Natural Frequency Estimation for an Euler-Bernoulli Beam Carrying a Mass with Rotary Inertia

Xutao Sun, Yusuke Mochida, Sinniah Ilanko

The School of Engineering, The University of Waikato

Gate 1 Knighton Road Private Bag

3105 Hamilton 3240 New Zealand

BrianSun2018@hotmail.com, yusuke@waikato.ac.nz, ilanko@waikato.ac.nz

Summary

An Euler-Bernoulli beam carrying a mass with rotary inertia is shown in Figure 1. The governing equation of motion of the beam can be expresses as

$$\bar{m}\ddot{u}(x,t) + EIu''''(x,t) = \left[-M\ddot{q}_M(t) - J\ddot{\theta}(t)\right]\delta(x - x_0) \tag{1}$$

where $\bar{m}$ is mass per unit length, EI is flexural rigidity, x_0 is the coordinate of the mass location, and $u(x,t)$ is the deflection of the beam. The overdot represents the derivative with respect to time and the prime represents the derivative with respect to x. M and J denote the translational inertia and rotary inertia of the mass, respectively. $q_M(t)$ and $\theta(t)$ are the vertical displacement and rotation of the mass, respectively. The mass is rigidly attached to the beam. For the nth mode, the motion of the mass can be written as

$$q_M(t) = u_n(x,t)|_{x_0} = u_n(x_0,t) \tag{2}$$

and

$$\theta(t) = u_n'(x,t)|_{x_0} = u_n'(x_0,t) \tag{3}$$

where $u_n(x,t)$ is the beam deflection for the nth mode.

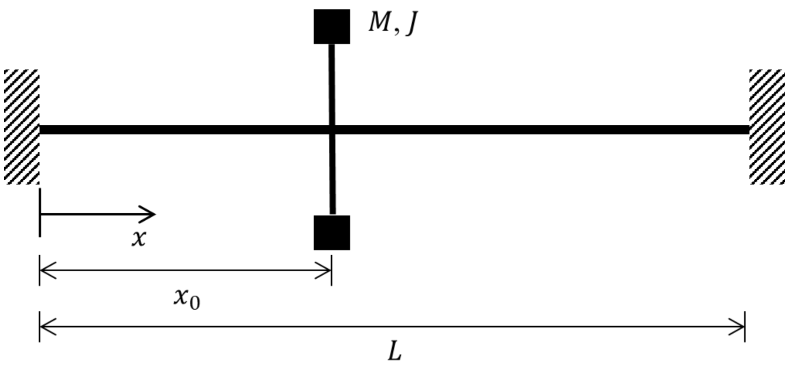


Figure 1. A beam carrying a roving mass with rotary inertia.

When M and J are small enough, the influence of the mass is insignificant. Therefore, assuming the mode shape stays the same after attaching the mass to the beam, $u_n(x,t)$ can be expressed as (assuming harmonic oscillation)

$$u_n(x,t) = \phi_n(x)e^{i\omega_n t} \tag{4}$$

where ω_n is the nth natural frequency of the beam carrying the mass, and $\phi_n(x)$ is the nth mode shape of the beam found by solving the following equation

$$EI\phi_n''''(x) - \bar{m}\omega_{bn}^2\phi_n(x) = 0 \tag{5}$$

where ω_{bn} is the nth natural frequency of the bare beam (i.e. the beam without carrying a mass). Substituting Equations (2)-(4) into Equation (1) and multiplying $\phi_n(x)$ on both sides and integrating over the whole beam yields

$$\begin{aligned} -\omega_n^2 \int_0^L \bar{m}\phi_n(x)\phi_n(x)dx + \int_0^L EI\phi_n(x)\phi_n''''(x)dx = \\ \omega_n^2 \int_0^L \phi_n(x)[M\phi_n(x_0) + J\phi_n'(x_0)]\delta(x - x_0)dx \end{aligned} \tag{6}$$

Considering the orthogonal property of mode shapes

$$\int_0^L \phi_n(x)\phi_m(x)dx = \begin{cases} 0 & (n \neq m) \\ \psi_n & (n = m) \end{cases} \tag{7}$$

the first term on the left-hand side of Equation (6) can be simplified as

$$-\omega_n^2 \int_0^L \bar{m}\phi_n(x)\phi_n(x)dx = -\omega_n^2\bar{m}\psi_n \tag{8}$$

Considering Equation (5) and Equation (7), the second term on the left-hand side of Equation (6) can be written as

$$\int_0^L EI\phi_n(x)\phi_n''''(x)dx = \bar{m}\omega_{bn}^2\psi_n \tag{9}$$

Using Equation (7) and the sifting property of δ function, the right-hand side of Equation (6) can be expressed as

$$\omega_n^2 \int_0^L \phi_n(x)[M\phi_n(x_0) + J\phi_n'(x_0)]\delta(x - x_0)dx = \omega_n^2\phi_n(x_0)[M\phi_n(x_0) + J\phi_n'(x_0)] \tag{10}$$

Therefore, rearranging and rewriting Equation (6) gives

$$\bar{m}\omega_{bn}^2\psi_n - \omega_n^2\{\bar{m}\psi_n + \phi_n(x_0)[M\phi_n(x_0) + J\phi_n'(x_0)]\} = 0 \tag{13}$$

The nth natural frequency of the beam carrying a mass at x_0 can be written as

$$\omega_n^2(x_0) = \frac{\omega_{bn}^2}{1 + \dfrac{M\phi_n^2(x_0) + J\phi_n(x_0)\phi_n'(x_0)}{\bar{m}\psi_n}} \tag{14}$$

To verify the accuracy of Equation (14), the natural frequency of a simply supported steel beam carrying a mass with rotary inertia is calculated using the dynamic stiffness method (DSM). The dimension of the beam and the mass location are shown in Figure 2. For the beam material: ρ=7850kg/m^3, E=200GPa, and ν=0.28. For the mass: $\tau = M/M_{beam}$ and $\varphi = J/J_{beam}$ where J_{beam}=1.1307kg·m^2 is the rotary inertia about the central axis O' of the beam.

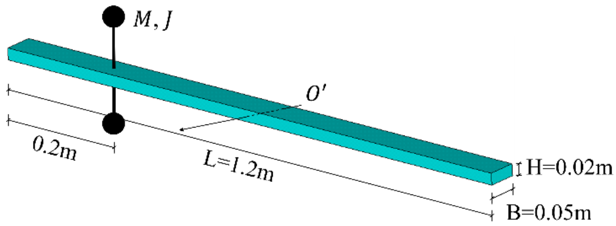
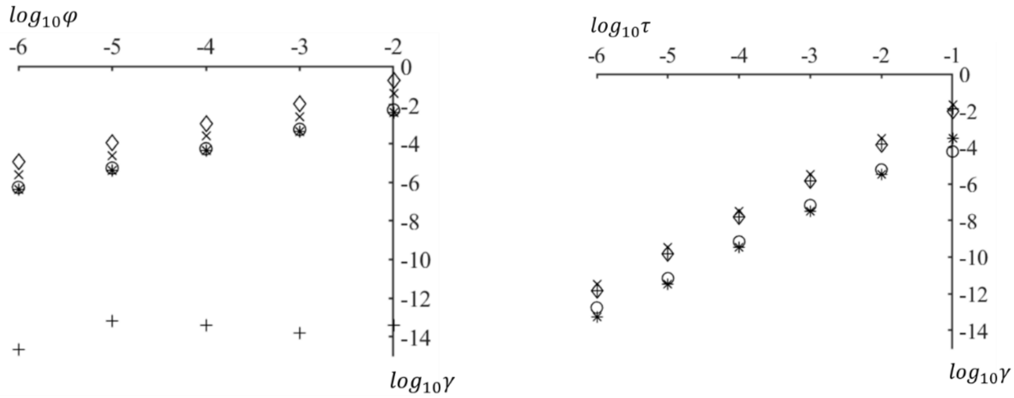


Figure 2. The geometry of the simply supported beam carrying a mass with rotary inertia.

When maintaining τ (or φ) equal to zero and adjusting φ (or τ), the relative frequency error is evaluated by the ratio between absolute frequency error and DSM frequency result, i.e.

$$\gamma = \frac{\left|\omega_{DSM} - \omega_{equation(14)}\right|}{\omega_{DSM}} \tag{15}$$

Figure 3 shows the variation of γ against φ (or τ) plotted in logarithm for the first five natural frequencies. The magnitude of γ is found to be very small when φ and τ are small, which means the assumption that the mode shape stays the same after attaching the mass to the beam does not cause significant error when φ and τ are small. In that case, Equation (14) gives very good natural frequency estimates. It is worth noting that in Figure 3(a), the magnitude of γ for ω_3 is consistently close to zero. This is because the mass is located at the extreme point of $\phi_3(x)$ featuring zero beam rotation. Hence the effect of rotary inertia is nullified.



(a) when τ=0 and $\varphi \neq 0$ (b) when φ=0 and $\tau \neq 0$

* ω_1 ○ ω_2 + ω_3 × ω_4 ◇ ω_5

Figure 3. The development of relative frequency error as φ (or τ) increases.

Similar equations for natural frequency estimation were found in [1-3], however, no explanation is given on the assumption for the approximation. The derivation in this summary addresses this issue. Equation (14) offers a straightforward way to estimate the natural frequency of a beam carrying a mass, especially for beams with standard boundary conditions of which the mode shape expressions are readily available in textbooks [4]. It is possible to generalise Equation (14) to consider the situation when multiple masses are distributed on the beam. In addition, Equation (14) also explicitly shows how M and J affect the natural frequency. The effect of a roving J has been discussed in a cracked beam scenario in [5]. From Equation (14), as J is engaged by multiplying $\phi_n(x_0)$ and $\phi_n'(x_0)$, the discontinuity in $\phi_n'(x_0)$ caused by a crack leads to a shift in ω_n, which can be used for crack detection.

References

[1] Y. Zhang, L. Wang, S.T. Lie, Z. Xiang, Damage detection in plates structures based on frequency shift surface curvature. J. Sound Vibr. 332 (25), 6665-6684, 2013.

[2] Y. Zhang and Z. Xiang, Frequency shift curve based damage detection method for beam structures. Computers, Materials, & Continua, 2011. 26(1): p. 19-35.

[3] X. Sun, Y. Mochida, I. Sinniah, The natural frequency estimation for a system composed of a beam and a spring-mass system, Noise and Vibration: Emerging Methods (NOVEM), Auckland, New Zealand, 2023.

[4] R.D. Blevins and R. Plunkett, Formulas for Natural Frequency and Mode Shape. Journal of Applied Mechanics, 1980. 47(2): p. 461-462.

[5] F. Cannizzaro, J. De Los Rios, S. Caddemi, I. Caliò, S. Ilanko, On the use of a roving body with rotary inertia to locate cracks in beams. J. Sound Vibr. 425, 275-300, 2018.