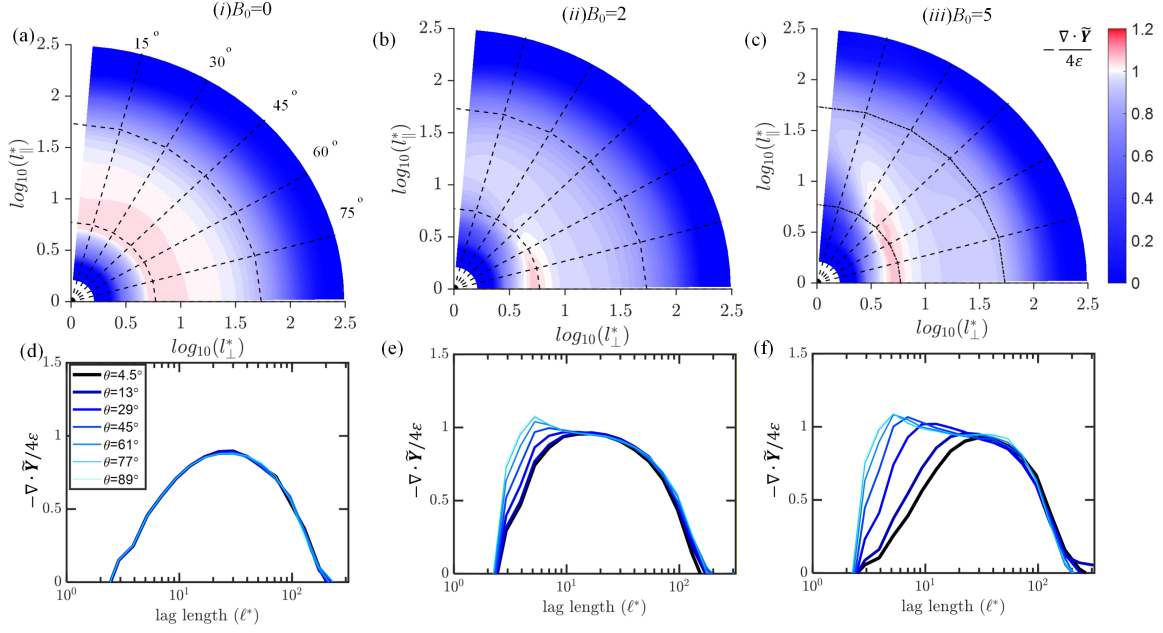


Figure 3. Top row: image plots of normalized divergence of (azimuthally averaged) energy flux $\tilde{Y}$. Bottom row: selected cuts at constant θ from the top row. Columns are for (i): $B_0 = 0$; (ii): $B_0 = 2$; (iii): $B_0 = 5$. Recall that θ is the angle between the lag ℓ and B_0 . Black dashed arcs indicate inertial range boundaries, i.e., normalized lag lengths $\ell_{\parallel}^*, \ell_{\perp}^* \in [6, 55]$, as identified using the direction-averaged form of the third-order law, i.e., Method III, Equation (10), with $-3(\overline{Y_l^+} + \overline{Y_l^-})/(8\varepsilon\ell)$ above a threshold, here 0.9.

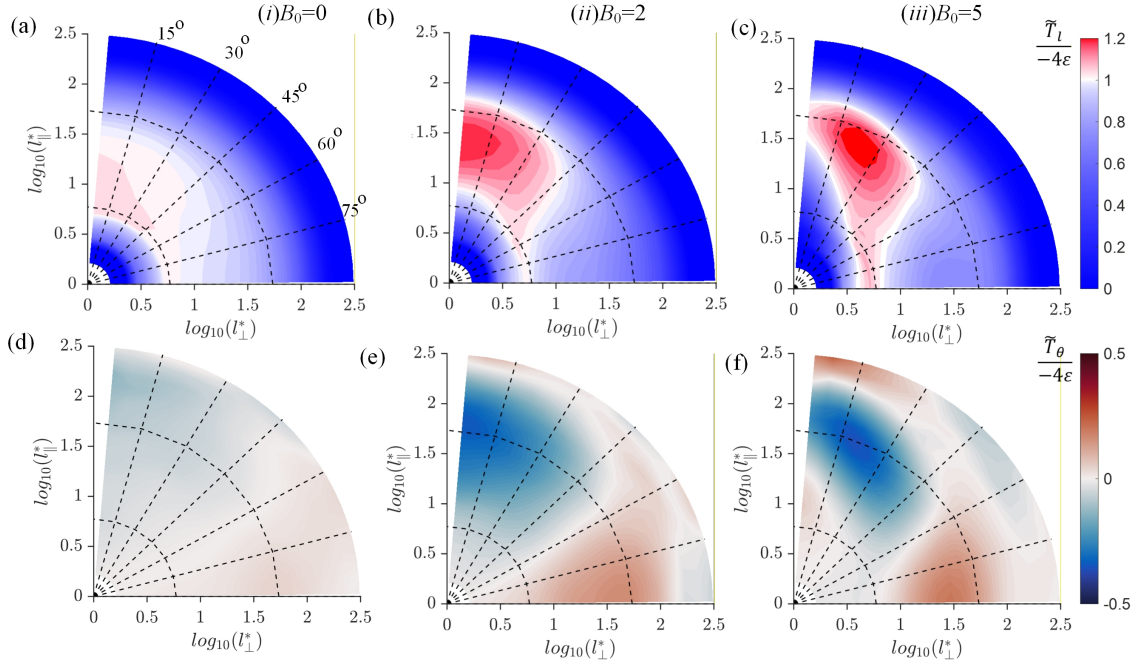


Figure 4. Image plots of the (top row) lag contribution $\tilde{T}_l$ and (bottom row) polar contribution $\tilde{T}_\theta$ to the normalized divergence of the (azimuthally averaged) energy flux for $B_0 = 0$ (panels (a), (d)), 2 (b), (e), and 5 (c), (f). Black dashed arcs indicate inertial range boundaries, i.e., normalized lag lengths $\ell_{\parallel}^*, \ell_{\perp}^* \in [6, 55]$, as identified using the direction-averaged form of the third-order law, i.e., Method III, Equation (8), with $-3(\overline{Y_l^+} + \overline{Y_l^-})/(8\varepsilon\ell)$ above a threshold, here 0.9. Plotted quantities have been averaged over time and ϕ .

section with those from standard dissipation cases. That analysis verifies that our main conclusions are still valid when hyperviscosity is employed.

Figure 4 shows the $\tilde{T}_l$ and $\tilde{T}_\theta$ additive contributions to the divergence of the energy flux vector; see Equation (3). The $\tilde{T}_\phi$ contribution (not shown here) is negligible, with its largest magnitude occurring for the strongest B_0 case. For the $\tilde{T}_l$

contribution, which is single signed, the maximum value shifts slightly away from the parallel direction with increasing B_0 . One also sees that the highest values with $\tilde{T}_l > 1$ are mainly located from $\theta \in [0^\circ, 45^\circ]$. The $\tilde{T}_\theta$ contribution can have positive and negative regions, and, for large enough B_0 , there is a negative peak at smaller angles with $\theta \in [0^\circ, 45^\circ]$, and a positive peak at larger angles with $\theta \in [60^\circ, 90^\circ]$. When $\tilde{T}_\theta$ is

added to $\tilde{T}_\ell$, the negative peak region of the former offsets the “overlarge” values of the latter in the $\theta \in [0^\circ, 45^\circ]$ region, consistent with the differences seen between Figures 3(f) and 2(a). Interestingly, at inertial range scales, $\tilde{T}_\theta$ is always approximately zero around the $\theta = 60^\circ$ line. Referring to Figure 2, we see this is the same angle for which $\tilde{Y}_\ell(\theta) \approx \bar{Y}_\ell$. Since the θ -average of $\tilde{T}_\theta$, essentially $\int_0^\pi \tilde{T}_\theta \sin \theta d\theta$, is zero, and $\tilde{T}_\theta$ is continuous, the mean value theorem of integrals means that there exists at least one θ in the range $[0, \pi]$ where $\tilde{T}_\theta \sin \theta$ is zero. The empirical evidence from Figure 4 is that this occurs at $\theta \approx 60^\circ$.

4.3. Possible General Form of the Energy Flux

Having identified the importance of the (time and azimuthally averaged) polar contribution to the divergence of the energy flux vector, $\tilde{T}_\theta = \frac{1}{\ell \sin \theta} \frac{\partial(\tilde{Y}_\theta \sin \theta)}{\partial \theta}$, we now examine it in more detail in an effort to explain what is responsible for the speciality of $\theta = 60^\circ$. J. Podesta et al. (2007) derived the general mathematical form of the energy flux vectors $\mathbf{Y}$ under the assumption that the turbulence is statistically axisymmetric for rotations about the direction of the mean magnetic field. Their result, expressed in *spherical* polar coordinates, is

$$Y_{\ell, \text{Podesta}} = \ell[A + B \cos^2 \theta], \quad (11)$$

$$Y_{\theta, \text{Podesta}} = -B\ell \sin \theta \cos \theta, \quad (12)$$

$$Y_{\phi, \text{Podesta}} = C\ell \sin \theta, \quad (13)$$

$$Y_{\rho, \text{Podesta}} = A\ell \sin \theta, \quad (14)$$

where A , B , and C are functions of $\cos \theta$, that we may attempt to determine empirically. Note that A and B are coupled by a differential equation, as can be seen by substitution of the model into Equation (2) (J. Podesta et al. 2007). Y_ρ is the component of the energy flux vector in the direction of $\mathbf{e}_z \times (\ell \times \mathbf{e}_z)$, which is parallel to the *cylindrical* polar coordinate radial unit vector that lies in the ℓ - $\mathbf{B}_0$ plane. For simplicity, here we consider the zero-order solution in which both A and B are constants and $C = 0$ (J. Podesta et al. 2007).

In a distinct approach, S. Galtier (2012) assumed a power-law relation between correlation lengths along and transverse to the local mean magnetic field direction, and proposed a model for $\mathbf{Y}$. In cylindrical polar coordinates (ρ, ϕ, z) it is

$$\mathbf{Y}_{\text{Galtier}}(\rho, z) = -\frac{4\varepsilon}{3+a}[\rho\mathbf{e}_\rho + (1+a)z\mathbf{e}_z], \quad (15)$$

where $a > 0$ or $a < 0$ represent convex or concave turbulence, respectively, and $\mathbf{e}_\rho$ denotes the unit (cylindrical) radial vector. Rewriting this in spherical coordinates, we obtain

$$Y_{\ell, \text{Galtier}} = -\frac{4\varepsilon\ell}{3+a}(1+a\cos^2 \theta), \quad (16)$$

$$Y_{\theta, \text{Galtier}} = \frac{4a\varepsilon\ell}{3+a}\sin \theta \cos \theta, \quad (17)$$

$$Y_{\rho, \text{Galtier}} = -\frac{4\varepsilon}{3+a}\ell \sin \theta. \quad (18)$$

We see that the zero-order model from J. Podesta et al. (2007) and the model from S. Galtier (2012) have the same polar angle dependence.

Table 2
Fitting Parameters in the Podesta and Galtier Models for $\mathbf{Y}$

Parameter	$B_0 = 0$	$B_0 = 2$	$B_0 = 5$
A	-2.53	-1.93	-1.96
B	-0.18	-1.42	-1.60
a	0	0.67	0.78

To help assess these models, we employ least-squares fitting of them to the plateau regions of the DNS-determined $-3\tilde{Y}_\rho/(4\varepsilon\ell)$ and $-3\tilde{Y}_\theta/(4\varepsilon\ell)$; that is, the fitting is only over the nominal inertial range scales. The fitting parameters are listed in Table 2, and the models based on them are plotted in Figure 5. One can see that the models for Y_θ and Y_ρ agree well with our simulation results, while there are more observable differences for Y_ℓ (solid curves), especially for the $B_0 = 5$ case. In the future, one may also test the higher-order solution of the model from J. Podesta et al. (2007), i.e., take into account the angular-dependent coefficients $A(\cos \theta)$, $B(\cos \theta)$, and $C(\cos \theta)$.

Figure 6 displays the vectors of the (time and azimuthally averaged) energy flux $\tilde{\mathbf{Y}}$ as a function of the parallel and perpendicular lag coordinates, for the $B_0 = 0, 2$, and 5 simulations. For anisotropic cases, $B_0 = 5$, with a fixed length scale, the arrow length is the longest around $\theta \approx 30^\circ$, consistent with the trend of Y_ℓ in Figure 5. Overall, we see that the energy flux vectors are well represented by the models of J. Podesta et al. (2007) and S. Galtier (2012). Some overestimation appears for the $B_0 = 5$ case, especially near the parallel direction and close to the dissipation range.

The model(s) for Y_θ can also provide an explanation for the speciality of the 60° polar angle discussed in Sections 4.1 and 4.2. Substituting the Podesta model for Y_θ , Equation (12), into T_θ (see Equation (3)) gives

$$T_\theta = \frac{1}{\ell \sin \theta} \frac{\partial(\sin \theta Y_\theta)}{\partial \theta} = -B(\cos 2\theta + \cos^2 \theta). \quad (19)$$

Setting $T_\theta = 0$, one finds $\theta = 54.7^\circ$. This supports our empirical results regarding the speciality of $\theta \approx 60^\circ$. By setting $T_\theta = 0$, assuming the negligible effect of azimuthal component on this divergence, and combining Equations (2) and (3), one can attain the relationship between the longitudinal third-order moments and the dissipation rate along $\theta = 54.7^\circ (\approx 60^\circ)$,

$$Y_{\ell, \theta \approx 60^\circ}^\pm = -\frac{4}{3}\varepsilon^\pm \ell. \quad (20)$$

Therefore, when the assumption that the turbulence is statistically axisymmetric about the direction of the mean magnetic field and the third-order moment is statistically independent from azimuthal direction is valid, one should expect to see the speciality of $\theta \approx 60^\circ$ in the inertial range.

4.4. Azimuthal Angle Dependence

In previous works, the turbulence has often been assumed to be statistically axisymmetric about the direction of the mean magnetic field (also called azimuthally symmetric or cylindrically symmetric). Having shown the speciality of 60° polar angle with the azimuthal average, in this subsection, we go beyond the axisymmetric model and demonstrate the azimuthal dependence of the third-order structure function.

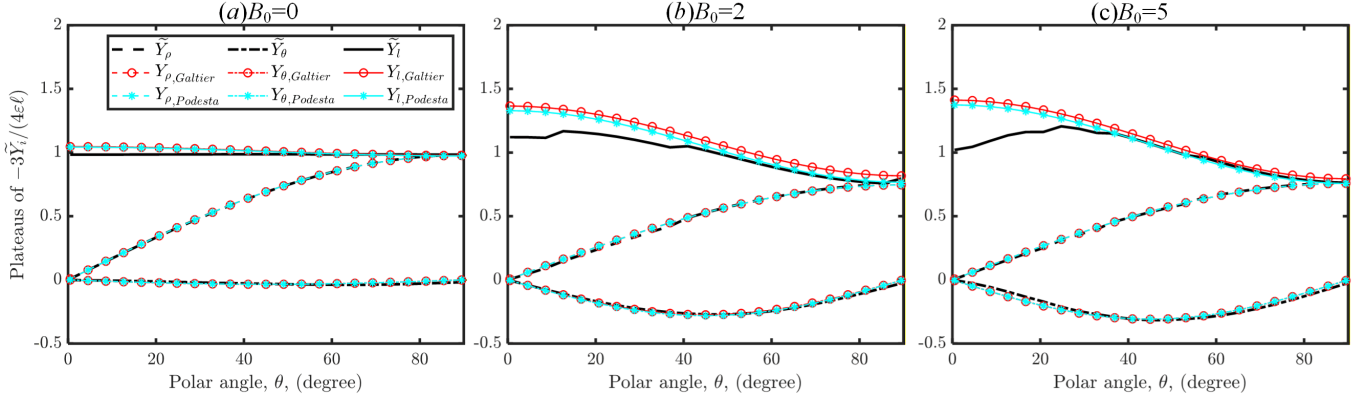


Figure 5. Verification of the Podesta (blue, stars) and Galtier (red, circles) models for Y . The dashed, dashed-dotted, and solid curves are results for the Y_ρ , Y_ℓ , and Y_θ components, respectively. Curves without symbols are obtained directly from DNS data. (a) $B_0 = 0$, (b) $B_0 = 2$, and (c) $B_0 = 5$.

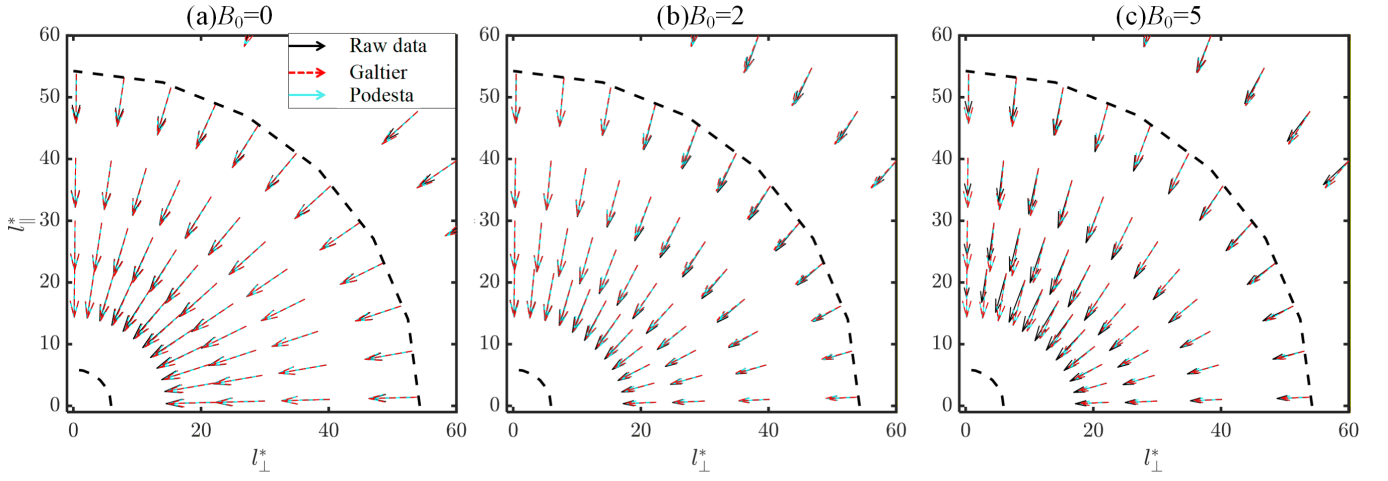


Figure 6. Vectors (in black) of the (azimuthally and time averaged) energy flux $\tilde{Y}$ in parallel and perpendicular lag space, as obtained using DNS data for (a) $B_0 = 0$, (b) $B_0 = 2$, and (c) $B_0 = 5$. Vectors are normalized by the lag length, l^* . Red dashed vectors are for the S. Galtier (2012) model, and light blue vectors for the J. Podesta et al. (2007) model. Black dashed arcs indicate inertial range boundaries, i.e., normalized lag length $l_\parallel^*, l_\perp^* \in [6, 55]$, as identified using the direction-averaged form of the third-order law, i.e., Method III, Equation (8), with $-3(\overline{Y_\ell^+} + \overline{Y_\ell^-})/(8\epsilon\ell)$ above a threshold, here 0.9.

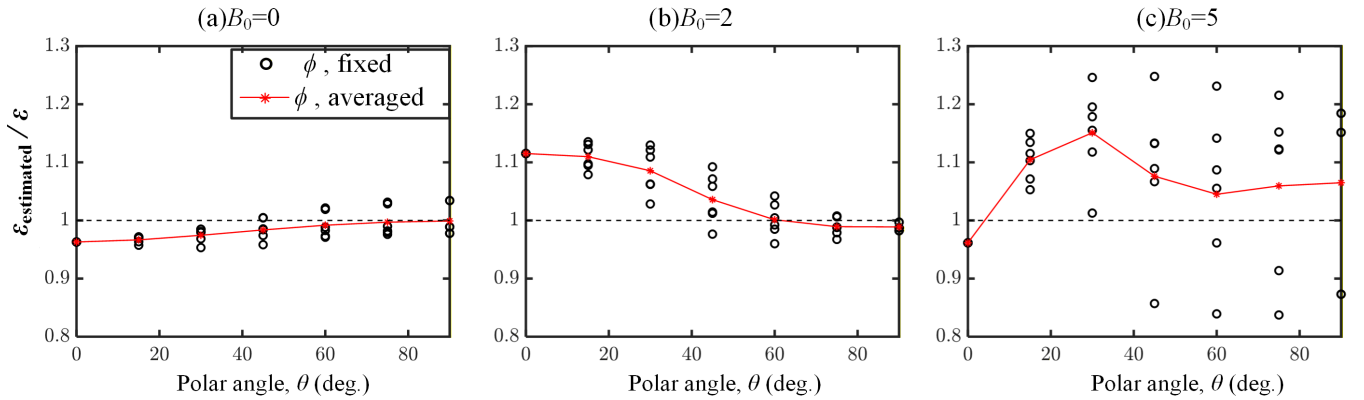


Figure 7. Dissipation rates estimated from normalized longitudinal third-order structure functions for various θ and ϕ , as obtained using Method I, Equation (8). Values are normalized to the total dissipation rate, ϵ . The solid red line with stars represents the azimuthally averaged profiles, as obtained using Method II, Equation (9). Data have been time averaged. (a): $B_0 = 0$; (b): $B_0 = 2$; (c): $B_0 = 5$.

Figure 7 displays the distribution of the estimated dissipation rates at different θ and ϕ , marked with dark circles, for three values of B_0 . The red curves with stars represent the azimuthal averages, and we see that the 60° polar angle gives the most accurate result in the anisotropic cases ($B_0 = 2, 5$),

with the values $\theta > 60^\circ$ also being quite accurate. It is evident that the distributions of estimated dissipation rates at fixed θ and varying ϕ are more scattered for larger B_0 . For $B_0 = 2$ and 5, the maximum estimated cascade rate can depart from the actual value by 10% and 25%, respectively. We expect this

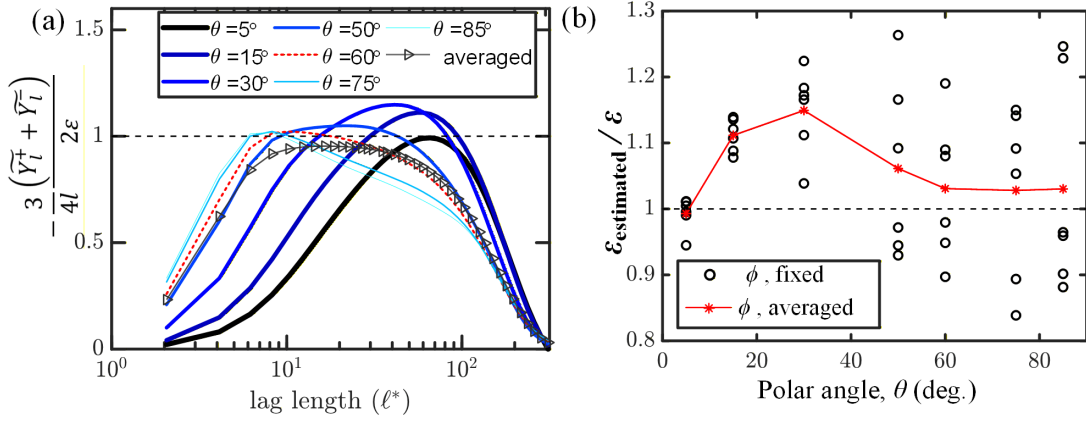


Figure 8. Results from virtual spacecraft measurements for the $B_0 = 5$ simulation at indicated values of θ : (a) Azimuthal average. The solid line with triangles is the “full” direction-averaged profile. (b) Dissipation rates estimated from normalized longitudinal third-order structure functions for various θ and ϕ . Values are normalized to the total dissipation rate, ε . The solid red curve with (red) stars represents the azimuthally averaged profiles. Data have been time averaged with the same snapshots as those in Figure 7(c).

departure to be even greater at larger B_0 . Nonetheless, as shown by the red curves with stars in Figures 7(b) and (c), after averaging over azimuth, the maximum departures from unity reduce to 3% and 15% for $B_0 = 2$ and 5, respectively. From a practical perspective, azimuthal coverage generally accompanies polar coverage in spacecraft observations. Moreover, given that it is almost impossible that all intervals will have the same ϕ when a large number of intervals are used, it may be feasible to achieve sufficient coverage over ϕ . More quantitative assessment on the ϕ dependence shall be done in our companion paper.

Theoretically, assuming axisymmetry holds, when the number of time snapshots employed is sufficient to compute a stable average, the averaged structure function should be independent of azimuthal angle. Thus, when we describe energy transfer as isotropic in the perpendicular plane, we are referring to statistically averaged transfer. When sampling and averaging are limited, e.g., a small number of snapshots (as in DNS) or sampling directions (as in observations), then a residual dependence on the azimuthal directions might persist in the estimates. This dependence could be associated with local spatial and temporal fluctuations caused by large-scale structures in the perpendicular plane at large B_0 , as illustrated in the image of current intensity; see Figure 1(f).

4.5. Verification with Virtual Space Observation

To verify our observation on the speciality of 60° polar angle, we also perform virtual spacecraft measurements. That is, we employ four spacecraft in tetrahedral configurations to fly through numerically generated turbulent fields, mimicking satellite (e.g., MMS and Cluster missions) flights through solar wind and magnetosheath turbulence. The relative positions of the virtual spacecraft are scaled to fit within the simulation domain, and the interspacecraft separation is set to be 37 times the (velocity) Kolmogorov length scale ($\eta_{k,v}$), i.e., in the inertial range. The trajectories of the virtual spacecraft are parallel lines with specified polar and azimuthal angles. Due to the periodic boundary conditions of the simulations, the spacecraft trajectories cross the simulation box several times; see Figure 2(a) in F. Pecora et al. (2023a).

The MHD simulation we are using here is the $B_0 = 5$ case. Temporal sampling is as indicated in Table 1, with 75

snapshots over about 75 large-eddy turnover times (T_e). The sampling polar angles are $\theta = [5^\circ, 15^\circ, 30^\circ, 50^\circ, 60^\circ, 75^\circ, 85^\circ]$, and azimuthal angles $\phi = [0^\circ: 60^\circ: 300^\circ]$. The mean number of data points for each spacecraft along one trajectory, i.e., one direction, is about 1.6×10^5 per snapshot. For each trajectory, the third-order structure function is calculated for 40 lag lengths spanning 2–314 times the $\eta_{k,v}$. The longitudinal third-order structure function for a fixed polar and azimuthal angle, $Y_\ell^\pm(\ell, \theta_i, \phi_j)$, is averaged from four spacecraft,

$$Y_\ell^\pm(\ell, \theta_i, \phi_j) = \frac{\sum_{k=1}^{N_k=4} \sum_{t=1}^{N_t} Y_\ell^\pm(\ell, \theta_i, \phi_j, t, k)}{N_k N_t}, \quad (21)$$

where N_k and N_t denote the number of virtual spacecraft and snapshots, respectively. Comparing Figure 8(a) with Figure 2(a), we see that the result at $\theta = 60^\circ$ is still the best approximation to the profile with the “full” direction-averaged method, especially for the plateau. The results from virtual spacecraft at different azimuthal angles, shown in Figure 8(b), are in general the same as the DNS results in Figure 7(c). Indeed, Figures 7 and 8 both indicate that there is a strong dependence of the estimated dissipation rate on the azimuthal angle ϕ . When employing spacecraft observations, the azimuthal average could be performed using different measurement intervals, similar to the polar averaging reported on in K. T. Osman et al. (2011) and R. Bandyopadhyay et al. (2018). Based on the results presented here, we suggest that the main objective should be the selection of intervals with $\theta \approx 60^\circ$, and a range of ϕ values (to average over).

5. Conclusions and Discussion

Anisotropy, a typical property of MHD-scale fluctuations in the solar wind, challenges the applicability of the third-order law with the isotropic assumption. The literature (e.g., K. Osman et al. 2010; R. Bandyopadhyay et al. 2018; Y. Wang et al. 2022) presents a direction-averaged form to resolve this challenge, although it requires full angle coverage and a large number of space datasets in solar wind measurement. Regarding various angle averages of the MHD longitudinal third-order moment, Y_ℓ , in this work, we have found the following:

1. The azimuthally averaged third-order structure function at $\theta = 60^\circ$ can predict the (full) direction-averaged results; that is, $\tilde{Y}_\ell(\theta \approx 60^\circ) \approx \bar{Y}_\ell$. The agreement holds for mean magnetic field strengths up to at least $B_0 = 5$. However, the deviation increases with increasing B_0 , but is still within 10% differences at $B_0 = 5$.
2. This speciality of 60° polar angle relative to the mean magnetic field direction can be explained by considering the divergence of energy flux vector, $\nabla_\ell \cdot \mathbf{Y}$. In the inertial range, the $\tilde{Y}_\theta$ contribution to the divergence, namely, $\tilde{T}_\theta$, integrates (wrt θ) to zero. Based on the mean value theorem of integrals, this suggests that there exists at least one θ in the range $[0, \pi]$ where $\tilde{T}_\theta$ is zero. Furthermore, we find that the θ -averaged $\tilde{T}_\theta \sin \theta$ is negative over $[0^\circ, 60^\circ]$ and positive over $[60^\circ, 90^\circ]$, which gives rise to this special angle 60° .
3. In the theory of the anisotropic form of the third-order moments and their relationship to the energy dissipation rates, the zero-order solution for the model from J. Podesta et al. (2007) and the model from S. Galtier (2012) are assessed, and support our observation on the speciality of 60° polar angle.
4. The dependence on azimuthal angles ϕ is assessed. We find that the distributions of estimated dissipation rates at fixed θ and varying ϕ are more scattered for larger B_0 . At least in part, we expect that this is related to the effects of insufficient sampling of large-scale structures present in the available data.
5. Based on (tetrahedral) virtual spacecraft measurements, analogous to the MMS and Cluster missions, the speciality of the 60° polar angle and azimuthal dependence is further verified. These results provide guidance for real spacecraft measurements.

As Figure 2 indicates, the estimation accuracy along the 60° polar angle decreases slightly with increasing B_0 with the present setups. This deviation is likely due to the increasing contribution of the azimuthal component to the divergence of the structure function as discussed in Sections 2 and 4.4. One may reason in this way: In a fully helical simulation with strongly magnetic forcing at large scales, A. Alexakis (2013) found that the energy is concentrated in large scales, similar to Figure 1(f), and the dynamics are controlled by magnetic helicity condensates. Therefore, the magnetic helicity and the nonlinear energy transfer between velocity and magnetic fields play a major role in the formation of the large-scale structures, and also impact the azimuthal dependence of the third-order moments.

It is worth emphasizing that our simulations in this work present some limitations and do not accommodate all the dynamical complexities of space plasmas. For example, the present simulations are limited in size, and periodic boundary conditions have been used. Moreover, other important parameters, such as values of plasma β , turbulence level, and Reynolds numbers, are more variable in the solar wind than in these simulations. Consequently, the results here must be regarded at present as a starting point for further detailed treatment using space plasma observations. Nevertheless, the result of the estimation along $\theta \approx 60^\circ$ in the $B_0/b_{\text{rms}} \approx 2$ case, typically of the order of the mean magnetic field in the solar wind, could be applicable to the space plasma observations.

A practical question is: in addition to the aforementioned effect of external mean magnetic fields, what other factors

could impact the accuracy of estimation along $\theta \approx 60^\circ$ in the context of the solar wind? Two potential factors are the size of the forcing scales and Reynolds number, and we now make some brief remarks regarding these. First, solar wind observations indicate that dual (forward and inverse) cascades may occur (R. Marino & J.-H. Xie 2025). In simulations, these could be realized through forcing at intermediate scales. Compared with large-scale forcing, intermediate-scale forcing may not introduce large-scale structures in the perpendicular plane, such as those evident in Figure 1. As such, the azimuthal dependence of the divergence of the third-order moments may be attenuated, and we conjecture that the speciality of $\theta \approx 60^\circ$ might be more observable. Second, in the Appendix, we show the robustness of the estimation along $\theta \approx 60^\circ$ with respect to the Re. However, in the solar wind, the large-scale magnetic Re_b is up to $10^5 \sim 10^7$ (e.g., D. Wrench et al. 2024), which is 3–5 orders higher than the present setup. At such a high Re_b , the robustness of the present estimation deserves further verification. The effects of these complexities on the speciality of the 60° polar angle deserve further verifications in the context of the solar wind observations, and we are exploring some of them with solar wind observations.

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Author Contributions

All authors contributed equally to the Terra Mater Collaboration.

Appendix

A.1. Effects of the Reynolds Number

Here, we compare results from two $B_0 = 5$ standard Laplacian dissipation simulations with different Reynolds numbers. These simulations employ either a 1024^3 grid (Case E) or a 512^3 grid (Case D); see Table 1. For both cases, the profile of the azimuthally averaged normalized structure function evaluated along $\theta = 60^\circ$ is the closest to the fully directional averaged one among all these polar angles, as shown in Figures 9(a) and (b), and the estimation errors of the energy dissipation rates are within 1%. Therefore, the estimated dissipation rate along $\theta = 60^\circ$ is robust with respect to the Re. Panel (c) indicates that the estimation along $\theta = 60^\circ$ at a higher Re (1024^3 grid) is more accurate than that for the

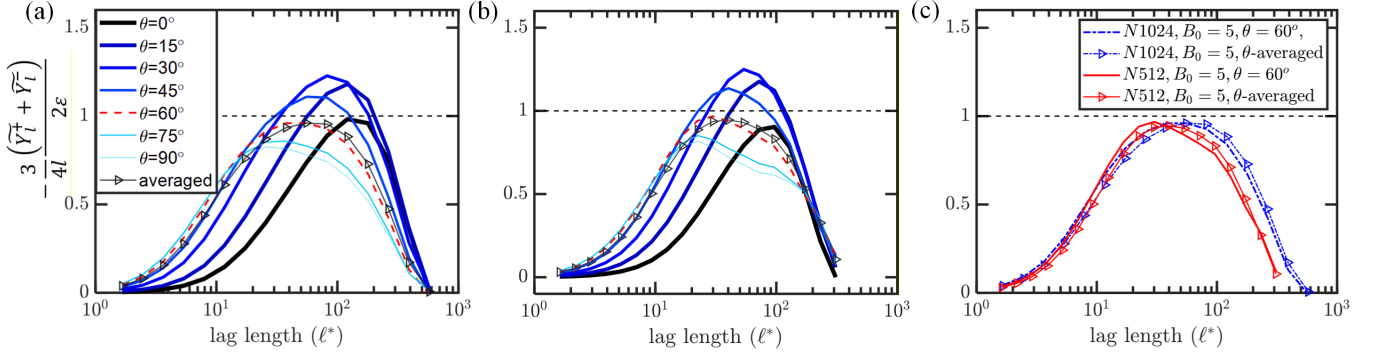


Figure 9. Normalized longitudinal third-order structure functions, $Y_\ell = (Y_\ell^+ + Y_\ell^-)/2$, for various types of angle averaging with standard Laplacian viscosity and $B_0 = 5$ for a (a) 1024^3 grid and (b) 512^3 grid. Lines without symbols are for azimuthal averaging using Method II, $\tilde{Y}_\ell$, at the indicated values of θ . The solid line with triangles represents the average of these and is equivalent to the “full” direction-averaged profile: $-3(\overline{Y_\ell^+} + \overline{Y_\ell^-})/(8\ell)$. (c) Direction-averaged (lines with triangle symbols) and azimuthally averaged at fixed $\theta = 60^\circ$ (lines without triangle symbols) for the $B_0 = 5$ simulations. Recall that θ is the angle between the lag vector and the external mean magnetic field, $B_0 e_z$. Time averaging has been employed in all cases; see Table 1.

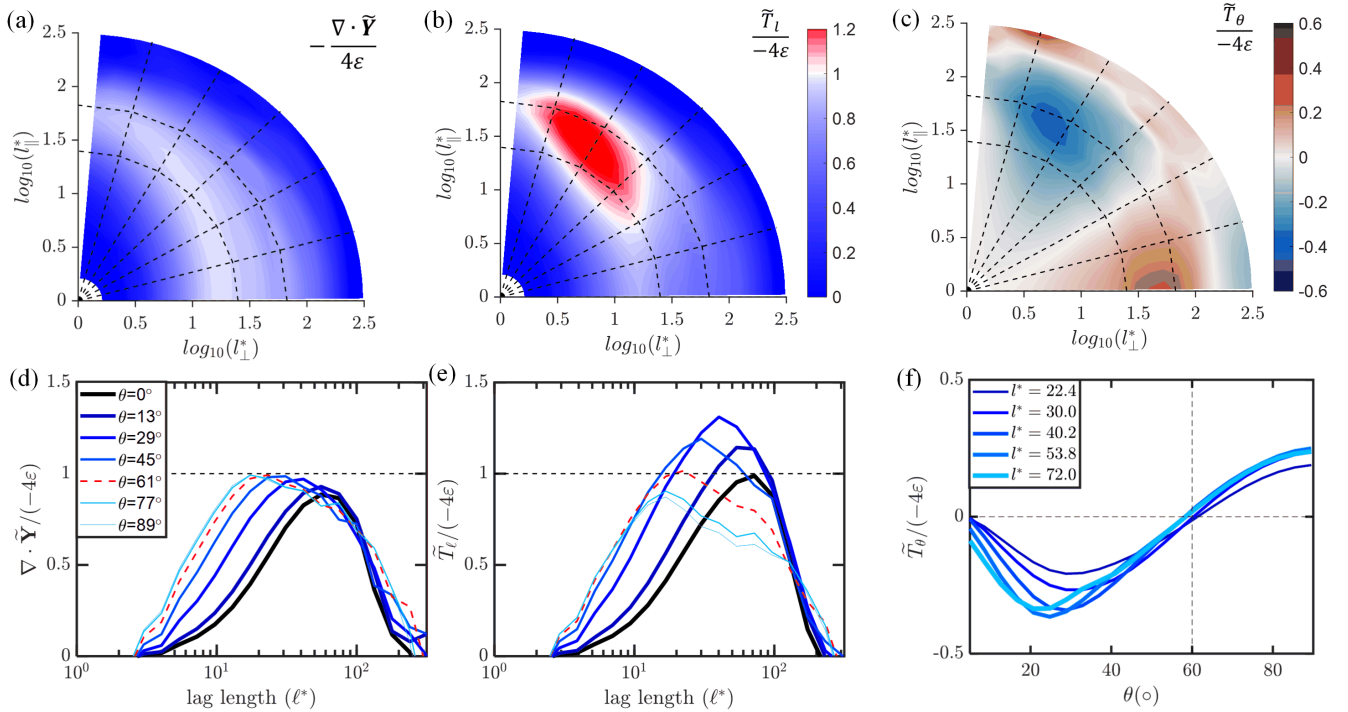


Figure 10. Results for standard Laplacian dissipation, Case (D) in Table 1 with $B_0 = 5$, 512^3 grids. The top row shows images, and the bottom row shows the curves extracted from the top row. Panels (a), (d): divergence of the (azimuthally averaged) energy flux vector. Panels (b), (e): longitudinal contribution $\tilde{T}_l$ to the normalized divergence of the (azimuthally averaged) energy flux. Panels (c), (f): polar contribution $\tilde{T}_\theta$ to the normalized divergence of the (azimuthally averaged) energy flux. Black dashed arcs represent the inertial range boundaries, i.e., lag length $\ell_\parallel^*, \ell_\perp^* \in [25, 67]$, as identified using the direction-averaged third-order law, i.e., with $-3(\overline{Y_\ell^+} + \overline{Y_\ell^-})/(8\ell)$ above a threshold, here 0.9.

low Re simulation (512^3 grid). Therefore, we anticipate that at a higher Re, the estimation accuracy may be improved.

A.2. Effects of Hyperviscosity

From Figures 2 and 9, we see the speciality of $\theta \approx 60^\circ$ for the estimation of dissipation rate with both hyperviscosity and standard Laplacian viscosity. To illustrate the detailed effects of hyperviscosity, we compare here standard Laplacian viscosity $B_0 = 5$ simulation results (Figure 10) with our hyperviscous results (shown in Figures 3(c), (f) and 4(c), (f)). From the divergence of normalized structure function in Figures 10(a) and (d), the location of the peak dissipation rate

shifts to larger scales for this standard viscous case, and the peak value associated with the parallel direction decreases. This indicates that, for the hyper-dissipative case, the energy transfer in the parallel direction is enhanced and the strength of perpendicular small-scale turbulence structures is increased. From polar components of the divergence in Figures 10(c) and (f), one can see a better collapse at $\theta = 60^\circ$ for the standard Laplacian viscous case than the hyperviscous case. The reason may be related to the prolonged saturation phase of the large-scale magnetic field in the hyperviscous case, which makes the magnetic helicity dissipate more slowly than the case with standard Laplacian viscosity (A. Brandenburg & G. R. Sarson 2002).

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