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# Evolution of energy-containing turbulent eddies in the solar wind

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**Abstract.** Previous theoretical treatments of fluid-scale turbulence in the solar wind have concentrated on describing the state and dynamical evolution of fluctuations in the inertial range, which are characterized by power law energy spectra. In the present paper a model for the evolution of somewhat larger, more energetic magnetohydrodynamic (MHD) fluctuations is developed by analogy with classical hydrodynamic turbulence in the quasi-equilibrium range. The model is constructed by assembling and extending existing phenomenologies of homogeneous MHD turbulence, as well as simple two-length-scale models for transport of MHD turbulence in a weakly inhomogeneous medium. A set of equations is presented for the evolution of the turbulence, including the transport and nonlinear evolution of magnetic and kinetic energy, cross helicity, and their correlation scales. Two versions of the model are derived, depending on whether the fluctuations are distributed isotropically in three dimensions or restricted to the two-dimensional plane perpendicular to the mean magnetic field. This model includes a number of potentially important physical effects that have been neglected in previous discussions of transport of solar wind turbulence. Numerical solutions are shown for several cases of interest that demonstrate the advantages of this approach. We suggest that this model may prove useful in studies of solar wind heating and acceleration, as well as in describing the response of interplanetary turbulence to wave energy injected by pickup ions and planetary upstream waves.

## 1. Introduction

Much of the attention in studies of solar wind fluctuations at magnetohydrodynamic (MHD) scales has focused on correlations and spectral characterizations at length scales corresponding to spacecraft frame periods of several seconds to several hours. In this range, power-law energy spectra are typically observed [*Coleman*, 1968], along with frequently occurring Alfvénic fluctuations [*Coleman*, 1968; *Belcher and Davis*, 1971]. A widespread, though perhaps not universally accepted, interpretation of these fluctuations is that they correspond to a Kolmogoroff-type inertial-range turbulence in an MHD setting. Considerable efforts have been devoted to developing this MHD turbulence perspective [*Jokipii and Coleman*, 1968; *Matthaeus and Goldstein*, 1982a, b; *Montgomery*, 1983; *Bavassano and Bruno*, 1989; *Grappin et al.*, 1990]. This near-power law range of scales is highly reminiscent of the inertial range in steady hydrodynamic turbulence and represents those features of solar wind turbulence that are in many respects the most reproducible in a statistical sense. However, extending the analogy with hydrodynamics further still, one also expects the evolution of the inertial range to be strongly influenced by a supply of energy transferred from somewhat larger scales. These larger, more energetic structures, known in classical turbulence theory as energy-containing eddies, require a theory distinct from that of the inertial range, for reasons to be discussed below. To develop a quantitative understanding of the turbulence as a whole, including the rate at which energy is supplied to the inertial range and the turbulent heating rate, one needs at least an approximate theory for the evolution of the energy-containing turbulent structures. Such models of solar wind turbulent fluctuations, for both energy containing and inertial ranges [e.g., *Tu et al.*, 1984; *Hollweg*, 1986] should deal simultaneously with the effects of spatial inhomogeneities due to large-scale variations in plasma and field parameters, as well as local turbulence properties. In the present paper we develop such a model for the evolution of the energy containing range of solar wind turbulence.

In hydrodynamic turbulence the relative concentration of turbulent energy at scales larger than the power law inertial range has caused the theory of the energy-

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containing range to occupy a position of great importance. The energy cascade from larger to smaller scales is accomplished by the interactions of eddies at each scale with all other eddies. In the inertial range the nature of the couplings is thought to be self-similar and is readily treated by a theory that invokes statistical equilibrium for turbulence in a steady state [*Kolmogoroff*, 1941; *Batchelor*, 1953]. The energy-containing structures in decaying turbulence are never in a steady state; nevertheless, they are responsible for supplying the energy that drives inertial-range transfer toward ever smaller scales. The energy-containing range cannot be self similar and requires an additional parameter, conveniently taken as a length scale, for its description. Unlike the inertial range, a spectral decomposition is typically not performed in simple treatments of an energy-containing range. Rather, the entire band of energy-containing eddies is dealt with at once, and the principle output of the model is the rate at which energy is fed into the inertial range. The inertial-range couplings mediate this spectral transfer but do not control it. At high wave numbers the inertial range gives way to the dissipation range, where turbulent energy is dissipated as heat. Thus it is clear that the energy-containing eddies, in large part, determine the structure of the inertial range and the rate of turbulent heating. In hydrodynamic turbulence theory [*Batchelor*, 1953] it is recognized that the energy-containing structures require a separate dynamical theory, generally known as quasi-equilibrium theory to distinguish it from the conditions of the steady inertial range.

To date, the need for an MHD theory of the energy-containing range has received less attention in the space physics community than has the inertial range. One reason is that solar wind observations have afforded ample opportunity to study the inertial range of MHD turbulence in great detail. Moreover, the approximation of local homogeneity [e.g., *Jokipii and Coleman*, 1968; *Matthaeus and Goldstein*, 1982b] works best at these relatively shorter scales. This assumption provides the basis for comparison of observations with both analytical theories [*Dobrowolny et al.*, 1980a, b; *Grappin et al.*, 1982, 1983] and numerical simulations [*Matthaeus et al.*, 1983; *Pouquet et al.*, 1986, 1988] of homogeneous MHD turbulence. These studies have indicated that a reasonable understanding of local features of solar wind MHD turbulence can be achieved by appealing to basic homogeneous turbulence theory.

However, observations have also demonstrated the need for a more sophisticated analytical framework in order to understand the radial evolution of solar wind turbulence on larger scales [*Bavassano et al.*, 1982a, b; *Bavassano and Bruno*, 1989; *Roberts et al.*, 1987a, b; *Bavassano and Smith*, 1986; *Grappin et al.*, 1990; *Tu et al.*, 1989a, b]. Turbulent heating also seems to require a more extensive treatment [e.g., *Barnes*, 1968, 1969; *Tu et al.*, 1989c; *Freeman*, 1988]; indeed, the original suggestion of *Coleman* [1968] clearly involved energy-containing structures and not solely inertial-range fluctuations. The subtlety of the turbulent heating issue is most apparent in theories of solar wind acceleration [*Holzer*, 1977; *Holzer and Leer*, 1980; *Leer and Holzer*, 1979, 1980; *Leer et al.*, 1982; *Hollweg*, 1978, 1986; *Hollweg and Johnson*, 1988; *Tu*, 1987; *Isenberg*, 1990], wherein the detailed properties of the turbulence sensitively influences the results.

The most widely used mathematical model for solar wind fluctuations has been WKB theory where the fluctuations are treated as noninteracting waves that respond to long wavelength variations in the background plasma and magnetic field parameters [*Parker*, 1965; *Belcher*, 1971; *Alazraki and Couturier*, 1971; *Barnes*, 1979; *Barnes and Hollweg*, 1974; *Jacques*, 1977; *Hollweg*, 1973, 1974; *Whang*, 1973, 1980]. The model of *Tu et al.* [1984] is essentially a WKB model into which *Coleman*'s [1968] idea of a Kolmogoroff-like turbulent cascade has been incorporated. More complete theoretical formalisms incorporating various MHD spectra, e.g., magnetic and kinetic energy and cross helicity, have been presented recently [*Zhou and Matthaeus*, 1989, 1990*a*; *Marsch and Tu*, 1989; *Tu and Marsch*, 1990]. Although quite involved, these theories have shown promise in explaining a number of features of observed inertial-range spectra [*Oughton and Matthaeus*, 1992; *Oughton*, 1993; *Marsch and Tu*, 1993]. In principle, the same spatial operators in these transport equations can also be used to describe the energy-containing fluctuations that govern the evolution of the turbulence as a whole, although they have not previously been used in this way.

The need for an improved transport theory of solar wind fluctuations can be motivated more directly by discussion of the *Tu et al.* [1984] model of inhomogeneous, WKB-like transport, which include local spectral transfer models originally developed to explain the power law inertial range in steady hydrodynamic turbulence [*Batchelor*, 1953]. The method succeeds in describing some aspects of inertial range evolution because fluctuations at small scales have characteristic turnover times that are much shorter than the overall decay rate. Such an approach is indefensible for slowly evolving structures whose decay rates are comparable to that of the total energy. The eddy turnover time increases with the linear dimension, so in freely decaying turbulence the assumption of statistical stationarity must break down for structures larger than some size. Furthermore, the power law decline of inertial-range energy density with increasing wave number means that total energy is dominated by fluctuations of relatively low wave number. Consequently, the kind of modeling used by *Tu et al.* [1984] is plausible for the local interactions within the inertial range but cannot describe the energy fed into the inertial range from its low wave number boundary. Below this boundary, taken to be roughly several times the reciprocal of the longitudinal correlation length, the turbulence can be described by the dynamics of the energy-containing eddies. The wave number associated with these scales lies near the maximum in the omnidirectional fluctuation spectrum. For models of the omnidirectional spectrum this peak occurs near the wave number at which the observed one-dimensional spectrum bends over from a relatively flat spectrum to the inertial range powerlaw. Consequently, inertial range models for solar wind transport need to be augmented by an energy-containing model appropriate to these scales.

A simple approach to the problem of transporting bulk turbulence has been employed by *Hollweg* [1986] and *Hollweg and Johnson* [1988] in the context of solar wind acceleration. Rather than consider the spectrally decomposed fluctuations, Hollweg suggests using a Kolmogoroff-type heating rate as described by *Coleman* [1968], which contributes an additional term to the WKB equation for the transport of the mean-square fluctuating magnetic field strength. This approach has the advantage of discarding unneeded inertial-range spectral information and simplifying the problem considerably. This point will be discussed further below. However, by assuming the Alfvénic condition (pure cross helicity) associated with outward traveling waves, the theory of Hollweg and coworkers eliminates a number of potentially important MHD effects on turbulent evolution, spectral transfer, and heating due to turbulent decay. Although the Hollweg model is limited in the scope of physical effects included and has been found to fall short of explaining solar wind observations [cf. *Isenberg*, 1990], to our knowledge it includes the first proper treatment of quasi-equilibrium range Kolmogoroff phenomenology in a quantitative solar wind model.

The purpose of the present paper is to develop a more complete phenomenological model for transport of energy and other rugged invariants such as cross helicity [*Frisch et al.*, 1975, *Kraichnan and Montgomery*, 1980; *Matthaeus and Goldstein*, 1982a] in the solar wind. There is no rigorous, solvable mathematical model of turbulence, so we merge elements from two types of approximate theories. First, we develop a single-point phenomenological closure model to describe the decay of energy-containing eddies in homogeneous MHD turbulence, including the possibility of forcing terms that drive excitations at relevant scales. This step represents an adaptation to MHD of the simplest Kolmogoroff-Oubukoff-Batchelor theory for the decay of the energy-containing eddies in hydrodynamic turbulence. Second, we use a reduced version of the spectral transport formalism of *Zhou and Matthaeus* [1990a] to describe the spatial transport of the energy-containing eddies. This method assumes a substantial separation between the turbulence length scales and the distance over which significant changes in the properties of the solar wind are experienced.

## 2. Review of Hydrodynamic Turbulence

The theory of turbulence developed by Kolmogoroff, Obukhoff, and others is well described in the classic text by *Batchelor* [1953]. We mention here a few basic results to be used later in the paper. We emphasize the distinction in even the simplest theoretical picture between the treatment of fluctuations in the inertial and energy-containing ranges. In particular, in the present model we do not use the methods proposed for modeling nonlinear MHD effects in the inertial range [*Tu et al.*, 1984; *Tu*, 1988; *Zhou and Matthaeus*, 1990b]. Those approaches are based on the equilibrium conditions of the inertial range and are inappropriate for the quasi-equilibrium conditions in the energy-containing range [*Batchelor*, 1953].

The distinction between the energy-containing, inertial, and dissipation ranges can be clarified by writing a general equation for the evolution of the energy spectrum [*Batchelor*, 1953; *Monin and Yaglom*, 1971, 1975]:

$$\frac{\partial E(k,t)}{\partial t} = T(k,t) + S(k,t) - \nu k^2 E(k,t). \qquad (1)$$

In this equation, $E(k,t)$ is the omnidirectional energy spectrum; $T(k,t)$ is the energy transfer function representing the changes in the spectrum due to the sum of all nonlinear turbulent interactions; $S(k,t)$ represents the source terms; and the final term, involving the kinematic viscosity $\nu$, is the rate of dissipation by heat. Because nonlinear interactions among eddies conserve total energy, $\int T(k,t)\, dk$ is always zero, so that $T(k)$ can generally be written as the divergence of a flux of energy in wave number space. For typical spectra, in both wind tunnels and the solar wind, the total energy $\int E(k,t)\, dk$ is dominated by low wave numbers (the energy-containing range) while the net heating rate $-\nu \int k^2\, E(k,t)\, dk$ is dominated by high wave numbers (the dissipation range). In highly developed turbulence these two limits are separated by the inertial range where the source and dissipation terms are locally negligible. However, equation (1) shows that the sources and dissipative sinks outside the inertial range provide boundary data on the energy flux at whatever values of $k$ are used to partition the spectrum into different wave number ranges. For driven turbulence a purely stationary solution of (1) exists where the heating rate equals the driving rate, and the Kolmogoroff analysis predicts an inertial-range energy spectrum proportional to $k^{-5/3}$. For freely decaying turbulence ($S = 0$), integrating (1) over all wave numbers shows that the total viscous heating rate is balanced by the time derivative of the total energy, which is essentially equal to the decay rate of the energy-containing eddies. The inertial range contains only a fraction of energy, and its main dynamic role is to mediate the passage of energy from low to high wave numbers.

### Homogeneous Kolmogoroff Theory

Fluctuations in the inertial range are characterized solely by their dimension (reciprocal of wave number $k$) and spectral energy $E(k)$. From these follow a characteristic velocity $u_k = [kE(k)]^{1/2}$ and timescale $\tau_k = (ku_k)^{-1}$ for decay due to nonlinear interactions.

Thus fluctuations at $k$ provide a flux of energy to higher wave numbers given by $\epsilon(k) \propto u_k^2/\tau_k = k^{3/5}E(k)^{3/2}$. The distinguishing feature of a steady inertial range is that the spectrum remains steady over times much longer than $\tau_k$ because energy is replenished by a cascade from larger fluctuations. This means that the energy flux $\epsilon$ is constant across the inertial range and not a function of $k$, which implies that $E(k) \propto k^{-5/3}$.

Naively, the above arguments can be used for the energy-containing eddies. However, as discussed in the introduction the lack of statistical equilibrium eliminates the justification for treating discrete shells of spectral energy individually. A less dubious approach is to group together the entire range of energy-containing eddies and describe them with a single length $\ell$ and flow speed $u$. In practice, these quantities are conveniently calculated as the longitudinal correlation length and the root-mean-square speed. The Kolmogoroff estimate for the turbulent decay rate of the energy-containing eddies is [*Batchelor*, 1953, chap. VII]

$$\frac{du^2}{dt} = -u^3/\ell + S, \tag{2}$$

where $S$ is a source (or driving) term that may supply energy to the turbulence. Unlike the inertial range, these eddies govern the decay of the turbulence and are not replenished by structures at lower $k$. This equation forms the starting point for many phenomenological treatments of the decay of hydrodynamic turbulence. It has also been employed by *Coleman* [1968] and *Hollweg* [1986] as an estimate for the decay of solar wind fluctuations.

The behavior of $\ell(t)$ is crucial, and one must make some assumption about its dynamics supported by observational information. One consistent possibility is that changes in $\ell$ are due solely to the value of $\ell$ and the eddy turnover time $\tau$, the simplest dimensionally correct expression of which is $d\ell/dt = \ell/\tau = u$. This equation together with (2) form a simple closed set that reproduces the physics of freely decaying turbulence to within order-unity constants. Wind tunnel experiments indicate that the turbulent energy behaves as $u^2 \propto (t - t_0)^{-1}$, where $t$ is the time and $t_0$ is a virtual origin in time. For consistency with (2), these observations require that $\ell \propto \sqrt{t - t_0}$, which has also been corroborated by experimental evidence [*Batchelor*, 1953]. To reproduce the observed temporal behavior the expression for $\ell$ must be adjusted by a factor of 2, so that $d\ell/dt = u/2$. For convenience, this factor is omitted in most subsequent discussions, but it will be present in the equations to be solved numerically.

For driven turbulence, energy is supplied to the large-scale eddies at rate $S$ and length scale $\ell_S$. A simple modification of the above equation for the evolution of $\ell$ is

$$d\ell/dt = u - (\ell_S S)^{1/3}. \tag{3}$$

This equation reduces to the standard decay phenomenology for $S = 0$, and when $S$ and $\ell_S$ are both nonzero a steady state solution exists with $\ell = \ell_S$ and $u^2 = (l_S S)^{2/3}$. Equations (2) and (3) with nonzero driving terms represent a simple and obvious extension to the purely decaying phenomenology described by *Batchelor* [1953]. This pair of equations is roughly equivalent to certain models of homogeneous turbulence used in engineering applications [see, e.g., *Bradshaw et al.*, 1981], as we shall discuss below.

## Inhomogeneous Turbulence Models

A standard method for constructing equations to describe the evolution of inhomogeneous turbulence is to decompose the flow into a mean and fluctuating part [*Reynolds*, 1883, 1894; *Tritton*, 1977; *Hinze*, 1975]. For incompressible hydrodynamics the procedure is to express the flow velocity as $\mathbf{U} = \mathbf{U}_0 + \mathbf{u}$, where the mean flow velocity $\mathbf{U}_0 = \langle \mathbf{U} \rangle$ (not necessarily uniform or constant in time) is determined by an appropriately defined ensemble averaging operation $\langle \ldots \rangle$. A similar method provides the basis for the mean field electrodynamic treatment of the MHD dynamo problem [*Steenbeck et al.*, 1966; *Krause and Rädler*, 1980; *Moffatt*, 1978] and for theories of turbulent transport coefficients [*Biskamp and Welter*, 1983; *Biskamp*, 1984; *Montgomery and Hatori*, 1984; *Montgomery and Chen*, 1984; *Chen and Montgomery*, 1987]. More complex closure schemes for inhomogeneous MHD turbulence modeling have been proposed by *Yoshizawa* [1985, 1988, 1990].

By applying the averaging operation to the incompressible Navier-Stokes equations, one computes the Reynolds equations for the time evolution of the mean velocity and the Reynolds stress tensor $\langle u_i u_j \rangle$, in Cartesian components. These equations have been well discussed in the literature [e.g., *Tritton*, 1977; *Bradshaw et al.*, 1981]. For illustration, we note here that they have the general structure

$$\frac{\partial U_{0i}}{\partial t} + (\ldots \text{terms involving } \mathbf{U}_0)\ldots = -\nabla_j \langle u_j u_i \rangle, \tag{4}$$

$$\begin{aligned}\frac{\partial \langle u_i u_j \rangle}{\partial t} &= \ldots + \langle u_i u_l \rangle \frac{\partial U_{0j}}{\partial x_l} \\ &- \nabla_k \langle u_i u_j u_k \rangle - 2\nu \langle \frac{\partial u_i}{\partial x_k} \frac{\partial u_j}{\partial x_k} \rangle.\end{aligned} \tag{5}$$

Only those terms relevant to the current discussion have been written explicitly. The spatial derivatives of the Reynolds stress tensor in (4) are of central importance in determining the feedback of the turbulence on the dynamics of the mean flow. The first term on the right side of (6) represents the influence of a shear in the mean flow back on the Reynolds stress. It is the only term that permits dynamical evolution of the mean flow, such as occurs by instabilities, to influence and supply energy to the turbulence. The second term, which involves the divergence of the triple correlation tensor, acts to eliminate inhomogeneities in the turbulence amplitudes, effectively spreading turbulence in space; it vanishes for strictly homogeneous turbulence. Finally, the third term describes viscous dissipation, which is strongly dominated by contributions from small-scale structures. In a broad band turbulence model, it is not usually possible to account for the full extent of the fluctuations in wave number, so the dissipation rate will need to be determined by other considerations, such as those described in the previous section.

Considerable attention in the hydrodynamics literature has been devoted to modeling the influence of the Reynolds stress. The simplest approach avoids the need to solve (6) at all by adopting an eddy viscosity closure of the kind attributed to *Prandtl* [1945], *Kolmogoroff* [1941], and *Bousinessq* [1877] (see also *Landahl and Mollo-Christensen*, [1986]). Specifically, one assumes that the components of Reynolds stress are proportional to the shear in the mean flow, the constant of proportionality being the eddy viscosity, which is modeled using a variety of different schemes. However, these models can be improved by using information derived from (6). For example, if the eddy viscosity is estimated as $u\lambda$, where $u \equiv \sqrt{\langle \mathbf{u}\cdot\mathbf{u}\rangle}$ and $\lambda$ is the integral or correlation scale of the turbulence, it can in principle be computed directly from solutions to (6). The correlation scale [*Batchelor*, 1953] is defined by $u^2\lambda = \int dr \langle \mathbf{u}\cdot\mathbf{u}'\rangle$, where $\mathbf{u}$ is evaluated at the position $\mathbf{x}$ and $\mathbf{u}'$ is evaluated at $\mathbf{x}' = \mathbf{x} + r\mathbf{a}$ for some offset direction specified by the unit vector $\mathbf{a}$. To compute the dynamics of the correlation scale, one first forms a differential equation for $\langle \mathbf{u}\mathbf{u}'\rangle$ that is similar in structure to (6), except that combinations of $u_i u_j'$ appear instead of simply $u_i u_j$. After integrating over $r$ and taking the trace, the result is an equation for the evolution of $u^2\lambda$. Having also solved for $u^2$, $\lambda$ is therefore determined, as is the eddy viscosity. This procedure is similar to the one we adopt in the following sections.

However, (6) is insufficient to close the set of dynamical equations because it involves the triple correlation $\langle u_i u_j u_k\rangle$ and the correlation of the derivatives $\langle \nabla_k u_i \nabla_k u_j\rangle$ of the velocity fluctuations. Neither of these quantities are determined by knowledge of the Reynolds stress itself. Therefore, in order to implement the above prescriptions for computing quantities like the turbulent energy $u^2$ and the eddy viscosity, one must consider methods for solving approximations to equation (6). Mathematically sophisticated approaches such as the direct interaction approximations [*Kraichnan*, 1959] are an active area of research [e.g., *Yoshizawa*, 1985]. However, there are many simpler and more practical prescriptions available for enforcing such approximations [*Bradshaw et al.*, 1981]. One of the most widely used approaches in the engineering community leads to what are known as $K$-$\epsilon$ models [*Jones and Launder*, 1972], which we discuss briefly to motivate our own developments for MHD in subsequent sections.

Models of the $K$-$\epsilon$ type describe the time evolution of the turbulent energy density $K \equiv u^2$ (density being normalized to unity) and the turbulent dissipation rate $\epsilon$. An example is the two-equation system

$$\frac{Du^2}{Dt} = \nu_T \left(\frac{\partial U_i}{\partial x_j} + \frac{\partial U_j}{\partial x_i}\right)\frac{\partial U_i}{\partial x_j} + \frac{\partial}{\partial x_j}\left(\frac{\nu_T}{\sigma_K}\frac{\partial u^2}{\partial x_j}\right) - \epsilon \tag{6}$$

$$\frac{D\epsilon}{Dt} = C_1 \nu_T \frac{\epsilon}{u^2}\left(\frac{\partial U_i}{\partial x_j} + \frac{\partial U_j}{\partial x_i}\right)\frac{\partial U_i}{\partial x_j} + \frac{\partial}{\partial x_j}\left(\frac{\nu_T}{\sigma_\epsilon}\frac{\partial \epsilon}{\partial x_j}\right) - C_2\frac{\epsilon^2}{u^2}, \tag{7}$$

where $\sigma_K$, $\sigma_\epsilon$, $C_1$, and $C_2$ are phenomenological order-unity constants that can be adjusted to fine tune the model and $D/Dt$ is the convective derivative with respect to the large-scale flow. The first terms on the right sides of (7) and (8) model the coupling of the Reynolds stress to the mean flow shear in a manner similar to the eddy viscosity closure of the Reynolds equations. Here eddy viscosity is modeled by using the simplest dimensionally correct expression $\nu_T = K^2/\epsilon$, which depends strictly on properties of the turbulence. The second term on the right side of each of these equations models the effects of the triple correlation term in (6), namely, the diffusion of turbulent energy and turbulent dissipation in space as induced by inhomogeneities in the turbulent intensities. The diffusion coefficient is assumed to be proportional to $\nu_T$. The final terms are phenomenological expressions employed to describe the effects of terms at higher order, such as to those that appear in the Reynolds equations. They are formulated from arguments similar to those used in the Kolmogoroff analysis, as well as consideration of dimensional analysis and examination of their influence on the structure of the equations.

We will not delve into solutions of the $K$-$\epsilon$ equations themselves in this paper, but their structure bears a relationship of some interest to the MHD model we develop here. If one drops all inhomogeneities from (7) and (8), the phenomenological model presented earlier is recovered. Changing variables from $K$ and $\epsilon$ to $u^2$ and $\ell$, the difference between the two models lies in the choice of the constant $C_2$. Pursuing this point, the phenomenological relations among $u$, $\epsilon$, and $\ell$ suggests developing other $K$-$\epsilon$ type models involving $u^2 = K$ and another variable. For example, in hydrodynamics a $K$-$\ell$ model has been proposed by *Shir* [1973], and a $K$-$K\ell$ has been developed by Rotta (see *Bradshaw et al.*, [1981]). Further discussion of models of this type is given by *Bradshaw et al.* [1981]. For technical reasons, the MHD model we will develop is of the $K$-$K\ell$ (or $u^2$-$u^2\ell$) type.

## 3. Extension to Homogeneous MHD Turbulence: Nonlinear Effects and Sources

A simple quasi-equilibrium model for the evolution of homogeneous MHD turbulence can be developed in analogy to the well-known hydrodynamic turbulence theory reviewed in section 2. This type of model is usually known as a one-point closure, referring to the use of information (correlation functions) at one point in time and space. Although there has been some discussion in the literature of one-point closure arguments for the evolution of cross helicity [e.g., *Dobrowolny et al.*, 1980b], most efforts in the development of MHD phenomenology have concentrated on more elaborate two-point closures [*Pouquet et al.*, 1976; *Grappin et al.*, 1982, 1983] that solve cumbersome spectral evolution equations. Simple one-point closure models may be less accurate than more involved theories, but the physics underlying the turbulence is modeled in a transparent way, while affording a description that can be readily applied to systems such as the solar wind. To our knowledge, a complete one-point phenomenology of homogeneous MHD turbulence has not been put forth previously. However, most of the required ingredients, such as estimates of timescales for spectral transfer and evaluation of approximate spectral fluxes, have been described elsewhere [*Kraichnan*, 1965; *Pouquet et al.*, 1976; *Dobrowolny et al.*, 1980b; *Grappin et al.*, 1982, 1983; *Matthaeus and Zhou*, 1989; *Mangeney et al.*, 1991]. Here we develop a phenomenology based on many of these previous ideas to describe the decay of MHD turbulence due to direct spectral transfer of energy to ever higher wave numbers. For the present, we ignore the important possibility that inverse spectral transfer to lower wave numbers may occur in MHD [*Frisch et al.*, 1975] especially when magnetic helicity is present.

It is convenient to write the MHD equations describing incompressible, homogeneous turbulence in the *Elsässer* [1950] representation, defining

$$\mathbf{z}^{\pm} \equiv \mathbf{v} \pm \frac{\mathbf{b}}{\sqrt{4\pi\rho}} = \mathbf{v} \pm \mathbf{v}_A, \tag{8}$$

where $\mathbf{v}$ is the velocity, $\mathbf{b}$ is the magnetic field, and $\mathbf{v}_A$ is the magnetic field in Alfvén speed units. The plasma density is assumed constant. The dynamical equations are then

$$\frac{\partial \mathbf{z}^{\pm}}{\partial t} \mp \mathbf{V}_A \cdot \nabla \mathbf{z}^{\pm} = -\frac{1}{\rho}\nabla p^T - \mathbf{z}^{\mp} \cdot \nabla \mathbf{z}^{\pm} \tag{9}$$

where $\mathbf{V}_A$ is the mean magnetic field $\langle \mathbf{B} \rangle$ in Alfvén speed units. Note that the mean velocity $\langle \mathbf{v} \rangle$ is assumed to be zero; this can always be accomplished for homogeneous conditions by a suitable Galilean transformation. Of particular relevance to the present discussion are estimates for the turbulent decay of the mean-square values of the Elsässer fields, $Z_+^2 \equiv \langle |\mathbf{z}^+|^2 \rangle$ and $Z_-^2 \equiv \langle |\mathbf{z}^-|^2 \rangle$, and of the mean value $D \equiv \langle \mathbf{z}^+ \cdot \mathbf{z}^- \rangle$, where $\langle \ldots \rangle$ denotes a volume average over a representative parcel of homogeneous turbulence. The kinetic and magnetic energies per unit mass of the fluctuations are $E_v = \langle |\mathbf{v}|^2 \rangle /2 \equiv u^2/2$ and $E_b = \langle |\mathbf{v}_A|^2 \rangle /2 \equiv v_A^2/2$, and the total turbulent energy per unit mass is $E = E_v + E_b$. The cross helicity is defined as $H_c = \langle \mathbf{v} \cdot \mathbf{v}_A \rangle /2$, and the difference between the kinetic and magnetic energies (i.e., the residual energy) is $E_D \equiv E_v - E_b = D/2$. Note that $Z_\pm^2 = 2(E \pm 2H_c)$. The quantities $Z_\pm^2$ and $D$ form a convenient set of variables to describe the amplitudes of (nonhelical) MHD turbulence. In addition, we will associate correlation lengths $\lambda_+$ and $\lambda_-$ with the $Z_+^2$ and $Z_-^2$ fluctuations, respectively. These lengths reflect the typical size of eddies in the energy-containing range.

As described by *Kraichnan* [1965], *Dobrowolny et al.* [1980b], and *Grappin et al.* [1982, 1983], the decay rates and spectral fluxes of $Z_\pm^2$ and $D$ may be estimated by examining the structure of (9). To do so, it is necessary to consider several characteristic timescales associated with nonlinear couplings among the turbulent MHD fields. First, the nonlinear timescale $\tau_{nl}^{\pm}$, analogous to the eddy turnover time in hydrodynamics, can be estimated as $\tau_{nl}^{\pm} = \lambda_\pm / Z_\mp$ [*Dobrowolny et al.*, 1980b]. A second important MHD timescale is associated with propagation of Alfvén waves. Packets of $\mathbf{z}^{\pm}$ tend to propagate in opposite directions relative to the local magnetic field direction, under the influence of the large-scale magnetic field. This propagation enhances the decay of triple correlations associated with spectral transfer [*Kraichnan*, 1965]. The contribution to this decorrelation due to the uniform component of the magnetic field, for each of the Elsässer fields, depends upon the corresponding timescales $\tau_A^{\pm} = \lambda_\pm / V_A$, where the uniform part of the Alfvén speed is denoted as $V_A$.

A third essential timescale is the characteristic lifetime of the triple correlations, which reflects the decay of the energy-containing eddies via the mechanism of spectral transfer [*Kraichnan*, 1965]. In ordinary hydrodynamics, this timescale is the same as the nonlinear timescale $\tau_{nl}^{\pm}$, but for MHD the triple lifetime may be governed by effects that propagate at the Alfvén speed [*Kraichnan*, 1965]. It is reasonable to assume that the triple correlations decay at a rate that is the sum of contributions due to both convection and Alfvén wave propagation. Using this argument, one estimates triple-correlation lifetimes $\tau_3^{\pm}$ for the energy-containing structures that satisfy $(\tau_3^{\pm})^{-1} = (\tau_{nl}^{\pm})^{-1} + (\tau_A^{\pm})^{-1}$ [*Kraichnan*, 1971; *Pouquet et al.*, 1976; *Matthaeus and Zhou*, 1989]. This expression reduces either to the usual hydrodynamic estimate or to *Kraichnan*'s [1965] estimate in the appropriate limit of the Alfvén speed.

A final important timescale is that associated with spectral transfer and therefore, in the quasi-equilibrium picture, with the turbulent decay of each MHD turbulence variable. In particular, the turbulent decay rate of $Z_\pm^2$ will be of the form $Z_\pm^2/\tau_s^\pm$, which defines the spectral transfer (or turbulent decay) time $\tau_s^\pm$ for the two Elsässer fields. In theories of spectral dynamics in MHD, rules relating the triple-correlation lifetime, the nonlinear timescale, and the spectral transfer rate have been proposed [e.g., *Dobrowolny et al.*, 1980b]. We adopt a more general form here [*Matthaeus and Zhou*, 1989; *Zhou and Matthaeus*, 1990b] that incorporates the composite triple-correlation lifetime described above. Adapting this rule to the energy-containing scales, one finds that $\tau_s^\pm = (\tau_{nl}^\pm)^2/\tau_3^\pm$.

The one-point estimates for the decay rate of the Elsässer fields can now be easily assembled. The decay rates of $Z_\pm^2$ are defined by

$$\epsilon_\pm = \frac{Z_\pm^2}{\tau_s^\pm}. \tag{10}$$

For purely decaying turbulence, one can then explicitly write an expression for the decay of the $Z_\pm^2$ amplitudes as

$$\frac{dZ_\pm^2}{dt} = -\frac{Z_\pm^2 Z_\mp^2}{\lambda_\pm(V_A + Z_\mp)}. \tag{11}$$

Equations to describe the time evolution of the correlation lengths $\lambda_\pm$ are also needed, and we can develop them in a manner analogous to the argument leading to (3). However, we choose not to write simply $d\lambda_\pm/dt = Z_\pm$, because this would lead to correlation lengths that continue to evolve when either one of the Elsässer fields vanishes. As is well known, all turbulent spectral transfer is halted in that case, and the surviving fluctuations continue to propagate as finite amplitude Alfvén waves [e.g., *Cowling*, 1976]. To resolve this difficulty, one can reinterpret the argument leading to (3) in the following way. Let $d\ell/dt$ depend on $\ell$ and $\epsilon = u^3/\ell$, rather than on $\ell$ and $u$. For freely decaying hydrodynamic turbulence, one arrives again at the result expressed in (3) that $d\ell/dt = (\ell\epsilon)^{1/3} = u$. In the same fashion we can compute $d\lambda_\pm/dt = (\lambda_\pm\epsilon_\pm)^{1/3}$, arriving at

$$\frac{d\lambda_\pm}{dt} = \left(\frac{Z_\pm^2\lambda_\pm}{\tau_s^\pm}\right)^{1/3} \tag{12}$$

as an acceptable description that gives steady values for both of the correlation lengths when either of the Elsässer fields vanish. We remind the reader that a factor of 1/2 must be inserted on the right of (12) for consistency with hydrodynamic experiments as discussed in section 2.

Next, we consider the evolution of the energy difference as represented by the variable $D = 2(E_v - E_b)$. Estimating its turbulent decay rate is not as straightforward as for the Elsässer fields, mainly because it is not an invariant of ideal MHD and therefore is not conserved by nonlinear interactions in either the energy-containing or the inertial range. However, on the basis of closure equations [*Pouquet et al.*, 1976] and other arguments [*Fyfe et al.*, 1977], one can argue that $E_D$ tends to relax toward zero. This conjectured tendency of MHD toward equipartition of energy between magnetic and kinetic disturbances is known as the Alfvén effect [*Kraichnan*, 1965; *Pouquet et al.*, 1976; *Fyfe et al.*, 1977]. However, this assumption of equipartition must be adopted with caution, because closures [*Pouquet et al.*, 1976, *Grappin et al.*, 1983], observations in the solar wind [*Matthaeus and Goldstein*, 1982a; *Roberts et al.*, 1987a, b], and direct simulations [*Matthaeus and Lamkin*, 1986; *Biskamp and Welter*, 1989] all suggest that the magnetic energy in the inertial range may be somewhat larger than the kinetic energy. (In addition, the very largest scale magnetic fluctuations may contain excess energy due to inverse cascade effects [*Frisch et al.*, 1975; *Fyfe and Montgomery*, 1976].) In fact, within the inertial range, $E_b$ is frequently of the order of $2E_v$, so including only fluctuations at the relevant scales, $E_D/E \approx -1/3$ [*Zhou and Matthaeus*, 1989, 1990a]. The physics of this phenomenon is not yet completely clear. For example, *Matthaeus and Lamkin* [1986] associated departures of inertial range equipartition with structures appearing in zones of magnetic reconnection, but closures [e.g., *Grappin et al.*, 1983] also show nonequipartition, although they presumably cannot include effects due to these coherent structures. Anticipating that a better understanding of departures from the Alfvén effect may be available in the future, we follow the development described by *Grappin et al.* [1983] and *Mangeney et al.* [1991] for assembling a simple phenomenology for the behavior of the energy difference.

First, the dynamical tendency for the energy difference to decay toward zero due to the Alfvén effect is modeled by a relaxation time $\tau_{AD} = \lambda_D/V_A$, where $\lambda_D$ is the correlation scale of $D$, in accord with the suggestion of *Pouquet et al.* [1976]. Next, the net effect of local wave number couplings upon the value of $D$ is estimated following the arguments of *Grappin et al.* [1983] for the inertial range. The essence of their reasoning is that $D$ is driven at a rate equal to the net spectral transfer rate of the total energy, which from (10) is given by $Z_+^2/\tau_s^+ + Z_-^2/\tau_s^-$. This amounts to asserting that as the $\mathbf{z}^\pm$ fields are carried by spectral transfer to higher wave number, so is their correlation $D = \langle \mathbf{z}^+ \cdot \mathbf{z}^- \rangle$. The sign of this coupling to $D$ is that associated with negative values of $D$ in the inertial range, in accord with simulations, closures, and observations [*Pouquet et al.*, 1976; *Mangeney et al.*, 1991], which leads to

$$\frac{dD}{dt} = -\frac{D}{\tau_{AD}} - \frac{Z_+^2}{\tau_s^+} - \frac{Z_-^2}{\tau_s^-}. \tag{13}$$

Unfortunately, despite the plausibility of these arguments, numerical trials using this equation yield solutions with unphysical characteristics. In particular, the kinematic constraint that the magnitude of $2D$ not exceed $Z_+^2 + Z_-^2$ is not consistently obeyed. However, the above expression is not determined uniquely by the arguments presented, and alternatives that are just as reasonable may be proffered. We attempt to retain the key features of these ideas by adopting the following expression to model the time development of $D$ due to homogeneous decay processes

$$\frac{dD}{dt} = -\frac{D}{\tau_{AD}} - \frac{u^2}{\tau^*}, \tag{14}$$

where $\tau^*$ is defined as the timescale for decay of the total energy, $\tau^* = (Z_+^2 + Z_-^2)/(\dot{Z}_+^2 + \dot{Z}_-^2)$. For the parameters used in the numerical solutions presented later this equation produces acceptable results.

Finally, to determine the relaxation timescale $\tau_{AD}$, the evolution of $\lambda_D$ is needed. In principle, $\lambda_D$ requires an additional dynamical equation. To keep the model as simple as possible, we assume that $\lambda_D$ is determined by the correlation scales $\lambda_\pm$ by combining them in a reasonable way and keep dynamical equations for these two length scales only. Rather than simply averaging the independent lengths, we weight them by the amount of the associated energy, and close the model using

$$\lambda_D = \frac{\lambda_+ Z_+^2 + \lambda_- Z_-^2}{Z_+^2 + Z_-^2} = \frac{L_+ + L_-}{Z_+^2 + Z_-^2}, \tag{15}$$

where we have introduced the energy-weighted lengths $L_\pm = Z_\pm^2 \lambda_\pm$. To complete our phenomenological model, we include the possibility of forcing terms that supply $Z_\pm^2$ and $D$ at rates $S_\pm$ and $S_D$, respectively. Also, the terms $S_\pm$ have associated injection length scales $l_\pm$. In view of the approximation regarding $\lambda_D$ introduced above, there is no need for a length scale for $S_D$. The forcing terms are incorporated into the equations for the amplitudes in analogy with the treatment of hydrodynamic forcing considered in section 2. That is, as $S_\pm$ drives $Z_\pm^2$ toward the steady values $Z_\pm^2 = S_\pm \tau_s^\pm$, we require that the correlation lengths $\lambda_\pm$ approach $l_\pm$, the scales characterizing the forcing terms.

The preceding arguments lead to a closed, one-point dynamical model for simultaneously decaying and driven quasi-equilibrium homogeneous MHD turbulence, including magnetic energy, kinetic energy, cross helicity, and two correlation lengths. Not all of the arguments, especially those dealing with the energy difference, are entirely satisfactory, and none of the development is mathematically rigorous. In addition, we have entirely neglected the role of the turbulent electric fields and the effect of helicities associated with the magnetic and velocity fields, as well as more exotic quantities such as the helicity of the electric field or the helicity of the cross helicity, both of which may in principle play a role in MHD turbulence [*Zhou and Matthaeus*, 1990a]. This approximation implies that we deal exclusively with the symmetric parts of the correlation tensors, which are then reduced to their traces. We assemble the above developments to arrive at the following set of equations:

$$\frac{dZ_\pm^2}{dt} = -\frac{Z_\pm^2}{\tau_s^\pm} + S_\pm \tag{16}$$

$$\frac{dD}{dt} = -\frac{D}{\tau_{AD}} - \frac{u^2}{\tau^*} + S_D \tag{17}$$

$$\frac{d\lambda_\pm}{dt} = (\lambda_\pm \epsilon_\pm)^{1/3} - (l_\pm S_\pm)^{1/3} \tag{18}$$

However approximate this model may be, we propose this set of five equations as a phenomenological description of the decay of MHD energy-containing eddies as a close analogy to the classic, hydrodynamic turbulence models reviewed in section 2.

## 4. Extension to MHD Turbulence: Inhomogeneous Effects

For inhomogeneous turbulence a decomposition of the MHD variables into mean and fluctuating parts is accomplished in the manner described for hydrodynamics in section 3. With a suitably defined averaging operation $\langle \ldots \rangle$, the mean values of the velocity, magnetic field, and density are respectively denoted $\mathbf{U}$, $\mathbf{B}_0$, and $\rho$, with corresponding fluctuating parts $\mathbf{v}$, $\mathbf{b}$, and $\delta\rho$. For example, the total velocity field is $\mathbf{U} + \mathbf{v}$. Using this decomposition, the equations for the evolution of the mean and fluctuating fields can be derived straightforwardly from the full equations for compressible MHD [e.g., *Whang*, 1980]. The result is the MHD analog of the Reynolds equations for hydrodynamic flow. The relevant terms in these equations can then be modeled according to various schemes [e.g., *Yoshizawa*, 1990] to give dynamical models for the reproducible mean fields, the turbulent fluctuations, and the interactions between these two components of the total flow.

For application to the solar wind, several additional approximations can be invoked at the onset. One is that length scales of the fluctuating fields are well separated from the characteristic distance over which the reproducible mean fields vary. This is a typical assumption for solar wind theories [e.g., *Zhou and Matthaeus*, 1990a, c], dating to the use of WKB theory [*Parker*, 1965] to describe propagation of noninteracting waves in the weakly inhomogeneous solar wind background plasma and fields. Scale separation amounts to expansion in a small parameter $\epsilon = \lambda/R$, where $\lambda$ is a correlation scale for the fluctuations, and $R$ typifies the scale for variations of the large-scale flow, density, and magnetic fields, which for the solar wind is well approximated by the heliocentric distance. Second, we invoke the approximation that the local turbulence in the solar wind is incompressible, or perhaps nearly incompressible [*Montgomery et al.*, 1987; *Matthaeus and Brown*, 1988; *Zank and Matthaeus*, 1990], so that the density fluctuation $\delta\rho$ can be safely neglected in a leading order theory. Scale separation gives rise to an ordering of spatial derivatives [*Zhou and Matthaeus*, 1990a], where the large-scale fields $\mathbf{U}$, $\mathbf{B}_0$, and $\rho$ depend only on the slowly varying, large-scale spatial coordinate, while the fluctuations depend on both the local fast-varying coordinate and the slow coordinate. Thus we have the convenient simplification that the correlation functions and spectra computed from the decomposed field equations are formally independent of the fast-varying coordinates that track the details of the local turbulent fluctuations.

For our phenomenological treatment of the energy-containing eddies we do not require equations for the evolution of the full correlation functions but only for the magnetic and kinetic energy densities and the cross helicity content. This is equivalent to specifying equations for the transport of energy-containing turbulent eddies interacting with a weakly inhomogeneous background solar wind plasma in terms of (1) the Elsässer "energies" of the fluctuating part of the flow, (2) the residual energy, and (3) their correlation scales. The most direct way to obtain the required equations is to start with the evolution equations for the correlation tensors of the Elsässer variables, take their traces, and set the spatial separation variable equal to zero. The relevant expressions are given by *Zhou and Matthaeus* [1990a] as equations (49) and (59).

It is convenient to express the mean magnetic field in terms of the mean Alfvén speed as $\mathbf{V}_A = \mathbf{B}_0/\sqrt{4\pi\rho}$. The symbols $\mathbf{z}^{\pm}$ now refer strictly to the fluctuating components of the Elsässer fields, $\mathbf{z}^{\pm} = \mathbf{v} \pm \mathbf{b}/\sqrt{4\pi\rho}$, and $Z_{\pm}^2$ and $D$ are twice the Elsässer energies and residual energy of those components, i.e., $Z_{\pm}^2 = \langle|\mathbf{z}^{\pm}|^2\rangle$ and $D = \langle\mathbf{z}^{+} \cdot \mathbf{z}^{-}\rangle$. Likewise, $\lambda_{\pm}$ and $\lambda_D$ are the correlation scales of $Z_{\pm}^2$ and $D$. To facilitate manipulation, we note the following relations between variables used by *Zhou and Matthaeus* [1990a] and those used in the present paper: $H_{ii}^{\pm}(r = 0) = Z_{\pm}^2$ and $\Lambda_{ii}(r = 0) = \tilde{\Lambda}_{ii}(r = 0) = D$.

To close the set of equations for $Z_{\pm}^2$, $D$, $\lambda_{\pm}$, and $\lambda_D$ describing the energy-containing MHD eddies, we again assume that the helicities of the magnetic field and velocity field vanish, as does the induced MHD electric field. Because the full transport equations involve tensor quantities, some assumption is needed about the angular distribution of the fluctuations to reduce the equations to a managable form. We assume that the turbulence is isotropic in either three or two dimensions. In the first case (3D), the directions of the turbulent fluctuations $\mathbf{v}$ and $\mathbf{b}$ are isotropically in three-dimensions, and all correlation tensors built from them are invariant with regard to rotation. For two-dimensional (2D) turbulence, the fluctuations are restricted to the plane normal to $\mathbf{B}_0$ but uniformly distributed in azimuth about it [*Fyfe and Montgomery*, 1976]. These assumptions and their implications are discussed further in the final section. After some simple manipulations of the correlation function equations referred to above, we find

$$\frac{\partial Z_{\pm}^2}{\partial t} + (\mathbf{U} \mp \mathbf{V}_A) \cdot \nabla Z_{\pm}^2 + Z_{\pm}^2 \nabla \cdot \left(\frac{\mathbf{U}}{2} \pm \mathbf{V}_A\right) + M^{\pm} D = N_E^{\pm} \tag{19}$$

$$\frac{\partial D}{\partial t} + \mathbf{U} \cdot \nabla D + D \nabla \cdot \frac{\mathbf{U}}{2} - \frac{1}{2}\left[M^{-} Z_{+}^2 - M^{+} Z_{-}^2\right] = N_D. \tag{20}$$

The terms on the right represent the nonlinear interactions that are responsible for spectral transfer. These will be modeled following the phenomenological treatment of the last section. The symbols $M^{\pm}$ are the so-called mixing terms, which depend upon the assumed symmetry of the small-scale turbulence. In particular, they depend on the form of the symmetric part of the energy-difference correlation tensor $R_{ij}^{Ds}(r = 0)$ [cf. *Zhou and Matthaeus*, 1990a]. If the turbulence is three dimensional, then the tensor is simply a multiple of the unit tensor $\mathbf{1}$, while for 2D turbulence in the plane perpendicular to $\mathbf{B}$, it is proportional to $\mathbf{1} - \hat{\mathbf{b}}\hat{\mathbf{b}}$. The explicit forms are

$$M_{3D}^{\pm} = \nabla \cdot \frac{\mathbf{U}}{6} \mp \nabla \cdot \mathbf{V}_A \tag{21}$$

$$M_{2D}^{\pm} = \nabla \cdot \left(\frac{\mathbf{U}}{2} \mp \mathbf{V}_A\right) - \hat{\mathbf{b}}\hat{\mathbf{b}} : \left(\nabla\mathbf{U} \pm \frac{\nabla\mathbf{B}_o}{\sqrt{4\pi\rho}}\right) \tag{22}$$

Note that in deriving (21), a correlation tensor that depends upon the induced electric field has been neglected, as it vanishes when all spectral and correlation functions are isotropic or 2D. In a broader context, this term would have enforced oscillations in the dynamical behavior of $D$ with a frequency proportional to $V_A$. On average, the result is the reduction of the energy difference and is, in fact, the contribution of the uniform magnetic field to the Alfvén effect. This phenomenon has already been incorporated into the present model, in the phenomenological treatment of the energy difference discussed in the previous section.

Rather than following the correlation lengths $\lambda_{\pm}$ and $\lambda_D$ directly, we consider the weighted quantities $L_{\pm} \equiv Z_{\pm}^2\lambda_{\pm}$ and $L_D \equiv D\lambda_D$. Equations describing their evolution are obtained by integrating the contraction of (49) of *Zhou and Matthaeus* [1990a] over the spatial separation variable in the definition of the Elsässer correlation functions, using the definitions

$$\lambda_{\pm} = \frac{\int dr\langle\mathbf{z}^{\pm} \cdot \mathbf{z}^{\pm\prime}\rangle}{\langle|\mathbf{z}^{\pm}|^2\rangle}, \tag{23}$$

$$\lambda_D = \frac{\int dr\langle\mathbf{z}^{\pm} \cdot \mathbf{z}^{\mp\prime}\rangle}{\langle\mathbf{z}^{\pm} \cdot \mathbf{z}^{\mp}\rangle}. \tag{24}$$

The primed and unprimed quantities are evaluated at spatial positions separated by distance $r$. The direction of integration is immaterial for 3D turbulence, and it is simply required to be orthogonal to $\mathbf{B}_0$ for 2D turbulence. The result for $L_{\pm}$ is

$$\frac{\partial L_{\pm}}{\partial t} + (\mathbf{U} \mp \mathbf{V}_A) \cdot \nabla L_{\pm} + L_{\pm} \nabla \cdot \left(\frac{\mathbf{U}}{2} \pm \mathbf{V}_A\right) + \int \Pi_{ii}^{\pm}(r) dr = N_{\lambda}^{\pm}, \tag{25}$$

and these equations contain the required dynamical information concerning the correlation scales $\lambda_\pm$. Following (51), (52), and (43) of *Zhou and Matthaeus* [1990a], we define

$$\begin{aligned}\Pi^{\pm}_{ii}(r) &= -\nabla\cdot\left(\frac{\mathbf{U}}{2}\pm\mathbf{V}_A\right)R^{Ds}_{ii}(r) \\ &+ \left(\frac{\partial U_i}{\partial x_j}\pm\frac{1}{\sqrt{4\pi\rho}}\frac{\partial B_{0i}}{\partial x_j}\right)2R^{Ds}_{ij}(r).\end{aligned} \quad (26)$$

Integration is over the local, small-scale separation variable $r$, so large-scale variables and their derivatives are treated as constant. Because the turbulence is locally isotropic, there is a constraint among the components of the symmetric part of any solenoidal tensor [*Orszag*, 1977], which $R^{Ds}_{ij}$ satisfies for locally incompressible flow. This leads to the convenient expressions

$$\int_0^\infty R^{Ds}_{ij}dr = L_D\frac{(\delta_{ij}+\hat{r}_i\hat{r}_j)}{4}, \quad (27)$$

for fully isotropic turbulence, and

$$\int_0^\infty R^{Ds}_{ij}dr = L_D\hat{r}_i\hat{r}_j, \quad (28)$$

for 2D turbulence distributed uniformly normal to the mean magnetic field (cf. equations (21) and (22)). The integration is carried out along the unit vector $\hat{\mathbf{r}}$, with $\hat{\mathbf{r}}\cdot\mathbf{B}=0$ in the 2D case, and $\delta_{ij}$ is the identity tensor. When these expressions are incorporated above, the evolution of either $L_\pm$ is explicitly coupled to $L_D$ and to the mean flow. The presence of $\hat{\mathbf{r}}$ in the evolution equations reflects the formal inconsistency of treating the turbulence as locally isotropic though the large-scale flow is nonisotropic. Evidently, the coupling of the small-scale structure to the large-scale flow depends upon whether the fluctuations are along or normal to the mean flow. Because the present examination has been developed with the assumption of isotropic turbulence, we defer consideration of this interesting point to a later paper.

In a similar fashion, the weighted correlation length for the energy difference $L_D$ is governed by integrating the contraction of (59) of *Zhou and Matthaeus* [1990a].

$$\begin{aligned}&\frac{\partial L_D}{\partial t}+\mathbf{U}\cdot\nabla L_D-2\hat{\mathbf{r}}\cdot\mathbf{V}_A D+\frac{L_D}{2}(\nabla\cdot\mathbf{U}) \\ &-\frac{(L_++L_-)}{4}(\nabla\cdot\mathbf{U})+\frac{(L_+-L_-)}{2}(\nabla\cdot\mathbf{V}_A) \\ &+\left[\nabla_iU_j-\frac{1}{\sqrt{4\pi\rho}}\nabla_iB_{oj}\right]\int H^+_{ji}(r)dr \\ &+\left[\nabla_iU_j+\frac{1}{\sqrt{4\pi\rho}}\nabla_iB_{oj}\right]\int H^-_{ji}(r)dr = N^D_\lambda.\end{aligned} \quad (29)$$

The $r$ integrations of $H^{\pm}_{ji}\equiv\langle z^\pm_i z^{\pm\prime}_j\rangle$ yield terms analogous to (27) and (28) with $L_D$ replaced by $L_\pm$.

## 5. Solar Wind Solutions

In the governing equations for inhomogeneous MHD turbulence derived in the last section, the nonlinear terms due to triple correlations among the fluctuating fields are grouped on the right sides of the equations and denoted symbolically by $N^\pm_E$, $N_D$, and $N^\pm_\lambda$. The models for homogeneous MHD turbulence given by (16), (17), and (18) in section 4 provide simple approximations to these nonlinear terms and permit us to close the equations. Substituting these expressions into (20), (21), (26), and (29) we arrive at the principal goal of this paper: a closed six-equation model for the evolution and transport of locally homogeneous, nearly incompressible MHD turbulence in externally specified, weakly inhomogeneous, large-scale background fields. The dependent variables are $Z^2_\pm$ and $D$, which differ by a factor of 2 from the mean Elsässer energies per unit mass $E^\pm=\langle|\mathbf{z}|^2\rangle/2$, and the residual energy (or energy difference) $E_D=\langle\mathbf{z}_+\cdot\mathbf{z}_-\rangle/2$, along with the weighted correlation scales $L_\pm=Z^2_\pm\lambda_\pm$ and $L_D=D\lambda_D$. The possibility of forcing is accounted for by source strengths $S_\pm$, which drive the $\pm$ fields at scales $\ell_\pm$, while the $D$ field may also be driven by the source $S_D$. Note that a separate homogeneous phenomenology was not developed for $L_D$, so that the nonlinear terms in (29), are the average of the $L_\pm$ terms.

For completeness, recall that the spectral transfer times can be written, according to the approximations leading to (10) and (11), as

$$\tau^\pm_s=\lambda_\pm\frac{V_A+Z_\mp}{Z^2_\mp}=L_\pm\frac{V_A+Z_\mp}{Z^2_+Z^2_-}. \quad (30)$$

One should note, however, that analytical estimates of $\tau^\pm_s$ might also be obtained from more elaborate theoretical treatments than the one described above. For example, an MHD eddy-viscosity theory could be developed using Heisenberg theory, renormalization methods [*Zhou et al.*, 1988, 1989; *Zhou and Vahala*, 1990], or MHD direct interaction approximations [e.g., *Yoshizawa*, 1988, 1990]. It would be straightforward to incorporate such results by using the appropriate expressions rather than (30). We also recall that, with the approximations cited in section 4, the Alfvénic relaxation time $\tau_{AD}$ of the energy difference is

$$\begin{aligned}\tau_{AD} &= \frac{\lambda_D}{\sqrt{V^2_A+v^2_A}} \\ &= \frac{L_++L_-}{(Z^2_++Z^2_-)\sqrt{V^2_A+(Z^2_++Z^2_--2D)/4}}\end{aligned} \quad (31)$$

Likewise, if an improved treatment of the decay of the energy difference becomes available, a more precise relaxation time could be used in place of this equation.

The model includes as large-scale parameters the velocity field $\mathbf{U}$, magnetic field $\mathbf{B}_0$, and density $\rho$, which must be input from solutions to the mean field equations for the solar wind. We adopt a fairly simply model in which the mean flow is assumed to vary only with radial distance, so the problem becomes essentially one-dimensional in the large-scale parameters. The bulk velocity $\mathbf{U} = U\hat{\mathbf{R}}$ represents spherically symmetric outflow at constant speed, while the mean magnetic field $\mathbf{B}_0$ is given by an outwardly directed Parker spiral. Density is constrained by continuity and the constant outflow speed to obey $\rho = \rho_0(R_0/R)^2$. Under these approximations the governing equations for $Z_\pm^2$ and $D$ may be written

$$\frac{\partial Z_\pm^2}{\partial t} + (U \mp V_{AR})\frac{\partial Z_\pm^2}{\partial R} + \frac{(U \pm V_{AR})}{R} Z_\pm^2 + M^\pm D = -\frac{Z_\pm^2}{\tau_s^\pm} + S_\pm, \tag{32}$$

$$\frac{\partial D}{\partial t} + U\frac{\partial D}{\partial R} + \frac{U}{R}D + \frac{1}{2}\left[Z_+^2 M^- + Z_-^2 M^+\right] = -\frac{D}{\tau_{AD}} - \frac{u^2}{\tau^*} + S_D, \tag{33}$$

where $V_{AR} = V_{AR0}(R_0/R)$ is the radial component of the large-scale Alfvén velocity. The mixing operators $M^\pm$ are now given by either

$$M_{3D}^\pm = \frac{1}{R}\left[\frac{U}{3} \mp V_{AR}\right] \tag{34}$$

or

$$M_{2D}^\pm = \frac{1}{R}\Big[U\cos^2\psi \pm V_{AR}(3\cos^2\psi - 2) \pm V_{AR0}\left(\frac{\Omega R_0}{U}\right)\left(1 - \frac{3R_0}{2R}\right)\sin\theta\sin 2\psi\Big] \tag{35}$$

and reflect the strength of the couplings among the small-scale fields. Note that they are entirely determined by the gradients of the large-scale fields. These mixing operators play the same role as the corresponding operators in our previous work on the scale-separated theory of the transport of inertial-range fluctuations in the solar wind [*Zhou and Matthaeus*, 1989, 1990a; *Matthaeus et al.*, 1992; *Oughton and Matthaeus*, 1992]. As in that case, they can be interpreted as representing the scattering of $\mathbf{z}^\pm$ fluctuations off large-scale gradients and inhomogeneities in the slowly varying mean fields. Because the $M^\pm$ operators are inversely proportional to $R$, their influence will be most important in the inner heliosphere (physically, the large-scale gradients of the mean fields are shallower in the outer heliosphere). We will have more to say below regarding the mixing operators.

The evolution equations for the correlation lengths have proven to be more problematic. The same reductions taken above yielded numerical solutions with unphysical features not present in the analytic formulation. Further work is proceeding on this matter, but in light of these difficulties, we now reduce the model complexity further by positing a single fundamental length scale $\lambda$ defined by

$$\lambda \equiv \frac{L_+ + L_-}{Z_+^2 + Z_-^2} = \frac{L}{E}, \tag{36}$$

where $L \equiv L_+ + L_-$ and $E \equiv Z_+^2 + Z_-^2$. The nonlinear evolution of $\lambda$ is presumed to be governed by $d\lambda/dt = \sqrt{Z_+ Z_-}$, which vanishes if either Elsässer variable vanishes, in accord with the discussion of section 4. Furthermore, the same length scale is used for the energy difference, so that $L_D \equiv \lambda D$ (note that this choice corresponds precisely to the definition of $\lambda_D$ in (15)).

To arrive at an equation for the evolution of $L$ in the 3D isotropic geometry we add equations (26). With large-scale parameters appropriate to the solar wind (as above) we arrive at the following equation

$$\frac{\partial L}{\partial t} + U\frac{\partial L}{\partial R} + \frac{U}{R}L = \lambda\left(\frac{Z_+^2}{\tau_s^+} + \frac{Z_-^2}{\tau_s^-}\right) + \left(Z_+^2 + Z_-^2\right)\sqrt{Z_+ Z_-}, \tag{37}$$

where $\hat{\mathbf{r}}$ has been chosen to be radially outward. In view of (27) the choice of $\hat{\mathbf{r}}$ means that $L_D$ does not appear in (38) because $\hat{\mathbf{r}} \cdot \nabla\mathbf{U} = 0$.

For 2D turbulence, $\hat{\mathbf{r}}$ must be normal to $\mathbf{B}$ so that the option chosen in the 3D case is no longer valid; we therefore take $\hat{\mathbf{r}}$ perpendicular to both $\mathbf{B}$ and $\mathbf{R}$, which does not change direction with $R$. Using $L_D = \lambda D$ and (28), this yields the following expression for 2D turbulence:

$$\frac{\partial L}{\partial t} + U\frac{\partial L}{\partial R} + \frac{U}{R}L\left(1 + \frac{2D}{E}\right) = \lambda\left(\frac{Z_+^2}{\tau_s^+} + \frac{Z_-^2}{\tau_s^-}\right) + \left(Z_+^2 + Z_-^2\right)\sqrt{Z_+ Z_-}. \tag{38}$$

Note that in this simplified, single length scale formulation, the large-scale Alfvén speed does not appear in the spatial transport terms of the $L$ equation. This is true for both the 2D and 3D isotropic turbulence cases

Equations (33), (34), and either (38) or (39) therefore constitute a reduced, four-equation version of our model. Given the complexity of the dynamical equations in the present model, it seems unlikely that analytic solutions will be forthcoming for the general case. To elucidate the physics of the model, we now discuss simplified versions of the steady-state equation for which analytical or numerical solutions have been obtained.

We first construct an analytic solution that can be used as a baseline for comparison with and test of numerical solutions. Because of the complexity of the mixing operators for 2D turbulence (35), analytic consideration will be restricted to the fully 3D case. All forcing terms and nonlinear interactions are neglected, and terms of $\mathcal{O}(V_{AR})$ are neglected relative to those of $\mathcal{O}(U)$. These approximations reduce the model to the following set of linear equations:

$$\frac{dZ_\pm^2}{dR} + \frac{Z_\pm^2}{R} + \frac{D}{3R} = 0, \quad (39)$$

$$\frac{dD}{dR} + \frac{D}{R} + \frac{Z_+^2 + Z_-^2}{6R} = 0, \quad (40)$$

$$\frac{dL_\pm}{dR} + \frac{L_\pm}{R} = 0. \quad (41)$$

Note that the (constant) flow velocity has scaled out of the equations completely and the $L_\pm$ are decoupled from the other fields. Equation (41) may be solved immediately, showing that $L_\pm$ decrease as $1/R$. Using the relations $Z_+^2+Z_-^2 = 4E$, $Z_+^2-Z_-^2 = 8H_c$, and $D = 2E_D$, the solutions for the energy and energy difference are

$$2E = \frac{E_0 + E_{D0}}{(R/R_0)^{4/3}} + \frac{E_0 - E_{D0}}{(R/R_0)^{2/3}} \quad (42)$$

$$2E = \frac{E_0 + E_{D0}}{(R/R_0)^{4/3}} - \frac{E_0 - E_{D0}}{(R/R_0)^{2/3}} \quad (43)$$

where a zero subscript indicates that the quantity is evaluated at $R = R_0$, which we choose to be the Alfvén critical radius. If desired, these solutions may be combined to yield explicit forms for $Z_\pm^2$.

The solution for the radial variation of the normalized bulk cross helicity for the energy-containing range is

$$\begin{aligned}\sigma_c &= \frac{2H_c}{E} \quad (44)\\ &= 2H_{c0}\left[\frac{E_0 + E_{D0}}{(R/R_0)^{1/3}} + \frac{E_0 - E_{D0}}{(R_0/R)^{1/3}}\right]^{-1}\end{aligned}$$

To evaluate this expression, we make the traditional assumption that only one mode exists at the critical point and set $Z_{+0}^2 = 0 = E_{D0}$. This assumption comes from the WKB formalism where sunward propogating disturbances are excluded at the critical point, although *Hollweg* [1990] has shown that it is not required by a general wave treament. It is not clear that a turbulence model should obey the same restriction, but we adopt it to allow comparisons with the earlier work. The normalized cross helicity then takes the form

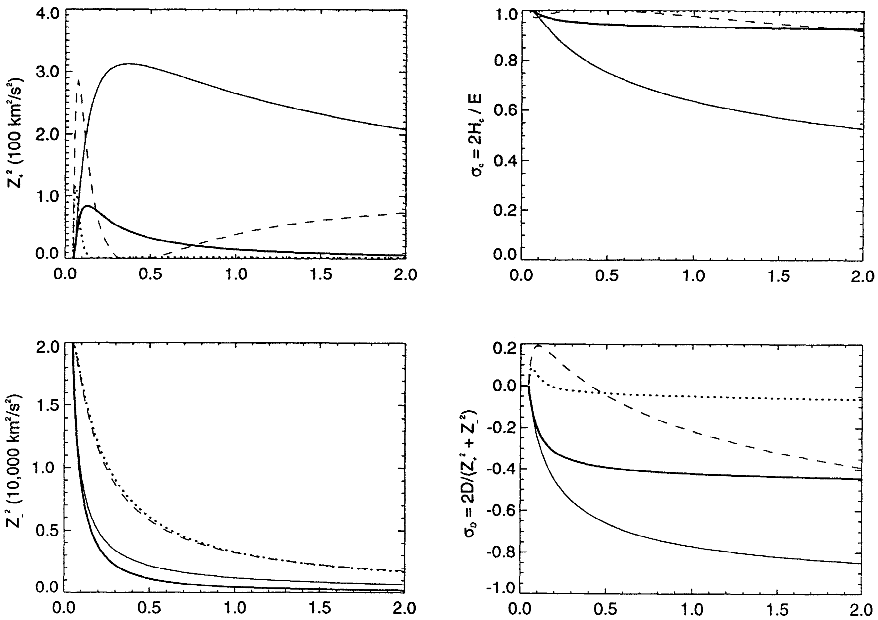
$$\frac{\sigma_c}{\sigma_{c0}} = \left[\left(\frac{R}{R_0}\right)^{1/3} + \left(\frac{R_0}{R}\right)^{1/3}\right]^{-1}, \quad (45)$$

which approaches zero as $R$ tends to infinity. The monotonic decay of $\sigma_c$ with increasing heliocentric distance demonstrates how an initial dominance of one mode of the energy-containing fluctuations can be destroyed via mixing effects in the absence of both a large-scale Alfvén velocity and nonlinear effects. Observational studies [e.g., *Roberts et al.*, 1987a] have shown that low values of $\sigma_c$ are the norm in the outer heliosphere, where $V_{AR}/U$ is indeed a small parameter. We shall have more to say about this in our discussion of other models of solar wind transport. However, for zero mean magnetic field the linear coupling between velocity and magnetic fluctuations vanishes, a point that is obscured by relying solely on the Elsässer representation [*Tu and Marsch*, 1993].

Having illustrated properties of a simplified, analytic version of our model, we turn to numerical solutions of the coupled set comprising equations (33), (34), and either (38) or (39). The four-equation model is solved using a pseudospectral method where the radial dependence of the fields is expanded in terms of Chebychev polynomials [*Gottlieb and Orszag*, 1977; *Canuto et al.*, 1988]. All source terms are set to zero, and the equations are integrated forward in time until a steady state solution is achieved. Boundary conditions for quantities that are nonzero at the critical point are chosen to be consistent with those currently deemed appropriate for the solar wind. For the present formulation the values of $Z_+^2(R_0)$ and $L_+(R_0)$ should be zero; in practice, they are given small positive values to avoid certain numerical difficulties. Note that because the decay terms are themselves nonlinear functions of $Z_\pm^2$, $D$, and $L_\pm$, the actual values of the boundary conditions are relevant, rather than simply their ratios. Further information concerning the numerical methods employed has been provided by *Oughton* [1993].

Figure 1 displays the results of four numerical solutions for fully isotropic, 3D turbulence. The solid (broken) curves represent cases where the nonlinear terms are absent (present), while solutions with thin or dashed (thick or dotted) curves neglect (include) a mean magnetic field. The thin unbroken line therefore treats the situation considered analytically, and the code does indeed reproduce the expressions given above. Those results will be included in all the plots shown to aid comparisons among runs with different parameters. No sources are included, and all other parameters are identical so that any differing results will be clearly attributable to the altered terms. The constant solar wind speed is set at 400 km/s and $R_0$ is 10 solar radii, which together determine the Alfvén speed of the mean magnetic field (when included). Again, because of the nonlinearities, significantly different solutions may occur when boundary conditions are changed.

For the boundary conditions chosen the nonlinear terms have very little effect on the radial variation of the intensity of the outward fluctuation as represented by $Z_-^2$. However, the inward fluctuations $Z_+^2$ exhibit a much more dramatic response. Examination of the modeled nonlinear terms (11) shows that the relative importance of spectral transfer on one of the Elsässer fields is roughly determined by the magnitude of the other [*Dobrowolny et al.*, 1980b]. The apparently unequal dependence on the nonlinear terms is therefore consistent with the domination of outward fluctuations over inward ones throughout the range consid-



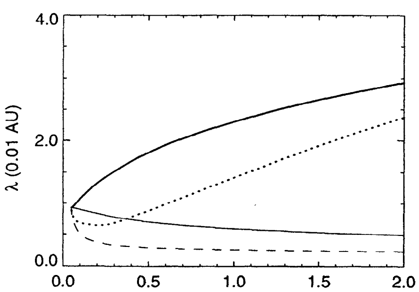




**Figure 1.** Results of four numerical solutions for fully isotropic, 3D turbulence for various versions of the four-equation model: linear, $\mathbf{B}_0 = 0$ (thin); linear, $\mathbf{B}_0 \neq 0$ (thick); nonlinear, $\mathbf{B}_0 = 0$ (dashed); nonlinear, $\mathbf{B}_0 \neq 0$ (dotted). For comparison, the thin curve also reproduces the analytic solutions in the text.

ered, which is itself a consequence of the adopted inner boundary conditions. Furthermore, the predominant suppression of the less energetic fluctuations manifests as a higher normalized cross helicity, which implies that mixing is not as strong as in the linear case. This phenomenon occurs whether the mean field is present or not.

Comparing cases where the mean magnetic field is present to those where it is absent, the initial growth rate of $Z_+^2$ is enhanced, and an additional minimum is present in the inner heliosphere. Furthermore, the precipitous initial drop of $Z_-^2$ from its value at $R = R_0$ is attenuated, though the shapes of the curves are qualitatively similar. The mean magnetic field causes these differences via its influence on the mixing operators (34). In particular, when the Alfvén velocity given by the radial component of $\mathbf{B}_0$ falls to one third the outflow speed, the sign of $M^+$ changes from negative to positive. The result is that the sense of some of the couplings between the $Z_\pm^2$ Elsässer fields and the normalized energy difference $D$ are reversed. With the mean outflow adopted here, this changeover occurs at $R = 3R_0$ (about 0.13 AU). In contrast, $M^-$ is always positive, but the presence of the mean magnetic field enhances it. Keeping in mind that $U - V_{AR}$ is always positive for $R > R_0$, equation (33) shows that $Z_+^2$ must decrease with distance when $M^+$ and $D$ are of the same sign. The energy difference $D$ is zero at the inner boundary, but the linear terms in (34) imply that its radial gradient is positive. Because $M^+$ is initially negative a positive $D$ near $R_0$ is consistent with the fact that a zero but manifestly non-negative quantity such as $Z_+^2$ can only grow. So long as $M^+$ remains negative, the mixing effects on $D$ due to the two $Z_\pm^2$ fields are of opposite sense; their net influence together with expansion and spectral transfer eventually forces $D$ to reach a maximum and then decrease through zero. In the region where $M^+$ and $D$ are of like sign, $Z_+^2$ must decrease, but it can increase once they have both changed sign, depending on the strength of the other terms. Note that once $D$ and $M^+$ both change sign, setting $D$ to zero in (34) would require that $\partial D/\partial R$ be negative, so $D$ cannot return to positive values. These solutions demonstrate the richness of possible behavior contained in the governing equations, particularly with regard to the structure and influence of the mixing terms.

Two-dimensional solutions were also obtained using (35) for the mixing operator. In these cases the mean magnetic field was always included for logical consistency. For the full solution including the nonlinear terms the behavior of $Z_-^2$ is qualitatively quite similar to that in the 3D cases, while the initial rise and subsequent fall of $Z_+^2$ are even more dramatic. The more involved form of the mixing operator actually leads to a monotonic decrease of $Z_+^2$ outward of its single maximum, and the changes in slope found in the fully isotropic cases are absent. This is a consequence of the single-signed nature of $M^+$.

Figure 2 plots the normalized cross helicities and energy differences for the four cases that include a mean magnetic field together with the analytic solution (dotted curve) for comparison. The two 3D cases (curves having both dots and dashes) have nearly maximal cross helicities, denoting a relatively high degree of Alfvénicity. Likewise, their normalized energy differences reveal greater equipartition than the 2D solutions. This is particularly true for the fully nonlinear solution (dash-dot) which differs from zero by less than 0.1 over the entire range considered. The linear 3D solution for $\sigma_D$ (dash-dot-dot-dot) declines steadily and most closed replicates the value observed at 1 AU of about −0.3. In contrast, the 2D linear solutions (dashed) rapidly saturate at extreme values that differ very strongly from anything observed. The corresponding solutions including the nonlinear terms (solid) display some interesting features near the inner boundary that are probably quite sensitive to boundary conditions and model details; however, they quickly settle down into gradual variation near fairly plausible values.

The solutions displayed in Figure 2 suggest that a fairly wide range of behavior can be produced by altering input parameters. Obviously, both physical insight

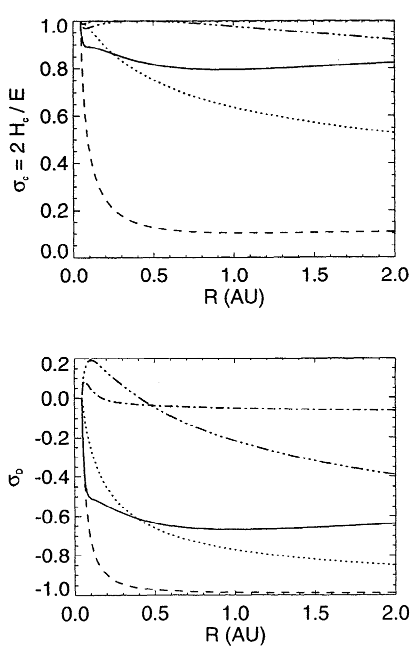



**Figure 2.** Normalized cross helicities and energy differences for the four cases that include a mean magnetic field ($\mathbf{B}_0 \neq 0$) together with the analytic solution (dotted curve) for comparison: 3D linear (dash-dot-dot-dot); 3D nonlinear (dash-dot); 2D linear (dashed); 2D nonlinear (solid).

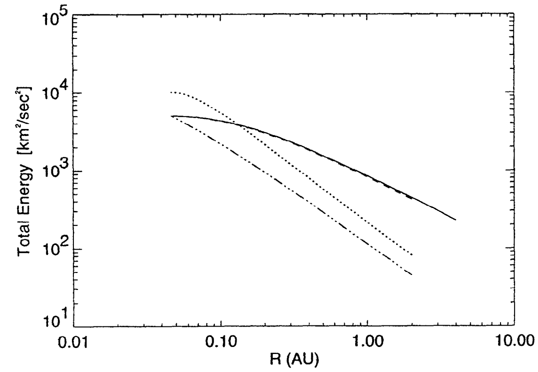



**Figure 3.** Variation of total energy $(Z_+^2 + Z_-^2)/4$ with heliocentric distance for various versions of the present model with the nonlinear terms included: 3D $\mathbf{B}_0 \neq 0$ (dashed); 3D $\mathbf{B}_0 = 0$ (dash-dot-dot-dot); 2D $\mathbf{B}_0 \neq 0$ (dotted). The WKB prediction is also include for comparison (solid).

and observational data must be employed to provide meaningful constraints on the model. However, given the complexity of the governing equations, it is not easy to predict in advance how a particular adjustment of the model will affect a relevant variable without actually carrying out the full numerical solution. We are currently carrying out additional numerical trials of the code to gain experience with its formal properties so that more physically accurate results can be obtained. However, the present results show quite clearly that mixing effects are stronger for 2D turbulence relative to 3D solutions, as indicated by reduced cross helicity and more negative energy difference. Furthermore, mixing is generally enhanced by suppressing either the nonlinearities or effects due to the mean magnetic field.

Finally, Figure 3 plots another important feature of the solutions, the variation of total energy $(Z_+^2 + Z_-^2)/4$ with increasing heliocentric distance for a number of cases. The solution for 3D turbulence with the mean field included (dashed) is practically indistinguishable from the WKB prediction (solid). The 3D case with no magnetic field (dash-dot-dot-dot) declines much faster and is lower by almost an order of magnitude by 1 AU. The steepest decline of total energy arises in the solution for 2D turbulence (dotted). Of course, the fluctuations in all these cases are freely decaying, and their behavior could be significantly changed if forcing were included. Again, we see that the most dramatic departures from wave-based WKB theory occur for 2D turbulence or for turbulence lacking the effects of the large-scale Alfvén speed.

## 6. Discussion and Comparison with Previous Work

The MHD turbulence model developed above is far from being an exact treatment of inhomogeneous turbulence in the solar wind. However, a number of important physical effects are incorporated here that have been neglected in previous treatments of the problem. Local turbulence is treated by one-point closure, including direct energy transfer for each of the $\mathbf{z}^{\pm}$ Elsässer fields separately, so that the influence of cross helicity is maintained at some reasonable level. The influence of a possible imbalance in the kinetic and magnetic energy densities is also included, albeit incompletely because we have only a crude estimate of the decay rate of the energy difference and the effect of the energy difference on the cascade of the Elsässer fields has not been included. Large-scale inhomogeneous effects treated in the model include convection, Alfvén wave propagation, expansion, large-scale density gradients, and mixing effects that couple the $\pm$ fields. Absent are the effects of induced electric fields and of magnetic and kinetic helicities. Additionally, the symmetry properties of the correlation tensors must be chosen from external considerations, although the formalism is sufficiently flexible to accommodate any choice.

Clearly, this model incorporates a variety of physical effects that may be of importance in explaining solar wind observations. Investigation of other effects, such as those associated with magnetic helicity must await development of more complete models. Compared to previous models of both hydrodynamic and MHD turbulence, the sophistication of the present model should probably be judged as roughly equivalent to the $K$-$\epsilon$ class of models used in engineering hydrodynamics. [*Jones and Launder*, 1972; *Bradshaw et al.*, 1981]. These were briefly discussed in section 3. In fact, the particular way that we dealt with the correlation scale is most like the $K$-$Kl$ model developed by Rotta (as described by *Bradshaw et al.*, [1981]). MHD models of the $K$-$\epsilon$ type have also been developed by *Yoshizawa* [1988, 1990] starting from much more elaborate two-scale expansions of the direct interaction approximation to the MHD equations. Like the $K$-$\epsilon$ models, the model described here can be fine-tuned by introducing order unity constants in several of our estimates of the nonlinear terms. Generally speaking, it may be necessary to optimize these parameters to obtain the best possible agreement of this kind of model with data from observations or numerical simulations. In the present model, we have usually ignored these constants for simplicity.

A critical assumption of the present model concerns the rotational symmetry of the fluctuations. Simple treatments of MHD turbulence [e.g., *Kraichnan*, 1965; *Frisch et al.*, 1975; *Pouquet et al.*, 1976] implicitly or explicitly assume that the fluctuations are strictly isotropic. Similarly, analyses of solar wind turbulence often make the same assumption [e.g., *Dobrowolny et al.*, 1980a, b; *Tu et al.*, 1984; *Zhou and Matthaeus*, 1990b]. Using an isotropic turbulence model is a useful first step in such treatments, despite the fact that a steady large-scale magnetic field provides a preferred direction for the small-scale fluctuations that may introduce violations of isotropy. In particular, simulations [*Shebalin et al.*, 1983] and analytic theory [*Montgomery*, 1983] of homogeneous turbulence, as well as solar wind observations [*Belcher and Davis*, 1971; *Klein et al.*, 1991; *Matthaeus et al.*, 1990], suggest the presence of anisotropy. The model of 2D turbulence [*Fyfe and Montgomery*, 1976] is a useful alternative, in which excited wave vectors and fluctuation vectors lie in the plane locally perpendicular to the applied mean magnetic field. The situation in the solar wind may be an intermediate case where the spectral tensor has components in all directions but is symmetric about the mean magnetic field. The two cases treated here therefore represent opposite limits to which the governing equations may tend.

The difference in assumed symmetry causes the final equations to be slightly different. In particular, the structure of the mixing terms depends upon the choice of the relative total amplitude of the three Cartesian components of the fluctuations. This feature is insensitive to the rotational symmetry of the spectral distribution of the fluctuations. In fact, details of spectral anisotropy would enter a model such as the present one mainly through the structure of the modeled nonlinear terms. However, at the current level of nonlinear modeling via a one point closure, the distinction between 2D and 3D isotropy cases is minimal. The basic nonlinearities are modeled in the same way, while the 2D model lacks the direct influence of the DC magnetic field through propagation. For more elaborate models of nonlinear effects we would expect a more detailed dependence on the spectral distribution relative to the large-scale magnetic field.

Another simpification is that the large-scale density, flow, and magnetic field have been treated as a priori specified functions of space and time. The simplest possible example was examined in the previous section, and in view of the energetic dominance of the nearly radial solar wind flow, that approximation is generally regarded as reasonable for many applications. However, to investigate transient phenomena, the effects of stream structure, etc., one would need to allow the large-scale fields to have both time dependence and more complicated space dependencies. In principle, the present model for computing the approximate evolution of the turbulence is general enough to be adapted for such extended treatments, provided that the local isotropy of the turbulence remains an acceptable approximation. Moreover, it is feasible to utilize our equations for inhomogeneous turbulence while allowing the large-scale fields to change according to a set of coupled dynamical equations, such that the influence of the turbulence on the large-scale dynamics might be incorporated. Such a model, which we envision as a future goal of our present effort, would represent a full solar wind heating and acceleration model, including small scale turbulence.

An important feature of the present model is the inclusion of driving or forcing terms, $S_{\pm}$ and $S_D$, acting at length scales $l_{\pm}$ and $l_D = (L_+ + L_-)/(Z_+^2 + Z_-^2)$.

These have been included in a fairly general way, provided that the agencies that supply magnetic energy, kinetic energy, and cross helicity to the turbulence can be reasonably associated with a known rate of supply and known length scales. Several possible applications of such driving terms come to mind. For example, Kelvin-Helmholtz shear-driven instabilities associated with high-speed streams have been implicated as sources of interplanetary MHD turbulence [*Coleman*, 1968; *Korzhov et al.*, 1984; *Roberts et al.*, 1987a, b; *Goldstein et al.*, 1989]. Studies have been presented in which this possibility is investigated by numerical simulation of the local effects of shear-driven turbulence on small-scale Alfvénic fluctuations [*Roberts et al.*, 1991, 1992], but one would like also to evaluate these suggestions in a formalism that allows for large-scale inhomogeneity. Similarly, in the upstream wave scenario [*Hoppe and Russell*, 1982; *Smith et al.*, 1983; *Goldstein et al.*, 1983], energy and cross helicity (as well as magnetic helicity) are injected into an MHD turbulent medium through resonant wave particle interactions. Another example is the injection of wave energy by the assimilation of photoionized interstellar pickup ions [*Isenberg*, 1987; *Lee and Ip*, 1987]. In these and similar situations the dynamical model introduced here may be useful for determining the response of the turbulence to the injected excitations once the appropriate driving terms and associated length scales are defined.

Canuto and coworkers have described a method for introducing driving due to instabilities into turbulence calculations [*Canuto and Goldman*, 1985; *Canuto et al.*, 1987]. The basic idea is that linear instabilities, characterized by a growth rate depending on wave number, enter into the turbulence model and act to balance turbulent spectral transfer as emulated by eddy viscosity effects. Although the model of Canuto et al. involves wave number dependence of both sources and eddy viscosities, one can simplify their treatment for application to the present model by assuming that the growth rates are peaked at certain spatial scales. The excitations of the turbulent fields are then supplied mainly at those scales $l_\pm$, and one arrives at simple expressions for the driving terms,

$$S_\pm = Z_\pm^2 \eta_\pm \tag{46}$$

$$S_D = D\eta_D, \tag{47}$$

for the rates at which $Z_\pm^2$ and $D$ are supplied due to the instability growth rates $\eta_\pm$ and $\eta_D$. The restriction of the current model to values of $l_D$ determined by $l_+$ and $l_-$ (cf. 15)) obviously limits the types of driving that can be represented.

The relationship of the present model for evolution of the energy-containing eddies to theoretical models for evolution of the inertial-range spectrum of MHD turbulence [*Zhou and Matthaeus*, 1989, 1990b; *Tu and Marsch*, 1990] was discussed in the introduction, but it is crucial enough to require further amplification. The general picture we envision is largely consistent with the theory of homogeneous hydrodynamic turbulence in the quasi-equilibrium range. Accordingly, most of the turbulent energy resides in the largest of the turbulent eddies, and these energy-containing structures may also be supplied by driving terms acting at their characteristic length scales. Nonlinear interactions of the turbulent energy-containing structures subsequently transfer excitations to smaller scales. Much of the dynamical behavior of the smaller-scale turbulence is in or near a steady state and can be treated as an inertial range where the equilibrium assumptions apply. Ultimately, excitations are transferred to the dissipation range, where MHD flow and magnetic-field energy are converted to heat. The present perspective is that the rate at which this spectral transfer occurs is ultimately controlled by the energy-containing eddies. We suggest that the dynamics of the energy-containing structures develop approximately according to the transport theory developed here, in which the wavelengths near the energy-containing scales are treated collectively, and disregard all spectral information pertaining to turbulence at longer wavelengths.

With regard to the turbulent eddies in the inertial range the important output of the present model is the set of decay rates $\epsilon_\pm$ and $\epsilon_D$, which provide boundary data at the low wave number end of the inertial range of wave number space. We have not assumed that the nonlinear interactions influencing the energy-containing structures are local in wave number [e.g., *Rose and Sulem*, 1978, *Zhou and Matthaeus*, 1990b]; however, the dynamics of the inertial range are usually treated this way. Consequently, apart from the effects of inhomogeneous spatial transport, the energy fluxes into the inertial range determine the structure of the equilibrium nearly power law inertial range fluctuations. A complete model of solar wind MHD turbulence would include a model such as the present one to describe the evolution and driving of energy-containing structures, along with a spectral transport model for the inertial range [*Zhou and Matthaeus*, 1990a, b]. We intend to employ such a coupled energy-containing range/inertial range model to achieve a more complete treatment of the transport of solar wind turbulence in the near future. Although the heating rate is associated with the transfer of energy out of the inertial range through its boundary with the dissipation range, this perspective views it as actually determined by decay rates that are established by the evolution of the large-scale turbulence, in complex interplay with the effects of spatial transport on the inertial range itself.

A fully self-consistent solar wind model would also need to compute the dynamical behavior of the large-scale density, velocity, and magnetic field, which would require a thermodynamic energy equation containing some approximation to the heat flux [e.g., *Hollweg*, 1986; *Hollweg and Johnson*, 1988]. Such a model would be quite useful for studying solar wind heating and acceleration. We note that in the quasi-equilibrium view-point adopted here, the heating problem requires solution for both the inertial-range fluctuations and the energy-containing eddies.

We are also led at this point to a discussion of previous treatments of the role of MHD turbulence in heating solar wind plasma, especially the approaches initiated by *Hollweg* [1986] and by *Tu* [1987]. In a series of papers [*Hollweg*, 1986; *Hollweg and Johnson*, 1988; *Isenberg*, 1990], Hollweg and colleagues have developed a model for solar wind acceleration and heating that includes turbulent spectral transfer as a source of enhanced heating. Rather than using spectral decomposition, the theory focuses on the energy-containing eddies and considers the turbulent decay rate to be given by the Kolmogoroff $u^3/\ell$ rate. This estimate for the solar wind heating rate was originally proposed by *Coleman* [1968]. The Hollweg model incorporates the Kolmogoroff heating rate as an additional term in what is otherwise a WKB equation for the evolution of purely outward traveling Alfvén waves. Such large-amplitude Alfvénic fluctuations have nearly maximal $|2H_c/E|$, so the use of hydrodynamic heating rates appears to be an overestimate. We suggest that more appropriate heating estimates are given by the cross-helicity dependent rates described in the present paper.

Also absent in the Hollweg models are effects of inward traveling fluctuation, an omission the present model does not share because both of the Elsässer fields $Z_{\pm}^2$ have been included. Even in the absence of instabilities that might generate inward-type fluctuations, a flow initially dominated by one Elsässer field can excite the opposite (or minority) Elsässer field via scattering off the gradients of the large-scale fields. This proclivity of turbulence to upset an initial pure disturbance can be deduced from examination of Eqs. (33) and (34). For example, neglect the nonlinear effects and driving terms in those equations and choose the inner heliospheric boundary condition at $R = R_0$ such that $Z_-^2(R_0) = 0$, $D(R_0) = 0$, and $Z_+^2 > 0$. The presence of $Z_+$ in (34) will cause $D$ to depart from zero, which in turn feeds into (33) and generates nonzero values of $Z_-^2$. This mixing between fluctuations having the sense of inward- and outward-traveling modes has been illustrated in some detail in the context of inertial-range spectral transport theories of solar wind fluctuations [*Zhou and Matthaeus*, 1989, 1990a; *Matthaeus et al.*, 1992; *Oughton and Matthaeus*, 1992; *Oughton*, 1993]. An important point to note here is that the purity imbalance of the outward waves near the inner boundary at $R_0$ corresponds to diminished turbulent heating rates (cf. equation (11)) and to high values of cross-helicity $H_c = (Z_+^2 - Z_-^2)/8$. However, the cross helicity can decrease relative to the energy, due either to the mixing effects or to sources that inject energy but not cross helicity. This kind of decrease of the cross helicity relative to the energy has been observed in the solar wind [*Roberts et al.*, 1987a, b]. Thus, the strength of MHD turbulence can be modulated by inhomogeneous transport effects in a way that may have interesting implications for the radial evolution of solar wind turbulence. This effect cannot arise in the Hollweg model, due to the use of a hydrodynamic form of the heating rate and the absence of the mixing effect in the purely WKB spatial transport operator.

Another heating model has been proposed by *Tu* [1987] as an extension of the inertial-range transport model developed by Tu and coworkers [*Tu et al.*, 1984; *Tu*, 1988]. This model also assumes purely outward traveling waves and maximal cross helicity, but it includes an estimate of nonlinear spectral transfer and heating as associated with a small admixture of inward waves. The spectral transfer rates are calculated according to either *Kraichnan*'s [1965] MHD theory or Kolmogoroff theory [see, e.g., *Zhou and Matthaeus*, 1990b]. The Tu heating theory makes no reference to the dynamics of the energy-containing eddies, and the spatial transport is essentially that of WKB theory. Upon integrating the spectral transport equations over wave number, the heating rate in the WKB transport equation is matched to the nonlinear energy flux at the dissipation edge of the inertial range, which they associate with the thermal ion gyroradius.

The same criticism can be made of the Tu heating model that was made above in connection with the Hollweg model concerning the use of only one Elsässer field and the lack of mixing terms in the spatial transport operator. To some extent this may be an acceptable approximation as the second field is less energetic, although the nonlinear decay rate does depends linearly on its magnitude. However, there appears to be another serious problem with their treatment of the low wave number part of the spectrum. As argued in our introduction, the notion of a local spectral energy flux must fail for sufficiently large fluctuations. In the original *Tu et al.* [1984] model there was no true low-frequency boundary. Rather, the solution for the spectrum at the smallest frequency plotted was found from the same governing equation used at all others, which tacitly assumes an influx of energy from unspecified lower frequencies. Thus, the solution can be expected to be valid only so long as the local spectral transfer model holds for immediately lower fluctuations. *Tu* [1987] adopted a second solution for the low-frequency range that starts from zero and meets the first solution at some slightly higher frequency. This procedure has the advantage that the cascade of energy into the dissipation range is limited by what is contained in the spectrum, but at the expense of imposing a solution at lower frequencies derived from a model that is inappropriate there. In truth, this maneuver is only an ad hoc resolution of the *Tu* [1984] model's logical inconsistency, for the high-frequency region of the spectrum remains unaffected. Even though the decay rate of low-frequency fluctuations must ultimately control the heating rate, it has absolutely no effect in this model. Therefore, in our judgment the Tu heating model is seriously flawed.

In conclusion, we have presented a closed six-equation model for the evolution of the energy-containing structures of MHD turbulence in the solar wind. Specialization to the particular cases in which solutions were presented involved reduction to a four-equation format. The model is highly approximate and has been derived without insistence on mathematical rigor, but instead relies on adaptations of previous models of MHD and hydrodynamic turbulence, as well as recent work on the spatial transport of turbulence in a weakly inhomogeneous medium. In spite of the severity of several simplifying assumptions we have adopted, e.g., neglect of the turbulent magnetic helicity and induced electric field, and specialization to isotropic turbulence or 2D turbulence, the model appears to represent a substantial increase in completeness relative to earlier treatments that relied almost exclusively on WKB theory and the assumption of purely outward traveling waves.

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