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JACOBI ELLIPTIC COORDINATES, FUNCTIONS OF HEUN AND LAMÉ TYPE AND THE NIVEN TRANSFORM

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Lamé and Heun functions arise via separation of the Laplace equation in general Jacobi ellipsoidal or conical coordinates. In contrast to hypergeometric functions that also arise via variable separation in the Laplace equation, Lamé and Heun functions have received relatively little attention, since they are rather intractable. Nonetheless functions of Heun type do have remarkable properties, as was pointed out in the classical book "Modern Analysis" by Whittaker and Watson who devoted an entire chapter to the subject. Unfortunately the beautiful identities appearing in this chapter have received little notice, probably because the methods of proof seemed obscure. In this paper we apply the modern operator characterization of variable separation and exploit the conformal symmetry of the Laplace equation to obtain product identities for Heun type functions. We interpret the Niven transform as an intertwining operator under the action of the conformal group. We give simple operator derivations of some of the basic formulas presented by Whittaker and Watson and then show how to generalize their results to more complicated situations and to higher dimensions.

Contents

1. Introduction

Among the most important contributions of Carl Gustav Jacobi to mechanics was his introduction of the "remarkable change of variables", the generalized elliptical coordinates x_j in n dimensions, [1]. These can be defined by the relations

$$1 + \sum_{k=1}^{n} \frac{q_k^2}{z - e_k} = \frac{\Pi_{j=1}^{n}(z - x_j)}{\Pi_{k=1}^{n}(z - e_k)}, \tag{1.1}$$

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where the q_k are Cartesian coordinates and the e_k are distinct constants. An equivalent definition is

$$q_k^2 = \frac{\Pi_{j=1}^n (e_k - x_j)}{\Pi_{j \neq k}(e_j - e_k)},$$

where $e_1 < x_1 < e_2 < \cdots < e_n < x_n$ and $k = 1, \cdots, n$. In the case that $n = 3, 4$ the elliptic coordinates admit expression in terms of Jacobi elliptic functions [2], [3]. For $n = 3$ we have

$$q_1 = k\mathrm{sn}\alpha\ \mathrm{sn}\beta\ \mathrm{sn}\gamma, \quad q_2 = i\frac{k}{k'}\mathrm{cn}\alpha\ \mathrm{cn}\beta\ \mathrm{cn}\gamma, \quad q_3 = \frac{1}{kk'}\mathrm{dn}\alpha\ \mathrm{dn}\beta\ \mathrm{dn}\gamma,$$

where we write $x_1 = \mathrm{sn}\alpha$, $x_2 = \mathrm{sn}\beta$ and $x_3 = \mathrm{sn}\gamma$ with normalized choice of e_i according to $e_1 = 0, e_2 = 1$ and $e_3 = k^{-2}$ with $k^2 < 1$, and the k dependence of the Jacobi elliptic functions has been suppressed, i. e., $\mathrm{sn}\delta = \mathrm{sn}(\delta, k)$. Typically the Jacobi elliptic function $\mathrm{sn}(\delta, k)$ is defined by

$$\delta = \int_0^{\mathrm{sn}(\delta,k)} \frac{1}{\sqrt{(1-t^2)(1-k^2t^2)}}dt.$$

These functions have properties analogous to trigonometric functions. The variables α, β, γ vary in the ranges $\alpha \in [-K, K], \beta \in [K - iK', K + iK']$ and $\gamma \in [iK' - K, iK' + K]$. In addition to elliptic coordinates in Euclidean space there are also elliptic coordinates x_j on the n-dimensional sphere. These are defined by relations

$$\sum_{k=1}^{n+1} \frac{s_k^2}{z - e_k} = \frac{\Pi_{j=1}^{n+1}(z - x_j)}{\Pi_{k=1}^{n+1}(z - e_k)}, \tag{1.2}$$

where $s_1^2 + \cdots + s_{n+1}^2 = 1$. The inverse relations are

$$s_k^2 = \frac{\Pi_{j=1}^n (e_k - x_j)}{\Pi_{j \neq k}(e_j - e_k)}$$

where $k = 1, \cdots, n+1$ and the coordinates satisfy $e_1 < x_1 < e_2 < \cdots < e_n < x_n < e_{n+1}$. These coordinates enable the ansatz of separation of variables to be used for problems on the sphere analogous to those solved in Euclidean space. If $n = 2$, the coordinates can also be written in terms of Jacobi elliptic functions according to [2], [3]

$$s_1 = k\ \mathrm{sn}\alpha\ \mathrm{sn}\beta, \quad s_2 = i\frac{k}{k'}\mathrm{cn}\alpha\ \mathrm{cn}\beta, \quad s_3 = \frac{1}{k'}\mathrm{dn}\alpha\ \mathrm{dn}\beta \tag{1.3}$$

with α and β varying in the same ranges as for Euclidean elliptical coordinates. The Jacobi elliptical coordinates enabled the problem of geodesic motion on an ellipsoid to be solved. It was on the basis of these investigations of Jacobi that subsequent investigations in the theory of separation of variables developed. Most notable among these were the mechanism of separation extended by Stäckel [4] to quite general systems of orthogonal coordinates. Moreover, for product separability of the Helmholtz or Schrödinger equation $\Delta\Psi + \lambda_1\Psi = 0$ on a space of constant curvature it was found that the ellipsoidal coordinates are generic. Every orthogonal separable coordinate system for these equations is some limiting form of the ellipsoidal coordinates. (For product R-separability of the Laplace equation $\Delta\Psi = 0$ the generic coordinates are fourth order surfaces called cyclides.)

Most functions commonly called "special" obey symmetry properties best described via group theory. They arise as solutions of the PDEs of mathematical physics and can be characterized in terms of transformation properties under the Lie symmetries of the equations. In particular this is true for functions of hypergeometric type that arise as solutions of the Laplace equation of Euclidean space via separation of variables [5], [6], [7], [8]: spherical harmonics, Laguerre polynomials, Jacobi polynomials, Bessel functions, confluent hypergeometric functions, etc. These functions all arise via separation of coordinates that are degenerate forms of Jacobi ellipsoidal coordinates. They have been very well studied and shown to obey differential recurrence relations, generating functions and other identities that relate to Lie symmetries and have explicit power series expansions determined by two-term recurrence relations. However, the special functions that arise via separation of the Laplace equation in more general ellipsoidal coordinates, e. g., Lamé functions and general functions of Heun type, have received much less attention since their power series are given via three-term recurrence formulas and they admit no differential recurrence relations and other properties characteristic of hypergeometric functions. Nonetheless functions of Heun type do have remarkable properties, e. g., [9], [10], [11]. The most notable presentation of results related to these functions in the last century is contained in the classical book of Whittaker and Watson [2]. These authors devoted an entire chapter to ellipsoidal harmonics and Lamé functions and they called attention to the earlier pioneering work of Niven [12], in particular the Niven transform. Unfortunately the beautiful product identities appearing in this chapter have received little notice in modern special function theory. We suspect that this is because the methods of proof seem obscure and the relevance to more modern special function theory is unclear.

In this paper we apply the modern theory of variable separation in which orthogonal separable coordinates for the n-variable Laplace equation are characterized by $n-1$ commuting second order conformal symmetry operators, e. g., [13]. This approach allows us to exploit the conformal symmetry of the Laplace equation and obtain beautiful identities for Heun type functions that have a transparent interpretation. This is particularly true for the Niven transform which we reinterpret as an intertwining operator under the action of the conformal group. We will give simple operator derivations of some of the basic formulas presented by Whittaker and Watson and then show how to generalize their results, and those of Niven, to more complicated situations and to higher dimensions.

Section 2 describes the operator characterization of variable separation for solutions of the Laplace equation and the Laplace-Beltrami eigenvalue equation on a pseudo-Riemannian manifold. The results are specialized to the Euclidean space Laplace equations in three or more variables. The action of the conformal symmetry group on the solution space of the Laplace equation is worked out and, following [7], lower variable models of this action are constructed. The detailed computations in this paper are usually carried out in these models, for simplicity. In Section 3 we introduce the Niven operator that maps harmonic functions to ellipsoidal functions, and show that it can be interpreted as an intertwining operator. We work out a number of identities for Lamé functions that follow from this action. In Section 4 we interpret product formulas for Lamé polynomials as mappings from a lower variable model to the solution space of the Laplace equation. Several of the special function identities in the first few sections of the paper can already be found in references such as [2], [3], though with more complicated proofs. In the later sections, however, we extend our models and the Niven operator to general Heun functions and to higher dimensional spaces and obtain many new results.

We acknowledge consultations with Vadim Kuznetsov, who provided the critical impetus to launch this research.

2. Symmetries and variable separation

Let Δ_n be a Laplace-Beltrami operator for a pseudo-Riemannian manifold V_n in n dimensions. The *Laplace-Beltrami eigenvalue equation* (with potential) for functions Ψ on V_n is $H\Psi(\mathbf{q}) = E\Psi(\mathbf{q})$. The *Laplace equation* is $H\Psi(\mathbf{q}) = 0$. The linear partial differential operator S is a *symmetry operator* for $(\Delta_n + V)\Phi = E\Phi$ if S maps local solutions Φ to local solutions $S\Phi$. Similarly, $\tilde{S}$ is a *conformal symmetry operator* for $(\Delta_n + V)\Phi = 0$ if $\tilde{S}$ maps local solutions Φ of this equation to local solutions $S\Phi$. The 1st-order symmetry operators for $(\Delta_n + V)\Phi = E\Phi$ form a Lie algebra, the *symmetry algebra* of this equation. The associated local *Lie symmetry group* maps solutions to solutions. There are similar definitions for conformal symmetries.

A set of orthogonal coordinates $\{x_\ell\}$ is *R-separable* for the Laplace-Beltrami equation if this equation admits solutions $\Psi = \exp(R(\mathbf{x}))\Pi_{i=1}^{n}\Psi_i(x^i) = e^R\Theta$, where $R(\mathbf{x})$ is a fixed function, independent of parameters, and the factors $\Psi_i(x^i)$ are the solutions of n ODEs (the separation equations) $\Psi_i'' + g_i(x^i)\Psi_i' - \left(f_i(x^i) + \sum_{j=1}^{n}\lambda_j s_{ij}(x^i)\right)\Psi_i = 0$, $i = 1, \cdots, n$ and $\lambda_1 = E$. The parameters λ_j are the *separation constants*. If $R \equiv 0$ we have *separation*, and if $R(x) = \sum_{i=1}^{n} R^{(i)}(x^i)$ we have *trivial* R-separation. There is a corresponding definition of R-separation for the Laplace equation with $E = 0$.

A basic result in the theory is [13] that every orthogonal R-separable coordinate system $\{x^i\}$ for $(\Delta_n + V)\Psi = E\Psi$ corresponds to a linearly independent set $\{S_1 = H = \Delta_n + V, S_2, \cdots, S_n\}$ of commuting 2nd-order partial differential symmetry operators. The R-separable solutions $\Psi_{\lambda_1,\cdots,\lambda_n}(\mathbf{x}) = \exp(R(\mathbf{x}))\Pi_{i=1}^{n}\Psi_i(x^i)$ are characterized as the simultaneous eigenfunctions of the commuting symmetry operators S_h: $S_h\Psi_{\lambda_1,\cdots,\lambda_n} = \lambda_h\Psi_{\lambda_1,\cdots,\lambda_n}$, $h = 1, \cdots, n$. If $E = 0$ the characterization is the same, except that the S_h are conformal symmetry operators.

Finding all orthogonal separable coordinate systems $\mathbf{q}$ for a given space V_n is difficult. However, for real n-dimensional Euclidean space, the n-sphere, and the n-hyperboloid of two sheets, we have a graphical procedure to classify and construct all possibilities, [14], [15].

Special functions of Heun type arise through variable separation of the Euclidean space Laplace equation in ellipsoidal and related coordinates, but have no simple transformation properties under the Lie symmetry algebra. For the Laplace equation

$$\left(\partial_X^2 + \partial_Y^2 + \partial_Z^2\right)\Psi = 0$$

orthogonal separation is possible in the 11 Helmholtz separable systems [16] and nontrivial R-separation in 6 additional systems [7]. Each system is characterized by a pair of commuting 2nd-order conformal symmetry operators for the Laplacian. The conformal symmetry algebra of this equation is 10-dimensional, with basis

$$P_X = \partial_X,\ M_{YX} = -M_{XY} = Y\partial_X - X\partial_Y,\ D = -\left(\frac{1}{2} + X\partial_X + Y\partial_Y + Z\partial_Z\right),$$

$$K_X = -2XD - R^2\partial_X$$

etc., where $R^2 = X^2 + Y^2 + Z^2$. The 55 2nd-order operators formed from this Lie algebra of differential operators satisfy 20 relations on the solution space, among which are

$$\mathbf{P}\cdot\mathbf{P} \equiv P_X^2 + P_Y^2 + P_Z^2 = 0,\ \mathbf{M}\cdot\mathbf{M} \equiv M_{YX}^2 + M_{XZ}^2 + M_{ZY}^2 = \frac{1}{4} - D^2. \tag{2.1}$$

Every R-separable solution set is characterized by a pair of 2nd-order commuting conformal symmetries. For ellipsoidal coordinates ($R = 0$) the operators can be chosen as $\mathbf{M}\cdot\mathbf{M} + (a-1)P_Y^2 + aP_Z^2$, $M_{XZ}^2 + aM_{YZ}^2 - aP_Z^2$, whereas for conical coordinates ($R = 0$) they are $\mathbf{M}\cdot\mathbf{M}$, $M_{XZ}^2 + aM_{YZ}^2$. The complicated characterizations suggest what is true, that the Lamé functions associated with these separable systems have no simple transformation properties under the symmetry algebra, [3].

Computations involving separable solutions of the Laplace equation are simplified by making use of a 2-variable model for the solution space: we represent solutions $\Psi(X, Y, Z)$ in an integral form

$$\Psi(X,Y,Z) = \int_{C_1} d\beta \int_{\tilde{C}_2} d\varphi\ h[\beta,\varphi]\exp\left[\beta(iX\cos\varphi + iY\sin\varphi - Z)\right] \equiv I(h),$$

where h is analytic on a complex domain that contains the integration contours $C_1 \times C_2$ and is chosen such that $I(h)$ converges absolutely, and arbitrary differentiation with respect to X, Y, Z is permitted under the integral sign. For each h, $\Psi = I(h)$ is a solution of the Laplace equation and the action of the conformal symmetries on the solution space corresponds to the operators

$$\mathcal{P}_X = i\beta w_1,\ \mathcal{P}_Y = i\beta w_2,\ \mathcal{P}_Z = -\beta,\ D = \beta\partial_\beta + \frac{1}{2},\ \mathcal{M}_{XY} = -w_2\partial_{w_1},$$

$$\mathcal{M}_{ZX} = iw_1\beta\partial_\beta + iw_2^2\partial_{w_1},\ \mathcal{M}_{ZY} = iw_2\beta\partial_\beta - iw_1w_2\partial_{w_1} \tag{2.2}$$

where $w_1^2 + w_2^2 = 1$. (Indeed, we can take $w_1 = \cos\varphi$, $w_2 = \sin\varphi$.) We shall not make use of the conformal symmetries K.

Let us find an integral representation for solutions Ψ that are eigenfunctions of the dilation operator D with eigenvalue $-\ell - \frac{1}{2}$. We choose C_1 and C_2 as unit circles in the β and $t = e^{i\varphi}$ complex planes, respectively, and require ℓ to be a non-negative integer. Setting $\mathcal{D}h = \left(-\ell - \frac{1}{2}\right) h$ we find $h(\beta, t) = \beta^{-\ell-1} j(t)$, $j(t) = \sum_{m=-\ell}^{\ell} a_m t^m$. Then we evaluate the β integral by residues to obtain

$$\Psi(X, Y, Z) = I(h) = \int_0^{2\pi} [X\cos\varphi + Y\sin\varphi + iZ]^\ell \, j(e^{i\varphi}) d\varphi.$$

Since $\mathbf{M}\cdot\mathbf{M} = \frac{1}{4} - D^2$ we have $\mathbf{M}\cdot\mathbf{M}\Psi = -\ell(\ell+1)\Psi$. For $j(t) = t^m$, $-\ell \leqslant m \leqslant \ell$ we have $M^0\Psi = m\Psi$ so Ψ must be a multiple of the solid harmonic $R^\ell Y_\ell^m(\theta, \phi)$, expressed in spherical coordinates. This model has an obvious extension to the Laplace equation in n dimensions.

It is easy to extend this analysis to Laplace equations in N dimensions. We give the pertinent details for $N = 4$:

$$(\partial_X^2 + \partial_Y^2 + \partial_Z^2 + \partial_Y^2)\Psi = 0. \tag{2.3}$$

The conformal symmetry algebra of this equation is 15-dimensional but we consider only the 11-dimensional scale-Euclidean subalgebra with basis

$$P_X = \partial_X, \quad P_Y = \partial_Y, \quad P_Z = \partial_Z, \quad P_T = \partial_T \quad M_{YX} = -M_{XY} = Y\partial_X - X\partial_Y,$$
$$M_{XZ} = Z\partial_Z - Z\partial_X, \quad M_{ZY} = Z\partial_Y - Y\partial_Z, \quad M_{TX} = T\partial_X - X\partial_T,$$
$$M_{TY} = T\partial_Y - Y\partial_T, \quad M_{TZ} = T\partial_Z - Z\partial_T, \quad D = -(X\partial_X + Y\partial_Y + Z\partial_Z + T\partial_T).$$

Each element of this Lie algebra maps a solution of (2.3) to another solution. Note the relations

$$\mathbf{P}\cdot\mathbf{P} \equiv P_X^2 + P_Y^2 + P_Z^2 + P_T^2 = 0,$$
$$\mathbf{M}\cdot\mathbf{M} \equiv M_{YX}^2 + M_{XZ}^2 + M_{ZY}^2 + M_{TX}^2 + M_{TY}^2 + M_{TZ}^2 = -D(D-2).$$

These identities hold only on the solution space of (2.3).

The various spherical and ellipsoidal solution sets for (2.3) are each characterized by a triplet of second-order commuting scale-Euclidean symmetry operators. Here computations involving separable solutions of the Laplace equation are simplified by making use of a 3 variable model for the solution space. We represent solutions $\Psi(X, Y, Z, T)$ in the integrable form

$$\Psi = \int_{C_1} d\beta \int\int_{\mathcal{D}} \exp\left[\beta(w_1 X + w_2 Y + w_3 Z + iT)\right] h(\beta, \mathbf{w}) \frac{dw_1\, dw_2}{w_3} \equiv I(h), \tag{2.4}$$

where $w_1^2 + w_2^2 + w_3^2 = 1$, h is analytic on a domain in the space of three complex variables that contains the integration domain $C_1 \times \mathcal{D}$, where C_1 is an analytic curve and $\mathcal{D}$ is a two-dimensional Riemann surface over $w_1 - w_2$ space, and is chosen such that $I(h)$ converges absolutely, and arbitrary differentiation with respect to X, Y, Z is permitted under the integral sign. Then it is easy to check that for each h, $\Psi = I(h)$ is a solution of the Laplace equation. Moreover, integrating by parts, we find that the action of the symmetries $P_X, \cdots, D$ on the solution space of the Laplace equation corresponds to the operators

$$\mathcal{P}_X = \beta w_1, \quad \mathcal{P}_Y = \beta w_2, \quad \mathcal{P}_Z = \beta w_3, \quad \mathcal{P}_T = i\beta, \quad \mathcal{D} = \beta\partial_\beta \tag{2.5}$$
$$\mathcal{M}_{XY} = w_1\partial_{w_2} - w_2\partial_{w_1}, \quad \mathcal{M}_{XZ} = -w_3\partial_{w_1}, \quad \mathcal{M}_{YZ} = -w_3\partial_{w_2},$$
$$\mathcal{M}_{TX} = -iw_1 + iw_1\beta\partial_\beta + i\left(1 - w_1^2\right)\partial_{w_1} - iw_1 w_2\partial_{w_2},$$
$$\mathcal{M}_{TY} = -iw_2 + iw_2\beta\partial_\beta - iw_2 w_1\partial_{w_1} + i\left(1 - w_2^2\right)\partial_{w_2},$$
$$\mathcal{M}_{TZ} = -iw_3 + iw_3\beta\partial_\beta - iw_3 w_1\partial_{w_1} - iw_3 w_2\partial_{w_2}.$$

Let us find an integral representation for solutions Ψ of the Laplace equation that are eigenfunctions of the dilation operator D with eigenvalue $-\ell$: $D\Psi = -\ell\Psi$. We choose C_1 as a unit circle about the origin in the β plane, and require ℓ to be a non-negative integer. Setting $\mathcal{D}h = -\ell h$ we find $h(\beta, \mathbf{w}) = \beta^{-\ell-1} j(\mathbf{w})$. Then we can evaluate the β integral by residues to obtain

$$\Psi = I(h) = \int\int_{\mathcal{D}} [w_1 X + w_2 Y + w_3 Z + iT]^{\ell}\, j(\mathbf{w}) \frac{dw_1\, dw_2}{w_3} \tag{2.6}$$

to within a multiplicative constant. Since $\mathbf{M}\cdot\mathbf{M} = -D(D-2)$ we have $\mathbf{M}\cdot\mathbf{M}\Psi = -\ell(\ell+2)\Psi$, so Ψ can also be considered as a function on the complex three-sphere.

3. Niven operators

Niven constructed an operator that maps harmonic functions, i. e., solutions of the $n = 3$ Laplace equation that are homogeneous of degree ℓ in X, Y, Z, into ellipsoidal solutions. Indeed, it maps a conical coordinate solution to an ellipsoidal solution, and is an infinite-order differential operator. A detailed technical construction is given in [2]. Here we give a much simpler treatment. Our theory extends to cover Niven operators in n dimensions and for many new coordinate systems.

Let H_ℓ be the space of solutions of the Laplace equation, homogeneous of degree ℓ. There is an operator F_ℓ, the Niven operator, such that relations

$$\begin{aligned} \left(\mathbf{M}\cdot\mathbf{M} + (a-1)P_Y^2 + aP_Z^2\right) F_\ell &= F_\ell(\mathbf{M}\cdot\mathbf{M}) \\ \left(M_{XZ}^2 + aM_{YZ}^2 + aP_Z^2\right) F_\ell &= F_\ell\left(M_{XZ}^2 + aM_{YZ}^2\right), \end{aligned} \tag{3.1}$$

$$F_\ell = {}_0F_1\left(\begin{matrix} - \\ -\ell - 1/2 \end{matrix}; \frac{1}{4}\left((a-1)P_Y^2 + aP_Z^2\right)\right)$$

hold on H_ℓ, where $\mathbf{M}\cdot\mathbf{M}$ is given by (2.1) and ${}_0F_1(\cdot)$ is a hypergeometric function [5]. Thus F_ℓ is an intertwining operator on H_ℓ between the spaces of separated conical solutions and of separated ellipsoidal solutions, each expressible in terms of Lamé functions [2], [3]. We verify (3.1) using the model, on the space $\mathcal{H}_\ell$ of functions $h(\beta, t) = \beta^{-\ell-1} j(t)$. Setting $t = e^{i\varphi}$ we have

$$(a-1)\mathcal{P}_Y^2 + a\mathcal{P}_Z^2 = \beta^2\left(a\sin^2\varphi + \cos^2\varphi\right).$$

Set $\mathcal{F}_\ell = \mathcal{F}_\ell(x)$, where $x = \beta^2\left(a\sin^2\varphi + \cos^2\varphi\right)$. Thus on $\mathcal{H}_\ell$ the Niven operator is just multiplication by an ordinary analytic function of x. The first equation (3.1) on $\mathcal{H}_\ell$ then reduces to a second-order ODE for $\mathcal{F}_\ell$:

$$4x\mathcal{F}_\ell'' + (-4\ell + 2)\mathcal{F}_\ell' - \mathcal{F}_\ell = 0. \tag{3.2}$$

The solution bounded at 0 is $\mathcal{F}_\ell = {}_0F_1(-\ell - 1/2; x/4)$, [5]. An independent solution for ℓ not a negative half-integer, is

$$\tilde{\mathcal{F}}_\ell = x^{\ell+\frac{3}{2}}{}_0F_1\left(\begin{matrix} - \\ \ell + \frac{5}{2} \end{matrix}; \frac{1}{4}x\right).$$

(Usually we will employ the solution bounded at $x = 0$.) Similarly, the second condition (3.1) is satisfied provided exactly the same equation (3.2) holds. Transferring this operator over to the solution space via $F_\ell\Psi = I(\mathcal{F}_\ell h)$ we obtain the required result. Note that a solution of the intertwining equations is given by $\Psi_E = I(h)$ where $h = \mathcal{F}_\ell \beta^{-\ell-1} j(t)$. Here, $j(t)$ satisfies the eigenvalue equation for Lamé functions, [3], $\left(\mathcal{M}_{XZ}^2 + a\mathcal{M}_{YZ}^2\right) j = \lambda_1 j$ for operators $\mathcal{M}$ on the space of functions corresponding to homogeneity of degree ℓ, exactly the same equation as satisfied by the conical coordinate eigenfunctions. Thus we obtain the classical result [2] that if $L_\ell^m(\alpha)$ are Lamé polynomials then

$$L_\ell^m(\alpha) L_\ell^m(\beta) L_\ell^m(\gamma) = c\int_{-2K}^{2K} P_\ell(\mu) L_\ell^m(\delta)\, d\delta,$$

$$\mu = k^2 \mathrm{sn}\alpha\ \mathrm{sn}\beta\ \mathrm{sn}\gamma\ \mathrm{sn}\delta - \left(k^2/k'^2\right) \mathrm{cn}\alpha\ \mathrm{cn}\beta\ \mathrm{cn}\gamma\ \mathrm{cn}\delta - \left(1/k'^2\right) \mathrm{dn}\alpha\ \mathrm{dn}\beta\ \mathrm{dn}\gamma\ \mathrm{dn}\delta$$

where $P_\ell(z)$ is a Legendre polynomial and c is a normalization constant.

Niven's operator can be extended to Laplace equations in N dimensions. We give the details of one extension for $N = 4$ and then exhibit the basic mechanism that will allow us to extend the construction to all N. The Laplace equation (2.3) admits separation in rotational conical coordinates

$$X + iY = R\sqrt{\frac{vw}{a}}e^{i\phi},\ Z = R\sqrt{\frac{(v-1)(w-1)}{1-a}},\ T = R\sqrt{\frac{(v-a)(w-a)}{a(a-1)}}, \tag{3.3}$$

where $R^2 = X^2 + Y^2 + Z^2 + T^2$, determined by symmetry operators

$$\Gamma_1 = M_{XY}^2, \quad \Gamma_2 = M_{XT}^2 + M_{YT}^2 + a\left(M_{XZ}^2 + M_{YZ}^2\right), \quad \Gamma_3 = \mathbf{M}\cdot\mathbf{M}, \tag{3.4}$$

and in rotational ellipsoidal coordinates

$$X + iY = \sqrt{\frac{uvw}{a}}e^{i\phi},\ Z = \sqrt{\frac{(u-1)(v-1)(w-1)}{1-a}},\ T = \sqrt{\frac{(u-a)(v-a)(w-a)}{a(a-1)}}, \tag{3.5}$$

determined by symmetry operators

$$\Gamma_1' = M_{XY}^2, \quad \Gamma_2' = M_{XT}^2 + M_{YT}^2 + a\left(M_{XZ}^2 + M_{YZ}^2\right) - a\left(P_X^2 + P_Y^2\right), \tag{3.6}$$

$$\Gamma_3' = \mathbf{M}\cdot\mathbf{M} + P_Z^2 + aP_T^2.$$

Let H_ℓ be the space of solutions of the Laplace equation (2.3) that are homogeneous of degree ℓ. There is an operator F_ℓ, the Niven operator, such that the identities

$$\Gamma_j' F_\ell = F_\ell \Gamma_j, \quad j = 1, 2, 3. \tag{3.7}$$

hold on H_ℓ. The operator can be chosen in the form

$$F_\ell = {}_0F_1\left(\begin{matrix} \\ -\ell-1 \end{matrix}; \frac{1}{4}\left(P_Z^2 + aP_T^2\right)\right).$$

Thus the Niven operator is an intertwining operator on H_ℓ between the spaces of separated rotational conical solutions and of separated rotational ellipsoidal solutions. Since F_ℓ is a function of a partial differential operator, it appears to be rigorously defined only for the case where ℓ is a non-negative integer. However, again, we can remove this requirement by making use of the three-variable model. Indeed we will verify (3.7) using model (2.4), on the space $\mathcal{H}_\ell$ of functions $h(\beta, \mathbf{w}) = \beta^{-\ell-1}j(\mathbf{w})$. In this model we have $\mathcal{P}_Z^2 + a\mathcal{P}_T^2 = -\beta^2(w_1^2 + w_2^2 + a - 1)$. Set $\mathcal{F}_\ell = \mathcal{F}_\ell(x)$, where $x = -\beta^2(w_1^2 + w_2^2 + a - 1)$. Thus on $\mathcal{H}_\ell$ the Niven operator is just multiplication by an ordinary analytic function of x. The first equation (3.7) on $\mathcal{H}_\ell$ is trivial; the second reduces to a second-order ODE for $\mathcal{F}_\ell$:

$$4x\mathcal{F}_\ell'' - 4\ell\mathcal{F}_\ell' - \mathcal{F}_\ell = 0. \tag{3.8}$$

For ℓ not a negative half-integer, a basis for the solution space is

$$\mathcal{F}_\ell = {}_0F_1\left(\begin{matrix} - \\ -\ell-1 \end{matrix}; \frac{1}{4}x\right), \quad \tilde{\mathcal{F}}_\ell = x^{\ell+2}{}_0F_1\left(\begin{matrix} - \\ \ell+3 \end{matrix}; \frac{1}{4}x\right).$$

(Usually we will employ the solution bounded at $x = 0$.)

We show that the second condition (3.7) is satisfied provided exactly the same equation (3.8) holds. Write $\mathcal{M}_{UV} = \mathcal{M}_{UV}^o + f_{UV}$ where $\mathcal{M}_{UV}^o$ is the pure differential operator part of $\mathcal{M}_{UV}$ and f_{UV}

is multiplication by a function. Thus, $f_{XT}=-iw_1$, $f_{YT}=-iw_2$, and $f_{XZ}=f_{YZ}=0$. Then we have the operator identity

$$\mathcal{M}^2_{UV}\mathcal{F}_\ell=[\mathcal{M}^o{}_{UV}x]^2\mathcal{F}''_\ell+[\mathcal{M}^{o2}_{UV}x]\mathcal{F}'_\ell+2\mathcal{F}'_\ell[\mathcal{M}^o{}_{UV}x]\mathcal{M}_{UV}+\mathcal{F}_\ell\mathcal{M}^2_{UV}. \tag{3.9}$$

Using the facts that

$$\begin{aligned}
&[\mathcal{M}^o{}_{XT}x]=-2ia\beta^2w_1, && [\mathcal{M}^{o2}_{XT}x]=2a\beta^2(1+w_1^2),\\
&[\mathcal{M}^o{}_{YT}x]=-2ia\beta^2w_2, && [\mathcal{M}^{o2}_{YT}x]=2a\beta^2(1+w_2^2),\\
&[\mathcal{M}^o{}_{XZ}x]=2\beta^2w_3w_1, && [\mathcal{M}^{o2}_{XZ}x]=2\beta^2(2w_1^2+w_2^2-1),\\
&[\mathcal{M}^o{}_{YZ}x]=2\beta^2w_3w_2, && [\mathcal{M}^{o2}_{YZ}x]=2\beta^2(w_1^2+2w_2^2-1),
\end{aligned}$$

we evaluate the operator expression

$$\big(\mathcal{M}^2_{XT}+\mathcal{M}^2_{YT}+a\mathcal{M}^2_{XZ}+a\mathcal{M}^2_{YZ}-a\mathcal{P}^2_X-a\mathcal{P}^2_Y\big)\mathcal{F}_\ell-\mathcal{F}_\ell\big(\mathcal{M}^2_{XT}+\mathcal{M}^2_{YT}+a\mathcal{M}^2_{XZ}+a\mathcal{M}^2_{YZ}\big)$$

and verify that it is the zero operator on H_ℓ, provided $4x\mathcal{F}''_\ell-4\ell\mathcal{F}'_\ell-\mathcal{F}_\ell=0$, the differential equation for the Niven operator for $N=4$. Transferring this operator over to the solution space of the Laplace equation via $F_\ell\Psi=I(\mathcal{F}_\ell h)$ we obtain our result.

To clarify the mechanism behind the Niven operator construction we consider the case of general ellipsoidal coordinates for the $N=4$ Laplace equation. General ellipsoidal separable solutions for the Laplace equation (2.3) are characterized by the commuting operators

$$\begin{aligned}
&\Gamma'_1=\mathbf{M}\cdot\mathbf{M}+\big((a+b+ab)P_X^2+(a+b)P_Y^2+(1+b)P_Z^2+(1+a)P_T^2\big),\\
&\Gamma'_2=(a+b)M^2_{XY}+(1+b)M^2_{XZ}+(1+a)M^2_{XT}+bM^2_{YZ}+aM^2_{YT}+M^2_{ZT}+\\
&\quad+\big((1+a+b)P_X^2+abP_Y^2+bP_Z^2+aP_T^2\big),\\
&\Gamma'_3=abM^2_{XY}+bM^2_{XZ}+aM^2_{XT}+abP_X^2.
\end{aligned}\tag{3.10}$$

Here, we have chosen the parameters for the ellipsoidal coordinates as

$$(e_1,e_2,e_3,e_4)=(0,1,a,b),\qquad 1<a<b.$$

The associated conical coordinates are characterized by the commuting operators

$$\begin{aligned}
&\Gamma_1=\mathbf{M}\cdot\mathbf{M}\\
&\Gamma_2=(a+b)M^2_{XY}+(1+b)M^2_{XZ}+(1+a)M^2_{XT}+bM^2_{YZ}+aM^2_{YT}+M^2_{ZT},\\
&\Gamma_3=abM^2_{XY}+bM^2_{XZ}+aM^2_{XT}.
\end{aligned}\tag{3.11}$$

Note that $\Gamma'_j=\Gamma_j+\Phi_j$, $j=1,2,3$, where the Φ_j are linear combinations of squares of linear momentum operators. Thus,

$$[\Gamma'_i,\Gamma'_j]=[\Gamma_i,\Gamma_j]=[\Phi_i,\Phi_j]=0,\quad i,j=1,2,3.$$

It follows that the commutivity of the Γ'_j implies the important commutation relations

$$[\Gamma_i,\Phi_j]+[\Phi_i,\Gamma_j]=0,\quad 1\leqslant i,j\leqslant 3. \tag{3.12}$$

Theorem 1. *Let H_ℓ be the space of solutions of the Laplace equation* (2.3) *that are homogeneous of degree ℓ. There exists a Niven operator F_ℓ, such that the identities*

$$\Gamma'_jF_\ell=F_\ell\Gamma_j,\quad j=1,2,3. \tag{3.13}$$

hold on H_ℓ. The operator can be chosen in the form

$$F_\ell = {}_0F_1\left(\begin{matrix} - \\ -\ell-1 \end{matrix} ; \frac{1}{4}\left((1+a+b)P_X^2 + (a+b)P_Y^2 + (1+b)P_Z^2 + (1+a)P_T^2\right)\right).$$

Proof. We will verify (3.13) using some of the relations (3.12) and the model (2.4). In this model we have

$$\begin{aligned} x = \Phi_1 &= (1+a+b)\mathcal{P}_X^2 + (a+b)\mathcal{P}_Y^2 + (1+b)\mathcal{P}_Z^2 + (1+a)\mathcal{P}_T^2 = \\ &= \beta^2\left\{(1+a+b)w_1^2 + (a+b)w_2^2 + (1+b)w_3^2 - 1 - a\right\}. \end{aligned}$$

Set $\mathcal{F}_\ell = \mathcal{F}_\ell(x)$. Thus on $\mathcal{H}_\ell$ the Niven operator is just multiplication by an ordinary analytic function of x. Since $\mathcal{M}\cdot\mathcal{M} = -\mathcal{D}^2 + 2$, the first equation $j = 1$ (3.13) on $\mathcal{H}_\ell$ reduces to the usual (for N=4) second-order ODE for $\mathcal{F}_\ell$:

$$4x\mathcal{F}_\ell'' - 4\ell\mathcal{F}_\ell' - \mathcal{F}_\ell = 0. \tag{3.14}$$

The solution bounded at 0 is

$$\mathcal{F}_\ell = {}_0F_1\left(\begin{matrix} - \\ -\ell-1 \end{matrix} ; \frac{1}{4}x\right).$$

We show that, due to relations (3.12), the second and third conditions (3.13) are also satisfied. Write $\mathcal{M}_{UV} = \mathcal{M}_{UV}^o + f_{UV}$ where $\mathcal{M}_{UV}^o$ is the pure differential operator part of $\mathcal{M}_{UV}$ and f_{UV} is multiplication by a function. Then, using the identities (3.9) we see that each of the last two operator expressions (3.13) takes the general form

$$\sum \alpha_i\left[\mathcal{M}_i^{o2}x\right]\mathcal{F}_\ell' + \sum \alpha_i\left[\mathcal{M}_i^o x\right]^2\mathcal{F}_\ell'' + 2\mathcal{F}_\ell'\left(\sum \alpha_i[\mathcal{M}_i^o x]M_i\right) + \sum \gamma_k\mathcal{P}_k^2\mathcal{F}_\ell = 0, \tag{3.15}$$

where $\Gamma_j = \sum \alpha_i M_i^2$, $\Phi_j = \sum \gamma_k P_k^2$ for $j = 2$ or $j = 3$. Our task is to verify that (3.15) holds. In the model, each of the operator identities $[\mathcal{M}\cdot\mathcal{M}, \Phi_j] + [x, \Gamma_j] = 0$ takes the form

$$4\left(\sum \gamma_k\mathcal{P}_k^2\right)(1 + \beta\partial_\beta) + 2\sum \alpha_i[\mathcal{M}_i^o x]\mathcal{M}_i + \sum \alpha_i[\mathcal{M}_i^{o2}x] = 0. \tag{3.16}$$

If we apply both sides of (3.16) to x we find the function identity

$$12x\sum \gamma_k\mathcal{P}_k^2 + 2\sum \alpha_i[\mathcal{M}_i^o x]^2 + 2\alpha_i[\mathcal{M}_i^o x]f_i x + \sum \alpha_i[\mathcal{M}_i^{o2}x]x = 0. \tag{3.17}$$

Note that the functional (non-differential operator) component of (3.16) is the identity

$$4\sum \gamma_k\mathcal{P}_k^2 + 2\sum \alpha_i[\mathcal{M}_i^o x]f_i + \sum \alpha_i[\mathcal{M}_i^{o2}x] = 0. \tag{3.18}$$

Substituting (3.18) into (3.17) we obtain

$$\sum \alpha_i[\mathcal{M}_i^o x]^2 = -4x\sum \gamma_k\mathcal{P}_k^2, \tag{3.19}$$

and restricting (3.16) to $\mathcal{H}_\ell$ we obtain

$$2\sum \alpha_i[\mathcal{M}_i^o x]\mathcal{M}_i + \sum \alpha_i\left[\mathcal{M}_i^{o2}x\right] = 4\ell\left(\sum \gamma_k\mathcal{P}_k^2\right). \tag{3.20}$$

Substituting (3.19), (3.20) in (3.15) we obtain

$$\left(\sum \gamma_k\mathcal{P}_k^2\right)\left(4x\mathcal{F}_\ell'' - 4\ell\mathcal{F}_\ell' - \mathcal{F}_\ell\right) = 0,$$

which is implied by the differential equation for the Niven operator for $N = 4$. ■

It is clear from this proof that the same construction goes through for all ellipsoidal coordinates and their conical counterparts for the Laplace equation in N dimensions. The only novelty is that the Niven operator in N dimensions takes the form

$$ {}_0F_1\left(\begin{matrix} - \\ -\ell - N/2 + 1 \end{matrix} ; \frac{D^2}{4}\right) $$

where D^2 is a linear combination of squares of the P_k. Moreover there is a Niven operator corresponding to any degenerate limit of these coordinates, such as the rotational ellipsoidal system considered previously. (Two other degenerate examples are prolate and oblate spheroidal separable solutions for $N = 3$, obtained by Niven operators applied to the spherical harmonics.)

Niven operators for the complex Laplace equation

It is obvious that the Niven construction can also be extended to the complex Laplace equation and its other real forms, such as the various real wave equations. We will use the two-variable model (2.2) for the case $N = 3$ to clarify the commuting operator foundation of the construction. Any separable system for the complex Laplace equation contained in the space of solutions that are homogeneous of degree ℓ is characterized as a system of simultaneous eigenfunctions of the commuting symmetries

$$ \Gamma_1 = \mathbf{M}\cdot\mathbf{M} \quad \Gamma_2 = \frac{1}{2}\sum_{jk=1}^{3} \alpha_{jk}(M_jM_k + M_kM_j), \tag{3.21} $$

where $\alpha_{jk} = \alpha_{kj}$ and these constants are not all zero. Here, $M_{XY} = M_3$ $M_{ZX} = M_2$, $M_{ZY} = M_1$. We always have $\mathbf{P}\cdot\mathbf{P} \equiv \sum_{j=1}^{3} P_j^2 = 0$ where $P_1 = P_X,\ P_2 = P_Y,\ P_3 = P_Z$.

The question is when do there exist quadratic forms

$$ \Phi_1 = A_1P_1^2 + B_1P_2^2 + C_1P_3^2 + D_1P_1P_2 + E_1P_1P_3 + F_1P_2P_3, $$

$$ \Phi_2 = A_2P_1^2 + B_2P_2^2 + C_2P_3^2 + D_2P_1P_2 + E_2P_1P_3 + F_2P_2P_3, $$

not identically 0, such that the operators

$$ \Gamma_1' = \Gamma_1 + \Phi_1, \Gamma_2' = \Gamma_2 + \Phi_2, \tag{3.22} $$

commute? (Since $\mathbf{P}\cdot\mathbf{P} = 0$, without loss of generality we can require $A_j + B_j + C_j = 0$, $j = 1, 2$. Furthermore, by adding an appropriate multiple of Γ_1' to Γ_1' we can always achieve $\alpha_{11} = 0$, and we shall assume this.) Note that since

$$ [\Gamma_j', \Gamma_k'] = [\Gamma_j, \Gamma_k] = [\Phi_j, \Phi_k] = 0, \quad j, k = 1, 2, $$

we must require the important commutation relations

$$ [\Gamma_1, \Phi_2] + [\Phi_1, \Gamma_2] = 0. \tag{3.23} $$

Now we assume that the coefficients α_{jk}, not all zero, are given, and we use the model (2.2) to compute the possibilities for the operators Φ_1, Φ_2. In the model, the operator Φ_1 becomes multiplication by the function

$$ x = \beta^2\left(a + bw_1 + cw_2 + dw_1^2 + ew_1w_2\right), $$

where $w_1^2 + w_2^2 = 1$. Here,

$$ a = C_1 - B_1,\ b = -iE_1,\ c = -iF_1,\ d = B_1 - A_1,\ e = -D_1. $$

Similarly, in the model Φ_2 becomes multiplication by a function of the form

$$y = \beta^2 \left(a' + b'w_1 + c'w_2 + d'w_1^2 + e'w_1w_2\right),$$

Since $[\mathcal{M} \cdot \mathcal{M}, y] = -6y - 4y\beta\partial_\beta$, the operator equation (3.23) becomes (with $\alpha_{11} = 0$)

$$6y + 4y\beta\partial_\beta + \sum_{jk} \alpha_{jk} \left([\mathcal{M}_j x]\mathcal{M}_k + [\mathcal{M}_k x]\mathcal{M}_j\right) + \frac{1}{2}\sum_{jk} \alpha_{jk} \left([\mathcal{M}_j\mathcal{M}_k x] + [\mathcal{M}_k\mathcal{M}_j x]\right) = 0. \tag{3.24}$$

Equating the coefficients of ∂_{w_1} on both sides of this operator equation, we obtain the 7 conditions

$$\begin{pmatrix} 0 & 2\alpha_{12} & \alpha_{22} & 2i\alpha_{23} & -2i\alpha_{13} \\ 0 & -\alpha_{22} & 2\alpha_{12} & 2i\alpha_{13} & 2i\alpha_{23} \\ 2\alpha_{12} & 2i\alpha_{23} & 2i\alpha_{13} & 0 & \alpha_{22} - \alpha_{33} \\ -2i\alpha_{23} & 2\alpha_{12} & -\alpha_{33} & -2i\alpha_{23} & -2i\alpha_{13} \\ -2i\alpha_{13} & -\alpha_{22} + \alpha_{33} & -2\alpha_{12} & 0 & -2i\alpha_{23} \\ 4\alpha_{12} & 0 & 0 & 2\alpha_{12} & \alpha_{22} - 2\alpha_{33} \\ -\alpha_{22} & 0 & 0 & -\alpha_{22} + \alpha_{33} & 0 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \\ d \\ e \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}. \tag{3.25}$$

Similarly, equating the coefficients of ∂_{w_1} on both sides of the operator equation we obtain the expression for y in terms of the expansion coefficients for x:

$$\begin{pmatrix} a' \\ b' \\ c' \\ d' \\ e' \end{pmatrix} = \begin{pmatrix} \frac{1}{2}e\alpha_{12} + \frac{1}{2}ib\alpha_{13} \\ \frac{1}{2}c\alpha_{12} + b\alpha_{22} - \frac{1}{2}ie\alpha_{23} \\ \frac{1}{2}b\alpha_{12} + \frac{1}{2}ie\alpha_{13} \\ -\frac{1}{2}ib\alpha_{13} + (a+d)\alpha_{22} - \frac{1}{2}ic\alpha_{23} \\ (2a+d)\alpha_{12} - \frac{1}{2}ic\alpha_{13} + \frac{1}{2}e\alpha_{22} + \frac{1}{2}ib\alpha_{23} \end{pmatrix}. \tag{3.26}$$

The equation obtained by equating the functional (non-operator) terms on both sides of the operator equation is redundant; it is already implied by our preceeding two relations.

It is straightforward to check that for not identically zero $\{\alpha_{jk}\}$, the rank of the 7×5 matrix in (3.25) is always 4. Hence the conditions (3.25) always have a nonzero solution, unique up to multiplication by a constant. Thus we have proved the following.

Theorem 2. *For each set of operators*

$$\Gamma_1 = \mathbf{M} \cdot \mathbf{M}, \qquad \Gamma_2 = \frac{1}{2}\sum_{jk=1}^{3} \alpha_{jk}(M_jM_k + M_kM_j)$$

where $\alpha_{jk} = \alpha_{kj}$ and these constants are not all zero, there is a set of quadratic forms

$$\Phi_1 = A_1P_1^2 + B_1P_2^2 + C_1P_3^2 + D_1P_1P_2 + E_1P_1P_3 + F_1P_2P_3,$$
$$\Phi_2 = A_2P_1^2 + B_2P_2^2 + C_2P_3^2 + D_2P_1P_2 + E_2P_1P_3 + F_2P_2P_3,$$

such that the operators $\Gamma_1' = \Gamma_1 + \Phi_1, \Gamma_2' = \Gamma_2 + \Phi_2$, are commuting symmetries of the complex equation $\left(P_1^2 + P_2^2 + P_3^2\right)\Psi = 0$. Φ_1 and Φ_2 are unique, up to multiplication by a common constant.

It follows from this theorem and our earlier results that we can construct Niven operators corresponding to all of these cases. Though we don't give the details here, we note that limiting forms of our Lamé and Heun identities yield nontrivial product formulas for hypergeometric functions, particularly Legendre and Bessel functions.

4. Lamé polynomials.

As we have seen, if $\Psi(X,Y,Z)$ satisfies Laplace's equation and is homogeneous of degree ℓ (an integer) then Ψ may be represented by the integral

$$\Psi = \int_0^{2\pi} (iZ + X\cos\varphi + Y\sin\varphi)^\ell f(\varphi) d\varphi$$

for a suitable 2π-periodic function $f(\varphi)$. If we set

$$X = R\sqrt{\frac{uv}{a}}, \quad Y = R\sqrt{\frac{(u-1)(v-1)}{(1-a)}}, \quad Z = R\sqrt{\frac{(u-a)(v-a)}{a(a-1)}}, \tag{4.1}$$

Laplace's equation can be written as

$$\left(\partial_R^2 + \frac{2}{R}\partial_R + \frac{1}{R^2}\left\{\frac{-4}{u-v}\left[\sqrt{P(u)}\partial_u\left(\sqrt{P(u)}\partial_u\right) - \sqrt{P(v)}\partial_v\left(\sqrt{P(v)}\partial_v\right)\right]\right\}\right)\Psi = 0$$

where $P(u) = u(u-1)(u-a)$. If $\Psi = R^\ell \Phi(u,v)$ we see that $\Phi(u,v)$ satisfies the partial differential equation

$$\left\{\frac{4}{u-v}\left[\sqrt{P(u)}\partial_u\left(\sqrt{P(u)}\partial_u\right) - \sqrt{P(v)}\partial_v\left(\sqrt{P(v)}\partial_v\right)\right] - \ell(\ell+1)\right\}\Phi(u,v) = 0.$$

Now if $\Phi(u,v) = U(u)V(v)$ then U and V satisfy the separation equations

$$\left(\sqrt{P(u)}\partial_u(\sqrt{P(u)}\partial_u) - \ell(\ell+1)u + \lambda\right)U(u) = 0.$$
$$\left(\sqrt{P(v)}\partial_v(\sqrt{P(v)}\partial_v) - \ell(\ell+1)v + \lambda\right)V(v) = 0.$$

Setting $u = \mathrm{sn}^2(\mu,k)$ and $a = 1/k^2$ we see that the equation satisfied by $\hat{U}(\mu) = U(u)$ is

$$\left(\partial_\mu^2 - k^2\mathrm{sn}^2(\mu,k) + \lambda\right)\hat{U}(\mu) = 0.$$

Similarly we can put $v = \mathrm{sn}^2(\nu,k)$ and obtain the differential equation $\left(\partial_\nu^2 - k^2\mathrm{sn}^2(\nu,k) + \lambda\right)\hat{V}(\nu) = 0$ where $\hat{V}(\nu) = V(v)$. The operator which characterizes λ is

$$\Lambda = \frac{1}{k^2}M_3^2 + M_2^2.$$

If we check the action of the rotation group in X, Y, Z we obtain the representation

$$\mathcal{M}_{XY} = \partial_\varphi, \quad \mathcal{M}_{XZ} = i(\ell+1)\cos\varphi + i\sin\varphi\partial_\varphi, \quad \mathcal{M}_{YZ} = i(\ell+1)\sin\varphi - i\cos\varphi\partial_\varphi.$$

We now look for eigenfunctions of the operator $L = \frac{1}{k^2}\mathcal{M}_{XY}^2 + \mathcal{M}_{XZ}^2$. In this representation they must satisfy

$$\left[(a-\sin^2\varphi)\partial_\varphi^2 - (2\ell+3)\sin\varphi\cos\varphi\partial_\varphi - (\ell+1)(\ell+2)\cos^2 t + \ell + 1 - \lambda\right]f(\varphi) = 0.$$

Here, f can be recognized as a Lamé function as follows. With $\sin\varphi = \mathrm{sn}(w,k)$ and $a = \frac{1}{k^2}$ this equation becomes

$$\left(\partial_w^2 + \frac{\ell(\ell+1)k'^2}{\mathrm{dn}^2(w,k)} - \ell(\ell+1) - \lambda\right)W(w) = 0$$

where $W(w)\mathrm{dn}(w,k)^{\ell+1} = f(\varphi)$. If we now choose the new variable $w = \omega + K$ this equation has the form of the Lamé equation

$$\left(\partial_\omega^2 - k^2\ell(\ell+1)\mathrm{sn}^2(\omega,k) - \lambda\right)\Omega(\omega) = 0.$$

This is exactly the separation one would obtain from the original Laplace equation that involved the use of coordinates u and v. Indeed if we choose the elliptic coordinates $u = \mathrm{sn}^2(\mu,k)$ and $v = \mathrm{sn}^2(\nu,k)$ the X, Y, Z coordinates can be taken to be

$$X = Rk\ \mathrm{sn}(\mu,k)\mathrm{sn}(\nu,k), \quad Y = R\frac{ik}{k'}\mathrm{cn}(\mu,k)\mathrm{cn}(\nu,k), \quad Z = R\frac{i}{k'}\mathrm{dn}(\mu,k)\mathrm{dn}(\nu,k).$$

We readily obtain the product formula

$$L(\mu,k)L(\nu,k) = \kappa\int\Big[-ikk'\mathrm{sn}(\mu,k)\mathrm{sn}(\nu,k)\mathrm{sn}(\omega,k) - \frac{k}{k'}\mathrm{cn}(\mu,k)\mathrm{cn}(\nu,k)\mathrm{cn}(\omega,k) + \\ + \frac{i}{k}\mathrm{dn}(\mu,k)\mathrm{dn}(\nu,k)\mathrm{dn}(\omega,k)\Big]^\ell L(\omega,k)d\omega, \tag{4.2}$$

where $L(z,k)$ is a solution of the Lamé equation

$$\left(\partial_z^2 - k^2\ell(\ell+1)\mathrm{sn}^2(z,k) - \lambda\right)L(z,k) = 0$$

and the integral is over the path $-2K \leqslant \omega \leqslant 2K$. We now look at other examples involving Heun functions. However, the basic method has been outlined in this first example.

5. Rotational Heun functions

Here we consider rotational types of coordinates giving rise to Heun functions that are not of Lamé type. Consider solutions of Laplace's equation in four dimensions with Cartesian coordinates X, Y, Z, and T. We choose new coordinates

$$X + iY = R\sqrt{\frac{uv}{a}}e^{i\varphi}, \quad Z = R\sqrt{\frac{(u-1)(v-1)}{1-a}}, \quad T = R\sqrt{\frac{(u-a)(v-a)}{a(a-1)}}. \tag{5.1}$$

In these coordinates the Laplace equation becomes

$$0 = \left(\partial_X^2 + \partial_Y^2 + \partial_Z^2 + \partial_T^2\right)\Psi =$$
$$= \left(\partial_R^2 + \frac{3}{R}\partial_R + \frac{1}{R^2}\left\{\frac{-4}{u-v}\left[\sqrt{\frac{P(u)}{u}}\partial_u\left(\sqrt{uP(u)}\partial_u\right) - \sqrt{\frac{P(v)}{v}}\partial_v\left(\sqrt{vP(v)}\partial_v\right) + \frac{a}{uv}\partial_\varphi^2\right]\right\}\right)\Psi.$$

Setting $\Psi = R^\ell\Phi(u,v,\varphi)$ we find that the equation for Φ is

$$\left\{\frac{4}{u-v}\left[\sqrt{\frac{P(u)}{u}}\partial_u\left(\sqrt{uP(u)}\partial_u\right) - \sqrt{\frac{P(v)}{v}}\partial_v\left(\sqrt{vP(v)}\partial_v\right)\right] - \frac{a}{uv}\partial_\varphi^2 - \ell(\ell+2)\right\}\Phi = 0.$$

The separation equations have the form

$$\sqrt{\frac{P(u)}{u}}\partial_u\left(\sqrt{uP(u)}\partial_u\right) - \frac{ap^2}{u} - u\ell(\ell+2) + \lambda)U(u) = 0,$$
$$\sqrt{\frac{P(v)}{v}}\partial_v\left(\sqrt{vP(v)}\partial_v\right) - \frac{ap^2}{v} - v\ell(\ell+2) + \lambda)V(v) = 0,$$

where we have chosen the φ dependence to be $\Phi(u,v,\varphi)=U(u)V(v)e^{ip\varphi}$. If we put $u=\mathrm{sn}^2(\mu,k)$ and take as usual $a=1/k^2$, then with $U(u)=(\mathrm{sn}(\mu,k))^{-1/2}\hat{U}(\mu)$, the differential equation satisfied by $\hat{U}$ is

$$\left(\partial_\mu^2+\frac{\frac14-p^2}{\mathrm{sn}^2(\mu,k)}-k^2\left(\ell+\frac12\right)\left(\ell+\frac32\right)\mathrm{sn}^2(\mu,k)+\frac14\left(1+k^2\right)+k^2\lambda\right)\hat{U}(\mu)=0.$$

An identical equation is satisfied by $\hat{V}(\nu)$ where $V(\nu)=(\mathrm{sn}(\nu,k))^{-1/2}\hat{V}(\nu)$. The operator characterizing the variable separation is

$$\Lambda=(a+1)M_{XY}^2+a\left(M_{XZ}^2+M_{YZ}^2\right)+M_{XT}^2+M_{YT}^2.$$

Just as in the preceding section, we can realize any homogeneous solution of degree ℓ as an integral transform from a model:

$$\Psi=\int\!\!\int\left(T+iX\sin\theta\cos\varphi'+iY\sin\theta\sin\varphi'+iZ\cos\theta\right)^{\ell}f\left(\theta,\varphi'\right)d\theta d\varphi'.$$

The generators of the rotations acting on the functions $f\left(\theta,\varphi'\right)$ have the form

$$\mathcal{M}_{XY}=\partial_{\varphi'},\quad \mathcal{M}_{XZ}=-\cos\varphi'\partial_\theta+\cot\theta\sin\varphi'\partial_{\varphi'}+\cos\varphi'\cot\theta,$$

$$\mathcal{M}_{YZ}=-\sin\varphi'\partial_\theta-\cot\theta\cos\varphi'\partial_{\varphi'}+\sin\varphi'\cot\theta,$$

$$\mathcal{M}_{YT}=i(\ell+1)\sin\theta\sin\varphi'-i\sin\theta\sin\varphi'\partial_\theta+i\frac{\cos\varphi'}{\sin\theta}\partial_{\varphi'}-i\frac{\sin\varphi'}{\sin\theta},$$

$$\mathcal{M}_{XT}=-i(\ell+1)\sin\theta\cos\varphi'+i\cos\theta\cos\varphi'\partial_\theta-i\frac{\sin\varphi'}{\sin\theta}\partial_{\varphi'}-i\frac{\cos\varphi'}{\sin\theta},$$

$$\mathcal{M}_{TZ}=-i(\ell+1)\cos\theta-i\sin\theta\partial_\theta.$$

We now seek eigenfunctions of Λ with eigenvalue λ. If we look for solutions of the form $f(\theta,\varphi)=$ $=W(w)e^{ip\varphi}$ where $\sin\theta=\mathrm{sn}(w,k)$ then with $W(w)=\mathrm{dn}(w,k)^{-\ell-3/2}\mathrm{cn}(w,k)^{1/2}\Omega(\omega)$ where $w=\omega+$ $+K$, we see that $\Omega(\omega)$ satisfies the differential equation

$$\left(\partial_\omega^2+\frac{\frac14-p^2}{\mathrm{sn}^2(\omega,k)}-k^2\left(\ell+\frac12\right)\left(\ell+\frac32\right)\mathrm{sn}^2(\omega,k)+\frac14\left(1+k^2\right)+k^2\lambda\right)\Omega(\omega)=0,$$

which is the exact same equation that arises from the separation of the Laplace equation. We can similarly give a suitable product formula. In fact

$$(\mathrm{sn}(\mu,k)\mathrm{sn}(\nu,k))^{-1/2}M(\mu,k)M(\nu,k)e^{ip\varphi}= \tag{5.2}$$

$$=\kappa\int\!\!\int\Big[-ik'\mathrm{sn}(\mu,k)\mathrm{sn}(\nu,k)\mathrm{cn}(\omega',k)\cos(\varphi-\varphi')-k\mathrm{cn}(\mu,k)\mathrm{cn}(\nu,k)\mathrm{sn}(\omega',k)+$$

$$+\frac{i}{k'}\mathrm{dn}(\mu,k)\mathrm{dn}(\nu,k)\mathrm{dn}(\omega',k)\Big]^{\ell}\mathrm{sn}\left(\omega',k\right)^{1/2}M\left(\omega',k\right)e^{ip\varphi'}d\omega'd\varphi',$$

where $M(z,k)$ is a solution of the ordinary differential equation

$$\left(\partial_z^2+\frac{\frac14-p^2}{\mathrm{sn}^2(z,k)}-k^2\left(\ell+\frac12\right)\left(\ell+\frac32\right)\mathrm{sn}^2(z,k)+\frac14\left(1+k^2\right)+k^2\lambda\right)M(z,k)=0.$$

and the integral is over the domain $0\leqslant\varphi'\leqslant 2\pi$, $0\leqslant\omega'\leqslant 2K$. The φ' integration of this product formula could in principle be calculated.

6. General Heun functions

We now develop a product formula for reasonably general Heun functions. We choose the coordinates

$$X+iY=R\sqrt{\frac{uv}{a}}e^{i\varphi},\quad Z+iT=R\sqrt{\frac{(u-1)(v-1)}{1-a}}e^{i\psi},\tag{6.1}$$

$$U+iV=R\sqrt{\frac{(u-a)(v-a)}{a(a-1)}}e^{i\theta}.$$

In these six coordinates the Laplace equation has the form

$$\left(\partial_X^2+\partial_Y^2+\partial_Z^2+\partial_T^2+\partial_U^2+\partial_V^2\right)\Psi=$$

$$=\left(\partial_R^2+\frac{5}{R}\partial_R+\frac{1}{R^2}\left\{\frac{-4}{u-v}\left[\partial_u(P(u)\partial_u)-\partial_v(P(v)\partial_v)+\frac{a}{uv}\partial_\varphi^2+\right.\right.\right.$$

$$\left.\left.\left.+\frac{1-a}{(u-1)(v-1)}\partial_\psi^2+\frac{a(a-1)}{(u-a)(v-a)}\partial_\theta^2\right]\right\}\right)\Psi=0.$$

We look for solutions of the form $\Psi=R^\ell f(u,v,\varphi,\psi,\theta)$ and find that f satisfies

$$\left[\frac{-4}{u-v}\left[P(u)\partial_u(P(u)\partial_u)-P(v)\partial_v(P(v)\partial_v)+\frac{a}{uv}\partial_\varphi^2+\right.\right.$$

$$\left.\left.+\frac{1-a}{(u-1)(v-1)}\partial_\psi^2+\frac{a(a-1)}{(u-a)(v-a)}\partial_\theta^2-\ell(\ell+4)\right]\right]f=0.$$

Assuming $f=U(u)V(v)e^{ip\varphi+iq\psi+ir\theta}$, the separation equations have the form

$$\left\{4\partial_u[u(u-1)(u-a)\partial_u]-\frac{a}{u}p^2-\frac{(1-a)}{u-1}q^2-\frac{a(a-1)}{u-a}r^2-u\ell(\ell+4)-\lambda\right\}U(u)=0,$$

$$\left\{4\partial_v[v(v-1)(v-a)\partial_v]-\frac{a}{v}p^2-\frac{(1-a)}{v-1}q^2-\frac{a(a-1)}{v-a}r^2-v\ell(\ell+4)-\lambda\right\}V(v)=0.$$

If we write $U(u)=[u(u-1)(u-a)]^{-1/4}\bar{U}(u)$ then $\hat{U}(u)$ satisfies

$$\left(4\sqrt{P(u)}\partial_u\left(\sqrt{P(u)}\partial_u\right)-\left(\ell+\frac{3}{2}\right)\left(\ell+\frac{5}{2}\right)u+\frac{5}{4}(a+1)+\frac{a}{u}\left(\frac{1}{4}-p^2\right)+\frac{(1-a)}{u-1}\left(\frac{1}{4}-q^2\right)+\right.$$

$$\left.+\frac{a(a-1)}{u-a}\left(\frac{1}{4}-r^2\right)-\lambda\right)\hat{U}(u)=0,$$

with a similar equation for $\hat{V}(v)$. If we use our standard substitution $u=\mathrm{sn}^2(\mu,k)$ then this equation has the form

$$\left(\partial_\mu^2-k^2\left(\ell+\frac{3}{2}\right)\left(\ell+\frac{5}{2}\right)\mathrm{sn}^2(\mu,k)+\frac{5}{4}\left(1+k^2\right)+\frac{1}{\mathrm{sn}^2(\mu,k)}\left(\frac{1}{4}-p^2\right)+\frac{k'^2}{\mathrm{cn}^2(\mu,k)}\left(\frac{1}{4}-q^2\right)+\right.$$

$$\left.+\frac{k'^2}{\mathrm{dn}^2(\mu,k)}\left(\frac{1}{4}-r^2\right)-k^2\lambda\right)\hat{U}(\mu)=0,$$

where $\hat{U}(\mu)=\hat{U}(u)$. There is a similar equation for $\hat{V}(\nu)$. The operator with eigenvalue λ is

$$\Lambda=a\left(M_{XZ}^2+M_{XT}^2+M_{YZ}^2+M_{YT}^2\right)+M_{XU}^2+M_{XV}^2+M_{YU}^2+M_{YV}^2+(a+1)M_{XY}^2+$$

$$+(a-1)M_{ZT}^2+(1-a)M_{UV}^2.$$

As usual we write our solution of Laplace's equation as a transform of a model:

$$\Psi = \int (V + iUs_1 + iZs_2 + iTs_3 + iYs_4 + iXs_5)^\ell f(s_1, s_2, s_3, s_4, s_5)\ d\mathbf{s}$$

where $s_1^2 + s_2^2 + s_3^2 + s_4^2 + s_5^2 = 1$, we take $d\mathbf{s}$ as the differential on the four-sphere, and integrate over the four-sphere. Again we need to calculate the operators of the rotations on the space of functions f in coordinates that we take to be

$$s_1 = \cos\theta, \quad s_2 = \sin\theta\cos\varphi\cos A, \quad s_3 = \sin\theta\cos\varphi\sin A,$$

$$s_4 = \sin\theta\sin\varphi\cos B, \quad s_5 = \sin\theta\sin\varphi\sin B.$$

In terms of these new variables we have $f(s_1, s_2, s_3, s_4, s_5) = f(\theta, \varphi, A, B)$. We do not give the expressions for these operators although they have been computed by Maple. If we look for eigenfunctions of $\mathcal{M}_{XY}, \mathcal{M}_{ZT}$ and $\mathcal{M}_{UV}$ we see that the function f must have the form

$$f(\theta, \varphi, A, B) = (\sin\theta)^{-\ell-1}\left(\frac{\sin\theta}{1+\cos\theta}\right)^{ir} e^{ipA+iqB} f(\varphi).$$

This follows from the expressions for the generators of the group acting on the functions $f(\varphi, \theta, A, B)$. We now seek the eigenfunctions of the operator Λ and obtain the ordinary differential equation for $f(\varphi)$ viz.

$$\left(\left(a - \cos^2\varphi\right)\partial_\varphi^2 + \sin\varphi\left(-2\ell\cos\varphi - \frac{a}{\cos\varphi} + \frac{(1-a)\cos\varphi}{\sin^2\varphi}\right)\partial_\varphi + \right.$$
$$\left. + \left(\ell^2 - r^2\right)\cos^2\varphi + p^2 - ar^2 - q^2 - \ell^2 - 2\ell - \frac{a}{\cos^2\varphi}p^2 + -\frac{a-1}{\sin^2\varphi}q^2 - \lambda\right) f(\varphi) = 0.$$

Substituting $\cos\varphi = \mathrm{sn}(w, k)$ and $a = 1/k^2$, and writing

$$f(\varphi) = (\mathrm{sn}(w, k)\mathrm{cn}(w, k))^{1/2}\mathrm{dn}(w, k)^{-\ell-5/2} g(w)$$

with $w = \omega + K$, we see that the differential equation for $g(\omega)$ has the form

$$\left[\partial_\omega^2 - k^2\left(\ell + \frac{3}{2}\right)\left(\ell + \frac{5}{2}\right)\mathrm{sn}^2(\omega, k) + \frac{5}{4}\left(1 + k^2\right) + \frac{1}{\mathrm{sn}^2(\omega, k)}\left(\frac{1}{4} - p^2\right) + \right.$$
$$\left. + \frac{k'^2}{\mathrm{cn}^2(\omega, k)}\left(\frac{1}{4} - q^2\right) + \frac{k'^2}{\mathrm{dn}^2(\omega, k)}\left(\frac{1}{4} - r^2\right) - k^2\lambda\right] g(\omega) = 0.$$

This is the same as the equation for the separable products derived earlier.

We can derive a product formula which looks quite symmetric:

$$(\mathrm{sn}(\mu, k)\mathrm{cn}(\mu, k)\mathrm{dn}(\mu, k)\mathrm{sn}(\nu, k)\mathrm{cn}(\nu, k)\mathrm{dn}(\nu, k))^{-1/2}\Lambda(\mu)\Lambda(\nu)e^{ip\varphi+iq\psi+ir\theta} =$$
$$= \kappa \int \cdots \int \frac{1}{k'^2}\mathrm{dn}(\mu, k)\mathrm{dn}(\nu, k)\mathrm{dn}(\omega, k)\cos(Q - \varphi) -$$
$$- \left(\frac{k}{k'}\right)^2 \mathrm{cn}(\mu, k)\mathrm{cn}(\nu, k)\mathrm{cn}(\omega, k)\cos(A - \psi) -$$
$$-ik\mathrm{sn}(\mu, k)\mathrm{sn}(\nu, k)\mathrm{sn}(\omega, k)\cos(B - \theta)(\mathrm{sn}(\omega, k)\mathrm{cn}(\omega, k)\mathrm{dn}(\omega, k))^{1/2} \times$$
$$\times \Lambda(\omega)e^{ipQ+iqA+irB} d\omega\ dA\ dB\ dQ =$$
$$= \kappa' \int \Psi(\alpha, \beta, \gamma)(\mathrm{sn}(\omega, k)\mathrm{cn}(\omega, k)\mathrm{dn}(\omega, k))^{1/2}\Lambda(\omega)d\omega, \tag{6.2}$$

where

$$\alpha = \frac{1}{k'^2}\mathrm{dn}(\mu,k)\mathrm{dn}(\nu,k)\mathrm{dn}(\omega,k), \quad \beta = \frac{k}{k'^2}\mathrm{cn}(\mu,k)\mathrm{cn}(\nu,k)\mathrm{cn}(\omega,k),$$

$$\gamma = ik\mathrm{sn}(\mu,k)\mathrm{sn}(\nu,k)\mathrm{sn}(\omega,k), \quad e^{iQ} = \tan\frac{\theta}{2}.$$

The domain is $0 \leqslant A \leqslant 2\pi,\ 0 \leqslant B \leqslant 2\pi,\ 0 \leqslant C \leqslant 2\pi,\ -K \leqslant \omega \leqslant K$.

To calculate the kernel function for this product formula we use the formula

$$\Psi(\alpha,\beta,\gamma) = \int\int\int_{\mathcal{C}\times\mathcal{C}\times\mathcal{C}} (\alpha\cos A + \beta\cos B + \gamma\cos C)^{\ell} e^{i(pA+qB+rC)} dA\ dB\ dC =$$

$$= \frac{1}{2\pi}\Gamma(\ell+1)\alpha\beta\gamma \int_{\mathcal{C}} I_{-p-1}\left(\frac{1}{2}\lambda\alpha\right) I_{-q-1}\left(\frac{1}{2}\lambda\beta\right) I_{-r-1}\left(\frac{1}{2}\lambda\gamma\right)\lambda^{\ell-4} d\lambda$$

where λ runs over the unit circle and $I_\nu(z)$ is a second kind Bessel function. This integral is evaluated for ℓ an integer and has an obvious analytical continuation for complex ℓ.

We now give an example analogous to that of Section 4. We choose coordinates in five dimensional space

$$X = R\sqrt{\frac{uv}{a}}\sin\theta\cos\varphi, \quad Y = R\sqrt{\frac{uv}{a}}\sin\theta\sin\varphi, \tag{6.3}$$

$$Z = R\sqrt{\frac{uv}{a}}\cos\theta, \quad T = R\sqrt{\frac{(u-1)(v-1)}{1-a}}, \quad U = R\sqrt{\frac{(u-a)(v-a)}{a(a-1)}}$$

and we look for solutions of the Laplace equation

$$\left(\partial_X^2 + \partial_Y^2 + \partial_Z^2 + \partial_T^2 + \partial_U^2\right)\Psi = 0.$$

In these coordinates this equation looks like

$$\left\{\partial_R^2 + \frac{4}{R}\partial_R + \frac{1}{R^2}\left(\frac{-4}{u-v}\left(\frac{\sqrt{P(u)}}{u}\partial_u\left(u\sqrt{P(u)}\partial_u\right) - \frac{\sqrt{P(v)}}{v}\partial_v\left(v\sqrt{P(v)}\partial_v\right) + \right.\right.\right.$$

$$\left.\left.\left. + \frac{a}{uv}\left(\partial_\theta^2 + \cot\theta\partial_\theta + \frac{1}{\sin^2\theta}\partial_\varphi^2\right)\right)\right)\right\}\Psi = 0.$$

For solutions of the form $\Psi = R^\ell P_J^M(\cos\theta)\, e^{iM\varphi}U(u)V(v)$ then $U(u)$ satisfies the ordinary differential equation

$$\left(4\sqrt{\frac{P(u)}{u}}\partial_u\left(u\sqrt{P(u)}\partial_u\right) - \frac{a}{u}J(J+1) - \ell(\ell+3)u + \lambda\right)U(u) = 0$$

and $V(v)$ satisfies a similar equation. If we write $U(u) = \hat{U}(u)u^{-1/2}$ then the differential equation for $\hat{U}(u)$ satisfies

$$\left(\sqrt{P(u)}\partial_u\left(\sqrt{P(u)}\partial_u\right) - \frac{a}{u}J(J+1) - (\ell+1)(\ell+2)u + \lambda + a + 1\right)\hat{U}(u) = 0.$$

With $u = \mathrm{sn}^2(\mu,k)$ where $a = 1/k^2$ this equation has the form

$$\left(\partial_\mu^2 - \frac{J(J+1)}{\mathrm{sn}^2(\mu,k)} - (\ell+1)(\ell+2)k^2\mathrm{sn}^2(\mu,k) + k^2\lambda + k^2 + 1\right)U(\mu) = 0.$$

The operator that characterizes λ is

$$\Lambda = a\left(M_{XT}^2 + M_{YT}^2 + M_{ZT}^2\right) + M_{XU}^2 + M_{YU}^2 + M_{ZU}^2 + (a+1)\left(M_{XY}^2 + M_{YZ}^2 + M_{XZ}^2\right),$$

and the separation constant $-J(J+1)$ is characterized by $L = \left(M_{XY}^2 + M_{YZ}^2 + M_{XZ}^2\right)$. Solutions of the Laplace equation can be obtained by writing

$$\int \left(U + iT\cos A + iZ\sin A\cos\theta' + iY\sin A\sin\theta'\sin\varphi' + iX\sin A\sin\theta'\cos\varphi'\right)^{\ell} \times$$

$$\times f\left(A,\theta',\varphi'\right) dA\; d\theta\; d\varphi = \Psi,$$

where $0 \leqslant A \leqslant \pi,\ 0 \leqslant \theta' \leqslant \pi,\ 0 \leqslant \varphi' \leqslant 2\pi$.

We adopt our usual procedure and look for eigenfunctions of Λ and L when acting on the model functions $f(A,\theta',\varphi')$. Solutions take the form

$$f\left(A,\theta',\varphi'\right) = \mathrm{cn}(w,k)\mathrm{dn}(w,k)^{-\ell-1}\Omega(\omega)\sin\theta' P_J^M\left(\cos\theta'\right)e^{iM\varphi'}$$

where $\cos A = \mathrm{sn}(w,k)$, $a = 1/k^2$ and $w = \omega + K$. The function $\Omega(\omega)$ satisfies the differential equation

$$\left(\partial_\omega^2 - \frac{J(J+1)}{\mathrm{sn}^2(\omega,k)} - (\ell+1)(\ell+2)k^2\mathrm{sn}^2(\omega,k) + \lambda + k^2 + 1\right)\Omega(\omega) = 0.$$

We can easily obtain a product formula for this particular case:

$$\begin{aligned} L(\mu,k)L(\nu,k)P_J^M(\cos\theta)e^{iM\varphi} = \kappa \int\!\!\int\!\!\int \Bigg[\left(\frac{1}{k'}\right)^2 &\mathrm{dn}(\mu,k)\mathrm{dn}(\nu,k)\mathrm{dn}(\omega,k) - \\ -\left(\frac{k}{k'}\right)^2 &\mathrm{cn}(\mu,k)\mathrm{cn}(\nu,k)\mathrm{cn}(\omega,k) - \\ - ik\,\mathrm{sn}(\mu,k)\mathrm{sn}(\nu,k)\mathrm{sn}(\omega,k)&\left(\cos\theta\cos\theta' + \sin\theta\sin\theta'\right)\cos\left(\varphi-\varphi'\right)\Bigg]^{\ell} \times \\ \times\mathrm{sn}(\omega,k)L(\omega,k)\sin\theta' P_J^M&\left(\cos\theta'\right)e^{iM\varphi'}d\omega\; d\theta'\; d\varphi'. \end{aligned} \tag{6.4}$$

The domain of integration is $0 \leqslant \theta' \leqslant \pi,\ 0 \leqslant \varphi' \leqslant 2\pi,\ -2K \leqslant \omega \leqslant 0$. This is apparently a new formula. Here $L(z,k)$ satisfies the differential equation

$$\left[\partial_z^2 - \frac{J(J+1)}{\mathrm{sn}^2(z,k)} - (\ell+1)(\ell+2)k^2\mathrm{sn}^2(z,k) + k^2\lambda + k^2 + 1\right]L(z,k) = 0.$$

7. The general case of Heun type product formulas

To obtain the most general product formulas we choose coordinates in $p_1 + p_2 + p_3 + 3$ dimensional Euclidean space of the form

$$\mathbf{X} = rU_1\mathbf{s}_{p_1},\ \mathbf{Y} = rU_2\mathbf{s}_{p_2},\ \mathbf{Z} = rU_3\mathbf{s}_{p_3}$$

where r is a radial coordinate, $\mathbf{s}_p$ are coordinates on a p dimensional sphere and

$$U_i^2 = \frac{(u-e_i)(v-e_i)}{(e_i-e_j)(e_i-e_k)}$$

for $i,j,k = 1,2,3$ and i,j,k pairwise distinct. The U_i are the most general form of 3D conical coordinates. We now look for separable solutions of Laplace's equation $(\partial_{\mathbf{X}}\cdot\partial_{\mathbf{X}} + \partial_{\mathbf{Y}}\cdot\partial_{\mathbf{Y}} + \partial_{\mathbf{Z}}\cdot\partial_{\mathbf{Z}})\Psi = 0$ of the form

$$\Psi = r^t U(u)V(v)\pi_{i=1}^3 P_{\ell_i}(\mathbf{s}_{p_i}), \quad \Delta_{\mathbf{s}_{p_i}} P_{\ell_i}(\mathbf{s}_{p_i}) = -\ell_i(\ell_i + p_i - 1)P_{\ell_i}(\mathbf{s}_{p_i})$$

where $\Delta_{\mathbf{s}_{p_i}}$ is the Laplace operator on the p_i-sphere. With $\Phi(r,u,v) = r^t U(u)V(v)$ this equation becomes

$$\left(\partial_r^2 + \frac{(p_1+p_2+p_3+2)}{r}\partial_r + \frac{1}{r^2}\left(-\frac{4}{(u-v)}\left[P(u)\left(\partial_u^2 + \frac{1}{2}\left(\frac{p_1+1}{u-e_1}+\frac{p_2+1}{u-e_2}+\frac{p_3+1}{u-e_3}\right)\partial_u\right) - \right.\right.\right. \tag{7.1}$$

$$\left.-P(v)\left(\partial_v^2 + \frac{1}{2}\left(\frac{p_1+1}{v-e_1}+\frac{p_2+1}{v-e_2}+\frac{p_3+1}{v-e_3}\right)\partial_v\right)\right] + \frac{(e_1-e_2)(e_1-e_3)}{(u-e_1)(v-e_1)}\ell_1(\ell_1+p_1-1)+$$

$$\left.\left.+\frac{(e_2-e_1)(e_2-e_3)}{(u-e_2)(v-e_2)}\ell_2(\ell_2+p_2-1)+\frac{(e_3-e_2)(e_3-e_1)}{(u-e_3)(v-e_3)}\ell_3(\ell_3+p_3-1)\right)\right)\Phi(r,u,v)=0$$

where $P(\lambda) = (\lambda-e_1)(\lambda-e_2)(\lambda-e_3)$. The separation equations for the functions $U(u)$ and $V(v)$ are

$$\left(-4P(\lambda)\left(\partial_\lambda^2 + \frac{1}{2}\left(\frac{p_1+1}{\lambda-e_1}+\frac{p_2+1}{\lambda-e_2}+\frac{p_3+1}{\lambda-e_3}\right)\partial_\lambda\right) + \ell(\ell+p_1+p_2+p_3-1)+\right. \tag{7.2}$$

$$+\frac{(e_1-e_2)(e_1-e_3)}{(\lambda-e_1)}\ell_1(\ell_1+p_1-1)+\frac{(e_2-e_1)(e_2-e_3)}{(\lambda-e_2)}\ell_2(\ell_2+p_2-1)+$$

$$\left.+\frac{(e_3-e_2)(e_3-e_1)}{(\lambda-e_3)}\ell_3(\ell_3+p_3-1)-\mu\right)\Lambda(\lambda)=0$$

where $\Lambda = U, V$ and $\lambda = u, v$. If we look for solutions of the form $\Lambda(\lambda) = \left[\Pi_{q=1}^3(\lambda-e_q)\right]^{\ell_q} L(\lambda)$ we see that $L(\lambda)$ satisfies the general Heun equation. We can in fact find polynomial solutions to (7.1) as is indicated in [17]. Indeed if we require

$$U(u)V(v) = \Pi_{j=1}^q\left(\frac{U_1^2}{\theta_j-e_1}+\frac{U_2^2}{\theta_j-e_2}+\frac{U_3^2}{\theta_j-e_3}\right)\left(\Pi_{i=1}^3 U_i^{\ell_i}\right)$$

then the solutions are in product form because of the identity

$$\frac{U_1^2}{\theta-e_1}+\frac{U_2^2}{\theta-e_2}+\frac{U_3^2}{\theta-e_3} = \frac{(\theta-u)(\theta-v)}{(\theta-e_1)(\theta-e_2)(\theta-e_3)}.$$

The θ_i satisfy the Niven equations

$$\sum_{i=1}^3\frac{(2\ell_i+p_i+1)}{\theta_i-e_i}+\sum_{j\neq i}\frac{4}{\theta_i-\theta_j}=0.$$

The separation constant has the value

$$\mu = \sum_{i,j,k \text{ pairwise} \neq}[-(e_i-e_j-e_k)\ell_i(\ell_i+p_i-1)+2e_i\ell_j\ell_k+(4q+1)\ell_i(e_j+e_k)+$$

$$+2qp_i(e_j+e_k)]+4q^2(e_1+e_2+e_3)-[8q-2-4(\ell_1+\ell_2+\ell_3)-2(p_1+p_2+p_3)]\sum_{i=1}^q\theta_i.$$

Reverting to $e_1=0, e_2=1, e_3=1/k^2$ we have the product formula

$$(\mathrm{sn}(\mu,k)\mathrm{sn}(\nu,k))^{\ell_1}(\mathrm{cn}(\mu,k)\mathrm{cn}(\nu,k))^{\ell_2}(\mathrm{dn}(\mu,k)\mathrm{dn}(\nu,k))^{\ell_3}\times$$

$$\times L(\mu)L(\nu)P_{\ell_1}(\mathbf{s}_1)P_{\ell_2}(\mathbf{s}_2)P_{\ell_3}(\mathbf{s}_3)=$$

$$=c\int_{\mathbf{s}_1}\int_{\mathbf{s}_2}\int_{\mathbf{s}_3}\left[\frac{1}{k'^2}\mathrm{dn}(\mu,k)\mathrm{dn}(\nu,k)\mathrm{dn}(\omega,k)\mathbf{s}_3\cdot\mathbf{s}'_3+\right.$$

$$\left.+\left(\frac{k}{k'}\right)^2\mathrm{cn}(\mu,k)\mathrm{cn}(\nu,k)\mathrm{cn}(\omega,k)\mathbf{s}_2\cdot\mathbf{s}'_2-ik\mathrm{sn}(\mu,k)\mathrm{sn}(\nu,k)\mathrm{sn}(\omega,k)\mathbf{s}_1\cdot\mathbf{s}'_1\right]^\ell\times$$

$$\times\mathrm{sn}(\omega,k)^{\ell_1}\mathrm{cn}(\omega,k)^{\ell_2}\mathrm{dn}(\omega,k)^{\ell_3}L(\omega)P_{\ell_1}(\mathbf{s}'_1)\,P_{\ell_2}(\mathbf{s}'_2)\,P_{\ell_3}(\mathbf{s}'_3)\,d\mathbf{s}'_1 d\mathbf{s}'_2 d\mathbf{s}'_3 d\omega. \tag{7.3}$$

Here ds_i is the area measure taken over the sphere $\mathbf{s}_i \cdot \mathbf{s}_i = 1$ and $-K \leqslant \omega \leqslant K$. This is the most general formula possible. It is symbolic in the sense that $P_m(\mathbf{s'}_j)$ stands for a spherical function that has to be determined in each special case, and $L(\mu)$ is a Heun function.

8. Heun function identities from higher dimensional Niven operators

Now we apply the results of Section 3 to obtain examples of Niven operator identities for Heun functions in more than three dimensions. Consider again the example of rotational ellipsoidal coordinates (3.5),

$$X + iY = \sqrt{\frac{uvw}{a}}e^{i\phi}, \quad Z = \sqrt{\frac{(u-1)(v-1)(w-1)}{1-a}}, \tag{8.1}$$
$$T = \sqrt{\frac{(u-a)(v-a)(w-a)}{a(a-1)}}.$$

We seek separated solutions of the Laplace equation

$$\left(\partial_X^2 + \partial_Y^2 + \partial_Z^2 + \partial_T^2\right)\Psi = 0$$

in these coordinates. A typical solution has the form

$$\Psi = (\mathrm{sn}(\alpha,k)\mathrm{sn}(\beta,k)\mathrm{sn}(\gamma,k))^{-1/2}M_m^\ell(\alpha)M_m^\ell(\beta)M_m^\ell(\gamma)e^{im\phi}$$

where $M_m^\ell(\theta)$ satisfies the differential equation

$$\left(\partial_\theta^2 + \frac{\frac{1}{4}-p^2}{\mathrm{sn}^2(\theta,k)} - k^2\left(\ell+\frac{1}{2}\right)\left(\ell+\frac{3}{2}\right)\mathrm{sn}^2(\theta,k) + \frac{1}{4}\left(1+k^2\right) + k^2\lambda\right)M_m^\ell(\theta) = 0 \tag{8.2}$$

and $u = \mathrm{sn}^2(\alpha,k), v = \mathrm{sn}^2(\beta,k)$ and $w = \mathrm{sn}^2(\gamma,k)$. Functions of the form

$$H_\ell(X,Y,Z,T) = \int_0^\pi\int_{-\pi}^\pi (X\sin\theta\cos\varphi + Y\sin\theta\sin\varphi + Z\cos\theta + iT)^\ell f(\theta,\varphi)d\theta d\varphi$$

are homogeneous solutions of the Laplace equation. Here, the Niven operator is as given in Section 3 with $D^2 = \partial_Z^2 + a\partial_T^2$. The relationship between $H_\ell(X,Y,Z,T)$ and $G_\ell(X,Y,Z,T)$ is

$$G_\ell(X,Y,Z,T) = \sum_r \frac{(-1)^r D^{2r}}{4^r r!(\ell+2-r)(\ell+3-r)\cdots(\ell+1)}H_\ell(X,Y,Z,T)$$

To make sense of this expression for general ℓ we need to employ our model. If ℓ is an integer then this expression could be written as

$$G_\ell(X,Y,Z,T) = \sum_{r=0}^{\frac{\ell}{2}} \frac{(\ell+1-r)!}{r!(\ell+1)!}(-1)^r\left(\frac{D^2}{4}\right)^r H_\ell(X,Y,Z,T). \tag{8.3}$$

If we take H_ℓ in the form given above then, with $x = \left(-\cos^2\theta + a\right)$ and $U = X\sin\theta\cos\varphi + {}$ $+ Y\sin\theta\sin\varphi + Z\cos\theta + iT$, the formula gives

$$G_\ell(X,Y,Z,T) = \int_0^\pi\int_{-\pi}^\pi C_\ell^1\left(\frac{U}{\sqrt{x}}\right)h(\theta,\varphi)d\theta d\varphi$$

where $h(\theta,\varphi) = x^{n/2}f(\theta,\varphi)$ and $C_\ell^1(z)$ is a Gegenbauer polynomial. (Here the function G_ℓ is the ellipsoidal harmonic that is derived from the corresponding harmonic H_ℓ via the analog of the Niven operator. This is the same notation as used in the last chapter of [2].) Setting $\cos\theta = -\mathrm{cn}(\delta,k)/\mathrm{dn}(\delta,k)$

and noting that $G_\ell(X,Y,Z,T)$ can be taken to be a separated solution of the rotational Heun differential equation

$$\left[\sqrt{\frac{P(z)}{z}}\partial_z\left(\sqrt{zP(z)}\partial_z\right)+\frac{am^2}{z}-z\ell(\ell+2)+\lambda\right]Z(z)=0$$

where $z=u,v,w$, then

$$G_\ell(X,Y,Z,T)=(\mathrm{sn}(\alpha,k)\mathrm{sn}(\beta,k)\mathrm{sn}(\gamma,k))^{-1/2}M_m^\ell(\alpha)M_m^\ell(\beta)M_m^\ell(\gamma)e^{im\phi}=$$
$$=\kappa\int_{-\pi}^{\pi}\int_{-2K}^{2K}C_\ell^1(\nu)(\mathrm{sn}(\delta,k))^{-1/2}M_m^\ell(\delta)e^{im\varphi}d\delta d\varphi$$

where

$$\nu=k^2\mathrm{sn}\alpha\ \mathrm{sn}\beta\ \mathrm{sn}\gamma\ \mathrm{sn}\delta\cos(\phi-\varphi)-\left(k^2/k'^2\right)\mathrm{cn}\alpha\ \mathrm{cn}\beta\ \mathrm{cn}\gamma\ \mathrm{cn}\delta-\left(1/k'^2\right)\mathrm{dn}\alpha\ \mathrm{dn}\beta\ \mathrm{dn}\gamma\ \mathrm{dn}\delta.$$

In terms of the coordinates

$$X+iY=\sqrt{\frac{uvw}{a}}e^{i\phi},\quad Z+iT=\sqrt{\frac{(u-1)(v-1)(w-1)}{1-a}}e^{i\psi}$$
$$U=\sqrt{\frac{(u-a)(v-a)(w-a)}{a(a-1)}},\tag{8.4}$$

separated solutions of the 5D Laplace equation

$$\left(\partial_X^2+\partial_Y^2+\partial_Z^2+\partial_T^2+\partial_U^2\right)\Psi=0$$

take the form $\Psi=L_{pq}^\ell(\alpha)L_{pq}^\ell(\beta)L_{pq}^\ell(\gamma)e^{i(p\phi+q\psi)}$. Here $L_{pq}^\ell(z)$ is solution of

$$\left\{4\partial_z[z(z-1)(z-a)\partial_z]+\frac{a}{z}p^2+\frac{(1-a)}{z-1}q^2-z\ell(\ell+4)-\lambda\right\}L(z)=0$$

where $z=u,v,w$. Homogeneous solutions of the Laplace equation take the form

$$H_\ell(X,Y,Z,T,U)=\int(Xs_1+Ys_2+Zs_3+Ts_4+iU)^\ell f(s_1,s_2,s_3,s_4)ds$$

where ds is the standard area measure on the four-sphere and we set

$$s_1=\sin\theta\cos\varphi,\quad s_2=\sin\theta\sin\varphi,\quad s_3=\cos\theta\cos\psi',\quad s_4=\cos\theta\sin\psi'.$$

This solution is homogeneous of degree ℓ . We now relate this to $G_\ell(X,Y,Z,T,U)$ functions in the form

$$G_\ell=\sum_{r=0}^{\frac{1}{2}\ell}\frac{(-1)^rD^{2r}}{4^rr!(\ell+5/2-r)(\ell+7/2-r)\ldots(\ell+3/2)}H_\ell(X,Y,Z,T,U)$$

where $D^2=\partial_Z^2+\partial_T^2+a\partial_U^2$. This formula can also be expressed as

$$G_\ell(X,Y,Z,T,U)=\frac{\pi(-1)^{\ell+\ 1}}{\Gamma\left(\frac{3}{2}+\ell\right)}\left(\frac{D}{2}\right)^{\ell+\frac{3}{2}}I_{-\ell-\frac{3}{2}}(D)H_\ell(X,Y,Z,T,U).$$

The explicit form is

$$G_\ell(X,Y,Z,T,U)=\kappa\int_{-2K}^{2K}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi}C_\ell^{\frac{3}{2}}(\eta)h\left(\delta,\varphi,\psi'\right)d\delta\ d\varphi\ d\psi',$$
$$\eta=k^2\mathrm{sn}\alpha\ \mathrm{sn}\beta\ \mathrm{sn}\gamma\ \mathrm{sn}\delta\ \ \cos(\phi-\varphi)-\left(k^2/k'^2\right)\mathrm{cn}\alpha\ \mathrm{cn}\beta\ \mathrm{cn}\gamma\ \mathrm{cn}\delta\ \ \cos\left(\psi-\psi'\right)-$$
$$-\left(1/k'^2\right)\mathrm{dn}\alpha\ \mathrm{dn}\beta\ \mathrm{dn}\gamma\ \mathrm{dn}\delta.$$

In fact we can derive the multiplication formula for G_ℓ:

$$L^\ell_{pq}(\alpha)L^\ell_{pq}(\beta)L^\ell_{pq}(\gamma)e^{i(p\phi+q\psi)} = \kappa\int_{-\pi}^{\pi}\int_{-\pi}^{\pi}\int_{-2K}^{2K} C_n^{\frac{3}{2}}(\eta)L^\ell_{pq}(\delta)e^{i(p\varphi+q\psi')}d\delta\, d\varphi\, d\psi'.$$

These results generalize in an obvious manner.

Now consider the general coordinate system

$$X=\sqrt{\frac{uvw}{a}}s_p,\ Y=\sqrt{\frac{(u-1)(v-1)(w-1)}{1-a}}s_q,\ Z=\sqrt{\frac{(u-a)(v-a)(w-a)}{a(a-1)}}s_r \tag{8.5}$$

The product formula takes the general form

$$G_n(X,Y,Z,T)=L(\alpha)L(\beta)L(\gamma)P_\ell(s_1)P_m(s_2)P_n(s_3)=$$

$$=\kappa\int_{s_1}\int_{s_2}\int_{s_3}\int_{-2K}^{2K} C_n^\epsilon(\rho)L(\delta)P_\ell\left(s_1'\right)P_m\left(s_2'\right)P_n\left(s_3'\right)d\delta ds_1'ds_2'ds_3'$$

where the integrals ds_i are over the sphere $s_i\cdot s_i=1$,

$$\rho=k^2\mathrm{sn}\alpha\ \mathrm{sn}\beta\ \mathrm{sn}\gamma\ \mathrm{sn}\delta\ s_1\cdot s_1'-\left(k^2/k'^2\right)\ \mathrm{cn}\alpha\ \mathrm{cn}\beta\ \mathrm{cn}\gamma\ \mathrm{cn}\delta\ s_2\cdot s_2'-$$

$$-\left(1/k'^2\right)\ \mathrm{dn}\alpha\ \mathrm{dn}\beta\ \mathrm{dn}\gamma\ \mathrm{dn}\delta\ s_3\cdot s_3'$$

and $\epsilon=(p+q+r)/2$.

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