Complete sets of functions for perturbations of Robertson–Walker cosmologies and spin 1 equations in Robertson–Walker-type space-times

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Crucial to a knowledge of the perturbations of Robertson–Walker cosmological models are complete sets of functions with which to expand such perturbations. For the open Robertson–Walker cosmology an answer to this question is given. In addition some observations concerning explicit solution by separation of variables of wave equations for spin 1 in a Riemannian space having an infinitesimal line element of which the Robertson–Walker models are a special case are made.

I. INTRODUCTION

The original investigations of Lifshitz[1] and Lifshitz and Khalatnikov[2] into the gravitational stability of the Robertson–Walker (RW) isotropic cosmological models[3] demonstrated the utility of scalar, vector, and tensor harmonics in giving a complete description of small perturbations. In particular these authors[1,2] showed that in the synchronous gauge all perturbations involving pressure, density, velocity, and metric fluctuations can be obtained once a complete set of such functions is found for S_3 (three-dimensional sphere), E_3 (Euclidean three-space), or H_3 (three-dimensional hyperbolic space).

In Sec. I of this article we discuss the functions necessary to expand a tensor field $h_{\alpha\beta}$ on these three manifolds. For E_3 and S_3 this is basically a review. However, for H_3 a complete set of such functions is derived using results of group theory culminating in the orthogonality and completeness relations (2.32). In Sec. II spin 1 field equations (which include Maxwell's equations) are solved by a separation of variables ansatz relative to a particular choice of frame for the metric given in local coordinates by the line element

$$ds^2 = dt^2 - a^2(t)(dx^2 + b^2(x)(dy^2 + c^2(y)dz^2)) . \quad (1.1)$$

This line element is a generalization of the Robertson–Walker one. What this section does is to provide further examples of spin equations that are solvable by the separation of variables ansatz but are not characterized as eigensolutions of first-order operators. These examples also show how a natural choice of frame can be intrinsically characterized in terms of Killing–Yano tensors and their duals.

II. VECTOR AND TENSOR HARMONICS ON THREE-DIMENSIONAL SPACES OF CONSTANT RIEMANNIAN CURVATURE

The choice of three-dimensional manifold is determined by whether the closed, flat, or open RW model is used. In the book by Landau and Lifshitz[3] a complete set of basis functions is derived for the conformally flat RW model in which a general tensor field $h_{\alpha\beta}$ on E_3 can be expanded in terms of three families of functions related to three-dimensional plane waves.

(1) Using the scalar function $Q = e^{i\mathbf{n}\cdot\mathbf{r}}$ the tensor functions

$$Q_{\alpha\beta} = \frac{1}{3} g_{\alpha\beta} Q, \quad P_{\alpha\beta} = \left(\frac{1}{3} g_{\alpha\beta} - \frac{n^\alpha n^\beta}{(\mathbf{n}\cdot\mathbf{n})}\right) Q, \quad P^\alpha{}_\alpha = O \quad (2.1)$$

are formed. These plane waves in the conformally flat model correspond to perturbations in which the gravitational field, velocity, and density vary.

(2) With the transverse vector waves $\mathbf{S} = \mathbf{s}e^{i\mathbf{n}\cdot\mathbf{r}}$, $\mathbf{s}\cdot\mathbf{n} = 0$ the tensor $S_{\alpha\beta} = n_\alpha S_\beta + n_\beta S_\alpha$ satisfies $S^\alpha{}_\alpha = 0$. These waves correspond to perturbations in which the gravitational field and velocity vary but not the density.

(3) The transverse tensor waves $G_{\alpha\beta} = U_{\alpha\beta} e^{i\mathbf{n}\cdot\mathbf{r}}$ where the symmetric tensor $U_{\alpha\beta}$ satisfies $U_\alpha{}^\beta n_\beta = 0$, $U_\alpha{}^\alpha = 0$. These waves correspond to gravitational waves.

The expansion of a symmetric tensor $h_{\alpha\beta}$ can then be given in terms of the three families of functions. In fact the various families can be invariantly characterized on E_3 according to

$$\Delta W_{\alpha\beta} = (\nabla^\gamma \nabla_\gamma) W_{\alpha\beta} = -n^2 W_{\alpha\beta} , \quad (2.2)$$

where

$$W_{\alpha\beta} = Q_{\alpha\beta}, P_{\alpha\beta}, S_{\alpha\beta}, G_{\alpha\beta} ,$$
$$\nabla^\alpha G_{\alpha\beta} = 0, S^\alpha{}_\alpha = G^\alpha{}_\alpha = P^\alpha{}_\alpha = 0 .$$

(This set of functions is not the only choice possible. Spherical coordinates could have been chosen for r and the components of the tensor $h_{\alpha\beta}$ expanded in a suitable complete set of spherical waves.) The isometry group of E_3 is the six-dimensional Euclidean group E_3 generated by translations and rotations in the Cartesian coordinates $x^\alpha, \alpha = 1,2,3$. A basis for the Lie algebra $\mathbf{E}_3$ of E_3 consists of infinitesmal translations P_α and infinitesmal rotations M_α whose nonzero commutation relations are

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$$[M_\alpha, M_\beta] = \epsilon_{\alpha\beta\gamma} M_\gamma, \quad [M_\alpha, P_\beta] = \epsilon_{\alpha\beta\gamma} P_\gamma,$$
$$\alpha,\beta = 1,2,3.$$

Furthermore

$$\int W^{\alpha\beta} \overline{W}^*_{\alpha\beta}\, d\mathbf{r} = 0, \tag{2.3}$$

when $W_{\alpha\beta}, \overline{W}_{\alpha\beta}$ are not from the same type and each contributing tensor harmonic satisfies

$$P_\alpha W_{\beta\gamma} = \partial_\alpha W_{\beta\gamma} = i n_\alpha W_{\beta\gamma}, \tag{2.4}$$

the P_α being the translation generators as above.

The problem for the closed RW universe has been solved by Gerlach and Sengupta.[4] A general tensor field on S_3 is expanded in terms of three families of functions in direct analogy with the flat space case.

(1) From scalar eigenfunctions of the Laplace operator Q on S_3, viz.,

$$\Delta Q = (\nabla^\gamma \nabla_\gamma) Q = -(n^2-1) Q \tag{2.5}$$

and for n an integer, the tensor fields

$$Q_{\alpha\beta} = \tfrac{1}{3} g_{\alpha\beta} Q, \quad P_{\alpha\beta} = [1/(n^2-1)] \nabla_\beta \nabla_\alpha Q + Q_{\alpha\beta},$$
$$P^\alpha{}_\alpha = 0 \tag{2.6}$$

are constructed.

(2) From vector eigenfunctions of the Laplace operator S_α which are divergenceless, a tensor $S_{\alpha\beta} = \nabla_\alpha S_\beta + \nabla_\beta S_\alpha$ can be constructed where

$$\Delta S_\alpha = -(n^2-2) S_\alpha, \quad \nabla^\alpha S_\alpha = 0. \tag{2.7}$$

(3) From tensor eigenfunctions of the Laplace operator $G_{\alpha\beta}$, one can construct solutions that are symmetric, divergenceless, traceless,

$$\Delta G_{\alpha\beta} = -(n^2-3) G_{\alpha\beta}, \quad \nabla^\alpha G_{\alpha\beta} = 0, \quad G^\alpha{}_\alpha = 0.$$

Gerlach and Sengupta[4] developed a complete set of solutions for tensors of these types in terms of an angular momentum basis. The results are correct but can be derived more neatly using a knowledge of the group representation theory of SO(4) acting on S_3. In the open RW model the problem of a complete set of basis functions has, as far as we know, yet to be fully elucidated. In this article we explicitly compute a basis with which to expand second-order tensors $h_{\alpha\beta}$ on H_3. This is done by using the completeness results due to Naimark[5] and Gelfand *et al.*[6] for the decomposition of the left (or right) regular representation of the Lorentz group into unitary irreducible constituents. The manifold H_3 is realized on the upper sheet of the two-sheeted hyperboloid:

$$\nu_0^2 - \nu_1^2 - \nu_2^2 - \nu_3^2 = 1, \quad \nu_0 > 1.$$

We choose spherical coordinates on the unit hyperboloid, viz.,

$$\begin{aligned} \nu &= (\nu_0,\nu_1,\nu_2,\nu_3) \\ &= (\cosh a, \sinh a \sin\theta\cos\phi, \\ &\quad \sinh a \sin\theta \sin\phi, \sinh a\cos\theta) \\ &\quad 0 < a < \infty, \quad 0 \leqslant \theta < \pi, \quad 0 \leqslant \phi < 2\pi \end{aligned} \tag{2.8}$$

with line element

$$ds^2 = da^2 + \sinh^2 a (d\theta^2 + \sin^2\theta\, d\phi^2). \tag{2.9}$$

In order to obtain a complete set of functions with which to expand second-order tensors we proceed as outlined above.

(1) Scalar functions Q that satisfy

$$\Delta Q = -(1+\rho^2) Q \tag{2.10}$$

are readily obtained. A complete set of such functions in the coordinate basis given above is

$$Q^\rho_{JM} = \Phi^{\rho 0}_{00J}(a) D^J_{0M}(0,\theta,(\pi/2)-\phi)$$
$$0 < \rho < \infty, \quad J = 1,2,\ldots, |M| \leqslant J, \tag{2.11}$$

where $D^J_{MN}(\psi,\theta,\phi)$ is a matrix element of the rotation group in the Euler parametrization and $\Phi^{\rho m}_{l\lambda J}(a)$ the matrix element of the group element $e^{-N_3 a}$ in an angular momentum basis for the unitary irreducible representation of SO(3,1) labeled by $[m, i\rho]$. (See the Appendix for definitions of the Lorentz group and its Lie algebra.) The completeness and orthogonality relations for these functions is well known[7] and follows from Eqs. (A9) for $m = 0$.viz.,

$$\int Q^\rho_{JM}(a,\theta,\phi) Q^{\rho'*}_{J'M'}(a,\theta,\phi) \frac{d\nu_1\, d\nu_2\, d\nu_3}{\nu_0}$$
$$= 2\pi(2J+1) N^{\rho 0}_{0J} \delta_{JJ'} \delta(\rho-\rho') \tag{2.12}$$

$$\sum_{J=0}^{\infty} \sum_{M=-J}^{J} \int_0^\infty Q^\rho_{JM}(a,\theta,\phi) Q^{\rho *}_{JM}(a',\theta',\phi') \frac{d\rho}{N^{\rho 0}_{0J}}$$
$$= \frac{1}{\sinh^2 a \sin\theta} \delta(a-a')\delta(\theta-\theta')\delta(\phi-\phi').$$

The representation space is the natural one

$$T_g Q(\nu) = Q(g^{-1}\nu)$$

for $g \in$ SO(3,1) and the inner product is

$$(Q,R) = \int Q R^* \frac{d\nu_1\, d\nu_2\, d\nu_3}{\nu_0},$$

where $(d\nu_1\, d\nu_2\, d\nu_3)\nu_0^{-1} = \sinh^2 a \sin\theta\, da\, d\theta\, d\phi$ is the invariant measure on H_3.

(2) Vector harmonics S_α. The functions we require in this case must be eigenfunctions of Δ and divergenceless. Taking the choice of coordinates given in (1.9) we may write

$$\begin{aligned} \nu &= (\nu_0,\nu_1,\nu_2,\nu_3) \\ &= R_3(\phi-\pi/2) R_1(-\theta) N_3(a) R_3(\alpha) R_1(\beta) R_3(\gamma) \bar{\nu} \\ &= R_3(\phi) R_1(\theta) N_3(a) \bar{\nu}, \end{aligned} \tag{2.13}$$

where $\bar{\nu} = (1,\mathbf{0})$ and α, β, γ are arbitrary. Given a relativistic vector field S_b, $b = 0,1,2,3$ the action on S_b induced by the Lorentz group is

$$T_g S_b(x) = D^{[0,2]}{}_b{}^c(g) S_c(g^{-1}x), \tag{2.14}$$

where

$$x = r\nu, \quad g \in \mathrm{SO}(3,1), \quad r > 0.$$

This is just the transformation law for relativistic fields. We define new vector fields by

$$S'_b(g) = D^{[0,2]}{}_b{}^c(g) S_c(g^{-1}\bar{\nu}). \tag{2.15}$$

These new fields transform according to

$$T_g S'_b(g) = D^{[0,2]}{}_b{}^c(gg') S_c(g'^{-1} g^{-1} \bar{\nu})$$
$$= S'_b(gg'), \tag{2.16}$$

i.e., the individual components of the new vector fields, S'_b

transform independently. For the Euler parametrization of a Lorentz group element given in (2.13) we can write $S'_b(g)$ as

$$S'_b(g) = D^{\{0,2\}}{}_b{}^c(R) S_c(a,\theta,\phi) ,$$
$$R = R_3(-\gamma) R_1(-\beta) R_3(-\alpha) . \tag{2.17}$$

The functions $S'_b(g)$ transform under the Lorentz group according to the regular representation and are of the specific form given in (2.17). From the decomposition of the regular representation of the Lorentz group into its unitary irreducible components, a complete set of basis functions can be taken as

$$D^l_{N\lambda}(-\gamma,-\beta,-\alpha)\Phi^{\rho m}_{l\lambda J}(a) D^J_{\lambda M}(0,\theta,(\pi/2)-\phi) ,$$
$$0<\rho<\infty; \quad m=0,\pm 1,\pm 2,...;$$
$$J,l=|m|,|m|+1,... ,$$
$$|N|\leqslant l, \quad |M|\leqslant J, \quad |\lambda|\leqslant \min(l,J) . \tag{2.18}$$

For functions of the form (2.17) the expansion functions for $S_b(a,\theta,\phi)$ are

$$\Phi^{\rho m}_{l\lambda J}(a) D^J_{\lambda M}(0,\theta,(\pi/2)-\phi) ,$$
$$0<\rho<\infty; \quad l,m=0,\pm 1; \quad J=|m|,|m|+1,...;$$
$$|\lambda|\leqslant \min(l,J), |M|\leqslant J . \tag{2.19}$$

In choosing a frame in space-time at each point we can without loss of generality take the arbitrary parameters $\alpha=\beta=\gamma=0$ and consider the vector fields $S_b(a,\theta,\phi)$ instead of $S_b(x)$. The above expansion functions (2.19) then form a complete set for a general vector field. Proca's equation (and hence Maxwell's equations) can be solved in these coordinates. Agamaliev, Atakashiev, and Verdiev[8] have indicated how this can be done in Minkowski space-time. Returning to the problem on the manifold H_3, we seek transverse fields corresponding to spin 1 as a result of the condition $\nabla^\alpha S_\alpha = 0$. These functions can be obtained from considerations in Minkowski space-time as follows. Consider a general point in Minkowski space-time as $x = rv$ and choose the frame of one forms

$$e_{(0)i}\, dx^i = dr, \quad e_{(1)i} dx^i = r\, da ,$$
$$e_{(2)i}\, dx^i = (1/\sqrt{2})\, r \sinh a (d\theta + i \sin\theta\, d\phi) , \tag{2.20}$$
$$e_{(3)i}\, dx^i = (1/\sqrt{2})\, r \sinh a (d\theta - i \sin\theta\, d\phi) .$$

Then the components of the vector field S_b referred to this frame, viz., S_b can be expanded in terms of the functions

$$S_0 = f_1(r)\Phi^{\rho 0}_{00J}(a) D^J_{0M}(0,\theta,(\pi/2)-\phi) ,$$
$$S_1 = f_2(r)\Phi^{\rho m}_{10J}(a) D^J_{0M}(0,\theta,(\pi/2)-\phi), \quad m=0,\pm 1 ,$$
$$S_2 = f_2(r)\Phi^{\rho m}_{11J}(a) D^J_{1M}(0,\theta,(\pi/2)-\phi), \quad m=0,\pm 1 ,$$
$$S_3 = f_2(r)\Phi^{\rho m}_{1-1J}(a) D^J_{-1M}(0,\theta,(\pi/2)-\phi),$$
$$m=0,\pm 1 , \tag{2.21}$$

the r dependence being chosen so as to obtain a complete set of functions on H_3. This is done by taking $f_1 = 0$ and choosing solutions of $\Delta' S_b = (\nabla^c\nabla_c) S_b = 0$ to have $f_2(r) = r^{i\rho}$. The vectors S_α are then solutions of

$$\Delta S_\beta = (\nabla^\alpha \nabla_\alpha) S_\beta = -(\rho^2+2) S_\beta, \quad \beta = 1,2,3, \tag{2.22}$$

and $\nabla^\beta S_\beta = 0$, i.e., a suitable basis for transverse vector functions relative to the frame $e_{(a)}$, $a=1,2,3$ consists of the functions

$$S_1 = \Phi^{\rho\pm 1}_{10J}(a) D^J_{0M}(0,\theta,(\pi/2)-\phi) ,$$
$$S_2 = \Phi^{\rho\pm 1}_{11J}(a) D^J_{1M}(0,\theta,(\pi/2)-\phi) , \tag{2.23}$$
$$S_3 = \Phi^{\rho\pm 1}_{1-1J}(a) D^J_{-1M}(0,\theta,(\pi/2)-\phi) ,$$

for

$$0<\rho<\infty; \quad J=1,2,...; \quad |M|\leqslant J .$$

Even and odd parity states can be constructed by realizing that the parity operation corresponds to the replacement $a\to -a$ and the matrix element functions $\Phi^{\rho m}_{l\lambda J}(a)$ satisfy

$$\Phi^{\rho m}_{l\lambda J}(a) = (-1)^{l-J}\Phi^{\rho -m}_{l\lambda J}(-a) . \tag{2.24}$$

(3) Tensor harmonics $G_{\alpha\beta}$. The functions we require in this case must be eigenfunctions of Δ, traceless and divergenceless. As with the case of vector harmonics we consider the relativistic tensor fields that transform under the Lorentz group according to

$$T_g G_{bc}(x) = D^{\{0,3\}}{}_{bc}{}^{de}(g) G_{de}(g^{-1}x) . \tag{2.25}$$

Defining new vector fields

$$G'_{bc}(g) = D^{\{0,3\}}{}_{bc}{}^{de}(g) G_{de}(g^{-1}\bar{v}) \tag{2.26}$$

then these fields transform according to

$$T_{g'} G'_{bc}(g) = D^{\{0,3\}}{}_{bc}{}^{de}(gg') G_{de}(g'^{-1}g^{-1}\bar{v})$$
$$= G'_{bc}(gg') . \tag{2.27}$$

Then writing

$$G'_{bc}(g) = D^{\{0,3\}}{}_{bc}{}^{de}(R) G_{ed}(a,\theta,\phi) , \tag{2.28}$$

where $R = R_3(-\gamma)R_1(-\beta)R_3(-\alpha)$, we argue just as we did in the vector case that the suitable basis of expansion functions for functions $G_{cd}(a,\theta,\phi)$ are as in (2.18), but with

$$0<\rho<\infty; \quad l,m=0,\pm 1,\pm 2; \quad J=|m|,|m|+1,...;$$
$$|\lambda|\leqslant \min(l,J), |M|\leqslant J .$$

If we fix a frame as before by taking $\alpha=\beta=\gamma=0$, we can identify $G_{cd}(a,\theta,\phi)$ as our set of tensor fields. In order to identify which components of $G_{cd}(a,\theta,\phi)$ enable the canonical action of the rotation group to be realized we use the tetrad defined by (2.20). A suitable choice of tensor harmonics is

$$G_{00} = f_3(r)\Phi^{\rho m}_{00J}(a) D^J_{0M}(0,\theta,(\pi/2)-\phi) ,$$
$$G_{11} = \left[\sqrt{\tfrac{2}{3}} f_1(r)\Phi^{\rho m}_{20J}(a) + \tfrac{1}{3} f_3(r)\Phi^{\rho m}_{00J}(a)\right] D^J_{0M} \times\left(0,\theta,\frac{\pi}{2}-\phi\right),$$
$$G_{01} = \left[\left(\frac{2}{3}\right) f_3(r)\Phi^{\rho m}_{00J}(a) - \left(\frac{1}{\sqrt{2}}\right) f_2(r)\Phi^{\rho m}_{10J}(a)\right] D^J_{0M} \times\left(0,\theta,\frac{\pi}{2}-\phi\right),$$
$$G_{02} = (i/\sqrt{2}) f_2(r)\Phi^{\rho m}_{1-1J}(a) D^J_{-1M}(0,\theta,(\pi/2)-\phi) ,$$
$$G_{03} = (i/\sqrt{2}) f_2(r)\Phi^{\rho m}_{11J}(a) D^J_{1M}(0,\theta,(\pi/2)-\phi) ,$$
$$G_{12} = (i/\sqrt{2}) f_1(r)\Phi^{\rho m}_{21J}(a) D^J_{1M}(0,\theta,(\pi/2)-\phi) ,$$

$$G_{13}=(i/\sqrt{2})f_1(r)\Phi^{\rho m}_{2-1J}(a)D^J_{-1M}(0,\theta,(\pi/2)-\phi),$$
$$G_{33}=f_1(r)\Phi^{\rho m}_{2-2J}(a)D^J_{-2M}(0,\theta,(\pi/2)-\phi),$$
$$G_{22}=f_1(r)\Phi^{\rho m}_{22J}(a)D^J_{2M}(0,\theta,(\pi/2)-\phi),$$
$$G_{23}=(i/3)G_{00}-(1/\sqrt{6})G_{11}. \tag{2.29}$$

Here, $m=0,\pm 1,\pm 2$ where appropriate. The functions f_i, $i=1,2,3$ are chosen in such a way as to make the orthogonality relations for the functions G_{bc} coincide with those conditions given in the Appendix. If we now seek divergence free solutions which satisfy $\nabla^b G_{bc}=0$ we take $G_{0a}=0$ for all a. Then we obtain the two independent solutions by taking $f_1=r^{-1+i\rho}$ which are solutions of

$$\Delta G_{\beta\gamma}=(\nabla^\alpha\nabla_\alpha)G_{\beta\gamma}=-(3+\rho^2)G_{\beta\gamma} \tag{2.30}$$

and $\nabla^\alpha G_{\alpha\beta}=0$. A suitable basis of functions is

$$G^{\rho m}_{11JM}=\sqrt{2/3}\Phi^{\rho m}_{20J}(a)D^J_{0M}(0,\theta,(\pi/2)-\phi),$$
$$G^{\rho m}_{12JM}=(i/\sqrt{2})\Phi^{\rho m}_{21J}(a)D^J_{1M}(0,\theta,(\pi/2)-\phi),$$
$$G^{\rho m}_{13JM}=(i/\sqrt{2})\Phi^{\rho m}_{2-1J}(a)D^J_{-1M}(0,\theta,(\pi/2)-\phi),$$
$$G^{\rho m}_{33JM}=\Phi^{\rho m}_{2-2J}(a)D^J_{-2M}(0,\theta,(\pi/2)-\phi),$$
$$G^{\rho m}_{22JM}=\Phi^{\rho m}_{22J}(a)D^J_{2M}(0,\theta,(\pi/2)-\phi),$$
$$G^{\rho m}_{23JM}=(-\tfrac{1}{2})G^{\rho m}_{11JM},\quad m\doteq\pm 2. \tag{2.31}$$

By using the forms of the transverse vector fields S_α and the scalar field Q, the traceless fields given previously and the recurrence formulas of the Appendix, all the traceless components in the expansion of the field $h_{\alpha\beta}$ are then given by allowing $m=0,\pm 1,\pm 2$ in (2.31). The remaining component having trace is simply $H_{\alpha\beta JM}{}^{\rho 0}=g_{\alpha\beta}\Phi^{\rho 0}_{00J}(a)D^J_{0M}(0,\theta,(\pi/2)-\phi)$. This then gives the complete set of functions with which to expand a tensor on H_3.

The orthogonality and completeness relations are

$$\int G^{\rho m}_{\alpha\beta JM}(a,\theta,\phi)G^{\alpha\beta\rho' m'}_{J'M'}(a,\theta,\phi)^*\frac{dv_1\,dv_2\,dv_3}{v_0}$$
$$=2\pi N^{\rho m}_{2J}(2J+1)\delta_{JJ'}\delta_{mm'}\delta_{MM'}\delta(\rho-\rho'),$$
$$\sum_{J=2}^{\infty}\sum_{M=-J}^{J}\sum_{m=-2}^{2}\int_0^\infty G^{\rho m}_{\alpha\beta JM}(a,\theta,\phi)$$
$$\times G^{\alpha\beta\rho m}_{JM}(a',\theta',\phi')^*\frac{d\rho}{N^{\rho m}_{2J}}$$
$$=\frac{1}{\sinh^2 a\sin\theta}\delta(a-a')\delta(\theta-\theta')\delta(\phi-\phi'),$$
$$\int H^{\rho 0}_{\alpha\beta JM}(a,\theta,\phi)G^{\alpha\beta\rho' m'}_{J'M'}(a,\theta,\phi)^*\frac{dv_1\,dv_2\,dv_3}{v_0}=0,$$
$$\int H^{\rho 0}_{\alpha\beta JM}(a,\theta,\phi)H^{\alpha\beta\rho' 0'}_{J'M'}(a,\theta,\phi)^*\frac{dv_1\,dv_2\,dv_3}{v_0}$$
$$=2\pi(2J+1)N^{\rho 0}_{0J}\delta_{JJ'}\delta_{MM'}\delta(\rho-\rho'),$$
$$\sum_{J=0}^{\infty}\sum_{M=-J}^{J}\int_0^\infty H^{\rho 0}_{\alpha\beta JM}(a,\theta,\phi)H^{\alpha\beta\rho 0}_{JM}(a',\theta',\phi')\frac{{}^*d\rho}{N^{\rho 0}_{0J}}$$
$$=\frac{1}{\sinh^2 a\sin\theta}\delta(a-a')\delta(\theta-\theta')\delta(\phi-\phi'),$$
$$m,m'=0,\pm 1,\pm 2, \tag{2.32}$$

with $N^{\rho m}_{lJ}$ as in (A9).

III. SEPARATION OF VARIABLES FOR GENERALIZATIONS OF ROBERTSON–WALKER-TYPE SPACE-TIMES

In addition to the problem of determining complete sets of functions for the expansion of vector and tensor fields on H_3 there has been considerable interest in the intrinsic characterization of solutions of the nonscalar equations of mathematical physics. Considerable attention has been paid to this topic and we mention, in particular, studies of the Dirac equation[8–11] and Maxwell's equations.[12] In this section we discuss some extensions of the results of Kamran and Fels.[13] These authors studied the metric given in local coordinates by the line element

$$ds^2=dt^2-a^2(t)(dx^2+b^2(x)(dy^2+c^2(y)dz^2))$$
$$=g_{ab}\,dx^a\,dx^b. \tag{3.1}$$

In the null frame specified by the one-forms

$$e_{(0)i}dx^i=\frac{1}{\sqrt{2}}(dt-a\,dx),\quad e_{(1)i}dx^i=\frac{1}{\sqrt{2}}(dt+a\,dx),$$
$$e_{(2)i}dx^i=\frac{1}{\sqrt{2}}ab(dy+ic\,dz), \tag{3.2}$$
$$e_{(3)i}dx^i=\frac{1}{\sqrt{2}}ab(dy-ic\,dz).$$

Kamran and Fels[12] demonstrated that the Dirac equation could be solved by a separation of variables procedure that is described by second-order symmetries. We demonstrate that Maxwell's equations in their spinor and vector potential forms also admit separable solutions in direct analogy with what happens for the RW metrics, but that for spin $s\geqslant 2$ the solution mechanism breaks down. The null frame can be intrinsically characterized by using the observation that the Riemannian space with line element (2.1) admits a valence two Killing–Yano having nonzero component $K^{yz}=1/(abc)$. If we look for simultaneous eigenvectors of K^{bc} and its dual $K^*_{bc}=\varepsilon_{bcde}K^{de}$ the corresponding eigenvectors are

$$e^i_{(0)}=(1,a,0,0),\quad e^i_{(1)}=(1,-a,0,0),$$
$$e^i_{(2)}=(0,0,1,i\sin\theta),\quad e^i_{(3)}=(0,0,1,i\sin\theta), \tag{3.3}$$

with eigenvalues given according to Table I.

The null frame specified by the forms (3.2) is the natural one for the spinorial form of Maxwell's equations. However, for the vector potential form the quasidiagonal tetrad is more suitable. This can be characterized intrinsically by

TABLE I. Eigenvalues corresponding to the eigenvectors given in Eq. (3.3).

	Eigenvalues of	Eigenvalues of
	K_{bc}	K^*_{bc}
$l_{(0)}$	o	$1/bc$
$l_{(1)}$	o	$-1/bc$
$l_{(2)}$	i	o
$l_{(3)}$	$-i$	o

realizing that there is also a Killing–Yano tensor of valence 3 for the Riemannian space with line element (3.1) with components $K_{bcd} = \varepsilon_{bcde} K^e$ where the only nonzero element of K_e is $K_t = a$. If we now look for simultaneous eigenvectors of $\widehat{K}_{bc} = K_b K_c - (\nabla^d K_d)\, g_{bc}$ and K_{bd} we recover eigenvectors in the quasidiagonal tetrad. In the case of the form of Maxwell's equations written in terms of the vector potential, we solve the more general problem of the massive spin 1 equation, viz.,

$$\Delta' A_b - R^c_{\ b} A_c = m^2 A_b, \quad \nabla^d A_d = 0\,. \tag{3.4}$$

If instead of the frame $e^i_{(a)}$ we choose the quasidiagonal frame specified by

$$E^i_{(0)} = e^i_{(0)} + e^i_{(1)},$$
$$E^i_{(1)} = e^i_{(0)} - e^i_{(1)}, \quad E^i_{(k)} = e^i_{(k)}\, k = 2,3$$

then Maxwell's equations have the form

$$\begin{aligned}
&\left[\Delta_{KG} + 3\left(\frac{a_{tt}}{a}\right) - 3\left(\frac{a_t}{a}\right)^2\right] A_0 + 2\left(\frac{a_t}{a}\right)\left(\partial_x + \frac{2b_x}{b}\right) A_1 \\
&\quad - \left(\frac{\sqrt{2}a_t}{a^2 b}\right)\left[\left(\partial_y - \frac{i}{c}\partial_z + \frac{c_y}{c}\right) A_2 \right.\\
&\quad \left. + \left(\partial_y + \frac{i}{c}\partial_z + \frac{c_y}{c}\right) A_3\right] = m^2 A_0\,,\\
&\left[\Delta_{KG} + \left(\frac{a_{tt}}{a}\right) + \left(\frac{a_t}{a}\right)^2 + \left(\frac{2}{a^2}\right)\left[\left(\frac{b_x}{b}\right)^2 - \left(\frac{b_{xx}}{b}\right)\right]\right] A_1 \\
&\quad + 2\left(\frac{a_t}{a}\right)\partial_x A_0 - \left(\frac{\sqrt{2}b_x}{a^2 b^2}\right)\left[\left(\partial_y - \frac{i}{c}\partial_z + \frac{c_y}{c}\right) A_2 \right.\\
&\quad \left. - \left(\partial_y + \frac{i}{c}\partial_z + \frac{c_y}{c}\right) A_3\right] = m^2 A_1\,,\\
&\left[\Delta_{KG} - \frac{2c_y}{a^2 b^2 c} i\partial_z + \left(\frac{a_{tt}}{a}\right) + \left(\frac{a_t}{a}\right)^2 \right.\\
&\quad \left. + \left(\frac{c_y}{abc}\right)^2 + \left(\frac{1}{a^2}\right)\left[\left(\frac{b_x}{b}\right)^2 - \left(\frac{b_{xx}}{b}\right)\right]\right] A_2 \\
&\quad - \left(\frac{\sqrt{2}a_t}{a^2 b}\right)\left(\partial_y + \frac{i}{c}\partial_z\right) A_0 \\
&\quad + \left(\frac{\sqrt{2}b_x}{a^2 b^2}\right)\left(\partial_y + \frac{i}{c}\partial_z\right) A_1 = m^2 A_2\,,\\
&\left[\Delta_{KG} + \left(\frac{2c_y}{a^2 b^2 c}\right) i\partial_z + \left(\frac{a_{tt}}{a}\right) + \left(\frac{a_t}{a}\right)^2 \right.\\
&\quad \left. + \left(\frac{c_y}{abc}\right)^2 + \left(\frac{1}{a^2}\right)\left[\left(\frac{bx}{b}\right)^2 - \frac{b_{xx}}{b}\right]\right] A_3 \\
&\quad - \left(\frac{\sqrt{2}a_t}{a^2 b}\right)\left(\partial_y - \frac{i}{c}\partial_z\right) A_0 \\
&\quad + \left(\frac{\sqrt{2}b_x}{a^2 b^2}\right)\left(\partial_y - \frac{i}{c}\partial_z\right) A_1 = m^2 A_3\,,\\
&\left(\partial_t + \frac{3a_t}{a}\right) A_0 - \left(\frac{1}{a}\right)\left(\partial_x + \frac{2b_x}{b}\right) A_1 \\
&\quad + \left(\frac{1}{\sqrt{2}ab}\right)\left[\left(\partial_y + \frac{-i}{c}\partial_z + \frac{c_y}{c}\right) A_2 \right.\\
&\quad \left. + \left(\partial_y + \frac{i}{c}\partial_z + \frac{c_y}{c}\right) A_3\right] = 0\,,
\end{aligned} \tag{3.5}$$

where

$$\Delta_{KG} = g^{ab}\partial_a \partial_b\,.$$

There are two families of solutions for these equations:

(1) For the first type of solution we write

$$A_0 = a_0 g_1(y) e^{-i\lambda z}, \quad A_1 = a_1 g_1(y) e^{-i\lambda z},$$
$$A_2 = (1/\sqrt{2}) a_2 g_0(y) e^{-i\lambda z}, \quad A_3 = (1/\sqrt{2}) a_3 g_2(y) e^{-i\lambda z}, \tag{3.6}$$

where the functions g_i, $i = 0,1,2,3$ satisfy the first-order system

$$\begin{aligned}
&(\partial_y - (\lambda/c) + (c_y/c)) g_0(y) = \lambda_4 g_1(y),\\
&(\partial_y - (\lambda/c)) g_1(y) = \lambda_3 g_2(y),\\
&(\partial_y + (\lambda/c)) g_1(y) = \lambda_2 g_0(y),\\
&(\partial_y + (\lambda/c) + (c_y/c)) g_2(y) = \lambda_1 g_1(y),
\end{aligned} \tag{3.7}$$

which is consistent if $\lambda_1 \lambda_3 = \lambda_2 \lambda_4$. Then for the x dependence of solutions of first type choose

$$a_1 = \hat{a} h_1, \quad a_2 = \tfrac{1}{2}\hat{a} h_0, \quad a_3 = \tfrac{1}{2}\hat{a} h_2, \quad a_0 = 0, \tag{3.8}$$

where

$$\begin{aligned}
&(\lambda_4/b) h_0 + (u + \partial_x - 2b_x/b) h_1 = 0,\\
&(\lambda_3/b) h_1 + (u + \partial_x - b_x/b) h_2 = 0,\\
&(u - \partial_x - b_x/b) h_0 + \lambda_2/b h_1 = 0,\\
&(u - \partial_x - 2b_x/b) h_1 + (\lambda_1/b) h_2 = 0.
\end{aligned} \tag{3.9}$$

Then the function $\hat{a}$ satisfies the differential equation

$$\left[\partial_t^2 + \left(\frac{3a_t}{a}\right)\partial_t + \left(\frac{\partial_{tt}}{a}\right) + \left(\frac{a_t}{a}\right)^2 + \left(\frac{u}{a^2}\right)\right]\hat{a} = m^2 \hat{a}. \tag{3.10}$$

(2) For the second type of solution choose the components of the vector field as

$$a_0 = \hat{a}_0 b_0 g_1 e^{-i\lambda z}, \quad a_1 = \hat{a}_1 b_1 g_1 e^{-i\lambda z},$$
$$a_2 = (1/\sqrt{2})\hat{a}_1 b_2 g_0 e^{-i\lambda z}, \quad a_3 = (1/\sqrt{2})\hat{a}_1 b_2 g_2 e^{-i\lambda z}, \tag{3.11}$$

and require that the functions b_i, $i = 0,1,2$ satisfy the consistent system of equations

$$\begin{aligned}
&\partial_x b_0 = -\epsilon b_1,\\
&(\partial_x + (2b_x/b)) b_1 = 3\epsilon b_0 + (u/b) b_2,\\
&\epsilon b_2 + (\lambda b_0/2b) = 0,\\
&\lambda = -\tfrac{1}{2}\lambda_2 = -\tfrac{1}{2}\lambda_3, \quad \lambda_3 = \lambda_4 = \tfrac{1}{2}u.
\end{aligned} \tag{3.12}$$

Then the $\hat{a}_i$ functions satisfy

$$\begin{aligned}
&(\partial_t + (3a_t/a))\hat{a}_0 - (3\epsilon/a)\hat{a}_1 = 0,\\
&\left[\partial_t^2 + \left(\frac{3a_t}{a}\right)\partial_t + \left(\frac{3\epsilon^2}{a^2}\right) + \left(\frac{3a_{tt}}{a}\right) - 3\left(\frac{a_t}{a}\right)^2\right]\hat{a}_0 \\
&\quad + \left(\frac{6a_t\epsilon}{a^2}\right)\hat{a}_1 = m^2 \hat{a}_0,
\end{aligned} \tag{3.13}$$

$$\begin{aligned}
&\left[\partial_t^2 + \left(\frac{3a_t}{a}\right)\partial_t + \left(\frac{3\epsilon^2}{a^2}\right) + \left(\frac{3a_{tt}}{a}\right) + \left(\frac{a_t}{a}\right)^2\right]\hat{a}_1 \\
&\quad - \left(\frac{2a_t\epsilon}{a^2}\right)\hat{a}_0 = m^2 \hat{a}_1.
\end{aligned}$$

In particular, if the metric is chosen in local coordinates to correspond to the open RW cosmological model, then

$$a = \sinh^2(\psi/2), \quad t = \tfrac{1}{2}(\sinh\psi - \psi),$$
$$b = \sinh x, \quad c = \sin y. \tag{3.14}$$

Identifying

$$A_0 = a_0 \Phi^{\rho m}_{00J}(x) D^J_{0M}(0,y,z),$$
$$A_1 = a_1 \Phi^{\rho m}_{10J}(x) D^J_{0M}(0,y,z),$$
$$A_2 = a_1 \Phi^{\rho m}_{11J}(x) D^J_{1M}(0,y,z),$$
$$A_3 = a_1 \Phi^{\rho m}_{1-1J}(x) D^J_{-1M}(0,y,z), \quad m = 0, \pm 1, \tag{3.15}$$

we find that the solutions of Maxwell's equations (mass = 0) are given by

$$a_0 = 0, a_1 = [\cosh(\psi/2)]^{-3/2} P^{\pm 1/2}_{-1/2+2i\rho}[\cosh(\psi/2)], \tag{3.16}$$

and

$$a_0 = [\cosh(\psi/2)]^{-3/2} P^{\pm 5/2}_{-1/2+2i\rho}[\cosh(\psi/2)],$$
$$a_1 = (1+\rho^2)^{-1/2}[\partial_\psi + 3\cosh(\psi/2)]a_0,$$

where $P^\mu_\nu(z)$ is a solution of Legendre's equation.[14]

The second solution does not represent electromagnetic waves and can be removed by a gauge-fixing transformation. This solution represents the solutions of Maxwell's equations in which the vector $A = (A_0, A_1, A_2, A_3)$ is simultaneously in the synchronous and de Donder gauges.

The systems of first-order differential equations (3.7), (3.9), (3.12) mimic the recurrence relations for the matrix elements $\Phi^{\rho m}_{l\lambda J}(a)$, $m = 0,1$ and $D^J_{\lambda M}$, $[0,\theta,(\pi/2)-\phi]$.

In fact if one examines the spinor equivalent of Maxwell's equations, which are a special case of massive equations due to Wünsch,[15] viz.,

$$\nabla^{AA'}\phi_{AB} = m\psi_B{}^{A'},$$
$$\nabla_{(AA'}\psi^{A'}_{B)} = -m\phi_{AB}, \tag{3.17}$$

where $\nabla^{AA'}$ is the spinor derivative as defined by Penrose and Rindler.[16] Then, relative to the null frame, $e^i_{(a)}$ solutions can be chosen such that

$$\phi_{00} = a_1 h_0 g_0 e^{-i\lambda z}, \quad \phi_{01} = a_1 h_1 g_1 e^{-i\lambda z},$$
$$\phi_{11} = a_1 h_2 g_2 e^{-i\lambda z},$$
$$\psi_{00'} = A_1 h_1 g_1 e^{-i\lambda z}, \quad \psi_{11'} = -A_1 h_0 g_0 e^{-i\lambda z},$$
$$\psi_{01'} = A_1 h_2 g_2 e^{-i\lambda z}, \quad \psi_{10'} = -A h_1 g_1 e^{-i\lambda z}, \tag{3.18}$$

where the functions a_1, A_1 satisfy the coupled equations

$$\left(\frac{-1}{\sqrt{2}}\right)\left(\partial_t + \frac{2a_t}{a}\right) - \left(\frac{u}{a}\right)a_1 = mA_1,$$
$$\left(\frac{1}{\sqrt{2}}\right)\left(\partial_t + \frac{a_t}{a} + \frac{u}{a}\right)A_1 = ma_1. \tag{3.19}$$

This separation of variables procedure does not work if an attempt is made to mimic the use of the recurrence formulas of the Lorentz group to obtain solutions of free-field equations for spin $\geqslant 2$. In fact, the procedure will only work if the background metric $dx^2 + b^2(x)(dy^2 + c^2(y)dz^2)$ corresponds to a three-dimensional Riemannian space of constant curvature, i.e., the case which includes the RW metrics. Rather than write out the equations in detail, we mention that the solution to the equation for gravitational waves in the simultaneous synchronous and de Donder gauges has the form (2.29) with $f_2 = f_3 = 0$, $m = \pm 2$ and f_1 given by

$$f_1 = [\cosh(\psi/2)]^{-3/2} P^{\pm 5i/2}_{-1/2+2i\rho}[\cosh\psi/2], \tag{3.20}$$

where $P^\mu_u(z)$ is a solution of Legendre's equation.[14]

For this choice of function f_i, $i = 1,2,3$ the tensor G_{ab} in (2.29) is a solution of

$$G_{ab} + 2R_{acbd}G^{cd} - 2R_{c(a}G_{b)}{}^c = 0,$$
$$\nabla^a G_{ac} = 0, \quad G^a{}_a = 0.$$

Any theory that explains exactly when a separation of variables procedure works would need to show exactly why it is that spin 1 equations in the case of infinitesmal distance (3.1) admit separable solutions whereas high spin equations do not. This problem does not occur in the case of RW cosmological models, as group theory guarantees the results.

This section has shown that two formulations of spin 1 free-field equations in a background space-time corresponding to the metric (3.1) have solutions by means of a separated solutions for the components in appropriately chosen frames. These frames are in the case of the spinor formulation (3.17) determined by the simultaneous eigenvectors of the Killing–Yano tensor K^{bc} and its dual K^{*bc}. In the vector formulation the quasidiagonal tetrad is determined by choosing eigenvectors of K^{bc} and the Killing tensor $\widehat{K}^{*bc}$. A full understanding of what intrinsic properties enable explicit solution of complete sets of solutions to free-field equations will be the subject of further study.

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APPENDIX: THE LORENTZ GROUP SO(3,1) AND COMPLETE SETS OF MATRIX ELEMENTS

We give here in summarized form, the relevant properties of the Lorentz group. We refer the reader to Gel'fand, Minlos, and Shapiro.[6]

If $R_i(t)$ is the rotation about the ith spatial axis and $N_i(t)$ the hyperbolic rotation in the $0i$ plane $i = 1,2,3$ then the generators of these one-parameter subgroups denoted by M_i, N_i, $i = 1,2,3$ satisfy the commutation relations

$$[M_i, M_j] = \epsilon_{ijk}M_K, \quad [M_i, M_j] = \epsilon_{ijk}N_K,$$
$$[N_i, N_j] = -\epsilon_{ijk}M_K. \tag{A1}$$

Each irreducible representation (IR) of SO(3,1) is labeled by a pair of numbers $[m,c]$ where c is complex and $|m|$ a positive integer. There are two invariant operators

$$\mathbf{K}_1 = \mathbf{M}^2 - \mathbf{N}^2, \quad \mathbf{K}_2 = \mathbf{M}\cdot\mathbf{N}, \tag{A2}$$

such that in a given IR

$$K_1 = 1 - c^2 - m^2, \quad K_2 = icm. \tag{A3}$$

The IRs of SO(3,1) are of two types:

1. Infinite dimensional class

In this class $c^2 \neq (|m| + n)^2$ for any positive integer n. The action of the generators of the Lie algebra on a canonical SO(3) basis $f_{l\lambda}$ is

$$\begin{aligned}
M_+ f_{l\lambda} &= \alpha^l_{\lambda+1} f_{l\lambda+1},\\
M_- f_{l\lambda} &= \alpha^l_\lambda f_{l\lambda-1},\\
iM_3 f_{l\lambda} &= \lambda f_{l\lambda},\\
N_+ f_{l\lambda} &= \alpha^l_{\lambda,\lambda+1} c_l f_{l-1,\lambda+1} - \alpha^l_{\lambda,-\lambda-1} A_l f_{l,\lambda+1}\\
&\quad + \alpha^{l+1}_{\lambda,-\lambda-1} c_{l+1} f_{l+1,\lambda+1},\\
N_- f_{l\lambda} &= \alpha^l_{\lambda,-\lambda+1} c_l f_{l-1,\lambda-1} - \alpha^l_{-\lambda,\lambda-1} A_l f_{l,\lambda-1}\\
&\quad - \alpha^{l+1}_{\lambda,\lambda-1} c_{l+1} f_{l+1,\lambda-1},\\
iN_3 f_{l\lambda} &= \alpha^l_{\lambda,-\lambda} c_l f_{l-1,\lambda} - \lambda A_l f_{l,\lambda}\\
&\quad - \alpha^{l+1}_{-\lambda,\lambda} c_{l+1} f_{l+1,\lambda},\\
M_\pm &= M_1 \pm iM_2, \quad N_\pm = N_1 \pm iN_2,
\end{aligned} \tag{A4}$$

where

$$A_l = \frac{imc}{l(l+1)}, \quad c_l = \frac{i}{l}\sqrt{\left[\frac{(l^2-m^2)(l^2-c^2)}{4l^2-1}\right]},$$

$$\alpha^l_{\lambda u} = \sqrt{(l-\lambda)(l-u)}.$$

The l,λ spectrum for the IR $[m,c]$ is

$$|\lambda| \leqslant l, \quad l = |m|, |m|+1, \ldots .$$

The representations are unitary if,

$$c = i\rho, \quad 0 \leqslant \rho < \infty, \quad m = 0, \pm\tfrac{1}{2}, \pm 1, \pm\tfrac{3}{2}, \ldots$$

(this is the principal series);

$$\operatorname{Im} c = 0, \quad 0 < c < 1, \quad m = 0$$

(this is the complementary series).

2. Finite-dimensional class

In this class $c^2 = (|m| + n)^2$ for some positive integer n. The action of the generators on a canonical SO(3) basis is as in (A4). The l,λ spectrum for the IR $[m,c]$ is

$$|\lambda| \leqslant l, \quad l = |m|, \quad |m|+1, \ldots |m|+n-1.$$

The unitary IR $[m,i\rho]$ can be realized on the space of functions on the two-dimensional sphere via the orthonormal angular momentum basis functions:

$$\begin{aligned}
f_{l\lambda} &= [(2l+1)/4\pi]^{1/2} D^l_{\lambda m}(\phi,\theta,0),\\
&\quad |\lambda| \leqslant l, \quad l = |m|, |m|+1, \ldots .
\end{aligned} \tag{A5}$$

The action of the Lorentz group can be induced from the action

$$\begin{aligned}
&T^{[m,i\rho]}(g)\Phi(z)\\
&\quad = (\beta z + \gamma)^{m+i\rho-1}(\beta^* z^* + \gamma^*)^{-m+i\rho-1}\\
&\qquad \times \Phi[(\alpha z + \delta)/(\beta z + \gamma)]
\end{aligned} \tag{A6}$$

via the identification

$$|z|^{-1} = \tan\tfrac{1}{2}\theta, \quad \arg z = \phi,$$

and with

$$f(\theta,\phi) = e^{-im\phi}[\sin^2(\theta/2)]^{i\rho-1}\phi(z), \tag{A7}$$

the matrix element of $N_3(a)$ in the angular momentum basis has the integral representation

$$\begin{aligned}
\Phi^{\rho m}_{l\lambda J}(a) &= \tfrac{1}{2}\sqrt{(2l+1)(2J+1)}\\
&\quad \times \int_{-1}^{1} dx(\cosh a + x \sinh a)^{i\rho-1},\\
&\quad \times d^l_{\lambda m}(x) d^J_{\lambda m}(x')\\
x' &= (x + \tanh a)/(1 + x\tanh a),
\end{aligned} \tag{A8}$$

where $N_i(c) = e^{N_i c}$ $i = 1,2,3$ and $d^l_{\lambda m}(\cos\theta) = D^l_{\lambda m}(0,\theta,0)$.

An explicit expression for these functions has been obtained by Dao Wong duc and Nguyen Van Hieu.[17] These functions satisfy the orthogonality relations

$$\begin{aligned}
&\sum_\lambda \int_0^\infty \Phi^{\rho m}_{l\lambda J}(a)\Phi^{\rho' m' *}_{l\lambda J}(a)\sinh^2 a\, da\\
&\quad = N^{\rho m}_{lJ}\delta_{mm'}\delta(\rho - \rho'),\\
&\sum_{m=-j}^{j} \int_0^\infty \Phi^{\rho m}_{l\lambda J}(a)\Phi^{\rho m *}_{l\lambda J}(a')d\rho\\
&\quad = N^{\rho m}_{lJ}\frac{\delta(a-a')}{\sinh^2 a}, \quad j = \min(j,J).
\end{aligned} \tag{A9}$$

The normalization factor is

$$\begin{aligned}
N^{\rho m}_{lJ} &= 2\pi\frac{(L-j)![2(l+1)!]^2(j+|m|)!(j-|m|)!}{(L+j)!(L+m+l)!(L-m)!(L+l-m)!}\\
&\quad \times \prod_{k=|m|+1}^{j}(\rho^2+k^2)\left|\frac{\Gamma(i\rho+|m|)}{\Gamma(i\rho+L+1)}\right|^2,
\end{aligned}$$

where

$$L = \max(l,J).$$

These functions obey the symmetry relations

$$\begin{aligned}
\Phi^{\rho m}_{l-\lambda J}(a) &= (-1)^{l-J}\Phi^{\rho -m}_{l\lambda J}(-a)\\
&= (-1)^{l-J}\Phi^{-\rho -m}_{l\lambda J}(a) = \Phi^{\rho -m}_{l-\lambda J}(a).
\end{aligned} \tag{A10}$$

By a straightforward extension of the arguments in Sec. I, any tensor field representing the solution of a Lorentz invariant equation can in a suitable frame be directly expandable in an appropriate choice of matrix elements [e.g., the components G_{ab} in (2.31)].

We know from the group theory arguments that each component of a Lorentz invariant equation must be expandable in an appropriate choice of matrix elements. Recurrence formulas for the functions $\Phi^{\rho m}_{l\lambda J}(a)$ can be deduced by realizing the matrix element $D^{[m,c]}_{l\lambda J\lambda'}(g)$ in the generalized Euler parametrization in the form

$$\begin{aligned}
D^{[m,i\rho]}_{l\lambda J\lambda'}(g) &= \sum_\mu D^l_{\lambda\mu}\left(\phi - \frac{\pi}{2}, -\theta, 0\right)\\
&\quad \times \Phi^{\rho m}_{l\mu J}(a) D^J_{\mu\lambda'}(\alpha,\beta,\gamma).
\end{aligned} \tag{A11}$$

For fixed J,λ' the matrix elements provide a realization of the unitary IR $[m,i\rho]$ by the left regular representation

$$T_g D^{[m,i\rho]}_{l\lambda J\lambda'}(g) = D^{[m,i\rho]}_{l\lambda j\mu}(g') D^{[m,i\rho]j\mu}_{J\lambda'}(g). \tag{A12}$$

Consequently invoking the canonical action of the infinitesmal operators in (A4) we deduce the recurrence relations that follow. These results are due to Strom:[18]

$$\sqrt{[(l+1)^2-\lambda^2]}(\partial_a - l\coth a)\Phi^{\rho m}_{l\lambda J}(a) + \frac{1}{2\sinh a}\left[\sqrt{(l-\lambda)(l-\lambda+1)(J-\lambda)(J+\lambda+1)}\Phi^{\rho m}_{l\lambda+1J}(a)\right.$$
$$\left.\times\sqrt{(l+\lambda)(l+\lambda+1)(J+\lambda)(J-\lambda+1)}\Phi^{\rho m}_{l\lambda+J}(a)\right]$$
$$= -\left[((l+1)^2-m^2)((l+1)^2+\rho^2)\left(\frac{2l+1}{2l+3}\right)\right]^{1/2}\Phi^{\rho m}_{l+1\lambda J}(a),$$
$$\sqrt{[l^2-\lambda^2]}(\partial_a + (l+1)\coth a)\Phi^{\rho m}_{l\lambda J}(a) + \frac{-1}{2\sinh a}\left[\sqrt{(l+\lambda)(l+\lambda+1)(J+\lambda)(J+\lambda+1)}\Phi^{\rho m}_{l\lambda+1J}(a)\right.$$
$$\left.+\sqrt{(l-\lambda)(l-\lambda+1)(J+\lambda)(J-\lambda+1)}\Phi^{\rho m}_{l\lambda+1J}(a)\right]$$
$$= \left[(l^2-m^2)(l^2+\rho^2)\left(\frac{2l+1}{2l-1}\right)\right]^{1/2}\Phi^{\rho m}_{l-1\lambda J}(a),$$
$$(\lambda\partial_\alpha + \lambda\coth a + im\rho)\Phi^{\rho m}_{l\lambda J}(a) = \frac{1}{2\sinh a}\left[\sqrt{(l+\lambda)(l-\lambda+1)(J+\lambda)(J-\lambda+1)}\Phi^{\rho m}_{l\lambda-1J}(a)\right.$$
$$\left.-\sqrt{(l-\lambda)(l+\lambda+1)(J+\lambda+1)(J-\lambda)}\Phi^{\rho m}_{l\lambda+1J}(a)\right],$$
$$\left(\partial_a^2 + 2\coth a\partial_a - \frac{[l(l+1)+J(J+1)]}{\sinh^2 a} + (1+\coth^2 a)\lambda^2 + 1 + \rho^2 - m^2\right)\Phi^{\rho m}_{l\lambda J}(a)$$
$$= -\frac{\coth a}{\sinh a}\left[\sqrt{(l+\lambda)(l-\lambda)(J+\lambda)(J-\lambda+1)}\Phi^{\rho m}_{l\lambda-1J}(a)\right.$$
$$\left.+\sqrt{(l+\lambda+1)(l-\lambda)(J+\lambda+1)(J-\lambda)}\Phi^{\rho m}_{l\lambda+1J}(a)\right]. \qquad \text{(A13)}$$

These relations enable the a dependence of solutions to relativistically invariant equations to be obtained in the form given in e.g. (1.31). The matrix elements

$$\Phi^{\rho m}_{l\lambda J}(a)D^J_{\lambda M}[0,\theta,(\pi/2)-\phi],$$
$$0<\rho<\infty, \quad m=0,\pm 1,\pm 2,...,$$
$$J,l=|m|,|m|+1,...; \quad |M|\leqslant J, \quad |\lambda|\leqslant\min(l,J),$$

then provide a suitable complete set of functions with which to expand spin s fields $0\leqslant l\leqslant s$, for further details see Refs. 7 and 19. There are, however, other systems of basis functions possible corresponding to a different choice of group parametrization and coordinates on the hyperboloid. These functions are the analogs on H_3 of vector and tensor expansion functions corresponding to spherical, or cylindrical waves in Euclidean three-space. We list below a brief summary of other important sets of basis functions that are possible, together with the corresponding group parametrizations and coordinates on H_3. In each case the new basis functions are eigenfunctions of a definite subgroup chain of SO(3,1). In the case of spherical coordinates (2.9) the basis consists of sets of eigenfunctions of the operators M^2 (angular momentum) and M_3 (its third component).

Two other coordinate systems on the hyperboloid are the following.

(1) Hyperboloid coordinates:

$$\nu = (\cosh a\cosh b, \cosh a\sinh bc,$$
$$\cosh a\sinh b\sin\phi, \sinh a)$$
$$-\infty<a<\infty, 0\leqslant b<\infty, 0\leqslant\phi<2\pi. \qquad \text{(A14)}$$

The corresponding group parametrization is

$$g = R_3(\phi)N_1(b)N_3(a)R_3(\alpha)R(\beta)R_3(\gamma). \qquad \text{(A15)}$$

The appropriate basis functions are denoted by

$$H^{\rho m\epsilon}_{l\lambda j}(a)D^{j\epsilon}_{\lambda N}(0,b,\phi), \quad \epsilon=\pm,$$

where $D^{j\epsilon}_{\lambda N}(\varphi,b,\phi)$ are the matrix elements of a general element of the SO(2,1) group given in terms of the Euler parametrization

$$g = R_3(\varphi)N_1(b)R_3(\phi)$$

and in the corresponding unitary irreducible representations labeled by $j,\epsilon=\pm$ where $\epsilon=+, |\lambda|\leqslant l$

$$j=-\tfrac{1}{2}+iq, \quad 0<q<\infty; \quad \lambda,N=0,\pm 1,\pm 2,...,$$
$$j=0,1,...,|m|-1; \quad \lambda,N=j+1,j+2,...,$$
$$\epsilon=-, |\lambda|\leqslant l,$$
$$j=-\tfrac{1}{2}+iq, \quad 0<q<\infty; \quad \lambda,N=0,\pm 1,\pm 2,...,$$
$$j=0,1,...,|m|-1; \quad \lambda,N=-j-1,-j-2,...,$$

and integer.

The functions $H^{\rho m\epsilon}_{l\lambda j}(a)$ have the integral representation:[20]

$$H^{\rho m+}_{l\lambda j}(a) = \frac{1}{2}\sqrt{\left(l+\frac{1}{2}\right)\left(j+\frac{1}{2}\right)}$$
$$\times\int_0^\infty (\cosh a\cosh b + \sinh a)^{i\rho-1}i^{M-\lambda}$$
$$\times d^j_{m\lambda}(\cosh b)d^l_{m\lambda}(\cosh\theta_g)\sinh b\, db, \qquad \text{(A16)}$$

where

$$\cos\theta_g = \frac{(\cosh b\sinh a + \cosh a)}{(\cosh b\cosh a + \sinh a)},$$
$$d^J_{\lambda m}(\cos\theta) = D^J_{\lambda m}(0,\theta,0),$$

and

$$H^{\rho m-}_{l\lambda j}(a) = (-1)^{l-\lambda}H^{\rho-m+}_{l\lambda j}(-a). \qquad \text{(A17)}$$

As expected, the recurrence formulas for these functions enable the complete decoupling of relativistically invariant equations from the dependence on a,b,ϕ in a frame corresponding to the one forms

$$e_{(1)i}\,dx^i = da,$$
$$e_{(2)i}\,dx^i = (1/\sqrt{2})\sinh a(db + i\sinh b d\phi), \qquad \text{(A18)}$$
$$e_{(3)i}\,dx^i = (1/\sqrt{2})\sinh a(db - i\sinh b\,d\phi).$$

Bearing in mind that if we consider spinor equations, the use of null trends is appropriate, the basis functions are eigenfunctions of $N_1^2 + N_2^2 - M_3^2$ and M_3 with eigenvalues $-j(j+1)$ and M, respectively.

(2) Horospherical coordinates:

$$v = (\tfrac{1}{2}r^2e^a + \cosh a, re^a\cosh\phi, re^a\sin\phi, \tfrac{1}{2}r^2e^a - \sinh a)$$
$$-\infty < a < \infty, 0 \leqslant r < \infty, 0 \leqslant \phi < 2\pi. \qquad \text{(A19)}$$

The corresponding group parametrization is

$$g = R_3(\phi)T_1(r)N_3(a)R_3(\alpha)R_1(\beta)R_3(\gamma),$$

where

$$T_1(r) = e^{(N_1 + M_2)r}. \qquad \text{(A20)}$$

The appropriate basis functions are denoted by

$$E^{\rho m}_{l\lambda x}(a)J_{\lambda - M}(Xr)e^{iM\phi},$$

where $J_\nu(z)$ is a Bessel function.[14] The functions $E^{\rho m}_{l\lambda x}(a)$ have the integral representation[16]

$$E^{\rho m}_{l\lambda x}(a) = \sqrt{l + \frac{1}{2}}\int_0^\pi \left(e^a\cos^2\frac{1}{2}\theta\right)^{i\rho - 1}$$
$$\times J_{m-\lambda}\,(e^{-a}X\tan\tfrac{1}{2}\theta)d^{\,l}_{m\lambda}(\cos\theta)\sin\theta\,d\theta. \qquad \text{(A21)}$$

The corresponding frame of one-forms in which complete decoupling of relativistically invariant equations occurs from the variables a,r,ϕ is

$$e_{(1)i}\,dx^i = da, \quad e_{(2)i}\,dx^i = (1/\sqrt{2})e^{-a}(dr + ir\,d\phi),$$
$$e_{(3)i}\,dx^i = (1/\sqrt{2})e^{-a}(dr - ir\,d\phi), \qquad \text{(A22)}$$

with suitale modification to include the use of null tetrads if spinor equations are included. The basis functions are eigenfunctions of $(N_1 + M_2)^2 + (N_2 - M_1)^2$ and M_3 with eigenvalues $-X^2$ and M, respectively. In fact, all possible subgroup chains for the Lorentz group are known and appropriate basis functions on H_3 for symmetric tensors can be computed in a suitable frame. For further details see Kalnins.[19]

Specifically for the case of perturbations of the RW cosmological models, we give the explicit expressions for the expansion functions in the coordinates. The functions $\Phi^{\rho m}_{l\lambda x}(a)$ satisfy the differential equation

$$[\partial_a^2 + 2(l+1)\cosh a\partial_a$$
$$+ ([l(l+1) - J(J+1)]/\sinh^2 a)$$
$$- 2im\rho\coth a + (l+1)^2 + \rho^2 - m^2]\Phi^{\rho m}_{llJ}(a) = 0 \qquad \text{(A23)}$$

and have the solution

$$\Phi^{\rho m}_{llJ}(a) = i^{J-l}(1 - e^{-2a})^{J-l}\exp[-(l + 1 - m - i\rho)a]$$
$$\times {}_2F_1(J + 1 - i\rho, J + 1 - m; 2J + 2, 1 - e^{-2a}). \qquad \text{(A24)}$$

The other external matrix element can be obtained from the symmetry condition

$$\Phi^{\rho m}_{l-lJ}(a) = \Phi^{\rho -m}_{llJ}(a). \qquad \text{(A25)}$$

The remaining functions can be obtained from the recurrence formulas as follows:

(1) m=0

$$\Phi^{\rho o}_{2-1J}(a) = \Phi^{\rho o}_{21J}(a) = \frac{2}{\sqrt{J(J+1) - 2}}$$
$$\times(\sinh a\partial_a + \cosh a)\Phi^{\rho o}_{22J}(a),$$
$$\Phi^{\rho o}_{20J}(a) = \frac{2}{\sqrt{3J(J+1)}}(\sinh a\partial_a + \cosh a)\Phi^{\rho o}_{21J}(a)$$
$$+ \sqrt{J(J+1) - 2}\,\Phi^{\rho o}_{22J}(a); \qquad \text{(A26)}$$

(2) m=1

$$\Phi^{\rho l}_{2\pm 1}(a) = \frac{2}{\sqrt{J(J+1) - 2}}$$
$$\times[\sinh a\partial_a + \cosh a \pm i\rho]\Phi^{\rho 1}_{2\pm 2J}(a),$$
$$\Phi^{\rho o}_{20J}(a) = \frac{2}{\sqrt{3J(J+1)}}[(\sinh a\partial_a + \cosh a) \pm i\rho]$$
$$\times\Phi^{\rho 1}_{2\pm 1J}(a) + \sqrt{J(J+1) - 2}\,\Phi^{\rho 1}_{2\pm 2J}(a); \qquad \text{(A27)}$$

(3) m=2

$$\Phi^{\rho 2}_{2\pm 1J}(a) = \frac{2}{\sqrt{J(J+1)} - 2}$$
$$\times[\sinh a\partial_a + \cosh a \pm i\rho]\Phi^{\rho 2}_{2\pm 2J}(a),$$
$$\Phi^{\rho 2}_{20J}(a) = \frac{2}{\sqrt{J(J+1)} - 2}[\sqrt{3}(\sinh a\partial_a + \cosh a)$$
$$\times\Phi^{\rho 2}_{2\pm 1J}(a) - \sqrt{3[J(J+1) - 2]}\,\Phi^{\rho 2}_{2\pm 2J}(a). \qquad \text{(A28)}$$

These expressions are deduced from the simplest recurrence relations that enable all other matrix elements $\Phi^{\rho m}_{l\lambda J}(a)$ to be deduced from the expressions for the external components.

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