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Scale locality of the energy cascade in magnetohydrodynamics

A thesis
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of the requirements for the Degree
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Morganna Sophie Nickless



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Abstract

Turbulence behaves differently in electrically charged fluids, such as plasmas, due to the turbulent motions being accompanied by magnetic field influences. This thesis looks at how the magnetic field in these fluids impacts energy transferred from large scales to small scales—the energy cascade—by analyzing data from numerical simulations of turbulence. We will be following Doan et al’s approach [6] of decomposing the fluctuations into large-scale and small-scale contributions using bandpass filtering, then plotting the flux of the energy and enstrophy in 2D and 3D magnetohydrodynamics. This will help quantify various aspects of the energy cascade, including the degree of scale-locality. The results will be compared to those from Doan et al’s analysis of non-electrically charged fluids.

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Chapter 1

Introduction

Magnetohydrodynamic turbulence, the study of turbulence in fluids comprised of charged particles inducing a self-interacting magnetic field within the fluid, does not have nearly as much attention as ordinary turbulence [2]. The energy cascade, where energy and enstrophy are transferred from large to small length scales, is an important principle in fluid dynamics. The equations used to analyse this cascade, particularly in the MHD case, quickly grow in complexity, and so approximating the bandpass filtering with numerical methods become vital.

Chapter 2 takes a look at Kolmogorov's 1941 theory of turbulence, introducing the Navier-Stokes equation for incompressible fluids and the corresponding vorticity equation derived from it. We look at how the Reynold's number characterises turbulent flows and how this relates to the energy cascade. An overview of scale-decomposition, techniques later used in Chapter 4, will also be discussed.

In Chapter 3, we investigate the numerical methods used in our bandpass filtering - the discrete Fourier transform (DFT) and the efficiencies in computation the fast Fourier transform (FFT) algorithm grants us. We then examine bandpass filtering, to be used in our simulations.

Magnetohydrodynamics is properly introduced in Chapter 4, beginning with a derivation of the MHD equations from the Navier-Stokes equation and

Maxwell's equations. We go over the types of waves MHD fluids exhibit, a generally important derivation when learning MHD, and which help explain a concept in Chapter 5. This chapter concludes with scale-decomposing the MHD equations and using the bandpass filter to scale-localise the results.

Chapter 5 looks at the simplifications a 2D MHD fluid applies to our equations, and plots the scale-localised transfer functions for various Reynold's numbers and length scales. Chapter 6 builds upon this with the 3-dimensional case.

Chapter 7 is a small extension, plotting the time evolution of the transfer functions in surface plots.

The thesis concludes in Chapter 8.

Chapter 2

Turbulence

Many of the results in this chapter have been presented in Doan et al [6] and Davidson et al [5]. Here we provide more detailed derivations.

Section 2.1, in particular, presents a common derivation of the Reynolds number and non-dimensionalisation of the Navier-Stokes equation in greater detail, supplemented with results from NASA [3].

2.1 Reynolds Number

The natural starting point for this thesis is the Navier-Stokes equation [13], essentially Newton's 2nd Law applied to a fluid, written in vector form:

$$\underbrace{\frac{\partial \vec{u}}{\partial t} + (\vec{u} \cdot \nabla) \vec{u}}_{\text{fluid acceleration}} = \underbrace{-\nabla P}_{\text{pressure term}} + \underbrace{\nu \nabla^2 \vec{u}}_{\text{viscous term}} + \underbrace{\vec{F}}_{\text{external forces}} \quad (2.1)$$

$$\nabla \cdot \vec{u} = 0 \quad (2.2)$$

where $\vec{u} = (u_x, u_y, u_z)$ is the velocity field of a fluid flow, $P = p/\rho$ is the pressure per unit mass, and \vec{F} (a term often omitted for convenience and simplicity) represents additional forces per unit mass such as gravity or the Lorentz force, and ν is the kinematic viscosity.

The kinematic viscosity may also be expressed in non-dimensionalised form as

$$\nu = \frac{\bar{u}L}{Re} \quad (2.3)$$

where Re is the Reynolds Number [16] and L is some characteristic length. The Reynolds number is a dimensionless quantity that dictates the complexity of turbulence within a fluid flow.

We can investigate this definition by starting with the Navier-Stokes equation (2.1) multiplied by the fluid density ρ , with the extra force term omitted for simplicity:

$$\rho \left(\frac{\partial \vec{u}}{\partial t} + (\vec{u} \cdot \nabla) \vec{u} \right) = -\nabla p + \rho \nu \nabla^2 \vec{u} \quad (2.4)$$

Each term in this equation has the dimensional units $[\text{Mass}][\text{Length}]^{-2}[\text{Time}]^{-2}$. To non-dimensionalise this equation, we multiply both sides of (2.3) with a factor that has the inverse of these units, for instance:

$$\frac{L}{\rho \bar{u}^2} \quad (2.5)$$

where \bar{u} is the mean velocity of the fluid.

$$\frac{L}{\bar{u}^2} \left(\frac{\partial \vec{u}}{\partial t} + (\vec{u} \cdot \nabla) \vec{u} \right) = -\nabla P \frac{L}{\bar{u}^2} + \frac{\nu L}{\bar{u}^2} \nabla^2 \vec{u} \quad (2.6)$$

We can further simplify this by defining dimensionless variables by choosing a combination of our starting variables and the terms added from (2.5) to non-dimensionalise them:

$$\frac{\partial}{\partial t'} = \frac{L}{\bar{u}} \frac{\partial}{\partial t}, \quad \vec{u}' = \frac{\vec{u}}{\bar{u}}, \quad \nabla' = L \nabla, \quad P' = \frac{P}{\bar{u}^2} \quad (2.7)$$

thereby acquiring

$$\frac{\partial \vec{u}'}{\partial t'} + (\vec{u}' \cdot \nabla') \vec{u}' = -\nabla' P' + \frac{\nu}{\bar{u}L} (\nabla')^2 \vec{u}' \quad (2.8)$$

The $\frac{\nu}{\bar{u}L}$ term in (2.8) is a rearranged form of (2.3), which we can see is $\frac{1}{Re}$.

The value of the Reynolds number, and by extension the kinematic viscosity, ν , controls the flow from laminar (steady or smooth) solutions to disordered and increasingly complex solutions (see Figure 2.1).

Due to its inverse relationship with viscosity, we can extend that to having a relationship with drag within a fluid too. Generally, as the Reynolds number increases, the viscosity of the fluid decreases. This viscosity is essentially a glue, inducing greater friction within the flow and hence causing drag. We can thereby ascertain that drag tends to decrease as the Reynolds number increases, as shown in Figure 2.2.

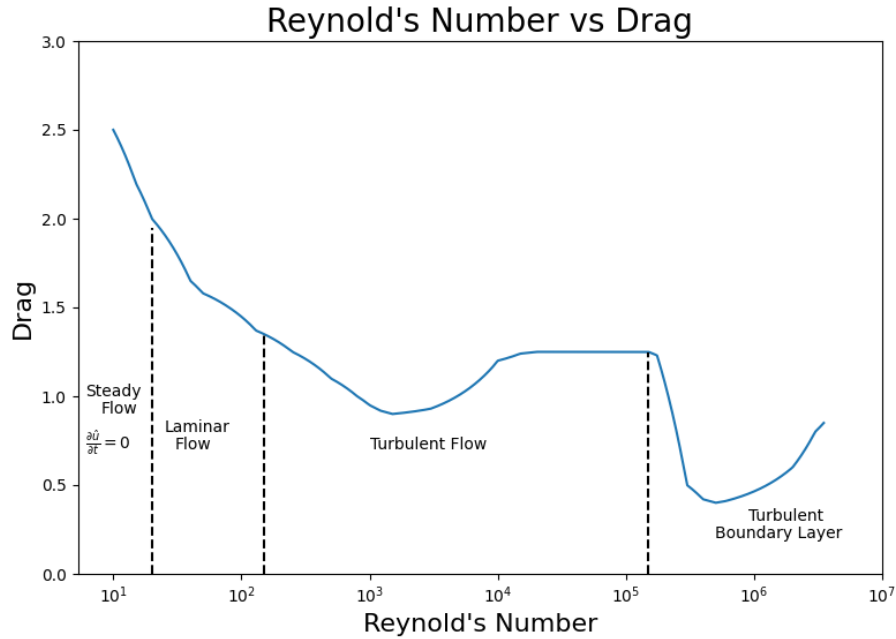


Figure 2.1: Relationship between Reynolds number and the drag in the fluid flow around a sphere (and by extension the viscosity), based on similar graphs by NASA [3].

[3] shows that the relationship between the Reynolds number and the fluid's drag is slightly more complicated, namely that

$$D = \frac{1}{2} C_d \rho |\vec{u}|^2 A \quad (2.9)$$

where D is the drag of the fluid, C_d is the drag coefficient of the fluid, and A is the reference area.

Rearranging and substituting (2.3) into this gives us the drag in terms of the Reynolds number:

$$D = \frac{1}{2} C_d \rho \frac{Re^2 \nu^2}{L^2} A \quad (2.10)$$

This added complexity is apparent in how the plot of the drag vs. Reynolds number starts off as a negative relationship for fluids that have steady or laminar flow, up until Reynolds numbers greater than 10^3 . Looking forward to Figure 2.2 shows that for Reynolds numbers beyond this, the fluid takes on more "roiling" characteristics and becomes more chaotic and irregular, inducing additional friction and drag in the fluid.

Due to this complexity, it is often easier to abstract these flows down to key elements, known as canonical flows—simple flows that characterise 1 or 2 features, such as the isotropic flows examined in this thesis. By studying these simpler cases, it allows us to gain a better understanding of the individual features and allows us to extrapolate this to more complex scenarios.

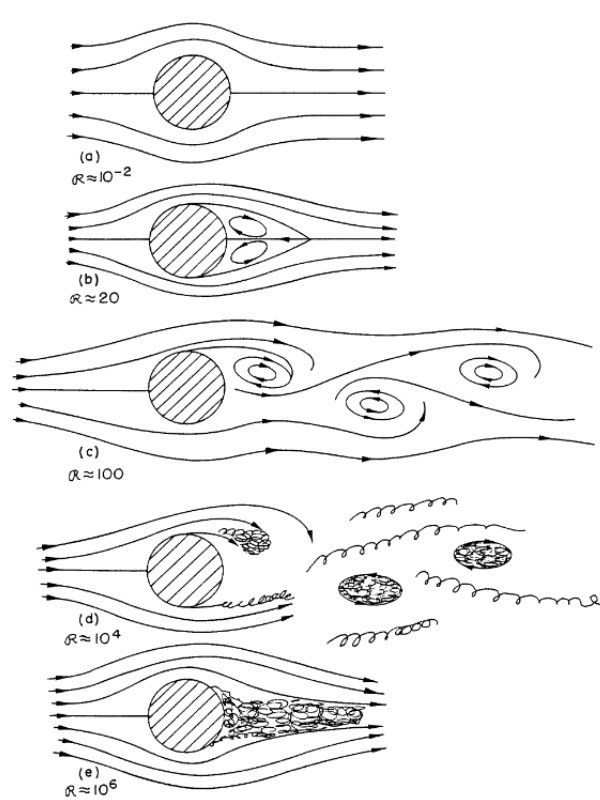


Figure 2.2: Flow past a cylinder for various Reynolds numbers [8].

As seen in Figure 2.1, as the Reynolds number increases and the flow becomes turbulent, turbulent eddies are produced as a consequence. These eddies are in a superposition of a wide range of sizes, from very large to very small.

2.2 The Energy Cascade

In 1941, Kolmogorov theorised that the kinetic energy of turbulent motion is distributed throughout a whole spectrum on different length scales, in fact cascading from the largest end of the spectrum down to the smallest end until such a point where the eddies dissipate directly to heat in the fluid due to viscosity.

We can examine some small periodic boundary region of a turbulent flow and measure the energy that these eddies have as they pass through the region, measuring from large-scale eddies to small-scale eddies.

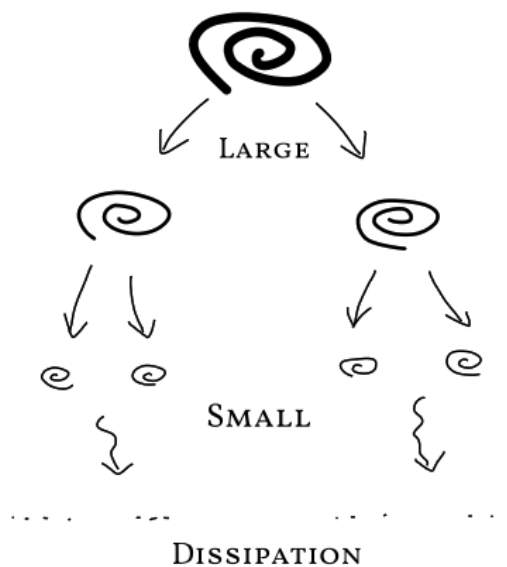


Figure 2.3: Diagram of an energy cascade from large scale eddies down to small scale and finally dissipating.

Consider a fluid with the velocity field $\vec{u} = (u_x, u_y, u_z)$. To decompose this fluid velocity field into different length scales, we take the Fourier transform of it. That is,

$$\hat{\vec{u}}(\vec{k}) = \frac{1}{(2\pi)^n} \int_{-\infty}^{\infty} e^{-i\vec{k}\cdot\vec{x}} \vec{u}(\vec{x}) d\vec{x} \quad (2.11)$$

where \vec{x} is a vector representing the spatial dimensions of \vec{u} , and n represents the number of dimensions.

Provided that the velocity field has a periodic motion, its Fourier transform converts it into a measure of spatial frequencies, represented by $\vec{k} = (k_x, k_y, k_z)$, which are the k -space analog to \vec{x} . \vec{k} can also be thought of as inversely proportional to the size; the larger the k -value, the greater the spatial frequency, and hence the smaller the turbulent eddy.

The energy spectrum, $E(k)$, or the mean kinetic energy per unit mass is then

$$\int_{k=0}^{\infty} E(\vec{k}) d\vec{k} = \frac{1}{2} \langle u_i u_i \rangle \quad (2.12)$$

where $i = 0, 1, 2$ denotes the index of $\hat{u}(k)$ in suffix notation, and the angled brackets denote a volume average.

$$\langle f \rangle = \frac{1}{L^n} \int_B f d^n x \quad (2.13)$$

where B denotes some boundary.

Some important features to note are the integral length scale

$$l_{corr} = \frac{3\pi}{4} \frac{\int_0^{\infty} k^{-1} E(k) dk}{\int_0^{\infty} E(k) dk} \quad (2.14)$$

which measures the correlation length of turbulence as a weighted average of the wavenumber, $k = |\vec{k}|$, inverted, along with the turbulence's rate of dissipation, ε , which relates to the Kolmogorov microscale, η , the length scale at which turbulence dissipates into the fluid as heat due to viscosity.

To find the rate of dissipation, ε , we start by taking the dot product of the velocity field \vec{u} with the Navier-Stokes equation (2.1), omitting the \vec{F} term for simplicity.

$$\vec{u} \cdot \frac{\partial \vec{u}}{\partial t} + \vec{u} \cdot (\vec{u} \cdot \nabla) \vec{u} = \vec{u} \cdot (-\nabla P) + \nu \vec{u} \cdot \nabla^2 \vec{u} \quad (2.15)$$

We can move the spatial derivative term on the left-hand side over to the right and factor with the gradient term:

$$\vec{u} \cdot \frac{\partial \vec{u}}{\partial t} = \vec{u} \cdot [-(\vec{u} \cdot \nabla) \vec{u} - \nabla P] + \nu \vec{u} \cdot \nabla^2 \vec{u} \quad (2.16)$$

Taking the mean or volume average results in this mixed middle term going to 0, and we are left with

$$\left\langle \vec{u} \cdot \frac{\partial \vec{u}}{\partial t} \right\rangle = \nu \langle \vec{u} \cdot \nabla^2 \vec{u} \rangle \quad (2.17)$$

$$\implies \frac{d}{dt} \left\langle \frac{1}{2} \vec{u} \cdot \vec{u} \right\rangle = \nu \langle |\nabla \vec{u}|^2 \rangle \quad (2.18)$$

The RHS is the negative of the dissipation, giving us:

$$\frac{d}{dt} \left\langle \frac{1}{2} |\vec{u}|^2 \right\rangle = -\varepsilon \quad (2.19)$$

This definition of the rate of dissipation is sufficient for x -space; however, as mentioned above, the energy spectrum is a function in k -space.

To do this, allow us to reconsider equation (2.16). We can take the Fourier transform of \vec{u} to get

$$\hat{u} \cdot \frac{\partial \hat{u}}{\partial t} = \hat{u} \cdot \left[-(\hat{u} \cdot \nabla) \hat{u} - \nabla P \right] + \nu \hat{u} \cdot \nabla^2 \hat{u} \quad (2.20)$$

Taking the volume average of (2.20) and following the process of (2.17)-(2.18) now yields

$$\frac{d}{dt} \left\langle \frac{1}{2} |\hat{u}|^2 \right\rangle = \nu \langle |\nabla \hat{u}|^2 \rangle \quad (2.21)$$

We can also use the fact that the spatial derivative of (2.9) gives us a factor of $-i\vec{k}$.

$$\nabla \hat{u}(\vec{k}) = -i\vec{k} \hat{u}(\vec{k}) \quad (2.22)$$

It is worth noting that in (2.19) the spatial derivative term is represented in k -space and is therefore

$$\nabla_k = \left(\frac{\partial}{\partial k_x}, \frac{\partial}{\partial k_y}, \frac{\partial}{\partial k_z} \right) \quad (2.23)$$

where the k index of ∇_k has been omitted for convenience.

Proof 2.1 Spatial Derivative of Fourier Transform. If the Fourier transform is given as per equation (2.9), then the spatial derivative (with \vec{k} given, as we

are now in k -space) is

$$\frac{\partial \vec{u}}{\partial \vec{k}} = \frac{\partial}{\partial \vec{k}} \left[\frac{1}{(2\pi)^n} \int_{-\infty}^{\infty} e^{-i\vec{k}\cdot\vec{x}} \vec{u}(\vec{x}) d\vec{x} \right] \quad (2.24)$$

$$= \frac{1}{(2\pi)^n} \int_{-\infty}^{\infty} \frac{\partial}{\partial \vec{k}} \left[e^{-i\vec{k}\cdot\vec{x}} \right] \vec{u}(\vec{x}) d\vec{x} \quad (2.25)$$

$$= \frac{1}{(2\pi)^n} \int_{-\infty}^{\infty} (-i\vec{k}) e^{-i\vec{k}\cdot\vec{x}} \vec{u}(\vec{x}) d\vec{x} \quad (2.26)$$

Factoring out the $-i\vec{k}$ term thus results in

$$\frac{\partial \vec{u}}{\partial \vec{k}} = (-i\vec{k}) \frac{1}{(2\pi)^n} \int_{-\infty}^{\infty} e^{-i\vec{k}\cdot\vec{x}} \vec{u}(\vec{x}) d\vec{x} \quad (2.27)$$

$$= -i\vec{k} \hat{u}(\vec{k}) \quad (2.28)$$

□

Substituting (2.19) into (2.18) now results in

$$\frac{d}{dt} \left\langle \frac{1}{2} |\hat{u}|^2 \right\rangle = -\nu k^2 \langle |\hat{u}|^2 \rangle \quad (2.29)$$

where $k = |\vec{k}|$ is the wavenumber or the magnitude of \vec{k} .

This now gives us

$$\varepsilon = -\nu k^2 \langle |\hat{u}|^2 \rangle = -\nu k^2 \langle \hat{u} \cdot \hat{u} \rangle \quad (2.30)$$

We can now integrate over all k and substitute (2.12), which finally yields our k -space expression for the rate of dissipation.

$$\varepsilon_k = -2\nu \int_0^k k^2 E(\vec{k}) d\vec{k} \quad (2.31)$$

In contrast, deriving the Kolmogorov microscale, η , requires only a bit of dimensional analysis. The dimensional units for the rate of dissipation, ε , and the kinematic viscosity, ν , are:

$$[\varepsilon] = \frac{L^2}{T^3}, \quad [\nu] = \frac{L^2}{T} \quad (2.32)$$

Clearly, since the Kolmogorov microscale is a measure of length, it has dimensional units of $[\eta] = L$.

$$\left(\frac{L^2}{T^3} \right)^a \left(\frac{L^2}{T} \right)^b = L \quad (2.33)$$

By expanding the brackets, we get

$$\frac{L^{2a+2b}}{T^{3a+b}} = L \quad (2.34)$$

Now we compare the units on both sides of the equation.

$$L^{2a+2b} = L \quad (2.35)$$

$$T^{3a+b} = 1 \quad (2.36)$$

Taking the logarithm of both sides and extracting the powers leaves us with

$$(2a + 2b) \log(L) = \log(L) \quad (2.37)$$

$$(3a + b) \log(T) = 0 \quad (2.38)$$

Since these are dimensional units, we can assume both L and T are not zero, hence leaving us with the simplified simultaneous equation.

$$a + b = \frac{1}{2} \quad (2.39)$$

$$3a + b = 0 \quad (2.40)$$

Solving this simultaneous equation finally leaves us with

$$a = \frac{3}{4}, \quad b = \frac{-1}{4} \quad (2.41)$$

And hence, the Kolmogorov microscale can be expressed as

$$\eta = \left(\frac{\nu^3}{\varepsilon} \right)^{\frac{1}{4}} \quad (2.42)$$

Now that we have all these characteristics of the turbulent flow, we can graph this energy cascade, $E(\vec{k})$, with respect to the wavenumber $k = |\vec{k}|$. As mentioned earlier, the wavenumber is inversely related to the length scale of the turbulent eddies, so as k increases, the eddies decrease in scale.

Since the energy of the individual eddies is small, it is convenient to plot this at log-scale.

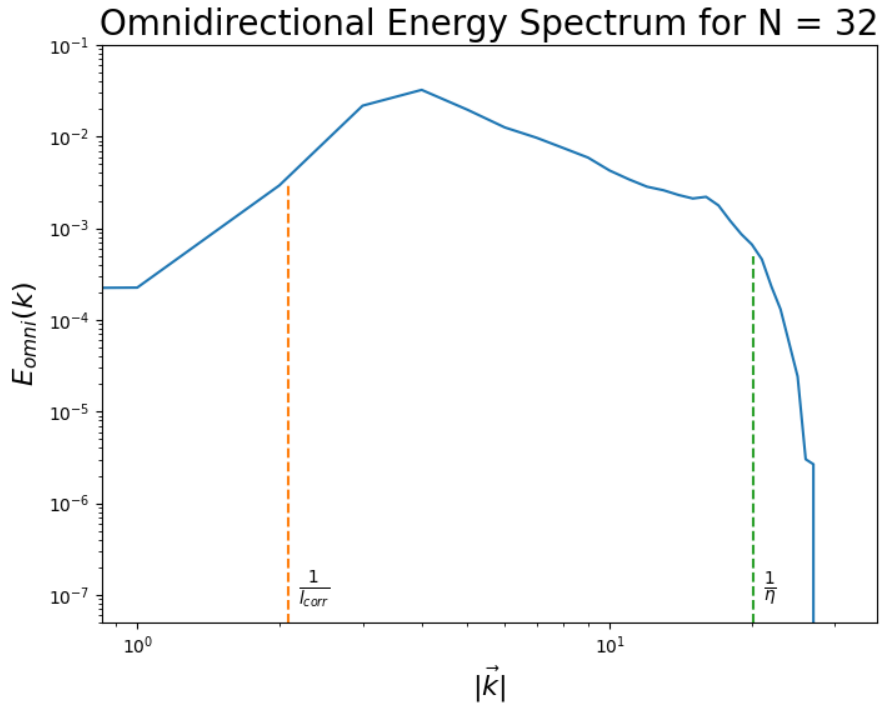


Figure 2.4: Energy spectra of a 3D simulated dataset, with N^3 datapoints, for $N = 32$. The code for this can be found in Appendix A.

The region in Figure 2.4 between the inverse of the integral length scale, $\frac{1}{l_{corr}}$, and the inverse of the Kolmogorov microscale, $\frac{1}{\eta}$, is where this cascade of energy happens. Some subregion within this, looking at only the intermediate length scales, is known as the inertial subrange.

The inertial subrange has a gradient proportional to $k^{-5/3}$. To the right of this are the length scales small enough that viscosity dominates, and thus the flow depends significantly on the viscosity term ν . Further right of this, past the $\frac{1}{\eta}$ term, is the scale at which dissipation occurs.

To the left of the inertial subrange, the length scales are large enough to maintain their energies and thus only depend on the flow's velocity, \vec{u} , and the size of the eddy, L .

These zones can be more explicitly seen in Figure 2.5, albeit with far more clutter than Figure 2.4.

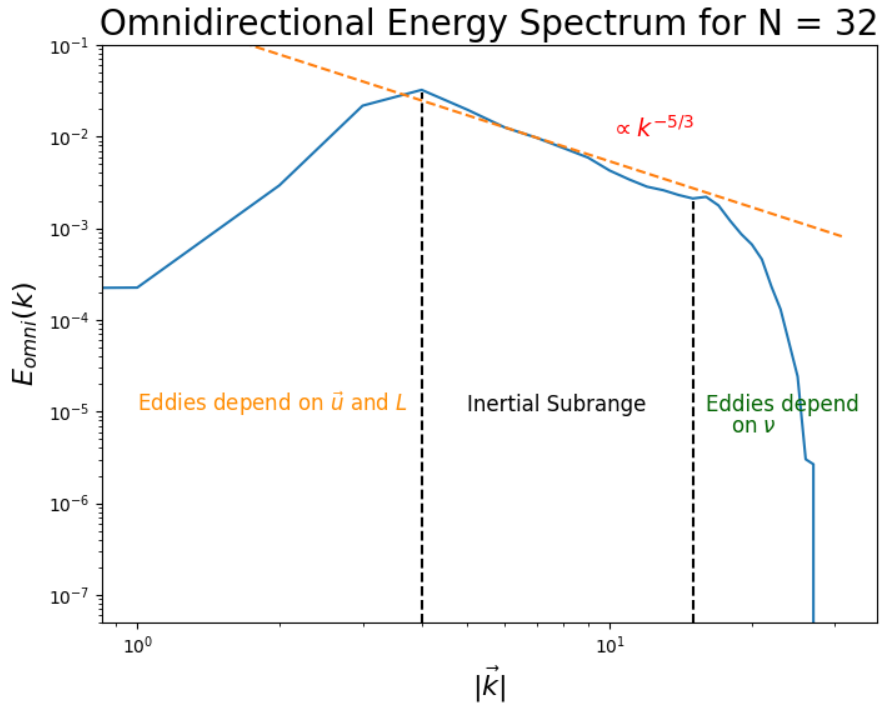


Figure 2.5: Energy spectra of a 3D simulated dataset, with N^3 datapoints, for $N = 32$, with the regions for inertial subrange and the dependencies for the turbulent eddies marked.

2.3 Scale Decomposition

Now that we have a formulation of the energy cascade and the inertial sub-range, we can begin to investigate the transfer of energy and enstrophy as turbulent eddies diminish from large scales to small scales. The enstrophy is related to kinetic energy and represents the dissipation effects in a fluid. In an incompressible flow, this is given by [7]:

$$\varepsilon(\vec{u}) = \int_{Volume} |\nabla \times \vec{u}|^2 dV \quad (2.43)$$

2.3.1 Transfer of Energy

We analyse these transfers by decomposing the velocity field, \vec{u} , into its large and small-scale components.

$$\vec{u} = \vec{u}^L + \vec{u}^S \quad (2.44)$$

where the superscripts L and S represent contributions from scales larger and smaller than some specific scale r . These scale decompositions are not unique and depend on which low-scale filter we use, as examined further in Chapter 3.

We now substitute (2.44) into (2.1), omitting the force term for convection, to get

$$\frac{\partial}{\partial t}(\vec{u}^L + \vec{u}^S) + ((\vec{u}^L + \vec{u}^S) \cdot \nabla)(\vec{u}^L + \vec{u}^S) = -\nabla P + \nu \nabla^2(\vec{u}^L + \vec{u}^S) \quad (2.45)$$

Since we know that energy is dependent on the volume average of the square of the velocity, similar to enstrophy, a good starting point would be taking the dot product of \vec{u}^L (for large scale energies) with (2.45):

$$\vec{u}^L \cdot \frac{\partial}{\partial t}(\vec{u}^L + \vec{u}^S) + \vec{u}^L \cdot ((\vec{u}^L + \vec{u}^S) \cdot \nabla)(\vec{u}^L + \vec{u}^S) = \vec{u}^L \cdot (-\nabla P) + \vec{u}^L \cdot \nu \nabla^2(\vec{u}^L + \vec{u}^S) \quad (2.46)$$

And now we take the volume average of the equation. By the summation rule of integration, this is identical to

$$\begin{aligned} \left\langle \vec{u}^L \cdot \frac{\partial}{\partial t}(\vec{u}^L + \vec{u}^S) \right\rangle + \left\langle \vec{u}^L \cdot ((\vec{u}^L + \vec{u}^S) \cdot \nabla)(\vec{u}^L + \vec{u}^S) \right\rangle &= \left\langle \vec{u}^L \cdot (-\nabla P) \right\rangle \\ &+ \left\langle \vec{u}^L \cdot \nu \nabla^2(\vec{u}^L + \vec{u}^S) \right\rangle \end{aligned} \quad (2.47)$$

Let us start with the time-derivative term of this:

$$\left\langle \vec{u}^L \cdot \frac{\partial}{\partial t}(\vec{u}^L + \vec{u}^S) \right\rangle \quad (2.48)$$

Expanding the dot product through the brackets, we get

$$\frac{\partial}{\partial t} \left\langle \frac{|\vec{u}^L|^2}{2} \right\rangle + \frac{\partial}{\partial t} \langle \vec{u}^L \cdot \vec{u}^S \rangle \quad (2.49)$$

By Parseval's identity [9]:

$$\langle fg \rangle = \sum_k \hat{f}_k \hat{g}_{-k} \quad (2.50)$$

We get that the large and small scales are orthogonal, provided they have the same wavenumber cutoff, r , [9].

$$\langle \vec{u}^S \cdot \vec{u}^L \rangle = \langle \hat{u}_r^S \cdot \hat{u}_r^L \rangle = 0 \quad (2.51)$$

Hence, (2.48) reduces down to

$$\left\langle \vec{u}^L \cdot \frac{\partial}{\partial t} (\vec{u}^L + \vec{u}^S) \right\rangle = \frac{\partial}{\partial t} \left\langle \frac{|\vec{u}^L|^2}{2} \right\rangle \quad (2.52)$$

The next easy-to-simplify term is the pressure term.

$$\langle \vec{u}^L \cdot (-\nabla P) \rangle \quad (2.53)$$

which can be converted to

$$\langle P \nabla \cdot \vec{u}^L \rangle \quad (2.54)$$

This reduces down to zero for incompressible flows: $\nabla \cdot \vec{u} = 0$.

Proof 2.2 $\langle \vec{u}^L \cdot (-\nabla P) \rangle = \langle P \nabla \cdot \vec{u}^L \rangle$. By extracting the negative sign out of the volume average and using the definition from (2.13), we obtain

$$-\langle \vec{u}^L \cdot \nabla P \rangle = -\langle u_i^L \partial_i P \rangle \quad (2.55)$$

$$= -\frac{1}{L^n} \int_B u_i^L \partial_i P d^n x \quad (2.56)$$

written in suffix notation.

From the product rule, we know that

$$\partial_i (u_i^L P) = u_i^L \partial_i P + P \partial_i u_i^L \quad (2.57)$$

Rearranging (2.57) and substituting back into (2.56) gives us

$$-\langle \vec{u}^L \cdot \nabla P \rangle = -\frac{1}{L^n} \int_B [\partial_i (u_i^L P) - P \partial_i u_i^L] d^n x \quad (2.58)$$

$$= -\frac{1}{L^n} \int_B \partial_i (u_i^L P) d^n x + \frac{1}{L^n} \int_B P \partial_i u_i^L d^n x \quad (2.59)$$

Since we are using a periodic boundary for our region, the volume average of the spatial gradient of a function is zero, that is, $\langle \partial_i f \rangle = -\frac{1}{L^n} \int_B \partial_i (f) d^n x = 0$.

This leaves us with

$$-\langle \vec{u}^L \cdot \nabla P \rangle = \frac{1}{L^n} \int_B P \partial_i u_i^L d^n x \quad (2.60)$$

$$= \langle P \partial_i u_i^L \rangle \quad (2.61)$$

$$= \langle P \nabla \cdot \vec{u}^L \rangle \quad (2.62)$$

□

The next easiest term to simplify would be the viscosity term

$$\langle \vec{u}^L \cdot \nu \nabla^2 (\vec{u}^L + \vec{u}^S) \rangle \quad (2.63)$$

We can factor out the kinematic viscosity, ν , and using vector identities, we get

$$\nu \langle \vec{u}^L \cdot \nabla^2 (\vec{u}^L + \vec{u}^S) \rangle = -\nu \langle (\nabla \times \vec{u}^L) \cdot (\nabla \times (\vec{u}^L + \vec{u}^S)) \rangle \quad (2.64)$$

provided that our flow is incompressible.

Expanding out the brackets gives us

$$\begin{aligned} -\nu \langle (\nabla \times \vec{u}^L) \cdot (\nabla \times (\vec{u}^L + \vec{u}^S)) \rangle &= -\nu \langle (\nabla \times \vec{u}^L) \cdot (\nabla \times \vec{u}^L) \rangle \\ &\quad - \nu \langle (\nabla \times \vec{u}^L) \cdot (\nabla \times \vec{u}^S) \rangle \end{aligned} \quad (2.65)$$

$$= -\nu \left[\langle |\nabla \times \vec{u}^L|^2 \rangle + \langle (\nabla \times \vec{u}^L) \cdot (\nabla \times \vec{u}^S) \rangle \right] \quad (2.66)$$

We now define a new term, the vorticity, $\vec{\omega} = \nabla \times \vec{u}$, which is the rotational velocity. The vorticity is solenoidal, that is

$$\nabla \cdot \vec{\omega} = \nabla \cdot (\nabla \times \vec{u}) = 0 \quad (2.67)$$

The vorticity can also be decomposed into large and small scales based on some specified scale r . That is, $\vec{\omega} = \vec{\omega}^L + \vec{\omega}^S$.

Proof 2.3 $\vec{\omega} = \vec{\omega}^L + \vec{\omega}^S$.

$$\vec{\omega} = \nabla \times \vec{u} \quad (2.68)$$

$$= \nabla \times (\vec{u}^L + \vec{u}^S) \quad (2.69)$$

$$= \nabla \times \vec{u}^L + \nabla \times \vec{u}^S \quad (2.70)$$

We can then simply define

$$\vec{\omega}^L = \nabla \times \vec{u}^L, \quad \vec{\omega}^S = \nabla \times \vec{u}^S \quad (2.71)$$

And thus

$$\vec{\omega} = \vec{\omega}^L + \vec{\omega}^S \quad (2.72)$$

□

Substituting (2.71) into (2.66), we get that

$$-\nu \langle (\nabla \times \vec{u}^L) \cdot (\nabla \times (\vec{u}^L + \vec{u}^S)) \rangle = -\nu [\langle |\omega^L|^2 \rangle + \langle \vec{\omega}^L \cdot \vec{\omega}^S \rangle] \quad (2.73)$$

By using Parseval's identity, we obtain a similar relationship as (2.51) with the mixed vorticity terms.

$$\langle \vec{\omega}^L \cdot \vec{\omega}^S \rangle = \langle \vec{\omega}_r^L \cdot \vec{\omega}_r^S \rangle = 0 \quad (2.74)$$

And thereby (2.63) simply reduces down to

$$\langle \vec{u}^L \cdot \nu \nabla^2 (\vec{u}^L + \vec{u}^S) \rangle = -\nu \langle |\omega^L|^2 \rangle \quad (2.75)$$

And finally, the most complicated term is:

$$\langle \vec{u}^L \cdot ((\vec{u}^L + \vec{u}^S) \cdot \nabla) (\vec{u}^L + \vec{u}^S) \rangle \quad (2.76)$$

First, let's expand everything out.

$$\begin{aligned} \langle \vec{u}^L \cdot ((\vec{u}^L + \vec{u}^S) \cdot \nabla) (\vec{u}^L + \vec{u}^S) \rangle &= \langle \vec{u}^L \cdot \vec{u}^L \cdot \nabla \vec{u}^L \rangle + \langle \vec{u}^L \cdot \vec{u}^L \cdot \nabla \vec{u}^S \rangle \\ &\quad + \langle \vec{u}^L \cdot \vec{u}^S \cdot \nabla \vec{u}^L \rangle + \langle \vec{u}^L \cdot \vec{u}^S \cdot \nabla \vec{u}^S \rangle \end{aligned} \quad (2.77)$$

$$\begin{aligned} &= \langle u_i^L u_j^L \partial_j u_i^L \rangle + \langle u_i^L u_j^L \partial_j u_i^S \rangle \\ &\quad + \langle u_i^L u_j^S \partial_j u_i^L \rangle + \langle u_i^L u_j^S \partial_j u_i^S \rangle \end{aligned} \quad (2.78)$$

The first and third terms of (2.78) are identically vanishing under a periodic ensemble average, and hence we are left with

$$\langle \vec{u}^L \cdot ((\vec{u}^L + \vec{u}^S) \cdot \nabla)(\vec{u}^L + \vec{u}^S) \rangle = \langle u_i^L u_j^L \partial_j u_i^S \rangle + \langle u_i^L u_j^S \partial_j u_i^S \rangle \quad (2.79)$$

Proof 2.4 $\langle u_i^L u_j^L \partial_j u_i^L \rangle = \langle u_i^L u_j^S \partial_j u_i^L \rangle = 0$. Starting with $\langle u_i^L u_j^L \partial_j u_i^L \rangle$, we can use the identity proved in Proof 2.2 to show that

$$\langle u_i^L u_j^L \partial_j u_i^L \rangle = -\langle \partial_j (u_i^L u_j^L) u_i^L \rangle \quad (2.80)$$

A simple product rule gives us

$$-\langle (\partial_j (u_i^L u_j^L)) u_i^L \rangle = -\langle (\partial_j u_j^L) u_i^L u_i^L \rangle - \langle (\partial_j u_i^L) u_j^L u_i^L \rangle \quad (2.81)$$

The first term in (2.81) becomes zero in an incompressible fluid ($\partial_j u_j = 0$), leaving us with just the second term, which we can manipulate with the chain rule.

$$-\langle (\partial_j u_i^L) u_j^L u_i^L \rangle = -\left\langle \left(\partial_j \frac{(u^L)^2}{2} \right) u_j^L \right\rangle \quad (2.82)$$

We can now do a rearrangement of the product rule to then give us

$$-\left\langle \left(\partial_j \frac{(u^L)^2}{2} \right) u_j^L \right\rangle = \frac{1}{2} \langle (u^L)^2 \partial_j u_j^L - \partial_j ((u^L)^2 u_j^L) \rangle \quad (2.83)$$

$$= \frac{1}{2} \langle (u^L)^2 \partial_j u_j^L \rangle - \frac{1}{2} \langle \partial_j ((u^L)^2 u_j^L) \rangle \quad (2.84)$$

Proof 2.2 also gave us the identity $\langle \partial_j f \rangle = 0$, that is, the volume average of the spatial gradient of a function is zero, which removes the 2nd term on (2.84), with the 1st term simply being the incompressibility condition.

Hence,

$$\langle u_i^L u_j^L \partial_j u_i^L \rangle = 0 \quad (2.85)$$

Similarly,

$$\langle u_i^L u_j^S \partial_j u_i^L \rangle = -\langle \partial_j (u_i^L u_j^S) u_i^L \rangle \quad (2.86)$$

$$= -\left\langle \underbrace{(\partial_j u_j^S)}_{=0} u_i^L u_i^L \right\rangle - \langle (\partial_j u_i^L) u_j^S u_i^L \rangle \quad (2.87)$$

$$= -\left\langle \left(\partial_j \frac{(u^L)^2}{2} \right) u_j^S \right\rangle \quad (2.88)$$

$$= \frac{1}{2} \left\langle (u^L)^2 \underbrace{\partial_j u_j^S}_{=0} \right\rangle - \frac{1}{2} \left\langle \underbrace{\partial_j ((u^L)^2 u_j^S)}_{=0} \right\rangle \quad (2.89)$$

So likewise,

$$\langle u_i^L u_j^S \partial_j u_i^L \rangle = 0 \quad (2.90)$$

□

To give a physical context to these vanishing terms, $\langle u_i^L u_j^L \partial_j u_i^L \rangle = 0$ means that interactions between large scale eddies cannot change the energies of those at small scales, and to a similar extent, $\langle u_i^L u_j^S \partial_j u_i^L \rangle = 0$ means that advection (or transmission of particals) of large scale eddies from eddies of small scales does not change the energies of eddies at large scales.

To continue on, we must define the Reynolds stress tensors for large and small scales.

$$\tau_{ij}^L = -u_i^L u_j^L, \quad \tau_{ij}^S = -u_i^S u_j^S \quad (2.91)$$

which describes the force per unit area at a point in the fluid. An important note is that the stress tensor is symmetric, that is

$$\tau_{ij} = \tau_{ji} \quad (2.92)$$

Now we can rewrite (2.79) as

$$\langle \vec{u}^L \cdot ((\vec{u}^L + \vec{u}^S) \cdot \nabla)(\vec{u}^L + \vec{u}^S) \rangle = \langle u_i^L u_j^L \partial_j u_i^S \rangle + \langle u_i^L u_j^S \partial_j u_i^S \rangle \quad (2.93)$$

$$= \langle u_i^L u_j^L \partial_j u_i^S \rangle + \langle u_i^L u_j^S \partial_j u_i^S \rangle \quad (2.94)$$

$$= \langle u_i^L u_j^L \partial_j u_i^S \rangle - \langle (\partial_j u_i^L) u_j^S u_i^S \rangle \quad (2.95)$$

$$= -\langle \tau_{ij}^L \partial_j u_i^S \rangle + \langle \partial_j u_i^L \tau_{ji}^S \rangle \quad (2.96)$$

Finally, we may rewrite this as

$$\langle \vec{u}^L \cdot ((\vec{u}^L + \vec{u}^S) \cdot \nabla)(\vec{u}^L + \vec{u}^S) \rangle = \langle S_{ij}^L \tau_{ij}^S - S_{ij}^S \tau_{ij}^L \rangle \quad (2.97)$$

where

$$S_{ij} = \frac{1}{2} \left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right) \quad (2.98)$$

is the strain-rate tensor, which measures the rate of change of strain in some material. In a fluid, this can be thought of as a velocity gradient between different points. Much like the stress tensor, it is also symmetric.

Proof 2.5 $\langle \partial_j u_i^L \tau_{ji}^S \rangle - \langle \tau_{ij}^L \partial_j u_i^S \rangle = \langle S_{ij}^L \tau_{ij}^S - S_{ij}^S \tau_{ij}^L \rangle$. Consider first $\partial_j u_i^L \tau_{ji}^S$. This is a Frobenius inner product (a double dot product used in dyadic algebra).

$$\vec{A} : \vec{B} = A_{ij} B_{ji} = \text{trace}(\vec{A} \vec{B}^T) \quad (2.99)$$

In our case, we have

$$\partial_j u_i^L \tau_{ji}^S = (\nabla \vec{u}^L) : \tau^S = \text{trace}((\nabla \vec{u}^L) (\tau^S)^T) \quad (2.100)$$

which is simply just $\text{trace}((\nabla \vec{u}^L) \tau^S)$ since τ is symmetric.

The trace of this, for the 3D case $i, j = 1, 2, 3$, then becomes

$$\begin{aligned} \text{trace}((\nabla \vec{u}^L) \tau^S) &= \partial_1 u_1^L \tau_{11}^S + \partial_2 u_2^L \tau_{22}^S + \partial_3 u_3^L \tau_{33}^S \\ &\quad + \partial_2 u_1^L \tau_{21}^S + \partial_3 u_1^L \tau_{31}^S + \partial_3 u_2^L \tau_{32}^S \\ &\quad + \partial_1 u_2^L \tau_{12}^S + \partial_1 u_3^L \tau_{13}^S + \partial_2 u_3^L \tau_{23}^S \end{aligned} \quad (2.101)$$

Using the symmetry of (2.92), we can factor the $i \neq j$ terms.

$$\begin{aligned} \text{trace}((\nabla \vec{u}^L) \tau^S) &= \partial_1 u_1^L \tau_{11}^S + \partial_2 u_2^L \tau_{22}^S + \partial_3 u_3^L \tau_{33}^S \\ &\quad + (\partial_2 u_1^L + \partial_1 u_2^L) \tau_{12}^S + (\partial_3 u_1^L + \partial_1 u_3^L) \tau_{13}^S \\ &\quad + (\partial_3 u_2^L + \partial_2 u_3^L) \tau_{23}^S \end{aligned} \quad (2.102)$$

Which, by definition of (2.98), we can see becomes

$$\begin{aligned} \text{trace}((\nabla \vec{u}^L) \tau^S) &= S_{11}^L \tau_{11}^S + S_{22}^L \tau_{22}^S + S_{33}^L \tau_{33}^S \\ &\quad + 2S_{12}^L \tau_{12}^S + 2S_{13}^L \tau_{13}^S + 2S_{23}^L \tau_{23}^S \end{aligned} \quad (2.103)$$

Allow us to now investigate the solution of $S_{ij}^L \tau_{ij}^S$ to examine if these are indeed the same. Consider the 3D matrices S^L and τ^S .

$$S^L = \begin{pmatrix} S_{11}^L & S_{12}^L & S_{13}^L \\ S_{21}^L & S_{22}^L & S_{23}^L \\ S_{31}^L & S_{32}^L & S_{33}^L \end{pmatrix}, \quad \tau^S = \begin{pmatrix} \tau_{11}^S & \tau_{12}^S & \tau_{13}^S \\ \tau_{21}^S & \tau_{22}^S & \tau_{23}^S \\ \tau_{31}^S & \tau_{32}^S & \tau_{33}^S \end{pmatrix} \quad (2.104)$$

Then, with simple matrix algebra, we can see that

$$S^L \tau^S = S^L (\tau^S)^T \quad (2.105)$$

$$= \begin{pmatrix} S_{11}^L \tau_{11}^S + S_{12}^L \tau_{21}^S + S_{13}^L \tau_{31}^S & \dots & \dots \\ \dots & S_{21}^L \tau_{12}^S + S_{22}^L \tau_{22}^S + S_{23}^L \tau_{32}^S & \dots \\ \dots & \dots & S_{31}^L \tau_{13}^S + S_{32}^L \tau_{23}^S + S_{33}^L \tau_{33}^S \end{pmatrix} \quad (2.106)$$

where the non-diagonal terms are inconsequential since the trace takes the sum of only the diagonal terms.

Then the trace of (2.106) becomes

$$\begin{aligned} \text{trace} \left(S^L (\tau^S)^T \right) &= S_{11}^L \tau_{11}^S + S_{12}^L \tau_{21}^S + S_{13}^L \tau_{31}^S \\ &\quad + S_{21}^L \tau_{12}^S + S_{22}^L \tau_{22}^S + S_{23}^L \tau_{32}^S \\ &\quad + S_{31}^L \tau_{13}^S + S_{32}^L \tau_{23}^S + S_{33}^L \tau_{33}^S \end{aligned} \quad (2.107)$$

Since both the strain-rate tensor, S^L , and the stress tensor, τ^S , are symmetric, this further reduces down to

$$\begin{aligned} \text{trace} \left(S^L (\tau^S)^T \right) &= S_{11}^L \tau_{11}^S + S_{22}^L \tau_{22}^S + S_{33}^L \tau_{33}^S \\ &\quad + 2S_{12}^L \tau_{12}^S + 2S_{13}^L \tau_{13}^S + 2S_{23}^L \tau_{23}^S \end{aligned} \quad (2.108)$$

which is identical to (2.103).

Therefore,

$$\partial_j u_i^L \tau_{ji}^S = S_{ij}^L \tau_{ij}^S \quad (2.109)$$

and by symmetry,

$$\partial_j u_i^S \tau_{ji}^L = S_{ij}^S \tau_{ij}^L \quad (2.110)$$

So the claim $\langle \partial_j u_i^L \tau_{ji}^S \rangle - \langle \tau_{ij}^L \partial_j u_i^S \rangle = \langle S_{ij}^L \tau_{ij}^S - S_{ij}^S \tau_{ij}^L \rangle$ is indeed valid. \square

And hence, (2.47) reduces down to:

$$\boxed{\frac{\partial}{\partial t} \left\langle \frac{|\vec{u}^L|^2}{2} \right\rangle = \langle S_{ij}^S \tau_{ij}^L - S_{ij}^L \tau_{ij}^S \rangle - \nu \langle |\vec{\omega}^L|^2 \rangle} \quad (2.111)$$

Similarly, for energy in the small-scale eddies, do the dot product in (2.46) with \vec{u}^S instead.

$$\boxed{\frac{\partial}{\partial t} \left\langle \frac{|\vec{u}^S|^2}{2} \right\rangle = \langle S_{ij}^L \tau_{ij}^S - S_{ij}^S \tau_{ij}^L \rangle - \nu \langle |\vec{\omega}^S|^2 \rangle} \quad (2.112)$$

It is also convenient to represent

$$\Pi_V(r) = \langle S_{ij}^L \tau_{ij}^S - S_{ij}^S \tau_{ij}^L \rangle \quad (2.113)$$

as the flux of energy from large scale eddies down to small scale, across some specified scale r .

2.3.2 Transfer of Enstrophy

If we recall from (2.43), the enstrophy is based on the curl of the velocity field, all squared. Since we know that the curl of the velocity field is the vorticity of the fluid, we may rewrite (2.43) as

$$\varepsilon(\vec{u}) = \int_{Volume} |\vec{\omega}|^2 dV \quad (2.114)$$

This suggests that, in a similar fashion to the scale-decomposed energy, we scale-decompose the vorticity equation and dot product it with $\vec{\omega}^L$ and $\vec{\omega}^S$ for large and small scale enstrophy, respectively.

To start, we take the curl of (2.1) to obtain the vorticity equation for incompressible flow:

$$\nabla \times \frac{\partial \vec{u}}{\partial t} + \nabla \times (\vec{u} \cdot \nabla) \vec{u} = \nabla \times (-\nabla P) + \nabla \times \nu \nabla^2 \vec{u} \quad (2.115)$$

It may also be helpful to know the identity.

$$(\vec{u} \cdot \nabla) \vec{u} = \nabla \left(\frac{u^2}{2} \right) - \vec{u} \times (\nabla \times \vec{u}) \quad (2.116)$$

$$= \nabla \left(\frac{u^2}{2} \right) - \vec{u} \times \vec{\omega} \quad (2.117)$$

Substituting (2.117) into (2.115) and simplifying gives

$$\frac{\partial \vec{\omega}}{\partial t} + \nabla \times \nabla \left(\frac{u^2}{2} \right) - \nabla \times (\vec{u} \times \vec{\omega}) = \nabla \times (-\nabla P) + \nu \nabla^2 \vec{\omega} \quad (2.118)$$

Using the fact that the curl of a gradient is zero

$$\nabla \times \nabla f = 0 \quad (2.119)$$

and the identity

$$\nabla \times (\vec{u} \times \vec{\omega}) = (\vec{\omega} \cdot \nabla) \vec{u} - (\vec{u} \cdot \nabla) \vec{\omega} - \underbrace{\vec{\omega} (\nabla \cdot \vec{u})}_{=0} \quad (2.120)$$

finally leaves us with the vorticity equation.

$$\boxed{\frac{\partial \vec{\omega}}{\partial t} - (\vec{\omega} \cdot \nabla) \vec{u} + (\vec{u} \cdot \nabla) \vec{\omega} = \nu \nabla^2 \vec{\omega}} \quad (2.121)$$

Scale decomposing (2.121) with (2.72) leads to

$$\frac{\partial}{\partial t} (\vec{\omega}^L + \vec{\omega}^S) - ((\vec{\omega}^L + \vec{\omega}^S) \cdot \nabla) \vec{u} + (\vec{u} \cdot \nabla) (\vec{\omega}^L + \vec{\omega}^S) = \nu \nabla^2 (\vec{\omega}^L + \vec{\omega}^S) \quad (2.122)$$

We do this only for $\vec{\omega}$ instead of also including the scale decomposition of \vec{u} because, while correct to do so, our final simplification merges the \vec{u}^L and \vec{u}^S terms back into \vec{u} , rendering the scale decomposition of it unnecessarily messy.

This is followed by, for large-scale enstrophy, taking the dot product with $\vec{\omega}^L$ and volume averaging.

$$\begin{aligned} \left\langle \vec{\omega}^L \cdot \frac{\partial}{\partial t} (\vec{\omega}^L + \vec{\omega}^S) \right\rangle - \left\langle \vec{\omega}^L \cdot ((\vec{\omega}^L + \vec{\omega}^S) \cdot \nabla) \vec{u} \right\rangle + \left\langle \vec{\omega}^L \cdot (\vec{u} \cdot \nabla) (\vec{\omega}^L + \vec{\omega}^S) \right\rangle \\ = \left\langle \vec{\omega}^L \cdot \nu \nabla^2 (\vec{\omega}^L + \vec{\omega}^S) \right\rangle \end{aligned} \quad (2.123)$$

Using an almost identical process as used to derive (2.52) and (2.75), we can further simplify this equation down to

$$\left\langle \frac{\partial (\vec{\omega}^L)^2}{\partial t} \right\rangle - \left\langle \vec{\omega}^L \cdot ((\vec{\omega}^L + \vec{\omega}^S) \cdot \nabla) \vec{u} \right\rangle + \left\langle \vec{\omega}^L \cdot (\vec{u} \cdot \nabla) (\vec{\omega}^L + \vec{\omega}^S) \right\rangle = -\nu \left\langle (\nabla \times \vec{\omega}^L)^2 \right\rangle \quad (2.124)$$

leaving only the middle two terms as requiring simplification.

Expanding the second of these yields

$$\left\langle \vec{\omega}^L \cdot ((\vec{\omega}^L + \vec{\omega}^S) \cdot \nabla) \vec{u} \right\rangle = \left\langle \vec{\omega}^L \cdot (\vec{u} \cdot \nabla \vec{\omega}^L) \right\rangle + \left\langle \vec{\omega}^L \cdot (\vec{u} \cdot \nabla \vec{\omega}^S) \right\rangle \quad (2.125)$$

The $\langle \vec{\omega}^L \cdot (\vec{u} \cdot \nabla \vec{\omega}^L) \rangle$ term of (2.125) becomes zero, reducing (2.125) to

$$\langle \vec{\omega}^L \cdot ((\vec{\omega}^L + \vec{\omega}^S) \cdot \nabla) \vec{u} \rangle = \langle \vec{\omega}^L \cdot (\vec{u} \cdot \nabla \vec{\omega}^S) \rangle \quad (2.126)$$

It is convenient to represent this term as

$$F(r) = \langle \vec{\omega}^L \cdot (\vec{u} \cdot \nabla \vec{\omega}^S) \rangle \quad (2.127)$$

where $F(r)$ is the transfer of enstrophy from across our specified scale r , from large scales to small scales.

$F(r)$ may also be written as

$$F(r) = \langle \vec{\omega}^L \cdot (\vec{u} \cdot \nabla \vec{\omega}^S) \rangle = - \langle \vec{\omega}^S \cdot (\vec{u} \cdot \nabla \vec{\omega}^L) \rangle \quad (2.128)$$

where the latter version comes when we take the dot product with $\vec{\omega}^S$ instead.

Proof 2.6 $\langle \vec{\omega}^L \cdot (\vec{u} \cdot \nabla \vec{\omega}^L) \rangle = 0$. Using the identity from Proof 2.2, we have

$$\langle \vec{\omega}^L \cdot (\vec{u} \cdot \nabla \vec{\omega}^L) \rangle = \langle \omega_i^L u_j \partial \omega_i^L \rangle \quad (2.129)$$

$$= - \langle (\partial_j (\omega_i^L u_j)) \omega_i^L \rangle \quad (2.130)$$

Taking the product rule of (2.130),

$$- \langle (\partial_j (\omega_i^L u_j)) \omega_i^L \rangle = - \left\langle \underbrace{(\partial_j u_j)}_{=0} \omega_i^L \omega_i^L \right\rangle - \langle (\partial_j \omega_i^L) u_j \omega_i^L \rangle \quad (2.131)$$

where the first term goes to zero due to the incompressibility condition. We can now apply the chain rule and a rearranged product rule to the remaining term.

$$- \langle (\partial_j \omega_i^L) u_j \omega_i^L \rangle = - \left\langle \left(\partial_j \frac{(\omega_i^L)^2}{2} \right) u_j \right\rangle \quad (2.132)$$

$$= \frac{1}{2} \left\langle (\omega_i^L)^2 \underbrace{\partial_j u_j}_{=0} \right\rangle - \frac{1}{2} \underbrace{\langle \partial_j ((\omega_i^L)^2 u_j) \rangle}_{=0} \quad (2.133)$$

where the last term is zero due to the volume average of a spatial gradient of a function being zero: $\langle \partial_j f \rangle = 0$.

Hence,

$$\langle \vec{\omega}^L \cdot (\vec{u} \cdot \nabla \vec{\omega}^L) \rangle = 0 \quad (2.134)$$

□

We can now move on to the first of the two middle terms from (2.124), $\langle \vec{\omega}^L \cdot ((\vec{\omega}^L + \vec{\omega}^S) \cdot \nabla) \vec{u} \rangle$. In this instance, much like \vec{u} , we want to merge the $\vec{\omega}^L + \vec{\omega}^S$ term back into $\vec{\omega}$.

This, then, allows us to define

$$G^L(r) = \langle \vec{\omega}^L \cdot (\vec{\omega} \cdot \nabla) \vec{u} \rangle \quad (2.135)$$

as the generation of enstrophy occurs in the fluid at large scales.

With the definitions from (2.127) and (2.135), along with our other simplifications, (2.123) finally simplifies down to the enstrophy equation for large scales.

$$\left\langle \frac{\partial (\vec{\omega}^L)^2}{\partial t} \frac{1}{2} \right\rangle = G^L(r) - F(r) - \nu \langle (\nabla \times \vec{\omega}^L)^2 \rangle \quad (2.136)$$

By symmetry, the enstrophy equation for small scales is then

$$\left\langle \frac{\partial (\vec{\omega}^S)^2}{\partial t} \frac{1}{2} \right\rangle = G^S(r) + F(r) - \nu \langle (\nabla \times \vec{\omega}^S)^2 \rangle \quad (2.137)$$

where

$$G^S(r) = \langle \vec{\omega}^S \cdot (\vec{\omega} \cdot \nabla) \vec{u} \rangle \quad (2.138)$$

with $G^S(0) = 0$ and

$$G^S(r) + G^L(r) = \langle \omega_i \omega_j S_{ij} \rangle \quad (2.139)$$

Proof 2.7 $G^S(r) + G^L(r) = \langle \omega_i \omega_j S_{ij} \rangle$. We begin by rewriting $G^S(r)$ and $G^L(r)$ in suffix notation.

$$G^S(r) = \langle \vec{\omega}^S \cdot (\vec{\omega} \cdot \nabla) \vec{u} \rangle = \langle \omega_i^S \omega_j \partial_j u_i \rangle \quad (2.140)$$

$$G^L(r) = \langle \vec{\omega}^L \cdot (\vec{\omega} \cdot \nabla) \vec{u} \rangle = \langle \omega_i^L \omega_j \partial_j u_i \rangle \quad (2.141)$$

Recalling from Proof 2.5, a double dot product (or Frobenius inner product) with $\partial_j u_i$ may be expressed as

$$G^S(r) = \langle \omega_i^S \omega_j S_{ij} \rangle, \quad G^L(r) = \langle \omega_i^L \omega_j S_{ij} \rangle \quad (2.142)$$

Adding these terms together finally leaves us with

$$G^S(r) + G^L(r) = \langle \omega_i^S \omega_j S_{ij} \rangle + \langle \omega_i^L \omega_j S_{ij} \rangle \quad (2.143)$$

$$= \langle \omega_i^S \omega_j S_{ij} + \omega_i^L \omega_j S_{ij} \rangle \quad (2.144)$$

$$= \langle (\omega_i^S + \omega_i^L) \omega_j S_{ij} \rangle \quad (2.145)$$

$$= \langle \omega_i \omega_j S_{ij} \rangle \quad (2.146)$$

□

Proof 2.8 $G^S(0) = 0$. $G^S(r)$ represents the generation of enstrophy induced by vortical stretching at a small length scale, defined as being below some chosen length scale, r .

As $r \rightarrow 0$, naturally the amount of vortices or eddies causing enstrophy at a scale below r decreases. In fact, as r decreases, $G^S(r)$ becomes smaller and $G^L(r)$ becomes larger. It thus makes sense that when $r = 0$ and the turbulence occurs in a physical world where it is impossible for us to have a negative length scale, there are no vortices at a scale below r to induce a generation of enstrophy. □

Corollary 2.1. *Davidson et al [5] stated that $G^S(0) = 0$. This is a rather soft boundary on $G^S(r)$ given that, of course, no enstrophy can be generated if there is no physical length scale to do so.*

From the explanations for Figures 2.3 and 2.4, given that below the Kolmogorov microscale, η , viscosity dominates and the vortices dissipate directly into heat in the fluid, a stronger boundary may be

$$r = \beta\eta > 0 \quad (2.147)$$

where $\beta < 1$ is some proportionality constant.

This is merely an interesting observation, and a proof of this and its validity is outside the scope of this thesis.

Chapter 3

The Bandpass Filter

The contents of this chapter are not new. The work in Section 3.2 has been presented in Leung et al [10] and Section 3.3 involves results presented in Doan et al [6]. This chapter provides greater detail in the derivations.

The definitions of the discrete and fast Fourier transforms are rather common, and not original, with the redundant mode simplifications coming from discussions with my supervisor, Prof. Oughton.

Bandpass filtering plays an important role in structure identification at different scales. It is a filter that creates a sub-region within our velocity or vorticity fields based on some desired length scale L . Everything within this subregion (or band) is permitted to pass through, and everything outside of this range around L is not, generally being set to zero. Hence, a filter where only certain lengths within a band may pass through.

Ideally, the bandpass filter would be entirely flat, like a step function; however, practically, this is not the case, creating a roll-off region where our filtered field gradually goes to zero outside our desired range.

The simplest bandpass filter that is very compact, thus reducing the size of the roll-off region, and is spherically symmetric in physical space is the Gaussian filter, defined as [10]:

$$G(L; r) = \left(\frac{1}{\sqrt{\pi}L} \right)^{\text{dimensions}} e^{-\left(\frac{r}{L}\right)^2}, \quad \int G(L; r) d\vec{r} = 1, \quad r = |\vec{r}| \quad (3.1)$$

Bandpass filters are generally used in areas involving frequency, such as signal processing in electronics or spectrum analysis in astronomy. For our purposes, by applying the bandpass filter to spatial frequencies using the Fourier transform, we can filter our fields by the length scale of the eddies.

3.1 The Discrete Fourier Transform

The continuous Fourier transform was briefly touched upon in (2.11) as a method to convert our velocity field (and by extension the vorticity field) into the frequency domain, often also called the Fourier space or k -space (as will be referred to throughout this thesis), provided our input field is periodic in some space.

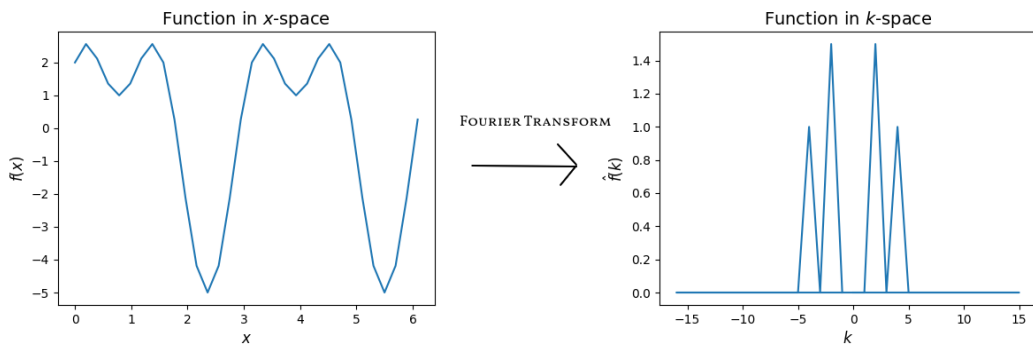


Figure 3.1: An example periodic function in x -space and its corresponding frequency in k -space. We can see our example function has a dominant frequency component of 2 and another, less dominant frequency component of 4. Considering the example function was $f(x) = 3 \sin(2x) + 2 \cos(4x)$, this makes sense.

The discrete Fourier transform is perfect when the input field is a finite periodic sequence of uniformly spaced points within a finite region, as is the case when performing simulations on a computer.

The discrete Fourier transform can be defined as:

$$\hat{f}(\vec{k}) = \frac{1}{N} \sum_{m=0}^{N-1} e^{-i\vec{k} \cdot \vec{x}_m} f(\vec{x}_m) \quad (3.2)$$

where

$$\vec{x}_m = \frac{2\pi\vec{m}}{N} \quad (3.3)$$

is the uniform step-size of the function, essentially $\Delta\vec{x}$ for a region of size 2π separated into $N \in \mathbb{N}$ data points, for steps, $m = 0, 1, \dots, N - 1$.

The inverse discrete Fourier transform, used to convert from k -space back to x -space, is thus given by

$$f(\vec{x}) = \sum_{k=0}^{N-1} e^{i\vec{k}\cdot\vec{x}_m} \hat{f}(\vec{k}) \quad (3.4)$$

These definitions are not unique; the scale factor of $\frac{1}{N}$ often appears in (3.4) instead of (3.2), and in some cases, each uses a scale factor of $\frac{1}{\sqrt{N}}$ instead, to name a few of the most common. The use of each of these common conventions typically depends on the use of the discrete Fourier transform. For instance, the $\frac{1}{\sqrt{N}}$ scale factor is symmetrically pleasing and is occasionally seen in quantum mechanics as a means to preserve energy in transformations.

The convention used in this thesis, see (3.2) and (3.4), is known as the wave propagation convention [15], and is the "standard" for transforms involving a position, x , and a wavenumber, k , as the exponential in the transform represents a wave moving in the positive x direction [15].

This may be generalised to n dimensions with

$$\hat{f}(\vec{k}) = \left(\frac{1}{N} \frac{1}{L} \dots \frac{1}{J} \right) \sum_{m=0}^{N-1} \sum_{l=0}^{L-1} \dots \sum_{j=0}^{J-1} f(\vec{x}) e^{-i(\vec{k}_m \cdot \vec{x}_m + \vec{k}_l \cdot \vec{x}_l + \dots + \vec{k}_j \cdot \vec{x}_j)} \quad (3.5)$$

$$f(\vec{x}) = \sum_{k_m=0}^{N-1} \sum_{k_l=0}^{L-1} \dots \sum_{k_j=0}^{J-1} \hat{f}(\vec{k}) e^{-i(\vec{k}_m \cdot \vec{x}_m + \vec{k}_l \cdot \vec{x}_l + \dots + \vec{k}_j \cdot \vec{x}_j)} \quad (3.6)$$

where $m, l, \dots, j \in \mathbb{N}$.

For a symmetrical field, as used in this thesis, this simplifies down to

$$\hat{f}(\vec{k}) = \frac{1}{(N)^n} \sum_{m=0}^{N-1} \sum_{m=0}^{N-1} \dots \sum_{m=0}^{N-1} f(\vec{x}) e^{-i\vec{k}_m \cdot \vec{x}_m} \quad (3.7)$$

$$f(\vec{x}) = \sum_{k=0}^{N-1} \sum_{k=0}^{N-1} \dots \sum_{k=0}^{N-1} \hat{f}(\vec{k}) e^{i\vec{k}_m \cdot \vec{x}_m} \quad (3.8)$$

3.1.1 The Fast Fourier Transform

When writing the discrete Fourier transform into a computer for n dimensions, the computation will require $n + 1$ for loops. This results in a complexity of $\mathcal{O}(N^2)$, which rapidly becomes expensive and slow to compute as the size of the dataset, N , increases.

This level of complexity can be reduced with the fast Fourier transform (FFT), which is a computationally efficient way to compute the discrete Fourier transform.

For simplicity, let us consider the 1D case. Higher dimensions follow the same method, just with extra tediousness.

Start by defining a vector \vec{f} sampled at regular and discrete values $(x_0, x_1, \dots, x_{N-1})$.

$$\vec{f} = \begin{bmatrix} f_0 \\ f_1 \\ \dots \\ f_{N-1} \end{bmatrix} \quad (3.9)$$

We may now rewrite the discrete Fourier transform as a matrix operation.

$$\begin{bmatrix} \hat{f}_0 \\ \hat{f}_1 \\ \dots \\ \hat{f}_{N-1} \end{bmatrix} = F_N \begin{bmatrix} f_0 \\ f_1 \\ \dots \\ f_{N-1} \end{bmatrix} \quad (3.10)$$

where F_N is an $N \times N$ matrix. The rows of F_N correspond to increments in k , and the columns correspond to increments in m , expanded to additional axes for higher dimensions.

Examining the elements of F_N , we can notice some patterns. For either $m = 0$ or $k = 0$, the power of the exponent reduces to zero, making the matrix element at these values become 1.

For $m \neq 0$ and $k \neq 0$, allow us to look at the unsimplified version of these elements:

$$(F_N)_{(m,k)} = e^{\frac{-2\pi i k m}{N}} \quad (3.11)$$

Consider the case where $m = k = 1$. Then (3.11) becomes

$$(F_N)_{(1,1)} = e^{\frac{-2\pi i}{N}} \quad (3.12)$$

Physically, this is known as the fundamental frequency, expressed as ω_N . We can see then that

$$(F_N)_{(1,2)} = (F_N)_{(2,1)} = e^{\frac{-4\pi i}{N}} = \omega_N^2 \quad (3.13)$$

$$(F_N)_{(1,3)} = (F_N)_{(3,1)} = e^{\frac{-6\pi i}{N}} = \omega_N^3 \quad (3.14)$$

$$(F_N)_{(2,2)} = (F_N)_{(1,4)} = (F_N)_{(4,1)} = e^{\frac{-8\pi i}{N}} = \omega_N^4 \quad (3.15)$$

...

$$(F_N)_{(N,N)} = e^{\frac{-2\pi i N^2}{N}} = \omega_N^{(N-1)^2} \quad (3.16)$$

We can now rewrite F_N in terms of the fundamental frequency, ω_N , and substitute into (3.10):

$$\begin{bmatrix} \hat{f}_0 \\ \hat{f}_1 \\ \hat{f}_2 \\ \dots \\ \hat{f}_{N-1} \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & \dots & 1 \\ 1 & \omega_N & \omega_N^2 & \dots & \omega_N^{N-1} \\ 1 & \omega_N^2 & \omega_N^4 & \dots & \omega_N^{2(N-1)} \\ \dots & \dots & \dots & \dots & \dots \\ 1 & \omega_N^{N-1} & \omega_N^{2(N-1)} & \dots & \omega_N^{(N-1)^2} \end{bmatrix} \begin{bmatrix} f_0 \\ f_1 \\ f_2 \\ \dots \\ f_{N-1} \end{bmatrix} \quad (3.17)$$

This is known as a Vandermonde matrix, a matrix where each row can be written as a geometric progression. Additionally, F_N can be plotted as a very interesting pattern, as seen in Appendix B as a bit of fun, albeit irrelevant to the thesis.

(3.17) is not much less expensive to compute, only in the sense that vectorisation is more efficient than for-loops; however, writing it in this way allows us to see that F_N has symmetries we can exploit to simplify the computation.

For further simplification, let us assume that N is some power of 2. This allows us to reorder our matrices into equally-sized even and odd components:

$$\begin{bmatrix} \hat{f}_0 \\ \hat{f}_1 \\ \hat{f}_2 \\ \dots \\ \hat{f}_{N-1} \end{bmatrix} = P \begin{bmatrix} 1 & 1 & \dots & 1 & 1 & 1 & 1 & 1 \\ 1 & \omega_N^2 & \dots & \omega_N^{N-2} & \omega_N & \omega_N^3 & \dots & \omega_N^{N-1} \\ 1 & \omega_N^4 & \dots & \omega_N^{2(N-2)} & \omega_N^2 & \omega_N^6 & \dots & \omega_N^{2(N-1)} \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 1 & \omega_N^{2(N-1)} & \dots & \omega_N^{(N-1)(N-2)} & \omega_N^{N-1} & \omega_N^{3(N-1)} & \dots & \omega_N^{(N-1)^2} \end{bmatrix} \begin{bmatrix} f_0 \\ f_2 \\ \dots \\ f_{N-2} \\ f_1 \\ f_3 \\ \dots \\ f_{N-1} \end{bmatrix} \quad (3.18)$$

where P is a permutation matrix that re-orders our even and odd split back to the regular order to match with the \hat{f} vector and satisfies $P = P^T$.

With how messy the F_N matrix in (3.18) is starting to look, we can avoid accidental mathematical hiccups by rewriting F_N in a nicer-to-read way:

$$F_N = \begin{bmatrix} E & O \end{bmatrix} \quad (3.19)$$

where

$$E = \sum_{m=0}^{N/2-1} f(2m) e^{\frac{-2\pi i(2m)k}{N}} \quad (3.20)$$

denotes the values corresponding to the even indices of f , and

$$O = \sum_{m=0}^{N/2-1} f(2m+1) e^{\frac{-2\pi i(2m+1)k}{N}} \quad (3.21)$$

likewise denotes the odd indices.

To start reducing the complexity of the computation of the discrete Fourier transform, we can decompose E and O into equally sized sets based on k . Namely,

$$E = E|_{k < \frac{N}{2}} + E|_{k \geq \frac{N}{2}} \quad (3.22)$$

$$O = O|_{k < \frac{N}{2}} + O|_{k \geq \frac{N}{2}} \quad (3.23)$$

Due to the periodicity of the Fourier transform, the k values greater than or equal to $\frac{N}{2}$ are identical to the k values less than this cutoff, added by $\frac{N}{2}$. Defining p to represent $k < \frac{N}{2}$, we can rewrite (3.22) and (3.23) as

$$E = E|_p + E|_{p+\frac{N}{2}} \quad (3.24)$$

$$O = O|_p + O|_{p+\frac{N}{2}} \quad (3.25)$$

Substituting (3.20) and (3.21) into (3.24) and (3.25), respectively, we get

$$E = \sum_{m=0}^{N/2-1} f(2m)e^{\frac{-2\pi i(2m)p}{N}} + \sum_{m=0}^{N/2-1} f(2m)e^{\frac{-2\pi i(2m)(p+\frac{N}{2})}{N}} \quad (3.26)$$

$$O = \sum_{m=0}^{N/2-1} f(2m+1)e^{\frac{-2\pi i(2m+1)p}{N}} + \sum_{m=0}^{N/2-1} f(2m+1)e^{\frac{-2\pi i(2m+1)(p+\frac{N}{2})}{N}} \quad (3.27)$$

Expanding brackets and using the fact that

$$e^{A+B} = e^A e^B, \quad e^{AB} = (e^A)^B \quad (3.28)$$

We may rewrite (3.26) and (3.27) as

$$E = \sum_{m=0}^{N/2-1} f(2m)e^{\frac{-2\pi i m p}{N/2}} + \sum_{m=0}^{N/2-1} f(2m)e^{\frac{-2\pi i m p}{N/2}} (e^{-i\pi})^{2m} \quad (3.29)$$

$$O = \sum_{m=0}^{N/2-1} f(2m+1)e^{\frac{-2\pi i m p}{N/2}} e^{\frac{-2\pi i p}{N}} + \sum_{m=0}^{N/2-1} f(2m+1)e^{\frac{-2\pi i m p}{N/2}} (e^{-i\pi})^{2m} e^{\frac{-2\pi i p}{N}} e^{-i\pi} \quad (3.30)$$

The $e^{\frac{-2\pi i p}{N}}$ terms in (3.30) do not depend on m , so can be factored out of the sum. Additionally, by Euler's formula

$$e^{i\theta} = \cos(\theta) + i \sin(\theta) \quad (3.31)$$

We can deduce that $e^{-i\pi} = -1$. Since $2m$ is strictly even for $m \in \mathbb{N}$, making $(-1)^{2m} = 1$, (3.29) and (3.30) can be simplified to

$$E = \sum_{m=0}^{N/2-1} f(2m)e^{\frac{-2\pi i m p}{N/2}} + \sum_{m=0}^{N/2-1} f(2m)e^{\frac{-2\pi i m p}{N/2}} \quad (3.32)$$

$$O = e^{\frac{-2\pi i p}{N}} \sum_{m=0}^{N/2-1} f(2m+1)e^{\frac{-2\pi i m p}{N/2}} - e^{\frac{-2\pi i p}{N}} \sum_{m=0}^{N/2-1} f(2m+1)e^{\frac{-2\pi i m p}{N/2}} \quad (3.33)$$

Recalling (3.11), the $e^{\frac{-2\pi imp}{N/2}}$ terms in (3.32) and (3.33) are simply $F_{\frac{N}{2}}$. Given that E and O were decomposed based on k , the column component, we can rewrite (3.19) as

$$F_N = \begin{bmatrix} F_{\frac{N}{2}}|_p & \Omega F_{\frac{N}{2}}|_p \\ F_{\frac{N}{2}}|_{p+\frac{N}{2}} & -\Omega F_{\frac{N}{2}}|_{p+\frac{N}{2}} \end{bmatrix} \quad (3.34)$$

where

$$\Omega = \omega_N^p = \left(e^{\frac{-2\pi i}{N}} \right)^p = e^{\frac{-2\pi ip}{N}} \quad (3.35)$$

Let us drop the $|p$ script for ease of readability. We can factor $F_{\frac{N}{2}}$ out of the matrix, leaving us with

$$F_N = \begin{bmatrix} I_{\frac{N}{2}} & \Omega \\ I_{\frac{N}{2}} & -\Omega \end{bmatrix} \begin{bmatrix} F_{\frac{N}{2}} & 0 \\ 0 & F_{\frac{N}{2}} \end{bmatrix} \quad (3.36)$$

Finally, we can substitute (3.36) into (3.18).

$$\begin{bmatrix} \hat{f}_0 \\ \hat{f}_1 \\ \hat{f}_2 \\ \dots \\ \hat{f}_{N-1} \end{bmatrix} = P \begin{bmatrix} I_{\frac{N}{2}} & \Omega \\ I_{\frac{N}{2}} & -\Omega \end{bmatrix} \begin{bmatrix} F_{\frac{N}{2}} & 0 \\ 0 & F_{\frac{N}{2}} \end{bmatrix} \begin{bmatrix} f_0 \\ f_2 \\ \dots \\ f_{N-2} \\ f_1 \\ f_3 \\ \dots \\ f_{N-1} \end{bmatrix} \quad (3.37)$$

The complexity of computation of the identity-containing matrix is $\mathcal{O}(N)$, the P matrix is virtually free as it is a simple reordering, and the $F_{\frac{N}{2}}$ matrix is half the cost of the F_N matrix.

Since we assumed N was a power of 2, we can recursively repeat this algorithm, known as the Cooley-Tukey FFT algorithm for the radix-2 decimation-in-time (DIT) case, on the $F_{\frac{N}{2}}$ matrix to reduce it down to F_2 .

The outcome is now a way to determine the discrete Fourier transform that has computational complexity $\mathcal{O}(N \log(N))$ rather than $\mathcal{O}(N^2)$. Due to exploiting symmetries in the F_N matrix rather than discarding data for simplicity, the output of the FFT is exactly the same as the regular way.

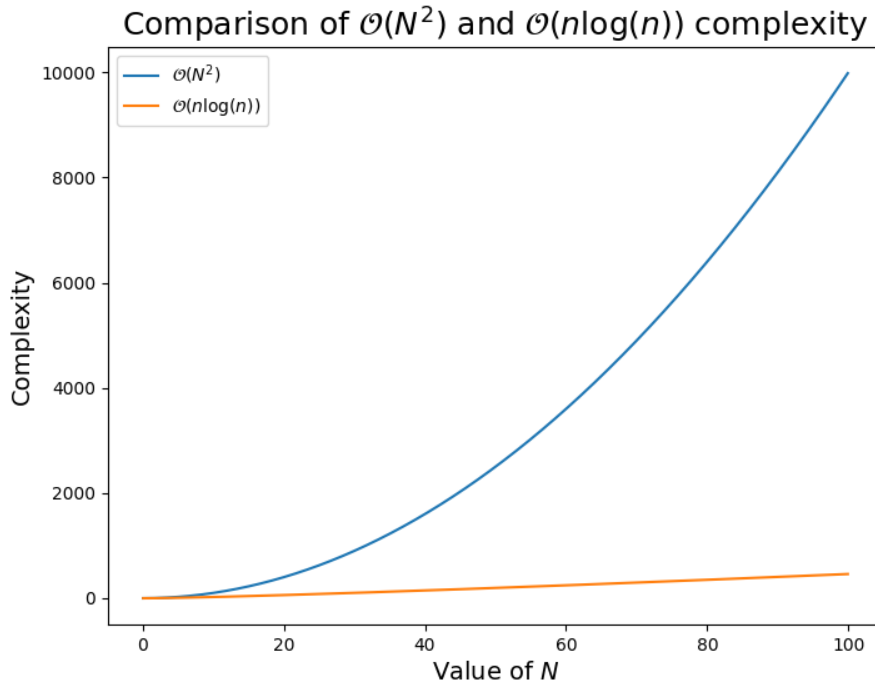


Figure 3.2: Comparison of how $\mathcal{O}(N^2)$ increases in complexity versus $\mathcal{O}(N \log(N))$.

For small values of N , there is not much difference between the two methods. However, as N increases, the complexity of the regular computation of the DFT rapidly increases exponentially, whereas for the FFT it maintains a slow and almost linear increase.

For values of N that are not integer powers of 2, the FFT is so efficient that it is computationally cheaper to inflate the matrices with unnecessary zeros until N is artificially an integer power of 2 and then perform the FFT.

The Cooley-Tukey algorithm is not the only way to develop an FFT, although it is the most common, and other, more efficient, methods do exist.

3.1.2 Optimisation for Real Functions

Recall from Figure 3.1 that the k -space graph had 4 spikes, corresponding to frequencies at $k = -4, k = -2, k = 2, k = 4$. Clearly, there is symmetry here! And one that may be exploited.

For a real x -space function, $f(\vec{x}) \in \mathbb{R}$, the negative k -values become redundant modes, and for the sake of computational efficiency and memory storage, we can disregard these values.

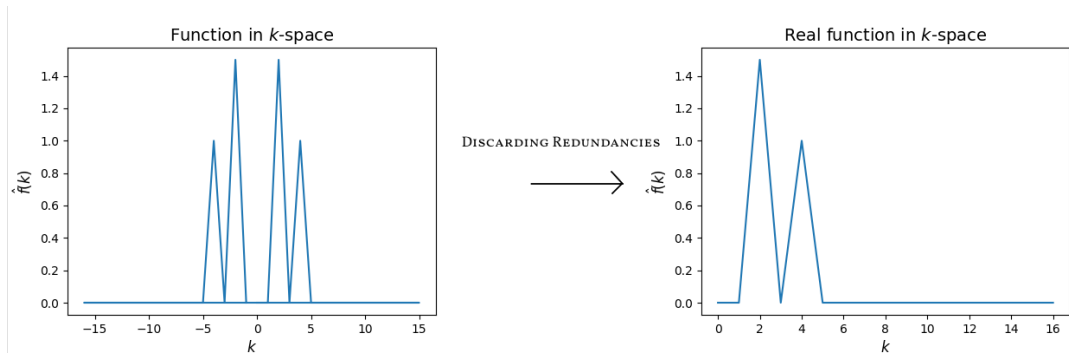


Figure 3.3: The k -space graph from Figure 3.1, with the redundant modes discarded.

While the redundancies are discarded, we are not throwing away important information and data. This is due to the fact that these negative k -values and the complex conjugates of the positive k -values, are what allow us to discard these redundancies in the first place.

Proof 3.1 $\hat{f}(-\vec{k}) = \hat{f}^*(\vec{k})$, for $f(\vec{x}) \in \mathbb{R}$. First, consider (3.7), where

$$\hat{f}(\vec{k}) = \frac{1}{(N)^n} \sum_{m=0}^{N-1} \sum_{m=0}^{N-1} \dots \sum_{m=0}^{N-1} f(\vec{x}) e^{-i\vec{k} \cdot \vec{x}_m} \quad (3.38)$$

Then, if and only if $f(\vec{x}) \in \mathbb{R}$, the complex conjugate of (3.38) is

$$\hat{f}^*(\vec{k}) = \frac{1}{(N)^n} \sum_{m=0}^{N-1} \sum_{m=0}^{N-1} \dots \sum_{m=0}^{N-1} f(\vec{x}) e^{i\vec{k} \cdot \vec{x}_m} \quad (3.39)$$

Using the fact that $1 = (-1)^2$ and some simple algebra, we are finally left with

$$\hat{f}^*(\vec{k}) = \frac{1}{(N)^n} \sum_{m=0}^{N-1} \sum_{m=0}^{N-1} \dots \sum_{m=0}^{N-1} f(\vec{x}) e^{(-1)^2 i \vec{k} \cdot \vec{x}_m} \quad (3.40)$$

$$= \frac{1}{(N)^n} \sum_{m=0}^{N-1} \sum_{m=0}^{N-1} \dots \sum_{m=0}^{N-1} f(\vec{x}) e^{(-1)^i i (-1) \vec{k} \cdot \vec{x}_m} \quad (3.41)$$

$$= \frac{1}{(N)^n} \sum_{m=0}^{N-1} \sum_{m=0}^{N-1} \dots \sum_{m=0}^{N-1} f(\vec{x}) e^{-i(-\vec{k}) \cdot \vec{x}_m} \quad (3.42)$$

$$= \hat{f}(-\vec{k}) \quad (3.43)$$

If $f(\vec{x}) \notin \mathbb{R}$, then the complex conjugate $f^*(\vec{x}) \neq f(\vec{x})$ and $\hat{f}^*(\vec{k}) \neq \hat{f}(-\vec{k})$. \square

As shown in Proof 3.1 above, these redundant modes appear in all dimensions of the transformed real function. Figure 3.3 shows this extension in 2 dimensions, with the negative k_x -values discarded as in the 1D case. The k_y -values behave a bit differently, however. Since our function is periodic, everything to the right of the dashed line is symmetric with the far left shaded area of the graph, much like how $\sin(x)$ is symmetric on intervals of 2π .

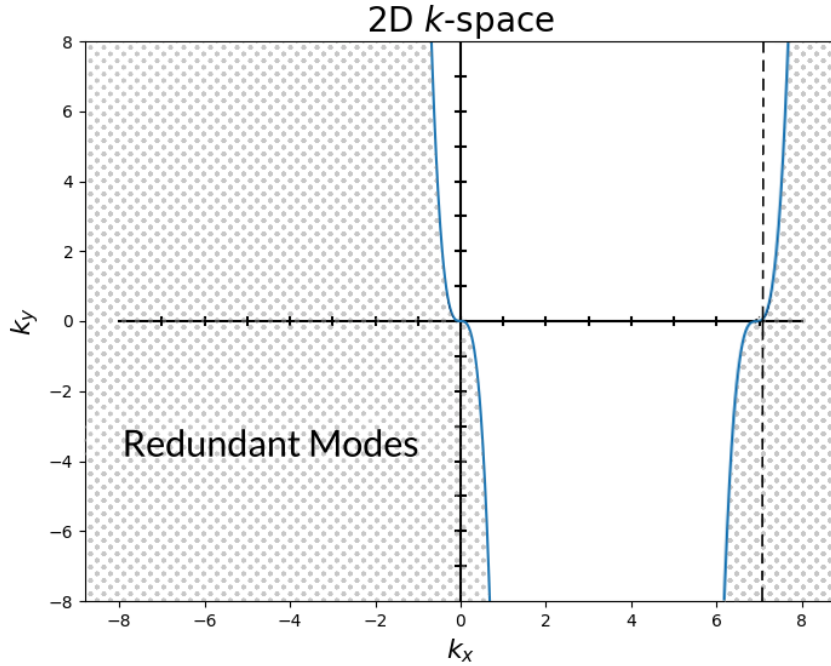


Figure 3.4: Visualisation of the redundant k -space modes in a 2D discrete Fourier transform, with the boundaries $[-8, 8]$ chosen arbitrarily for scale.

Between the k_y -axis and the dashed line, we have additional redundancies for $k_y < 0$. These are the modes that are complex conjugate to $k_x = 0, k_y > 0$ and $k_x = N/2, k_y > 0$.

It is also worth noting that while Figure 3.4 shows these redundant modes separated off by a smooth function, this was merely done for visual clarity in showing which values on an axis are redundant and which are not. Additionally, the way redundant modes are separated here in Figure 3.4 is not unique, and other configurations can be chosen so long as they adhere to the condition that $\hat{f}(-\vec{k}) = \hat{f}^*(\vec{k})$, for $f(\vec{x}) \in \mathbb{R}$.

The 3D case is similar, with redundant modes for (k_x, k_y, k_z) at

$$\begin{aligned}
 & (k_x < 0, k_y, k_z) \\
 & (0, k_y, k_z < 0) \\
 & (0, k_y < 0, 0) \\
 & (0, k_y < 0, N/2) \\
 & (N/2, k_y, k_z < 0) \\
 & (N/2, k_y < 0, 0) \\
 & (N/2, k_y < 0, N/2)
 \end{aligned} \tag{3.44}$$

The (k_y, k_z) planes at $k_x = 0$ and $N/2$ resemble Figure 3.4, with the switchings $k_x \mapsto k_y$ and $k_y \mapsto k_z$. Meanwhile, the rest may be visualised as a cube.

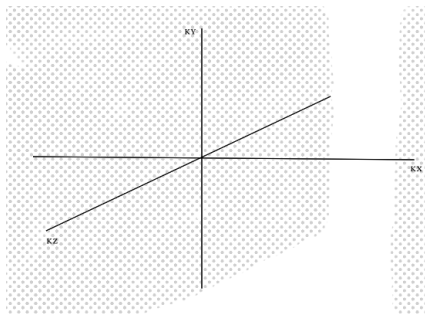


Figure 3.5: Rough visualisation of a k_x, k_y, k_z cube with shaded redundant modes specified in (3.44).

3.2 Bandpass Filtering

A low-pass filter for the velocity field, at some scale L , can be defined as the convolution:

$$\vec{u}^L(\vec{x}) = \{u \star G\}(\vec{x}) = \int \vec{u}(\vec{x} - \vec{r})G(L; r)d\vec{r} \quad (3.45)$$

where $G(L; r)$ is as defined in (3.1).

By definition, as the name suggests, a low pass filter eliminates all frequencies above some k , only keeping those below k . Since we are interested in the behaviour as we go from functions greater than a frequency k to smaller than k , it is useful to consider the x -space derivative, $L \frac{\partial \vec{u}^L}{\partial L}$, which in contrast is centered about a length scale L . Thus, it is useful to define the bandpass-filtered velocity field as [10]:

$$\vec{u}_b^L = \frac{\alpha}{\sqrt{L}}L \frac{\partial \vec{u}^L}{\partial L} = \alpha\sqrt{L} \frac{\partial \vec{u}^L}{\partial L} \quad (3.46)$$

where α is some dimensionless scaling quantity.

As discussed in Section 3.1, the Fourier transform is used to convert our function into k -space to filter the velocity based on length scale. We can use the convolution theorem.

$$\mathcal{F}\{f \star g\} = \mathcal{F}\{f\}\mathcal{F}\{g\} \quad (3.47)$$

on (3.45) to obtain our low-pass filtered velocity field in k -space.

$$\hat{\vec{u}}^L(\vec{k}) = T(\kappa)\hat{\vec{u}}(\vec{k}) \quad (3.48)$$

We can define a transfer function, T , for our Gaussian filter as

$$T(\kappa) = e^{-\kappa^2} = (2\pi)^{\text{dimensions}}\hat{G}(\vec{k}) \quad (3.49)$$

where

$$\kappa = \frac{kL}{2} \quad (3.50)$$

and $k = |\vec{k}|$.

For 2D and 3D, the dimensions we look at in our numerical simulations, respectively, this becomes

$$T(\kappa) = 4\pi^2\hat{G}(\vec{k}), \quad T(\kappa) = 8\pi^3\hat{G}(\vec{k}) \quad (3.51)$$

Proof 3.2 $T(\kappa) = e^{-\kappa^2} = (2\pi)^d \hat{G}(\vec{k})$ is the transfer function of our Gaussian filter.

$T(\kappa)$ is defined to be $e^{-\kappa^2}$, so the only thing to prove is that this is also equal to $(2\pi)^d \hat{G}(\vec{k})$.

We start by analytically deriving the Fourier transform of $G(L; r)$ for some dimension, d , namely

$$\hat{G}(\vec{k}) = \frac{1}{\sqrt{\pi}L} \left(\frac{1}{2\pi} \right)^d \int_{\mathbb{R}} e^{-\frac{r^2}{L^2}} e^{-i\vec{k}\cdot\vec{r}} d\vec{r} \quad (3.52)$$

Differentiating both sides leads to

$$\frac{d\hat{G}}{d\vec{k}} = \frac{1}{\sqrt{\pi}L} \left(\frac{1}{2\pi} \right)^d \int_{\mathbb{R}} e^{-\frac{r^2}{L^2}} (-i\vec{r}) e^{-i\vec{k}\cdot\vec{r}} d\vec{r} \quad (3.53)$$

This \vec{r} part in the $-i\vec{r}$ term can be merged into the derivative of the first exponential:

$$(-i\vec{r}) e^{-\frac{r^2}{L^2}} = \frac{-iL^2}{2} \left(\frac{d}{d\vec{r}} e^{-\frac{r^2}{L^2}} \right) \quad (3.54)$$

Substituting (3.54) into (3.53) then gives us

$$\frac{d\hat{G}}{d\vec{k}} = \frac{-iL}{2\sqrt{\pi}} \left(\frac{1}{2\pi} \right)^d \int_{\mathbb{R}} \left(\frac{d}{d\vec{r}} e^{-\frac{r^2}{L^2}} \right) (-i\vec{r}) e^{-i\vec{k}\cdot\vec{r}} d\vec{r} \quad (3.55)$$

We may now apply integration by parts to the integral, taking the smart choice of $u = \frac{d}{d\vec{r}} e^{-\frac{r^2}{L^2}}$ and $dv = e^{-i\vec{k}\cdot\vec{r}} d\vec{r}$, which returns

$$\int_{\mathbb{R}} \left(\frac{d}{d\vec{r}} e^{-\frac{r^2}{L^2}} \right) (-i\vec{r}) e^{-i\vec{k}\cdot\vec{r}} d\vec{r} = \left[e^{-\frac{r^2}{L^2}} e^{-i\vec{k}\cdot\vec{r}} \right]_{\mathbb{R}} + i\vec{k} \int_{\mathbb{R}} e^{-\frac{r^2}{L^2}} e^{-i\vec{k}\cdot\vec{r}} d\vec{r} \quad (3.56)$$

The first term on the RHS goes to zero, leaving us simply with

$$\int_{\mathbb{R}} \left(\frac{d}{d\vec{r}} e^{-\frac{r^2}{L^2}} \right) (-i\vec{r}) e^{-i\vec{k}\cdot\vec{r}} d\vec{r} = i\vec{k} \int_{\mathbb{R}} e^{-\frac{r^2}{L^2}} e^{-i\vec{k}\cdot\vec{r}} d\vec{r} \quad (3.57)$$

We now substitute (3.57) back into (3.55), yielding

$$\frac{d\hat{G}}{d\vec{k}} = \frac{\vec{k}L}{2\sqrt{\pi}} \left(\frac{1}{2\pi} \right)^d \int_{\mathbb{R}} e^{-\frac{r^2}{L^2}} e^{-i\vec{k}\cdot\vec{r}} d\vec{r} \quad (3.58)$$

(3.58) now very much looks like our analytic $G(\vec{k})$ in (3.52). We can express this with the differential equation:

$$\frac{d\hat{G}}{d\vec{k}} = \frac{\vec{k}L}{2}\hat{G}(\vec{k}) \quad (3.59)$$

This differential equation has the solution.

$$\hat{G}(\vec{k}) = Ae^{-\frac{\kappa^2 L^2}{4}} \quad (3.60)$$

where A is some proportionality constant, determined by solving the initial condition $A = \hat{G}(0)$.

Using the simplification in (3.50), (3.60) just becomes

$$\hat{G}(\vec{k}) = Ae^{-\kappa^2} \quad (3.61)$$

We can find A by simply solving the IVP.

$$A = \hat{G}(0) = \frac{1}{\sqrt{\pi}L} \left(\frac{1}{2\pi}\right)^d \int_{\mathbb{R}} e^{-\frac{r^2}{L^2}} d\vec{r} \quad (3.62)$$

The RHS is a Gaussian integral with the known solution.

$$\int_{\mathbb{R}} e^{-ax^2} d\vec{x} = \sqrt{\frac{\pi}{a}} \quad (3.63)$$

Substituting (3.63) into (3.62) now gives us our A .

$$A = \frac{1}{\sqrt{\pi}L} (\sqrt{\pi}L) \left(\frac{1}{2\pi}\right)^d = \left(\frac{1}{2\pi}\right)^d \quad (3.64)$$

(3.61) now becomes

$$\hat{G}(\vec{k}) = \left(\frac{1}{2\pi}\right)^d e^{-\kappa^2} \quad (3.65)$$

where clearly

$$T(\kappa) = e^{-\kappa^2} = (2\pi)^d \left(\frac{1}{2\pi}\right)^d e^{-\kappa^2} = (2\pi)^d \hat{G}(\vec{k}) \quad (3.66)$$

□

With this new definition of $T(\kappa)$, let us insert it into (3.46) using (3.48):

$$\hat{u}_b^L = -\frac{\alpha\kappa}{\sqrt{L}} \frac{dT}{d\kappa} \hat{u}(\vec{k}) \quad (3.67)$$

In a similar fashion to how we defined (3.46), we can define our bandpass-filtered Gaussian transfer function centered on a scale L as

$$T_b(\kappa) = -\kappa \frac{dT}{d\kappa} = 2\kappa^2 e^{-\kappa^2} \quad (3.68)$$

Naturally, (3.67) now simplifies down to

$$\hat{u}_b^L = \frac{\alpha}{\sqrt{L}} T_b(\kappa) \hat{u}(\vec{k}) \quad (3.69)$$

Since we know that the energy spectrum is given by $E(k) = \frac{1}{2} (\vec{u}(\vec{k}) \cdot \vec{u}(\vec{k}))$, we can take the dot product of (3.69) with itself to derive the bandpass-filtered energy spectrum [10]:

$$E_b^L(L; k) = \frac{\alpha^2}{L} T_b^2(\kappa) E(k) \quad (3.70)$$

In order to satisfy the condition

$$\int_0^\infty E_b^L(L; r) dL = E(k) \quad (3.71)$$

we have

$$\int_0^\infty \frac{\alpha^2}{L} T_b^2(\kappa) E(k) dL = E(k) \quad (3.72)$$

Since $E(k)$ and α^2 have no L -dependence, we can factor these out. $T_b^2(\kappa)$ is dependent on L by definition, due to (3.50), so must remain within the integral.

$$\alpha^2 E(k) \int_0^\infty \frac{1}{L} T_b^2(\kappa) dL = E(k) \quad (3.73)$$

Applying a change of variables to both L and dL into $\kappa = kL/2$, (3.73) becomes

$$\alpha^2 E(k) \int_0^\infty \frac{1}{\kappa} T_b^2(\kappa) d\kappa = E(k) \quad (3.74)$$

Expanding out $T_b^2(\kappa)$, we get

$$4\alpha^2 E(k) \int_0^\infty \kappa^4 e^{-2\kappa^2} d\kappa = E(k) \quad (3.75)$$

Solving the integral and rearranging for α then allows us to find

$$4\alpha^2 E(k) \left(\frac{1}{8}\right) = E(k) \quad (3.76)$$

$$\alpha^2 \frac{E(k)}{2} = E(k) \quad (3.77)$$

$$\alpha^2 = 2 \quad (3.78)$$

$$\alpha = \sqrt{2} \quad (3.79)$$

Thus, our bandpass-filtered velocity field in k -space becomes

$$\boxed{\hat{u}_b^L(\vec{k}) = 2\sqrt{\frac{2}{L}}\kappa^2 e^{-\kappa^2} \hat{u}(\vec{k})} \quad (3.80)$$

3.3 Scale Locality

Having constructed the bandpass filter, we can now apply it to our scale-decomposed transfer functions for energy and enstrophy in Chapter 1, namely (2.113) and (2.127), respectively.

The bandpass-filtered energy flux can be modelled by

$$\boxed{\Pi_{V,b}^r(r) = \langle S_{ij}^L \tau_{ij}^S - S_{ij}^S \tau_{ij}^L \rangle} \quad (3.81)$$

where

$$S_{ij}^r = \frac{1}{2} \left(\frac{\partial u_{b,i}^r}{\partial x_j} + \frac{\partial u_{b,j}^r}{\partial x_i} \right), \quad \tau_{ij}^r = -u_{b,i}^r u_{b,j}^r \quad (3.82)$$

Likewise, the bandpass-filtered enstrophy flux can be modelled by

$$\boxed{F_b^r(r) = \langle \vec{\omega}_b^L \cdot (\vec{u} \cdot \nabla \vec{\omega}_b^S) \rangle} \quad (3.83)$$

where

$$\hat{\omega}_b^L(\vec{k}) = 2\sqrt{\frac{2}{L}}\kappa^2 e^{-\kappa^2} \hat{\omega}(\vec{k}) \quad (3.84)$$

We can plot the transfer functions (3.81) and (3.83) with respect to a scaling S/L value, the ratio of the smaller length scale S compared to the larger L as it increases.

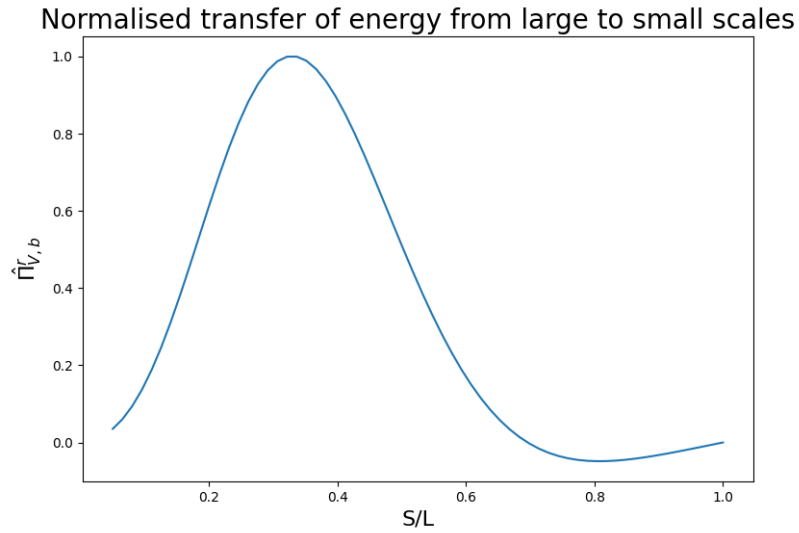


Figure 3.6: Transfer of energy of a hydrodynamic fluid, plotted with $N = 32^3$ simulation data.

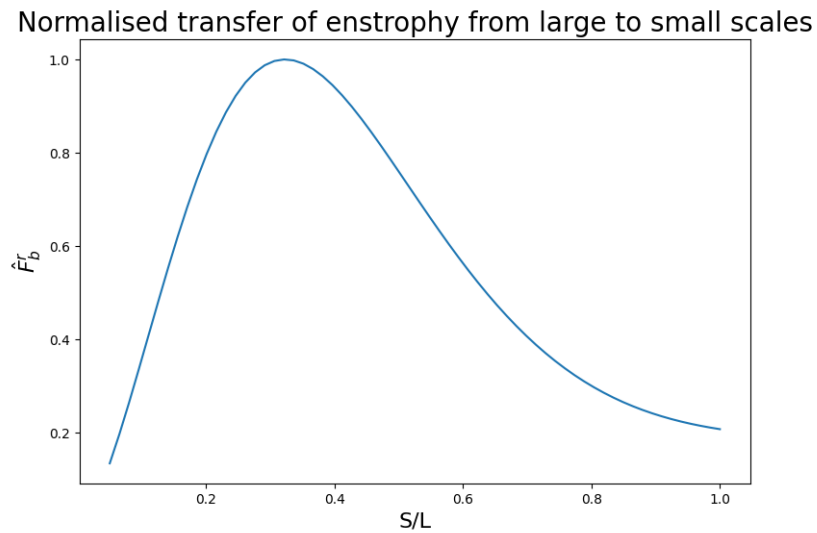


Figure 3.7: Transfer of enstrophy of a hydrodynamic fluid, plotted with $N = 32^3$ simulation data.

Here, $\hat{\Pi}_{V,b}^r$ and \hat{F}_b^r denote the normalised transfer functions, defined as

$$\hat{\Pi}_{V,b}^r = \frac{\Pi_{V,b}^r}{\max\{\Pi_{V,b}^r\}}, \quad \hat{F}_b^r = \frac{F_b^r}{\max\{F_b^r\}} \quad (3.85)$$

In Figure 3.6, we can see that most energy is transferred to turbulent structures at a length scale of $\sim 0.3L$. Similarly for Figure 3.7, with the most entropy transferred to a length scale of around $0.3L$.

These results resemble the results and outcome in Doan et al [6].

3.3.1 Verification of Numerical Bandpass

A verification of using numerical methods to approximate this analytic bandpass filter can be found in Appendix C, using a spherical example from Leung [10].

Chapter 4

Magnetohydrodynamics

Sections 4.1 and 4.2 include results from Biskamp [2], the derivation presented in greater detail in this chapter.

Section 4.3 is original work. While the theory of scale-decomposition is not a new one, the method and the results of deriving the scale-decomposed MHD transfer functions using the regular hydrodynamic scale-decomposition method used in results from Doan et al [6] are.

Sometimes a fluid has electrically conductive properties, and as a result, the fluid generates a magnetic field that interacts with itself. This phenomenon is rare on Earth, often only seen in laboratory nuclear-fusion research, which exhibits negligible dynamics [2]; however, beyond our planet, plasma, a gas consisting of ionised or electrically charged particles, is the most dominant form of visible matter in the universe [4].

The study of these magnetic fluids, magnetohydrodynamics (MHD), does not have nearly as much attention as ordinary turbulence [2], discussed in Chapter 2, and as such, not much is known about MHD turbulence.

While scale remains important for turbulent behaviour, magnetic field interactions only occur at a macroscopic, but non-relativistic, level [2].

Due to the relationship between magnetohydrodynamic turbulence and ordinary hydrodynamic turbulence, we can derive the MHD equations from them.

4.1 The MHD Equations

We begin by stating Maxwell's equations in differential form, the equations that, along with Ohm's law and the Lorenz force, form the fundamentals of our understanding of classical electromagnetism.

In SI units, these are:

$$\nabla \cdot \vec{E} = \frac{\rho}{\varepsilon_0}, \quad \text{Gauss' Law} \quad (4.1)$$

$$\nabla \cdot \vec{B} = 0, \quad \text{Gauss' Law for Magnetism} \quad (4.2)$$

$$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}, \quad \text{Faraday's Law of Induction} \quad (4.3)$$

$$\nabla \times \vec{B} = \mu_0 \vec{J} + \mu_0 \varepsilon_0 \frac{\partial \vec{E}}{\partial t}, \quad \text{Ampère-Maxwell Equation} \quad (4.4)$$

where \vec{E} and \vec{B} are the electric and magnetic fields, respectively, \vec{J} is the electric current density, ρ is the electric charge density, ε_0 is the permittivity in a vacuum, and μ_0 is the permeability in a vacuum.

While there are 8 equations in total (reduced from 20 in Maxwell's original paper) [11], the remaining four are less commonly used and are irrelevant to this thesis.

Gauss' Law (4.1) describes how an electric field interacts with charges, namely that it is attracted to negative charges and repelled from positive charges, similar to how attempting to put the same poles of two magnets together forces them apart and opposite poles bring them together.

From (4.2), we have the condition that magnetic monopoles, predicted in string theory, do not exist and that North and South magnetic poles cannot exist in isolation; a magnetic field must be dipolar, with the same amount of magnetic field lines entering a region as there are leaving it.

Faraday's Law of Induction (4.3) describes how the rotational motion in an electric field on a closed loop corresponds to the rate of change of the magnetic field through an enclosed surface.

(4.4) is in fact an alteration of Ampère's Law, with Maxwell's inclusion of the displacement current density, which is described by the rate of change of the electric field. It states that when the electric field changes, it causes a rotation in the magnetic field, resulting in the existence of electromagnetic waves, which travel at the speed of light $c = \frac{1}{\sqrt{\mu_0 \epsilon_0}}$.

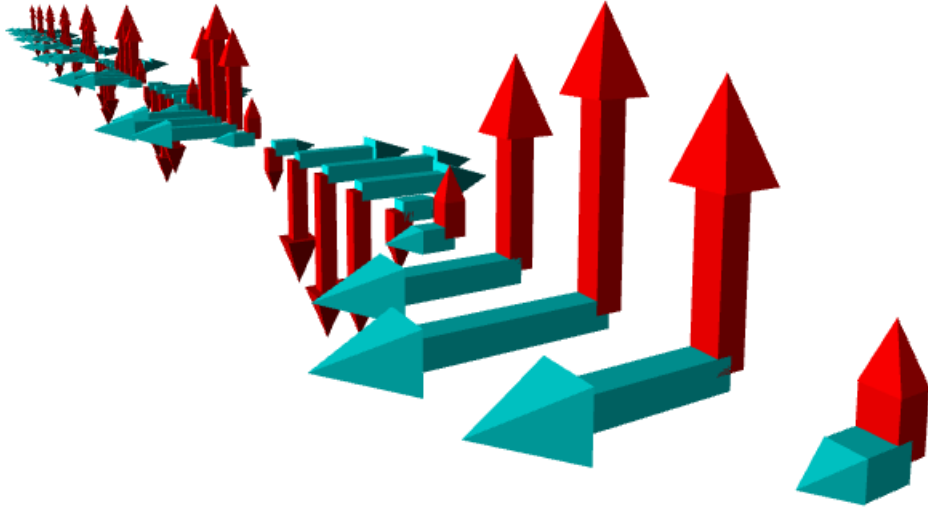


Figure 4.1: An example of an electromagnetic wave.

Taking into account that the fluid has a magnetic field, \vec{B} , and is moving at a velocity, \vec{u} , we can express Ohm's Law as

$$\vec{J} = \sigma \left(\vec{E} + \vec{u} \times \vec{B} \right) \quad (4.5)$$

where σ is the electrical conductivity, which is physically inverse to the resistivity.

As a consequence of this, we can roughly see that $\vec{E} \approx -\vec{u} \times \vec{B}$. The displacement term in (4.4) approximates some velocity term, which, when divided by $c^2 = \frac{1}{\mu_0 \epsilon_0}$ becomes zero for non-relativistic velocities, $u \ll c$.

This means the displacement term in (4.4) can be neglected for non-relativistic velocities, and the equation can be simplified down to

$$\frac{1}{\mu_0} \nabla \times \vec{B} = \vec{J} \quad (4.6)$$

Rearranging Ohm's Law (4.5) and substituting (4.6) in place of \vec{J} leaves us with

$$\vec{E} = \frac{1}{\sigma\mu_0} \nabla \times \vec{B} - \vec{u} \times \vec{B} \quad (4.7)$$

Taking the curl of both sides results in

$$\nabla \times \vec{E} = \nabla \times \left(\frac{1}{\sigma\mu_0} \nabla \times \vec{B} - \vec{u} \times \vec{B} \right) \quad (4.8)$$

$$= \nabla \times \left(\frac{1}{\sigma\mu_0} \nabla \times \vec{B} \right) - \nabla \times (\vec{u} \times \vec{B}) \quad (4.9)$$

$$= \frac{1}{\sigma\mu_0} \left(\nabla(\nabla \cdot \vec{B}) - \nabla^2 \vec{B} \right) - \nabla \times (\vec{u} \times \vec{B}) \quad (4.10)$$

From (4.2), we know that the divergence of the magnetic field is zero. Defining $\eta = \frac{1}{\sigma\mu_0}$ as the magnetic diffusivity, (4.10) becomes

$$\nabla \times \vec{E} = -\eta \nabla^2 \vec{B} - \nabla \times (\vec{u} \times \vec{B}) \quad (4.11)$$

Finally, using Faraday's Law (4.3), we arrive at the induction equation for MHD:

$$\boxed{\frac{\partial \vec{B}}{\partial t} = \nabla \times (\vec{u} \times \vec{B}) + \eta \nabla^2 \vec{B}} \quad (4.12)$$

The momentum equation for MHD is fairly trivial to derive. In equation (2.1), we substitute \vec{F} for the Lorenz force.

$$\vec{F} = \rho \vec{E} + \vec{J} \times \vec{B} \quad (4.13)$$

where q is the charge of a particle in the fluid. Similarly to the displacement term in (4.4), this $\rho \vec{E}$ can be neglected for non-relativistic flows. This, then, gives us

$$\frac{\partial \vec{u}}{\partial t} + (\vec{u} \cdot \nabla) \vec{u} = -\nabla P + \nu \nabla^2 \vec{u} + \vec{J} \times \vec{B} \quad (4.14)$$

We can rewrite (4.14) using (4.6), and hence arrive at the momentum equation for MHD:

$$\boxed{\frac{\partial \vec{u}}{\partial t} + (\vec{u} \cdot \nabla) \vec{u} = -\nabla P + \nu \nabla^2 \vec{u} + \frac{(\nabla \times \vec{B}) \times \vec{B}}{\mu_0}} \quad (4.15)$$

The third MHD equation is the mass continuity equation from ordinary hydrodynamics.

$$\boxed{\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \vec{u}) = 0} \quad (4.16)$$

which states that matter can neither be created nor destroyed. For an incompressible fluid, the density ρ is unchanging, and (4.16) reduces down to the incompressibility condition $\nabla \cdot \vec{u} = 0$.

Finally, for an adiabatic flow, that is, a flow where heat cannot be transferred into or out of the fluid, we have the adiabatic energy equation for MHD:

$$\boxed{\frac{D}{Dt} (P \rho^{-\gamma}) = 0} \quad (4.17)$$

where $\gamma = \frac{C_p}{C_v}$ is the ratio of specific heats, normally taken as $\frac{5}{3}$ [2], and

$$\frac{D}{Dt}(A) = \frac{\partial A}{\partial t} + \vec{u} \cdot \nabla A \quad (4.18)$$

is defined as the Lagrangian derivative (also known as the material or advection derivative) for some $A = A(\vec{x}, t)$. This appears in the LHS of the Navier-Stokes equation, meaning it too can be written in terms of the Lagrangian derivative $\frac{D\vec{u}}{Dt} = -\nabla P + \nu \nabla^2 \vec{u}$.

The equations (4.12), (4.15), (4.16), and (4.17) are known as the magnetohydrodynamic (MHD) equations.

$$\begin{aligned} \frac{\partial \vec{B}}{\partial t} &= \nabla \times (\vec{u} \times \vec{B}) + \eta \nabla^2 \vec{B} \\ \frac{\partial \vec{u}}{\partial t} + (\vec{u} \cdot \nabla) \vec{u} &= -\nabla P + \nu \nabla^2 \vec{u} + \frac{(\nabla \times \vec{B}) \times \vec{B}}{\mu_0} \\ \frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \vec{u}) &= 0 \\ \frac{D}{Dt} (P \rho^{-\gamma}) &= 0 \end{aligned}$$

4.2 MHD Waves

In nature, it is usually observed that a plasma has strong fluctuations about some mean state [2]. Since such wave phenomena are a fundamental component of turbulence, it is important to understand the behaviour of these waves.

In regular hydrodynamics, these waves take the form of eddies. However, in MHD, we have several types of waves, as shown in this section.

First, we consider that our flow has some equilibrium.

$$\bar{u} = 0 \tag{4.19}$$

Strictly speaking, as a vector field, the notation ought to be $\vec{\bar{u}}$; however, for many terms such as this, the equations will quickly become messy, so the vector arrow is omitted for readability.

It is also convenient to consider a uniform plasma, that is, one where the mean pressure and mean magnetic field are constant throughout the entire fluid.

$$\bar{P} = \text{const}, \quad \bar{B} = \text{const} \tag{4.20}$$

where $\tilde{P} \ll \bar{P}$ and $\tilde{B} \ll \bar{B}$ are sufficiently small perturbations.

We can now express our variables as a perturbation, dependent on position and time, about some mean:

$$\vec{B} = \bar{B} + \tilde{B}(\vec{x}, t) \tag{4.21}$$

$$\vec{u} = \bar{u} + \tilde{u}(\vec{x}, t) = \tilde{u}(\vec{x}, t) \tag{4.22}$$

$$P = \bar{P} + \tilde{P}(\vec{x}, t) \tag{4.23}$$

$$\rho = \bar{\rho} + \tilde{\rho}(\vec{x}, t) \tag{4.24}$$

Since we are assuming sufficiently small perturbations, we can substitute (4.21)-(4.24) into the MHD equations and linearise.

$$\frac{\partial}{\partial t} (\bar{B} + \tilde{B}) = \nabla \times (\tilde{u} \times (\bar{B} + \tilde{B})) + \eta \nabla^2 (\bar{B} + \tilde{B}) \quad (4.25)$$

$$\frac{\partial \tilde{u}}{\partial t} + (\tilde{u} \cdot \nabla) \tilde{u} = -\nabla (\bar{P} + \tilde{P}) + \nu \nabla^2 \tilde{u} + \frac{(\nabla \times (\bar{B} + \tilde{B})) \times (\bar{B} + \tilde{B})}{\mu_0} \quad (4.26)$$

$$\frac{\partial}{\partial t} (\bar{\rho} + \tilde{\rho}) + \nabla \cdot ((\bar{\rho} + \tilde{\rho}) \tilde{u}) = 0 \quad (4.27)$$

$$\frac{D}{Dt} \left((\bar{P} + \tilde{P}) (\bar{\rho} + \tilde{\rho})^{-\gamma} \right) = 0 \quad (4.28)$$

Neglecting higher-order terms and recognising the derivative of a const is zero; (4.25)-(4.28) cleanly reduce down to

$$\frac{\partial \tilde{B}}{\partial t} = \nabla \times (\tilde{u} \times \bar{B}) + \eta \nabla^2 \tilde{B} \quad (4.29)$$

$$\frac{\partial \tilde{u}}{\partial t} = -\nabla \tilde{P} + \frac{1}{\mu_0} \bar{B} \times (\nabla \times \tilde{B}) - \nu \nabla^2 \tilde{u} \quad (4.30)$$

$$\frac{\partial \tilde{\rho}}{\partial t} + \bar{\rho} \nabla \cdot \tilde{u} = 0 \quad (4.31)$$

$$\frac{\partial \tilde{P}}{\partial t} - \frac{\gamma \bar{P}}{\bar{\rho}} \frac{\partial \tilde{\rho}}{\partial t} = 0 \quad (4.32)$$

We can choose the perturbed values to be $\propto e^{i\vec{k}\vec{x} - i\bar{\omega}t}$, where the notation $\bar{\omega}$ is used as convenience to avoid confusion with vorticity or fundamental frequency. This choice in perturbed values reduces the differential operators into products, turning the differential equation into an algebraic one:

$$-i\bar{\omega} \tilde{B} = i\vec{k} \times \tilde{u} \times \tilde{B} - k^2 \eta \tilde{B} \quad (4.33)$$

$$-i\bar{\omega} \tilde{u} = -i\vec{k} \tilde{P} - \frac{1}{\mu_0} (i\vec{k} \times \tilde{B}) \times \bar{B} + k^2 \nu \tilde{u} \quad (4.34)$$

$$-i\bar{\omega} \tilde{\rho} + \bar{\rho} i\vec{k} \cdot \tilde{u} = 0 \quad (4.35)$$

$$-i\bar{\omega} \tilde{P} + \frac{\gamma \bar{P}}{\bar{\rho}} i\bar{\omega} \tilde{\rho} = 0 \quad (4.36)$$

We can decouple the density perturbation by substituting (4.35) into (4.36):

$$-i\bar{\omega} \tilde{P} + \gamma \bar{P} i\vec{k} \cdot \tilde{u} = 0 \quad (4.37)$$

Now the remaining three equations (4.33), (4.34), and (4.37) can be reduced to a singular equation. First, start by rearranging (4.33) so that we can get an expression for \tilde{B} .

$$i\bar{\omega}\tilde{B} + k^2\eta\tilde{B} = i\vec{k} \times \tilde{u} \times \tilde{B} \quad (4.38)$$

$$= (i\vec{k} \cdot \bar{B})\tilde{u} - (i\vec{k} \cdot \tilde{u})\bar{B} \quad (4.39)$$

Factoring out \tilde{B} and dividing yields

$$\tilde{B} = \frac{(i\vec{k} \cdot \bar{B})\tilde{u} - (i\vec{k} \cdot \tilde{u})\bar{B}}{-i\bar{\omega} + k^2\eta} \quad (4.40)$$

We can also rearrange (4.37) to obtain an expression for \tilde{P} .

$$\tilde{P} = \frac{-i\gamma\bar{P}\vec{k} \cdot \tilde{u}}{-i\bar{\omega}} \quad (4.41)$$

$$= \frac{\gamma\bar{P}\vec{k} \cdot \tilde{u}}{\bar{\omega}} \quad (4.42)$$

By substituting (4.40) and (4.42) into (4.34) and neglecting the dissipation term, $k^2\eta$, we get a single linearised equation:

$$-i\bar{\omega}\tilde{u} = \left(\frac{\gamma\bar{P}\vec{k} \cdot \tilde{u}}{\bar{\omega}} \right) (\vec{k} \cdot \tilde{u}) - \frac{1}{\mu_0} \left(i\vec{k} \times \left(\frac{(i\vec{k} \cdot \bar{B})\tilde{u} - (i\vec{k} \cdot \tilde{u})\bar{B}}{-i\bar{\omega}} \right) \right) \times \bar{B} \quad (4.43)$$

Multiplying $i\bar{\omega}$ to both sides gives us

$$\bar{\omega}^2\tilde{u} = (\vec{k}\gamma\bar{P})(\vec{k} \cdot \tilde{u}) - \frac{1}{\mu_0} \left(i\vec{k} \times \left((i\vec{k} \cdot \bar{B})\tilde{u} - (i\vec{k} \cdot \tilde{u})\bar{B} \right) \right) \times \bar{B} \quad (4.44)$$

$$= (\vec{k}\gamma\bar{P})(\vec{k} \cdot \tilde{u}) + \frac{1}{\mu_0} \left(\vec{k} \times \left((\vec{k} \cdot \bar{B})\tilde{u} - (\vec{k} \cdot \tilde{u})\bar{B} \right) \right) \times \bar{B} \quad (4.45)$$

$$= (\vec{k}\gamma\bar{P})(\vec{k} \cdot \tilde{u}) + \frac{\bar{B} \times \vec{k} \times \bar{B}}{\mu_0} (\vec{k} \cdot \tilde{u}) - \frac{\vec{k} \times \tilde{u} \times \bar{B}}{\mu_0} (\vec{k} \cdot \bar{B}) \quad (4.46)$$

$$(4.47)$$

Through some factorising and rearranging, we obtain

$$\bar{\omega}^2\tilde{u} = \underbrace{\left(\vec{k}\gamma\bar{P} + \frac{\bar{B} \times \vec{k} \times \bar{B}}{\mu_0} \right)}_{\text{longitudinal waves}} (\vec{k} \cdot \tilde{u}) - \underbrace{\frac{1}{\mu_0} \vec{k} \cdot \bar{B} (\vec{k} \times \tilde{u}) \times \bar{B}}_{\text{transverse waves}} \quad (4.48)$$

(4.48) clearly shows that plasma, or magnetohydrodynamics in general, generates longitudinal waves proportional to $\vec{k} \cdot \tilde{u}$, travelling along the fluid flow by stretching and compressing, and transverse waves proportional to $\vec{k} \times \tilde{u}$, travelling perpendicular or shear to the fluid flow.

Allow us to choose the coordinate system.

$$\vec{B} = \bar{B} \hat{e}_z, \quad \vec{k} = k_{\perp} \hat{e}_y + k_{\parallel} \hat{e}_z \quad (4.49)$$

where we briefly reintroduced the vector arrows to distinguish between the field vectors, their magnitudes, and the unit vectors.

(4.48) can now be written in matrix form along with the coordinate system from (4.49).

$$\begin{aligned} \bar{\omega}^2 \begin{bmatrix} \tilde{u}_x \\ \tilde{u}_y \\ \tilde{u}_z \end{bmatrix} &= \frac{1}{\mu_0} \left(\begin{bmatrix} 0 \\ 0 \\ \bar{B} \end{bmatrix} \times \begin{bmatrix} 0 \\ k_{\perp} \\ k_{\parallel} \end{bmatrix} \times \begin{bmatrix} 0 \\ 0 \\ \bar{B} \end{bmatrix} + \gamma \bar{P} \begin{bmatrix} 0 \\ k_{\perp} \\ k_{\parallel} \end{bmatrix} \right) \cdot \begin{bmatrix} 0 \\ k_{\perp} \\ k_{\parallel} \end{bmatrix} \cdot \begin{bmatrix} \tilde{u}_x \\ \tilde{u}_y \\ \tilde{u}_z \end{bmatrix} \\ &\quad - \frac{1}{\mu_0} \begin{bmatrix} 0 \\ k_{\perp} \\ k_{\parallel} \end{bmatrix} \cdot \begin{bmatrix} 0 \\ 0 \\ \bar{B} \end{bmatrix} \left(\begin{bmatrix} 0 \\ k_{\perp} \\ k_{\parallel} \end{bmatrix} \times \begin{bmatrix} \tilde{u}_x \\ \tilde{u}_y \\ \tilde{u}_z \end{bmatrix} \right) \times \begin{bmatrix} 0 \\ 0 \\ \bar{B} \end{bmatrix} \end{aligned} \quad (4.50)$$

Moving all terms to the RHS, we can simplify the non- $\bar{\omega}^2$ terms using coordinate vector notation to save space:

$$\begin{aligned} &\frac{-1}{\mu_0} k_{\perp} \bar{B}^2 (k_{\perp} \tilde{u}_y + k_{\parallel} \tilde{u}_z) \hat{y} - \frac{\gamma \bar{P}}{\mu_0} k_{\perp} (k_{\perp} \tilde{u}_y + k_{\parallel} \tilde{u}_z) \hat{y} \\ &- \frac{\gamma \bar{P}}{\mu_0} k_{\parallel} (k_{\perp} \tilde{u}_y + k_{\parallel} \tilde{u}_z) \hat{z} - \frac{\bar{B}^2}{\mu_0} k_{\parallel}^2 \tilde{u}_x \hat{x} - \frac{k_{\parallel} \bar{B}^2}{\mu_0} (k_{\perp} \tilde{u}_z - k_{\parallel} \tilde{u}_y) \hat{y} \end{aligned} \quad (4.51)$$

$$\begin{aligned} &= \frac{-1}{\mu_0} k_{\perp}^2 \bar{B}^2 \tilde{u}_y \hat{y} - \frac{1}{\mu_0} k_{\perp} k_{\parallel} \bar{B}^2 \tilde{u}_z \hat{y} - \frac{\gamma \bar{P}}{\mu_0} k_{\perp}^2 \tilde{u}_y \hat{y} - \frac{\gamma \bar{P}}{\mu_0} k_{\perp} k_{\parallel} \tilde{u}_z \hat{y} \\ &- \frac{\gamma \bar{P}}{\mu_0} k_{\parallel} k_{\perp} \tilde{u}_y \hat{z} - \frac{\gamma \bar{P}}{\mu_0} k_{\parallel}^2 \tilde{u}_z \hat{z} - \frac{k_{\parallel}^2 \bar{B}^2}{\mu_0} \tilde{u}_x \hat{x} - \frac{k_{\parallel} k_{\perp} \bar{B}^2}{\mu_0} \tilde{u}_z \hat{y} + \frac{k_{\parallel}^2 \bar{B}^2}{\mu_0} \tilde{u}_y \hat{y} \end{aligned} \quad (4.52)$$

Adding (4.52) back into the RHS and now including the $\bar{\omega}^2$ terms and factoring out \tilde{u} , (4.50) simply becomes

$$\begin{bmatrix} \bar{\omega}^2 - \frac{k_{\parallel}^2 \bar{B}^2}{\mu_0} & 0 & 0 \\ 0 & \bar{\omega}^2 - \frac{1}{\mu_0} k_{\perp}^2 \bar{B}^2 - \frac{\gamma \bar{P}}{\mu_0} k_{\perp}^2 - \frac{k_{\parallel}^2 \bar{B}^2}{\mu_0} & -\frac{\gamma \bar{P}}{\mu_0} k_{\perp} k_{\parallel} \\ 0 & \frac{\gamma \bar{P}}{\mu_0} k_{\parallel} k_{\perp} & \bar{\omega}^2 - \frac{\gamma \bar{P}}{\mu_0} k_{\parallel}^2 \end{bmatrix} \begin{bmatrix} \tilde{u}_x \\ \tilde{u}_y \\ \tilde{u}_z \end{bmatrix} = 0 \quad (4.53)$$

We can define $u_A = \frac{\bar{B}}{\sqrt{\mu_0}}$ as the Alfvén velocity, the velocity of a low-frequency and dispersionless oscillation in a plasma, $c_s = \sqrt{\frac{\gamma \bar{P}}{\mu_0}}$ as the speed of sound, and $k^2 = k_{\perp}^2 + k_{\parallel}^2$, simplifying (4.53) down to

$$\begin{bmatrix} \bar{\omega}^2 - k_{\parallel}^2 v_A^2 & 0 & 0 \\ 0 & \bar{\omega}^2 - k^2 v_A^2 - k_{\perp}^2 c_s^2 & -k_{\perp} k_{\parallel} c_s^2 \\ 0 & k_{\parallel} k_{\perp} c_s^2 & \bar{\omega}^2 - k_{\parallel}^2 c_s^2 \end{bmatrix} \begin{bmatrix} \tilde{u}_x \\ \tilde{u}_y \\ \tilde{u}_z \end{bmatrix} = 0 \quad (4.54)$$

From (4.54), we can find the dispersion relation of the Alfvén waves by constructing the eigenvalue equation.

$$(\bar{\omega}^2 - k_{\parallel}^2 v_A^2) (\bar{\omega}^4 - \bar{\omega}^2 k^2 (c_s^2 + v_A^2) + k^2 c_s^2 k_{\parallel}^2 v_A^2) = 0 \quad (4.55)$$

Solving (4.55) gives us 3 solutions, and hence 3 types of MHD waves.

The first is known as the shear Alfvén wave, describing an incompressible fluid, $\vec{k} \cdot \tilde{u} = 0$, where the velocity is perpendicular to the mean magnetic field, \bar{B} . This wave is given the notation $\bar{\omega}_A^2$:

$$\bar{\omega}_A^2 = k_{\parallel}^2 v_A^2 \quad (4.56)$$

The next two solutions come by factorising $\bar{\omega}^4 - \bar{\omega}^2 k^2 (c_s^2 + v_A^2) + k^2 c_s^2 k_{\parallel}^2 v_A^2$ with some substitution $W = \bar{\omega}^2$ and solving the quadratic equation:

$$\bar{\omega}^2 = \frac{k^2 (c_s^2 + v_A^2) \pm \sqrt{k^4 (c_s^2 + v_A^2)^2 + 4k^2 c_s^2 v_A^2}}{2} \quad (4.57)$$

$$= \frac{k^2}{2} \left(c_s^2 + v_A^2 \pm \sqrt{(c_s^2 + v_A^2)^2 + 4c_s^2 v_A^2 / k^2} \right) \quad (4.58)$$

Taking the positive case of (4.58), we obtain the compressible Alfvén wave, often known as the fast magnetosonic wave, with the velocity being in the range $c_s^2 + v_A^2 \geq \frac{\bar{\omega}^2}{k^2} \geq v_A^2$

$$\bar{\omega}_{\text{fast}}^2 = \frac{k^2}{2} \left(c_s^2 + v_A^2 + \sqrt{(c_s^2 + v_A^2)^2 + 4c_s^2 v_A^2 / k^2} \right) \quad (4.59)$$

Similarly, for the negative case of (4.58) with the velocity in the range $c_s^2 \geq \frac{\bar{\omega}^2}{k^2} \geq 0$, we have the slow magnetosonic wave:

$$\bar{\omega}_{\text{fast}}^2 = \frac{k^2}{2} \left(c_s^2 + v_A^2 - \sqrt{(c_s^2 + v_A^2)^2 + 4c_s^2 v_A^2 / k^2} \right) \quad (4.60)$$

4.3 Scale Decomposition of MHD Equations

In much the same way as regular hydrodynamic turbulence, the MHD equations can also be scale-decomposed about some wavenumber cutoff, r . The equations that are of particular importance for this are the induction equation (4.12) and the momentum equation (4.15).

We start by decomposing the velocity field, \vec{u} , and the magnetic field, \vec{B} , into their large and small-scale components.

$$\vec{u} = \vec{u}^L + \vec{u}^S \quad (4.61)$$

$$\vec{B} = \vec{B}^L + \vec{B}^S \quad (4.62)$$

As mentioned in Chapter 2, these scale decompositions are not unique and depend on our low-scale filter and our specified scale, r .

4.3.1 Transfer of Kinetic Energy

We can now substitute (4.61) and (4.62) into the momentum equation (4.15).

$$\begin{aligned} \frac{\partial}{\partial t} (\vec{u}^L + \vec{u}^S) + ((\vec{u}^L + \vec{u}^S) \cdot \nabla) (\vec{u}^L + \vec{u}^S) = & -\nabla P + \nu \nabla^2 (\vec{u}^L + \vec{u}^S) \\ & + \frac{\nabla \times (\vec{B}^L + \vec{B}^S) \times (\vec{B}^L + \vec{B}^S)}{\mu_0} \end{aligned} \quad (4.63)$$

Taking the dot product of \vec{u}^L with (4.63) and volume averaging the results, we can use the simplifications from Section 2.3.1 to obtain:

$$\begin{aligned} \frac{\partial}{\partial t} \left\langle \frac{|\vec{u}^L|^2}{2} \right\rangle &= \langle S_{ij}^S \tau_{ij}^L - S_{ij}^L \tau_{ij}^S \rangle - \nu \langle |\omega^L|^2 \rangle \\ &+ \left\langle \vec{u}^L \cdot \left(\frac{\nabla \times (\vec{B}^L + \vec{B}^S) \times (\vec{B}^L + \vec{B}^S)}{\mu_0} \right) \right\rangle \end{aligned} \quad (4.64)$$

where only the final term needs simplification. For $\vec{B} = 0$, we see that (4.64) reduces down to (2.111), as expected.

Using the vector analysis identity

$$(\vec{A} \cdot \nabla) \vec{A} = (\nabla \times \vec{A}) \times \vec{A} - \nabla \left(\frac{1}{2} A^2 \right) \quad (4.65)$$

The final term can be reduced to

$$\begin{aligned} &\left\langle \vec{u}^L \cdot \left(\frac{\nabla \times (\vec{B}^L + \vec{B}^S) \times (\vec{B}^L + \vec{B}^S)}{\mu_0} \right) \right\rangle \\ &= \left\langle \vec{u}^L \cdot \left(\frac{((\vec{B}^L + \vec{B}^S) \cdot \nabla) (\vec{B}^L + \vec{B}^S) - \nabla \left(\frac{1}{2} B^2 \right)}{\mu_0} \right) \right\rangle \end{aligned} \quad (4.66)$$

Expanding out the brackets gives us

$$\begin{aligned} \frac{1}{\mu_0} \left\langle \left(\vec{u}^L \cdot (\vec{B}^L \cdot \nabla) \vec{B}^L + \vec{u}^L \cdot (\vec{B}^S \cdot \nabla) \vec{B}^L + \vec{u}^L \cdot (\vec{B}^L \cdot \nabla) \vec{B}^S + \vec{u}^L \cdot (\vec{B}^S \cdot \nabla) \vec{B}^S \right) \right\rangle \\ - \frac{1}{\mu_0} \left\langle \vec{u}^L \cdot \nabla \left(\frac{1}{2} B^2 \right) \right\rangle \end{aligned} \quad (4.67)$$

To simplify the first term in (4.67), we can start by applying the fact that an volume average of a sum is the sum of volume averages and rewriting it in suffix notation:

$$\begin{aligned} \frac{1}{\mu_0} \left(\left\langle \vec{u}^L \cdot (\vec{B}^L \cdot \nabla) \vec{B}^L \right\rangle + \left\langle \vec{u}^L \cdot (\vec{B}^S \cdot \nabla) \vec{B}^L \right\rangle + \left\langle \vec{u}^L \cdot (\vec{B}^L \cdot \nabla) \vec{B}^S \right\rangle + \left\langle \vec{u}^L \cdot (\vec{B}^S \cdot \nabla) \vec{B}^S \right\rangle \right) \\ = \frac{1}{\mu_0} \left(\langle u_i^L B_j^L \partial_j B_i^L \rangle + \langle u_i^L B_j^S \partial_j B_i^L \rangle + \langle u_i^L B_j^L \partial_j B_i^S \rangle + \langle u_i^L B_j^S \partial_j B_i^S \rangle \right) \end{aligned} \quad (4.68)$$

Next, we apply the identity from Proof 2.2, namely $-\langle(\partial_j f)g\rangle = \langle f\partial_j g\rangle$, to turn our expression into derivatives of a product and thereby apply the product rule:

$$\frac{-1}{\mu_0} (\langle\partial_j(u_i^L B_j^L)B_i^L\rangle + \langle\partial_j(u_i^L B_j^S)B_i^L\rangle + \langle\partial_j(u_i^L B_j^L)B_i^S\rangle + \langle\partial_j(u_i^L B_j^S)B_i^S\rangle) \quad (4.69)$$

$$\begin{aligned} &= \frac{-1}{\mu_0} ([\langle B_i^L u_i^L \partial_j B_j^L \rangle + \langle B_i^L B_j^L \partial_j u_i^L \rangle] + [\langle B_i^L u_i^L \partial_j B_j^S \rangle + \langle B_i^L B_j^S \partial_j u_i^L \rangle]) \\ &+ [\langle B_i^S u_i^L \partial_j B_j^L \rangle + \langle B_i^S B_j^L \partial_j u_i^L \rangle] + [\langle B_i^S u_i^L \partial_j B_j^S \rangle + \langle B_i^S B_j^S \partial_j u_i^L \rangle]) \quad (4.70) \end{aligned}$$

The $\partial_j B_j^r$ terms, for some $r = L, S$, are identically zero due to Gauss' Law for Magnetism (4.2), meaning (4.70) simplifies to

$$\frac{-1}{\mu_0} (\langle B_i^L B_j^L \partial_j u_i^L \rangle + \langle B_i^L B_j^S \partial_j u_i^L \rangle + \langle B_i^S B_j^L \partial_j u_i^L \rangle + \langle B_i^S B_j^S \partial_j u_i^L \rangle) \quad (4.71)$$

This expression is equivalent to the double dot product of the magnetic stress, τ_{ij}^b , with the large-scale strain-rate tensor, S_{ij}^L .

$$\frac{1}{\mu_0} \langle \tau_{ij}^b S_{ij}^L \rangle = \frac{-1}{\mu_0} \langle \vec{B}\vec{B} : \nabla\vec{u} \rangle \quad (4.72)$$

where $\tau_{ij}^b = -B_i B_j$.

Proof 4.1 $\frac{-1}{\mu_0} \langle \vec{B}\vec{B} : \nabla\vec{u} \rangle$ is the same as (4.71). From (2.99), we know that the double dot product is

$$\vec{B}\vec{B} : \nabla\vec{u}^L = (BB)_{ij} (\nabla u^L)_{ji} = \text{trace} \left((\vec{B}\vec{B}) (\nabla\vec{u}^L)^T \right) \quad (4.73)$$

For $(BB)_{ij}$, we have

$$(BB)_{ij} = \begin{bmatrix} B_x B_x & B_y B_x & B_z B_x \\ B_x B_y & B_y B_y & B_z B_y \\ B_x B_z & B_y B_z & B_z B_z \end{bmatrix} \quad (4.74)$$

and for $(\nabla u^L)_{ij}$

$$(\nabla u^L)_{ij} = \begin{bmatrix} \nabla_x u_x^L & \nabla_y u_x^L & \nabla_z u_x^L \\ \nabla_x u_y^L & \nabla_y u_y^L & \nabla_z u_y^L \\ \nabla_x u_z^L & \nabla_y u_z^L & \nabla_z u_z^L \end{bmatrix} \quad (4.75)$$

Taking the transpose of (4.75) yields

$$(\nabla u^L)_{ji} = \begin{bmatrix} \nabla_x u_x^L & \nabla_x u_y^L & \nabla_x u_z^L \\ \nabla_y u_x^L & \nabla_y u_y^L & \nabla_y u_z^L \\ \nabla_z u_x^L & \nabla_z u_y^L & \nabla_z u_z^L \end{bmatrix} \quad (4.76)$$

We can now multiply (4.74) by (4.76) and take the transpose, which is merely the sum of the diagonals of the $(BB)_{ij}(\nabla u^L)_{ji}$ matrix:

$$\begin{aligned} \text{trace} \left((\vec{B}\vec{B})(\nabla \vec{u}^L)^T \right) &= B_x B_x \nabla_x u_x^L + B_y B_y \nabla_y u_x^L + B_z B_x \nabla_z u_x^L \\ &\quad + B_x B_y \nabla_x u_y^L + B_y B_y \nabla_y u_y^L + B_z B_y \nabla_z u_y^L \\ &\quad + B_x B_z \nabla_x u_z^L + B_y B_z \nabla_y u_z^L + B_z B_z \nabla_z u_z^L \end{aligned} \quad (4.77)$$

Collecting the like-terms simplifies (4.77) down to

$$\begin{aligned} \text{trace} \left((\vec{B}\vec{B})(\nabla \vec{u}^L)^T \right) &= B_x B_x \nabla_x u_x^L + B_y B_y \nabla_y u_y^L + B_z B_z \nabla_z u_z^L \\ &\quad + B_x B_y (\nabla_x u_y^L + \nabla_y u_x^L) + B_x B_z (\nabla_x u_z^L + \nabla_z u_x^L) \\ &\quad + B_y B_z (\nabla_y u_z^L + \nabla_z u_y^L) \end{aligned} \quad (4.78)$$

which, from (2.98) we can see are components of S_{ij}^L .

$$\begin{aligned} \text{trace} \left((\vec{B}\vec{B})(\nabla \vec{u}^L)^T \right) &= B_x B_x S_{xx}^L + B_y B_y S_{yy}^L + B_z B_z S_{zz}^L \\ &\quad + 2B_x B_y S_{xy}^L + 2B_x B_z S_{xz}^L + 2B_y B_z S_{yz}^L \end{aligned} \quad (4.79)$$

$$= (BB)_{ij} S_{ij}^L \quad (4.80)$$

The ordering of the indices of S_{ij}^L does not inherently matter since S_{ij} is a symmetric tensor. We can also denote $\tau_{ij}^b = -B_i B_j$.

Furthermore, we can scale decompose $\frac{-1}{\mu_0} \langle \vec{B} \vec{B} : (\nabla \vec{u}^L) \rangle$ for \vec{B} , from which we obtain

$$\frac{-1}{\mu_0} \left(\langle \vec{B}^L \vec{B}^L : (\nabla \vec{u}^L) \rangle + \langle \vec{B}^L \vec{B}^S : (\nabla \vec{u}^L) \rangle + \langle \vec{B}^S \vec{B}^L : (\nabla \vec{u}^L) \rangle + \langle \vec{B}^S \vec{B}^S : (\nabla \vec{u}^L) \rangle \right) \quad (4.81)$$

which is the vector notation form of (4.71). \square

Moving on to the 2nd term in (4.67), we can apply the product rule to expand this to

$$\frac{-1}{\mu_0} \left\langle \vec{u}^L \cdot \nabla \left(\frac{1}{2} B^2 \right) \right\rangle = - \left\langle \nabla \cdot \left(\frac{1}{2} B^2 \vec{u}^L \right) \right\rangle - \left\langle \frac{1}{2} B^2 \nabla \cdot \vec{u}^L \right\rangle = 0 \quad (4.82)$$

The first term meets the condition mentioned in Section 2.3 that the volume average of the spatial derivative of a function is zero, $\langle \partial_j f \rangle = 0$, and the second term becomes zero in the case of an incompressible fluid, $\nabla \cdot \vec{u} = 0$

With (4.81) and (4.82), (4.64) finally reduces down to

$$\begin{aligned} \frac{\partial}{\partial t} \left\langle \frac{|\vec{u}^L|^2}{2} \right\rangle &= \langle S_{ij}^S \tau_{ij}^L - S_{ij}^L \tau_{ij}^S \rangle - \nu \langle |\omega^L|^2 \rangle - \frac{1}{\mu_0} \left\langle \vec{B}^L \vec{B}^L : (\nabla \vec{u}^L) \right\rangle \\ &\quad - \frac{\left(\langle \vec{B}^L \vec{B}^S : (\nabla \vec{u}^L) \rangle + \langle \vec{B}^S \vec{B}^L : (\nabla \vec{u}^L) \rangle + \langle \vec{B}^S \vec{B}^S : (\nabla \vec{u}^L) \rangle \right)}{\mu_0} \end{aligned} \quad (4.83)$$

or, more simply,

$$\boxed{\frac{\partial}{\partial t} \left\langle \frac{|\vec{u}^L|^2}{2} \right\rangle = -\Pi_r^u - \frac{1}{\mu_0} \langle B_i^L B_j^L S_{ij}^L \rangle - \nu \langle |\omega^L|^2 \rangle} \quad (4.84)$$

where

$$\Pi_r^u = \langle S_{ij}^L \tau_{ij}^S - S_{ij}^S \tau_{ij}^L \rangle + \frac{\langle B_i^L B_j^S S_{ij}^L + B_i^S B_j^L S_{ij}^L + B_i^S B_j^S S_{ij}^L \rangle}{\mu_0} \quad (4.85)$$

represents the cascade terms in the scale-decomposed kinetic energy equation, that is, the terms where mixed-scale and low scale stress interact with the large-scale strain-rate tensor, acting as a sink for length scales $L > r$ and as a source for length scales $S < r$, for some specified length scale r .

By symmetry, we have, for small scales:

$$\boxed{\frac{\partial}{\partial t} \left\langle \frac{|\vec{u}^S|^2}{2} \right\rangle = \Pi_r^u - \frac{1}{\mu_0} \langle B_i^S B_j^S S_{ij}^S \rangle - \nu \langle |\omega^S|^2 \rangle} \quad (4.86)$$

The $\langle B_i^L B_j^L S_{ij}^L \rangle$ and $\langle B_i^S B_j^S S_{ij}^S \rangle$ terms in (4.84) and (4.86) respectively are terms that deal purely with fields of large length scale or purely with fields of small length scale and thus do not contribute to the energy transfer across length scales.

4.3.2 Transfer of Magnetic Energy

Moving onto the induction equation, we substitute (4.61) and (4.62) into (4.12).

$$\frac{\partial}{\partial t} (\vec{B}^L + \vec{B}^S) = \nabla \times (\vec{u}^L + \vec{u}^S) \times (\vec{B}^L + \vec{B}^S) + \eta \nabla^2 (\vec{B}^L + \vec{B}^S) \quad (4.87)$$

In the same process as before, we now perform the dot product, this time of the large-scale magnetic field, \vec{B}^L , with (4.63), and take the volume average of the result:

$$\begin{aligned} \left\langle \vec{B}^L \cdot \frac{\partial}{\partial t} (\vec{B}^L + \vec{B}^S) \right\rangle &= \left\langle \vec{B}^L \cdot \nabla \times (\vec{u}^L + \vec{u}^S) \times (\vec{B}^L + \vec{B}^S) \right\rangle \\ &\quad + \left\langle \vec{B}^L \cdot \eta \nabla^2 (\vec{B}^L + \vec{B}^S) \right\rangle \end{aligned} \quad (4.88)$$

The final term in (4.88) is the easiest to simplify. First, factor the magnetic diffusivity, η , out of the volume average.

$$\left\langle \vec{B}^L \cdot \eta \nabla^2 (\vec{B}^L + \vec{B}^S) \right\rangle = \eta \left\langle \vec{B}^L \cdot \nabla^2 (\vec{B}^L + \vec{B}^S) \right\rangle \quad (4.89)$$

Now, with the use of a volume-averaged vector identity [9],

$$\langle f \cdot \nabla^2 g \rangle = - \langle (\nabla \times f) \cdot (\nabla \times g) \rangle, \quad \text{if } \nabla \cdot f = 0 \quad (4.90)$$

The result in (4.89) becomes

$$-\eta \left\langle (\nabla \times \vec{B}^L) \cdot (\nabla \times (\vec{B}^L + \vec{B}^S)) \right\rangle \quad (4.91)$$

Expanding out the brackets in (4.91)

$$-\eta \left(\left\langle (\nabla \times \vec{B}^L) \cdot (\nabla \times \vec{B}^L) \right\rangle + \left\langle (\nabla \times \vec{B}^L) \cdot (\nabla \times \vec{B}^S) \right\rangle \right) \quad (4.92)$$

We can substitute (4.6) in place of \vec{B} , using the fact that since $\mu_0 \vec{J} = \nabla \times \vec{B}$, \vec{J} is solenoidal, and thus a scale decomposition in \vec{B} implies a scale decomposition in \vec{J} . A proof of this for a similar case is in Proof 2.3.

So (4.92) now simplifies to

$$-\eta \left(\left\langle \mu_0^2 |\vec{J}^L|^2 \right\rangle + \left\langle \mu_0^2 (\vec{J}^L) \cdot (\vec{J}^S) \right\rangle \right) \quad (4.93)$$

By Parseval's identity [9], as seen in (2.51), we know that

$$\left\langle \vec{J}^L \cdot \vec{J}^S \right\rangle = 0 \quad (4.94)$$

And hence, (4.93) reduces down to

$$-\eta \left\langle \mu_0^2 |\vec{J}^L|^2 \right\rangle \quad (4.95)$$

Using the same trick with Parseval's identity, we may also simplify the first term in (4.88) to

$$\left\langle \vec{B}^L \cdot \frac{\partial}{\partial t} (\vec{B}^L + \vec{B}^S) \right\rangle = \frac{\partial}{\partial t} \left\langle \frac{|\vec{B}^L|^2}{2} \right\rangle \quad (4.96)$$

Finally, moving onto the middle term in (4.88), we can begin by simplifying the curl of the cross product by using vector identities:

$$\begin{aligned} \nabla \times (\vec{u}^L + \vec{u}^S) \times (\vec{B}^L + \vec{B}^S) &= ((\vec{B}^L + \vec{B}^S) \cdot \nabla)(\vec{u}^L + \vec{u}^S) \\ &\quad - ((\vec{u}^L + \vec{u}^S) \cdot \nabla)(\vec{B}^L + \vec{B}^S) \\ &\quad + (\vec{u}^L + \vec{u}^S)(\nabla \cdot (\vec{B}^L + \vec{B}^S)) \\ &\quad - (\vec{B}^L + \vec{B}^S)(\nabla \cdot (\vec{u}^L + \vec{u}^S)) \end{aligned} \quad (4.97)$$

where the last two terms are zero due to Gauss' Law for Magnetism (4.2) and the condition of incompressible flow.

Now reimplementing the dot of \vec{B}^L alongside the volume average to (4.97), this becomes

$$\left\langle \vec{B}^L \cdot ((\vec{B}^L + \vec{B}^S) \cdot \nabla)(\vec{u}^L + \vec{u}^S) \right\rangle - \left\langle \vec{B}^L \cdot ((\vec{u}^L + \vec{u}^S) \cdot \nabla)(\vec{B}^L + \vec{B}^S) \right\rangle \quad (4.98)$$

For ease of derivation, we rewrite (4.98) into suffix notation as

$$\langle B_i^L (B^L + B^S)_j \partial_j (u^L + u^S)_i \rangle - \langle B_i^L (u^L + u^S)_j \partial_j (B^L + B^S)_i \rangle \quad (4.99)$$

Allow us to first investigate the term $\langle B_i^L (B^L + B^S)_j \partial_j (u^L + u^S)_i \rangle$, of which we can expand out its brackets.

$$\begin{aligned} \langle B_i^L (B^L + B^S)_j \partial_j (u^L + u^S)_i \rangle &= \langle B_i^L B_j^L \partial_j u_i^L \rangle + \langle B_i^L B_j^L \partial_j u_i^S \rangle \\ &\quad + \langle B_i^L B_j^S \partial_j u_i^L \rangle + \langle B_i^L B_j^S \partial_j u_i^S \rangle \end{aligned} \quad (4.100)$$

which is identical to (4.71) without the factor of $\frac{-1}{\mu_0}$. Thus, we obtain

$$\langle B_i^L B_j^L S_{ij}^L \rangle + \langle B_i^L B_j^L S_{ij}^S \rangle + \langle B_i^L B_j^S S_{ij}^L \rangle + \langle B_i^L B_j^S S_{ij}^S \rangle \quad (4.101)$$

Finally, for the last part of simplification, we can now look at and expand out the $-\langle B_i^L (u^L + u^S)_j \partial_j (B^L + B^S)_i \rangle$ term from (4.99), from which we have

$$\begin{aligned} -\langle B_i^L (u^L + u^S)_j \partial_j (B^L + B^S)_i \rangle &= -\langle B_i^L u_j^L \partial_j B_i^L \rangle - \langle B_i^L u_j^L \partial_j B_i^S \rangle \\ &\quad - \langle B_i^L u_j^S \partial_j B_i^L \rangle - \langle B_i^L u_j^S \partial_j B_i^S \rangle \end{aligned} \quad (4.102)$$

Using the identity

$$\langle (\partial_j f) g \rangle = -\langle f \partial_j g \rangle \quad (4.103)$$

from Proof 2.2, (4.102) may be rearranged into

$$\langle (\partial_j B_i^L u_j^L) B_i^L \rangle + \langle (\partial_j B_i^L u_j^L) B_i^S \rangle + \langle (\partial_j B_i^L u_j^S) B_i^L \rangle + \langle (\partial_j B_i^L u_j^S) B_i^S \rangle \quad (4.104)$$

We can perform the product rule on each term, discarding the terms containing $\partial_j u_j^L$ as these are identically zero due to the incompressibility condition.

$$\langle B_i^L u_j^L \partial_j B_i^L \rangle + \langle B_i^S u_j^L \partial_j B_i^L \rangle + \langle B_i^L u_j^S \partial_j B_i^L \rangle + \langle B_i^S u_j^S \partial_j B_i^L \rangle \quad (4.105)$$

For the first and third terms in (4.105), we can apply the chain rule to bring the outside B_i^L term inside the derivative:

$$\langle B_i^L u_j^L \partial_j B_i^L \rangle = \frac{1}{2} \langle u_j^L \partial_j (|B^L|^2) \rangle \quad (4.106)$$

$$\langle B_i^L u_j^S \partial_j B_i^L \rangle = \frac{1}{2} \langle u_j^S \partial_j (|B^L|^2) \rangle \quad (4.107)$$

Applying the product rule to these reduces them further.

$$\frac{1}{2} \langle u_j^L \partial_j (|B^L|^2) \rangle = \frac{1}{2} \underbrace{\langle \partial_j (u_j^L |B^L|^2) \rangle}_{=0} - \frac{1}{2} \underbrace{\langle |B^L|^2 \partial_j u_j^L \rangle}_{=0} \quad (4.108)$$

$$\frac{1}{2} \langle u_j^S \partial_j (|B^L|^2) \rangle = \frac{1}{2} \underbrace{\langle \partial_j (u_j^S |B^L|^2) \rangle}_{=0} - \frac{1}{2} \underbrace{\langle |B^L|^2 \partial_j u_j^S \rangle}_{=0} \quad (4.109)$$

which become zero due to incompressibility and $\langle \partial_j f \rangle = 0$ [9].

So (4.105) reduces down to

$$\langle B_i^S u_j^L \partial_j B_i^L \rangle + \langle B_i^S u_j^S \partial_j B_i^L \rangle \quad (4.110)$$

Rewriting (4.110) back into vector form, this reads as

$$\langle \vec{B}^S \cdot (\vec{u}^L \cdot \nabla) \vec{B}^L \rangle + \langle \vec{B}^S \cdot (\vec{u}^S \cdot \nabla) \vec{B}^L \rangle = \langle \vec{B}^S \cdot [(\vec{u}^L \cdot \nabla) + (\vec{u}^S \cdot \nabla)] \vec{B}^L \rangle \quad (4.111)$$

$$= \langle \vec{B}^S \cdot [(\vec{u}^L + \vec{u}^S) \cdot \nabla] \vec{B}^L \rangle \quad (4.112)$$

$$= \langle \vec{B}^S \cdot (\vec{u} \cdot \nabla) \vec{B}^L \rangle \quad (4.113)$$

(4.113) looks very similar in form to $F(r)$ in (2.127), suggesting this term deals with the flux of magnetic energy across some scale, r .

Therefore, using (4.95), (4.96), (4.101), and (4.113), we can simplify (4.88) to

$$\begin{aligned} \frac{\partial}{\partial t} \left\langle \frac{|\vec{B}^L|^2}{2} \right\rangle &= \left\langle \vec{B}^S \cdot (\vec{u} \cdot \nabla) \vec{B}^L \right\rangle + \left\langle B_i^L B_j^L S_{ij}^L \right\rangle + \left\langle B_i^L B_j^L S_{ij}^S \right\rangle \\ &\quad + \left\langle B_i^L B_j^S S_{ij}^L \right\rangle + \left\langle B_i^L B_j^S S_{ij}^S \right\rangle - \eta \left\langle \mu_0^2 |\vec{J}^L|^2 \right\rangle \end{aligned} \quad (4.114)$$

We can define a similar Π_r^b for the cascade terms between scales for the scale-decomposed magnetic equation and finally obtain

$$\boxed{\frac{\partial}{\partial t} \left\langle \frac{|\vec{B}^L|^2}{2} \right\rangle = \Pi_r^b + \left\langle B_i^L B_j^L S_{ij}^L \right\rangle - \eta \left\langle \mu_0^2 |\vec{J}^L|^2 \right\rangle} \quad (4.115)$$

where

$$\Pi_r^b = \left\langle \vec{B}^S \cdot (\vec{u} \cdot \nabla) \vec{B}^L \right\rangle + \left\langle B_i^L B_j^L S_{ij}^S \right\rangle + \left\langle B_i^L B_j^S S_{ij}^L \right\rangle + \left\langle B_i^L B_j^S S_{ij}^S \right\rangle \quad (4.116)$$

Likewise, the $\left\langle B_i^L B_j^L S_{ij}^L \right\rangle$ term was omitted from the cascade term as it is solely comprised of large-length scale terms and thus cannot describe effects across different scales.

Interestingly, this term is a source (positive) in (4.115) and a sink in the scale-decomposed kinetic energy equation (4.84). This represents kinetic energy being expended at large scales to influence the structure of the large-scale magnetic field lines [1].

By symmetry, we also have the scale-decomposed magnetic energy equation for small scales:

$$\boxed{\frac{\partial}{\partial t} \left\langle \frac{|\vec{B}^S|^2}{2} \right\rangle = -\Pi_r^b + \left\langle B_i^S B_j^S S_{ij}^S \right\rangle - \eta \left\langle \mu_0^2 |\vec{J}^S|^2 \right\rangle} \quad (4.117)$$

where we note that

$$\left\langle \vec{B}^S \cdot (\vec{u} \cdot \nabla) \vec{B}^L \right\rangle = - \left\langle \vec{B}^L \cdot (\vec{u} \cdot \nabla) \vec{B}^S \right\rangle \quad (4.118)$$

similar to (2.127).

4.3.3 Transfer of Kinetic Enstrophy

From (2.114), we know that the enstrophy is derived from the vorticity, $\vec{\omega}$, the curl of the velocity field. We can use the curl of the momentum equation (4.15) to derive the vorticity equation for MHD.

$$\nabla \times \frac{\partial \vec{u}}{\partial t} + \nabla \times (\vec{u} \cdot \nabla) \vec{u} = \nabla \times (-\nabla P) + \nabla \times \nu \nabla^2 \vec{u} + \nabla \times \left(\frac{(\nabla \times \vec{B}) \times \vec{B}}{\mu_0} \right) \quad (4.119)$$

We can use our knowledge from (2.121) to immediately reduce (4.119) to

$$\frac{\partial \vec{\omega}}{\partial t} - (\vec{\omega} \cdot \nabla) \vec{u} + (\vec{u} \cdot \nabla) \vec{\omega} = \nu \nabla^2 \vec{\omega} + \nabla \times \left(\frac{(\nabla \times \vec{B}) \times \vec{B}}{\mu_0} \right) \quad (4.120)$$

By substituting (4.6), we get

$$\frac{\partial \vec{\omega}}{\partial t} - (\vec{\omega} \cdot \nabla) \vec{u} + (\vec{u} \cdot \nabla) \vec{\omega} = \nu \nabla^2 \vec{\omega} + \nabla \times (\vec{J} \times \vec{B}) \quad (4.121)$$

We can further simplify this final term using vector identities, as we did for (4.97):

$$\nabla \times (\vec{J} \times \vec{B}) = (\vec{B} \cdot \nabla) \vec{J} - (\vec{J} \cdot \nabla) \vec{B} + \underbrace{\vec{J}(\nabla \cdot \vec{B})}_{=0} - \underbrace{\vec{B}(\nabla \cdot \vec{J})}_{=0} \quad (4.122)$$

Clearly, $\nabla \cdot \vec{B} = 0$ due to (4.2). $\nabla \cdot \vec{J} = 0$ is not immediately obvious but can be understood by knowing \vec{J} is solenoidal.

Proof 4.2 $\nabla \cdot \vec{J} = 0$. From (4.6), we know that

$$\vec{J} = \frac{1}{\mu_0} \nabla \times \vec{B} \quad (4.123)$$

which is the definition of a function being solenoidal.

Taking the divergence of both sides results in

$$\nabla \cdot \vec{J} = \frac{1}{\mu_0} \nabla \cdot \nabla \times \vec{B} \quad (4.124)$$

The divergence of a curl is, by definition, zero. Therefore

$$\nabla \cdot \vec{J} = 0 \quad (4.125)$$

□

We have now arrived at the vorticity equation for magnetohydrodynamics:

$$\boxed{\frac{\partial \vec{\omega}}{\partial t} - (\vec{\omega} \cdot \nabla) \vec{u} + (\vec{u} \cdot \nabla) \vec{\omega} = (\vec{B} \cdot \nabla) \vec{J} - (\vec{J} \cdot \nabla) \vec{B} + \nu \nabla^2 \vec{\omega}} \quad (4.126)$$

Applying scale decomposition leads to

$$\begin{aligned} & \frac{\partial}{\partial t} (\vec{\omega}^L + \vec{\omega}^S) - ((\vec{\omega}^L + \vec{\omega}^S) \cdot \nabla) (\vec{u}^L + \vec{u}^S) + ((\vec{u}^L + \vec{u}^S) \cdot \nabla) (\vec{\omega}^L + \vec{\omega}^S) \\ &= ((\vec{B}^L + \vec{B}^S) \cdot \nabla) (\vec{J}^L + \vec{J}^S) - ((\vec{J}^L + \vec{J}^S) \cdot \nabla) (\vec{B}^L + \vec{B}^S) + \nu \nabla^2 (\vec{\omega}^L + \vec{\omega}^S) \end{aligned} \quad (4.127)$$

From (2.136), we know that taking the dot product of $\vec{\omega}^L$ with (4.127), the resulting equation takes the form

$$\begin{aligned} \left\langle \frac{\partial}{\partial t} \frac{(\vec{\omega}^L)^2}{2} \right\rangle &= \langle \vec{\omega}^L \cdot (\vec{\omega} \cdot \nabla) \vec{u} \rangle - \langle \vec{\omega}^L \cdot (\vec{u} \cdot \nabla) \vec{\omega}^S \rangle - \nu \langle (\nabla \times \vec{\omega}^L)^2 \rangle \\ &+ \langle \vec{\omega}^L \cdot ((\vec{B}^L + \vec{B}^S) \cdot \nabla) (\vec{J}^L + \vec{J}^S) \rangle \\ &- \langle \vec{\omega}^L \cdot ((\vec{J}^L + \vec{J}^S) \cdot \nabla) (\vec{B}^L + \vec{B}^S) \rangle \end{aligned} \quad (4.128)$$

Allow us to first investigate the $\langle \vec{\omega}^L \cdot ((\vec{B}^L + \vec{B}^S) \cdot \nabla) (\vec{J}^L + \vec{J}^S) \rangle$ term. We can start by expanding out the brackets.

$$\langle \vec{\omega}^L \cdot \vec{B}^L \cdot \nabla \vec{J}^L \rangle + \langle \vec{\omega}^L \cdot \vec{B}^L \cdot \nabla \vec{J}^S \rangle + \langle \vec{\omega}^L \cdot \vec{B}^S \cdot \nabla \vec{J}^L \rangle + \langle \vec{\omega}^L \cdot \vec{B}^S \cdot \nabla \vec{J}^S \rangle \quad (4.129)$$

and then rewrite (4.129) out in suffix notation.

$$\langle \omega_i^L B_j^L \partial_j J_i^L \rangle + \langle \omega_i^L B_j^L \partial_j J_i^S \rangle + \langle \omega_i^L B_j^S \partial_j J_i^L \rangle + \langle \omega_i^L B_j^S \partial_j J_i^S \rangle \quad (4.130)$$

Applying the identity (4.103), (4.130) becomes

$$- \langle (\partial_j \omega_i^L B_j^L) J_i^L \rangle - \langle (\partial_j \omega_i^L B_j^L) J_i^S \rangle - \langle (\partial_j \omega_i^L B_j^S) J_i^L \rangle - \langle (\partial_j \omega_i^L B_j^S) J_i^S \rangle \quad (4.131)$$

Applying the product and ignoring the terms containing $\partial_j B_j^r$ for some $r = S, L$, due to (4.2), we get

$$- \langle J_i^L B_j^L \partial_j \omega_i^L \rangle - \langle J_i^S B_j^L \partial_j \omega_i^L \rangle - \langle J_i^L B_j^S \partial_j \omega_i^L \rangle - \langle J_i^S B_j^S \partial_j \omega_i^L \rangle \quad (4.132)$$

which we know can be expressed as the double dot product.

$$\begin{aligned}
-\langle \vec{J}\vec{B} : \nabla\vec{\omega}^L \rangle &= -\langle \vec{J}^L\vec{B}^L : \nabla\vec{\omega}^L \rangle - \langle \vec{J}^S\vec{B}^L : \nabla\vec{\omega}^L \rangle \\
&\quad - \langle \vec{J}^L\vec{B}^S : \nabla\vec{\omega}^L \rangle - \langle \vec{J}^S\vec{B}^S : \nabla\vec{\omega}^L \rangle
\end{aligned} \tag{4.133}$$

Next, we shall investigate the $-\langle \vec{\omega}^L \cdot ((\vec{J}^L + \vec{J}^S) \cdot \nabla)(\vec{B}^L + \vec{B}^S) \rangle$ term once again by expanding out the brackets into

$$\begin{aligned}
&-\langle \vec{\omega}^L \cdot \vec{J}^L \cdot \nabla\vec{B}^L \rangle - \langle \vec{\omega}^L \cdot \vec{J}^L \cdot \nabla\vec{B}^S \rangle \\
&-\langle \vec{\omega}^L \cdot \vec{J}^S \cdot \nabla\vec{B}^L \rangle - \langle \vec{\omega}^L \cdot \vec{J}^S \cdot \nabla\vec{B}^S \rangle
\end{aligned} \tag{4.134}$$

In suffix notation, this reads as

$$-\langle \omega_i^L J_j^L \partial_j B_i^L \rangle - \langle \omega_i^L J_j^L \partial_j B_i^S \rangle - \langle \omega_i^L J_j^S \partial_j B_i^L \rangle - \langle \omega_i^L J_j^S \partial_j B_i^S \rangle \tag{4.135}$$

Applying the identity (4.103) brings us to

$$\langle (\partial_j \omega_i^L J_j^L) B_i^L \rangle + \langle (\partial_j \omega_i^L J_j^L) B_i^S \rangle + \langle (\partial_j \omega_i^L J_j^S) B_i^L \rangle + \langle (\partial_j \omega_i^L J_j^S) B_i^S \rangle \tag{4.136}$$

Using the product rule and simplifying with (4.125), that is, $\partial_j J_j^r$, for some $r = S, L$, yields

$$\langle B_i^L J_j^L \partial_j \omega_i^L \rangle + \langle B_i^S J_j^L \partial_j \omega_i^L \rangle + \langle B_i^L J_j^S \partial_j \omega_i^L \rangle + \langle B_i^S J_j^S \partial_j \omega_i^L \rangle \tag{4.137}$$

which we can see again results in the double dot product or Frobenius inner product of

$$\begin{aligned}
\langle \vec{B}\vec{J} : \nabla\vec{\omega}^L \rangle &= \langle \vec{B}^L\vec{J}^L : \nabla\vec{\omega}^L \rangle + \langle \vec{B}^S\vec{J}^L : \nabla\vec{\omega}^L \rangle \\
&\quad + \langle \vec{B}^L\vec{J}^S : \nabla\vec{\omega}^L \rangle + \langle \vec{B}^S\vec{J}^S : \nabla\vec{\omega}^L \rangle
\end{aligned} \tag{4.138}$$

It is worth noting that (4.138) does not cancel out (4.133), as, generally, in matrix algebra, $\vec{J}\vec{B} \neq \vec{B}\vec{J}$. With these, we may obtain our scale-decomposed kinetic enstrophy equation.

Substituting (4.133) and (4.138) into (4.128), we get

$$\begin{aligned}
\left\langle \frac{\partial (\vec{\omega}^L)^2}{\partial t} \frac{1}{2} \right\rangle &= \langle \vec{\omega}^L \cdot (\vec{\omega} \cdot \nabla) \vec{u} \rangle - \langle \vec{\omega}^L \cdot (\vec{u} \cdot \nabla \vec{\omega}^S) \rangle - \nu \langle (\nabla \times \vec{\omega}^L)^2 \rangle \\
&\quad - \langle \vec{J}^L \vec{B}^L : \nabla \vec{\omega}^L \rangle - \langle \vec{J}^S \vec{B}^L : \nabla \vec{\omega}^L \rangle - \langle \vec{J}^L \vec{B}^S : \nabla \vec{\omega}^L \rangle \\
&\quad - \langle \vec{J}^S \vec{B}^S : \nabla \vec{\omega}^L \rangle + \langle \vec{B}^L \vec{J}^L : \nabla \vec{\omega}^L \rangle + \langle \vec{B}^S \vec{J}^L : \nabla \vec{\omega}^L \rangle \\
&\quad + \langle \vec{B}^L \vec{J}^S : \nabla \vec{\omega}^L \rangle + \langle \vec{B}^S \vec{J}^S : \nabla \vec{\omega}^L \rangle
\end{aligned} \tag{4.139}$$

We can simplify this down into

$$\begin{aligned}
\left\langle \frac{\partial (\vec{\omega}^L)^2}{\partial t} \frac{1}{2} \right\rangle &= G^{L,u}(r) - F^u(r) - \nu \langle (\nabla \times \vec{\omega}^L)^2 \rangle \\
&\quad - \langle \vec{J}^L \vec{B}^L : \nabla \vec{\omega}^L \rangle + \langle \vec{B}^L \vec{J}^L : \nabla \vec{\omega}^L \rangle
\end{aligned} \tag{4.140}$$

where $G^{L,u}(r)$ is as defined in (2.135) with the u index to distinguish it from its magnetic counterpart, and

$$\begin{aligned}
F^u(r) &= \langle \vec{\omega}^L \cdot (\vec{u} \cdot \nabla \vec{\omega}^S) \rangle + \langle \vec{J}^S \vec{B}^L : \nabla \vec{\omega}^L \rangle + \langle \vec{J}^L \vec{B}^S : \nabla \vec{\omega}^L \rangle \\
&\quad + \langle \vec{J}^S \vec{B}^S : \nabla \vec{\omega}^L \rangle - \langle \vec{B}^S \vec{J}^L : \nabla \vec{\omega}^L \rangle \\
&\quad - \langle \vec{B}^L \vec{J}^S : \nabla \vec{\omega}^L \rangle - \langle \vec{B}^S \vec{J}^S : \nabla \vec{\omega}^L \rangle
\end{aligned} \tag{4.141}$$

is the cascade term denoting the transfer of flux of kinetic enstrophy (hence the u index) from large scales to small scales across some length scale, r . When no magnetic influences are present, that is, $\vec{B} = 0$, this reduces down to (2.128), as expected.

Again, we see the $-\langle \vec{J}^L \vec{B}^L : \nabla \vec{\omega}^L \rangle + \langle \vec{B}^L \vec{J}^L : \nabla \vec{\omega}^L \rangle$ not contained within $F^u(r)$, since these terms are purely of large length scales and thus cannot describe the behaviour of enstrophy across length scales.

By symmetry, for small scales, we have:

$$\begin{aligned}
\left\langle \frac{\partial (\vec{\omega}^S)^2}{\partial t} \frac{1}{2} \right\rangle &= G^{S,u}(r) + F^u(r) - \nu \langle (\nabla \times \vec{\omega}^L)^2 \rangle \\
&\quad + \langle \vec{J}^S \vec{B}^S : \nabla \vec{\omega}^S \rangle - \langle \vec{B}^S \vec{J}^S : \nabla \vec{\omega}^S \rangle
\end{aligned} \tag{4.142}$$

4.3.4 Transfer of the Mean Square Vector Potential

While it may be obvious to repeat the same steps as kinetic enstrophy, by taking the curl of the MHD induction equation to make it a function of the electric current density, \vec{J} , it is actually convention to take the uncurl instead, using the fact that

$$\vec{B} = \nabla \times \vec{a} \quad (4.143)$$

where \vec{a} is the momentum of free electricity per unit volume, a similar quantity to \vec{J} with equivalent cascade terms across a scale r . a^2 is known as our mean square vector potential, and is a more common units used in MHD than \vec{J} .

This also shows that the magnetic field, \vec{B} , is solenoidal, further reinforcing Gauss' Law for Magnetism (4.2). As a result, since \vec{B} can be scale-decomposed, we can do the same to \vec{a} :

$$\vec{a} = \vec{a}^L + \vec{a}^S \quad (4.144)$$

Rewriting the MHD induction equation (4.12) in terms of (4.143) gives us

$$\frac{\partial}{\partial t}(\nabla \times \vec{a}) = \nabla \times (\vec{u} \times (\nabla \times \vec{a})) + \eta \nabla^2(\nabla \times \vec{a}) \quad (4.145)$$

uncurling This simply results in

$$\frac{\partial \vec{a}}{\partial t} = \vec{u} \times (\nabla \times \vec{a}) + \eta \nabla^2 \vec{a} \quad (4.146)$$

We can simplify the middle term using suffict notation, namely:

$$[\vec{u} \times (\nabla \times \vec{a})]_i = \epsilon_{ijk} u_j \epsilon_{kmn} \partial_m a_n \quad (4.147)$$

$$= \epsilon_{kij} \epsilon_{kmn} u_j \partial_m a_n \quad (4.148)$$

$$= (\delta_{im} \delta_{jn} - \delta_{in} \delta_{jm}) u_j \partial_m a_n \quad (4.149)$$

$$= u_j \partial_i a_j - u_j \partial_j a_i \quad (4.150)$$

In vector form, this is written as

$$\vec{u} \times (\nabla \times \vec{a}) = \nabla_a (\vec{u} \cdot \vec{a}) - (\vec{u} \cdot \nabla) \vec{a} \quad (4.151)$$

where ∇_a is known as Feynman notation and means that only the variation due to \vec{a} is considered for that term.

Now (4.146) reads as

$$\frac{\partial \vec{a}}{\partial t} = \nabla_a(\vec{u} \cdot \vec{a}) - (\vec{u} \cdot \nabla)\vec{a} + \eta \nabla^2 \vec{a} \quad (4.152)$$

We can move onto the scale decomposition of (4.152), giving us

$$\frac{\partial}{\partial t}(\vec{a}^L + \vec{a}^S) = \nabla_a((\vec{u}^L + \vec{u}^S) \cdot (\vec{a}^L + \vec{a}^S)) - ((\vec{u}^L + \vec{u}^S) \cdot \nabla)(\vec{a}^L + \vec{a}^S) + \eta \nabla^2(\vec{a}^L + \vec{a}^S) \quad (4.153)$$

To find the mean square potential for large length scales, we take the dot product of \vec{a}^L with (4.153) and volume average the result:

$$\begin{aligned} \left\langle \vec{a}^L \cdot \frac{\partial}{\partial t}(\vec{a}^L + \vec{a}^S) \right\rangle &= \left\langle \vec{a}^L \cdot \nabla_a((\vec{u}^L + \vec{u}^S) \cdot (\vec{a}^L + \vec{a}^S)) \right\rangle \\ &\quad - \left\langle \vec{a}^L \cdot ((\vec{u}^L + \vec{u}^S) \cdot \nabla)(\vec{a}^L + \vec{a}^S) \right\rangle \\ &\quad + \left\langle \vec{a}^L \cdot \eta \nabla^2(\vec{a}^L + \vec{a}^S) \right\rangle \end{aligned} \quad (4.154)$$

Based on our previous derivations, clearly the first and last terms can be simplified to

$$\left\langle \vec{a}^L \cdot \frac{\partial}{\partial t}(\vec{a}^L + \vec{a}^S) \right\rangle = \frac{\partial}{\partial t} \left\langle \frac{|\vec{a}^L|^2}{2} \right\rangle \quad (4.155)$$

$$\left\langle \vec{a}^L \cdot \eta \nabla^2(\vec{a}^L + \vec{a}^S) \right\rangle = \eta \left\langle (\nabla \times \vec{a}^L)^2 \right\rangle \quad (4.156)$$

$$= \eta \left\langle |\vec{B}^L|^2 \right\rangle \quad (4.157)$$

We may now investigate the $-\left\langle \vec{a}^L \cdot ((\vec{u}^L + \vec{u}^S) \cdot \nabla)(\vec{a}^L + \vec{a}^S) \right\rangle$ term, expanding out the brackets.

$$-\left\langle \vec{a}^L \cdot \vec{u}^L \cdot \nabla \vec{a}^L \right\rangle - \left\langle \vec{a}^L \cdot \vec{u}^L \cdot \nabla \vec{a}^S \right\rangle - \left\langle \vec{a}^L \cdot \vec{u}^S \cdot \nabla \vec{a}^L \right\rangle - \left\langle \vec{a}^L \cdot \vec{u}^S \cdot \nabla \vec{a}^S \right\rangle \quad (4.158)$$

This can then be rewritten in suffix notation.

$$-\left\langle a_i^L u_j^L \partial_j a_i^L \right\rangle - \left\langle a_i^L u_j^L \partial_j a_i^S \right\rangle - \left\langle a_i^L u_j^S \partial_j a_i^L \right\rangle - \left\langle a_i^L u_j^S \partial_j a_i^S \right\rangle \quad (4.159)$$

Applying (4.103) to each term, (4.159) may be expressed as

$$\langle (\partial_j a_i^L u_j^L) a_i^L \rangle + \langle (\partial_j a_i^L u_j^L) a_i^S \rangle + \langle (\partial_j a_i^L u_j^S) a_i^L \rangle + \langle (\partial_j a_i^L u_j^S) a_i^S \rangle \quad (4.160)$$

The product rule can now be done on (4.160), and, after omitting the $\partial_j u_j^r = 0$ terms for some $r = S, L$, we get as a result

$$\langle a_i^L u_j^L \partial_j a_i^L \rangle + \langle a_i^S u_j^L \partial_j a_i^L \rangle + \langle a_i^L u_j^S \partial_j a_i^L \rangle + \langle a_i^S u_j^S \partial_j a_i^L \rangle \quad (4.161)$$

The first and third terms in (4.161) can be further simplified with the chain rule, namely

$$\langle a_i^L u_j^L \partial_j a_i^L \rangle = \frac{1}{2} \langle u_j^L \partial_j (a^L)^2 \rangle \quad (4.162)$$

$$\langle a_i^L u_j^S \partial_j a_i^L \rangle = \frac{1}{2} \langle u_j^S \partial_j (a^L)^2 \rangle \quad (4.163)$$

Applying the product rule to each (4.162) and (4.163), these become

$$\frac{1}{2} \langle u_j^L \partial_j (a^L)^2 \rangle = -\frac{1}{2} \langle (a^L)^2 \partial_j u_j^L \rangle + \frac{1}{2} \langle \partial_j (u_j^L (a^L)^2) \rangle \quad (4.164)$$

$$\frac{1}{2} \langle u_j^S \partial_j (a^L)^2 \rangle = -\frac{1}{2} \langle (a^L)^2 \partial_j u_j^S \rangle + \frac{1}{2} \langle \partial_j (u_j^S (a^L)^2) \rangle \quad (4.165)$$

Both (4.164) and (4.165) are identically zero due to the incompressibility condition, $\partial_j u_j^r = 0$, for some $r = S, L$, and the condition that the volume average of the spatial derivative of a function is zero, $\langle \partial_j f \rangle = 0$.

As a result, the $-\langle \vec{a}^L \cdot ((\vec{u}^L + \vec{u}^S) \cdot \nabla)(\vec{a}^L + \vec{a}^S) \rangle$ term in (4.154) simply becomes

$$\langle a_i^S u_j^L \partial_j a_i^L \rangle + \langle a_i^S u_j^S \partial_j a_i^L \rangle \quad (4.166)$$

which, when written in vector form, reads as

$$\langle \vec{a}^S \cdot (\vec{u}^L \cdot \nabla \vec{a}^L) \rangle + \langle \vec{a}^S \cdot (\vec{u}^S \cdot \nabla \vec{a}^L) \rangle = \langle \vec{a}^S \cdot (\vec{u} \cdot \nabla \vec{a}^L) \rangle \quad (4.167)$$

(4.167) looks to be a similar analog to (2.128), suggesting the derivation is on the right path.

Now investigating the more complicated term in (4.154)

$$\langle \vec{a}^L \cdot \nabla_a((\vec{u}^L + \vec{u}^S) \cdot (\vec{a}^L + \vec{a}^S)) \rangle \quad (4.168)$$

We can start by expanding out the brackets to yield

$$\langle \vec{a}^L \cdot \nabla_a(\vec{u}^L \cdot \vec{a}^L) \rangle + \langle \vec{a}^L \cdot \nabla_a(\vec{u}^L \cdot \vec{a}^S) \rangle + \langle \vec{a}^L \cdot \nabla_a(\vec{u}^S \cdot \vec{a}^L) \rangle + \langle \vec{a}^L \cdot \nabla_a(\vec{u}^S \cdot \vec{a}^S) \rangle \quad (4.169)$$

Writing this out in suffix notation, we get

$$\langle a_i^L u_j^L \partial_i a_j^L \rangle + \langle a_i^L u_j^L \partial_i a_j^S \rangle + \langle a_i^L u_j^S \partial_i a_j^L \rangle + \langle a_i^L u_j^S \partial_i a_j^S \rangle \quad (4.170)$$

Using the identity (4.103) on each term, (4.170) may be expressed as

$$- \langle (\partial_i a_i^L u_j^L) a_j^L \rangle - \langle (\partial_i a_i^L u_j^L) a_j^S \rangle - \langle (\partial_i a_i^L u_j^S) a_j^L \rangle - \langle (\partial_i a_i^L u_j^S) a_j^S \rangle \quad (4.171)$$

Applying the product rule now gives us

$$\begin{aligned} & - [\langle a_j^L u_j^L \partial_i a_i^L \rangle + \langle a_j^L a_i^L \partial_i u_j^L \rangle] - [\langle a_j^S u_j^L \partial_i a_i^L \rangle + \langle a_j^S a_i^L \partial_i u_j^L \rangle] \\ & - [\langle a_j^L u_j^S \partial_i a_i^L \rangle + \langle a_j^L a_i^L \partial_i u_j^S \rangle] - [\langle a_j^S u_j^S \partial_i a_i^L \rangle + \langle a_j^S a_i^L \partial_i u_j^S \rangle] \end{aligned} \quad (4.172)$$

It is not immediately clear whether we can say $\partial_i a_i^r = 0$ since \vec{a} has gauge freedom—that is, we do not know if \vec{a} is a solenoidal quantity, $\vec{a} = \nabla \times \vec{f}$. However, the simulation data used in Chapter 5 takes \vec{a} to be a Coulomb gauge, mainly for ease of computation, resulting in the simplification $\partial_i a_i^r = 0$ in this particular instance. To remain consistent with the equations used in Chapter 5's simulations, (4.172) reduces down to

$$- \langle \vec{a}^L \cdot (\vec{a}^L \cdot \nabla \vec{u}^L) \rangle - \langle \vec{a}^S \cdot (\vec{a}^L \cdot \nabla \vec{u}^L) \rangle - \langle \vec{a}^L \cdot (\vec{a}^L \cdot \nabla \vec{u}^S) \rangle - \langle \vec{a}^S \cdot (\vec{a}^L \cdot \nabla \vec{u}^S) \rangle \quad (4.173)$$

(4.173), of course, simplifies down to a more compact version.

$$-\langle \vec{a} \cdot (\vec{a}^L \cdot \nabla \vec{u}) \rangle = \langle \vec{a}^L \cdot (\vec{a} \cdot \nabla \vec{u}) \rangle \quad (4.174)$$

Proof 4.3 (4.173) simplifies to $-\langle \vec{a} \cdot (\vec{a}^L \cdot \nabla \vec{u}) \rangle$. Since we have, in suffix notation, two $a_i^L a_j^L$ and two $a_i^S a_j^L$ terms, we can factor (4.173) into these

$$-\langle a_i^L a_j^L (\partial_j u_i^L + \partial_j u_i^S) \rangle - \langle a_i^S a_j^L (\partial_j u_i^L + \partial_j u_i^S) \rangle \quad (4.175)$$

Using the fact that differentiation is a linear operator, that is, the derivative of a sum is the sum of the derivatives, (4.175) can be rewritten as

$$-\langle a_i^L a_j^L \partial_j (u_i^L + u_i^S) \rangle - \langle a_i^S a_j^L \partial_j (u_i^L + u_i^S) \rangle \quad (4.176)$$

By (2.44), the derivative terms are clearly derivatives of \vec{u} .

$$-\langle a_i^L a_j^L \partial_j u_i \rangle - \langle a_i^S a_j^L \partial_j u_i \rangle \quad (4.177)$$

Both terms contain $a_j^L \partial_j u_i$, allowing us to factor this out.

$$-\langle (a_i^L + a_i^S) a_j^L \partial_j u_i \rangle \quad (4.178)$$

And by (4.144), we can see that (4.178) is simply

$$-\langle \vec{a} \cdot (\vec{a}^L \cdot \nabla \vec{u}) \rangle \quad (4.179)$$

□

Finally, we now have the scale-decomposed mean square potential equation for MHD in a simplified form:

$$\boxed{\frac{\partial}{\partial t} \left\langle \frac{|\vec{a}^L|^2}{2} \right\rangle = G^{L,b}(r) - F^b(r) + \eta \left\langle |\vec{B}^L|^2 \right\rangle} \quad (4.180)$$

where

$$F^b(r) = -\langle \vec{a}^S \cdot (\vec{u} \cdot \nabla \vec{a}^L) \rangle \quad (4.181)$$

is the flux of the mean square vector potential from large to small length scales across some specified length scale, r , and

$$G^{L,b}(r) = \langle \vec{a}^L \cdot (\vec{a} \cdot \nabla \vec{u}) \rangle \quad (4.182)$$

is the generator term, representing the generation of the mean square potential induced by the stretching of large and small-scale vortices.

Considering both $F^b(r)$ and $G^{L,b}(r)$ have similar forms to (2.128) and (2.135), this suggests our derivation has likely yielded the correct solution.

By symmetry, we may also write the scale-decomposed mean square potential equation for small scales.

$$\boxed{\frac{\partial}{\partial t} \left\langle \frac{|\vec{a}^S|^2}{2} \right\rangle = G^{S,b}(r) + F^b(r) + \eta \left\langle |\vec{B}^S|^2 \right\rangle} \quad (4.183)$$

where

$$F^b(r) = \langle \vec{a}^L \cdot (\vec{u} \cdot \nabla \vec{a}^S) \rangle \quad (4.184)$$

and

$$G^{S,b}(r) = \langle \vec{a}^S \cdot (\vec{a} \cdot \nabla \vec{u}) \rangle \quad (4.185)$$

In a similar process to Proof 2.7 and Proof 2.8, we can see that $G^{L,b}(r)$ and $G^{S,b}(r)$ meet the conditions

$$G^{L,b}(r) + G^{S,b}(r) = \langle a_i a_j S_{ij} \rangle \quad (4.186)$$

$$G^{S,b}(0) = 0 \quad (4.187)$$

4.3.5 The Scale-Localised Transfer Functions

To investigate how the transfer functions behave in only a set region, such as the inertial subrange where the energy cascade occurs, we can apply the band-pass filtering techniques introduced in Chapter 3 to derive (3.80) to localise them about a specific scale, r , namely

$$\begin{aligned} \hat{u}_b^L &= 2\sqrt{\frac{2}{L}}\kappa^2 e^{-\kappa^2} \hat{u}(\vec{k}), & \hat{\omega}_b^L &= 2\sqrt{\frac{2}{L}}\kappa^2 e^{-\kappa^2} \hat{\omega}(\vec{k}) \\ \hat{a}_b^L &= 2\sqrt{\frac{2}{L}}\kappa^2 e^{-\kappa^2} \hat{a}(\vec{k}), & \hat{B}_b^L &= 2\sqrt{\frac{2}{L}}\kappa^2 e^{-\kappa^2} \hat{B}(\vec{k}), & \hat{J}_b^L &= 2\sqrt{\frac{2}{L}}\kappa^2 e^{-\kappa^2} \hat{J}(\vec{k}) \end{aligned} \quad (4.188)$$

Hence, the scaled-localised cascade functions describing the transfer of kinetic energy, magnetic energy, kinetic enstrophy, and the mean square potential across some scale, r , can be expressed as

$$\Pi_{r,b}^u = \langle S_{ij}^L \tau_{ij}^S - S_{ij}^S \tau_{ij}^L \rangle + \frac{\langle B_{b,i}^L B_{b,j}^S S_{ij}^L + B_{b,i}^S B_{b,j}^L S_{ij}^L + B_{b,i}^S B_{b,j}^S S_{ij}^L \rangle}{\mu_0} \quad (4.189)$$

$$\Pi_{r,b}^b = \langle \vec{B}_b^S \cdot (\vec{u} \cdot \nabla) \vec{B}_b^L \rangle + \langle B_{b,i}^L B_{b,j}^L S_{ij}^S \rangle + \langle B_{b,i}^L B_{b,j}^S S_{ij}^L \rangle + \langle B_{b,i}^L B_{b,j}^S S_{ij}^S \rangle \quad (4.190)$$

$$\begin{aligned} F_b^u(r) &= \langle \vec{\omega}_b^L \cdot (\vec{u} \cdot \nabla \vec{\omega}_b^S) \rangle + \langle \vec{J}_b^S \vec{B}_b^L : \nabla \vec{\omega}_b^L \rangle + \langle \vec{J}_b^L \vec{B}_b^S : \nabla \vec{\omega}_b^L \rangle \\ &\quad + \langle \vec{J}_b^S \vec{B}_b^S : \nabla \vec{\omega}_b^L \rangle - \langle \vec{B}_b^S \vec{J}_b^L : \nabla \vec{\omega}_b^L \rangle \\ &\quad - \langle \vec{B}_b^L \vec{J}_b^S : \nabla \vec{\omega}_b^L \rangle - \langle \vec{B}_b^S \vec{J}_b^S : \nabla \vec{\omega}_b^L \rangle \end{aligned} \quad (4.191)$$

$$F_b^b(r) = - \langle \vec{a}_b^S \cdot (\vec{u} \cdot \nabla \vec{a}_b^L) \rangle \quad (4.192)$$

Chapter 5

2D Scale Locality in MHD

Sections 5.1 and 5.2 in this chapter are simple and known results in the field, with the derivations in Section 5.1 conducted in detail. Section 5.3 is original work, building on the original results at the end of Chapter 4 with 2D simulations.

5.1 Simplifications in 2D MHD

While turbulence is a 3D phenomenon outside of some very special cases [14], it can often be useful to approximate some fluids to 2 dimensions, such as modelling the behaviour of Earth's atmosphere. To humans, the atmosphere is very clearly a 3D structure, but compared to the scale of Earth as a whole, the atmosphere is a thin, approximately 2D shell.

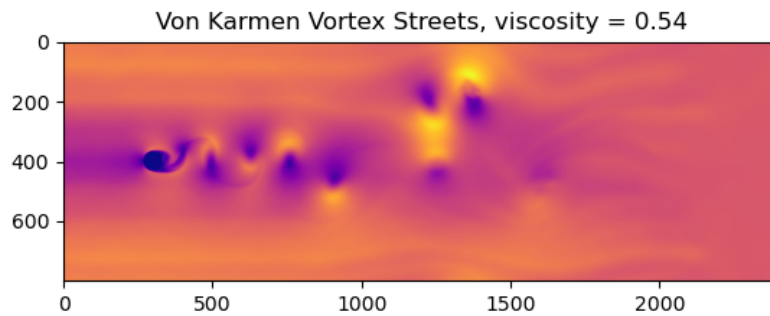


Figure 5.1: 2D simulation of von Karmen vortex streets, displaying an approximate pattern for the real-world phenomenon in Figure 5.2

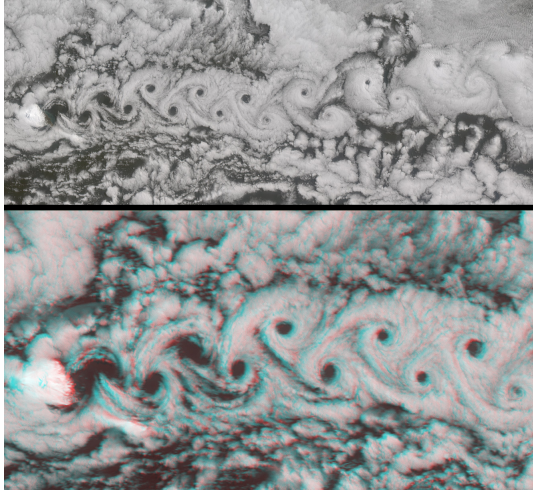


Figure 5.2: Turbulent atmospheric flow of stratocumulus clouds forming von Karman vortex streets over the Arctic, taken by MSIR [12].

By approximating our magnetohydrodynamic flows as 2-dimensional, we can simplify some of the scale-localised transfer equations, allowing for easier and more cost-effective computation.

Consider a 2D flow with an accompanying 2D magnetic field.

$$\vec{u} = u_x(x, y)\hat{x} + u_y(x, y)\hat{y}, \quad \vec{B} = B_x(x, y)\hat{x} + B_y(x, y)\hat{y} \quad (5.1)$$

where \hat{x}, \hat{y} are unit vectors in the x and y directions, respectively.

Then the corresponding vorticity and current density can be expressed as

$$\vec{\omega} = \omega(x, y)\hat{z}, \quad \vec{J} = J(x, y)\hat{z} \quad (5.2)$$

where \hat{z} is a unit vector in the z direction.

Proof 5.1 If $\vec{u} = u_x(x, y)\hat{x} + u_y(x, y)\hat{y}$, then $\vec{\omega} = \omega(x, y)\hat{z}$. From (2.68), we know that

$$\vec{\omega} = \nabla \times \vec{u} \quad (5.3)$$

We can express this in matrix form as

$$\vec{\omega} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ u_x & u_y & 0 \end{vmatrix} \quad (5.4)$$

where the vertical brackets denote the determinant of the matrix.

Expanding out the determinant yields

$$\vec{\omega} = \left(\frac{\partial}{\partial y}(0) - \frac{\partial u_y}{\partial z} \right) \hat{x} + \left(\frac{\partial u_x}{\partial z} - \frac{\partial}{\partial x}(0) \right) \hat{y} + \left(\frac{\partial u_y}{\partial x} - \frac{\partial u_x}{\partial y} \right) \hat{z} \quad (5.5)$$

Since \vec{u} is not a function of z , the derivatives with respect to z are zero.

Hence,

$$\vec{\omega} = \left(\frac{\partial u_y}{\partial x} - \frac{\partial u_x}{\partial y} \right) \hat{z} = \omega_z \hat{z} \quad (5.6)$$

This also proves $\vec{J} = J(x, y) \hat{z}$. \square

Additionally, we can see that

$$\vec{a} = a(x, y) \hat{z} \quad (5.7)$$

Proof 5.2 $\vec{a} = a_z \hat{z}$. Taking \vec{B} as our starting point, we know from Proof 5.1 that

$$\nabla \times \vec{B} = \nabla \times (\nabla \times \vec{a}) = J \hat{z} \quad (5.8)$$

We can now use the vector identity.

$$\nabla \times (\nabla \times \vec{F}) = \nabla(\nabla \cdot \vec{F}) - \nabla^2 \vec{F} \quad (5.9)$$

to write

$$J \hat{z} = \nabla(\nabla \cdot \vec{a}) - \nabla^2 \vec{a} \quad (5.10)$$

In Section 4.3.4, we noted that for the purposes of simulation, we would take \vec{a} to be a Coulomb gauge, resulting in

$$\nabla \cdot \vec{a} = 0 \quad (5.11)$$

Substituting (5.11) into (5.10), we get

$$J \hat{z} = -\nabla^2 \vec{a} \quad (5.12)$$

Further use of vector identities allows us to rewrite (5.12) as

$$J \hat{z} = -\nabla \cdot (\nabla \vec{a}) = -\nabla \cdot \begin{bmatrix} \frac{\partial a_x}{\partial x} & \frac{\partial a_y}{\partial x} & \frac{\partial a_z}{\partial x} \\ \frac{\partial a_x}{\partial y} & \frac{\partial a_y}{\partial y} & \frac{\partial a_z}{\partial y} \\ \frac{\partial a_x}{\partial z} & \frac{\partial a_y}{\partial z} & \frac{\partial a_z}{\partial z} \end{bmatrix} \begin{bmatrix} \hat{x} \\ \hat{y} \\ \hat{z} \end{bmatrix} \quad (5.13)$$

where we are assuming, for the sake of the proof, a generalised form of \vec{a} :

$$\vec{a} = a_x \hat{x} + a_y \hat{y} + a_z \hat{z} \quad (5.14)$$

It follows that

$$\begin{aligned} J\hat{z} &= - \left(\frac{\partial^2 a_x}{\partial x^2} + \frac{\partial^2 a_x}{\partial y^2} + \frac{\partial^2 a_x}{\partial z^2} \right) \hat{x} \\ &\quad - \left(\frac{\partial^2 a_y}{\partial x^2} + \frac{\partial^2 a_y}{\partial y^2} + \frac{\partial^2 a_y}{\partial z^2} \right) \hat{y} \\ &\quad - \left(\frac{\partial^2 a_z}{\partial x^2} + \frac{\partial^2 a_z}{\partial y^2} + \frac{\partial^2 a_z}{\partial z^2} \right) \hat{z} \end{aligned} \quad (5.15)$$

Since the LHS is purely a function in the z direction, then the RHS must be too. Thus, all other terms must be zero.

$$J\hat{z} = - \left(\frac{\partial^2 a_z}{\partial x^2} + \frac{\partial^2 a_z}{\partial y^2} + \frac{\partial^2 a_z}{\partial z^2} \right) \hat{z} = -\nabla^2 a_z \hat{z} \quad (5.16)$$

The z here denotes not that a_z is dependent on z , but rather that it was the component originally in the z direction, $a(x, y)\hat{z}$.

Therefore, \vec{a} must be a function in the z direction only.

$$\vec{a} = a(x, y)\hat{z} \quad (5.17)$$

□

From (5.1), (5.2), and (5.7), it follows that

$$\vec{\omega} \cdot \nabla \vec{u} = 0, \quad \vec{u} \cdot \vec{a} = 0, \quad \vec{J} \cdot \nabla \vec{B} = 0 \quad (5.18)$$

Proof 5.3 *The terms in (5.18) become zero in 2D flows.* Consider, for a 2D flow:

$$\vec{u} = (u_x, u_y, 0) \quad (5.19)$$

$$\vec{\omega} = (0, 0, \omega) \quad (5.20)$$

$$\vec{a} = (0, 0, a_z) \quad (5.21)$$

$$\vec{J} = (0, 0, J_z) \quad (5.22)$$

$$\nabla = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, 0 \right) \quad (5.23)$$

Then $\vec{\omega} \cdot \nabla \vec{u}$ becomes

$$\vec{\omega} \cdot \nabla \vec{u} = \left(0 \times \frac{\partial}{\partial x} + 0 \times \frac{\partial}{\partial y} + \omega_z \times 0 \right) \vec{u} \quad (5.24)$$

$$= 0 \times \vec{u} \quad (5.25)$$

$$= 0 \quad (5.26)$$

However, the opposite, $\vec{u} \cdot \nabla \vec{\omega} = 0$ is false!

$$\vec{u} \cdot \nabla \vec{\omega} = \left(u_x \frac{\partial}{\partial x} + u_y \frac{\partial}{\partial y} + 0 \times 0 \right) \vec{\omega} \quad (5.27)$$

$$= \left(u_x \frac{\partial \omega_z}{\partial x} + u_y \frac{\partial \omega_z}{\partial y} \right) \hat{z} \quad (5.28)$$

Similarly, for $\vec{J} \cdot \nabla \vec{B}$.

$$\vec{J} \cdot \nabla \vec{B} = \left(0 \times \frac{\partial}{\partial x} + 0 \times \frac{\partial}{\partial y} + J_z \times 0 \right) \vec{B} = 0 \quad (5.29)$$

Likewise, $\vec{B} \cdot \nabla \vec{J}$ is non-zero.

Finally,

$$\vec{a} \cdot \vec{u} = \vec{u} \cdot \vec{a} = (u_x \times 0 + u_y \times 0 + 0 \times a_z) = 0 \quad (5.30)$$

□

As a result of (5.18), we may simplify the magnetohydrodynamic vorticity equation (4.126) and the uncurled MHD induction equation (4.152) to

$$\boxed{\frac{\partial \vec{\omega}}{\partial t} + (\vec{u} \cdot \nabla) \vec{\omega} = (\vec{B} \cdot \nabla) \vec{J} + \nu \nabla^2 \vec{\omega}} \quad (5.31)$$

$$\boxed{\frac{\partial \vec{a}}{\partial t} + (\vec{u} \cdot \nabla) \vec{a} = \eta \nabla^2 \vec{a}} \quad (5.32)$$

Since there is a reduction of terms in the 2D vorticity and anti-curled induction equations, it stands to reason that these produce simplified versions of the scale-localised transfer functions.

The simplifications to (5.32) result in the generation of magnetic enstrophy, $G^{r,b}(r)$, to become zero, leaving the transfer of magnetic enstrophy function, $F_b^b(r)$, the same as before.

However, for (5.31), the removal of the $\vec{J} \cdot \nabla \vec{B}$ term removes (4.138) from the solution. This simplifies (4.191) down to

$$\begin{aligned} F_b^u(r) = & \langle \vec{\omega}_b^L \cdot (\vec{u} \cdot \nabla \vec{\omega}_b^S) \rangle + \langle \vec{J}_b^S \vec{B}_b^L : \nabla \vec{\omega}_b^L \rangle \\ & + \langle \vec{J}_b^L \vec{B}_b^S : \nabla \vec{\omega}_b^L \rangle + \langle \vec{J}_b^S \vec{B}_b^S : \nabla \vec{\omega}_b^L \rangle \end{aligned} \quad (5.33)$$

In Section 4.2, we investigated MHD waves and discovered that the introduction of magnetism in a fluid caused the flow to generate three different types of wave behaviours, rather than just the shear behaviour in regular hydrodynamics.

5.2 Estimations for Multiples of η

When plotting the 2D simulations (and 3D in Chapter 6), it is useful to only examine certain multiples of η , the Kolmogorov dissipation scale.

It is known that

$$\frac{k_{\text{diss}}}{N/2} < \frac{3}{2} \quad (5.34)$$

where

$$k_{\text{diss}} = \frac{1}{\eta} \quad (5.35)$$

is the k -value where dissipation occurs. Values greater than this k_{diss} are no longer in the inertial subrange for the energy cascade, and the transfer functions instead dissipate into heat. $N/2$ is chosen due to the real fast Fourier transform discarding the negative k -values for efficiency (see Section 3.1).

We can estimate multiples of η to investigate in our simulation plots by substituting (5.35) into (5.34) and solving for η .

$$\frac{2}{\eta N} < \frac{3}{2} \quad (5.36)$$

$$\eta > \frac{N}{3} \quad (5.37)$$

Equation (5.37) provides us with a minimum multiple where dissipation occurs. Therefore, for our simulation plots to investigate the inertial subrange, our η multiples cannot exceed this value.

For our 2D and 3D simulations with datasets of size $N = \{32, 256, 512, 2048\}$, we have:

N	Maximum Multiple	Floor of Maximum Multiple
32	10.67η	10η
256	85.33η	85η
32	170.67η	170η
32	682.67η	682η

Table 5.1: Table of maximum η multiples for simulations of size N .

5.3 2D Graphs

We start by defining the normalised transfer functions:

$$\hat{\Pi}_r^u = \frac{\Pi_{r,b}^u}{\max\{\Pi_{r,b}^u\}}, \quad \hat{\Pi}_r^b = \frac{\Pi_{r,b}^b}{\max\{\Pi_{r,b}^b\}}, \quad \hat{F}_r^u = \frac{F_{r,b}^u}{\max\{F_{r,b}^u\}}, \quad \hat{F}_r^b = \frac{F_{r,b}^b}{\max\{F_{r,b}^b\}} \quad (5.38)$$

where we omit the b -notation denoting bandpass filtering to avoid confusion with the b denoting the magnetic variables.

The graphs are plotted at 3 different length scales, corresponding to multiples of the Kolmogorov dissipation length, η , as done in Doan et al [6]. These multiples of η were chosen on a case-by-case basis for each simulated dataset, based on the results in Table 5.1.

The graphs were also plotted at various snapshots in time. Some simulations, such as the 2048^2 case, only had data for a singular snapshot in time, whereas others had multiple.

Normalised Transfer Functions at $t=0.0$

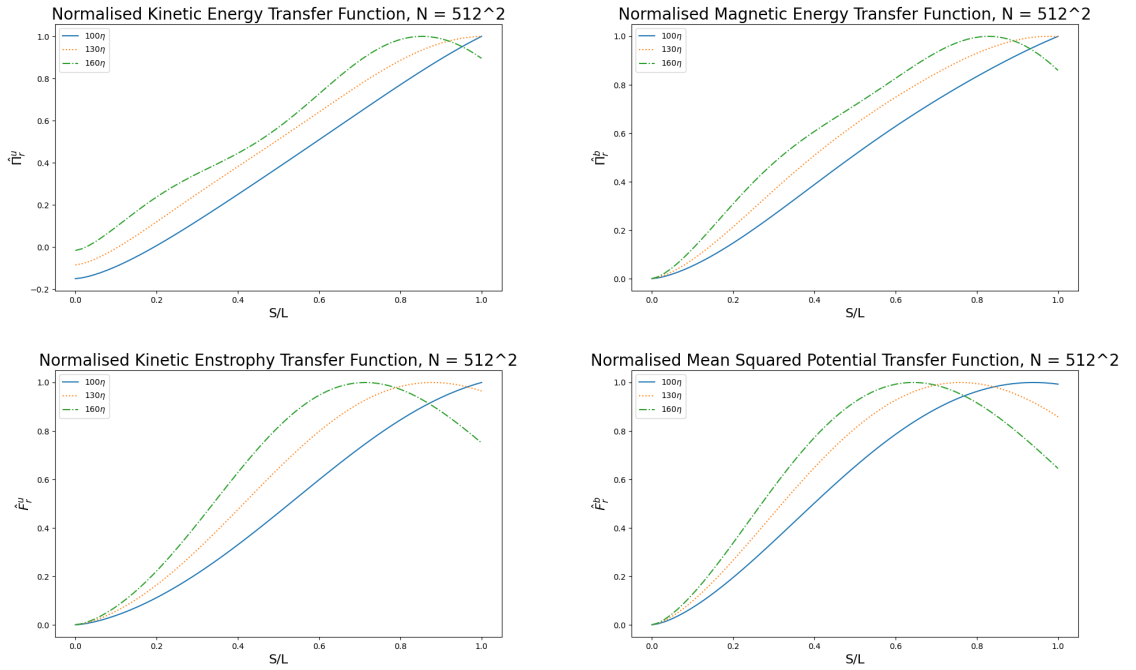


Figure 5.3: Normalised transfer functions at snapshot $t = 0.0$.

In Figure 5.3, at $t = 0.0$, the transfer functions are very broad. This means that while the scale with the peak transfer is $\sim 0.9L$ for the kinetic and magnetic energies, and $\sim 0.7L$ for the kinetic enstrophy and mean squared vector potential, a large proportion is transferred to a wide range of adjacent scales.

For Figure 5.4 at $t = 0.25$, the transfer functions are very sharp for the kinetic and magnetic energies and the kinetic enstrophy, meaning that the scale with the peak transfer is $\sim 0.1L$ for these.

The mean squared potential, on the other hand, is exceptionally broad, with a bimodal distribution at $\sim 0.3L$ and $\sim 0.5L$.

Negative values in these graphs suggest scales that lose transferred function values.

In Figure 5.5, at $t = 0.5$, the transfer functions are very sharp, with the mean squared potential being slightly broader. This means that most of the energy, enstrophy and MSVP is transferred at a scale of $\sim 0.1L$, with minimal transfer to adjacent scales.

Normalised Transfer Functions at $t=0.25$

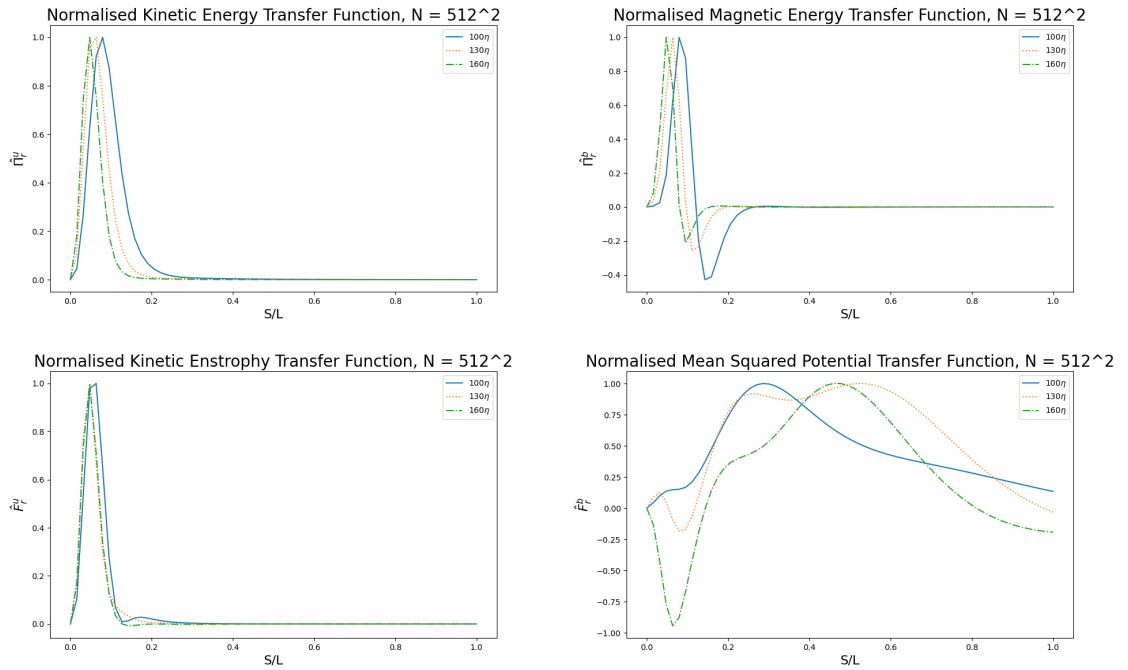


Figure 5.4: Normalised transfer functions at snapshot $t = 0.25$.

Normalised Transfer Functions at $t=0.5$

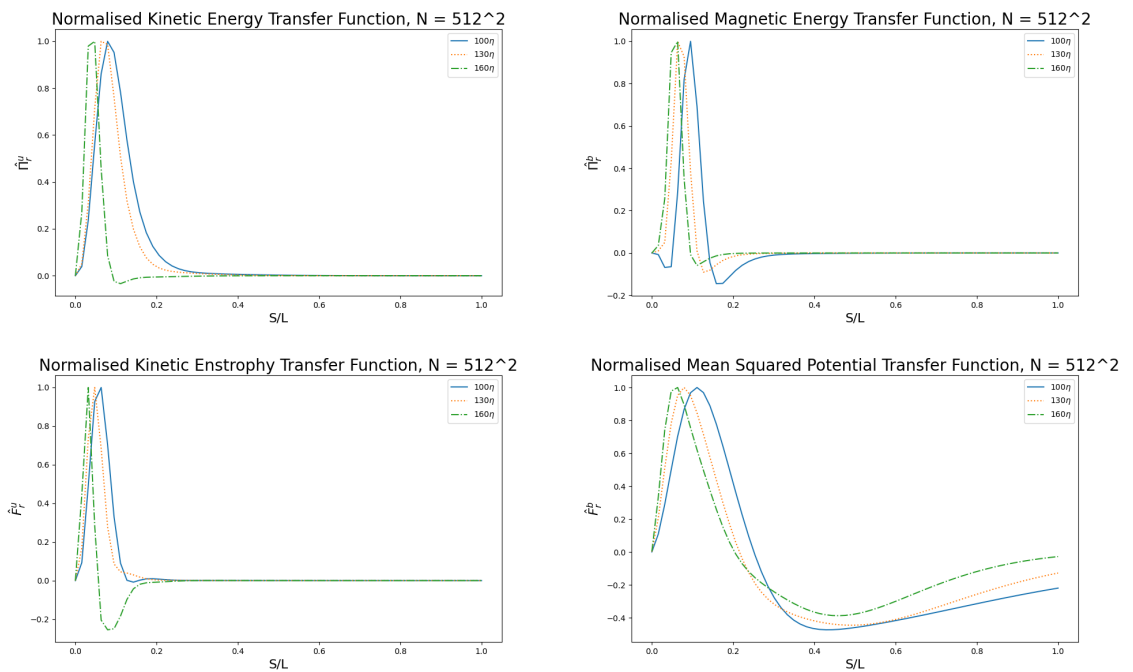


Figure 5.5: Normalised transfer functions at snapshot $t = 0.5$.

Normalised Transfer Functions at $t=0.75$

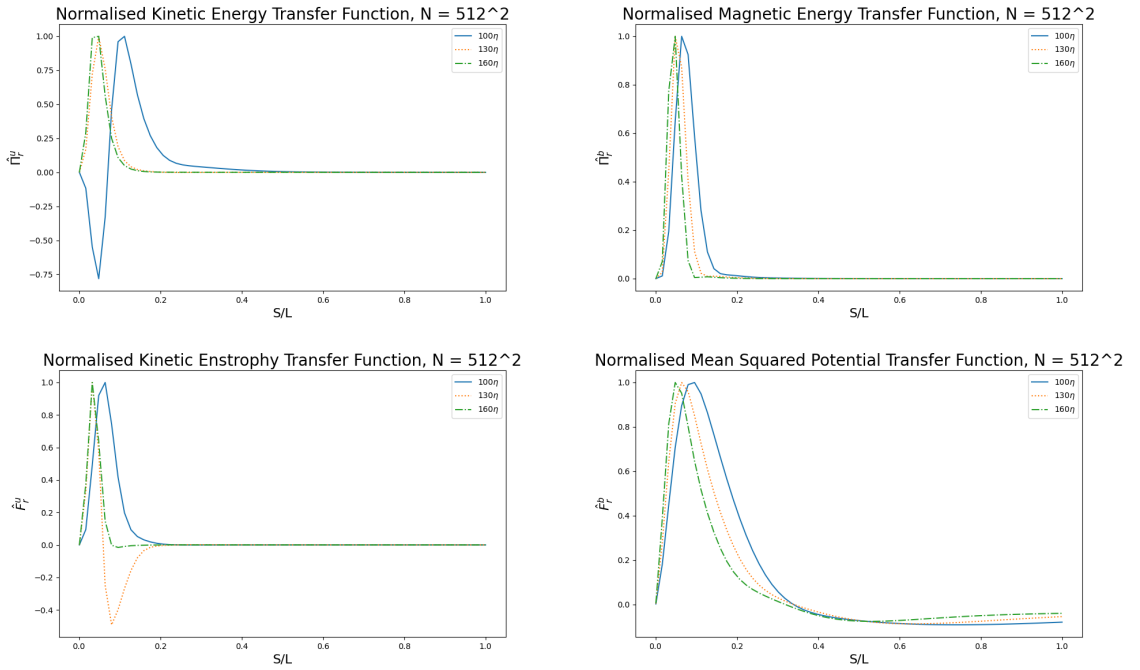


Figure 5.6: Normalised transfer functions at snapshot $t = 0.75$.

In Figure 5.6, at the snapshot $t = 0.75$, the transfer functions resemble the results in Figure 5.4, with peak transfer also occurring at a scale of $\sim 0.1L$.

We may also define the total energy transfer as

$$\Pi_{Total} = \Pi_{r,b}^u + \Pi_{r,b}^b \quad (5.39)$$

This lets us then plot the total energy at various snapshots, as seen in Figure 5.7.

In Figure 5.7, we see an initial transfer to higher length scales $\sim 0.9L$ at $t = 0.0$, with the peak transfer quickly shifting to $\sim 0.1L$ for subsequent time snapshots.

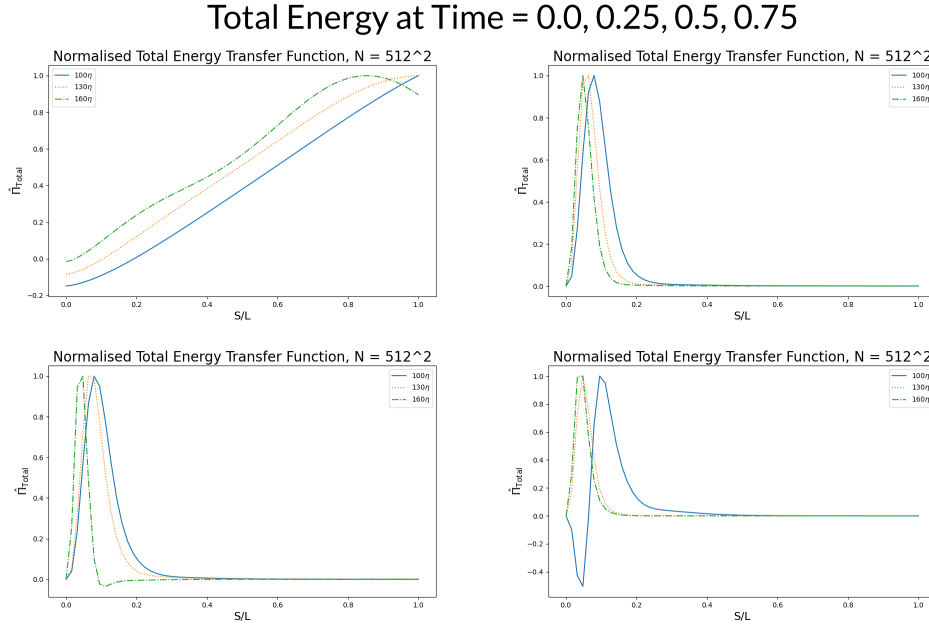


Figure 5.7: The total energy transfer from large to small scales at time values $[0.0, 0.25, 0.5, 0.75]$, read left-to-right and top-to-bottom.

For our simulation of 2048^2 grid points, we only have one time snapshot at $t = 0.75$. The η values were chosen such that they do not exceed the value in Table 5.1.

Figure 5.8 displays a similar pattern to the 512^2 simulation, with sharp and narrow peak transfer to scales of $\sim 0.1L$, and the mean squared potential having a slightly broader transfer distribution.

This means that most of the transfer happens at $\sim 0.1L$, with very little transfer to adjacent scales.

Figure 5.9 then shows the transfer of the total energy for this simulation. Similar to the 512^2 case, this has peak transfer going to scales at $\sim 0.1L$.

It is interesting to observe that as the multiple of η tends towards the dissipation range, the peaks of the transfer functions becomes narrower with less transferral to adjacent scales.

The fact that all the graphs, barring the initial value at $t = 0.0$, have a peak transfer at $\sim 0.1L$ suggests there is a scale locality at this scale, for a variety of Reynolds numbers.

Normalised Transfer Functions at $t=0.75$

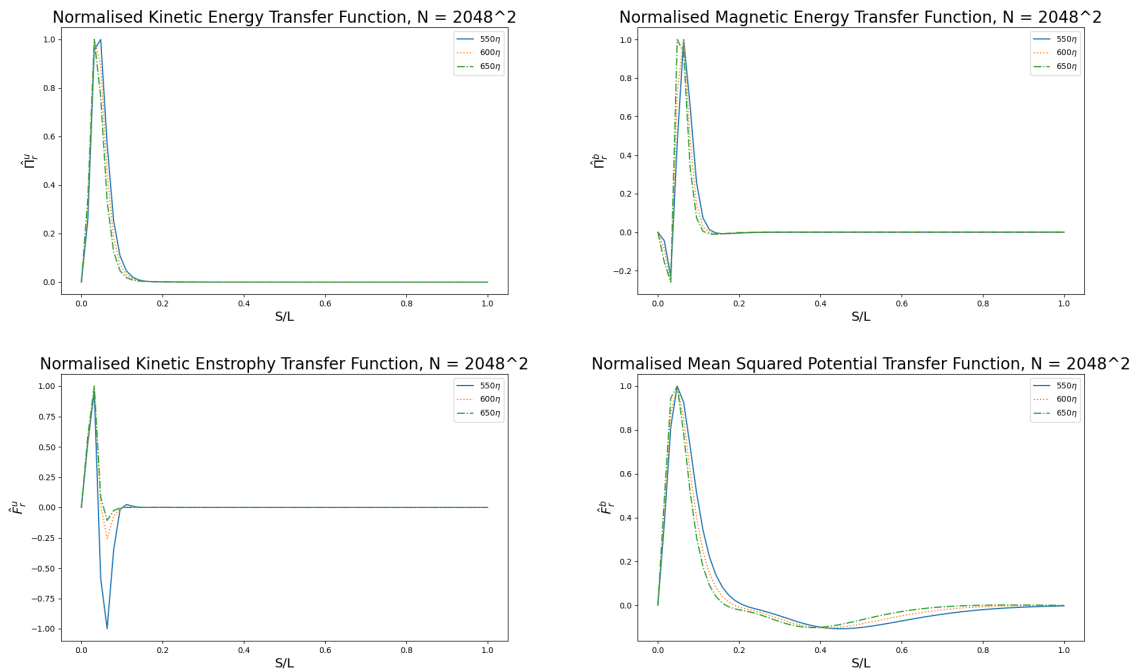


Figure 5.8: Normalised transfer functions at snapshot $t = 0.75$.

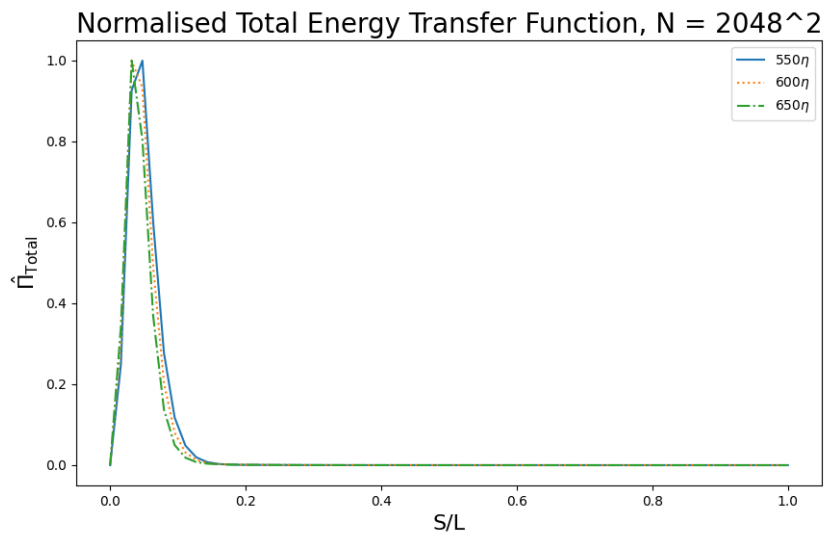


Figure 5.9: The total energy transfer from large to small scales at time value [0.75].

Chapter 6

3D Scale Locality in MHD

This chapter is original work, building on the original results presented at the end of Chapter 4 in 3D simulations.

We can start by defining our normalised transfer functions and the total energy, as we did in (5.34) and (5.35).

Similar to the 2D cases, the 3D cases are plotted for various multiples of η , as outlined in Table 5.1.

Normalised Transfer Functions at $t=0.0$

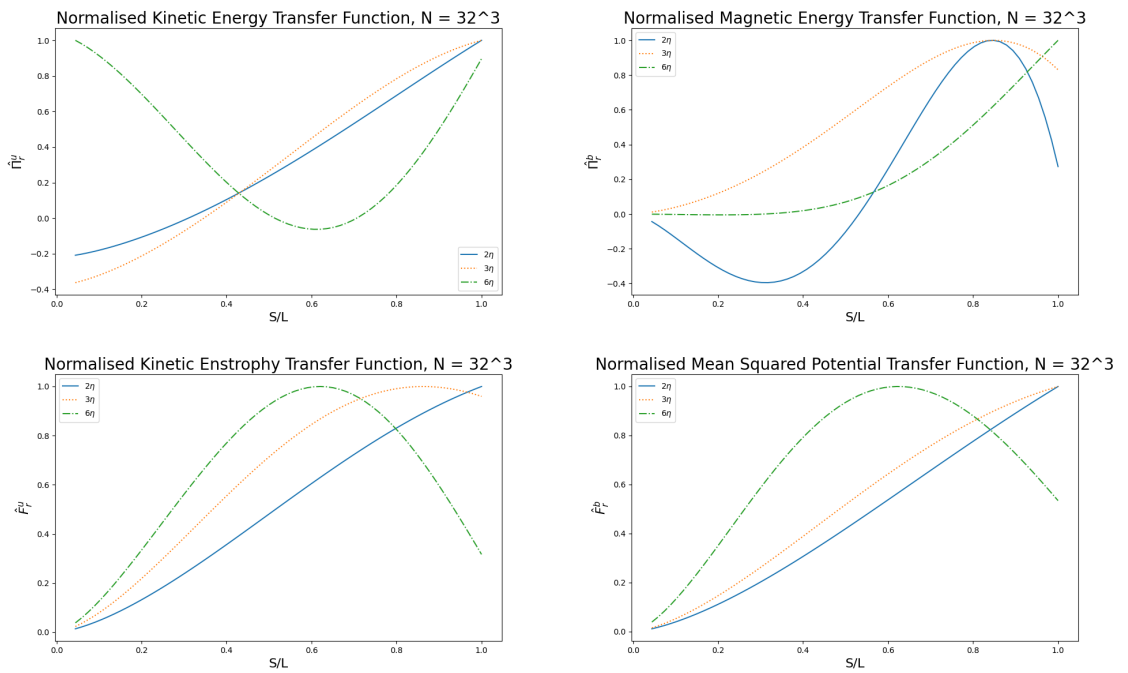


Figure 6.1: Normalised transfer functions at snapshot $t = 0.0s$.

Normalised Transfer Functions at $t=1.0$

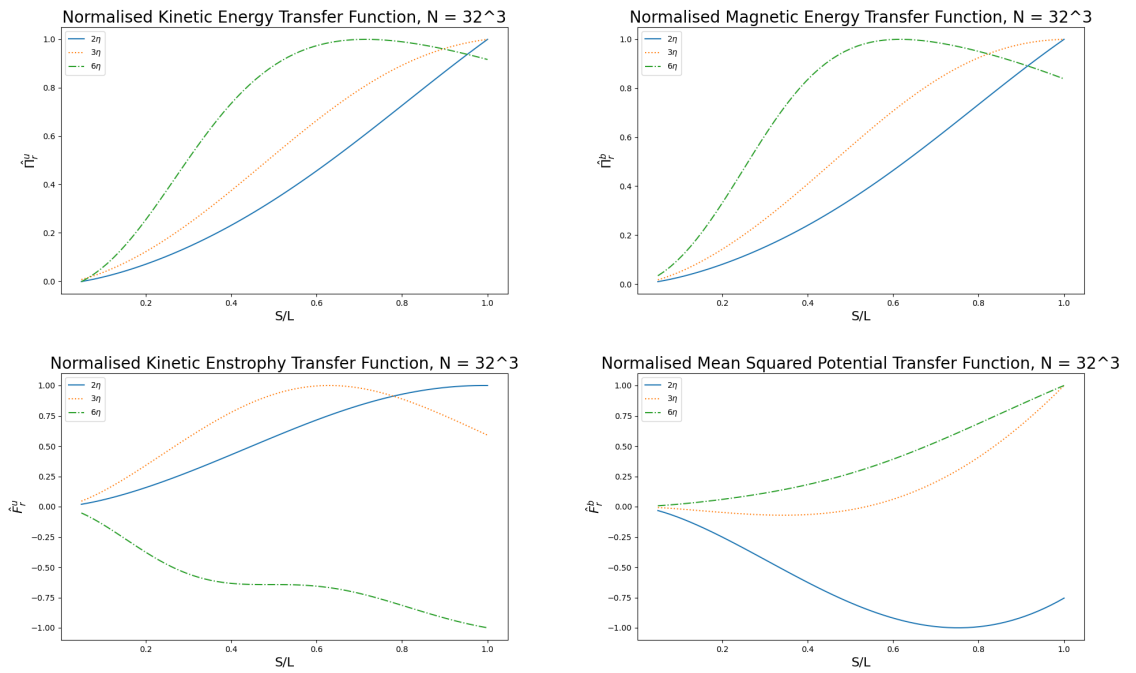


Figure 6.2: Normalised transfer functions at snapshot $t = 1.0s$.

Total Energy at Time = 0.0, 1.0

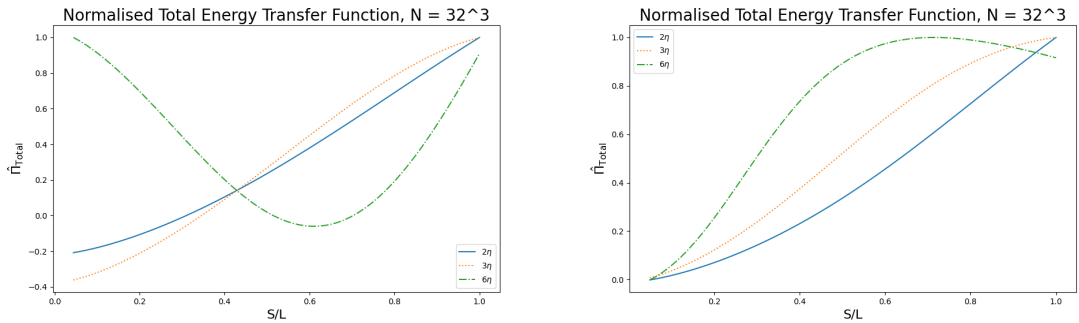


Figure 6.3: The total energy transfer from large to small scales at time values $[0.0, 1.0]$, read left-to-right.

The behaviours of the transfer function in Figures 6.1-6.3 show very broad transfer ranges and do not fully match with the results seen in Section 5.2 and with Figures 6.4-6.6.

This is due to the fact that 32^3 grid points are simply too small a quantity with not enough detail for observing turbulent behaviours. Ideally, the

minimum number of grid points to have enough detail to notice turbulent behaviours corresponds with $N = 256$.

The 32^3 case was mainly included as a test scenario.

Normalised Transfer Functions at $t=0.0$

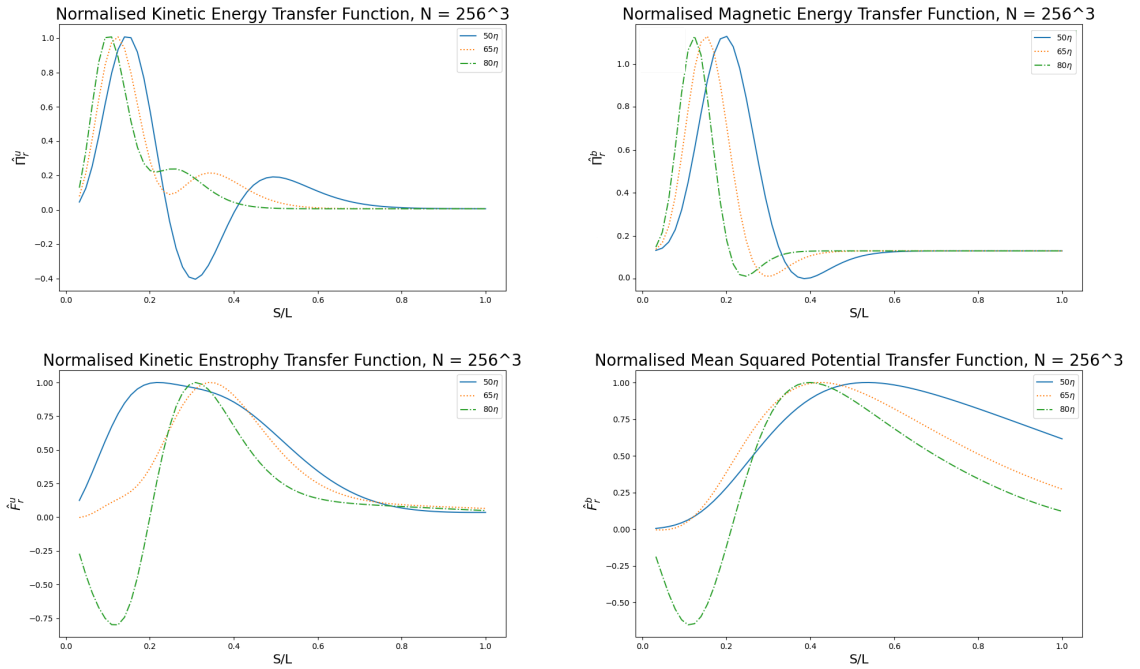


Figure 6.4: Normalised transfer functions at snapshot $t = 0.0$.

In Figure 6.4, at the time snapshot of $t = 0.0$ for the 256^3 , the kinetic and magnetic energy display peak transfers to a scale of $\sim 0.2L$ with a loss of kinetic energy occurring around $0.3L$. This could be due to the scale-localised MHD equations in Chapter 4 having terms that take energy and entropy from the kinetic values and distribute them among the magnetic ones.

This could also explain the magnetic energy and mean squared potential being broader than their kinetic counterparts.

For the time snapshot at $t = 0.6$ in Figure 6.5, we see a general approach to a peak transfer to the scale at $\sim 0.2L$. These transfers are quite broad, meaning that while $\sim 0.2L$ is the scale with the peak transfer, the scales adjacent to it experience a transfer too. For the mean squared vector potential, this is especially apparent.

Normalised Transfer Functions at $t=0.6$

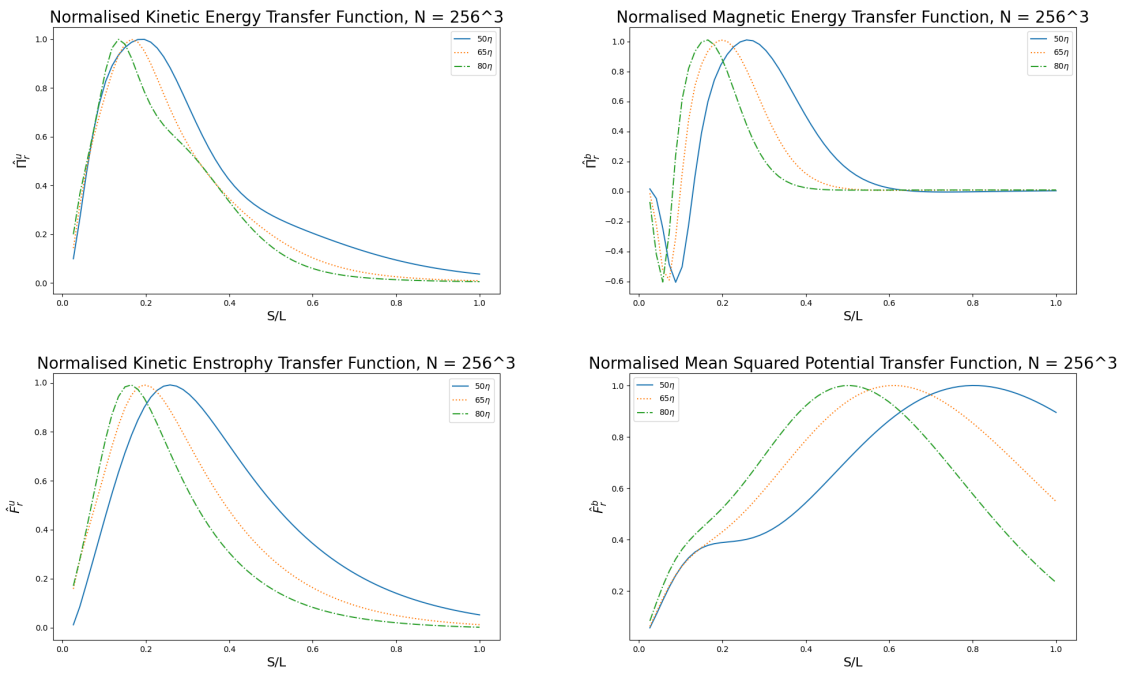


Figure 6.5: Normalised transfer functions at snapshot $t = 0.6$.

Total Energy at Time = 0.0, 0.6

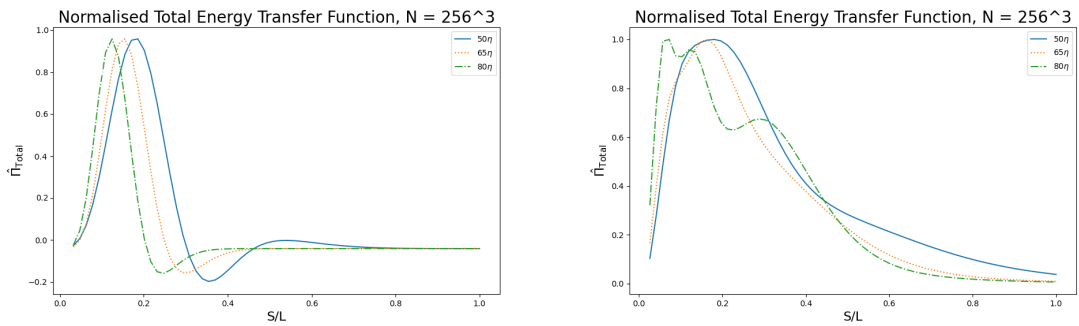


Figure 6.6: The total energy transfer from large to small scales at time values $[0.0, 0.6]$, read left-to-right.

Finally, Figure 6.6 shows the total energy transfer, occurring consistently at a scale of $\sim 0.2L$, with a broad range of adjacent transferral.

It is interesting to observe that as multiple of η tends towards the dissipation range, the peaks of the transfer functions became narrower with less transferral to adjacent scales.

Chapter 7

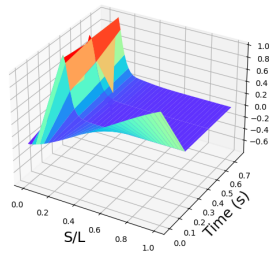
Time Evolution

The contents of this chapter are original work, acting as an extension to plot the transfer functions as time evolution surface plots, since it may be interesting to see how our transfer functions change between time snapshots.

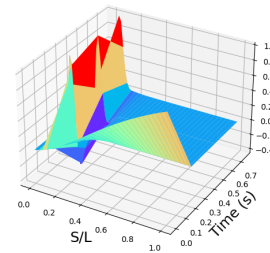
Since the 512^2 simulation contained 4 snapshots in time, it seems natural to choose this.

Surface Plots for 512^2 at 100η

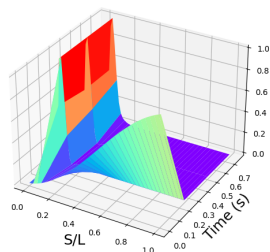
Normalised Kinetic Energy Transfer Function, 100η



Normalised Magnetic Energy Transfer Function, 100η



Normalised Kinetic Enstrophy Transfer Function, 100η



Normalised Mean Squared Potential Transfer Function, 100η

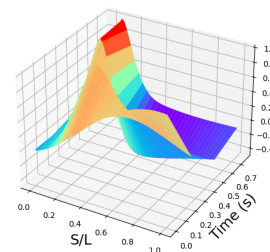
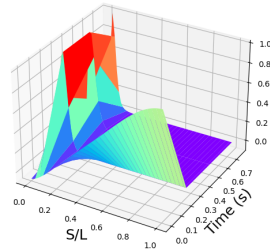


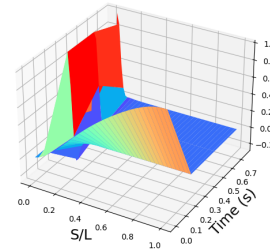
Figure 7.1: The transfer functions at a scale of 100η for time snapshots 0.0-0.75.

Surface Plots for 512^2 at 130η

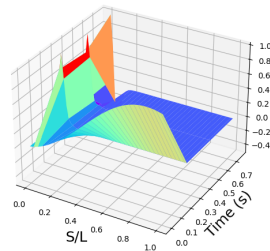
Normalised Kinetic Energy Transfer Function, 130η



Normalised Magnetic Energy Transfer Function, 130η



Normalised Kinetic Enstrophy Transfer Function, 130η



Normalised Mean Squared Potential Transfer Function, 130η

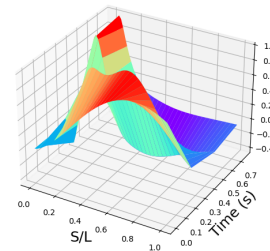
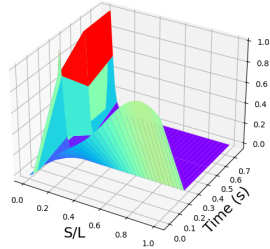


Figure 7.2: The transfer functions at a scale of 130η for time snapshots 0.0-0.75.

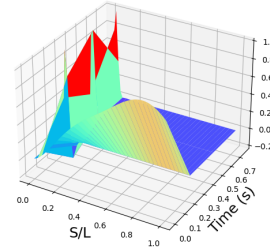
Surface plots of the 256^3 case were attempted, however the resulting plots were visually dull due to the consistency of the peaks. The 32^3 case was only a test case and exhibited a lack of turbulent behaviour, so a surface plot of this was deemed unimportant. Lastly, the 2048^2 case contained only a single snapshot in time, rendering a time evolution surface plot unviable.

Surface Plots for 512^2 at 160η

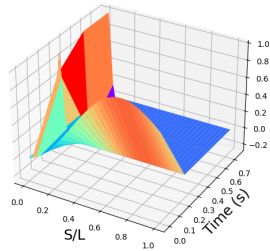
Normalised Kinetic Energy Transfer Function, 160η



Normalised Magnetic Energy Transfer Function, 160η



Normalised Kinetic Enstrophy Transfer Function, 160η



Normalised Mean Squared Potential Transfer Function, 160η

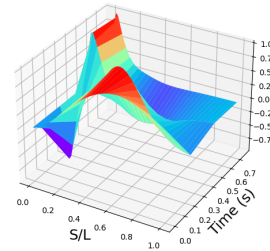
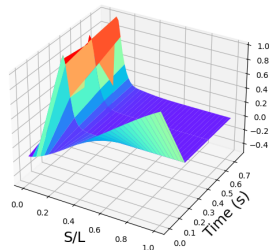


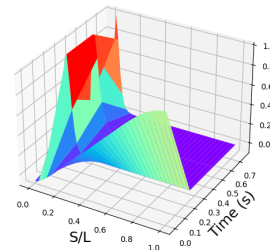
Figure 7.3: The transfer functions at a scale of 160η for time snapshots 0.0-0.75.

Surface Plots of Total Energy for 512^2 at 100η , 130η , 160η

Normalised Total Energy Transfer, 100η



Normalised Total Energy Transfer, 130η



Normalised Total Energy Transfer, 160η

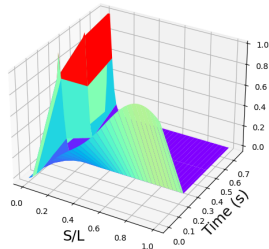


Figure 7.4: The total energy for $[100\eta, 130\eta, 160\eta]$ for time snapshots 0.0-0.75.

Chapter 8

Conclusion

This thesis explored the energy cascade in magnetohydrodynamic turbulence, the study of turbulent structures in an electrically charged fluid inducing its own magnetic field. While scale-localisation is not a new concept, the process in Chapter 4 takes Doan et al's [6] approach to scale-localising hydrodynamics and applies it in an original derivation for the MHD case, matching other scale-localised results obtained through different derivations. The graphs then produced in Chapters 4-6 are original work, using simulation provided by Prof. Sean Oughton.

It is clear that, while having some influence from the magnetic forces, the behaviour of the transfer functions in 2D MHD is rather underwhelming.

The graphs for the 3D case, in particular, deviate from the result of regular hydrodynamics [6], shown in the scale-decomposed MHD equations, where a magnetic term in the kinetic equations takes energy or enstrophy from the kinetic equations and applies it to their corresponding magnetic equations.

An extension to the research conducted in this thesis could be to investigate compressible flows, that is, flows where $\nabla \cdot \vec{u} \neq 0$. The incompressibility condition simplified many equations in this thesis, and the added complexity of removing these simplifications would likely be of great research value.

Another direction would be the string theory approach, where $\nabla \cdot \vec{B} \neq 0$, given that string theory suggests the existence of magnetic monopoles; how-

ever, this would be more for mathematical generalisation and pure curiosity since magnetic monopoles have yet to be discovered.

A topic that recently piqued my interest is extending this to superfluids or putting plasmas under such pressure that the very low scales, $r \ll l_{corr}$, become relevant and the plasma becomes a quark-gluon plasma, involving the combination of the Navier-Stokes and MHD equations with the Schrödinger equation. Both of these topics have relevance to neutron stars, a current active field of research.

Appendix A

Code

All the code for my thesis can be found at:

<https://github.com/morgannamaths/ThesisCode2024>

A.1 Index ordering in Numpy

Index ordering can be confusing when trying to use the Numpy library in Python to plot x, y or x, y, z functions—or, in fact, n D functions, where $n > 1$, $n \in \mathbb{N}$. From a mathematician's perspective, it is only natural to use the ordering (x, y, \dots) . However, for arrays larger than 1 dimension, Numpy reads the number of rows first, corresponding to changes in y . This means Numpy uses the ordering (y, x, \dots) . Failing to take this into account will lead to rotated or transposed outputs, as was the case in my code for many months.

This confusion is further amplified by the fact that the function `numpy.meshgrid()`, used to construct surfaces and planes, is the exception to this, where it takes the input (x, y) and gives the output (X, Y) , and then for even more confusion, for dimensions 3 or greater, takes the input (x, y, z, \dots) and gives the output (Y, X, Z, \dots) .

Fortunately, the only point for which this function was used was for (Kx, Ky, Kz) , which was symmetrical for my simulations.

Appendix B

Interesting Fourier Pattern

```
import numpy as np
import matplotlib.pyplot as plt

def FourierPattern(N):
    omega_N = np.exp((- 2j * np.pi)/N)
    M, K = np.meshgrid(np.arange(N), np.arange(N))
    F_N = omega_N**(M*K)
    return np.real(F_N)

def Graphing(N1, N2):
    fig, axes = plt.subplots(1, 2)
    fig.set_size_inches(10, 4)
    axes[0].set_title(r"Interesting Pattern in  $F_N$ , for  $N =$ " + f"{N1}", fontsize=16)
    axes[0].imshow(FourierPattern(N1))
    axes[1].set_title(r"Interesting Pattern in  $F_N$ , for  $N =$ " + f"{N2}", fontsize=16)
    axes[1].imshow(FourierPattern(N2))
    plt.show()

if __name__ == "__main__":
    Graphing(32, 128)
```

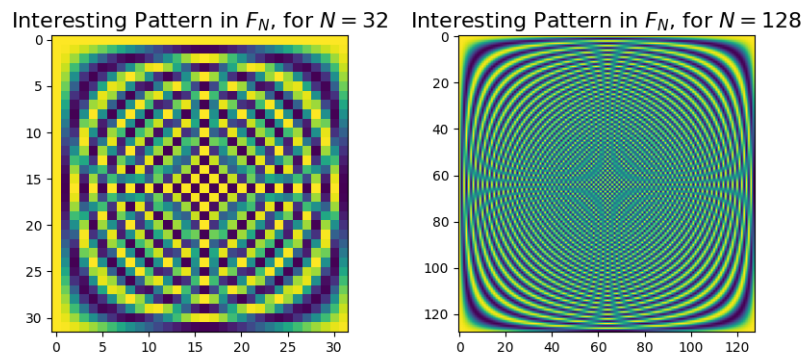


Figure B.1: An interesting pattern occurs in F_N , plotted for $N = 32$ and $N = 128$ respectively.

Appendix C

Bandpass Verification

Consider the bandpass filter of an axisymmetric turbulent vortex for a length scale $l = l(r, \theta, z)$, using the Gaussian filter defined in (3.1), with a velocity field as described in the appendix of Leung et al [10].

$$\vec{u}(r, \theta, z) = \Omega r e^{-\frac{2(r^2+z^2)}{l^2}} \hat{e}_\theta \quad (\text{C.1})$$

A graphical representation of this fluid flow can be seen in Figure C.1 below.

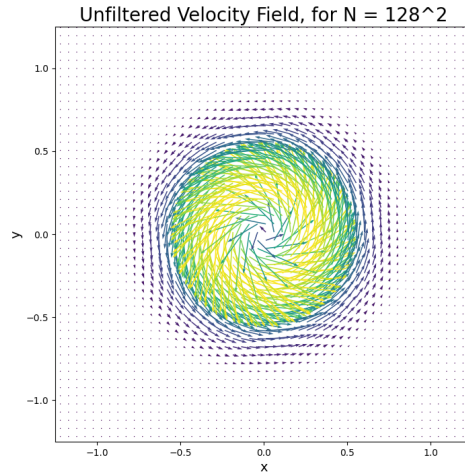


Figure C.1: The unfiltered velocity field, $\vec{u}(r, \theta, z)$, as defined in (C.1), with the plane cut at an axis normal to z .

It can be shown with (3.69) that the bandpass-filtered velocity field is then

given by

$$\vec{u}_b^L = \frac{L^2 l^5}{4\sqrt{L}(L^2 + \frac{l^2}{2})^{\frac{7}{2}}} \Omega r e^{\frac{-2(r^2+z^2)}{L^2 + \frac{l^2}{2}}} \left[5 - \frac{-2(r^2+z^2)}{L^2 + \frac{l^2}{2}} \right] \hat{e}_\theta \quad (\text{C.2})$$

The numerical derivation of the bandpass filter, using the fast Fourier transform, is plotted alongside the analytical derivation of (C.2) in Figure C.2. The outputs of both match a small numerical error, only seen outside the boundary of the flow.

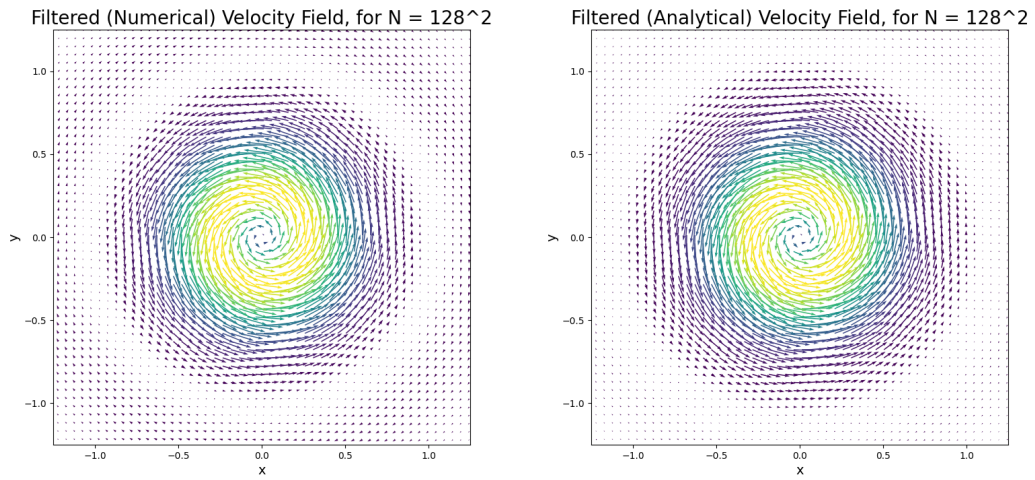


Figure C.2: The bandpass-filtered velocity field, $\vec{u}_b^L(r, \theta, z)$, derived both numerically [left] using the FFT and analytically [right] using the solution in (C.2), with the plane cut at an axis normal to z .

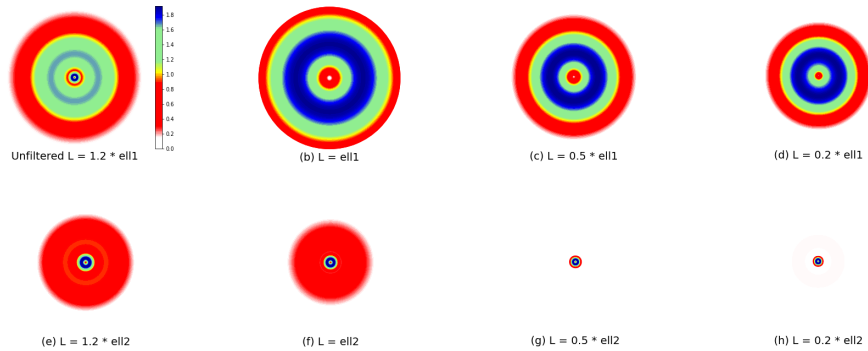


Figure C.3: Polar plots of the numerical bandpass filtering for various scales of L , with the plane cut at an axis normal to z , for the velocity field (C.2).

Barring a slight discrepancy in the colourbar, due to Leung et al not using a Matplotlib in-built colourmap meaning I had to try my best to replicate it with a custom colourmap, the results in Figure C.3 match with Figure 19 in Leung et al [10].

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