MIXED MUTUALLY ORTHOGONAL FREQUENCY SQUARES

A thesis
submitted in partial fulfilment
of the requirements for the Degree
of
Master of Science (Research)
at
The University of Waikato
by
Jilmy Poulose

THE UNIVERSITY OF
WAIKATO
Te Whare Wānanga o Waikato

2025

Abstract

A frequency square of type $F(n; \lambda_1, \lambda_2, \lambda_3, \ldots, \lambda_t)$ is an array $n \times n$ filled with symbols a_i, where each a_i occurs λ_i times in every row and column, and $n = \sum_{i=1}^{t} \lambda_i$.

Two such frequency squares $F_1(n; \lambda_1, \lambda_2, \lambda_3, \ldots, \lambda_t)$ and $F_2(n; \mu_1, \mu_2, \mu_3, \ldots, \mu_s)$ are mutually orthogonal, when superimposed, each of the ts possible ordered pairs (i, j), where $1 \leq i \leq t$ and $1 \leq j \leq s$, occurs exactly $\lambda_i \mu_j$ times.

This thesis generalizes the classical theory of mutually orthogonal Latin squares to mixed frequency settings, where symbols may appear with different frequencies. The non-uniform frequencies lead to a wider range of combinatorial structures with new methods of construction. Then, the thesis investigates maximizing the set of mixed mutually orthogonal frequency squares (MMOFS), focusing on theoretical methods rather than computational tools.

The following lemmas and theorems are new results presented in Chapter 3. In Lemma 4.1, we define mappings from two Latin squares to form two MOFS. Then in Theorem 4.7, we apply this to explore the maximum size of the sets of higher-order MMOFS by using mixed orthogonal arrays. Moreover, Section 2.2 gives an original alternative proof of an existing upper bound for sets of MOFS.

Furthermore, we identify sets of MMOFS for small orders in the final chapter by using the new results and the previous theories. These new results include Corollary 4.5, Lemma 5.1, Example 22, Corollary 5.6 and Example 24.

Acknowledgements

First and foremost, I am grateful to Almighty God, for his constant blessings and strength without which this thesis would not have been possible.

I wish to express my deepest gratitude to my supervisor, Associate Professor Nicholas Cavenagh, The University of Waikato, New Zealand, for his invaluable guidance and support throughout the course of my thesis. His expertise, patience, and insightful feedback have been fundamental in shaping the direction of this work and in helping me develop as a researcher.

I am humbled and forever indebted to my husband, Josemon, and to our beloved son, Derick, whose unwavering love, support, and sacrifices have enabled me to complete this thesis. They have been my pillars of strength, and I dedicate this work to them with heartfelt appreciation.

I also wish to acknowledge the enduring support and encouragement of my parents, brothers, sisters, and the rest of my family, as well as friends in both New Zealand and India. Their continued faith in me has been a source of motivation throughout this journey.

Contents

1 Background

In this chapter, we introduce the fundamental concepts and definitions that form the basis of this study, including Latin squares, frequency squares, MOLS, MOFS, and MMOFS. We also discuss the conditions required for orthogonality and review the classical theorems from various textbooks and research papers that provide upper bounds for the size of sets of MOLS, MOFS, and their extension to MMOFS.

1.1 Latin squares

Definition 1. *A Latin square of order n is an $n \times n$ array filled with n distinct symbols so that no symbols are repeated in any row or column.*

An example of a Latin square of order 4 is as follows.

1	2	3	4
3	4	1	2
4	3	2	1
2	1	4	3

1.1.1 Orthogonal Latin squares

Definition 2. *A pair of orthogonal Latin squares of order n is a pair of Latin squares of n distinct symbols so that we can form n^2 distinct ordered pairs by combining the corresponding entries of those Latin squares.*

An example of a pair of orthogonal Latin squares and the sixteen distinct ordered pairs are shown below.

$L_1 =$

2	4	3	1
1	3	4	2
4	2	1	3
3	1	2	4

$L_2 =$

4	3	2	1
3	4	1	2
2	1	4	3
1	2	3	4

(2,4)	(4,3)	(3,2)	(1,1)
(1,3)	(3,4)	(4,1)	(2,2)
(4,2)	(2,1)	(1,4)	(3,3)
(3,1)	(1,2)	(2,3)	(4,4)

Table 1: The superimposed ordered pairs of L_1 and L_2

.

1.1.2 Mutually Orthogonal Latin squares (MOLS)

Definition 3. *A set of Latin squares of the same order in which every pair is orthogonal is known as a set of mutually orthogonal Latin squares or* MOLS.

If a set of MOLS attains the maximum possible number for order n, it is called a ***complete*** *set of MOLS of order n.*

Theorem 1.1. *For $n \geq 2$, there exist at most $n-1$ mutually orthogonal Latin squares of order n.*

Proof. This proof is based on the one given in [14].

Suppose that there exist n MOLS of order n. We can independently permute the symbols in each Latin square without changing the orthogonality. Without loss of generality, we permute the symbols in all squares so that the first row in each square is $1, 2, 3, \ldots, n$ in order.

Consider the first column of each square. The first position is already fixed as 1 since the first entry in the first row is 1. The second position in this column must be filled with the remaining $2, 3, 4, \ldots, n$. But, if we want to create n squares, then there exist two squares with the same symbol at the second position of the first column which can create the ordered pair (i, i), $i \in \{2, 3, 4, \ldots, n\}$. However, (i, i), $i \in \{1, 2, 3, \ldots, n\}$ already appear in the first row. This violates orthogonality, and we cannot have n MOLS of order n.

Hence, for n, there exist at most $n - 1$ MOLS of order n. $\square$

The bound in Theorem 1.1 can be met when n is a prime number. We can generate $n - 1$ MOLS using finite field theory in which the first square can be formed by addition modulo(n), and other squares are formed by multiplication modulo(n). Since there are $n - 1$ non-zero entries in F_n, we get $n - 1$ MOLS.

Theorem 1.2. *For a prime number $p \geq 2$, there exists $p - 1$ MOLS of order n.*

Proof. This proof is also based on the one given in [14].

Define the Latin square S_k for $k \in \{1, 2, 3, 4, \ldots, p - 1\}$, and the entry in row i and column j of S_k is $(i - 1)k + (j - 1) \pmod{p}$. Since k and p are relatively prime, multiplying by k permutes the numbers in $\{1, 2, 3, 4, \ldots, p - 1\}$. The same result occurs when we vary j. The addition of $j - 1$, shifts the entries by ensuring no repetition within a column. We can show that every ordered pair (a, b), where $a \in S_i$ and $b \in S_j$ appears exactly once at the positions of the squares.

Suppose S_i and S_j share the same pair (a, b) in two different positions, say row k_1, column m_1, and row k_2, column m_2. From the definition of S_i, $S_j \in S_k$, we have

$$(k_1 - 1)i + (m_1 - 1) \equiv (k_2 - 1)i + (m_2 - 1) \pmod{p}, \quad \text{and}$$

$$(k_1 - 1)j + (m_1 - 1) \equiv (k_2 - 1)j + (m_2 - 1) \pmod{p}.$$

Thus,

$$k_1 i + m_1 \equiv k_2 i + m_2 \pmod{p}, \quad \textit{and} \quad k_1 j + m_1 \equiv k_2 j + m_2 \pmod{p}$$

$$\Rightarrow m_2 - m_1 \equiv k_2 i - k_1 i = (k_2 - k_1) i \equiv k_2 j - k_1 j = (k_2 - k_1) j \pmod{p}$$

$$\Rightarrow (k_2 - k_1)(i - j) \equiv 0 \pmod{p}.$$

Since $i \neq j$ and p is prime, $i - j$ is not divisible by p.

$$\Rightarrow (k_2 - k_1) \equiv 0 \pmod{p}.$$

Hence, $k_1 = k_2$. Similarly, we can prove $m_1 = m_2$.

Therefore, (a, b) cannot appear in two distinct positions, which shows that all S_k for $k \in \{1, 2, 3, 4, \ldots, p-1\}$ is indeed a set of $p - 1$ MOLS of order p.

$\square$

Theorem 1.2 can be extended to the prime power $q = p^k$ by working in the field F_q.

Theorem 1.3. *Let $q = p^k$ be a prime power, where p is a prime number and $k \geq 1$. Then, there exist $q - 1$ MOLS of order q* [15].

A complete set of MOLS of order n consists of exactly $n - 1$ Latin squares. When n is prime or prime power, such complete sets always exist. For example,

- For $n = 3$, there exist 2 MOLS.
- For $n = 4$, there exist 3 MOLS.
- For $n = 5$, there exist 4 MOLS.
- For $n = 7$, there exist 6 MOLS.

- For $n = 8$, there exist 7 MOLS.

However, when n is composite, the existence of a complete set of $n-1$ MOLS is not guaranteed, and in many cases it is still unknown how many MOLS exist. For instance,

- For $n = 6$, there are no two orthogonal Latin squares.
- For $n = 10$ not even the number of MOLS of order 10 is fully determined, but it is known that there are at least 2 MOLS for this order.
- For $n = 15$, the maximum number of MOLS is not fully determined.

The study of pairs of MOLS has an important place in the history of Latin squares. Leonhard Euler observed that for $n = 6$, there are no two orthogonal Latin squares. From this and similar observations, he proposed that no pair of orthogonal Latin squares exist when $n \equiv 2 \pmod 4$, that is, for orders $n = 6, 10, 14, 18, \ldots$. This became known as Euler conjecture. However, Euler conjecture was disproved in 1959 by Bose, Shrikhande, and Parker, who gave counterexamples for large values of $n \equiv 2 \pmod 4$ [5].

Remark 1.4.

- *If* n *is a product of distinct primes,* $n-1$ *MOLS does not always exist. For example, for* $n = 6 = 2 \times 3$*, we know only one Latin square of order 6. No pair of MOLS exists.*
- *The existence of MOLS depends on the divisibility and congruence conditions. If* $n \equiv 2 \pmod 4$*,* $n-1$ *MOLS cannot exist* [13]. For example, $n = 6$ and 10.

1.2 Frequency squares

A frequency square of order n is a $n \times n$ array, in which each element appears with constant frequency in each row and column.

More formally, a frequency square of order n, filled with elements a_i with respective frequencies f_i, where $i = 1, 2, 3, \dots, m$ such that $f_1 + f_2 + f_3 + \cdots + f_m = n$ is denoted by $F(n; f_1, f_2, f_3, \dots f_m)$.

An example of a frequency square of order 4 is as follows.

1	2	2	1
1	2	2	1
2	1	1	2
2	1	1	2

A frequency square is said to be in standard form if all the occurrences of a_i precede those of a_j whenever $i < j$ in the first row and column.

Remark 1.5.

- *A frequency square of order n with m symbols is* ***regular*** *if the frequencies $f_1 = f_2 = f_3 = \cdots = f_m = \frac{n}{m}$.*
- *An $n \times n$ array of ordered pairs obtained by superimposing two n^{th} order frequency squares F_i and F_j is represented as $[(F_i, F_j)]$.*
- *The number of times an ordered pair (i, j) is appearing can be represented by $N(i, j)$.*

1.2.1 Orthogonal Frequency squares

Definition 4. *A frequency square of type $F(n; f_1, f_2, f_3, \dots f_m)$ filled with elements a_i, where $i = 1, 2, 3, \dots, m$, and another frequency square $F(n; g_1, g_2, g_3, \dots g_k)$ filled with elements b_j, where $j = 1, 2, 3, \dots, k$ are said to be orthogonal if $N(a_i, b_j) = f_i g_j$, when F_1 is superimposed on F_2. The frequency square of order n, filled with symbols appearing with the same frequency λ, is denoted by $F(n; \lambda)$.*

The following is an example of orthogonal frequency squares with two elements, each with a frequency of two, where each ordered pair appears exactly four times.

$$F_1 = \begin{array}{|c|c|c|c|}\hline 2 & 3 & 2 & 3 \\ \hline 2 & 3 & 2 & 3 \\ \hline 3 & 2 & 3 & 2 \\ \hline 3 & 2 & 3 & 2 \\ \hline \end{array} \qquad F_2 = \begin{array}{|c|c|c|c|}\hline 2 & 2 & 3 & 3 \\ \hline 2 & 2 & 3 & 3 \\ \hline 3 & 3 & 2 & 2 \\ \hline 3 & 3 & 2 & 2 \\ \hline \end{array}$$

$$[(F_1, F_2)] = \begin{array}{|c|c|c|c|}\hline (2,2) & (3,2) & (2,3) & (3,3) \\ \hline (2,2) & (3,2) & (2,3) & (3,3) \\ \hline (3,3) & (2,3) & (3,2) & (2,2) \\ \hline (3,3) & (2,3) & (3,2) & (2,2) \\ \hline \end{array}$$

Table 2: The superimposed ordered pairs of F_1 and F_2

.

Remark 1.6.

- *A frequency square of type* $F(n; 1)$ *is a Latin square.*
- *It is possible that two frequency squares with the same order but different parameters are orthogonal.*

Example 1. *Consider the following frequency squares* F_1 *and* F_2 *of order* 4*, having different parameters, where* F_1 *is of type* $F(4; 1)$ *and the* F_2 *is of type* $F(4; 2)$*:*

$$F_1 = \begin{array}{|c|c|c|c|}\hline 1 & 2 & 3 & 4 \\ \hline 3 & 4 & 1 & 2 \\ \hline 4 & 3 & 2 & 1 \\ \hline 2 & 1 & 4 & 3 \\ \hline \end{array} \qquad F_2 = \begin{array}{|c|c|c|c|}\hline 2 & 2 & 3 & 3 \\ \hline 2 & 2 & 3 & 3 \\ \hline 3 & 3 & 2 & 2 \\ \hline 3 & 3 & 2 & 2 \\ \hline \end{array}$$

$$[(F_1, F_2)] = \begin{array}{|c|c|c|c|}
\hline
(1,2) & (2,2) & (3,3) & (4,3) \\
\hline
(3,2) & (4,2) & (1,3) & (2,3) \\
\hline
(4,3) & (3,3) & (2,2) & (1,2) \\
\hline
(2,3) & (1,3) & (4,2) & (3,2) \\
\hline
\end{array} .$$

Each ordered pair appears exactly two times, which confirms the orthogonality of F_1 *and* F_2*.*

1.2.2 Mutually Orthogonal Frequency squares (MOFS)

Definition 5. *A set of frequency squares of the same order is mutually orthogonal (MOFS) if every pair is orthogonal. A set of* k *mutually orthogonal frequency squares of order* n *will be referred to as a set of* k*-MOFS of order* n *or simply a set of* k*-MOFS.*

A set $\{F_1, F_2, F_3, \ldots, F_t\}$ *of MOFS is said to be* ***maximal*** *if there is no frequency square* F *orthogonal to* F_i*, where* $1 \leq i \leq t$*.*

1.2.3 Equivalent MOFS

Two sets of MOFS are said to be equivalent if one can be transformed into another through a sequence of rearrangements which do not alter the orthogonality and symbol frequencies of the squares. The equivalences are as follows:

- **Row Permutations**: Permute the rows of each square in a set of MOFS and apply the same row permutation to all the squares in the set. This does not change the frequency of entries in each row and column. Changing the order of rows simply rearranges the cells in squares, which does not affect orthogonality.

- **Column Permutations**: Similarly to row permutations, the columns of each square in a set can be permuted without affecting the orthogonality and frequency of the symbols.

- **Transpose**: Here we reflect the square across the diagonal so that rows become columns and vice versa. Since we transpose all squares in the set together, it affects neither the orthogonality nor the frequency distribution.

- **Symbol permutation**: Relabeling symbols having the same frequency within the same square does not affect the structure of the square, and hence the orthogonality. Note we can apply a symbol permutation to one square only.

Example 2. *Consider the two mutually orthogonal frequency squares from Table* 2 as follows:

$$F_1 = \begin{array}{|c|c|c|c|}\hline 2 & 3 & 2 & 3 \\ \hline 2 & 3 & 2 & 3 \\ \hline 3 & 2 & 3 & 2 \\ \hline 3 & 2 & 3 & 2 \\ \hline \end{array} \qquad F_2 = \begin{array}{|c|c|c|c|}\hline 2 & 2 & 3 & 3 \\ \hline 2 & 2 & 3 & 3 \\ \hline 3 & 3 & 2 & 2 \\ \hline 3 & 3 & 2 & 2 \\ \hline \end{array}.$$

- **Row Pemutation:** Swap row 1 and 3 in both F_1 and F_2. Then,

$$F_1' = \begin{array}{|c|c|c|c|}\hline 3 & 2 & 3 & 2 \\ \hline 2 & 3 & 2 & 3 \\ \hline 2 & 3 & 2 & 3 \\ \hline 3 & 2 & 3 & 2 \\ \hline \end{array} \qquad F_2' = \begin{array}{|c|c|c|c|}\hline 3 & 3 & 2 & 2 \\ \hline 2 & 2 & 3 & 3 \\ \hline 2 & 2 & 3 & 3 \\ \hline 3 & 3 & 2 & 2 \\ \hline \end{array}.$$

The frequency of each symbol is unaffected in each square and the orthogonality can be verified from the following table.

$[(F_1^{'}, F_2^{'})] =$

(3,3)	(2,3)	(3,2)	(2,2)
(2,2)	(3,2)	(2,3)	(3,3)
(2,3)	(3,2)	(2,3)	(3,3)
(3,3)	(2,3)	(3,2)	(2,2)

.

The entries of this table is equivalent to the Table 2, only the positions of entries of 1^{st} and 3^{rd} are swapped.

- **Column Pemutation:** Swap Column 2 and 3 in both F_1 and F_2. Then,

$F_1^{''} =$

2	2	3	3
2	2	3	3
3	3	2	2
3	3	2	2

$F_2^{''} =$

2	3	2	3
2	3	2	3
3	2	3	2
3	2	3	2

.

Similar to the row permutation, the frequency distribution and orthogonality are unaffected in column permutation.

$[(F_1^{''}, F_2^{''})] =$

(2,2)	(2,3)	(3,2)	(3,3)
(2,2)	(2,3)	(3,2)	(3,3)
(3,3)	(3,2)	(2,3)	(2,2)
(3,3)	(3,2)	(2,3)	(2,2)

.

- **Transpose:** Consider the transpose of F_1 and F_2.

$$F_1^T = \begin{array}{|c|c|c|c|}\hline 2 & 2 & 3 & 3 \\ \hline 3 & 3 & 2 & 2 \\ \hline 2 & 2 & 3 & 3 \\ \hline 3 & 3 & 2 & 2 \\ \hline \end{array} \qquad F_2^T = \begin{array}{|c|c|c|c|}\hline 2 & 2 & 3 & 3 \\ \hline 2 & 2 & 3 & 3 \\ \hline 3 & 3 & 2 & 2 \\ \hline 3 & 3 & 2 & 2 \\ \hline \end{array} .$$

Since each symbol appears the same number of times in rows and columns, transposing the squares preserves both orthogonality and symbol frequency.

$$[(F_1^T, F_2^T)] = \begin{array}{|c|c|c|c|}\hline (2,2) & (2,2) & (3,3) & (3,3) \\ \hline (3,2) & (3,2) & (2,3) & (2,3) \\ \hline (2,3) & (2,3) & (3,2) & (3,2) \\ \hline (3,3) & (3,3) & (2,2) & (2,2) \\ \hline \end{array} .$$

- **Symbol Permutation:** Swap the symbols 2 and 3 in F_1 and F_2. Then, the new squares are as follows:

$$F_1^* = \begin{array}{|c|c|c|c|}\hline 3 & 2 & 3 & 2 \\ \hline 3 & 2 & 3 & 2 \\ \hline 2 & 3 & 2 & 3 \\ \hline 2 & 3 & 2 & 3 \\ \hline \end{array} \qquad F_2^* = \begin{array}{|c|c|c|c|}\hline 3 & 3 & 2 & 2 \\ \hline 3 & 3 & 2 & 2 \\ \hline 2 & 2 & 3 & 3 \\ \hline 2 & 2 & 3 & 3 \\ \hline \end{array} .$$

Swapping symbols uniformly across each square does not change the orthogonality and frequency structure.

$$[(F_1^*, F_2^*)] = \begin{array}{|c|c|c|c|}\hline (3,3) & (2,3) & (3,2) & (2,2) \\ \hline (3,3) & (2,3) & (3,2) & (2,2) \\ \hline (2,2) & (3,2) & (2,3) & (3,3) \\ \hline (2,2) & (3,2) & (2,3) & (3,3) \\ \hline \end{array} .$$

Therefore, F_1 and F_2 are equivalent MOFS of type $F(4;2,2)$.

Remark 1.7. *Two sets of MOFS with the same number of F squares are said to be* ***isomorphic*** *if one set can be transformed into the squares of the other by a series of row and column permutations on all F squares in the first set.*

For example, there are exactly three ***non-isomorphic*** *sets of MOFS of type $F(4;2)$ where each set contains nine mutually orthogonal frequency squares* [5].

Theorem 1.8. *If there are t MOFS of type $F(n;n/m)$* [5], then the number of t MOFS satisfies the following inequality:

$$mt - t \quad \leq \quad (n-1)^2 \quad \text{or} \quad t \leq \frac{(n-1)^2}{m-1}. \tag{1}$$

Remark 1.9. *If equality holds in Theorem 1.8, then the set of MOFS is* ***complete*** [5] .

Let's check the theorem holds for some values of n.

Example 3. *For $n = 4$, it is known that there are 9 MOFS of type $F(4;2)$ [5]. The inequality (1) gives,*

$$mt - t \leq (n-1)^2$$

$$\Leftrightarrow 9 \times 2 - 9 \leq (4-1)^2$$

$$\Leftrightarrow 18 - 9 \leq 9$$

$$\Leftrightarrow 9 \leq 9,$$

which is true, and the set of MOFS of order 4 is complete.

Example 4. *For $n = 6$, there are* 17 *MOFS of type* $F(6; 2)$ [4],

According to the inequality,

$$2 \times 17 - 17 \leq (5 - 1)^2$$

$$\Leftrightarrow 34 - 17 \leq 25.$$

Therefore, this set of MOFS of order 6 satisfies the inequality, but it is not a complete set.

Computational research has been conducted to identify the largest possible sets of MOFS for specific values of n. For instance, when $n = 6$, there is a type maximal set of k MOFS of type $F(6; 2)$ if and only if k is 1, 17 or between 5 and 15 [4].

These findings offer valuable insight into the possible sizes that MOFS sets can be and illustrate how complex it is to find the largest possible sets.

In view of Theorem 1.8, it is natural to ask:

Question: How does the concept of mutual orthogonality extend to a set of Latin squares and frequency squares? What is the maximum number of frequency squares to make it complete in such cases?

Example 5. *Suppose that we check the orthogonality of a Latin square with a set of* **t** *frequency squares of type* $F(4; 2)$.

As per the inequality from Theorem 1.8,

$$4 + 2(t - 1) - t \leq 9$$

$$\Leftrightarrow 4 + 2t - 2 - t \leq 9$$

$$\Leftrightarrow t \leq 7.$$

What if it is a set of two Latin squares and **t** frequency squares?

$$8 + 2(t-2) - t \leq 9$$

$$\Leftrightarrow t \leq 5.$$

1.2.4 Mixed Mutually Orthogonal Frequency Squares (MMOFS)

Definition 6. *When symbols are allowed to have varying frequencies, we get frequency squares of mixed-type. A frequency square of type $F(n; \lambda_1, \lambda_2, \lambda_3, \dots, \lambda_t)$ is an array $n \times n$ filled with symbols a_i, where each a_i occurs λ_i times in every row and column, and $n = \sum_{i=1}^{t} \lambda_i$.*

The two frequency squares $F_1(n; \lambda_1, \lambda_2, \lambda_3, \dots, \lambda_t)$ and $F_2(n; \mu_1, \mu_2, \mu_3, \dots, \mu_s)$ are mutually orthogonal, when superimposed, each of the ts possible ordered pairs (i, j), where $1 \leq i \leq t$ and $1 \leq j \leq s$, occurs exactly $\lambda_i \mu_j$ times.

*A set of MMOFS $\{F_1, F_2, F_3, \dots, F_t\}$ is said to be **type maximal** if there is no additional frequency square F, of type already present in the set, orthogonal to F_i, where $1 \leq i \leq t$.*

Remark 1.10. *A set of MMOFS of the form $\{k \times F(n; \lambda_1, \lambda_2, \dots, \lambda_t); j \times F(n; \mu_1, \mu_2, \dots, \mu_s)\}$ refers to a set of k frequency squares of the type $F(n; \lambda_1, \lambda_2, \dots, \lambda_t)$ and j frequency squares of the type $F(n; \mu_1, \mu_2, \dots, \mu_s)$.*

We can verify the equivalence of two MMOFS through a sequence of operations similar to that in the Example 2.

Example 6. *Consider two MMOFS of type $F(4; 3, 1)$ and $F(4; 2, 2)$ as follows.*

$$M_1 = \begin{array}{|c|c|c|c|}\hline 2 & 2 & 2 & 3 \\ \hline 2 & 2 & 3 & 2 \\ \hline 2 & 3 & 2 & 2 \\ \hline 3 & 2 & 2 & 2 \\ \hline \end{array} \qquad M_2 = \begin{array}{|c|c|c|c|}\hline 2 & 3 & 2 & 3 \\ \hline 2 & 3 & 2 & 3 \\ \hline 3 & 2 & 3 & 2 \\ \hline 3 & 2 & 3 & 2 \\ \hline \end{array}.$$

Then,

$[(M_1;M_2)] =$

(2,2)	(2,3)	(2,2)	(3,3)
(2,2)	(2,3)	(3,2)	(2,3)
(2,3)	(3,2)	(2,3)	(2,2)
(3,3)	(2,2)	(2,3)	(2,2)

.

Table 3: Superimposed pairs of M_1 and M_2

The orthogonality is clear in the frequency sense. Now, we can apply the four operations to M_1 *and* M_2 *to examine the equivalence between them.*

- ***Row Permutation:*** *Swap row 1 and 3 in both* M_1 *and* M_2*. Then,*

$M_1' =$

2	3	2	2
2	2	3	2
2	2	2	3
3	2	2	2

$M_2' =$

3	2	3	2
2	3	2	3
2	3	2	3
3	2	3	2

.

The frequency of each symbol is unaffected in each square and the orthogonality can be verified from the following table.

$[(M_1', M_2')] =$

(2,3)	(3,2)	(2,3)	(2,2)
(2,2)	(2,3)	(3,2)	(2,3)
(2,2)	(2,3)	(2,2)	(3,3)
(3,3)	(2,2)	(2,3)	(2,2)

.

The entries of this table is equivalent to the Table 3 except the positions of entries of 1^{st} and 3^{rd} are swapped.

- **Column Pemutation:** Swap Column 2 and 3 in both F_1 and F_2. Then,

$$M_1'' = \begin{array}{|c|c|c|c|}\hline 2 & 2 & 2 & 3 \\ \hline 2 & 3 & 2 & 2 \\ \hline 2 & 2 & 3 & 2 \\ \hline 3 & 2 & 2 & 2 \\ \hline \end{array} \qquad M_2'' = \begin{array}{|c|c|c|c|}\hline 2 & 2 & 3 & 3 \\ \hline 2 & 2 & 3 & 3 \\ \hline 3 & 3 & 2 & 2 \\ \hline 3 & 3 & 2 & 2 \\ \hline \end{array} .$$

  Similar to the row permutation, the frequency distribution and orthogonality are unaffected in column permutation.

$$[(M_1'', M_2'')] = \begin{array}{|c|c|c|c|}\hline (2,2) & (2,2) & (2,3) & (3,3) \\ \hline (2,2) & (3,2) & (2,3) & (2,3) \\ \hline (2,3) & (2,3) & (3,2) & (2,2) \\ \hline (3,3) & (2,3) & (2,2) & (2,2) \\ \hline \end{array} .$$

  The entries of this table is equivalent to the Table 3 except the positions of entries of 2^{nd} and 3^{rd} are swapped.

- **Transpose:** Consider the transpose of M_1 and M_2.

$$M_1^T = \begin{array}{|c|c|c|c|}\hline 2 & 2 & 2 & 3 \\ \hline 2 & 2 & 3 & 2 \\ \hline 2 & 3 & 2 & 2 \\ \hline 3 & 2 & 2 & 2 \\ \hline \end{array} \qquad M_2^T = \begin{array}{|c|c|c|c|}\hline 2 & 2 & 3 & 3 \\ \hline 2 & 2 & 3 & 3 \\ \hline 3 & 3 & 2 & 2 \\ \hline 3 & 3 & 2 & 2 \\ \hline \end{array} .$$

  Since each symbol appears the same number of times in rows and columns, transposing the squares preserves both orthogonality and symbol frequency.

$$[(M_1^T, M_2^T)] = \begin{array}{|c|c|c|c|}\hline (2,2) & (2,2) & (2,3) & (3,3) \\ \hline (2,2) & (2,2) & (3,3) & (2,3) \\ \hline (2,3) & (3,3) & (2,2) & (2,2) \\ \hline (3,3) & (2,3) & (2,2) & (2,2) \\ \hline \end{array}.$$

This table is the transpose of Table 3.

- **Symbol Permutation:** Swap the symbols 2 and 3 in M_1 and M_2. Then, the new squares are as follows:

$$M_1^* = \begin{array}{|c|c|c|c|}\hline 3 & 3 & 3 & 2 \\ \hline 3 & 3 & 2 & 3 \\ \hline 3 & 2 & 3 & 3 \\ \hline 2 & 3 & 3 & 3 \\ \hline \end{array} \qquad M_2^* = \begin{array}{|c|c|c|c|}\hline 3 & 3 & 2 & 2 \\ \hline 3 & 3 & 2 & 2 \\ \hline 2 & 2 & 3 & 3 \\ \hline 2 & 2 & 3 & 3 \\ \hline \end{array}.$$

Swapping symbols uniformly across each square does not change the orthogonality and frequency structure.

$$[(M_1^*, M_2^*)] = \begin{array}{|c|c|c|c|}\hline (3,3) & (3,3) & (3,2) & (2,2) \\ \hline (3,3) & (3,3) & (2,2) & (3,2) \\ \hline (3,2) & (2,2) & (3,3) & (3,3) \\ \hline (2,2) & (3,2) & (3,3) & (3,3) \\ \hline \end{array}.$$

Therefore, M_1 and M_2 are equivalent MMOFS.

Remark 1.11. *A set of MMOFS is said to be **regular** if each frequency square is regular.*

1.2.5 Upper Bound for the Set of MMOFS

Now, Theorem1.8 also holds for MMOFS.

Theorem 1.12. *A set of MMOFS of order n, $\{F_1, F_2, F_3, \ldots, F_k\}$ on the number of symbols $m_1, m_2, m_3, \ldots, m_s$ respectively* [3]. Then, the number of k MMOFS satisfies the following inequality

$$\sum_{s=1}^{k} m_s - k \quad \leq \quad (n-1)^2. \tag{2}$$

To establish the result, we first demonstrate the linear independence of the set of MMOFS, which we present as a lemma.

Lemma 1.13. *Let $\{A_1, A_2, \ldots, A_K\} \in R^{m \times n}$ be matrices such that none of the A_i is the zero matrix, and for each $i \neq j$, $A_i \cdot A_j = 0$, that is, the matrices are pairwise orthogonal. Then the set $\{A_1, A_2, \ldots, A_K\}$ is linearly independent.*

Proof. Suppose that we have a linear combination:

$$\lambda_1 A_1 + \lambda_2 A_2 + \cdots + \lambda_k A_k = 0_{m \times n}.$$

We want to show that all $\lambda_i = 0$.

Take the Frobenius inner product of both sides of the linear combination above with A_j, where $j \in \{1, 2, \ldots, k\}$.

$$\left(\sum_{i=1}^{k} \lambda_i A_i \right) \cdot A_j = 0$$

Using the linearity of the Frobenius inner product:

$$\sum_{i=1}^{k} \lambda_i (A_i \cdot A_j) = 0$$

Since all $A_j's$ are non zero,

$$A_j \cdot A_j = A_j^2 > 0,$$

and we have $A_i \cdot A_j = 0$ for $i \neq j$.

Therefore, all terms in the sum vanish except $i = j$.

$$\Rightarrow \lambda_j A_j^2 = 0$$

$$A_j^2 > 0 \Rightarrow \lambda_j = 0$$

Since this holds for every $j = 1, 2, \ldots, k$, we conclude that all $\lambda_i = 0$.

Hence, the matrices $\{A_1, A_2, \ldots, A_k\}$ are linearly independent.

□

Now, we can prove Theorem 1.12.

Proof. We construct a set of linearly independent $n \times n$ matrices as vectors in a vector space of n^2. Define three sets $n \times n$ of indicator matrices as follows.

Row indicator matrices, R_r, for $1 \leq r \leq n$

$$R_r[i,j] = \begin{cases} 1, & \mathit{rows} \quad i = r \\ 0, & \mathit{otherwise}. \end{cases}$$

These matrices track the individual rows.

Column indicator matrices, C_c, for $1 \leq c \leq n$

$$C_c[i,j] = \begin{cases} 1, & \mathit{columns} \quad j = c \\ 0, & \mathit{otherwise}. \end{cases}$$

These matrices track the individual columns.

Symbol indicator matrices, $S_{s,t}$, for $1 \leq s \leq m_s$ and $1 \leq t \leq k$

$$S_{s,t}[i,j] = \begin{cases} 1, & F_t[i,j] \quad = s \\ 0, & otherwise. \end{cases}$$

These matrices track where each symbol m_s appears in F_t. Also, we have J_n, which is an n^{th} order matrix of all one.

We propose that the following set of matrices is linearly independent.

$$\{J_n\} \cup \{R_r : 1 \leq r \leq n-1\} \cup \{C_c : 1 \leq c \leq n-1\} \cup \{S_{s,t} : 1 \leq s \leq m_s, 1 \leq t \leq k\}. \quad (3)$$

The total number of matrices in this set is

$$1 + 2(n-1) + \sum_{s=1}^{k}(m_s - 1).$$

Since the space of $n \times n$ matrices has dimension n^2, the total number of matrices cannot exceed n^2. Hence,

$$1 + 2n - 2 + \sum_{s=1}^{k} m_s - k \leq n^2$$

$$\Rightarrow \sum_{s=1}^{k} m_s - k \leq n^2 - 2n + 1$$

$$\Rightarrow \sum_{s=1}^{k} m_s - k \leq (n-1)^2.$$

The linear independence of the above set establishes the complete result because,

- Each matrix in the set tells us something different about the MOFS: one keeps track of rows, another tracks columns, and other shows where different symbols appear.
- No matrix in the set is just a combination of others, which means they all give unique information.

- If we have these matrices, we can fully rebuild the MOFS from them without missing anything.
- These matrices act like building blocks that describe the MOFS completely.
- Checking their independence confirm the construction of MOFS.
- since 16 is the number of values required to describe a 4 × 4 matrix, the total number of these matrices shouldn't be more than 16.

To ensure independence, we define the transformed version of our matrices:

$$R'_r = nR_r - J_n, \quad C'_c = nC_c - J_n, \quad S'_{s,t} = \frac{n}{\lambda_{s,t}} S_{s,t} - J_n$$

These transformations normalize the matrices and ensure that they remain independent of J_n. We now check the orthogonality by verifying that the inner products (entry-wise sum of products) between different matrices in our set are zero:

Row-Column interaction: Since rows and columns are structurally orthogonal, their transformed matrices remain independent.

$$\Rightarrow R'_r \circ C'_c = 0.$$

Row-Symbol interaction: Symbol matrices track individual symbols and do not interfere with the row indicators.

$$\Rightarrow R'_r \circ S'_{s,t} = 0.$$

Column-Symbol interaction: Similarly, symbol matrices do not affect column indicators.

$$\Rightarrow C'_c \circ S'_{s,t} = 0.$$

Symbol matrices for different squares: Since different squares track independent sets of symbols, their indicator matrices remains orthogonal.

$$\Rightarrow S'_{s,t} \circ S'_{s',t'} = 0, \quad \text{for } t \neq t'$$

for all r, c, s, t, s', t' provided $t \neq t'$. $\square$

Remark 1.14. *Theorem* 1.12 *is a generalization of Theorem* 1.8.

Let us illustrate the theorem with an example.

Example 7. *We take two MOFS of order $n = 4$, where F_1 has two symbols, and F_2 has four symbols.*

$$F_1 = \begin{bmatrix} 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 \end{bmatrix} \qquad F_2 = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 4 & 3 \\ 3 & 4 & 1 & 2 \\ 4 & 3 & 2 & 1 \end{bmatrix}.$$

The row indicator matrices (R_r):

$$R_1 = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \qquad R_2 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$R_3 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \qquad R_4 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 \end{bmatrix}.$$

The column indicator matrices (*C_c*):

$$C_1 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \qquad C_2 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$$

$$C_3 = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \qquad C_4 = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

The Symbol indicator matrices (*$S_{s,t}$*)*, which represent the matrix for each symbol s in* F_t

$$S_{0,1} = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix} \qquad S_{1,1} = \begin{bmatrix} 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 \end{bmatrix}$$

$$S_{1,2} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad S_{2,2} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

$$S_{3,2} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \qquad S_{4,2} = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}.$$

Now, we define the transformed matrices by using the rule.

$$R'_r = 4R_r - J_4, \quad C'_c = 4C_c - J_4, \quad S'_{s,t} = \frac{4}{\lambda_{s,t}} S_{s,t} - J_4$$

Transformed row indicator matrices (R'_r):

$$R'_1 = \begin{bmatrix} 3 & 3 & 3 & 3 \\ -1 & -1 & -1 & -1 \\ -1 & -1 & -1 & -1 \\ -1 & -1 & -1 & -1 \end{bmatrix} \qquad R'_2 = \begin{bmatrix} -1 & -1 & -1 & -1 \\ 3 & 3 & 3 & 3 \\ -1 & -1 & -1 & -1 \\ -1 & -1 & -1 & -1 \end{bmatrix}$$

$$R'_3 = \begin{bmatrix} -1 & -1 & -1 & -1 \\ -1 & -1 & -1 & -1 \\ 3 & 3 & 3 & 3 \\ -1 & -1 & -1 & -1 \end{bmatrix} \qquad R'_4 = \begin{bmatrix} -1 & -1 & -1 & -1 \\ -1 & -1 & -1 & -1 \\ -1 & -1 & -1 & -1 \\ 3 & 3 & 3 & 3 \end{bmatrix}.$$

Transformed column indicator matrices (C'_c):

$$C'_1 = \begin{bmatrix} 3 & -1 & -1 & -1 \\ 3 & -1 & -1 & -1 \\ 3 & -1 & -1 & -1 \\ 3 & -1 & -1 & -1 \end{bmatrix} \qquad C'_2 = \begin{bmatrix} -1 & 3 & -1 & -1 \\ -1 & 3 & -1 & -1 \\ -1 & 3 & -1 & -1 \\ -1 & 3 & -1 & -1 \end{bmatrix}$$

$$C'_3 = \begin{bmatrix} -1 & -1 & 3 & -1 \\ -1 & -1 & 3 & -1 \\ -1 & -1 & 3 & -1 \\ -1 & -1 & 3 & -1 \end{bmatrix} \qquad C'_4 = \begin{bmatrix} -1 & -1 & -1 & 3 \\ -1 & -1 & -1 & 3 \\ -1 & -1 & -1 & 3 \\ -1 & -1 & -1 & 3 \end{bmatrix}.$$

Transformed Symbol indicator matrices *($S_{s,t}$), which represents the matrix for each symbol s in F_t, where each $\lambda_{s,t}$ represents the frequency of each symbol s in F_t.*

$$S'_{0,1} = \begin{bmatrix} 1 & 1 & -1 & -1 \\ -1 & -1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ -1 & -1 & 1 & 1 \end{bmatrix} \qquad S'_{1,1} = \begin{bmatrix} -1 & -1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ -1 & -1 & 1 & 1 \\ 1 & 1 & -1 & -1 \end{bmatrix}$$

$$S'_{1,2} = \begin{bmatrix} 3 & -1 & -1 & -1 \\ -1 & 3 & -1 & -1 \\ -1 & -1 & 3 & -1 \\ -1 & -1 & -1 & 3 \end{bmatrix} \qquad S_{2,2} = \begin{bmatrix} -1 & 3 & -1 & -1 \\ 3 & -1 & -1 & -1 \\ -1 & -1 & -1 & 3 \\ -1 & -1 & 3 & -1 \end{bmatrix}$$

$$S'_{3,2} = \begin{bmatrix} -1 & -1 & 3 & -1 \\ -1 & -1 & -1 & 3 \\ 3 & -1 & -1 & -1 \\ -1 & 3 & -1 & -1 \end{bmatrix} \qquad S'_{4,2} = \begin{bmatrix} -1 & -1 & -1 & 3 \\ -1 & -1 & 3 & -1 \\ -1 & 3 & -1 & -1 \\ 3 & -1 & -1 & -1 \end{bmatrix}.$$

It is clear that $R'_r \circ C'_c = 0$.

For instance,

$$R'_1 \circ C'_1 = 9 + -3 + -3 + -3 + -3 + 1 + 1 + 1 + -3 + 1 + 1 + 1 + -3 + 1 + 1 + 1 = 0.$$

Similarly, $R'_r \circ S'_{s,t} = 0$, $C'_c \circ S_{s,t} = 0$ *and* $S_{s,t} \circ S'_{s,t} = 0$ *for* $t \neq t'$.

From the above matrices we can confirm:

- *Linear independence: The set* $\{J_4\} \cup \{R_r\} \cup \{C_c\} \cup \{S_{s,t}\}$ *is linearly independent in the vector space of the* 4×4 *matrices.*

- *Calculating the sum,*

$$\sum_{s=1}^{k} m_s - k = 2 + 4 - 2 = 4$$

$$(n-1)^2 = (4-1)^2 = 9.$$

Since $4 \leq 9$*, satisfying the theorem.*

1.3 Thesis outline

This thesis aims to explore the size of sets of MMOFS for different orders. Chapter 2 focuses on constructing sets of MOFS by using Hadamard matrices and finite field theory. In the following chapter, we introduce permutation arrays and equidistant permutation arrays, which have been used to construct sets of MOFS.

Later, in Chapter 4, we define different mappings from MOLS to frequency squares and examine how these mappings preserve orthogonality. Moreover, we extend these mappings on MOLS to construct MMOFS of higher orders using mixed orthogonal arrays.

Finally, we apply the theories and constructions in the preceding chapters to identify the sets of MMOFS for small orders.

2 Construction of maximal sets of MOFS

In this chapter, we show how maximal sets of MOFS can be constructed using Hadamard matrices and the Kronecker product. We then adopt an alternative approach, based on finite field theory, to construct a set of maximal MOFS of order 8.

2.1 Hadamard Matrices

Definition 7. *A Hadamard matrix* H_n *is a square matrix of order* n *whose entries are either* $+1$ *or* -1 *and whose rows are mutually orthogonal. It means, each pair of rows has matching entries in exactly half of their columns and mismatched entries in the remaining columns, and it satisfies the equation:*

$$H_n \times H_n^T = nI_n$$

Example 8. *An example of a fourth-order Hadamard matrix is as follows.*

1	1	1	1
1	-1	1	-1
1	1	-1	-1
1	-1	-1	1

Table 4: Hadamard matrix of order 4

.

Let us compare the first and second rows. Within these two rows, the first and third columns match, and the second and fourth column mismatch. This property is a consequence of the mutual orthogonality of the rows. The dot product of two distinct rows is zero.

For example, if we compute the dot product of the first two rows,

$$(1 \times 1) + (1 \times -1) + (1 \times 1) + (1 \times -1) = 0$$

It is evident that if H_n is a Hadamard matrix, then the matrix obtained by permuting its rows (or columns) and negating any rows (or columns) will also be a Hadamard matrix. Using such transformations, we can always transform H_n into an equivalent matrix in which all the entries in the first row and the first column are equal to +1. This matrix is called a **normalised Hadamard matrix**. Example 8 is a normalized Hadamard matrix of order 4.

Later we transform the Hadamard matrix of order n into frequency squares of order n. The orthogonality of rows of the Hadamard matrix will ensure the mutual orthogonality of the constructed frequency squares.

2.1.1 Kronecker products

Definition 8. *The Kronecker product, sometimes denoted by ⊗, is an operation on two matrices of arbitrary size that produces a block matrix.*

Generally given two matrices:

$$A = [a_{ij}] \in \mathbb{R}^{m \times n}, \quad B \in \mathbb{R}^{p \times q}$$

The Kronecker product of A and B is denoted by

$$A \otimes B,$$

and is defined as

$$A \otimes B = \begin{bmatrix} a_{11}B & a_{12}B & \dots & a_{1n}B \\ a_{21}B & a_{22}B & \dots & a_{2n}B \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1}B & a_{m2}B & \dots & a_{mn}B \end{bmatrix},$$

where each entry a_{ij} in the matrix A is scaled by the entire matrix B. The resulting matrix has dimensions:

$$mp \times nq.$$

Here is an example of the Kronecker product derived from two 2 × 2 matrices.

Example 9.

$$\begin{bmatrix} 1 & 2 \\ 4 & 3 \end{bmatrix} \otimes \begin{bmatrix} 3 & 1 \\ 5 & 2 \end{bmatrix}.$$

$$= \left[\begin{array}{cc|cc} 1\times 3 & 1\times 1 & 2\times 3 & 2\times 1 \\ 1\times 5 & 1\times 2 & 2\times 5 & 2\times 2 \\ \hline 4\times 3 & 4\times 1 & 3\times 3 & 3\times 1 \\ 4\times 5 & 4\times 2 & 3\times 5 & 3\times 2 \end{array}\right].$$

$$= \left[\begin{array}{cc|cc} 3 & 1 & 6 & 2 \\ 5 & 2 & 10 & 4 \\ \hline 12 & 4 & 9 & 3 \\ 20 & 8 & 15 & 6 \end{array}\right].$$

The resulting matrix has dimensions

$$2 \cdot 2 \times 2 \cdot 2 = 4 \times 4.$$

The Kronecker product serves as a fundamental tool in constructing large-order Hadamard matrices from smaller ones. For instance, if H_1 and H_2 are Hadamard matrices of order m and n respectively, then their Kronecker product $H = H_1 \otimes H_2$ results in a Hadamard matrix of order $m \times n$. Sylvester construction [1] utilizes this principle by recursively defining

Hadamard matrices as follows:

$$H_{2^{k+1}} = \begin{bmatrix} H_{2^k} & H_{2^k} \\ H_{2^k} & -H_{2^k} \end{bmatrix} = H_2 \otimes H_{2^k},$$

where $2 \leq k \in N$, and

$$H_2 = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$

is the Hadamard matrix of order 2.

In this manner, Sylvester [19] constructed Hadamard matrices of order 2^k for every non-negative integer k.

This is known as the **Doubling construction** for Hadamard matrices.

Consider the Hadamard matrices of order 2 and 4,

$$H_2 = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix},$$

and

$$H_4 = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \end{bmatrix}.$$

Then,

$$H_8 = H_2 \otimes H_4 = \left[\begin{array}{cccc|cccc} +1 & +1 & +1 & +1 & +1 & +1 & +1 & +1 \\ +1 & -1 & +1 & -1 & +1 & -1 & +1 & -1 \\ +1 & +1 & -1 & -1 & +1 & +1 & -1 & -1 \\ +1 & -1 & -1 & +1 & +1 & -1 & -1 & +1 \\ \hline +1 & +1 & +1 & +1 & -1 & -1 & -1 & -1 \\ +1 & -1 & +1 & -1 & -1 & +1 & -1 & +1 \\ +1 & +1 & -1 & -1 & -1 & -1 & +1 & +1 \\ +1 & -1 & -1 & +1 & -1 & +1 & +1 & -1 \end{array}\right].$$

$$= \begin{bmatrix} H_4 & H_4 \\ H_4 & -H_4 \end{bmatrix}.$$

Why does this work?

A Hadmard matrix satisfies the property that any two distinct rows are orthogonal, which means that their dot product is zero. This property is preserved under the Kronecker product because:

- Each entry a_{ij} of the original Hadamard matrix H_n is replaced by $a_{ij}B$ where B is another Hadamard matrix.

- Since B is another Hadamard matrix, the orthogonality of the original rows is preserved for the final matrix.

- The resulting matrix maintains the Hadamard properties because it still satisfies condition $HH^T = nI$.

Thus, the Kronecker product of two Hadamard matrix is again another Hadamard matrix.

Since the Hadamard matrix guarantees that any two rows are orthogonal, the function defined by these rows are mutually orthogonal. This property allows us to construct the mutually orthogonal frequency squares (MOFS) using the Kronecker product of Hadamard matrices. This construction [5] can be iterated to produce $(n-1)^2$ MOFS of order n, where n is a power of 2.

Let H_n be a Hadamard matrix of order $n = 2^k$. The Kronecker product $H_n \otimes H_n$ results a Hadamard matrix of order n^2, which preserves the orthogonality of rows and columns. This matrix is a $n \times n$ array of $n \times n$ submatrices, each of which is $\pm H_n$.

By excluding uniform first rows and column blocks, the remaining $(n-1)^2$ blocks can be converted into frequency squares of order n with two elements. The orthogonality of rows of H_n ensures the mutual orthogonality of the n frequency squares. Hence, for every $n = 2^k$, we can construct $(n-1)^2$ MOFS of order n by the Kronecker product. See Example 10.

Theorem 2.1. *If there exists a Hadamard matrix of order n, then there exists a complete set of MOFS(n) (Theorem* 1 [4]).

The Hadamard conjecture suggests that Hadamard matrices exist for every size $4n$. Suppose that the conjecture is true. This would confirm the existence of a Hadamard matrix of order $4n$ for every positive integer n. Then, there would exist a complete set of $(4n-1)^2$ MOFS of type $F(4n; 2n)$ (Theorem 5 in [9]).

On the other hand, Theorem 4.6 in [11] gives an estimate of how large a set of MOFS can be when n is a multiple of 4 , roughly as n^2 (with a small correction).

Conjecture 1. *Hadamard conjecture*

There is a Hadamard matrix of order $4n$ *for every positive integer* n.

Theorem 2.2. *(Theorem* 4.6 *in* [11]) For every $\varepsilon > 0$, there is an integer N such that for any integer $\lambda > N$, there are

$$\ell \geq \frac{16}{9}\lambda^2(1-\varepsilon)$$

mutually orthogonal F-squares, $F(4\lambda, 2\lambda)$.

However, when $n \equiv 2 \pmod 4$, no such lower bound is known.

Although Hadamard matrices enable the construction of a complete set of MOFS of certain orders, there are cases where such cases do not exist. Specifically, there is no complete set of MOFS of order n when $\frac{n}{2}$ is odd (Corollary 11 in [4]).

Example 10. *Consider the Hadamard matrix of order* 4 *and taking the Kronecker product of* H_4 *with itself, we obtain:*

$$H_4 \otimes H_4 = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \end{bmatrix} \otimes \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \end{bmatrix}$$

$$
= \left[\begin{array}{cccc|cccc|cccc|cccc}
+1 & +1 & +1 & +1 & +1 & +1 & +1 & +1 & +1 & +1 & +1 & +1 & +1 & +1 & +1 & +1 \\
+1 & -1 & +1 & -1 & +1 & -1 & +1 & -1 & +1 & -1 & +1 & -1 & +1 & -1 & +1 & -1 \\
+1 & +1 & -1 & -1 & +1 & +1 & -1 & -1 & +1 & +1 & -1 & -1 & +1 & +1 & -1 & -1 \\
+1 & -1 & -1 & +1 & +1 & -1 & -1 & +1 & +1 & -1 & -1 & +1 & +1 & -1 & -1 & +1 \\
\hline
+1 & +1 & +1 & +1 & -1 & -1 & -1 & -1 & +1 & +1 & +1 & +1 & -1 & -1 & -1 & -1 \\
+1 & -1 & +1 & -1 & -1 & +1 & -1 & +1 & +1 & -1 & +1 & -1 & -1 & +1 & -1 & +1 \\
+1 & +1 & -1 & -1 & -1 & -1 & +1 & +1 & +1 & +1 & -1 & -1 & -1 & -1 & +1 & +1 \\
+1 & -1 & -1 & +1 & -1 & +1 & +1 & -1 & +1 & -1 & -1 & +1 & -1 & +1 & +1 & -1 \\
\hline
+1 & +1 & +1 & +1 & +1 & +1 & +1 & +1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 \\
+1 & -1 & +1 & -1 & +1 & -1 & +1 & -1 & -1 & +1 & -1 & +1 & -1 & +1 & -1 & +1 \\
+1 & +1 & -1 & -1 & +1 & +1 & -1 & -1 & -1 & -1 & +1 & +1 & -1 & -1 & +1 & +1 \\
+1 & -1 & -1 & +1 & +1 & -1 & -1 & +1 & -1 & +1 & +1 & -1 & -1 & +1 & +1 & -1 \\
\hline
+1 & +1 & +1 & +1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & +1 & +1 & +1 & +1 \\
+1 & -1 & +1 & -1 & -1 & +1 & -1 & +1 & -1 & +1 & -1 & +1 & +1 & -1 & +1 & -1 \\
+1 & +1 & -1 & -1 & -1 & -1 & +1 & +1 & -1 & -1 & +1 & +1 & +1 & +1 & -1 & -1 \\
+1 & -1 & -1 & +1 & -1 & +1 & +1 & -1 & -1 & +1 & +1 & -1 & +1 & -1 & -1 & +1
\end{array}\right].
$$

Since the first block consists of uniform columns and first row in each block has uniform entries, we can skip it and pick the column-wise entries of the next three blocks to construct the 9 *MOFS of order* 4*. By replacing* $+1$ *by* 1 *and* -1 *by* 0*, we obtain the* 9 *MOFS of order* 4 *as follows.*

$$\begin{bmatrix}1&0&1&0\\0&1&0&1\\1&0&1&0\\0&1&0&1\end{bmatrix}\begin{bmatrix}1&1&0&0\\0&0&1&1\\1&1&0&0\\0&0&1&1\end{bmatrix}\begin{bmatrix}1&0&0&1\\0&1&1&0\\1&0&0&1\\0&1&1&0\end{bmatrix}\begin{bmatrix}1&0&1&0\\1&0&1&0\\0&1&0&1\\0&1&0&1\end{bmatrix}\begin{bmatrix}1&1&0&0\\1&1&0&0\\0&0&1&1\\0&0&1&1\end{bmatrix}$$

$$\begin{bmatrix}1&0&0&1\\1&0&0&1\\0&1&1&0\\0&1&1&0\end{bmatrix}\begin{bmatrix}1&0&1&0\\0&1&0&1\\0&1&0&1\\1&0&1&0\end{bmatrix}\begin{bmatrix}1&1&0&0\\0&0&1&1\\0&0&1&1\\1&1&0&0\end{bmatrix}\begin{bmatrix}1&0&0&1\\0&1&1&0\\0&1&1&0\\1&0&0&1\end{bmatrix}$$

Table 5: MOFS of order 4

.

Determining whether the Hadamard conjecture holds for every positive integer n remains an open problem in mathematics, but extensive progress has been made in the construction of Hadamard matrices of specific order. The origin of such constructions dates back to Sylvester's method from 1867, which provided Hadamard matrices of order 2^k for $k \geq 0$. which results in matrices of order $2, 4, 8, 16, etc.$ In 1893 Jacques Hadamard constructed matrices of orders 12 and 20 [7].

In 1933, Raymond Paley discovered the Paley construction using finite fields, which produces a Hadamard matrix of order $2(q + 1)$ when q is a prime power that is congruent to 1 modulo 4 [16]. The smallest order that cannot be constructed by a combination of the Sylvester and Paley methods is 92. Baumert, Golomb, and Hall found a Hadamard matrix of this order using a computer in 1962 at JPL [2].

In 2005, Hadi Kharaghani and Behruz Tayfeh-Rezaie published their construction of a Hadamard matrix of order 428 [10]. In 2014 [6], there were 12 multiples of 4 less than 2000 for which no Hadamard matrix of that order was known, which are:

$$668, 716, 892, 1132, 1244, 1388, 1436, 1676, 1772, 1916, 1948, 1964.$$

2.2 Construction MOFS from Finite Fields

In this section, we present an alternative approach for proving Theorem 2.2, focusing on case $n = 8$, using concepts from the finite field theory. This construction was found independently and helps to explain the algebraic ideas behind the result.

Definition 9. *The finite field $\mathbb{F}_2$ is a field with 2 elements $\{0, 1\}$ with two operations:*

- ***Addition*** $\pmod 2$*:*

$$0 \oplus 0 = 0, \quad 0 \oplus 1 = 1, \quad 1 \oplus 0 = 0, \quad 1 \oplus 1 = 0.$$

- ***Multiplication*** $\pmod 2$*:*

$$0 \cdot 0 = 0, \quad 0 \cdot 1 = 0, \quad 1 \cdot 0 = 0, \quad 1 \cdot 1 = 1.$$

The n^{th} dimensional vector space over $\mathbb{F}_2$ is denoted by $\mathbb{F}_2^n$ is the set of all vectors of length n:

$$\mathbb{F}_2^n = \{(a_1, a_2, a_3, \ldots, a_n) \mid a_i \in \mathbb{F}_2\}.$$

- **Vector addition** is done coordinate-wise modulo 2:

$$(a_1, a_2, \ldots, a_n) \oplus (b_1, b_2, \ldots, b_n) = (a_1 \oplus b_1, a_2 \oplus b_2, \ldots, a_n \oplus b_n).$$

- **Scalar multiplication** is also done coordinate-wise. For $\lambda \in \mathbb{F}_2$,

$$\lambda \cdot (a_1, a_2, \ldots, a_n) = (\lambda a_1, \lambda a_2, \ldots, \lambda a_n).$$

The vector space $\mathbb{F}_2^n$ has exactly 2^n elements.

For example, for $n = 3$,

$$\mathbb{F}_2^3 = \{(0,0,0), (0,0,1), (1,0,0), (0,1,0), (1,1,0), (1,0,1), (0,1,1), (1,1,1)\}.$$

Definition 10. *For each non-zero vector $a_k \in \mathbb{F}_2^3$, where $k \in \{1, 2, 3, \ldots, 7\}$ let us define 8×8 matrix M_k over $\mathbb{F}_2^3 \times \mathbb{F}_2^3$ by a function $f_{a_k} : \mathbb{F}_2^3 \to \mathbb{F}_2$ as follows:*

For each $u, v \in \mathbb{F}_2^3$, the entry in the row u and column v of M_k is given by

$$M_k(u, v) = f_{a_k}(u \oplus v) = a_k \cdot (u \oplus v) \pmod 2.$$

Example 11. *Consider the vector space*

$$\mathbb{F}_2^3 = \{(0,0,0), (0,0,1), (1,0,0), (0,1,0), (1,1,0), (1,0,1), (0,1,1), (1,1,1)\}.$$

Define the vector addition $\pmod 2$ *among these* 7 *vectors as follows.*

$\oplus$	(0,0,0)	(0,0,1)	(0,1,0)	(0,1,1)	(1,0,0)	(1,0,1)	(1,1,0)	(1,1,1)
(0,0,0)	(0,0,0)	(0,0,1)	(0,1,0)	(0,1,1)	(1,0,0)	(1,0,1)	(1,1,0)	(1,1,1)
(0,0,1)	(0,0,1)	(0,0,0)	(0,1,1)	(0,1,0)	(1,0,1)	(1,0,0)	(1,1,1)	(1,1,0)
(0,1,0)	(0,1,0)	(0,1,1)	(0,0,0)	(0,0,1)	(1,1,0)	(1,1,1)	(1,0,0)	(1,0,1)
(0,1,1)	(0,1,1)	(0,1,0)	(0,0,1)	(0,0,0)	(1,1,1)	(1,1,0)	(1,0,1)	(1,0,0)
(1,0,0)	(1,0,0)	(1,0,1)	(1,1,0)	(1,1,1)	(0,0,0)	(0,0,1)	(0,1,0)	(0,1,1)
(1,0,1)	(1,0,1)	(1,0,0)	(1,1,1)	(1,1,0)	(0,0,1)	(0,0,0)	(0,1,1)	(0,1,0)
(1,1,0)	(1,1,0)	(1,1,1)	(1,0,0)	(1,0,1)	(0,1,0)	(0,1,1)	(0,0,0)	(0,0,1)
(1,1,1)	(1,1,1)	(1,1,0)	(1,0,1)	(1,0,0)	(0,1,1)	(0,1,0)	(0,0,1)	(0,0,0)

.

$$f_{(0,0,1)} = (0,0,1) \cdot \begin{bmatrix} (0,0,0) & (0,0,1) & (0,1,0) & (0,1,1) & (1,0,0) & (1,0,1) & (1,1,0) & (1,1,1) \\ (0,0,1) & (0,0,0) & (0,1,1) & (0,1,0) & (1,0,1) & (1,0,0) & (1,1,1) & (1,1,0) \\ (0,1,0) & (0,1,1) & (0,0,0) & (0,0,1) & (1,1,0) & (1,1,1) & (1,0,0) & (1,0,1) \\ (0,1,1) & (0,1,0) & (0,0,1) & (0,0,0) & (1,1,1) & (1,1,0) & (1,0,1) & (1,0,0) \\ (1,0,0) & (1,0,1) & (1,1,0) & (1,1,1) & (0,0,0) & (0,0,1) & (0,1,0) & (0,1,1) \\ (1,0,1) & (1,0,0) & (1,1,1) & (1,1,0) & (0,0,1) & (0,0,0) & (0,1,1) & (0,1,0) \\ (1,1,0) & (1,1,1) & (1,0,0) & (1,0,1) & (0,1,0) & (0,1,1) & (0,0,0) & (0,0,1) \\ (1,1,1) & (1,1,0) & (1,0,1) & (1,0,0) & (0,1,1) & (0,1,0) & (0,0,1) & (0,0,0) \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \end{bmatrix}.$$

$$f_{(0,1,0)} = (0,1,0) \cdot \begin{bmatrix} (0,0,0) & (0,0,1) & (0,1,0) & (0,1,1) & (1,0,0) & (1,0,1) & (1,1,0) & (1,1,1) \\ (0,0,1) & (0,0,0) & (0,1,1) & (0,1,0) & (1,0,1) & (1,0,0) & (1,1,1) & (1,1,0) \\ (0,1,0) & (0,1,1) & (0,0,0) & (0,0,1) & (1,1,0) & (1,1,1) & (1,0,0) & (1,0,1) \\ (0,1,1) & (0,1,0) & (0,0,1) & (0,0,0) & (1,1,1) & (1,1,0) & (1,0,1) & (1,0,0) \\ (1,0,0) & (1,0,1) & (1,1,0) & (1,1,1) & (0,0,0) & (0,0,1) & (0,1,0) & (0,1,1) \\ (1,0,1) & (1,0,0) & (1,1,1) & (1,1,0) & (0,0,1) & (0,0,0) & (0,1,1) & (0,1,0) \\ (1,1,0) & (1,1,1) & (1,0,0) & (1,0,1) & (0,1,0) & (0,1,1) & (0,0,0) & (0,0,1) \\ (1,1,1) & (1,1,0) & (1,0,1) & (1,0,0) & (0,1,1) & (0,1,0) & (0,0,1) & (0,0,0) \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 \end{bmatrix}.$$

$$f_{(0,1,1)} = (0,1,1)\cdot \begin{bmatrix} (0,0,0) & (0,0,1) & (0,1,0) & (0,1,1) & (1,0,0) & (1,0,1) & (1,1,0) & (1,1,1) \\ (0,0,1) & (0,0,0) & (0,1,1) & (0,1,0) & (1,0,1) & (1,0,0) & (1,1,1) & (1,1,0) \\ (0,1,0) & (0,1,1) & (0,0,0) & (0,0,1) & (1,1,0) & (1,1,1) & (1,0,0) & (1,0,1) \\ (0,1,1) & (0,1,0) & (0,0,1) & (0,0,0) & (1,1,1) & (1,1,0) & (1,0,1) & (1,0,0) \\ (1,0,0) & (1,0,1) & (1,1,0) & (1,1,1) & (0,0,0) & (0,0,1) & (0,1,0) & (0,1,1) \\ (1,0,1) & (1,0,0) & (1,1,1) & (1,1,0) & (0,0,1) & (0,0,0) & (0,1,1) & (0,1,0) \\ (1,1,0) & (1,1,1) & (1,0,0) & (1,0,1) & (0,1,0) & (0,1,1) & (0,0,0) & (0,0,1) \\ (1,1,1) & (1,1,0) & (1,0,1) & (1,0,0) & (0,1,1) & (0,1,0) & (0,0,1) & (0,0,0) \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 1 & 1 & 0 & 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 & 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 & 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 & 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 & 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 & 0 & 1 & 1 & 0 \end{bmatrix}.$$

$$f_{(1,0,0)} = (1,0,0)\cdot \begin{bmatrix} (0,0,0) & (0,0,1) & (0,1,0) & (0,1,1) & (1,0,0) & (1,0,1) & (1,1,0) & (1,1,1) \\ (0,0,1) & (0,0,0) & (0,1,1) & (0,1,0) & (1,0,1) & (1,0,0) & (1,1,1) & (1,1,0) \\ (0,1,0) & (0,1,1) & (0,0,0) & (0,0,1) & (1,1,0) & (1,1,1) & (1,0,0) & (1,0,1) \\ (0,1,1) & (0,1,0) & (0,0,1) & (0,0,0) & (1,1,1) & (1,1,0) & (1,0,1) & (1,0,0) \\ (1,0,0) & (1,0,1) & (1,1,0) & (1,1,1) & (0,0,0) & (0,0,1) & (0,1,0) & (0,1,1) \\ (1,0,1) & (1,0,0) & (1,1,1) & (1,1,0) & (0,0,1) & (0,0,0) & (0,1,1) & (0,1,0) \\ (1,1,0) & (1,1,1) & (1,0,0) & (1,0,1) & (0,1,0) & (0,1,1) & (0,0,0) & (0,0,1) \\ (1,1,1) & (1,1,0) & (1,0,1) & (1,0,0) & (0,1,1) & (0,1,0) & (0,0,1) & (0,0,0) \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

$$f_{(1,0,1)} = (1,0,1) \cdot \begin{bmatrix} (0,0,0) & (0,0,1) & (0,1,0) & (0,1,1) & (1,0,0) & (1,0,1) & (1,1,0) & (1,1,1) \\ (0,0,1) & (0,0,0) & (0,1,1) & (0,1,0) & (1,0,1) & (1,0,0) & (1,1,1) & (1,1,0) \\ (0,1,0) & (0,1,1) & (0,0,0) & (0,0,1) & (1,1,0) & (1,1,1) & (1,0,0) & (1,0,1) \\ (0,1,1) & (0,1,0) & (0,0,1) & (0,0,0) & (1,1,1) & (1,1,0) & (1,0,1) & (1,0,0) \\ (1,0,0) & (1,0,1) & (1,1,0) & (1,1,1) & (0,0,0) & (0,0,1) & (0,1,0) & (0,1,1) \\ (1,0,1) & (1,0,0) & (1,1,1) & (1,1,0) & (0,0,1) & (0,0,0) & (0,1,1) & (0,1,0) \\ (1,1,0) & (1,1,1) & (1,0,0) & (1,0,1) & (0,1,0) & (0,1,1) & (0,0,0) & (0,0,1) \\ (1,1,1) & (1,1,0) & (1,0,1) & (1,0,0) & (0,1,1) & (0,1,0) & (0,0,1) & (0,0,0) \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 1 & 0 & 1 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 1 & 0 & 1 & 0 \end{bmatrix}.$$

$$f_{(1,1,0)} = (1,1,0)\cdot \begin{bmatrix} (0,0,0) & (0,0,1) & (0,1,0) & (0,1,1) & (1,0,0) & (1,0,1) & (1,1,0) & (1,1,1) \\ (0,0,1) & (0,0,0) & (0,1,1) & (0,1,0) & (1,0,1) & (1,0,0) & (1,1,1) & (1,1,0) \\ (0,1,0) & (0,1,1) & (0,0,0) & (0,0,1) & (1,1,0) & (1,1,1) & (1,0,0) & (1,0,1) \\ (0,1,1) & (0,1,0) & (0,0,1) & (0,0,0) & (1,1,1) & (1,1,0) & (1,0,1) & (1,0,0) \\ (1,0,0) & (1,0,1) & (1,1,0) & (1,1,1) & (0,0,0) & (0,0,1) & (0,1,0) & (0,1,1) \\ (1,0,1) & (1,0,0) & (1,1,1) & (1,1,0) & (0,0,1) & (0,0,0) & (0,1,1) & (0,1,0) \\ (1,1,0) & (1,1,1) & (1,0,0) & (1,0,1) & (0,1,0) & (0,1,1) & (0,0,0) & (0,0,1) \\ (1,1,1) & (1,1,0) & (1,0,1) & (1,0,0) & (0,1,1) & (0,1,0) & (0,0,1) & (0,0,0) \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 & 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 & 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 \end{bmatrix}.$$

$$f_{(1,1,1)} = (1,1,1)\cdot \begin{bmatrix} (0,0,0) & (0,0,1) & (0,1,0) & (0,1,1) & (1,0,0) & (1,0,1) & (1,1,0) & (1,1,1) \\ (0,0,1) & (0,0,0) & (0,1,1) & (0,1,0) & (1,0,1) & (1,0,0) & (1,1,1) & (1,1,0) \\ (0,1,0) & (0,1,1) & (0,0,0) & (0,0,1) & (1,1,0) & (1,1,1) & (1,0,0) & (1,0,1) \\ (0,1,1) & (0,1,0) & (0,0,1) & (0,0,0) & (1,1,1) & (1,1,0) & (1,0,1) & (1,0,0) \\ (1,0,0) & (1,0,1) & (1,1,0) & (1,1,1) & (0,0,0) & (0,0,1) & (0,1,0) & (0,1,1) \\ (1,0,1) & (1,0,0) & (1,1,1) & (1,1,0) & (0,0,1) & (0,0,0) & (0,1,1) & (0,1,0) \\ (1,1,0) & (1,1,1) & (1,0,0) & (1,0,1) & (0,1,0) & (0,1,1) & (0,0,0) & (0,0,1) \\ (1,1,1) & (1,1,0) & (1,0,1) & (1,0,0) & (0,1,1) & (0,1,0) & (0,0,1) & (0,0,0) \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 1 & 1 & 0 & 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 & 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 & 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 & 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 & 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 & 0 & 1 & 1 & 0 \end{bmatrix}.$$

The matrices in Example 11 are frequency squares of order 8. Since 8 is a prime power, by Theorem 1.2, there are at most 7 MOFS of order 8. We prove this below.

Lemma 2.3. *For any non-zero vector $a_k \in \mathbb{F}_2^3$, where $k \in \{1, 2, 3, \dots, 7\}$ the matrix M_k defined by*

$$M_k(u, v) = f_{a_k}(u \oplus v) = a_k \cdot (u \oplus v) \pmod 2$$

forms a square of type $F(8; 4, 4)$; that is, each row and column of M_k contains exactly four 0*s and four* 1*s.*

Proof. The function f_{a_k} outputs 0 or 1 depending on whether the number of overlapping 1s in a_k and $u \oplus v$ is even or odd. Since u and v range uniformly over $\mathbb{F}_2^3$, the value $u \oplus v$ also ranges uniformly over $\mathbb{F}_2^3$.

Since f_{a_k} is a non-zero linear functional over $\mathbb{F}_2^3$, and it is balanced. Then, it outputs 0 for exactly half of the inputs and 1 for the other half. That is, over the 8 possible values of $u \oplus v \in \mathbb{F}_2^3$, exactly 4 will satisfy $f_{a_k}(u \oplus v) = 1$ and 4 will satisfy $f_{a_k}(u \oplus v) = 0$.

Fixing a row (fixing u), and letting v vary over $\mathbb{F}_2^3$, the values of $u \oplus v$ range over all of $\mathbb{F}_2^3$. Hence, each row will contain four 1s and four 0s. Similarly, fixing a column (fixing v), and letting u vary over the values of $u \oplus v$ again range over all of $\mathbb{F}_2^3$.

Therefore, M_k is a matrix with each row and column containing exactly four 1s and four 0s, that is a frequency square of type $F(8; 4, 4)$. $\square$

Lemma 2.4. *For any two non-zero vectors* $a_k, a_l \in \mathbb{F}_2^3$ *and* $a_k \neq a_l$*, the functions* $f_{a_k}, f_{a_l} : \mathbb{F}_2^3 \to \mathbb{F}_2$ *defined by* $f_{a_k}(u \oplus v) = a_k \cdot (u \oplus v) \pmod 2$, $\forall u, v \in \mathbb{F}_2^3$ *are linearly independent.*

Proof. Suppose the contradiction that the functions f_{a_k}, f_{a_l} are linearly dependent. Then, there exists scalars $\lambda_1, \lambda_2 \in \mathbb{F}_2$, not both are zeros, such that:

$$\lambda_1 f_{a_k}(u \oplus v) + \lambda_2 f_{a_l}(u \oplus v) = 0, \forall u, v \in \mathbb{F}_2^3.$$

$$\Rightarrow \lambda_1 a_k \cdot (u \oplus v) + \lambda_2 a_l \cdot (u \oplus v) = 0, \forall u, v \in \mathbb{F}_2^3,$$

$$\Rightarrow (\lambda_1 a_k + \lambda_2 a_l) \cdot (u \oplus v) = 0, \forall u, v \in \mathbb{F}_2^3,$$

which is true if and only if:

$$\lambda_1 a_k + \lambda_2 a_l = 0.$$

Since $u \oplus v$ ranges over all $\mathbb{F}_2^3$, as u, v range over $\mathbb{F}_2^3$, we have

$$\lambda_1 a_k = -\lambda_2 a_l.$$

$$\Rightarrow a_k = \frac{-\lambda_2}{\lambda_1} a_l \quad \text{or} \quad a_l = \frac{-\lambda_1}{\lambda_2} a_k.$$

case 1: $\lambda_1 = \lambda_2 = 1 \Rightarrow a_k = -a_l$.

$\Rightarrow a_k = a_l$, by the definition of a_k and a_l.

case 2: $\lambda_1 = 0$ and $\lambda_2 = 1$ or $\lambda_1 = 1$ and $\lambda_2 = 0$.

This is not possible since the division by 0 is undefined, and a_k and a_l are non-zero vectors. Hence, this occurs only when $\lambda_1 = \lambda_2 = 0$, contradicting our assumption that not both λ_1 and λ_2 are zeros.

Therefore, the functions f_{a_k}, f_{a_l} are linearly independent. $\square$

Definition 11. *Let* $A = \begin{bmatrix} a_{ij} \end{bmatrix}$ *and* $B = \begin{bmatrix} a_{ij} \end{bmatrix}$ *be two matrices of the same dimensions* $m \times n$*, where* $a_{ij}, b_{ij} \in \mathbb{R}$*. The elementwise product (or Hadamard product) of* A *and* B *is the matrix defined by*

$$A \circ B = \sum_{i=1}^{2m} \sum_{j=1}^{2m} a_{ij} \cdot b_{ij},$$

for all $1 \leq i \leq m$ *and* $1 \leq j \leq n$ [8].

Lemma 2.5. *Let* A *and* B *be two frequency squares of type* $F(2m; m, m)$ *with symbol set {0,1}. Suppose, the Hadamard product* $A \circ B$ *satisfies:*

$$A \circ B = m^2$$

Then, A *and* B *are orthogonal frequency squares; that is the ordered pairs* $(A(i,j), B(i,j)) \in \{(0,0), (0,1), (1,0), (1,1)\}$ *occur exactly* m^2 *times each in the superimposed square.*

Proof. Since A and B are frequency squares of type $F(2m; m, m)$, each row and column contains exactly m number of 0s and $m$number of 1s.

Hence, total number of 1s in each matrix is

$$m \cdot 2m = 2m^2.$$

Now, consider the four possible combinations in the superimposed matrix $(A(i,j), B(i,j))$. Let:

- $N(0,0)$ - number of positions where $A(i,j) = 0$ and $B(i,j) = 0$.
- $N(0,1)$ - number of positions where $A(i,j) = 0$ and $B(i,j) = 1$.

- $N(1,0)$ - number of positions where $A(i,j) = 1$ and $B(i,j) = 0$.
- $N(1,1)$ - number of positions where $A(i,j) = 1$ and $B(i,j) = 1$.

The total number of positions in the superimposed square is

$$2m \cdot 2m = 4m^2.$$

We have $A(i,j) \cdot B(i,j) = 1$ when $A(i,j) = 1$ and $B(i,j) = 1$, and given that

$$A \circ B = m^2.$$

$$\Rightarrow N(1,1) = m^2.$$

Moreover, A has $2m^2$ 1s and $N(1,1)$ overlap with 1s in B m^2 number of times, the remaining $2m^2 - m^2 = m^2$ positions where $A(i,j) = 1$ must have $B(i,j) = 0$.

$$\Rightarrow N(1,0) = m^2.$$

Similarly, B has $2m^2$ 1s. So:

$$\Rightarrow N(0,1) = 2m^2 - N(1,1) = m^2.$$

Finally,

$$N(0,0 = 4m^2 - (N(1,1) + N(0,1) + N(1,0))$$

$$= 4m^2 - 3m^2 = m^2.$$

Hence, all four ordered pairs $(0,0), (0,1), (1,0), (1,1)$ occurs exactly m^2 times in the superimposed matrix of A and B.

Therefore, A and B are orthogonal frequency squares.

□

Theorem 2.6. *Let $M_1, M_2, \ldots, M_7$ be the seven frequency squares in $\mathbb{F}_2^3 \times \mathbb{F}_2^3$ over the symbols $\{0, 1\}$ defined by the non-zero vector $a_k \in \mathbb{F}_2^3$ for each $u, v \in \mathbb{F}_2^3$.*

Then, the seven frequency squares $M_1, M_2, \ldots, M_7$ are mutually orthogonal.

Proof. Since each M_k , where $k \in \{1, 2, \ldots, 7\}$ is defined by a linear function over $\mathbb{F}_2^3 \times \mathbb{F}_2^3$, the number of $0s$ and $1s$ is exactly 4 in each row and column by Lemma 2.3.

Hence, each M_k is a frequency square of type $F(8; 4, 4)$.

Now for $k, l \in \{1, 2, \ldots, 7\}$ and $k \neq l$, consider the Hadamard product $M_k \circ M_l$. Each entry in this product is equal to 1 if and only if $M_k = M_l = 1$.

Thus, the sum of all entries in $M_k \circ M_l$ counts the number of positions where both matrices have a 1. M_k and M_l are matrices with $2m^2 = 32$ ones in total (for $m = 4$), and each function f_{a_k}, defined on $\mathbb{F}_2^3 \times \mathbb{F}_2^3$, are linearly independent functions by Lemma 2.4.

Then, the number of positions where both M_k and M_l can have a 1 can be calculated by using the multiplication rule of calculating the probability of intersection of two independent events, which gives:

$$\frac{(\text{number of ones in} \quad M_k) \times (\text{number of ones in} \quad M_l)}{\text{total number of entries}}$$

$$= \frac{32 \cdot 32}{64}$$

$$= 16$$

$$= m^2, \text{ since m=4.}$$

Therefore, by Lemma 2.5, any pair of M_k, where $k \in \{1, 2, \ldots, 7\}$ are orthogonal. i.e., $M_1, M_2, \ldots, M_7$ are mutually orthogonal. $\square$

3 MOFS via Permutation Arrays

In this chapter, we discuss permutation arrays (PA) and equidistant permutation arrays (EPA), and listed some examples from [5] in the first section. We then demonstrate how MOFS can be constructed using EPA.

3.1 Permutation Arrays

Definition 12. *Let X be a nonempty set with cardinality n. A permutation array of length n and minimum distance d, denoted by* $\mathrm{PA}(n, d)$, is an $m \times n$ array with the following properties:

- Each row is a permutation of X.
- For any two distinct rows, the number of positions in which they differ is at least d, and d is called minimum Hamming distance.

The integer m, the number of rows in the array, is called the size of the permutation array. The maximum possible size of a $\mathrm{PA}(n, d)$ is denoted by $M(n, d)$.

Example 12. *Consider the set $X = \{1, 2, 3, 4, 5\}$. A* $\mathrm{PA}(5, 3)$ formed out of X is as follows:

$$\begin{bmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 1 & 4 & 5 & 3 \\ 4 & 5 & 2 & 3 & 1 \\ 3 & 4 & 5 & 1 & 2 \\ 5 & 3 & 1 & 2 & 4 \\ 2 & 3 & 4 & 1 & 5 \end{bmatrix},$$

which has the following properties:

- Each row is a permutation of $\{1, 2, 3, 4, 5\}$.

- The Hamming distance between any two rows is at least 3. For instance, R_1 and R_2 differ in 4 positions, while R_1 and R_3 differ in 5 positions, and so on. The minimum distance is between R_2 and R_6, which is 3.

Hence, it is a valid PA$(5,3)$ with size 6 and minimum distance is 3.

Remark 3.1.

- *The maximum size of* PA(n,d) satisfies the inequality[5]:

$$M(n,d) \geq max\{M(n-1,d), M(n+1,d)\}.$$

- If all $n!$ permutations of X are taken as rows, then:

$$M(n,1) = M(n,2) = n!.$$

- From a Latin square of order n, one can obtain a PA(n,n), such that:

$$M(n,n) = n.$$

Example 13. *Consider the set* $X = \{1,2,3,4\}$. *A* PA$(4,4)$ formed out of X is as follows:

$$\begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 4 & 3 \\ 3 & 4 & 1 & 2 \\ 4 & 3 & 2 & 1 \end{bmatrix}.$$

Each row is a permutation of {1,2,3,4}, and the Hamming distance between any two rows is 4.

3.1.1 Equidistant Permutation Arrays

Definition 13. *A* $\mathrm{PA}(n, d)$ in which every pair of distinct rows differs in exactly d positions is called an equidistant permutation array (EPA). The maximum size of $\mathrm{EPA}(n, d)$ is denoted by $M'(n, d)$.

Example 14. *Consider the permutation array on the symbol set* $\{1, 2, 3, 4, 5, 6, 7\}$ *as follows:*

$$\begin{bmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 1 & 3 & 2 & 5 & 4 & 7 & 6 \\ 1 & 6 & 7 & 3 & 2 & 4 & 5 \\ 7 & 5 & 1 & 4 & 2 & 3 & 6 \end{bmatrix}.$$

Here $n = 7$ *and* $m = 4$*. Any two distinct rows agree in exactly one position. For example,* R_1 *and* R_2 *agree in column* 1*,* R_1 *and* R_4 *agree at column* 4*. Thus, every pair of rows differs in exactly* $d = 6$ *positions.*

Therefore, it is an $\mathrm{EPA}(7, 6)$.

Example 15. [5] Consider an EPA$(7,6)$ with $M'(7,6) = 13$.

$$\begin{bmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 1 & 3 & 2 & 5 & 4 & 7 & 6 \\ 1 & 6 & 7 & 3 & 2 & 4 & 5 \\ 1 & 7 & 5 & 2 & 6 & 3 & 4 \\ 2 & 6 & 5 & 4 & 3 & 7 & 1 \\ 7 & 1 & 5 & 3 & 4 & 6 & 2 \\ 7 & 3 & 6 & 2 & 5 & 4 & 1 \\ 3 & 2 & 5 & 1 & 7 & 4 & 6 \\ 3 & 6 & 1 & 2 & 4 & 5 & 7 \\ 3 & 7 & 4 & 5 & 2 & 6 & 1 \\ 4 & 3 & 5 & 6 & 2 & 1 & 7 \\ 5 & 2 & 7 & 6 & 4 & 3 & 1 \\ 7 & 5 & 1 & 4 & 2 & 3 & 6 \end{bmatrix}.$$

3.2 Construction of MOFS using EPA

For an $n \times n$ array A, the entry in row i column j can be represented as $(A)_{i,j}$. Then, we use the notation

$$A = [(A)_{i,j}].$$

Given a permutation R, we can construct a permutation array $n \times n$ $P(R) = [p_{i,j}]$ as follows.

$$p_{i,j} = \begin{cases} 1, & \text{if } R(j) = i \\ 0, & \text{otherwise} \end{cases}$$

Each row and column of $P(R)$ contains exactly one entry equal to 1 and ($n-1$ zeros). Hence,

$P(R)$ is a frequency square of type $F(n; n-1, 1)$.

In turn, if $A = \{R_1, R_2, R_3, \ldots, R_m\}$ is an EPA$(n, n-1)$, then the frequency squares $\{P(R_1), P(R_2), \ldots, P(R_n)\}$ are a set of MOFS of type $F(n; n-1, 1)$.

This process is reversible, as demonstrated in the following theorem, which constructs an EPA$(n, n-1)$ from a given set of MOFS of type $F(n; n-1, 1)$.

Theorem 3.2. *Let* $\{F_1, F_2, \ldots, F_m\}$ *be a set of MOFS of type* $F(n; n-1, 1)$ [5]. For each $t \in \{1, 2, 3, \ldots, m\}$ and each row $i \in [n]$, there is a unique column j with $(F_t)_{i,j} = 1$.

Define a map $R_t : [n] \to [n]$ by

$$R_t(i) = j \quad \text{iff} \quad (F_t)_{i,j} = 1.$$

Then,

For every t, R_t is a permutation of $[n]$, and for every $r \neq s$, the permutations R_s and R_s agree in exactly one coordinate. Consequently, $\{R_1, R_2, \ldots .R_m\}$ is an EPA$(n, n-1)$.

Proof. The definition of $R_t : [n] \to [n]$ as $R_t(i) = j \quad \text{iff} \quad (F_t)_{i,j} = 1$ gives a bijection from rows to columns. Hence, R_t is a permutation of $[n]$.

Consider R_r and R_s for $r \neq s$. Since F_r and F_s are orthogonal, the ordered pair $(1, 1)$ which is $((F_r)_{i,j}, (F_s)_{i,j})$ occurs exactly once in the array at the position of (i, j). By the definition of R_r and R_s, this corresponds to the unique row i where $R_t(i) = R_s(i)$. Hence, R_t and R_s agree in exactly one column.

Therefore R_t, where $t \in \{1, 2, , 3, \ldots, m\}$, is a permutation of $[n]$ in which any two rows agrees in exactly one column. Consequently, $\{R_1, R_2, \ldots .R_m\}$ is an EPA$(n, n-1)$. ☐

Example 16. *Consider the* $EPA(7, 6)$ *in Example* 14.

$$R_1 = (1, 2, 3, 4, 5, 6, 7).$$

$$R_2 = (1,3,2,5,4,7,6).$$

$$R_3 = (1,6,7,3,2,4,5).$$

$$R_4 = (7,5,1,4,2,3,6).$$

We can construct P_1, P_2, P_3 and P_4 by the definition of $[p_{ij}]$.

$$P_1 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} P_2 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

$$P_3 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix} P_4 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}.$$

Each P_t, where $t = 1,2,3,4$ is a frequency square of type $F(7;6,1)$. To verify the orthogonality, we count the number of ordered airs by superimposing any two permutation arrays P_t and P_s, where $t, s \in \{1,2,3,4\}$.

Since $\{R_1, R_2, R_3, R_4\}$ forms an EPA$(7,6)$, each R_t coincides in exactly one column with R_s. Thus, when superimposing the corresponding permutation arrays P_t and P_s, we get:

- Exactly 1 ordered pair $(1,1)$ in the column where they coincides each other.
- 6 remaining 0's of P_t with single 1 of P_s produce 6 $(0,1)$ pairs, and the single 1 of P_t with 6 0's of P_s produce 6 $(1,0)$ pairs.
- 6 0's of P_t and 6 0's of P_s produce 36 $(0,0)$ pairs.

All these together forms the 49 pairs of 7×7 arrays P_s and P_t. Hence, P_s and P_t are orthogonal in the frequency sense. This result is true for any two pairs of P_s and P_t. Therefore, $\{P_1, P_2, P_3, P_4\}$ is a set of MOFS of order 7.

3.2.1 Lower and Upper Bound for $M^{'}(n, n-1)$

In an EPA$(n, n-1)$, any two rows differ in exactly $n-1$ columns. Hence, we can construct $M^{'}(n, n-1)$ MOFS of type $F(n; n-1, 1)$ as the construction in the Example 16. Let $t = M^{'}(n, n-1)$, and each of these frequency squares has 2 symbols in each row and column. By Theorem 1.8, t satisfies the inequality:

$$2t - t \leq (n-1)^2$$

$$\Leftrightarrow t \leq (n-1)^2$$

$$\Rightarrow M^{'}(n, n-1) \leq (n-1)^2,$$

which is an upper bound for $M^{'}(n, n-1)$.

We have some known exact values of the lower bounds for $M^{'}(n, n-1)$ [5]. The known exact values of the lower and corresponding upper bounds for $M^{'}(n, n-1)$ are listed in the following table.

n	Lower bound of $M^{'}(n,n-1)$	Upper bound of $M^{'}(n,n-1)$
4	5	9
5	10	16
6	13	25
7	16	36

Table 6: Lower and Upper bounds for $M^{'}(n,n-1)$

.

For $n \geq 8$, the exact values of lower bound of $M^{'}(n,n-1)$ are not known. The values listed in Table 6 correspond to known constructions, and these constructions are not complete sets.

4 Construction of MMOFS via Mappings

In this chapter, we define mappings from Latin squares, and establish the orthogonality of the resultant frequency squares by using the lemmas presented in the first section. Then, we demonstrate how mixed orthogonal arrays can be utilized to construct MMOFS of higher order by applying these lemmas.

4.1 Mappings and Orthogonality

Definition 14. *Let L be a Latin square of order n, and let f be a function defined on the set of symbols of L. We define an array $M = M(i,j)$ as follows.*

$$M(i,j) = f(L(i,j)), \quad \textit{for all} \quad 1 \leq i,j \leq n,$$

where M is the image of L under the function f, denoted by $f(L)$.

Lemma 4.1. *Let L_1 and L_2 be orthogonal Latin squares of order n, and let f_1 and f_2 be any functions defined on the symbol sets of L_1 and L_2, respectively. Let $M_1 = f_1(L_1)$ and $M_2 = f_2(L_2)$. Then M_1 and M_2 are orthogonal.*

Proof. Let S be the symbol set of L_1 and T be the symbol set of L_2. Consider the functions $f_1 : S \to Im(f_1)$ and $f_2 : T \to Im(f_2)$.

For every symbol $y \in M_1$ and $z \in M_2$, let us define $\lambda_1(y) = \left|f_1^{-1}(y)\right|$ and $\lambda_2(z) = \left|f_2^{-1}(z)\right|$. Here, $\lambda_1(y)$ gives the number of times y occurs in each row and column of M_1, and $\lambda_2(z)$ gives the number of times y occurs in each row and column of M_2.

Now we want to count the ordered pairs in $[(M_1, M_2)]$. This occurs precisely when there exist symbols $s \in S$ and $t \in T$ such that $f_1(s) = y$ and $f_2(t) = z$, and the pair $(s,t) \in S \times T$ appears in the original superimposed array $[(L_1, L_2)]$. Since L_1 and L_2 are orthogonal Latin squares, each pair pair $(s,t) \in S \times T$ appears exactly once in $[(L_1, L_2)]$.

Hence, the number of times that the pair (y, z) appears in M is exactly the number of pairs $(s, t) \in S \times T$ such that $f_1(s) = y$ and $f_2(t) = z$, which is given by:

$$|\{s \in S \mid f_1(s) = y\}| \cdot |\{t \in T \mid f_2(s) = z\}|$$

$$\Rightarrow \lambda_1(y) \cdot \lambda_2(z).$$

Hence by Definition 4, M_1 and M_2 are orthogonal. $\square$

Remark 4.2. *If the functions defined above are bijective, the case of orthogonality is trivial because the mapping results in a perfect reshuffling of the elements of the Latin squares. However, if the functions are not bijective, orthogonality is still preserved in a frequency sense by transforming Latin squares into frequency squares.*

Example 17. *Consider the following orthogonal Latin squares as follows.*

$$L_1 = \begin{array}{|c|c|c|c|} \hline 1 & 2 & 3 & 4 \\ \hline 2 & 3 & 4 & 1 \\ \hline 3 & 4 & 1 & 2 \\ \hline 4 & 1 & 2 & 3 \\ \hline \end{array} \quad L_2 = \begin{array}{|c|c|c|c|} \hline 1 & 3 & 4 & 2 \\ \hline 2 & 4 & 1 & 3 \\ \hline 3 & 1 & 2 & 4 \\ \hline 4 & 2 & 3 & 1 \\ \hline \end{array}.$$

Let us define the functions $f_1, f_2 : \{1, 2, 3, 4\} \to \{1, 2\}$ *as*

$$f_1(1) = 1, f_1(2) = 1, f_1(3) = 2, f_1(4) = 2.$$

$$f_2(1) = 1, f_2(2) = 2, f_2(3) = 1, f_1(4) = 2.$$

Then,

$$f_1(L_1) = \begin{array}{|c|c|c|c|} \hline 1 & 1 & 2 & 2 \\ \hline 1 & 2 & 2 & 1 \\ \hline 2 & 2 & 1 & 1 \\ \hline 2 & 1 & 1 & 2 \\ \hline \end{array} \quad f_2(L_2) = \begin{array}{|c|c|c|c|} \hline 1 & 1 & 2 & 2 \\ \hline 2 & 2 & 1 & 1 \\ \hline 1 & 1 & 2 & 2 \\ \hline 2 & 2 & 1 & 1 \\ \hline \end{array}.$$

Then,

$$[(f_1(L_1), f_2(L_2))] = \begin{array}{|c|c|c|c|}\hline (1,1) & (1,1) & (2,2) & (2,2) \\ \hline (1,2) & (2,2) & (2,1) & (1,1) \\ \hline (2,1) & (2,1) & (1,2) & (1,2) \\ \hline (2,2) & (1,2) & (1,1) & (2,1) \\ \hline \end{array}\ .$$

Clearly, orthogonality is preserved in the frequency sense as described. The $f_1(L_1)$ and $f_2(L_2)$ are two orthogonal frequency squares. Here, the resultant frequency squares are regular.

Now, let us define two functions in L_1 and L_2 that result in frequency squares that are not regular.

Example 18. *Let us define the functions* $f_1, f_2 : \{1,2,3,4\} \to \{1,2\}$ *for the same* L_1 *and* L_2 *as*

$$f_1(1) = 1, f_1(2) = 2, f_1(3) = 2, f_1(4) = 2.$$

$$f_2(1) = 1, f_2(2) = 2, f_2(3) = 1, f_1(4) = 2.$$

Then,

$$f_1(L_1) = \begin{array}{|c|c|c|c|}\hline 1 & 1 & 1 & 2 \\ \hline 1 & 1 & 2 & 1 \\ \hline 1 & 2 & 1 & 1 \\ \hline 2 & 1 & 1 & 1 \\ \hline \end{array} \quad f_2(L_2) = \begin{array}{|c|c|c|c|}\hline 1 & 1 & 2 & 2 \\ \hline 2 & 2 & 1 & 1 \\ \hline 1 & 1 & 2 & 2 \\ \hline 2 & 2 & 1 & 1 \\ \hline \end{array}\ .$$

Then,

$$[(f_1(L_1), f_2(L_2))] = \begin{array}{|c|c|c|c|}\hline (1,1) & (1,1) & (1,2) & (2,2) \\ \hline (1,2) & (1,2) & (2,1) & (1,1) \\ \hline (1,1) & (2,1) & (1,2) & (1,2) \\ \hline (2,2) & (1,2) & (1,1) & (1,1) \\ \hline \end{array} .$$

Again, the orthogonality is preserved in the frequency sense, but this time $f_1(L_1)$ *and* $f_2(L_2)$ *are orthogonal mixed frequency squares.*

We next explore applying functions to the same Latin square to form mutually orthogonal frequency squares. We give an example in the following.

Example 19. *Consider the following Latin square.*

$$L = \begin{array}{|c|c|c|c|}\hline 1 & 2 & 3 & 4 \\ \hline 2 & 3 & 4 & 1 \\ \hline 3 & 4 & 1 & 2 \\ \hline 4 & 1 & 2 & 3 \\ \hline \end{array} .$$

Let us define two functions $f_1 : \{1, 2, 3, 4\} \rightarrow \{1, 2\}$ *as*

$$f_1(1) = 1, f_1(2) = 1, f_1(3) = 2, f_1(4) = 2,$$

and $f_2 : \{1, 2, 3, 4\} \rightarrow \{3, 4\}$ *as*

$$f_2(1) = 3, f_2(2) = 4, f_2(3) = 3, f_2(4) = 4.$$

Then the image squares are given by,

$$f_1(L) = \begin{array}{|c|c|c|c|}\hline 1 & 1 & 2 & 2 \\ \hline 1 & 2 & 2 & 1 \\ \hline 2 & 2 & 1 & 1 \\ \hline 2 & 1 & 1 & 2 \\ \hline \end{array} \quad f_2(L) = \begin{array}{|c|c|c|c|}\hline 3 & 4 & 3 & 4 \\ \hline 4 & 3 & 4 & 3 \\ \hline 3 & 4 & 3 & 4 \\ \hline 4 & 3 & 4 & 3 \\ \hline \end{array}.$$

Then,

$$[(f_1(L), f_2(L))] = \begin{array}{|c|c|c|c|}\hline (1,3) & (1,4) & (2,3) & (2,4) \\ \hline (1,4) & (2,3) & (2,4) & (1,3) \\ \hline (2,3) & (2,4) & (1,3) & (1,4) \\ \hline (2,4) & (1,3) & (1,4) & (2,3) \\ \hline \end{array}.$$

Each ordered pair appears exactly four times in the array, which confirms the orthogonality of the frequency squares $f_1(L)$ and $f_2(L)$.

We generalize this idea in the next subsection using Hadamard matrices.

4.2 MMOFS from Hadamard matrices

Lemma 4.3. *Let L_1 and L_2 be two orthogonal Latin squares of order $4n$ on a symbol set S, and let $f, g : S \Rightarrow \{0, 1\}$ such that*

$$A = \{s \in S : f(s) = 0, g(s) = 0\},$$

$$B = \{s \in S : f(s) = 0, g(s) = 1\},$$

$$C = \{s \in S : f(s) = 1, g(s) = 0\},$$

$$D = \{s \in S : f(s) = 1, g(s) = 1\},$$

with $|A| = |B| = |C| = |D| = n$.

Define $F = f(L_2)$ *and* $G = g(L_2)$ *by replacing each symbol* s *in* L_2. *Then,* F *and* G *are frequency squares of type* $F(4n; 2n, 2n)$, *and* $\{L_1, F, G\}$ *is a set of MMOFS.*

Proof. Since L_2 is a Latin square, each row r of L_2 contains every symbol of S exactly once. The number of symbols mapped to 0 by f is $|A| + |B| = n + n = 2n$, and by g is $|A| + |C| = n + n = 2n$. Hence, in each row r of F and G, 0 appears exactly $2n$ times. Similarly, 1 appears $2n$ times in each row of F and G. The same holds for each column of F and G by the property of Latin squares. Therefore, F and G are squares of type $F(4n; 2n, 2n)$.

Now, the ordered pairs of $[(F, G)]$ result in one of $(0, 0)$, $(0, 1)$, $(1, 0)$, $(1, 1)$, corresponds to the symbols A, B, C and D. Since $|A| = |B| = |C| = |D| = n$, and each row r of L_2 is a permutation of S, the number of entries in row r whose symbol lie in A, B, C, or D is n. Consequently, the row r of $[(F, G)]$ contains the pairs $(0, 0), (0, 1), (1, 0), (1, 1)$, exactly n times.

Similar result hold for the columns of the superimposed set. Hence, F and G are orthogonal. Moreover, by Lemma 4.1, L_1 is orthogonal to both F and G. Therefore, $\{L_1, F, G\}$ is a set of MMOFS. □

Theorem 4.4. *Let* L_1 *and* L_2 *are two orthogonal Latin squares of order* $4n$. *If there exists a Hadamard matrix of order* $4n$, *then there are* $4n - 1$ *frequency squares* $F_1, F_2, \ldots, F_{4n-1}$, *each of type* $F(4n; 2n, 2n)$ *such that* $\{L_1, F_1, ...F_{4n-1}\}$ *is a set of MMOFS.*

Proof. Take a row r except the first row of a Hadamard matrix H of order $4n$, where H is in the standard form. Define $f_r(c) = 1$ for row r and column c of H as follows.

$$f_r(c) = \begin{cases} 1, & \text{if and only if there is a } 1 \text{ in row } r \text{ and column } c \text{ of } H \\ 0, & \text{otherwise.} \end{cases}$$

Since each non-first row of H has exactly $2n$ entries equal to 1 and $2n$ entries equal to -1, we have $|{f_r}^{-1}(1)| = 2n$ and $|{f_r}^{-1}(0)| = 2n$.

Define the square $F_r = f_r(L_2)$ by replacing each symbol$s$ in L_2 with $f_r(s)$. As each row and column of L_2 is a permutation of the set of symbols, each row and column of F_r contains exactly $2n$ zeros and $2n$ ones. Hence, F_r is a frequency square of type $F(4n; 2n, 2n)$. Since there are $4n-1$ rows in L_2 except the first row, we get $4n-1$ frequency squares of type $F(4n; 2n, 2n)$. By lemma 4.3, all these frequency squares are mutually orthogonal, and by Lemma 4.1, L_1 is orthogonal to each of the frequency squares $F_1, F_2, \ldots, F_{4n-1}$ of type $F(4n; 2n, 2n)$.

Therefore, $\{L_1, F_1, F_2, \ldots, F_{4n-1}\}$ is a set of MMOFS. $\square$

Corollary 4.5. *There is a complete set of MMOFS of the form*

$$\{6 \times F(8; 1, 1, 1, 1, 1, 1, 1, 1); 7 \times F(8; 4, 4)\}.$$

Proof. Consider the 7 Latin squares of order 8 as follows.

$L_1 =$

0	1	2	3	4	5	6	7
2	3	0	1	6	7	4	5
3	2	1	0	7	6	5	4
4	5	6	7	0	1	2	3
5	4	7	6	1	0	3	2
6	7	4	5	2	3	0	1
7	6	5	4	3	2	1	0
1	0	3	2	5	4	7	6

$L_2 =$

0	1	2	3	4	5	6	7
3	2	1	0	7	6	5	4
4	5	6	7	0	1	2	3
5	4	7	6	1	0	3	2
6	7	4	5	2	3	0	1
7	6	5	4	3	2	1	0
1	0	3	2	5	4	7	6
2	3	0	1	6	7	4	5

$$L_3 = \begin{array}{|c|c|c|c|c|c|c|c|}\hline 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ \hline 4 & 5 & 6 & 7 & 0 & 1 & 2 & 3 \\ \hline 5 & 4 & 7 & 6 & 1 & 0 & 3 & 2 \\ \hline 6 & 7 & 4 & 5 & 2 & 3 & 0 & 1 \\ \hline 7 & 6 & 5 & 4 & 3 & 2 & 1 & 0 \\ \hline 1 & 0 & 3 & 2 & 5 & 4 & 7 & 6 \\ \hline 2 & 3 & 0 & 1 & 6 & 7 & 4 & 5 \\ \hline 3 & 2 & 1 & 0 & 7 & 6 & 5 & 4 \\ \hline \end{array} \qquad L_4 = \begin{array}{|c|c|c|c|c|c|c|c|}\hline 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ \hline 5 & 4 & 7 & 6 & 1 & 0 & 3 & 2 \\ \hline 6 & 7 & 4 & 5 & 2 & 3 & 0 & 1 \\ \hline 7 & 6 & 5 & 4 & 3 & 2 & 1 & 0 \\ \hline 1 & 0 & 3 & 2 & 5 & 4 & 7 & 6 \\ \hline 2 & 3 & 0 & 1 & 6 & 7 & 4 & 5 \\ \hline 3 & 2 & 1 & 0 & 7 & 6 & 5 & 4 \\ \hline 4 & 5 & 6 & 7 & 0 & 1 & 2 & 3 \\ \hline \end{array}$$

$$L_5 = \begin{array}{|c|c|c|c|c|c|c|c|}\hline 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ \hline 6 & 7 & 4 & 5 & 2 & 3 & 0 & 1 \\ \hline 7 & 6 & 5 & 4 & 3 & 2 & 1 & 0 \\ \hline 1 & 0 & 3 & 2 & 5 & 4 & 7 & 6 \\ \hline 2 & 3 & 0 & 1 & 6 & 7 & 4 & 5 \\ \hline 3 & 2 & 1 & 0 & 7 & 6 & 5 & 4 \\ \hline 4 & 5 & 6 & 7 & 0 & 1 & 2 & 3 \\ \hline 5 & 4 & 7 & 6 & 1 & 0 & 3 & 2 \\ \hline \end{array} \qquad L_6 = \begin{array}{|c|c|c|c|c|c|c|c|}\hline 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ \hline 7 & 6 & 5 & 4 & 3 & 2 & 1 & 0 \\ \hline 1 & 0 & 3 & 2 & 5 & 4 & 7 & 6 \\ \hline 2 & 3 & 0 & 1 & 6 & 7 & 4 & 5 \\ \hline 3 & 2 & 1 & 0 & 7 & 6 & 5 & 4 \\ \hline 4 & 5 & 6 & 7 & 0 & 1 & 2 & 3 \\ \hline 5 & 4 & 7 & 6 & 1 & 0 & 3 & 2 \\ \hline 6 & 7 & 4 & 5 & 2 & 3 & 0 & 1 \\ \hline \end{array}$$

$$L_7 = \begin{array}{|c|c|c|c|c|c|c|c|} \hline 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ \hline 1 & 0 & 3 & 2 & 5 & 4 & 7 & 6 \\ \hline 2 & 3 & 0 & 1 & 6 & 7 & 4 & 5 \\ \hline 3 & 2 & 1 & 0 & 7 & 6 & 5 & 4 \\ \hline 4 & 5 & 6 & 7 & 0 & 1 & 2 & 3 \\ \hline 5 & 4 & 7 & 6 & 1 & 0 & 3 & 2 \\ \hline 6 & 7 & 4 & 5 & 2 & 3 & 0 & 1 \\ \hline 7 & 6 & 5 & 4 & 3 & 2 & 1 & 0 \\ \hline \end{array}.$$

By Theorem 3.41 in [5], this is a complete set of MOLS of order 8.

Let $S = \{0,1,2,3,4,5,6,7\}$, and define functions $f_1, f_2, f_3, \ldots, f_7 : S \Rightarrow \{0,1\}$ on the Latin square L_7 as follows:

$$f_1(0) = f_1(1) = f_1(2) = f_1(3) = 0,$$

$$f_1(4) = f_1(5) = f_1(6) = f_1(7) = 1.$$

Then,

$$F_1 = f_1(L_7) = \begin{array}{|c|c|c|c|c|c|c|c|} \hline 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ \hline 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ \hline 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ \hline 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ \hline 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ \hline 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ \hline 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ \hline 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ \hline \end{array}.$$

$$f_2(0) = f_2(2) = f_2(4) = f_2(6) = 0,$$

$$f_2(1) = f_2(3) = f_2(5) = f_2(7) = 1.$$

Then,

$$F_2 = f_2(L_7) = \begin{array}{|c|c|c|c|c|c|c|c|} \hline 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \\ \hline 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \\ \hline 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \\ \hline 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \\ \hline 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \\ \hline 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \\ \hline 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \\ \hline 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \\ \hline \end{array}.$$

$$f_3(0) = f_3(1) = f_3(4) = f_3(5) = 0,$$

$$f_3(2) = f_3(3) = f_3(6) = f_3(7) = 1.$$

Then,

$$F_3 = f_3(L_7) = \begin{array}{|c|c|c|c|c|c|c|c|}\hline 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 \\ \hline 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 \\ \hline 1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 \\ \hline 1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 \\ \hline 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 \\ \hline 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 \\ \hline 1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 \\ \hline 1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 \\ \hline \end{array}\,.$$

$$f_4(0) = f_4(3) = f_4(4) = f_4(7) = 0,$$

$$f_4(1) = f_4(2) = f_4(5) = f_4(6) = 1.$$

Then,

$$F_4 = f_4(L_7) = \begin{array}{|c|c|c|c|c|c|c|c|}\hline 0 & 1 & 1 & 0 & 0 & 1 & 1 & 0 \\ \hline 1 & 0 & 0 & 1 & 1 & 0 & 0 & 1 \\ \hline 1 & 0 & 0 & 1 & 1 & 0 & 0 & 1 \\ \hline 0 & 1 & 1 & 0 & 0 & 1 & 1 & 0 \\ \hline 0 & 1 & 1 & 0 & 0 & 1 & 1 & 0 \\ \hline 1 & 0 & 0 & 1 & 1 & 0 & 0 & 1 \\ \hline 1 & 0 & 0 & 1 & 1 & 0 & 0 & 1 \\ \hline 0 & 1 & 1 & 0 & 0 & 1 & 1 & 0 \\ \hline \end{array}\,.$$

$$f_5(0) = f_5(3) = f_5(5) = f_5(7) = 0,$$

$$f_5(1) = f_5(2) = f_5(4) = f_5(6) = 1.$$

Then,

$$F_5 = f_5(L_7) = \begin{array}{|c|c|c|c|c|c|c|c|}\hline 0 & 1 & 0 & 1 & 1 & 0 & 1 & 0 \\ \hline 1 & 0 & 1 & 0 & 0 & 1 & 0 & 1 \\ \hline 0 & 1 & 0 & 1 & 1 & 0 & 1 & 0 \\ \hline 1 & 0 & 1 & 0 & 0 & 1 & 0 & 1 \\ \hline 1 & 0 & 1 & 0 & 0 & 1 & 0 & 1 \\ \hline 0 & 1 & 0 & 1 & 1 & 0 & 1 & 0 \\ \hline 1 & 0 & 1 & 0 & 0 & 1 & 0 & 1 \\ \hline 0 & 1 & 0 & 1 & 1 & 0 & 1 & 0 \\ \hline \end{array} .$$

$$f_6(0) = f_6(1) = f_6(6) = f_6(7) = 0,$$

$$f_6(2) = f_6(3) = f_6(4) = f_6(5) = 1.$$

Then,

$$F_6 = f_6(L_7) = \begin{array}{|c|c|c|c|c|c|c|c|}\hline 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 \\ \hline 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 \\ \hline 1 & 1 & 0 & 0 & 0 & 0 & 1 & 1 \\ \hline 1 & 1 & 0 & 0 & 0 & 0 & 1 & 1 \\ \hline 1 & 1 & 0 & 0 & 0 & 0 & 1 & 1 \\ \hline 1 & 1 & 0 & 0 & 0 & 0 & 1 & 1 \\ \hline 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 \\ \hline 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 \\ \hline \end{array} .$$

$$f_7(0) = f_7(3) = f_7(5) = f_7(6) = 0,$$

$$f_7(1) = f_7(2) = f_7(4) = f_7(7) = 1.$$

Then,

$$F_7 = f_7(L_7) = \begin{array}{|c|c|c|c|c|c|c|c|}\hline 0 & 1 & 1 & 0 & 1 & 0 & 0 & 1 \\ \hline 1 & 0 & 0 & 1 & 0 & 1 & 1 & 0 \\ \hline 1 & 0 & 0 & 1 & 0 & 1 & 1 & 0 \\ \hline 0 & 1 & 1 & 0 & 1 & 0 & 0 & 1 \\ \hline 1 & 0 & 0 & 1 & 0 & 1 & 1 & 0 \\ \hline 0 & 1 & 1 & 0 & 1 & 0 & 0 & 1 \\ \hline 0 & 1 & 1 & 0 & 1 & 0 & 0 & 1 \\ \hline 1 & 0 & 0 & 1 & 0 & 1 & 1 & 0 \\ \hline \end{array} .$$

Hence by Theorem 4.4, $\{L_i, F_1, F_2, ..., F_7\}$ is a set of MMOFS, where $i = 1, 2, 3, \ldots, 6$. But, $\{L_1, L_2, L_3, L_4, L_5, L_6, L_7\}$ is a set of MOLS, which implies

$$\{L_1, L_2, L_3, L_4, L_5, L_6, F_1, F_2, F_3, F_4, F_5, F_6, F_7\}$$

is a set of MMOFS.

Therefore, $\{6 \times F(8; 1, 1, 1, 1, 1, 1, 1, 1); 7 \times F(8; 4, 4)\}$ is a set of MMOFS.

Now, the inequality from Theorem 1.12 implies:

$$8 \times 6 + 2 \times 7 - 13 \leq (8 - 1)^2$$

$$\Leftrightarrow 49 \leq 49,$$

Hence, $\{6 \times F(8;1,1,1,1,1,1,1,1); 7 \times F(8;4,4)\}$ is a complete set of MMOFS. $\square$

Remark 4.6. *We can define functions* $g_1, g_2, g_3, \ldots, g_7 : S \to \{0,1\}$ *on the Latin square* L_6 *to form another set of* 7 *frequency squares* $G_1, G_2, G_3, \ldots, G_7$ *as in Corollary* 4.5. Then by Theorem 4.4,

$$\{L_1, L_2, L_3, L_4, L_5, F_1, F_2, F_3, F_4, F_5, F_6, F_7, G_1, G_2, G_3, G_4, G_5, G_6, G_7\}$$

is a set of MMOFS.

Now, the inequality from Theorem 1.12 implies:

$$8 \times 5 + 2 \times 14 - 19 \leq (8-1)^2$$

$$\Leftrightarrow 49 \leq 49,$$

Hence,

$$\{5 \times F(8;1,1,1,1,1,1,1,1); 14 \times F(8;4,4)\}$$

is a complete set of MMOFS.

Similarly, we get 4 other sets of MMOFS such as:

$$\{4 \times F(8;1,1,1,1,1,1,1,1); 21 \times F(8;4,4)\},$$

$$\{3 \times F(8;1,1,1,1,1,1,1,1); 28 \times F(8;4,4)\},$$

$$\{2 \times F(8;1,1,1,1,1,1,1,1); 35 \times F(8;4,4)\},$$

$$\{1 \times F(8;1,1,1,1,1,1,1,1); 42 \times F(8;4,4)\},$$

each of which is a complete set of MMOFS.

4.3 Mixed Orthogonal Arrays

Definition 15. *A mixed orthogonal array or* $OA(n, x_1^{y_1}, x_2^{y_2}, \ldots\ldots, x_k^{y_k}, t)$ *is an array with* n *columns and the number of rows equal to* $y_1 + y_2 + \cdots + y_k$*, and satisfies the following conditions:*

- *Each row* y_i *contains* x_i *distinct symbols, where* $i \in \{1, 2, 3, \ldots, k\}$.
- *Each symbol of the row* y_i *has a frequency* n/x_i.
- *If any* t *distinct rows are chosen, then every* t*-tuple of symbols appears same number of times. The parameter* t *is called the* ***strength*** *of the array.*

Example 20. *Consider the orthogonal array* $OA(12, 2^4, 3^1, 2)$ [12].

$$\begin{array}{cccccccccccc}
0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 \\
0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \\
0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 & 1 & 0 & 0 & 1 \\
0 & 1 & 0 & 1 & 1 & 0 & 0 & 1 & 1 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 2 & 2 & 2 & 2
\end{array}$$

There are 12 columns. There are 4 rows with 2 symbols, each have frequency 6, and one row with 3 symbols, each have frequency 4. The strength of the array is 2.

A library of mixed orthogonal arrays with strength 2 up to 100 columns are maintained by N. J. A. Sloane in [18].

4.4 Higher Order MMOFS using MOLS

Theorem 4.7. *Let* L_1 *and* L_2 *be two Latin squares of order* n*, and let* $OA(n, x_1^{y_1}, x_2^{y_2}, ..., x_k^{y_k}, 2)$ *is a mixed orthogonal array with* n *columns,* $Y = y_1 + y_2 + \cdots + y_k$ *rows and strength* 2*. Then, there exists a set of MMOFS* $\{F_0, F_1, \ldots, F_Y\}$*, where* F_0 *is of type* $F(n; 1^n)$*,* $\{F_1, F_2, \ldots, F_{y_1}\}$ *each are of type* $F(n, (n/x_1)^{x_1})$*, the next* y_2 *squares are of type* $F(n; (n/x_2)^{x_2})$*, ..., the last* y_k *squares are of type* $F(n; (n/x_k)^{x_k})$*.*

Proof. Let $S = \{1, 2, \ldots, n)$ be the set of symbols for L_1 and L_2, and let $F_0 = L_1$.

For each row $i \in \{1, 2, \ldots, Y\}$ of the orthogonal array, define $\phi_i : \{1, 2, \ldots, n\} \to \{0, 1, 2, \ldots, x_{i-1}\}$ represents the entry in row i and column j of the array such that $\phi_i(j)$ is the entry in row i and column j of the orthogonal array. Then, construct the frequency squares of order n from L_2 as follows:

$$F_i(r, c) = \phi_i(L_2(r, c)), \quad i = \{1, 2, \ldots, Y\},$$

where $L_2(r, c) \in \{1, 2, \ldots, n\}$.

By the definition of mixed orthogonal arrays, each symbol x_i of OA$(n, x_1^{y_1}, x_2^{y_2}, ..., x_k^{y_k}, 2)$ has the frequency n/x_1 in the first y_1 rows, the frequency n/x_2 in the next y_2 rows,..., and the frequency n/x_k in the last y_k rows. Hence, ϕ_i maps the symbols in S to a set of x_i symbols, each of which appears n/x_i times in every row of the $n \times n$ array F_i.

Therefore, each F_i is of type $F(n, (n/x_i)^{x_i})$.

Since, L_1 and L_2 are MOLS of order n, by Lemma 4.1, F_0 is orthogonal to each of the F_i, where $i = \{1, 2, \ldots, Y\}$.

Now, the strength of the OA$(n, x_1^{y_1}, x_2^{y_2}, ..., x_k^{y_k}, 2)$ is 2 so that the ordered pairs of the symbols of any two rows occur the same number of times across n columns. This, in turn, guarantees that every ordered pair of symbols in the superimposed array of any two frequency squares F_i and F_j, where $i, j = \{1, 2, \ldots, Y\}$ and $i \neq j$, appears uniformly, since frequency squares F_i is derived from a row of the mixed orthogonal array $\mathrm{OA}(n, x_1^{y_1}, x_2^{y_2}, ..., x_k^{y_k}, 2)$.

Therefore, $\{F_0, F_1, F_2, F_3, \ldots, F_Y\}$ is a set of MMOFS by Lemma 4.1.

$\square$

Example 21. *Consider the following Latin squares of order* 12 *with symbol set*

$$S = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12\}.$$

$$L_1 = F_0 = \begin{array}{|c|c|c|c|c|c|c|c|c|c|c|c|}
\hline
1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 \\
\hline
2 & 1 & 4 & 3 & 6 & 5 & 8 & 7 & 10 & 9 & 12 & 11 \\
\hline
3 & 4 & 1 & 2 & 7 & 8 & 5 & 6 & 11 & 12 & 9 & 10 \\
\hline
4 & 3 & 2 & 1 & 8 & 7 & 6 & 5 & 12 & 11 & 10 & 9 \\
\hline
5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 1 & 2 & 3 & 4 \\
\hline
6 & 5 & 8 & 7 & 10 & 9 & 12 & 11 & 2 & 1 & 4 & 3 \\
\hline
7 & 8 & 5 & 6 & 11 & 12 & 9 & 10 & 3 & 4 & 1 & 2 \\
\hline
8 & 7 & 6 & 5 & 12 & 11 & 10 & 9 & 4 & 3 & 2 & 1 \\
\hline
9 & 10 & 11 & 12 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\
\hline
10 & 9 & 12 & 11 & 2 & 1 & 4 & 3 & 6 & 5 & 8 & 7 \\
\hline
11 & 12 & 9 & 10 & 3 & 4 & 1 & 2 & 7 & 8 & 5 & 6 \\
\hline
12 & 11 & 10 & 9 & 4 & 3 & 2 & 1 & 8 & 7 & 6 & 5 \\
\hline
\end{array}\,.$$

$$L_2 = \begin{array}{|c|c|c|c|c|c|c|c|c|c|c|c|}
\hline
1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 \\ \hline
3 & 4 & 1 & 2 & 7 & 8 & 5 & 6 & 11 & 12 & 9 & 10 \\ \hline
4 & 3 & 2 & 1 & 8 & 7 & 6 & 5 & 12 & 11 & 10 & 9 \\ \hline
5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 1 & 2 & 3 & 4 \\ \hline
6 & 5 & 8 & 7 & 10 & 9 & 12 & 11 & 2 & 1 & 4 & 3 \\ \hline
7 & 8 & 5 & 6 & 11 & 12 & 9 & 10 & 3 & 4 & 1 & 2 \\ \hline
8 & 7 & 6 & 5 & 12 & 11 & 10 & 9 & 4 & 3 & 2 & 1 \\ \hline
9 & 10 & 11 & 12 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ \hline
10 & 9 & 12 & 11 & 2 & 1 & 4 & 3 & 6 & 5 & 8 & 7 \\ \hline
11 & 12 & 9 & 10 & 3 & 4 & 1 & 2 & 7 & 8 & 5 & 6 \\ \hline
12 & 11 & 10 & 9 & 4 & 3 & 2 & 1 & 8 & 7 & 6 & 5 \\ \hline
2 & 1 & 4 & 3 & 6 & 5 & 8 & 7 & 10 & 9 & 12 & 11 \\ \hline
\end{array}.$$

Consider the following mixed orthogonal array $OA(12, 2^4, 3^1, 2)$

$$\begin{array}{cccccccccccc}
0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 \\
0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \\
0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 & 1 & 0 & 0 & 1 \\
0 & 1 & 0 & 1 & 1 & 0 & 0 & 1 & 1 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 2 & 2 & 2 & 2
\end{array}.$$

Define frequency squares F_1, F_2, F_3, F_4, F_5 *from* L_2 *by the following functions:*

$$\phi_1(s) = \begin{cases} 0, s \in \{1, 2, 5, 6, 9, 10\} \\ 1, s \in \{3, 4, 7.8, 11, 12\}, \end{cases}$$

$$\phi_2(s) = \begin{cases} 0, s \in \{1,3,5,7,9,11\} \\ 1, s \in \{2,4,6.8,10,12\}, \end{cases}$$

$$\phi_3(s) = \begin{cases} 0, s \in \{1,2,7,8,10,11\} \\ 1, s \in \{3,4,5,6,9,12\}, \end{cases}$$

$$\phi_4(s) = \begin{cases} 0, s \in \{1,3,6,7,10,12\} \\ 1, s \in \{2,4,5,8,9,10\}, \end{cases}$$

$$\phi_5(s) = \begin{cases} 0, s \in \{1,2,3,4\} \\ 1, s \in \{5,6,7,8\} \\ 2, s \in \{9.10.11,12\}. \end{cases}$$

Then, we have the frequency squares F_1, F_2, F_3, F_4, F_5 *as follows:*

$$F_1 = \begin{array}{|c|c|c|c|c|c|c|c|c|c|c|c|}
\hline
0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 \\ \hline
1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 \\ \hline
1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 \\ \hline
0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 \\ \hline
0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 \\ \hline
1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 \\ \hline
1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 \\ \hline
0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 \\ \hline
0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 \\ \hline
1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 \\ \hline
1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 \\ \hline
0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 \\ \hline
\end{array} .$$

$$F_2 = \begin{array}{|c|c|c|c|c|c|c|c|c|c|c|c|}
\hline
0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \\ \hline
0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \\ \hline
1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \\ \hline
0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \\ \hline
1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \\ \hline
0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \\ \hline
1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \\ \hline
0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \\ \hline
1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \\ \hline
0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \\ \hline
1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \\ \hline
1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \\ \hline
\end{array} .$$

$$F_3 = \begin{array}{|c|c|c|c|c|c|c|c|c|c|c|c|}\hline
0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 & 1 & 0 & 0 & 1 \\ \hline
1 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 1 & 1 & 0 \\ \hline
1 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 1 \\ \hline
1 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 1 \\ \hline
1 & 1 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 1 & 1 \\ \hline
0 & 0 & 1 & 1 & 0 & 1 & 1 & 0 & 1 & 1 & 0 & 0 \\ \hline
0 & 0 & 1 & 1 & 1 & 0 & 0 & 1 & 1 & 1 & 0 & 0 \\ \hline
1 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 \\ \hline
0 & 1 & 1 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 \\ \hline
0 & 1 & 1 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 1 & 1 \\ \hline
1 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 1 & 1 \\ \hline
0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 1 & 1 & 0 \\ \hline
\end{array}\ .$$

$$F_4 = \begin{array}{|c|c|c|c|c|c|c|c|c|c|c|c|}\hline
0 & 1 & 0 & 1 & 1 & 0 & 0 & 1 & 1 & 0 & 1 & 0 \\ \hline
0 & 1 & 0 & 1 & 0 & 1 & 1 & 0 & 1 & 0 & 1 & 0 \\ \hline
1 & 0 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 & 1 \\ \hline
1 & 0 & 0 & 1 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & 1 \\ \hline
0 & 1 & 1 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 1 & 0 \\ \hline
0 & 1 & 1 & 0 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & 1 \\ \hline
1 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 1 & 0 & 1 & 0 \\ \hline
1 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 0 & 1 \\ \hline
0 & 1 & 0 & 1 & 1 & 0 & 1 & 0 & 0 & 1 & 1 & 0 \\ \hline
1 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 1 & 0 \\ \hline
0 & 1 & 0 & 1 & 1 & 0 & 1 & 0 & 1 & 0 & 0 & 1 \\ \hline
1 & 0 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 0 & 1 \\ \hline
\end{array}\ .$$

$$F_5 = \begin{array}{|c|c|c|c|c|c|c|c|c|c|c|c|}
\hline
0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 2 & 2 & 2 & 2 \\
\hline
0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 2 & 2 & 2 & 2 \\
\hline
0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 2 & 2 & 2 & 2 \\
\hline
1 & 1 & 1 & 1 & 2 & 2 & 2 & 2 & 0 & 0 & 0 & 0 \\
\hline
1 & 1 & 1 & 1 & 2 & 2 & 2 & 2 & 0 & 0 & 0 & 0 \\
\hline
1 & 1 & 1 & 1 & 2 & 2 & 2 & 2 & 0 & 0 & 0 & 0 \\
\hline
1 & 1 & 1 & 1 & 2 & 2 & 2 & 2 & 0 & 0 & 0 & 0 \\
\hline
2 & 2 & 2 & 2 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\
\hline
2 & 2 & 2 & 2 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\
\hline
2 & 2 & 2 & 2 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\
\hline
2 & 2 & 2 & 2 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\
\hline
0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 2 & 2 & 2 & 2 \\
\hline
\end{array}.$$

Therefore, $\{F_0, F_1, F_2, F_3, F_4, F_5\}$ *is a set of MMOFS by Lemma* 4.1.

5 Results for small orders

In this chapter, we identify several sets of MMOFS for small orders, some of which form complete sets. The orthogonality of these frequency squares is verified using the previous theories. Specifically, we obtain sets of MMOFS of regular and irregular frequency squares of order 4.

5.1 Regular frequency squares of order 4

Lemma 5.1. *There exists a complete set of MMOFS of the form* $\{2 \times F(4;1,1,1,1); 3 \times F(4;2,2)\}$.

Example 22. *Consider two MOLS* L_1 *and* L_2*, i.e, the frequency squares of type* $F(4;1,1,1,1)$*, and the MOFS* F_1*,* F_2 *and* F_3 *of type* $F(4;2,2)$ *from Example 10 as follows.*

$$L_1 = \begin{array}{|c|c|c|c|}\hline 1 & 2 & 3 & 4 \\ \hline 2 & 1 & 4 & 3 \\ \hline 3 & 4 & 1 & 2 \\ \hline 4 & 3 & 2 & 1 \\ \hline \end{array} \qquad L_2 = \begin{array}{|c|c|c|c|}\hline 1 & 2 & 3 & 4 \\ \hline 3 & 4 & 1 & 2 \\ \hline 4 & 3 & 2 & 1 \\ \hline 2 & 1 & 4 & 3 \\ \hline \end{array} .$$

$$F_1 = \begin{array}{|c|c|c|c|}\hline 0 & 1 & 1 & 0 \\ \hline 0 & 1 & 1 & 0 \\ \hline 1 & 0 & 0 & 1 \\ \hline 1 & 0 & 0 & 1 \\ \hline \end{array} \quad F_2 = \begin{array}{|c|c|c|c|}\hline 0 & 0 & 1 & 1 \\ \hline 1 & 1 & 0 & 0 \\ \hline 0 & 0 & 1 & 1 \\ \hline 1 & 1 & 0 & 0 \\ \hline \end{array} \quad F_3 = \begin{array}{|c|c|c|c|}\hline 0 & 1 & 0 & 1 \\ \hline 1 & 0 & 1 & 0 \\ \hline 1 & 0 & 1 & 0 \\ \hline 0 & 1 & 0 & 1 \\ \hline \end{array} .$$

Now, we only need to verify the orthogonality of each frequency square of type $F(4;1,1,1,1)$ *with frequency squares of type* $F(4;2,2)$*. Consider the superimposed squares as follows.*

$$[(L_1, F_1)] = \begin{array}{|c|c|c|c|}\hline (1,0) & (2,1) & (3,1) & (4,0) \\ \hline (2,0) & (1,1) & (4,1) & (3,0) \\ \hline (3,1) & (4,0) & (1,0) & (2,1) \\ \hline (4,1) & (3,0) & (2,0) & (1,1) \\ \hline \end{array} .$$

$$[(L_1, F_2)] = \begin{array}{|c|c|c|c|}\hline (1,0) & (2,0) & (3,1) & (4,1) \\ \hline (2,1) & (1,1) & (4,0) & (3,0) \\ \hline (3,0) & (4,0) & (1,1) & (2,1) \\ \hline (4,1) & (3,1) & (2,0) & (1,0) \\ \hline \end{array} .$$

$$[(L_1, F_3)] = \begin{array}{|c|c|c|c|}\hline (1,0) & (2,1) & (3,0) & (4,1) \\ \hline (2,1) & (1,0) & (4,1) & (3,0) \\ \hline (3,1) & (4,0) & (1,1) & (2,0) \\ \hline (4,0) & (3,1) & (2,0) & (1,1) \\ \hline \end{array} .$$

$$[(L_2, F_1)] = \begin{array}{|c|c|c|c|}\hline (1,0) & (2,1) & (3,1) & (4,0) \\ \hline (3,0) & (4,1) & (2,1) & (1,0) \\ \hline (4,1) & (3,0) & (2,0) & (1,1) \\ \hline (2,1) & (1,0) & (4,0) & (3,1) \\ \hline \end{array} .$$

$$[(L_2, F_2)] = \begin{array}{|c|c|c|c|}\hline (1,0) & (2,0) & (3,1) & (4,1) \\ \hline (3,1) & (4,1) & (2,0) & (1,0) \\ \hline (4,0) & (3,0) & (2,1) & (1,1) \\ \hline (2,1) & (1,1) & (4,0) & (3,0) \\ \hline \end{array} .$$

$$[(L_2, F_3)] = \begin{array}{|c|c|c|c|}\hline (1,0) & (2,1) & (3,0) & (4,1) \\ \hline (3,1) & (4,0) & (2,1) & (1,0) \\ \hline (4,1) & (3,0) & (2,1) & (1,0) \\ \hline (2,0) & (1,1) & (4,0) & (3,1) \\ \hline \end{array}.$$

$N(1,0) = 2,\ N(1,1) = 2,\ N(2,0) = 2,\ N(2,1) = 2,\ N(3,0) = 2,\ N(3,1) = 2,\ N(4,0) = 2,\ N(4,1) = 2.$

By Definition 6, $\{2\times F(4;1,1,1,1); 3\times F(4;2,2)\}$ *is a set of MMOFS. Now, the inequality from Theorem* 1.12 implies:

$$4 + 4 + 2 + 2 + 2 - 5 \leq (4-1)^2$$

$$\Leftrightarrow 9 \leq 9,$$

Hence, $\{2 \times F(4;1,1,1,1); 3 \times F(4;2,2)\}$ is a complete set.

5.2 Irregular (not regular) frequency squares of order 4

It is not always guaranteed to have a complete set of MMOFS for regular frequency squares. There are also many frequency squares with unequal frequencies that meet the bound of Theorem 1.12.

Example 23. $\{1 \times F(4;3,1); 6 \times F(4;2,2)\}$*, where the* $6 \times F(4;2,2)\}$ *are* 6 *MOFS from Example 10. Thus, we have to verify only the orthogonality of* $F(4;3,1)$ *to each* $F(4;2,2)$*.*

$$M = \begin{array}{|c|c|c|c|}\hline 0 & 0 & 0 & 1 \\ \hline 0 & 0 & 1 & 0 \\ \hline 0 & 1 & 0 & 0 \\ \hline 1 & 0 & 0 & 0 \\ \hline\end{array} \qquad F_1 = \begin{array}{|c|c|c|c|}\hline 0 & 1 & 1 & 0 \\ \hline 0 & 1 & 1 & 0 \\ \hline 1 & 0 & 0 & 1 \\ \hline 1 & 0 & 0 & 1 \\ \hline\end{array} \qquad F_2 = \begin{array}{|c|c|c|c|}\hline 0 & 0 & 1 & 1 \\ \hline 1 & 1 & 0 & 0 \\ \hline 0 & 0 & 1 & 1 \\ \hline 1 & 1 & 0 & 0 \\ \hline\end{array} \qquad F_3 = \begin{array}{|c|c|c|c|}\hline 0 & 1 & 0 & 1 \\ \hline 0 & 1 & 0 & 1 \\ \hline 1 & 0 & 1 & 0 \\ \hline 1 & 0 & 1 & 0 \\ \hline\end{array}$$

$$F_4 = \begin{array}{|c|c|c|c|}\hline 0 & 0 & 1 & 1 \\ \hline 1 & 1 & 0 & 0 \\ \hline 1 & 1 & 0 & 0 \\ \hline 0 & 0 & 1 & 1 \\ \hline\end{array} \qquad F_5 = \begin{array}{|c|c|c|c|}\hline 0 & 1 & 1 & 0 \\ \hline 1 & 0 & 0 & 1 \\ \hline 0 & 1 & 0 & 1 \\ \hline 1 & 0 & 1 & 0 \\ \hline\end{array} \qquad F_6 = \begin{array}{|c|c|c|c|}\hline 0 & 1 & 0 & 1 \\ \hline 1 & 0 & 1 & 0 \\ \hline 1 & 0 & 0 & 1 \\ \hline 0 & 1 & 1 & 0 \\ \hline\end{array}.$$

$$[(M,F_1)] = \begin{array}{|c|c|c|c|}\hline (0,0) & (0,1) & (0,1) & (0,0) \\ \hline (0,0) & (0,1) & (1,1) & (0,0) \\ \hline (0,1) & (1,0) & (0,0) & (0,1) \\ \hline (1,1) & (0,0) & (0,0) & (0,1) \\ \hline\end{array} \qquad [(M,F_2)] = \begin{array}{|c|c|c|c|}\hline (0,0) & (0,0) & (0,1) & (1,1) \\ \hline (0,1) & (0,1) & (1,0) & (0,0) \\ \hline (0,0) & (1,0) & (0,1) & (0,1) \\ \hline (1,1) & (0,1) & (0,0) & (0,0) \\ \hline\end{array}$$

$$[(M,F_3)] = \begin{array}{|c|c|c|c|}\hline (0,0) & (0,1) & (0,0) & (1,1) \\ \hline (0,0) & (0,1) & (1,0) & (0,1) \\ \hline (0,1) & (1,0) & (0,1) & (0,0) \\ \hline (1,1) & (0,0) & (0,1) & (0,0) \\ \hline\end{array} \qquad [(M,F_4)] = \begin{array}{|c|c|c|c|}\hline (0,0) & (0,0) & (0,1) & (1,1) \\ \hline (0,1) & (0,1) & (1,0) & (0,0) \\ \hline (0,1) & (1,1) & (0,0) & (0,0) \\ \hline (1,0) & (0,0) & (0,1) & (0,1) \\ \hline\end{array}$$

$$[(M,F_5)] = \begin{array}{|c|c|c|c|}\hline (0,0) & (0,1) & (0,0) & (1,1) \\ \hline (0,1) & (0,0) & (1,1) & (0,0) \\ \hline (0,1) & (1,0) & (0,0) & (0,1) \\ \hline (1,0) & (0,1) & (0,1) & (0,0) \\ \hline\end{array} \qquad [(M,F_6)] = \begin{array}{|c|c|c|c|}\hline (0,0) & (0,1) & (0,1) & (1,0) \\ \hline (0,1) & (0,0) & (1,0) & (0,1) \\ \hline (0,0) & (1,1) & (0,0) & (0,1) \\ \hline (1,1) & (0,0) & (0,1) & (0,0) \\ \hline\end{array}.$$

Each ordered pair appears as similar to Example 22, and by Definition 6, $\{1\times F(4;3,1); 6\times F(4;2,2)\}$ *is a set of MMOFS.*

Now, the inequality from Theorem 1.12 implies:

$$3+2+2+2+2+2+2-7 \leq (4-1)^2.$$

$$\Leftrightarrow 8 \leq 9.$$

. *Now, we can establish the orthogonality in an alternative way. Consider the following MOLS,*

$$L_1 = \begin{array}{|c|c|c|c|}\hline 1&2&3&4\\ \hline 2&1&4&3\\ \hline 3&4&1&2\\ \hline 4&3&2&1\\ \hline \end{array}\ , L_2 = \begin{array}{|c|c|c|c|}\hline 1&2&3&4\\ \hline 3&4&1&2\\ \hline 4&3&2&1\\ \hline 2&1&4&3\\ \hline \end{array}\ , L_3 = \begin{array}{|c|c|c|c|}\hline 1&2&3&4\\ \hline 4&3&2&1\\ \hline 2&1&4&3\\ \hline 3&4&1&2\\ \hline \end{array}\ .$$

Define the functions f *on* L_1, $f:\{1,2,3,4\}\to\{0,1\}$ *as*

$f(1)=0,\ f(2)=0,\ f(3)=0,\ f(4)=1,$

and f_1 *on* L_3, $f_1:\{1,2,3,4\}\to\{0,1\}$ *as*

$f_1(1)=0,\ f_1(2)=1,\ f_1(3)=1,\ f_1(4)=0.$ *Then,*

$$M = f(L_1) = \begin{array}{|c|c|c|c|}\hline 0&0&0&1\\ \hline 0&0&1&0\\ \hline 0&1&0&0\\ \hline 1&0&0&0\\ \hline \end{array}\ .$$

$$F_1 = f_1(L_3) = \begin{array}{|c|c|c|c|}\hline 0 & 1 & 1 & 0 \\ \hline 0 & 1 & 1 & 0 \\ \hline 1 & 0 & 0 & 1 \\ \hline 1 & 0 & 0 & 1 \\ \hline \end{array}\,.$$

Then, by Lemma 4.1, M and F_1 are orthogonal.

Similarly define another function f_2 on L_3, $f_2 : \{1, 2, 3, 4\} \to \{0, 1\}$ as

$f_2(1) = 0,\ f_2(2) = 0,\ f_2(3) = 1,\ f_2(4) = 1.$

$$F_2 = f_2(L_3) = \begin{array}{|c|c|c|c|}\hline 0 & 0 & 1 & 1 \\ \hline 1 & 1 & 0 & 0 \\ \hline 0 & 0 & 1 & 1 \\ \hline 1 & 1 & 0 & 0 \\ \hline \end{array}\,.$$

Hence, again by Lemma 4.1, M and F_2 are orthogonal. This can be repeated by defining different functions on the MOLS L_1, L_2, and L_3.

For instance, define f_3 on L_2, $f_3 : \{1, 2, 3, 4\} \to \{0, 1\}$ as

$f_3(1) = 0,\ f_2(2) = 1,\ f_2(3) = 0,\ f_2(4) = 1.$ Then,

$$F_3 = f_3(L_2) = \begin{array}{|c|c|c|c|}\hline 0 & 1 & 0 & 1 \\ \hline 0 & 1 & 0 & 1 \\ \hline 1 & 0 & 1 & 0 \\ \hline 1 & 0 & 1 & 0 \\ \hline \end{array}\,.$$

and f_4 on L_2, $f_4 : \{1, 2, 3, 4\} \to \{0, 1\}$ as

$f_4(1) = 0,\ f_4(2) = 0,\ f_2(3) = 1,\ f_2(4) = 1.$ Then,

$$F_4 = f_4(L_2) = \begin{array}{|c|c|c|c|}\hline 0 & 0 & 1 & 1 \\ \hline 1 & 1 & 0 & 0 \\ \hline 1 & 1 & 0 & 0 \\ \hline 0 & 0 & 1 & 1 \\ \hline \end{array} .$$

Therefore, Lemma 4.1 establishes the orthogonality of M and F_3, and M and F_4. Likewise, we can prove the orthogonality of M with F_5 and F_6.

This set is not complete, but it satisfies the bound given in Theorem 1.12. In the following lemmas, we aim to show that no additional frequency squares of same type can be included to enlarge this set.

Lemma 5.2. *There does not exist a set of MMOFS* $\{1 \times F(4;3,1); 7 \times F(4;2,2)\}$.

Proof. Let G be a frequency square of type $F(4;3,1)$ with two symbols 0 and 1, where 0 has frequency 3 and 1 has frequency 1, and $A_i = F(4;2,2)$, for $i = 1, 2, \dots, 7$ with same symbols 0 and 1, each having frequency 2.

Suppose, for the sake of contradiction, $\{G, A_1, \dots, A_7\}$ is a set of MMOFS.

Without loss of generality, $P = \{(1,1), (2,2), (3,3), (4,4)\}$ are 4 positions where 1 appears in G. In order for G to be orthogonal to each A_i, for $i = 1, 2, \dots, 7$, exactly two of the cells in P must be labeled 0 and the other two must be labeled 1 in A_i, for $i = 1, 2, \dots, 7$.

Hence, for each A_i, for $i = 1, 2, \dots, 7$, the 4 positions P must have a unique $2-2$ coloring. Since, there are only $\binom{4}{2} = 6$ such colorings, only 6 frequency squares A_i can be pairwise orthogonal to G, using a different $2-2$ coloring. Thus at least two of the squares, say A_1 and A_2, must use the same $2-2$ coloring at the positions of P by **Pigeonhole principle** [17]. Without loss of generality, by rearranging the rows and columns, we may assume G as follows:

$$G = \begin{array}{|c|c|c|c|}\hline 1 & 0 & 0 & 0 \\ \hline 0 & 1 & 0 & 0 \\ \hline 0 & 0 & 1 & 0 \\ \hline 0 & 0 & 0 & 1 \\ \hline \end{array}\,.$$

For each of the remaining A_i's, we can swap the entries 0 and 1 within any of these squares without changing orthogonality. So without loss of generality, we can assume that cell $(1, 1)$ contains 1 in each of the A_i's. Therefore there are only 3 choices for the splitting.

Without loss of generality, assume that both A_1 and A_2 have the following pattern on the positions of 1 in G.

$$\begin{array}{|c|c|c|c|}\hline 1 & * & * & * \\ \hline * & 1 & * & * \\ \hline * & * & 0 & * \\ \hline * & * & * & 0 \\ \hline \end{array}\,.$$

Then, up to isomorphism and by looking at transpose, there are only two possibilities for A_1 and A_2, as follows:

$$A_1 = \begin{array}{|c|c|c|c|}\hline 1 & 0 & 1 & 0 \\ \hline 0 & 1 & 0 & 1 \\ \hline 1 & 0 & 0 & 1 \\ \hline 0 & 1 & 1 & 0 \\ \hline \end{array} \qquad A_2 = \begin{array}{|c|c|c|c|}\hline 1 & 0 & 0 & 1 \\ \hline 0 & 1 & 1 & 0 \\ \hline 0 & 1 & 0 & 1 \\ \hline 1 & 0 & 1 & 0 \\ \hline \end{array}$$

or,

$$A_1 = \begin{array}{|c|c|c|c|}\hline 1 & 0 & 0 & 1 \\ \hline 0 & 1 & 1 & 0 \\ \hline 0 & 1 & 0 & 1 \\ \hline 1 & 0 & 1 & 0 \\ \hline \end{array} \qquad A_2 = \begin{array}{|c|c|c|c|}\hline 1 & 0 & 1 & 0 \\ \hline 0 & 1 & 0 & 1 \\ \hline 1 & 0 & 0 & 1 \\ \hline 0 & 1 & 1 & 0 \\ \hline \end{array}\,.$$

The permutation (12)(34) applied to both rows and columns fixes the squares A_1 and A_2, that is, they admit an autotopism.

However, this is true for each of the 3 possible $2-2$ splittings of the positions in P, and in each splitting, at most 2 squares can use it without violating the orthogonality. Thus, maximum number of such frequency squares is $3 \times 2 = 6 < 7$, is a contradiction.

Therefore, there doesn't exist a set of MMOFS $\{1 \times F(4;3,1); 7 \times F(4;2,2)\}$.

□

Lemma 5.3. *In a frequency square of type $F(4;2,2)$ contains the following substructure:*

$$\begin{array}{|c|c|c|c|}\hline 1 & * & * & * \\ \hline * & 1 & * & * \\ \hline * & * & 0 & * \\ \hline * & * & * & 0 \\ \hline \end{array}\,.$$

Then, it must have the following structure (up to isomorphism):

$$\begin{array}{|c|c|c|c|}\hline 1 & 0 & * & * \\ \hline 0 & 1 & * & * \\ \hline * & * & 0 & 1 \\ \hline * & * & 1 & 0 \\ \hline \end{array}\,.$$

Proof. Consider a frequency square of type $F(4;2,2)$ having the following substructure:

$$F = \begin{array}{|c|c|c|c|}\hline 1 & * & * & * \\ \hline * & 1 & * & * \\ \hline * & * & 0 & * \\ \hline * & * & * & 0 \\ \hline \end{array}.$$

Suppose, the entry at the position $(1,2)$ is 1. Then,

$$F = \begin{array}{|c|c|c|c|}\hline 1 & 1 & * & * \\ \hline * & 1 & * & * \\ \hline * & * & 0 & * \\ \hline * & * & * & 0 \\ \hline \end{array}.$$

Then, we are forced to have:

$$F = \begin{array}{|c|c|c|c|}\hline 1 & 1 & 0 & 0 \\ \hline * & 1 & * & * \\ \hline * & * & 0 & * \\ \hline * & * & * & 0 \\ \hline \end{array}.$$

Then, we are forced to have:

$$F = \begin{array}{|c|c|c|c|}\hline 1 & 1 & 0 & 0 \\ \hline * & 1 & 1 & 1 \\ \hline * & * & 0 & 1 \\ \hline * & * & 1 & 0 \\ \hline \end{array},$$

which is a contradiction to $F(4;2,2)$ since there are three 1s in the second row. Hence, the entry at the position $(1,2)$ must be 0.

Therefore, F is forced to have the form:

$$F = \begin{array}{|c|c|c|c|}\hline 1 & 0 & * & * \\ \hline 0 & 1 & * & * \\ \hline * & * & 0 & 1 \\ \hline * & * & 1 & 0 \\ \hline \end{array}.$$

□

Lemma 5.4. *There does not exist a set of MMOFS* $\{2 \times F(4;3,1); 4 \times F(4;2,2)\}$.

Proof. Suppose that there are 4 MOFS of type $F(4;2,2)$ say A_1, A_2, A_3, A_4 that are orthogonal to MMOFS G_1 and G_2 of type $F(4;3,1)$. Without loss of generality, assume that

$$G_1 = \begin{array}{|c|c|c|c|}\hline 1 & 0 & 0 & 0 \\ \hline 0 & 1 & 0 & 0 \\ \hline 0 & 0 & 1 & 0 \\ \hline 0 & 0 & 0 & 1 \\ \hline \end{array}.$$

Without loss of generality, by rearranging rows and columns, we can assume that $(1,1)$ is the position where G_1 and G_2 overlap. Then, the other possibilities of $F(4;3,1)$ are as follows;

$$\begin{array}{|c|c|c|c|}\hline 1 & 0 & 0 & 0 \\ \hline 0 & 0 & 1 & 0 \\ \hline 0 & 0 & 0 & 1 \\ \hline 0 & 1 & 0 & 0 \\ \hline \end{array} \quad \text{or} \quad \begin{array}{|c|c|c|c|}\hline 1 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 1 \\ \hline 0 & 1 & 0 & 0 \\ \hline 0 & 0 & 1 & 0 \\ \hline \end{array},$$

where one is the transpose of the other. Without loss of generality, we can assume

$$G_2 = \begin{array}{|c|c|c|c|}\hline 1 & 0 & 0 & 0 \\ \hline 0 & 0 & 1 & 0 \\ \hline 0 & 0 & 0 & 1 \\ \hline 0 & 1 & 0 & 0 \\ \hline \end{array} .$$

Furthermore, since 0's and 1's can be swapped within a frequency square of type $F(4; 2, 2)$, without loss of generality, we can assume that each A_i has 1 in the cell $(1, 1)$. Let P_1 and P_2 be the set of three positions in G_1 and G_2, where 1 appears on them respectively, ignoring the first row. Hence, $P_1 = \{(2, 2), (3, 3), (4, 4)\}$ and $P_2 = \{(2, 3), (3, 4), (4, 2)\}$.

Since each A_i for $i = 1, 2, 3, 4$ is orthogonal to both G_1 and G_2 and the cell $(1, 1)$ is fixed as 1 in each A_i, there is exactly one more 1 in a cell which is an element of P_1, and similarly in a cell which is an element of P_2. We have the positions of 1 are $\{(1, 1), (2, 2), (3, 3), (4, 4)\}$ and $\{(1, 1), (2, 3), (3, 4), (4, 2)\}$ in G_1 and G_2 respectively. If we apply the permutation (234) or (243) to the rows and columns, then

$$(2, 3) \Rightarrow (3, 4) \quad \text{and} \quad (3, 4) \Rightarrow (4, 2),$$

or

$$(2, 3) \Rightarrow (4, 2) \quad \text{and} \quad (3, 4) \Rightarrow (2, 3).$$

In any case, the positions of 1 remain the same. Hence, G_1 and G_2 are invariant under the permutations of (234) or (243).

Let X be any frequency of type $F(4; 2, 2)$ that has a fixed 1 in the cell $(1, 1)$ and orthogonal to both G_1 and G_2.

Case 1: 1 is in cell $(2, 2)$ of X. $(2, 2) \in P_1$. Then, without loss of generality,

$$X = \begin{array}{|c|c|c|c|}\hline 1 & * & * & * \\ \hline * & 1 & * & * \\ \hline * & * & 0 & * \\ \hline * & * & * & 0 \\ \hline \end{array}.$$

By Lemma 5.3, X is forced to be:

$$X = \begin{array}{|c|c|c|c|}\hline 1 & 0 & * & * \\ \hline 0 & 1 & * & * \\ \hline * & * & 0 & 1 \\ \hline * & * & 1 & 0 \\ \hline \end{array}.$$

Hence, considering P_2 as above, since 1 is in cell $(3,4) \in P_2$, we must have 0 in cells $(2,3)$ and $(4,2)$ for orthogonality with G_2.

Thus, there is only one possibility in this case, and let it be A_1. Hence,

$$A_1 = \begin{array}{|c|c|c|c|}\hline 1 & 0 & 1 & 0 \\ \hline 0 & 1 & 0 & 1 \\ \hline 0 & 1 & 0 & 1 \\ \hline 1 & 0 & 1 & 0 \\ \hline \end{array}.$$

This matrix has two 1s and 0s in each row and column, and has 1 in $(1,1),(2,2)$ in G_1 and $(1,1),(3,4)$ in G_2. Hence, X is orthogonal to both G_1 and G_2.

Case 2: 1 is in cell $(3,3)$ of X. Hence,

$$X = \begin{array}{|c|c|c|c|}\hline 1 & * & * & * \\ \hline * & 0 & * & * \\ \hline * & * & 1 & * \\ \hline * & * & * & 0 \\ \hline \end{array}.$$

By Lemma 5.3 X must follow:

$$X = \begin{array}{|c|c|c|c|}\hline 1 & * & 0 & * \\ \hline * & 0 & * & 1 \\ \hline 0 & * & 1 & * \\ \hline * & 1 & * & 0 \\ \hline \end{array}.$$

Therefore, we are forced to choose $(4, 2)$ from P_2, and there is only one possibility for this case, and let it be A_2. Hence,

$$A_2 = \begin{array}{|c|c|c|c|}\hline 1 & 0 & 0 & 1 \\ \hline 1 & 0 & 0 & 1 \\ \hline 0 & 1 & 1 & 0 \\ \hline 0 & 1 & 1 & 0 \\ \hline \end{array}.$$

Case 3**:** Let 1 in the cell $(4, 4)$ of X. Hence,

$$X = \begin{array}{|c|c|c|c|}\hline 1 & * & * & * \\ \hline * & 0 & * & * \\ \hline * & * & 0 & * \\ \hline * & * & * & 1 \\ \hline \end{array}.$$

By Lemma 5.3, X must follow:

$$X = \begin{array}{|c|c|c|c|}\hline 1 & * & * & 0 \\ \hline * & 0 & 1 & * \\ \hline * & 1 & 0 & * \\ \hline 0 & * & * & 1 \\ \hline \end{array}.$$

Therefore, we are forced to choose $(3, 4)$ from P_2, and there is only one possibility for this case, and let it be A_3. Hence,

$$A_3 = \begin{array}{|c|c|c|c|}\hline 1 & 1 & 0 & 0 \\ \hline 1 & 0 & 1 & 0 \\ \hline 0 & 1 & 0 & 1 \\ \hline 0 & 0 & 1 & 1 \\ \hline \end{array}.$$

Now, there are no remaining positions in P_1 and P_2 for a second occurrence of 1 in a 4^{th} square A_4, without overlapping at least one of A_1, A_2, and A_3. Hence, a 4^{th} frequency square A_4 is orthogonal to both G_1 and G_2, if and only if it must share a position of 1 in P_1 or P_2 with at least one of A_1, A_2 and A_3. This violates their mutual orthogonality as similar to the case in Lemma 5.2.

Hence, we get at most 3 A_i of type $F(4; 2, 2)$, which are orthogonal to both G_1 and G_2.This contradicts our assumption that there are 4 MOFS A_1, A_2, A_3, A_4 of type $F(4; 2, 2)$, which are orthogonal to MMOFS G_1 and G_2 of type $F(4; 3, 1)$.

Therefore, there does not exist a set of MMOFS $\{2 \times F(4; 3, 1); 4 \times F(4; 2, 2)\}$. $\square$

Remark 5.5. *It is not possible to add a frequency square of type* $F(4; 3, 1)$ *in order to increase the size of the set* $\{1 \times F(4; 3, 1); 6 \times F(4; 2, 2)\}$. *By Lemma* 5.4, there does not exist a set of MMOFS $\{2 \times F(4; 3, 1); 4 \times F(4; 2, 2)\}$. Consequently, it is impossible to obtain a set of the form $\{2 \times F(4; 3, 1); 6 \times F(4; 2, 2)\}$. Therefore, by Lemma 5.2 and Lemma 5.4, no additional frequency squares of the types $F(4;3,1)$ and $F(4;2,2)$ can be added to the set $\{1 \times F(4;3,1); 6 \times F(4;2,2)\}$, establishing that this set is type maximal.

Corollary 5.6. *There is a unique set of MMOFS of the form* $\{2 \times F(4;3,1); 3 \times F(4;2,2)\}$.

Example 24. $\{2 \times F(4;3,1); 3 \times F(4;2,2)\}$, *where* $3 \times F(4;2,2)\}$ *are* 3 *MOFS from Example* 10.

$$M_1 = \begin{array}{|c|c|c|c|} \hline 1 & 0 & 0 & 0 \\ \hline 0 & 1 & 0 & 0 \\ \hline 0 & 0 & 1 & 0 \\ \hline 0 & 0 & 0 & 1 \\ \hline \end{array} \qquad M_2 = \begin{array}{|c|c|c|c|} \hline 1 & 0 & 0 & 0 \\ \hline 0 & 0 & 1 & 0 \\ \hline 0 & 0 & 0 & 1 \\ \hline 0 & 1 & 0 & 0 \\ \hline \end{array}$$

$$F_1 = \begin{array}{|c|c|c|c|} \hline 1 & 0 & 1 & 0 \\ \hline 0 & 1 & 0 & 1 \\ \hline 0 & 1 & 0 & 1 \\ \hline 1 & 0 & 1 & 0 \\ \hline \end{array} \quad F_2 = \begin{array}{|c|c|c|c|} \hline 1 & 0 & 0 & 1 \\ \hline 1 & 0 & 0 & 1 \\ \hline 0 & 1 & 1 & 0 \\ \hline 0 & 1 & 1 & 0 \\ \hline \end{array} \quad F_3 = \begin{array}{|c|c|c|c|} \hline 1 & 1 & 0 & 0 \\ \hline 0 & 0 & 1 & 1 \\ \hline 1 & 1 & 0 & 0 \\ \hline 0 & 0 & 1 & 1 \\ \hline \end{array}.$$

Define the functions f_1 on L_1, $f_1 : \{1,2,3,4\} \to \{0,1\}$ as
$f_1(1) = 1,\ f_1(2) = 0,\ f_1(3) = 0,\ f_1(4) = 0,$
and f_2 on L_2, $f_2 : \{1,2,3,4\} \to \{0,1\}$ as
$f_2(1) = 1,\ f_2(2) = 0,\ f_2(3) = 0,\ f_2(4) = 0.$ Then,

$$M_1 = f_1(L_1) = \begin{array}{|c|c|c|c|} \hline 0 & 0 & 0 & 1 \\ \hline 0 & 0 & 1 & 0 \\ \hline 0 & 1 & 0 & 0 \\ \hline 1 & 0 & 0 & 0 \\ \hline \end{array}.$$

$$M_2 = f_2(L_3) = \begin{array}{|c|c|c|c|}\hline 0 & 0 & 0 & 1 \\ \hline 1 & 0 & 0 & 0 \\ \hline 0 & 0 & 1 & 0 \\ \hline 0 & 1 & 0 & 0 \\ \hline \end{array}.$$

Similarly, define functions f_3 on L_3, $f_3 : \{1, 2, 3, 4\} \to \{0, 1\}$ as

$f_3(1) = 1,\ f_2(2) = 0,\ f_2(3) = 1,\ f_2(4) = 0,$

f_4 on L_3, $f_4 : \{1, 2, 3, 4\} \to \{0, 1\}$ as

$f_4(1) = 1,\ f_4(2) = 0,\ f_2(3) = 0,\ f_2(4) = 1,$

and f_5 on L_3, $f_5 : \{1, 2, 3, 4\} \to \{0, 1\}$ as

$f_5(1) = 1,\ f_5(2) = 1,\ f_5(3) = 0,\ f_5(4) = 0.$ Then,

$$F_3 = f_3(L_2) = \begin{array}{|c|c|c|c|}\hline 0 & 1 & 0 & 1 \\ \hline 0 & 1 & 0 & 1 \\ \hline 1 & 0 & 1 & 0 \\ \hline 1 & 0 & 1 & 0 \\ \hline \end{array}$$

$$F_4 = f_4(L_2) = \begin{array}{|c|c|c|c|}\hline 0 & 0 & 1 & 1 \\ \hline 1 & 1 & 0 & 0 \\ \hline 1 & 1 & 0 & 0 \\ \hline 0 & 0 & 1 & 1 \\ \hline \end{array}$$

$$F_5 = f_5(L_2) = \begin{array}{|c|c|c|c|}\hline 0 & 1 & 1 & 0 \\ \hline 1 & 0 & 0 & 1 \\ \hline 0 & 1 & 1 & 0 \\ \hline 1 & 0 & 0 & 1 \\ \hline \end{array}.$$

Hence, $\{2 \times F(4;3,1); 3 \times F(4;2,2)\}$ is a set of MMOFS by Lemma 4.1.

Now, the inequality from Theorem 1.12 implies:

$$2+2+2+2+2-5 \leq (4-1)^2.$$

$$\Leftrightarrow 5 \leq 9,$$

This set is not complete but it is a type maximal of the combination of types $F(4;3,1)$ and $F(4;2,2)$ by Lemma 5.4. We may consider adding frequency squares of other types of order 4 to obtain a maximal set of MMOFS.

Example 25. *Consider the set* $\{1 \times F(4;2,1,1); 6 \times F(4;2,2)\}$ *as follows.*

$$M = \begin{array}{|c|c|c|c|}\hline 0 & 0 & 1 & 2 \\ \hline 1 & 2 & 0 & 0 \\ \hline 0 & 0 & 2 & 1 \\ \hline 2 & 1 & 0 & 0 \\ \hline \end{array} \quad F_1 = \begin{array}{|c|c|c|c|}\hline 0 & 1 & 1 & 0 \\ \hline 0 & 1 & 1 & 0 \\ \hline 1 & 0 & 0 & 1 \\ \hline 1 & 0 & 0 & 1 \\ \hline \end{array} \quad F_2 = \begin{array}{|c|c|c|c|}\hline 0 & 0 & 1 & 1 \\ \hline 0 & 0 & 1 & 1 \\ \hline 1 & 1 & 0 & 0 \\ \hline 1 & 1 & 0 & 0 \\ \hline \end{array} \quad F_3 = \begin{array}{|c|c|c|c|}\hline 0 & 0 & 1 & 1 \\ \hline 1 & 1 & 0 & 0 \\ \hline 1 & 1 & 0 & 0 \\ \hline 0 & 0 & 1 & 1 \\ \hline \end{array}$$

$$F_4 = \begin{array}{|c|c|c|c|}\hline 0 & 1 & 0 & 1 \\ \hline 1 & 0 & 1 & 0 \\ \hline 0 & 1 & 0 & 1 \\ \hline 1 & 0 & 1 & 0 \\ \hline \end{array} \quad F_5 = \begin{array}{|c|c|c|c|}\hline 0 & 1 & 0 & 1 \\ \hline 1 & 0 & 1 & 0 \\ \hline 1 & 0 & 1 & 0 \\ \hline 0 & 1 & 0 & 1 \\ \hline \end{array} \quad F_6 = \begin{array}{|c|c|c|c|}\hline 0 & 1 & 1 & 0 \\ \hline 1 & 0 & 0 & 1 \\ \hline 0 & 1 & 1 & 0 \\ \hline 1 & 0 & 0 & 1 \\ \hline \end{array}.$$

Let's define function functions f on L_3, $f : \{1,2,3,4\} \to \{0,1,2\}$ as $f(1) = 0$, $f(2) = 0$, $f(3) = 1$, $f(4) = 2$,

f_1 on L_3, $f_1 : \{1,2,3,4\} \to \{0,1\}$ as $f_1(1) = 0$, $f_1(2) = 1$, $f_1(3) = 1$, $f_1(4) = 0$,

f_2 on L_1, $f_2 : \{1,2,3,4\} \to \{0,1\}$ as $f_2(1) = 0$, $f_2(2) = 0$, $f_2(3) = 1$, $f_2(4) = 1$,

f_3 on L_2, $f_3 : \{1,2,3,4\} \to \{0,1\}$ as $f_2(1) = 0$, $f_2(2) = 0$, $f_2(3) = 1$, $f_2(4) = 1$,

f_4 *on* L_1, $f_4 : \{1,2,3,4\} \to \{0,1\}$ *as* $f_4(1) = 0,\ f_4(2) = 1,\ f_4(3) = 0,\ f_4(4) = 1,$

f_5 *on* L_3, $f_5 : \{1,2,3,4\} \to \{0,1\}$ *as* $f_5(1) = 0,\ f_5(2) = 1,\ f_5(3) = 0,\ f_5(4) = 1,$

f_6 *on* L_2, $f_6 : \{1,2,3,4\} \to \{0,1\}$ *as* $f_6(1) = 0,\ f_6(2) = 1,\ f_6(3) = 1,\ f_6(4) = 0.$

Therefore, $f(L_3) = M,\ f(L_3) = F_1,\ f_2(L_1) = F_2,\ f_3(L_2) = F_3,\ f_4(L_1) = F_4,\ f_5(L_3) = F_5,$ *and* $f_6(L_2) = F_6$.

Hence, by Lemma 4.1, the set $\{1 \times F(4;2,1,1); 6 \times F(4;2,2)\}$ is a set of MMOFS, and satisfies the bound as follows.

$$3 + 2 + 2 + 2 + 2 + 2 + 2 - 7 \leq (4-1)^2.$$

$$\Leftrightarrow 8 \leq 9.$$

Although, the set $\{1 \times F(4;2,1,1); 6 \times F(4;2,2)\}$ satisfies the bound in Theorem 1.12 more closely, it remains incomplete. Future studies could investigate the addition of frequency squares of types $F(4;2,1,1)$ and $F(4;2,2)$ to determine whether the set is a type maximal. Moreover, we could try adding frequency squares of other types of order 4 to investigate a maximal set of MMOFS.

Example 26. *Consider the following set* $\{2 \times F(4;2,1,1); 3 \times F(4;2,2)\}$,

$$M_1 = \begin{array}{|c|c|c|c|} \hline 0 & 0 & 1 & 2 \\ \hline 1 & 2 & 0 & 0 \\ \hline 0 & 0 & 2 & 1 \\ \hline 2 & 1 & 0 & 0 \\ \hline \end{array} \qquad M_2 = \begin{array}{|c|c|c|c|} \hline 0 & 1 & 2 & 0 \\ \hline 2 & 0 & 0 & 1 \\ \hline 0 & 2 & 1 & 0 \\ \hline 1 & 0 & 0 & 2 \\ \hline \end{array}$$

$$F_1 = \begin{array}{|c|c|c|c|}\hline 0 & 1 & 0 & 1 \\ \hline 1 & 0 & 1 & 0 \\ \hline 1 & 0 & 1 & 0 \\ \hline 0 & 1 & 0 & 1 \\ \hline \end{array} \qquad F_2 = \begin{array}{|c|c|c|c|}\hline 0 & 0 & 1 & 1 \\ \hline 0 & 0 & 1 & 1 \\ \hline 1 & 1 & 0 & 0 \\ \hline 1 & 1 & 0 & 0 \\ \hline \end{array} \qquad F_3 = \begin{array}{|c|c|c|c|}\hline 0 & 1 & 1 & 0 \\ \hline 0 & 1 & 1 & 0 \\ \hline 1 & 0 & 0 & 1 \\ \hline 1 & 0 & 0 & 1 \\ \hline \end{array}.$$

Define the functions f_1 *on* L_3, $f_1 : \{1,2,3,4\} \to \{0,1,2\}$ *as* $f_1(1) = 0$, $f_1(2) = 0$, $f_1(3) = 1$, $f_1(4) = 2$,

f_2 *on* L_2, $f_2 : \{1,2,3,4\} \to \{0,1,2\}$ *as* $f_2(1) = 0$, $f_2(2) = 1$, $f_2(3) = 2$, $f_2(4) = 0$,

f_3 *on* L_3, $f_3 : \{1,2,3,4\} \to \{0,1\}$ *as* $f_3(1) = 0$, $f_2(2) = 1$, $f_2(3) = 0$, $f_2(4) = 1$,

f_4 *on* L_1, $f_4 : \{1,2,3,4\} \to \{0,1\}$ *as* $f_4(1) = 0$, $f_4(2) = 0$, $f_2(3) = 1$, $f_2(4) = 1$, *and*

f_5 *on* L_3, $f_5 : \{1,2,3,4\} \to \{0,1\}$ *as* $f_5(1) = 0$, $f_5(2) = 1$, $f_5(3) = 1$, $f_5(4) = 0$.

Then, $f_1(L_3) = M_1$, $f_2(L_2) = M_2$, $f_3(L_3) = F_1$, $f_4(L_1) = F_2$, $f_5(L_3) = F_3$. *Therefore,* $\{1 \times F(4;2,1,1); 6 \times F(4;2,2)\}$ *is a set of MMOFS by Lemma* 4.1.

$$3 + 3 + 2 + 2 + 2 - 5 \leq (4-1)^2$$

$$\Leftrightarrow \quad 7 \leq 9.$$

Example 27. *Consider the following set* $\{2 \times F(4;2,1,1); 1 \times F(4;3,1)\}$,

$$M_1 = \begin{array}{|c|c|c|c|}\hline 0 & 0 & 1 & 2 \\ \hline 0 & 0 & 2 & 1 \\ \hline 1 & 2 & 0 & 0 \\ \hline 2 & 1 & 0 & 0 \\ \hline \end{array} \qquad M_2 = \begin{array}{|c|c|c|c|}\hline 0 & 1 & 0 & 2 \\ \hline 2 & 0 & 1 & 0 \\ \hline 1 & 0 & 2 & 0 \\ \hline 0 & 2 & 0 & 1 \\ \hline \end{array} \qquad M_3 = \begin{array}{|c|c|c|c|}\hline 0 & 0 & 0 & 1 \\ \hline 0 & 1 & 0 & 0 \\ \hline 1 & 0 & 0 & 0 \\ \hline 0 & 0 & 1 & 0 \\ \hline \end{array}.$$

Define the functions f_1 *on* L_1, $f_1 : \{1,2,3,4\} \to \{0,1,2\}$ *as* $f_1(1) = 0$, $f_1(2) = 0$, $f_1(3) = 1$, $f_1(4) = 2$,

f_2 *on* L_3, $f_2 : \{1,2,3,4\} \to \{0,1,2\}$ *as* $f_2(1) = 0$, $f_2(2) = 1$, $f_2(3) = 0$, $f_2(4) = 2$,

f_3 on L_2, $f_3 : \{1,2,3,4\} \to \{0,1\}$ as $f_3(1) = 0$, $f_2(2) = 0$, $f_2(3) = 0$, $f_2(4) = 1$,

Then, $f_1(L_1) = M_1$, $f_2(L_3) = M_2$, $f_3(L_2) = M_3$.

Hence, by Lemma 4.1, the set $\{2 \times F(4;2,1,1); 1 \times F(4;3,1)\}$ is a set of MMOFS, and satisfies the bound as follows.

$$3 + 3 + 2 + -3 \leq (4-1)^2.$$

$$\Leftrightarrow 5 \leq 9.$$

This set is clearly not complete, and it is not yet verified whether it is maximal or type maximal. Future research could involve adding more frequency squares of the same order, including both the same type and of different types, in order to examine whether the size of the set can be extended.

6 Conclusion

In this conclusion, we focus on several open problems and potential directions for future research in mixed mutually orthogonal frequency squares (MMOFS). The constructions and results presented in this thesis lay a strong foundation for understanding the structure of MMOFS. However, many important questions remain unanswered in the construction of MMOFS, particularly regarding their completeness and optimality.

If more time was available, the next steps would involve a more detailed investigation into mixed orthogonal arrays and functional mappings as tools to generate MMOFS. Theorem 4.7 in Chapter 4 shows that mixed orthogonal arrays can be used to construct sets of MMOFS by using the mappings on Latin squares.

Now, how do the variations in the array parameters influence the structure and orthogonality of MMOFS, and lead to new constructions? Additionally, it is important to explore whether optimal mixed orthogonal arrays always produce complete sets of MMOFS. Computational and theoretical studies could provide insight into these questions and help extend the range of known MMOFS.

Moreover, we can extend the study on MMOFS of small orders, building on Hadamard matrices and finite field construction in Chapter 2 and equidistant permutation arrays (EPAs) in Chapter 3. In Chapter 5, we investigated sets of MMOFS of small orders. We were able to identify a complete set as well as a type-maximal set. However, for several sets, it remains unclear whether they are maximal or type maximal. Thus there is further work to be done.

References

[1] B. Arazi. “Some properties of Hadamard matrices generated recursively by Kronecker products”. In: *Linear Algebra and its Applications* 25 (1979), pp. 27–39.

[2] L. Baumert, S. W. Golomb, and M. Hall Jr. “Discovery of an Hadamard matrix of order 92”. In: (1962).

[3] C. Bodkin and I. M. Wanless. “Mutually orthogonal binary frequency squares of mixed type”. In: *International Conference on New Advances in Designs, Codes and Cryptography*. Springer. 2022, pp. 1–23.

[4] T. Britz, N. J. Cavenagh, A. Mammoliti, and I. M. Wanless. “Mutually orthogonal binary frequency squares”. In: *The Electronic Journal of Combinatorics* 27 (2020), P3.7.

[5] C. J. Colbourn. *CRC handbook of combinatorial designs*. CRC Press, 2010.

[6] D. Z. Djokovic, O. Golubitsky, and I. S. Kotsireas. “Some new orders of Hadamard and Skew-Hadamard matrices”. In: *Journal of Combinatorial Designs* 22.6 (2014), pp. 270–277.

[7] J. Hadamard. “Résolution d’une question relative aux déterminants”. In: *Bull. des sciences math.* 2 (1893), pp. 240–246.

[8] R. A. Horn and C. R. Johnson. *Matrix analysis*. Cambridge University Press, 2012.

[9] J. Jedwab and T. Popatia. “A new representation of mutually orthogonal frequency squares”. In: *J. Combin. Math. Combin. Comput.* 116 (2020), pp. 287–301.

[10] H. Kharaghani and B. Tayfeh-Rezaie. “A Hadamard matrix of order 428”. In: *Journal of Combinatorial Designs* 13.6 (2005), pp. 435–440.

[11] W. de Launey. “On the asymptotic existence of partial complex Hadamard matrices and related combinatorial objects”. In: *Discrete applied mathematics* 102.1-2 (2000), pp. 37–45.

[12] C. D. Lin and J. Stufken. “Orthogonal Arrays: A Review”. In: *Wiley Interdisciplinary Reviews: Computational Statistics* 17.2 (2025).

[13] C. C. Lindner and C. A. Rodger. *Design theory.* Chapman and Hall/CRC, 2017.

[14] J. Morris. *Combinatorics: Enumeration, Graph Theory, and Design Theory.* 2017.

[15] G. L. Mullen and D. Panario. *Handbook of finite fields.* Vol. 17. CRC Press Boca Raton, 2013.

[16] R. E. Paley. “On orthogonal matrices”. In: *Journal of Mathematics and Physics* 12.1-4 (1933), pp. 311–320.

[17] K. H. Rosen. *Discrete Mathematics and Its Applications, 7^{th}Edition.* 2019.

[18] N. Sloane, R. Hardin, and W. Smith. “Neil ja sloane: Home page”. In: *Online: http://neilsloane.com* (2000).

[19] J. J. Sylvester. “LX. Thoughts on inverse orthogonal matrices, simultaneous signsuccessions, and tessellated pavements in two or more colours, with applications to Newton’s rule, ornamental tile-work, and the theory of numbers”. In: *The London, Edinburgh, and Dublin Philosophical Magazine and Journal of Science* 34.232 (1867), pp. 461–475.