Differential-Stäckel matrices

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We show that additive separation of variables for linear homogeneous equations of all orders is characterized by differential-Stäckel matrices, generalizations of the classical Stäckel matrices used for multiplicative separation of (second-order) Schrödinger equations and additive separation of Hamilton–Jacobi equations. We work out the principal properties of these matrices and demonstrate that even for second-order Laplace equations additive separation may occur when multiplicative separation does not.

I. INTRODUCTION

Our motivation for this study of additive separation of variables for linear differential equations was the following example in Ref. 1:

$$(x_1+x_2)(\partial_{11}u+\partial_{22}u)-2(\partial_1 u+\partial_2 u)=E.$$

This equation admits a five-parameter separable solution in the coordinates x_1, x_2:

$$u=(\alpha x_1^3+\beta x_1^2+\gamma x_1-\tfrac{1}{2}Ex_1)+(-\alpha x_2^3+\beta x_2^2-\gamma x_2+\delta).$$

The mechanism of separation was puzzling to us until we realized that the appropriate separation equations are

$$\begin{aligned}
&\partial_1 u + E/2 - \gamma - 2\beta x_1 - 3\alpha x_1^2 = 0,\\
&\partial_{11}u - 2\beta - 6\alpha x_1 = 0,\\
&\partial_2 u + \gamma - 2\beta x_2 + 3\alpha x_2^2 = 0,\\
&\partial_{22}u - 2\beta + 6\alpha x_2 = 0.
\end{aligned}$$

The associated "Stäckel matrix" responsible for the separation is

$$\begin{bmatrix} \frac{1}{2} & -1 & -2x_1 & -3x_1^2\\ 0 & 0 & -2 & -6x_1\\ 0 & 1 & -2x_2 & 3x_2^2\\ 0 & 0 & -2 & 6x_2 \end{bmatrix}.$$

This is not a true Stäckel matrix since more than one row depends on a given variable x_I (Refs. 2 and 3). Moreover, the second and fourth rows are the derivatives of the first and third rows, respectively. It is a nontrivial example of a differential-Stäckel matrix.

In this paper we show that the above example is not isolated. All additive separation of nth-order linear differential equations $L=E$ or $L=0$ is associated with differential-Stäckel matrices. In Sec. II we derive, in the form of a coupled system of nonlinear partial differential equations, necessary and sufficient conditions that a linear differential equation admits additive separation in a given coordinate system. In Sec. III we develop the principal properties of the matrices inverse to differential-Stäckel matrices, and in Sec. IV we find all solutions of the separability conditions and show that they correspond to differential-Stäckel matrices. Our method is an extension of Eisenhart's study of true Stäckel matrices.[3] In Sec. V we comment on the relation between multiplicative separation and additive separation for Laplace equations on Riemannian manifolds, and we give an example to show that additive separation may occur for a Laplace equation in a given coordinate system even when multiplicative separation is absent.

All functions appearing in this paper are assumed to be locally real analytic.

II. ADDITIVE SEPARABILITY FOR LINEAR DIFFERENTIAL EQUATIONS

In Ref. 1 the authors introduced a general definition of additive separation of variables for a partial differential equation

$$H(x_I,u,u_I,u_{IJ},\ldots)=E, \tag{2.1}$$

in the coordinates $x_1,\ldots,x_N$. Here u is the dependent variable, $u_I=\partial_{x_I}u$, $u_{IJ}=\partial_{x_I}\partial_{x_J}u$, etc., and E is a parameter. A separable solution of (2.1) is a solution of the form $u=\Sigma_{J=1}^N S^{(J)}(x_J,E)$. We briefly review this definition (a generalization of that of Levi-Civita[4] and its simple consequences. (See Ref. 5 for a discussion of other definitions of separability.)

For convenience we suppose H is a polynomial in the derivatives $u_I,u_{IJ},\ldots$. Furthermore, there is no loss of generality in setting all mixed partial derivatives identically equal to zero (since $u_{IJ}=0$ for $I\neq J$ if u is a separable solution) and writing (2.1) in the form

$$H(x_I,u,u_I,u_{II},\ldots)=E. \tag{2.2}$$

We introduce the new notation $u_{I,1}=u_I$, $u_{I,i+1}=\partial_{x_I}u_{I,i}$, $i=1,2,\ldots$, and define n_I to be the largest number l such that $\partial u_{I,l}H=H_{u_{I,l}}\not\equiv 0$. To avoid discussion of degenerate cases we require $n_I>0$ for $I=1,\ldots,N$.

Let the truncated differentiation operator $\hat{D}_I$ be defined by

$$\hat{D}_I=\partial_{x_I}+u_{I,1}\,\partial_u+\cdots+u_{I,n_I}\,\partial_{u_{i,n_I-1}}.$$

In Ref. 1 we showed that every separable solution u of (2.2) satisfies the integrability conditions

$$\begin{aligned}
&H_{u_{i,n_I}}H_{u_{J,n_J}}(\hat{D}_I\hat{D}_J H)+H_{u_{I,n_I}u_{J,n_J}}(\hat{D}_I H)(\hat{D}_J H)\\
&\quad-H_{u_{J,n_J}}(\hat{D}_I H)(\hat{D}_J H_{u_{I,n_I}})-H_{u_{I,n_I}}(\hat{D}_J H)(\hat{D}_I H_{u_{J,n_J}})\\
&=0,\quad 1\leqslant I<J\leqslant N.
\end{aligned} \tag{2.3}$$

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If (2.3) is an identity in the dependent variables $u, u_{K,k}$ then we say that $\{x_I\}$ is a regular separable coordinate system. In this case the separable solutions involve $\Sigma_{J=1}^N n_J + 1$ independent parameters; that is, at a fixed point $\mathbf{x}^0$ the separable solutions are uniquely determined by prescribing $u(\mathbf{x}^0)$ and the $\Sigma_{J=1}^N n_J$ derivatives $u_{I,i}(\mathbf{x}^0)$, $1 \leqslant I \leqslant N$, $1 \leqslant i \leqslant n_I$. If the integrability conditions (2.3) do not hold identically then the separation is nonregular; separable solutions may exist but they will involve (strictly) fewer parameters than the regular case. In the following when we speak of variable separation we mean regular separation. [Note that multiplicative separation can easily be treated by the preceding definition since $v = \Pi_{J=1}^N T^{(J)}(x_J)$ is multiplicatively separable if and only if $u = \ln v$ is additively separable.]

For Laplacelike equations

$$H(x_I, u, u_I, u_{II}, \ldots) = 0, \tag{2.4}$$

there is a minor modification of the integrability conditions. Denoting by F_{IJ} the left-hand side of Eqs. (2.3) we can state the integrability conditions for (2.4) in the form

$$F_{IJ} = P_{I,J} H, \quad 1 \leqslant I < J < N, \tag{2.5}$$

where $P_{I,J}(x_K, u, u_{K,k})$ are polynomials in $u_{K,k}$. If (2.5) is satisfied identically in the dependent variables $u, u_{J,j}$ we say that $\{x_K\}$ is a regular separable coordinate system. In this case the separable solutions depend on $\Sigma_{J=1}^N n_J$ independent parameters. For nonregular separation the separable solutions depend on fewer parameters.

Now we will apply these criteria to determine additive separability conditions for the linear equations

$$L = E \tag{2.6}$$

and

$$L = 0, \tag{2.7}$$

where

$$L = \sum_{J=1}^{N} \sum_{j=1}^{n_J} H_{(J,j)}(x) u_{J,j}, \quad H_{(J,n_J)} \neq 0. \tag{2.8}$$

Introducing the abbreviation $H_J \equiv H_{(J,n_J)}$ we can write the integrability conditions (2.3) for $L = E$ in the form

$$\hat{D}_I \hat{D}_J L - \hat{D}_I L\, \partial_J \ln H_I - \hat{D}_J L\, \partial_I \ln H_J = 0, \quad I \neq J, \tag{2.9}$$

where $\partial_I = \partial_{x_I}$. Equating to zero the coefficients of the derivatives $u_{J,j}$ on the left-hand side of (2.9) we obtain the following necessary and sufficient conditions for regular separation:

$$\partial_{IJ} H_{(P,p)} - \partial_I H_{(P,p)}\, \partial_J \ln H_I - \partial_J H_{(P,p)}\, \partial_I \ln H_J = 0,$$
$$P \neq I, J, \quad p = 1, \ldots, n_P, \tag{2.10a}$$

$$\partial_{IJ} H_{(J,j)} - \partial_I H_{(J,j)}\, \partial_J \ln H_I - \partial_J H_{(J,j)}\, \partial_I \ln H_J$$
$$= H_{(J,j-1)}\, \partial_I \ln H_J - \partial_I H_{(J,j-1)}, \tag{2.10b}$$
$$j = 1, \ldots, n_J.$$

Here $I \neq J$, $H_{(J,0)} \equiv 0$. In terms of the linear operators

$$A_{IJ} = \partial_{IJ} - \partial_J \ln H_I \partial_I - \partial_I \ln H_J\, \partial_J,$$
$$B_{IJ} = -\partial_I + \partial_I \ln H_J, \quad I \neq J, \tag{2.11}$$

these conditions can be written as

$$A_{IJ} H_{(P,p)} = 0, \quad P \neq I, J,$$
$$A_{IJ} H_{(J,j)} = B_{IJ} H_{(J,j-1)}, H_{(J,0)} = 0, \quad I \neq J. \tag{2.12}$$

[The possibility that L, Eq. (2.8), has an additive term of the form $H(x)u$ can be treated as a special case of our considerations. Formally Eqs. (2.10) are still the separation conditions for $L = E$ where now $H_{(J,0)} = H$, $H_{(J,-1)} = 0$, $J = 1, \ldots, N$, and the index j takes the values $0, 1, \ldots, n_J$.]

Similarly, the integrability conditions (2.5) for the homogeneous equation $L = 0$ take the form

$$A_{IJ} H_{(P,p)} = Q_{IJ}(x) H_{(P,p)}, \quad P \neq I, J,$$
$$A_{IJ} H_{(J,j)} = B_{IJ} H_{(J,j-1)} + Q_{IJ}(x) H_{(J,j)}, \tag{2.13}$$
$$H_{(J,0)} = 0, \quad I \neq J,$$

where Q_{IJ} is a function of the independent variables x_K alone.

The two sets of integrability conditions are closely related.

Lemma 1: If the functions $\{H_{(I,i)}\}$ satisfy (2.12) and $R(x)$ is nonzero then the functions $\{H'_{(I,i)} = R H_{(I,i)}\}$ satisfy (2.13). Indeed

$$-Q_{IJ} = 2\, \partial_I \ln R \partial_J \ln R$$
$$+ \partial_I \ln R\, \partial_J \ln H_I + \partial_J \ln R \partial_I \ln H_J.$$

Suppose the functions $\{H_{(I,i)}\}$, $I = 1, \ldots, N$, $i = 1, \ldots, n_I$ are not all zero and set $H_{I,0} = 0$. Then there must exist some $H_{(K,k)} \neq 0$ such that $H_{(K,k-1)} = 0$. Let $H'_{(I,i)} = H_{(I,i)}/H_{(K,k)}$.

Lemma 2: The functions $\{H_{(I,i)}\}$ satisfy conditions (2.13) if and only if the functions $\{H'_{(I,i)}\}$ satisfy (2.12), i.e.,

$$\partial_{IJ} H'_{(P,p)} - \partial_J \ln H'_I \partial_I H'_{(P,p)} - \partial_I \ln H'_J \partial_J H'_{(P,p)} = 0,$$
$$P \neq I, J, \tag{2.14}$$

$$\partial_{IJ} H'_{(J,j)} - \partial_J \ln H'_I \partial_I H'_{(J,j)} - \partial_I \ln H'_J\, \partial_J H'_{(J,j)}$$
$$= H'_{(J,j-1)}\, \partial_I \ln H'_J - \partial_I H'_{(J,j-1)},$$

for $I \neq J$.

It follows from these lemmas that all sets of functions $\{H_{(I,i)}\}$ satisfying conditions (2.13) are of the form $H_{(I,i)} = R H'_{(I,i)}$, where the $\{H'_{(I,i)}\}$ satisfy conditions (2.12). In the next section we will show how to find all solutions of Eqs. (2.12).

III. D-STÄCKEL MATRICES

Consider a coordinate set $x_1, \ldots, x_N$ and let $n_1, \ldots, n_N$ be positive integers with $n = \Sigma_{I=1}^n n_I$. Let $S = (S_{(I,i),l}(x_I))$ be an $n \times n$ matrix with the properties following.

(1) $S_{(I,i),l}(x_I) = \dfrac{d^{\,i-1}}{dx_I^{\,i-1}} S_{(I,1),l}(x_I)$,

$$i = 1, 2, \ldots, n_I. \tag{3.1}$$

[Here, the rows of S are designated by the index (I,i), where $I = 1, \ldots, N$, $i = 1, \ldots, n_I$. The columns of S are designated by the index $l = 1, 2, \ldots, n$. Thus row (I,i) depends only on the variable x_I and is the $i-1$ derivative of row $(x,1)$.]

(2) $\det S \neq 0$.

(3) $T^{1,(J,j)} \neq 0$, $J = 1, \ldots, N$, $h = 1, \ldots, n_J$, where $T = S^{-1}$, i.e.,

$$\sum_{l=1}^{n} S_{(I,i),l}(x_I)T^{l,(J,j)} = \delta_{(I,i)}^{(J,j)} . \qquad (3.2)$$

We say that a matrix S satisfying properties (1)–(3) is a differential-Stäckel matrix (D-Stäckel matrix). If $n_1 = \cdots = n_N = 1$ then S is simply the usual Stäckel matrix.[2,3] In order to obtain results about D-Stäckel matrices that are useful for separation of variables we need to characterize the inverse matrix T. For this we generalize Eisenhart's study of Stäckel matrices.[3,6]

Differentiating (3.2) with respect to x_I, we obtain

$$\sum_l S_{(I,i+1),l}(x_I)T^{l,(J,j)} + \sum_l S_{(I,i),l}\, \partial_{x_I} T^{l,(J,j)} = 0, \qquad (3.3)$$

where we adopt the convention $S_{(I,n_I+1),l} = 0$. Since S is nonsingular it follows that

$$\partial_{x_I} T^{l,(J,j)} = f_I^{(J,j)}\, T^{l,(I,n_I)} - T^{l,(J,j-1)}\delta_I^J, \qquad (3.4)$$

where $f_I^{(J,j)}$ is a function and we adopt the convention $T^{l,(J,0)} = 0$.

Now set

$$T^{l,(J,j)} = \rho_{(J,j)}^l H_{(J,j)}, \quad \rho_{(J,j)}^1 = 1. \qquad (3.5)$$

In particular, $T^{1,(J,j)} = H_{(J,j)}$. We will characterize T in terms of the "roots" $\rho_{(J,j)}^l$ and the $H_{(J,j)}$. Substituting (3.5) in (3.4) we obtain

$$\partial_I\rho_{(J,j)}^l H_{(J,j)} + \rho_{(J,j)}^l\, \partial_I H_{(J,j)} = f_I^{(J,j)}\rho_{(I,n_I)}^l H_{(I,n_I)} - \rho_{(J,j-1)}^l H_{(J,j-1)}\delta_I^J, \qquad (3.6)$$

where $\rho_{(J,0)}^l = H_{(J,0)} = 0$. For $l = 1$, Eq. (3.6) reduces to

$$\partial_I H_{(J,j)} = f_I^{(J,j)} H_{(I,n_I)} - H_{(J,j-1)}\delta_I^J, \qquad (3.7)$$

in view of (3.5). Solving this expression for $f_I^{(J,j)}H_{(I,n_I)}$ and substituting into (3.6) we obtain the desired characterization

$$\partial_I\rho_{(J,j)}^l = (\rho_{(I,n_I)}^l - \rho_{(J,j)}^l)\partial_I \ln H_{(J,j)} + (\rho_{(I,n_I)}^l - \rho_{(J,j-1)}^l)\frac{H_{(J,j-1)}}{H_{(J,j)}}\,\delta_I^J, \qquad (3.8)$$

$I, J = 1,...,N,\ h = 1,...,n_J$.

At this point we have shown that if S is a D-Stäckel matrix then the system of equations

$$\partial_I\, \rho_{(J,j)} = (\rho_{(I,n_I)} - \rho_{(J,j)})\partial_I \ln H_{(J,j)} + (\rho_{(I,n_I)} - \rho_{(J,j-1)})\frac{H_{(J,j-1)}}{H_{(J,j)}}\,\delta_I^J, \qquad (3.9)$$

$I,J = 1,...,N,\ h = 1,...,n_J$, where $H_{(J,j)} = T^{1,(J,j)}$ admits a full linearly independent set of n vector-valued solutions $\{\rho_{(J,j)}^l\}, l = 1,...,n$.

Conversely, suppose we are given a set of n nonzero functions $\{H_{(J,j)}\}$ such that the system (3.9) admits a full linearly independent set of n vector-valued solutions $\{\rho_{(J,j)}^l\}$. Since $\rho_{(J,j)} \equiv 1$, all J, j, is a solution, without loss of generality we can include it in our basis set and assume $\rho_{(J,j)}^1 = 1$. It follows that the $n\times n$ matrix T defined by (3.5) is invertible. Let $S = T^{-1}$, i.e.,

$$\sum_{l=1}^{n} S_{(I,i),l}T^{l,(J,j)} = \delta_{(I,i)}^{(J,j)}. \qquad (3.10)$$

It follows from (3.9) and (3.5) that (3.4) holds with

$$f_I^{(J,j)} = H_{(I,n_I)}^{-1}(\partial_I H_{(J,j)} + H_{(J,j-1)}\delta_I^J).$$

Differentiating both sides of (3.10) with respect to x_K and using (3.4), we find

$$\sum_{l=1}^{n} \partial_K S_{(I,i),l}T^{l,(J,j)} = \delta_K^J\delta_{(I,i)}^{(J,j-1)} - f_K^{(J,j)}\delta_{(I,i)}^{(K,n_K)}. \qquad (3.11)$$

It follows that $\partial_K S_{(I,i),l} = 0$ if $K \neq I$ and $\partial_I S_{(I,i),l} = S_{(I,i+1),l}$ for $i = 1,...,n_I - 1$. Thus S is a D-Stäckel matrix.

Theorem 1: Let $\{H_{(J,j)}\}$ $(J = 1,...,N,\ j = 1,...n_J,\ \Sigma_J n_J = n)$ be a set of n nonzero functions of the N variables x_I. There exists an $n\times n$ D-Stäckel matrix $S = (S_{(I,i),l}(x_I))$ with inverse $T = (T^{l,(J,j)})$ such that $H_{(J,j)} = T^{1,(J,j)}$ if and only if $\{H_{J,j)}\}$ satisfies Eqs. (2.10a) and (2.10b).

Proof: It is straightforward to verify that (2.10a) and (2.10b) are simply the integrability conditions $\partial_K(\partial_I\,\rho_{(J,j)}) = \partial_I(\partial_K\,\rho_{(J,j)}), K \neq I$, for the system (3.9). Thus if $\{H_{(J,j)}\}$ satisfies the integrability conditions then (3.9) has n independent vector-valued solutions and we can construct a D-Stäckel matrix S such that $H_{(J,j)} = T^{1,(J,j)}$.

Conversely, if $H_{(J,j)} = T^{1,(J,j)}$ and $T^{-1} = S$ for some D-Stäckel matrix S then the system (3.9) admits n independent vector-valued solutions and the integrability conditions (2.10) must be satisfied.

We now have a partial answer to the problems posed in Sec. II. Consider the equation $L = E$ in N independent variables x_I, where

$$L = \sum_{J=1}^{N}\sum_{j=1}^{n_J} H_{(J,j)}(x)u_{J,j},$$

and suppose each of the $H_{(J,j)}$ is nonzero. This equation admits (regular) additively separable solutions provided the conditions (2.10) are satisfied. These conditions imply the existence of a D-Stäckel matrix S such that $H_{(J,j)} = T^{1,(J,j)}$. The separation equations are evident

$$u_{J,j} + \sum_{l=1}^{n} S_{(J,j),l}(x_J)\lambda_l = 0, \qquad (3.12)$$

$$1 \leqslant J \leqslant N, \quad 1 \leqslant j \leqslant n_J, \quad \lambda_1 = -E.$$

Here there are n separation parameters λ_l. The separable solutions u are obtained by integrating the N first-order ordinary differential equations

$$u_{J,1} + \sum_{l=1}^{n} S_{(J,1),l}(x_J)\lambda_l = 0. \qquad (3.13)$$

The remaining $n - N$ equations are redundant since they are obtained by differentiating the basic set (3.13). The number of parameters in the solution u is $n + \Sigma_J n_J + 1$, in agreement with the prediction in Sec. II. Multiplying the separation equation (3.12) for $u_{J,j}$ by $T^{1,(J,j)}$ and summing over the index (J,j) we once again obtain $L = E$ for $E = -\lambda_1$.

The treatment for the equation $L = 0$ is similar. Suppose

$$L = \sum_{J=1}^{N}\sum_{j=1}^{n_J} H'_{(J,j)}(x)u_{J,j},$$

where none of the $H'_{(J,j)}$ is zero and suppose these functions satisfy the separability conditions (2.14). Then there is a nonzero function $R(x)$ such that $H'_{(J,j)} = RH_{(J,j)}$, where the $H_{(J,j)}$ satisfy conditions (2.10) and, thus, determine a D-Stäckel matrix S. The separation equations are (3.12) with $\lambda_1 = 0$. There are $n - 1$ separation parameters and a separated solution contains $n = \Sigma_J n_J$ parameters. Multiplying the separation equation (3.12) for $u_{J,j}$ by $RT^{1,(J,j)}$ and summing over the index (J,j), we rederive $L = 0$, since $\lambda_1 = 0$.

IV. ANALYSIS OF THE SEPARATION EQUATIONS

We do not as yet have a complete solution of the integrability conditions characterizing regular separation for the linear equation $L = E$

$$A_{IJ} H_{(P,p)} = 0, \quad P \neq I,J,$$
$$A_{IJ} H_{(J,j)} = B_{IJ} H_{(J,j-1)}, \quad H_{(J,0)} = 0, \quad I \neq J, \tag{4.1}$$

where

$$A_{IJ} = \partial_{IJ} - \partial_J \ln H_I \partial_I - \partial_I \ln H_J \partial_J,$$
$$B_{IJ} = -\partial_I + \partial_I \ln H_J, \quad H_J = H_{(J,n_J)} \neq 0, \tag{4.2}$$
$$1 \leqslant J \leqslant N, \quad 1 \leqslant j \leqslant n_J, \quad \sum_J n_J = n.$$

In order that Theorem 1 can be applied to obtain a D-Stäckel matrix we must have all $H_{(J,j)} \neq 0$. However, we are assuming only that $H_J \neq 0$. Furthermore, it is easy to construct examples of separable systems where at least one of the $H_{(J,j)}$ vanishes.

A more detailed analysis of the structure of Eqs. (4.1) will resolve the difficulty. Suppose we are given N nonzero functions H_J satisfying $A_{IJ} H_P = 0$ for $P \neq I,J, I \neq J$. Our task will be to construct a finite set of functions $H_{(J,j)}$ with $H_{(J,n_J)} = H_J$ such that Eqs. (4.1) are satisfied. (We do not require that the $H_{(J,j)}$ are all nonzero.) Initially we will not know the values of the integers n_J.

The construction process is based on the second equation of (4.1), which we can write in the form

$$\partial_I [H_{(K,k-1)}/H_K] = (\partial_K \ln H_I\, \partial_I H_{(K,k)} + \partial_I \ln H_K \partial_K H_{(K,k)} - \partial_{IK} H_{(K,k)})/H_K, \quad I \neq K. \tag{4.3}$$

If $H_{(K,k)}$ is known then we can construct $H_{(K,k-1)}$ from (4.3) by quadrature.

Lemma 5: Suppose the N nonzero functions H_P satisfy $A_{IJ} H_P = 0$ for $P \neq I, J, I \neq J$ and suppose the function $H_{(K,k)}$ (fixed K,k) satisfies $A_{IJ} H_{(K,k)} = 0, K \neq I,J, I \neq J$. Then the $N-1$ equations (4.3) are compatible and have the general solution

$$H_{(K,k-1)} = \widetilde{H}_{(K,k-1)} + f^{(k-1)}(x_K) H_K, \tag{4.4}$$

where $\widetilde{H}_{(K,k-1)}$ is a particular solution and $f^{(k-1)}$ is an arbitrary function of x_K. The solution satisfies

$$A_{IJ} H_{(K,k-1)} = 0, \quad K \neq I,J, \quad I \neq J. \tag{4.5}$$

Proof: The compatibility requirement $\partial_J(\partial_I(H_{(K,k-1)}/H_K)) = \partial_I(\partial_J(H_{(K,k-1)}/H_K)), \; I,J \neq K$ and (4.5) are straightforward consequences of (4.3) and the conditions $A_{IJ} H_P = 0, A_{IJ} H_{(K,k)} = 0$.

It follows from Lemma 5 that for each K we can always construct functions $H_{(K,k-1)}$ through a step-by-step procedure using the second of Eqs. (4.1), such that the first of Eqs. (4.1) is automatically satisfied. At each step the solution $H_{(K,k-1)}$ is arbitrary up to the additive term $f^{(k-1)}(x_K) H_K$ and we simply choose one of these solutions. Thus we generate an infinite sequence $\{H_{(K,k)} = H_K^{(l)}\}$, $l = 0,1,2,\ldots$, where $n_{K-l} = k$ (but n_K is unknown)

$$A_{IK} H_K^{(l)} = B_{IK} H_K^{(l+1)}, \quad I \neq K, \quad H_K = H_K^{(0)}. \tag{4.6}$$

The following properties of the operators A_{IK}, B_{IK} will prove useful:

$$B_{IK} F(x) = 0, \text{ for all } I \neq K$$
$$\Leftrightarrow F(x) = f(x_K) H_K, \tag{4.7}$$
$$A_{IK}(f(x_K) H_K^{(l)}) = B_{IK}(f H_K^{(l+1)} - f' H_K^{(l)}), \tag{4.8}$$

where $f' = \partial_K f$.

Suppose there is a smallest finite positive integer m_K for which functions $f_{(i)}(x_K)$ exist such that

$$H_K^{(m_K)} = \sum_{i=0}^{m_K-1} f_{(i)}(x_K) H_K^{(i)}. \tag{4.9}$$

Lemma 6: Each $H_K^{(m_K+s)}$, $s = 0,1,2,\ldots$, is a linear condition of the finite set $\{H_K^{(l)}: l = 0,\ldots,m_K - 1\}$ with coefficients that are functions of x_K.

Proof: The proof is by induction on s. The statement is clearly true for $s = 0$. We assume it holds for $s = t$

$$H_K^{(m_K+t)} = \sum_{i=0}^{m_K-1} g_{(i)}(x_K) H_K^{(i)}.$$

Now

$$B_{IK} H_K^{(m_K+t-1)} = A_{IK} H_K^{(m_K+t)} = A_{IK}\left(\sum_0^{m_K-1} g_{(i)} H_K^{(i)}\right)$$
$$= B_{IK}\left(\sum_0^{m_K-1} g_{(i)} H_K^{(i+1)} - \sum_0^{m_K-1} g'_{(i)} H_K^{(i)}\right).$$

Hence, by (4.7) there is a function $g(x_K)$ such that

$$H_K^{(m_K+t+1)} = \sum_{i=0}^{m_K-1} h_{(i)}(x_K) H_K^{(i)},$$
$$h_{(i)}(x_K) = \begin{cases} g_{(i-1)} - g'_{(i)}, & 1 \leqslant i \leqslant m_K - 1, \\ g - g'_{(0)}, & i = 0. \end{cases} \qquad \text{Q.E.D.}$$

Let $\{\mathscr{H}_K^{(l)}\}$, $\{\mathscr{k}_K^{(l)}\}$, $l = 0,1,2,\ldots$, be two sequences constructed by the procedure (4.6).

Lemma 7: There is a sequence of functions $g_1(x_K), g_2(x_K),\ldots$, and expressions $L_{i,j}(g_1,g_2,\ldots,g_{i-j-1})$ with $L_{i0} = 0$, $L_{i,i-1} = 0$, and $L_{i+1,j} = L_{i,j-1} + g'_{i-j} - L'_{i,j}$ such that

$$\mathscr{H}_K^{(i)} = \mathscr{k}_K^{(i)} + \sum_{j=0}^{i-1} (g_{i-j}(x_K) - L_{i,j}(x_K)) \mathscr{k}_K^{(j)},$$
$$i = 0,1,2,\ldots. \tag{4.10}$$

Any such sequence $\{g_l(x_K)\}$ together with $\{\mathscr{k}_K^{(l)}\}$ determines a new sequence of solutions $\{\mathscr{H}_K^{(i)}\}$ of (4.6). The induction proof of this result is similar to that of the preceding lemma.

Now let $\{H_K^{(l)}\}$ be the solution sequence treated in Lemma 6 and consider the relation (4.9). Set $\hbar_K^{(i)} = H_K^{(i)}$ in (4.10) and choose $g_1,\dots,g_{m_K-1}$ recursively such that

$$-f_{(j)} = g_{m_K-j} - L_{m_K,j}, \quad j = 0,1,\dots,m_K-1.$$

Then $\mathscr{H}_K^{(m_K)} = 0$. We see that there is a solution sequence $\{\mathscr{H}_K^{(l)}\}$ with $\mathscr{H}_K^{(0)},\dots,\mathscr{H}_K^{(m_K-1)}$ nonzero and all further terms zero. According to Lemma 7 all other solution sequences are linear combinations of these m_K nonzero terms.

Lemma 8: The integer m_K, if it exists, is unique.

In general, there is no finite integer m_K for which (4.9) holds. [An example is $N=2$, $H_1 = 1$, $H_2 = \exp(x_1x_2)$. Here $m_1 = 1$, but m_2 does not exist.] However, if the $H_{(J,j)}$ satisfy equations (4.1), i.e., if they correspond to a regular separable system for the equation $L=E$ then the integers m_J always exist and $1 \leqslant m_J \leqslant n_J$. Thus there is a set of $\Sigma_{J=1}^N m_J$ functions $\{\hbar_{(J,j)}\}$, $1 \leqslant j \leqslant m_J$ satisfying (4.1) such that $H_J = \hbar_{(J,m_J)}$ and each $\hbar_{(J,j)}$ is nonzero. Using Lemma 7 we can express the equation $L=E$ in terms of the new functions $\hbar_{(J,j)}$.

Lemma 9:

$$L = \sum_{K=1}^{N}\sum_{k=1}^{n_K} H_{(K,k)}u_{K,k} = \sum_{K=1}^{N}\sum_{k=1}^{m_K} \hbar_{(K,k)}U_{K,k},$$

where

$$U_{K,k}(u_{k,l},x_K) = u_{K,n_K-m_K+k} + \sum_{s=1}^{n_K-m_K+k-1}(g_{n_K-m_K-s+k}(x_K) - L_{n_K-s,m_K-k}(x_K))u_{K,s}.$$

In particular,

$$\partial_K U_{K,k} = U_{K,k+1}, \quad 1 \leqslant k \leqslant m_K - 1.$$

It follows from this result and Theorem 1 that when $L=E$ is separable then there exists a set of $m = \Sigma_{J=1}^N m_J$ nonzero functions $\hbar_{(J,j)}$ and an associated $m\times m$ D-Stäckel matrix S such that $\hbar_{(J,j)} = T^{1,(J,j)}$, where $T = S^{-1}$ and the separation equations for $L=E$ take the form

$$U_{K,k} + \sum_{l=1}^{m} S_{(K,k),l}(x_K)\lambda_l = 0, \quad K=1,\dots,N, \tag{4.11}$$
$$1 \leqslant K \leqslant N, \quad 1 \leqslant k \leqslant m_K \leqslant n_K, \quad \lambda_1 = -E.$$

There are m separation parameters λ_l. The separable solutions u are determined by solving the N ordinary differential equations

$$U_{K,1} + \sum_{l=1}^{m} S_{(K,1),l}(x_K)\lambda_l = 0. \tag{4.12}$$

The remaining $m-N$ equations (4.11) are redundant, since they are obtained by differentiating the basic set (4.12). The highest derivative term in $U_{K,1}$ is u_{K,n_K-m_K+1} so each equation (4.12) is of order $n_K - m_K + 1$. The number of parameters in the solution u is $m + \Sigma_K(n_K - m_K) + 1 = n+1$. We now have the complete solution of the separation of Eqs. (2.10).

V. SEPARATION OF LAPLACE EQUATIONS

Suppose Δ_N is the Laplace–Beltrami operator on a local pseudo-Riemannian manifold V^N. In local orthogonal coordinates x_I we have

$$\Delta_N = \frac{1}{h}\sum_{I=1}^{N}\partial_I(hH_I\,\partial_I), \tag{5.1}$$

where

$$ds^2 = \sum_{I=1}^{N} H_I^{-1}\,dx_I^2, \quad h^2 = \prod_I H_I^{-1}.$$

It is of interest to determine the relationships between the well-developed theory of multiplicative R separation for the Laplace equation $\Delta_N u = 0$ (Refs. 7–9) and additive separation. Recall that multiplicative R separation in the orthogonal coordinates x_I leads to solutions for the Laplace equation of the form

$$u = e^{R(x)}\prod_{I=1}^{N} u^{(I)}(x_I), \tag{5.2}$$

where the fixed function R is independent of the separation parameters. Similarly we can introduce additive R separation

$$u = e^{R(x)}\Big(\sum_{I=1}^{N} u^{(I)}(x_I)\Big). \tag{5.3}$$

The following is a straightforward consequence of the principal results of this paper.

Theorem 2: If the Laplace equation $\Delta_N U = 0$ is multiplicatively R separable in the orthogonal coordinates x_I then it is additively R separable in these same coordinates if and only if $e^{-R}\Delta_N e^R = cH_I$, $I=1,\dots,N$, where c is a constant.

In each of these cases a true Stäckel matrix determines the separation; no nontrivial D-Stäckel matrices appear. Note that true multiplicative separation ($R\equiv 1$) always leads to additive separation.

On the other hand, Laplace equations may admit additive separation in an orthogonal coordinate system for which no multiplicative R separation is possible. For example, consider the three-dimensional manifold with metric coefficients

$$H_1 = H_2 = (x_1+x_2)^5, \quad H_3 = (x_1+x_2)^4. \tag{5.4}$$

Then

$$\Delta_3 = (x_1+x_2)^4[(x_1+x_2)(\partial_{11}+\partial_{22}) - 2(\partial_1+\partial_2) + \partial_{33}], \tag{5.5}$$

and since (5.4) is not conformal to a Stäckel form metric, no multiplicative R separation is possible. However, since

$$(x_1+x_4)^{-4}\Delta_3 u = (x_1+x_2)(u_{1,2}+u_{2,2}) - 2(u_{1,1}+u_{2,1}) + u_{3,2} = 0, \tag{5.6}$$

we have

$$H_1' = H_2' = x_1 + x_2, \quad H_3' = 1,$$

$$H'_{(1,1)} = H'_{(2,1)} = -2,$$

which satisfies Eqs. (2.10) [or (2.14)] with $n_1 = n_2 = 2$, $n_3 = 1$. Thus the Laplace equation admits additive separation in these coordinates corresponding to a 5×5 D-Stäckel matrix.

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[1]E. G. Kalnins and W. Miller, "Intrinsic characterization of variable separation for the partial differential equations of mechanics," in *Proceedings of IUTAM-ISIMM Symposium on Modern Development in Analytical Mechanics*, Turin, 1982 (Atti della Academia delle Scienze di Torino, Turin, 1983), Supplement to Vol. 117, pp. 511–533.
[2]P. Stäckel, *Über die Integration der Hamilton-Jacobischen Differentialgleichung Mittels Separation der Variabelen* (Habitilatationsschrift, Halle, 1891).
[3]L. P. Eisenhart, Ann. Math. **35**, 284 (1934).
[4]T. Levi-Civita, Math. Ann. **59**, 383 (1904).
[5]T. W. Koornwinder, "A precise definition of separation of variables," in *Lecture Notes in Mathematics*, Vol. 810 (Springer, Berlin, 1980).
[6]L. P. Eisenhart, *Riemannian Geometry* (Princeton U. P., Princeton, 1949), Appendix 13.
[7]P. Moon and D. E. Spencer, J. Franklin Inst. **253**, 585 (1952).
[8]V. N. Shapovalov, Differential Equations **16**, 1864 (1980); **10**, 1212 (1981).
[9]E. G. Kalnins and W. Miller, Trans. Am. Math. Soc. **244**, 241 (1978).