R-separation of variables for the time-dependent Hamilton–Jacobi and Schrödinger equations

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The theory of R-separation of variables is developed for the time-dependent Hamilton–Jacobi and Schrödinger equations on a Riemannian manifold V^n where time-dependent vector and scalar potentials are permitted. As an application it is shown how to obtain all R-separable coordinates for the n-sphere and Euclidean n-space.

I. INTRODUCTION AND TECHNICAL CONSIDERATIONS

In the study of additive separation or R-separation of variables for Hamilton–Jacobi equations on pseudo-Riemannian manifolds one naturally distinguishes three types of equations:

$$\text{(I)}\quad \sum_{l,m} g^{lm} W_{x^l} W_{x^m} = E,$$

$$\text{(II)}\quad \sum_{l,m} g^{lm} W_{x^l} W_{x^m} + 2\lambda W_t = 0,$$

$$\text{(III)}\quad \sum_{l,m} g^{lm} W_{x^l} W_{x^m} = 0.$$

Here (g^{lm}) is the contravariant metric tensor with respect to the coordinate system $\{x^i\}$ on a Riemannian or pseudo-Riemannian manifold and E, λ are nonzero parameters. (We can also add vector and scalar potentials to the left-hand sides of each of these equations, since this is only a minor complication from the viewpoint of variable separation.) See Refs. 1 and 2 for discussions of the relevance of these equations to classical mechanics. Although (I) can be considered as a special case of (II), and (II) as a special case of (III) (in a space of two more dimensions), the three types of equations exhibit distinct forms of behavior. In particular, (II) has proved much more difficult to analyze from the viewpoint of variable separation than have (I) and (III).

For a given Hamilton–Jacobi equation, variable separation research has typically divided into three categories: (a) explicit determination of separable systems and application of these results to derive explicit solutions of the equation; (b) intrinsic, i.e., coordinate-free, characterizations of separable coordinate systems and their relation to completely integrable Hamiltonian systems; and (c) studies of the "quantization problem," the relationship between additively separable solutions of the Hamilton–Jacobi equation and multiplicatively R-separable solutions of the associated Schrödinger equation,

$$\text{(I}'\text{)}\quad \Delta\psi = E\psi,$$

$$\text{(II}'\text{)}\quad \Delta\psi + 2\lambda i\psi_t = 0,$$

$$\text{(III}'\text{)}\quad \Delta\psi = 0,$$

where Δ is the Laplace–Beltrami operator on the pseudo-Riemannian manifold.

For equations of types (I) and (III) considerable recent progress has been made in all three of the preceding categories. (See Refs. 3 and 4 for reviews of this work.) The present paper is a contribution to category (a) for equations of type (II). Shapovalov has already announced the solution to the category (b) problem for all these equations, see Ref. 3.

In the latter half of this section we point out the sense in which (II) (with added time-dependent vector and scalar potentials) is a special case of (III) and use this connection to work out the technical conditions for a coordinate system to be R separable for (II). We show that corresponding to each R-separable coordinate system $\{y^i\}$ for (II) on a Riemannian manifold V^n there is associated a unique "time" coordinate y^1 and that the transformed equation in these new coordinates is again in Hamilton–Jacobi form (II) on the same manifold V^n. The transformed Hamiltonian and potential may, however, depend on the new time coordinate y^1. If there is no dependence of the Hamiltonian and potential on y^1 (the *regular* case) then we can use Lie theoretic methods to analyze such coordinates.[5–7]

In Sec. II we turn to the principal topic of this paper, the case where the transformed Hamiltonian $\mathscr{H}(y^1)$ is strictly y^1 dependent. We determine all such time-dependent Hamiltonians for the n-sphere S^n and Euclidean n-space E^n, and in Sec. III we show how to compute all of the associated R-separable coordinate systems for II (with added time-dependent potentials) on these manifolds.

The solution of the regular case for S^n and E^n is taken up in Sec. IV. In Sec. V it is shown that all our results extend to the time-dependent Schrödinger equations on S^n and E^n. Finally, in Sec. VI we give an intrinsic characterization of those equations of type (III) for which coordinates $\{t,x^i\}$ can be chosen such that (III) restricts to (II). All functions appearing in this paper are assumed to be locally analytic.

Technically our task is to analyze the possible R-separable solutions for the time-dependent Hamilton–Jacobi equation

$$2\lambda W_t + \sum_{l,m=1}^{n} g^{lm}(\mathbf{x}) W_{x^l} W_{x^m} + 2\lambda \sum_{l=1}^{n} A^l(\mathbf{x},t) W_{x^l} + \lambda^2 V(\mathbf{x},t) = 0, \quad g^{lm} = g^{ml}. \tag{1.1}$$

Here λ is a parameter, $\{x^i\}$ is a local coordinate system, and $g^{lm}(\mathbf{x})$ the contravariant metric tensor on the Riemannian

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manifold V^n. In particular, the matrix $\{g^{lm}\}$ is positive definite. A solution of (1.1) is a function $W=W(t,x)$ that satisfies this equation.

We must first state precisely which transformations will be permitted in the search for separable solutions. We will do this by considering (1.1) as a special case of the (conformal) Hamilton–Jacobi equation

$$\sum_{u,v=1}^{n+2} K^{uv}(z) Z_{z^u} Z_{z^v} = 0, \tag{1.2}$$

where

$$z^i = x^i,\quad i=1,\dots,n,\quad z^{n+1}=t,\quad z^{n+2}=\tau,$$
$$Z=\lambda\tau + W,\quad K^{i,n+2}=K^{n+2,i}=A^i,$$
$$K^{ij}=g^{ij},\quad 1\leqslant i,j\leqslant n,$$
$$K^{n+1,n+2}=K^{n+2,n+1}=1,\quad K^{n+2,n+2}=V,$$

and all other matrix elements of K^{uv} vanish. Thus, the solutions of (1.1) can be identified with those solutions Z of (1.2) for which $Z_\tau=\lambda$ (after which we set $\tau=0$).

The general theory of variable separation for the Hamilton–Jacobi equation (1.2) (and its relation to Lie symmetries) is well understood[8,9] and we need only modify this theory to the special requirement $Z_\tau=\lambda$. In the following paragraphs we present the modification.

We pass to separable coordinates $y^1,\dots,y^{n+1}$, μ, where

$$x^k = x^k(\mathbf{y}),\quad k=1,\dots,n,$$
$$t=t(\mathbf{y}),\quad \tau=\mu-R(\mathbf{y}), \tag{1.3}$$

and R is a function to be determined. Then (1.2) transforms to

$$2Z_\mu([\xi^i+\mathscr{A}^i]Z_i + G^{ij}R_iZ_j) + G^{ij}Z_iZ_j$$
$$+(2[\xi^i+\mathscr{A}^i]R_i + V + G^{ij}R_iR_j)Z_\mu Z_\mu = 0, \tag{1.4}$$

where we observe the Einstein summation convention, the variables i,j take the values $1,2,\dots,n+1$; $Z_i=Z_{y_i}$, $R_i=R_{y_i}$, and $A^l(\partial y^i/\partial x^l)=\mathscr{A}^i$. Note that

$$\sum_{l,m=1}^{n} g^{lm}(\mathbf{x}) W_{x^l} W_{x^m} = G^{ij}(\mathbf{y}) Z_i Z_j,\quad Z_\mu=\lambda. \tag{1.5}$$

The separable coordinates $\mathbf{y},\mu$ are of three types: there are n_1 first kind variables y^a, n_2 second kind variables y^r, and n_3 ignorable variables y^α and μ; $n_1+n_2+n_3=n+2$. The contravariant metric tensor for Eq. (1.4), expressed in these coordinates, must be of the form

$$\begin{array}{c|cccc} & n_1 & n_2 & n_3-1 & \mu \\ \hline n_1 & QH_a^{-2}\delta^{ab} & 0 & 0 & 0 \\ n_2 & 0 & 0 & Qf_r^\beta(y^r)H_r^{-2} & Qk_r(y^r)H_r^{-2} \\ n_3-1 & 0 & Qf_s^\alpha(y^s)H_s^{-2} & Q\sum_i K_i^{\alpha\beta}(y^i)H_i^{-2} & Q\sum_i F_i^\alpha(y^i)H_i^{-2} \\ \mu & 0 & Qk_s(y^s)H_s^{-2} & Q\sum_i F_i^\beta(y^i)H_i^{-2} & Q\sum_i \mathscr{F}_i(y^i)H_i^{-2} \end{array} . \tag{1.6}$$

Here there is no summation on n repeated indices unless explicitly indicated, and the indicated sums are $i=1,\dots,n_1+n_2$. (This expression follows immediately from Theorem 5 of Ref. 9.) The metric

$$ds'^2 = \sum_{a=1}^{n_1} H_a^2(y^b,y^s)(dy^a)^2 + \sum_{r=n_1+1}^{n_1+n_2} H_r^2(y^b,y^s)(dy^r)^2 \tag{1.7}$$

must be in Stäckel form. The matrix elements are independent of the ignorable variables y^α,μ.

Comparing (1.4) with (1.6) we have the following conditions:

$$G^{ab}=QH_a^{-2}\delta^{ab},\quad G^{rs}=G^{ar}=0,$$
$$G^{\alpha\beta}=Q\sum_i K_i^{\alpha\beta}H_i^{-2},\quad G^{a\alpha}=0, \tag{1.8}$$
$$G^{r\alpha}=Qf_r^\alpha H_r^{-2},$$

and

$$\xi^a+\mathscr{A}^a+QH_a^{-2}R_a=0,$$
$$\xi^r+\mathscr{A}^r+QH_r^{-2}\sum_\beta f_r^\beta R_\beta = Qk_rH_r^{-2},$$
$$\xi^\alpha+\mathscr{A}^\alpha+Q\sum_{\beta,i}K_i^{\alpha\beta}H_i^{-2}R_\beta + Q\sum_s f_s^\alpha H_s^{-2}R_s = Q\sum_i F_i^\alpha H_i^{-2}, \tag{1.9}$$
$$2\sum_i[\xi^i+\mathscr{A}^i]R_i + V + \sum_{i,j}G^{ij}R_iR_j = Q\sum_i \mathscr{F}_i H_i^{-2}.$$

Here $Q=Q(\mathbf{y})\neq 0$ and each of f_r^β, k_r, $K_i^{\alpha\beta}$, F_i^α, $\mathscr{F}_i$ depend only on the variable denoted by the subscript. Finally, we have that ξ^ip_i is a Killing vector for the Hamiltonian $G^{ij}p_ip_j$, i.e.,

$$\{\xi^ip_i, G^{ij}p_ip_j\}=0, \tag{1.10}$$

where $\{\cdot,\cdot\}$ is the Poisson bracket in the canonical coordinates $y^i,\mu;p_i,p_\mu$, and there is a closed one-form $dt=df=f_i\,dy^i$ such that

$$f_i\xi^i=1,\quad f_iG^{ij}=0,\quad j=1,\dots,n+1. \tag{1.11}$$

Conditions (1.7)–(1.11) are necessary and sufficient for R-separation of (1.1) in the coordinates y^i. The R-separable solutions take the form

$$W = +\lambda R(\mathbf{y}) + \sum_{i=1}^{n+1} W^{(i)}(y^i), \tag{1.12}$$

where R satisfies conditions (1.9). (It is the presence of a possibly nontrivial R which leads to the term *R-separable*; if $\partial_{y^iy^j}R = 0$, $i \neq j$, the system $\{y^k\}$ is *separable*.)

We can simplify our problem somewhat by noting from (1.5) and (1.8) and the requirement (g^{lm}) positive definite that $n_2 \leqslant 1$.

Theorem 1: If the time-dependent Hamilton–Jacobi equation (1.1) is R-separable in the coordinates $\{y^i\}$ then the transformed equation (1.4) can be put in time-dependent Hamilton–Jacobi form (with potential)

$$2\lambda Z_f + \sum_{i,j\neq f} G^{ij}Z_iZ_j + \lambda \sum_{\alpha \neq f} U^\alpha Z_\alpha + \lambda^2 U = 0, \tag{1.13}$$

where $df = dt$. There are two possibilities:

Case 1: $n_2 = 0$. Then $f = y^\delta$ is ignorable and $\partial_f G^{ij} = \partial_f U^\alpha = \partial_f U = 0$. The metric (G^{ij}), $i,j \neq \delta$ determines the *same* Riemannian space V^n as does (g^{lm}).

Case 2: $n_2 = 1$. Then $f = y^s$ is a second kind coordinate and at least one of G^{ij}, U^α, U has nontrivial f dependence. For each fixed value of y^s the metric $(G^{ij}(y^s))$, $i,j \neq s$, determines the *same* Riemannian space V^n as does (g^{lm}).

Proof: Suppose (1.1) is R-separable in the coordinates $\{y^i\}$. Then from (1.11) there is a function f such that $df = dt = f_i\, dy^i$, where $f_i \xi^i = 1$ and

$$f_i G^{ij} = 0, \quad j = 1,...,n+1. \tag{1.14}$$

It follows immediately from (1.8) and (1.14) that $f_a = 0$ for $a = 1,...,n_1$. Furthermore, since $\mathrm{Rank}(G^{ij}) = n$ and $\partial_{y^\beta} G^{ij} = 0$ for each ignorable variable y^β, we must have $\partial_{y^\beta}(f_i/f_j) = 0$ whenever $f_j \neq 0$. It follows that f must be of the form $f = h(u,v)$ where

$$u = \sum_\alpha C_\alpha(y^r)y^\alpha, \quad v = y^r.$$

(If $n_2 = 1$ then f, C_α may depend on the single second kind coordinate y^r; if $n_2 = 0$ then the C_α must be constants.)

Now suppose $n_2 = 1$ and $\partial_u h = 0$. Then $f = f(y^r)$ and from the requirement $f_i \xi^i = f_r \xi^r = 1$ we see that $\partial_{y^i}\xi^r = 0$ for $i \neq r$. Thus, by a change of second kind coordinate $y^{r'} = k(y^r)$ if necessary (which preserves separation), we can assume $\xi^r \equiv 1$ and $\mathscr{A}^r = 0$. Further, the condition $f_r G^{rj} = 0$ implies $G^{rj} = 0$ for $j = 1,...,n+1$, so the $n \times n$ matrix (G^{ij}), $i,j \neq r$ is nonsingular. Thus Eq. (1.4) takes the form (1.13) and since $f = y^r$ is not ignorable, at least one of G^{ij}, U^α, U has nontrivial f dependence.

Next, suppose $n_2 = 1$ and $\partial_u h \neq 0$ where at least one of the C_α is nonzero. Without loss of generality we can assume $C_\gamma = 1$ for fixed ignorable variable y^γ. Then

$$\partial_{y^\beta}\left(\frac{f_r}{f_\gamma}\right) = \partial_{y^\beta}\left(\frac{h_u C'_\delta y^\delta + h_v}{h_u}\right)$$

$$= C'_\beta + C_\beta\, \partial_u\left(\frac{h_v}{h_u}\right) = 0$$

since f_r/f_γ is independent of each ignorable variable y^β. Setting $\beta = \gamma$ we have $\partial_u(h_v/h_u) = 0$. Thus $C'_\beta \equiv dC_\beta/dy^r = 0$ for each β and the C_β are constants. Further, we have $f = h(u,v) = H(u + K(v))$ for some function K. We can now pass to a new set $\{y^{i'}\}$ of equivalent R-separable coordinates such that $y^{\delta'} = C_\beta y^\beta + K(y^r)$. (See Ref. 10 for a discussion of the pseudogroup of transformations taking separable coordinates into equivalent systems of separable coordinates.) Dropping the primes, we have $f = f(y^\delta)$ and, using (1.11), $\xi^\delta = \xi^\delta(y^\delta) \neq 0$, $G^{\delta i} = 0$, $i = 1,...,n+1$ and $\mathscr{A}^\delta = 0$. From the third equation of (1.9) we have

$$\xi^\delta = Q \sum_{i=1}^{n_1+n_2} F_i^\delta(y^i) H_i^{-2} = QF \neq 0.$$

Since F is a Stäckel multiplier (see Ref. 9) we can pass to an equivalent Stäckel form $\widetilde{H}_i^2 = FH_i^2$ so that $\widetilde{Q} = \xi^\delta$. Dividing (1.4) by the common factor $\xi^\delta = \widetilde{Q}$ [see (1.8) and (1.9)] we obtain (1.13), where each term is independent of $f = y^\delta$.

Finally, suppose $n_2 = 0$, so $f = h(u)$ with $\delta_u h \neq 0$. Then a simplification of the argument in the preceding paragraph shows that we can take $f = y^\delta$ and obtain (1.13), where each term is independent of y^δ. Q.E.D.

We have shown that corresponding to each R-separable coordinate system $\{y^j\}$ for the time-dependent Hamilton–Jacobi equation on a Riemannian manifold V^n there is associated a unique time coordinate $f = y^\delta$ or $f = y^r$. The transformed equation in the $\{y^i\}$ coordinates is again in time-dependent Hamilton–Jacobi form for a Hamiltonian on V^n. The transformed Hamiltonian is strictly time dependent if and only if $f = y^r$.

In the following we will regard the problem of finding all *R-separable* solutions of a given time-dependent equation (1.1) as solved once we reduce it to the problem of finding all *separable* solutions of explicit time-independent Hamilton–Jacobi equations of the form

$$\sum_{i,j=1}^{n} G^{ij}Z_iZ_j + \lambda \sum_{i=1}^{n} U^iZ_i + \lambda^2 U = E, \tag{1.15}$$

where (G^{ij}) is the metric on V^n.

For $f = y^\delta$ this problem was solved in Ref. 7. There we studied all mappings of the form

$$t = T(f,\mathbf{y}), \quad \mathbf{x} = X(f,\mathbf{y}), \quad W = Z + \lambda h(f,\mathbf{y}) \tag{1.16}$$

that take (1.1) into another evolution equation (1.13), a "related" evolution equation. It was shown that there is a one-to-one correspondence between (equivalence classes of) related Hamilton–Jacobi equations and conformal symmetries for (1.1) of the form $\mathscr{L} = q(t,\mathbf{x})p_t + \gamma^j(t,\mathbf{x})p_{x^j} + \lambda k(t,\mathbf{x})$, where $q \neq 0$; alternatively,

$$L = q\,\partial_t + \gamma^j\,\partial_{x^j} + \lambda k\,\partial_W. \tag{1.17}$$

If L is a conformal symmetry with $q \neq 0$ then one can show that $\partial_{x^j} q = 0$ and that we can introduce new coordinates $f,\mathbf{y}$ and a new dependent variable z such that

$$\begin{aligned} &\partial_f = q\,\partial_t + \gamma^j\,\partial_{x^j}, \\ &t = T(f), \quad x^i = X^i(f,\mathbf{y}), \\ &W = Z + \lambda h(f,\mathbf{y}), \end{aligned} \tag{1.18}$$

and (1.1) transforms to the related Hamilton–Jacobi equation (1.13) with f-independent Hamiltonian.

Conversely, if case 1 of Theorem 1 occurs for (1.1) then the variable $f=y^\delta$ is ignorable. This means that the operator ∂_{y_δ} is a conformal symmetry for (1.13) which in turn transforms to a conformal symmetry for (1.1) of the form $q\partial_t + \gamma^j\partial_{x^j} + \lambda k\,\partial_w$ with $q\neq 0$.

Note that when f is ignorable we can require $Z_f = -E/2\lambda$ and reduce (1.13) to the time-independent equation (1.15). Thus, to find all R-separable coordinate systems $\{f,\mathbf{y}\}$ for (1.1) corresponding to case 1 we first enumerate the conjugacy classes of symmetry operators in the conformal symmetry algebra $\mathscr{G}$ for (1.1).[11] Choosing a representative operator L in each conjugacy class we make the transformation (1.18) of (1.1) into a related evolution equation and then find all R-separable systems $\{y^j\}$ for the reduced equation (1.15). (Of course, for some choices of L the reduced equation is not R-separable in any coordinate system.) See Ref. 7 for more information concerning this procedure, and Ref. 4 for a more general point of view.

We can thus regard case 1 of the preceding theorem as well understood from the viewpoint of Lie symmetries.

II. TIME-DEPENDENT HAMILTONIANS

We now turn our attention to case 2 of Theorem 1, the case where $f=y^r$ and the transformed equation (1.13) has a time-dependent Hamiltonian. Our aim, not entirely achieved, will be to enumerate the instances where this type of R-separation occurs for the Hamilton–Jacobi equation (1.1).

For simplicity we will limit ourselves to coordinate systems that are orthogonal on V^n. In other words, the metric tensor (G^{ij}), $i,j\neq f$, in (1.13) should be diagonal. (For many spaces, such as Euclidean spaces or spaces of constant curvature, only orthogonal separation can occur, so this is no restriction at all.[12]) Since orthogonal ignorable coordinates can always be considered as special cases of type 1 coordinates, without loss of generality we can assume that the separable coordinates are labeled y^a $(a=1,\dots,n)$, y^r, μ; $n_1=n$, $n_2=n_3=1$. [This is true so long as we restrict attention to (G^{ij}) and ignore the vector and scalar potentials.]

With the above assumptions our problem simplifies substantially. The second equation in (1.9) becomes $1=Qk_rH_r^{-2}$. Replacing the Stäckel form H_r^{-2}, H_a^{-2} by the new Stäckel form $H'^{-2}_r = k_rH_r^{-2}, H'^{-2}_a = H_a^{-2}$, we can assume $k_r=1$. Furthermore since H'^{-2}_r is a Stäckel multiplier[5] we can pass to a new Stäckel form with $H''^{-2}_r=1$, H''^{-2}_a/H'^{-2}_r, $Q''=QH'^{-2}_r=1$. Thus in terms of the coordinates y^a, y^r, μ we have (dropping the primes)

$$H_r^{-2}=1,\quad H_a^{-2}=H_a^{-2}(y^b,y^r),\quad Q=1. \tag{2.1}$$

The transformation from "standard" to separable coordinates becomes

$$\begin{aligned} &x^k = x^k(y^a,y^r),\quad k=1,\dots,n,\\ &t=y^r,\quad \tau=\mu-R(y^a,y^r), \end{aligned} \tag{2.2}$$

and the metric becomes

$$\sum_{i,j=1}^n g^{ij}p_{x^i}p_{x^j} = \sum_{a=1}^n H_a^{-2}(y^b,t)p^2_{y^a}, \tag{2.3}$$

so for each fixed t the right-hand side of (2.3) defines a Stäckel-form metric on V^n.

To analyze this one-parameter metric we recall a few facts about Stäckel form metrics. An $n\times n$ nonsingular matrix $S(\mathbf{y})$ is said to be in Stäckel form if $S_{ij}=S_{ij}(y^i)$ and each of the elements $(S^{-1})^{1l}$, $l=1,\dots,n$, is nonzero. Set

$$\mathscr{H}_i(\mathbf{y},\mathbf{p}) = \sum_{l=1}^n (S^{-1})^{il}p_l^2. \tag{2.4}$$

Then

$$\{\mathscr{H}_i,\mathscr{H}_j\}=0,\quad i,j=1,\dots,n, \tag{2.5}$$

where $\{\cdot,\cdot\}$ is the Poisson bracket on the $2(n+1)$-dimensional symplectic manifold with canonical coordinates (y^i,p_i,t,p_r). Here $\mathscr{H}_1$ is the Hamiltonian associated with the Stäckel matrix S.

Theorem 2: Let

$$\mathscr{H}' = H_r^{-2}p_r^2 + \sum_{a=1}^n H_a^{-2}(\mathbf{y},t)p_a^2$$

be a Stäckel form Hamiltonian with $H_r^{-2}=1$ and $n\geqslant 2$, and suppose t_0 is in the domain of t. Let

$$\mathscr{H}(t) = \sum_{a=1}^n H_a^{-2}(\mathbf{y},t)p_a^2.$$

Then there exists an $n\times n$ Stäckel matrix $S(\mathbf{y})$ such that

$$\mathscr{H}(t) = \sum_{k=1}^n g_k(t)\mathscr{H}_k, \tag{2.6}$$

where the $\mathscr{H}_k$ are defined by (2.4) and the g_k are scalar-valued functions with $g_k(t_0)=\delta_{1k}$.

Proof: Since $\mathscr{H}'$ is a Stäckel form Hamiltonian there exists an $(n+1)\times(n+1)$ Stäckel matrix $T'(\mathbf{y},t)$ such that

$$T' = \begin{pmatrix} T_{00}(t) & T_{01}(t)\cdots T_{0n}(t)\\ T_{10}(y^1) & T_{11}(y^1)\cdots T_{1n}(y^1)\\ \vdots & \\ T_{n0}(y^n) & T_{n1}(y^n)\cdots T_{nn}(y^n)\end{pmatrix} \tag{2.7}$$

and $(T'^{-1})^{00}=H_r^{-1}=1$, $(T'^{-1})^{0a}=H_a^{-2}$, $a=1,\dots,n$. It follows that

$$T'' = \begin{pmatrix} 1 & T_{01} & \cdots & T_{0n}\\ 0 & T_{11} & & T_{1n}\\ \vdots & \vdots & & \vdots\\ 0 & T_{n1} & & T_{nn}\end{pmatrix}$$

is also a Stäckel matrix for $\mathscr{H}'$, since $(T'^{-1})^{0i}=(T''^{-1})^{0i}$, $i=0,1,\dots,n$. We can multiply column i of T'' by a nonzero constant c and column j by c^{-1} where $i\neq j$, i,j $i,j>0$, and obtain another Stäckel matrix for $\mathscr{H}'$. Furthermore, the interchange of two such columns $\mathbf{T}_i$, $\mathbf{T}_j$ or the replacement of $\mathbf{T}_i$ by $\mathbf{T}_i + c'\mathbf{T}_j$ again leads to a Stäckel matrix for $\mathscr{H}'$. It follows that there is a Stäckel matrix for $\mathscr{H}'$ of the form

$$T = \begin{pmatrix} 1 & -g_1(t) & \cdots & -g_n(t)\\ 0 & S_{11}(y^1) & \cdots & S_{1n}(y^1)\\ \vdots & \vdots & & \vdots\\ 0 & S_{n1}(y^n) & \cdots & S_{nn}(y^n)\end{pmatrix}, \tag{2.8}$$

where $g_a(t_0)=\delta_{a1}$. Thus $\mathscr{H}(t)$ is given by (2.6) where the

$\mathscr{H}_k$ are computed from the $n\times n$ Stäckel matrix $(S_{ab}(y^a))$. Q.E.D.

Corollary: $\{\mathscr{H}(t_1), \mathscr{H}(t_2)\} = 0$.

Since $\mathscr{H}(t_0) = \mathscr{H}_1$ and

$$\sum_{i,j=1}^{n} g^{ij} p_{x^i} p_{x^j} = \mathscr{H}(t) \tag{2.9}$$

we see that $\mathscr{H}_1$ is a Stäckel form Hamiltonian for V^n and that $\mathscr{H}(t)$ is a one-parameter family of such Hamiltonians. The requirements that $\mathscr{H}(t)$ corresponds to V^n is a very strong condition on the functions $g_i(t)$ for any choice of separable Hamiltonian $\mathscr{H}_1$.

To show how restrictive these conditions are we consider the "generic" separable coordinates in Euclidean space E^n and on the unit sphere S^n,

$$\mathscr{H}_1 = \sum_{i=1}^{n} \frac{f(y^i)}{\pi_{i\neq j}(y^i - y^j)} p_i^2 = \sum_{i=1}^{n} H_i^{-2} p_i^2, \tag{2.10}$$

where f is a polynomial with distinct real roots. This is a separable Hamiltonian on S^n iff $\deg f = n+1$ (Jacobi elliptic coordinates), and on E^n iff $\deg f = n$ (ellipsoidal coordinates) or $\deg f = n-1$ (paraboloidal coordinates). The related $n\times n$ Stäckel matrix is[12]

$$S_{ij} = (y^i)^{n-j}/f(y^i), \quad i,j = 1,\ldots,n. \tag{2.11}$$

In general, the orthogonal coordinates $\{x^i\}$ are separable on E^n provided the metric $ds^2 = \Sigma_{a=1}^n H_a^2(\mathbf{x})\,(dx^i)^2$ is in Stäckel form, i.e.,

$$\partial_{jk} \log H_i^2 - \partial_j \log H_i^2 \partial_k \log H_i^2 + \partial_j \log H_i^2 \partial_k \log H_j^2 + \partial_k \log H_i^2 \partial_j \log H_k^2 = 0, \quad j\neq k, \tag{2.12}$$

and $R_{hijk} = 0$ where R is the Riemann curvature tensor. For S^n (of constant curvature -1) this last condition is replaced by

$$\begin{aligned} R_{ijji} &= -H_i^2 H_j^2, \quad i\neq j, \\ R_{hiik} &= 0, \quad h,i,k \text{ distinct}. \end{aligned} \tag{2.13}$$

Eisenhart[13] (p. 269) has shown that for both E^n and S^n these conditions imply

$$\partial_{jk} \log H_i^{-2} = 0, \quad i,j,k \text{ distinct}. \tag{2.14}$$

Suppose $n\geqslant 3$. Then condition (2.4) applied to $\mathscr{H}(t)$, (2.6), where $\mathscr{H}_1$ is given by (2.10), becomes

$$\partial_{jk} \log\left(\sum_{l=1}^{n} g_l(t) \prod_{\substack{i_1 < i_2 < \cdots < i_{l-1} \\ i_h \neq i}} y^{i_1} \cdots y^{i_{l-1}} \right) = 0, \quad i,j,k \text{ distinct}. \tag{2.15}$$

The solution is

$$H_i^{-2}(\mathbf{y},t) = \frac{f(y^i)}{\pi_{i\neq j}(y^i - y^j)} g_1(t) \prod_{i\neq j} (1 + h(t) y^j), \quad i = 1,\ldots,n.$$

Clearly, $g_1 \neq 0$. Suppose $h\neq 0$. Under the change of coordinates $x^i = y^i/(1+hy^i)$ the metric transforms to

$$\widetilde{H}_i^{-2}(\mathbf{x}) = \frac{f(x^i/(1-hx^i))}{\pi_{i\neq j}(x^i - x^j)} (1-hx^i)^{n+3} g_1(t),$$

which is again of the form (2.10), except that the polynomial in the numerator is of order $n+3$, which does not correspond to E^n or S^n. Thus $h = 0$ and

$$H_i^{-2}(\mathbf{y},t) = \frac{f(y^i) g_1(t)}{\pi_{i\neq j}(y^i - y^j)}.$$

It is well known that for $g_1(t) \neq 1$ and $V^n = S^n$ the factor g_1 changes the curvature, so that the transformed metric is not one on S^n. We conclude that for Jacobi elliptic coordinates on S^n the only possibility is $\mathscr{H}(t) = \mathscr{H}_1$ whereas for ellipsoidal or paraboloidal coordinates on E^n the only possibilities are dilatations $\mathscr{H}(t) = g_1(t)\mathscr{H}_1$. For $n = 2$ a similar but simpler argument than the preceding one yields the same result.

We can now treat the most general separable coordinate system on S^n. In Ref. 12 it is shown that the most general separable system can be constructed by "nesting" collections of the generic Jacobi elliptic coordinates. The infinitesimal distance on S^n, expressed in a separable system, can always be written in the form

$$d\omega^2 = \sum_{I=1}^{p} d\omega_I^2 \left[\frac{\pi_{l=1}^{n_1}(y^l - e_I)}{\pi_{m\neq I}(e_m - e_I)} \right] - \frac{1}{4} \sum_{i=1}^{n_1} \frac{\pi_{j\neq i}(y^i - y^j)}{\pi_{j=1}^{n_1+1}(y^i - e_j)} (dy^i)^2. \tag{2.16}$$

Here the $\{y^i\}$ are Jacobi elliptic coordinates on S^{n_1} and each $d\omega_I^2$ is the infinitesimal distance of a S^{P_I} where $\Sigma_{I=1}^p P_I + n_1 = n$, and $p \leqslant n_1 + 1$. The coordinates on each S^{P_I} are again separable and the metrics $d\omega_I^2$ can be expressed in terms of separable coordinates by using (2.16) recursively. [The case of Jacobi elliptic coordinates on S^n corresponds to $p = 0$, $n_1 = n$ in (2.16).] In Ref. 12 a graphical procedure is presented to elucidate that construction, and the separation equations for the Hamilton–Jacobi equation are written explicitly. Thus for every separable system on S^n it is straightforward to compute the Stäckel matrix and to construct the quadratic forms $\mathscr{H}_k$, (2.6). The y coordinates in (2.6), since they are separable and $\mathscr{H}(t)$ is analytic in t, must be of the same type (2.16) as the coordinates of $\mathscr{H}_1 = \mathscr{H}(t_0)$. Though the details are somewhat tedious, it is not difficult to use the argument of (2.15), and its following paragraphs, recursively in (2.16). The results of this argument, followed by imposition of the curvature conditions (2.13), is the following.

Theorem 3: Let S be a Stäckel matrix corresponding to a separable coordinate system on S^n, $n\geqslant 2$, and define the corresponding Hamiltonian $\mathscr{H}_1$ and constants of the motion $\mathscr{H}_i$, $i = 2,\ldots,n$, by (2.4). Then $\mathscr{H}(t) = \Sigma_{i=1}^n g_i(t)\mathscr{H}_i$ is an S^n Hamiltonian with $\mathscr{H}(t_0) = \mathscr{H}_1$ iff $g_i(t) = \delta_{1i}$.

The most general separable coordinate system on E^n is also determined in Ref. 12. It is shown there that in the coordinates of such a system the metric ds^2 on E^n can be expressed as

$$ds^2 = \sum_{I=1}^{Q} ds_I^2, \tag{2.17}$$

where each ds_I^2 is an Euclidean space metric itself. In turn we have

$$ds_I^2 = \sum_{i=1}^{n_I} \frac{\pi_{l=1}^{N_I}(y^l - e_i^I)}{\pi_{j\neq i}(e_j^I - e_i^I)} d\omega_i^2 + d\sigma_I^2, \tag{2.18}$$

where $d\sigma_I^2$ is the infinitesimal distance corresponding to ellipsoidal or paraboloidal coordinates $\{y^l\}$ for E^{N_I}, and each $d\omega_i^2$ is the infinitesimal distance corresponding to a separable system on the sphere S^{p_I}. Here $n = \Sigma_{I=1}^{Q}(N_I + p_I)$ and $n_I \leqslant N_I$ for $d\sigma_I^2$ ellipsoidal, $n_I < N_I$ for $d\sigma_I^2$ paraboloidal. Thus the most general separable system on E^n is constructed by first decomposing E^n as a direct sum of Q mutually orthogonal Euclidean subspaces and then in each subspace nesting collections of Jacobi elliptic coordinates into either an ellipsoidal or a paraboloidal system. The generic ellipsoidal or paraboloidal systems for E^n correspond to the case $Q=1, n_1 = 0, N_1 = n$.

From the results of Ref. 12 it is straightforward to compute the Stäckel matrix corresponding to each separable system for E^n and to construct the quadratic forms $\mathscr{H}_k$. Again the y coordinates (2.6) must be of the same type (2.18) as the coordinates of $\mathscr{H}_1 = \mathscr{H}(t_0)$. It is tedious, though not difficult, to use the argument of (2.15) and its following paragraphs recursively in (2.18) and (2.16), followed by imposition of the Euclidean space curvature conditions to obtain the following.

Theorem 4: Let S be a Stäckel matrix corresponding to a separable coordinate system on E^n, $n \geqslant 2$, and let

$$ds^2 = \sum_{I=1}^{Q} ds_I^2$$

be the associated decomposition of the infinitesimal distance on E^n into distances ds_I^2 on Q mutually orthogonal Euclidean subspaces. Let $\mathscr{H}_1^{(I)}$ be the Hamiltonian on the Ith subspace so that $\mathscr{H}_1 = \mathscr{H}(t_0) = \Sigma_{I=1}^{Q}\mathscr{H}_1^{(I)}$. Then $\mathscr{H}(t)$ is an E^n Hamiltonian for all t iff it can be expressed in the form

$$\mathscr{H}(t) = \sum_{I=1}^{Q} h_I(t)\mathscr{H}_1^{(I)},$$

with $h_I(t_0) = 1, I = 1,...,Q$.

To date we have been unsuccessful in proving Theorems 3 and 4 without using the explicit list of all separable coordinate systems on S^n and E^n.

III. COORDINATES ON S^n AND E^n

Continuing our study of case 2 of Theorem 1, let $\{y^a, y^\alpha\}$ be an orthogonal separable coordinate system on the Riemannian space V^n such that the associated infinitesimal distance

$$ds^2 = \sum_{i=1}^{n} H_i^2(dy^i)^2 = \sum_a H_a^2(dy^a)^2 + \sum_\alpha H_\alpha^2(dy^\alpha)^2$$

is in Stäckel form [with Stäckel matrix (2.5)]. Here $H_\alpha^{-2} = \Sigma_{a=1}^{n_1} K_a^{\alpha\alpha}(y^a) H_a^{-2}$. Let $\mathscr{H}_1$ be the Hamiltonian in these coordinates and suppose we have determined a one-parameter family $\mathscr{H}(t) = \Sigma_{i=1}^{n} g_i(t)\mathscr{H}_i$ of Hamiltonians on V^n such that $\mathscr{H}(t_0) = \mathscr{H}_1$. We now study the remaining conditions on $g_i(t)$, $\{y^a, y^\alpha\}$ so that $\{y^r, y^a, y^\alpha\}$ will lead to R-separation of the time-dependent Hamilton–Jacobi equation

$$2\lambda W_t + \sum_{l,m=1}^{n} g^{lm}(\mathbf{x}) W_{x^l} W_{x^m} + 2\lambda \sum_{l=1}^{n} A^l(\mathbf{x}) W_{x^l} + \lambda^2 V(\mathbf{x}) = 0. \tag{3.1}$$

Here $\{x^i\}$ is a given coordinate system on V^n and we must have

$$x^i = x^i(y^a, y^r, y^\alpha), \quad i = 1,...,n, \quad t = y^r. \tag{3.2}$$

The remaining conditions to be satisfied are

$$\frac{\partial y^a}{\partial t} + \mathscr{A}^a = -H_a^{-2}R_a, \quad a = 1,...,n_1, \tag{3.3a}$$

$$\frac{\partial y^\alpha}{\partial t} + \mathscr{A}^\alpha = -H_\alpha^{-2}R_\alpha + \sum_{a=1}^{n_1} \mathscr{F}_a^\alpha(y^a)H_a^{-2} + \mathscr{F}_r^\alpha(y^r)$$
$$= -H_\alpha^{-2}R_\alpha + \mathscr{B}^\alpha, \quad \alpha = n_1,...,n, \tag{3.3b}$$

$$2R_r + V - \sum_{i=1}^{n} H_i^{-2}R_i^2 + 2\sum_\alpha R_\alpha \mathscr{B}^\alpha$$
$$= \sum_a \mathscr{F}_a(y^a)H_a^{-2} + \mathscr{F}_r(y^r), \tag{3.3c}$$

where $R_a = \partial_{y^a}R(y^r, y^a)$ and R_r, R_α are defined similarly. Each $\mathscr{F}_i$, $\mathscr{F}_i^\alpha$ is a function of a single variable y^i. We will discuss the solution of these equations with special emphasis on the important examples S^n and E^n.

First note that (3.3a) and (3.3b) can be written in covariant form,

$$\frac{\partial z^i}{\partial t} = -G^{ij}(\mathbf{z},t)R_{z^j} - \mathscr{A}^i + \mathscr{B}^i, \quad i = 1,...,n, \tag{3.4}$$

where (G^{ij}) is the metric for V^n in the coordinates z^i. Here, $z^i = z^i(y^a, y^\alpha)$. We can choose the initial coordinates $\{x^i\}$ and the $\{z^i\}$ to be in a convenient standard form and such that $z^i = Z^i(x^j, t)$, $i = 1,...,n$, with $Z^i(x^j, t_0) = x^i$. We will use the integrability conditions for (3.4) to determine the possible forms of the functions Z^i, and then express the $\{z^i\}$ in terms of separable $\{y^a, y^\alpha\}$ coordinates.

Consider first the space S^n. It is convenient to identify S^n with the unit sphere $\mathbf{x}\cdot\mathbf{x} = \Sigma_{j=1}^{n+1}(x^j)^2 = 1$ in E^{n+1} where $\{x^1,...,x^{n+1}\}$ are standard Cartesian coordinates. We choose $\{z^1,...,z^{n+1}\}$ to be Cartesian coordinates of the same type. Since the motion group of S^n is $O[n+1]$ (see Ref. 13, p. 23) it is clear that

$$\mathbf{z}(t) = O(t)\mathbf{x}, \quad O(t) \in O[n+1], \quad O(t_0) = I, \tag{3.5}$$

where I is the identity matrix. Equations (3.4) become

$$\sum_{l,j=1}^{n+1} \dot{O}_{ij} O_{lj} z^l = -R_{z^i} - \mathscr{A}^i + \mathscr{B}^i,$$
$$i = 1,...,n+1, \quad \mathbf{z}\cdot\mathbf{z} = 1. \tag{3.6}$$

The integrability conditions for (3.6) imply

$$\mathscr{A}^i - \mathscr{B}^i = -R_{z^i} + \sum_{s=1}^{n+1} X_{si}(t)z^s, \quad X = \dot{O}O^{-1}. \tag{3.7}$$

This can be regarded as a necessary condition on the vector potential in order that it permit variable separation. The second term on the right-hand side of (3.7) is "trivial" in the sense that it can always be removed by transformation to an appropriate rotating frame $\hat{\mathbf{x}}(t)=\hat{O}(t)\mathbf{x}$, $\hat{O}\in O[n+1]$. [Thus every separable potential is via the foregoing transformation equivalent to a vector potential which is a gradient $\mathscr{A}^i-\mathscr{B}^i=-\partial_{x^i}R(\mathbf{x},t)$ in Cartesian coordinates.] Assuming the equivalent vector potential is a gradient we can remove it by an appropriate R transformation and then have $z^i\equiv x^i$, $t=y^r$, $\mathscr{A}^i=\mathscr{B}^i$, $R=R(t)$.

Thus Eq. (3.1) transforms to

$$2\lambda Z_t+\sum_{i=1}^{n}H_i^{-2}(y^a)Z_i^2+2\lambda\sum_{\alpha,a}F_a^\alpha(y^a)H_a^{-2}Z_\alpha$$
$$+\sum_\alpha\mathscr{F}_a(y^a)H_a^{-2}=0, \tag{3.8}$$

where $\{y^a,y^\alpha\}$ is a separable system for the time-independent Hamilton–Jacobi equation on S^n.[14]

Theorem 5: Nontrivial R-separation corresponding to case 2 does not occur for the time-dependent Hamilton–Jacobi equation on S^n, i.e., every such separable system arises from a separable system for the time-independent equation.

Now we examine the same problem for E^n. As standard coordinates $\{x^1,\ldots,x^n\}$ we choose Cartesian coordinates. By Theorem 4, corresponding to a set of separable coordinates for E^n, there is a decomposition of this space into Q mutually orthogonal Euclidean subspaces $E_J^{N_J}$. Let $\{z_J^1,\ldots,z_J^{N_J}\}$ be Cartesian coordinates on the Jth such subspace (of dimension N_J). The Hamiltonian on $E_J^{N_J}$ is $\mathscr{H}_1^{(J)}=\Sigma_{i=1}^{N_J}(p_{z_J^i})^2$ and we have

$$\mathscr{H}_1^{(J)}=\sum_{J=1}^{Q}h_J^2(t)\mathscr{H}_1^{(J)}, \tag{3.9}$$

where $h_J(t_0)=1$. Since the motion group for E^n is the Euclidean group[13] (p. 23), it follows that the two coordinate systems are related by a t-dependent Euclidean transformation,

$$D^{-1}(t)\mathbf{z}(t)=O(t)\mathbf{x}+\mathbf{c}(t), \tag{3.10}$$

where O is an $n\times n$ orthogonal matrix, $\mathbf{c}$ is an n vector, and D is an $n\times n$ diagonal matrix whose diagonal term corresponding to z_J^i is $h_J(t)$. Equations (3.4) take the form

$$(\dot{D}D^{-1}+D\dot{O}O^{-1}D^{-1})\mathbf{z}-D\dot{O}O^{-1}\mathbf{c}+D\dot{\mathbf{c}}$$
$$=-D^2R_{\mathbf{z}}-\mathscr{A}^{\mathbf{z}}+\mathscr{B}^{\mathbf{z}}. \tag{3.11}$$

The integrability conditions for (3.11) imply that there exists a function $q(\mathbf{z},t)$ such that

$$(\mathscr{A}^i-\mathscr{B}^i)D_i^{-2}=q_{z^i}-D_i^{-1}\sum_{l,k=1}^{n}\dot{O}_{il}O_{kl}D_k^{-1}z^k, \tag{3.12}$$

where

$$-D_i^2(R_{z^i}+q_{z^i})$$
$$=\dot{D}_iD_i^{-1}z^i+D_i\dot{c}_i+D_i\sum_{l,k=1}^{n}\dot{O}_{il}O_{kl}c_k. \tag{3.13}$$

Here $D_{ij}(t)=D_i(t)\delta_{ij}$. This implies that in the standard Cartesian coordinates the vector potential has the form

$$A^j(\mathbf{x},t)-B^j(\mathbf{x},t)$$
$$=-\sum_{i,s=1}^{n}O_{ij}(t)\dot{O}_{is}(t)x^s+H_{x^j}(\mathbf{x},t). \tag{3.14}$$

Just as in the S^n case we can remove the first term on the right-hand side of (3.14) through a coordinate transformation $\hat{x}=\hat{O}(t)x$, where $\hat{O}^{-1}\dot{\hat{O}}=-O^{-1}\dot{O}$. Thus any separable vector potential is equivalent to a potential in gradient form ($A^j-B^j=H_{x^j}$), so that $(\mathscr{A}^i-\mathscr{B}^i)D_i^{-2}=q_{z^i}$ and we can assume $O(t)\equiv \mathrm{I}$ in (3.12) and (3.13). Then by means of an R transformation (which does not affect separability) we can take $q=0$ and $\mathscr{A}^i=\mathscr{B}^i$.

Thus

$$h_J^{-1}(t)z_J^i(t)=x_J^i+c_J^i(t),\quad i=1,\ldots,N_J,\quad J=1,\ldots,Q, \tag{3.15}$$

where we have adopted the same notation for the vectors $\mathbf{x},\mathbf{c}$ as for $\mathbf{z}$. Substituting (3.15) back into (3.11) we obtain

$$(\dot{h}_J/h_J^3)z_J^i+\dot{c}_J^i/h_J=-R_{z_J^i},$$

so that

$$R=-\sum_{J=1}^{Q}\sum_{i=1}^{N_J}\left[\frac{1}{2}\frac{\dot{h}_J}{h_J^3}(z_J^i)^2+\frac{\dot{c}_J^i}{h_J}z_J^i\right]+f(t). \tag{3.16}$$

We consider first the special case where the original vector and scalar potentials vanish: $\mathscr{A}^i=V=0$. Then $\mathscr{B}^i=0$ and we can assume $n_3=1$. Substituting (3.16) and (3.9) into the remaining condition (3.3c) we find

$$\sum_{J=1}^{Q}\sum_{i=1}^{N_J}\left[\frac{1}{2}(\ddot{h}_J^{-2})(z_J^i)^2\right.$$
$$\left.-2\frac{d}{dy^r}\left(\frac{\dot{c}_J^i}{h_J}\right)z_J^i-h_J^2\left(\frac{\dot{h}_J}{h_J^3}z_J^i+\frac{\dot{c}_J^i}{h_J}\right)^2\right]$$
$$=\sum_{J,a}h_J^2(\mathscr{F}_{a,J}(y_J^a)H_{a,J}^{-2}(y_J^b)+\mathscr{F}_J(y^r)). \tag{3.17}$$

Since, for each fixed J, the N_J coordinates z_J^i are functions of the N_J coordinates y_J^b, a necessary condition for (3.17) to hold is that the coefficients of $(z_J^i)^2$ and z_J^i on the left-hand side are constants times h_J^2,

$$h_J^{-2}(\ddot{h}_J^{-2})-2\dot{h}_J^2h_J^{-6}=\alpha_J, \tag{3.18a}$$
$$\ddot{c}_J^i/h_J^3=\beta_J^i. \tag{3.18b}$$

It follows that under the R transformation the original Hamilton–Jacobi equation

$$2\lambda W_t+\sum_{i=1}^{n}W_{x^i}^2=0 \tag{3.19}$$

maps to

$$2\lambda Z_r+\sum_{J=1}^{Q}\sum_{i=1}^{N_J}\left(Z_{z_J^i}^2+\frac{\alpha_J}{2}(z_J^i)^2-2\beta_J^iz_J^i\right)h_J^2=0, \tag{3.20}$$

where the α_J,β_J^i are constants. Note that the original Hamiltonian "decouples" into Q Hamiltonians

$$\mathscr{H}_1^{(J)}=\sum_{i=1}^{N_J}\left[p_{z_J^i}^2+\frac{\alpha_J}{2}(z_J^i)^2-2\beta_J^iz_J^i\right].$$

The separable coordinates $\{y_J^b\}$ are just those that separate the time-independent Hamilton–Jacobi equations $\mathscr{H}_1^{(J)}=E$, $J=1,\ldots,Q$.

Equation (3.18a) is equivalent to $(\dddot{h}_J^{-2})=0$ and has the general solution

$$h_J^2(t)=(b_1^Jt^2+b_2^Jt+b_3^J)^{-1},$$
$$2b_1^Jb_3^J-\tfrac{1}{2}(b_2^J)^2=\alpha_J. \tag{3.21}$$

We can simplify the ensuing argument by identifying coordinate systems that are equivalent under the action of the Galilean (and dilatation) symmetries of (3.19), see Ref. 2, Chap. 2. Thus, in addition to the Euclidean symmetries already employed, we identify systems related by dilatations $t\to\alpha^2 t$, $\mathbf{x}\to\alpha\mathbf{x}$, time translations $t\to t+\beta$ and velocity transformations $\mathbf{x}\to\mathbf{x}+\mathbf{c}t$. Then, in case $\alpha_J\neq 0$ for fixed J we can perform a translation of the $\{z_J^i\}$ coordinates to achieve $\beta^i_J=0$, $c^i_J=0$, $i=1,\ldots,N_J$. Thus the decoupled Hamiltonian is

$$\mathscr{H}_1^{(J)}=\sum_{i=1}^{N_J}\left[p^2_{z^i_J}+\frac{\alpha_J}{2}(z^i_J)^2\right],\tag{3.22}$$

the "harmonic oscillator" for $\alpha_J>0$ and the "repulsive oscillator" for $\alpha_J<0$. Here

$$h_J^{-1}(t)z^i_J(y^a_J)=x^i_J$$

and the possible orthogonal separable coordinates y^a_J are just those that separate $\mathscr{H}_1^{(J)}=E$.

In case $\alpha_J=0$ for fixed J we can perform a rotation of coordinates $\{z^i_J\}$ to achieve $\beta^i_J=0$ for $i=2,\ldots,N_J$. The corresponding decoupled Hamiltonian is

$$\mathscr{H}_1^{(J)}=\sum_{i=1}^{N_J}\left[p^2_{z^i_J}-2\beta^1_J z^1_J\right],\tag{3.23}$$

the "free fall" Hamiltonian. Here

$$h_J^{-1}(t)z^i_J(y^a_J)=x^i_J+\delta^{li}c^1_J(t),$$

where

$$h_J(t)=((b^J_2/2)t+1)^{-1},$$

$$c^1_J(t)=\begin{cases}\beta^1_J(t^2/2), & b^J_2=0,\\ [2\beta^1_J/(b^J_2)^2]((b^J_2/2)t+1)^{-1}, & b^J_2\neq 0.\end{cases}\tag{3.24}$$

We have used the property that (by a suitable time translation if necessary) we can assume $t_0=0$, $h_J(0)=1$. The possible orthogonal separable coordinates y^a_J are those that separate $\mathscr{H}_1^{(J)}=E$.

Although we will not state our results as a theorem, we have reduced the problem of finding all R-separable coordinate systems for the time-dependent Euclidean equation (3.19) (with zero potential) to the problem of finding all separable coordinate systems for the time-independent Hamiltonians (3.22) and (3.23). The answer to this last set of problems is known.[15]

In the general case where the vector and scalar potentials do not both vanish we have $\mathscr{A}^i=\mathscr{B}^i$ so that in the coordinates $\{y^a,y^\alpha,y^r\}$, $\mathscr{A}^a=\mathscr{A}^r=0$,

$$\mathscr{A}^\alpha=\mathscr{B}^\alpha=\sum_{a=1}^{n_1}\mathscr{F}^\alpha_a(y^a)H_a^{-2}+\mathscr{F}^\alpha_r(y^r).$$

Then Eq. (3.1) transforms to

$$2\lambda Z_t+\sum_{J,i}h_J^2(t)H_{J,i}^{-2}(y^b_J)Z^2_{J,i}$$
$$+2\lambda\sum_\alpha\left(\sum_{J,a}\mathscr{F}^\alpha_{J,a}(y^a_J)h^2_J(t)H_{J,a}^{-2}(y^b_J)+\mathscr{F}^\alpha_r(t)\right)Z_\alpha$$
$$+\sum_{J,a}\mathscr{F}_{J,a}(y^a_J)h^2_J(t)H_{J,a}^{-2}=0,\tag{3.25}$$

where

$$V=-2\sum_\alpha\left(\sum_{J,a}\mathscr{F}^\alpha_{J,a}(y^a_J)h^2_J(t)H_{J,a}^{-2}(y^b_J)+\mathscr{F}^\alpha_r(t)\right)\partial_\alpha R$$
$$-\sum_{J=1}^{Q}\sum_{i=1}^{N_J}\left[\frac{\alpha_J}{2}h^2_J(t)(z^i_J)^2-2h^2_J(t)\beta^i_J z^i_J-(\dot c^i_J)^2\right]$$
$$+\sum_{J,a}h^2_J(t)\mathscr{F}_{a,J}(y^a_J)H_{a,J}^{-2}(y^b_J),\tag{3.26}$$

and α_J, β^i_J are defined by (3.18). Since, for each fixed J, the N_J coordinates z^i_J are functions of the N_J coordinates y^b_J, condition (3.26) is a strong restriction on V. Clearly, given any separable system $\{y^b_J,y^a_J\}$ for E^n there always exist potentials V for which (3.26) is satisfied. The $h_J(t)$ must be determined from this functional equation.

IV. COMMENTS AND EXAMPLES

We can also use the results of the preceding section to find all R-separable coordinate systems for the time-dependent Hamilton–Jacobi equations on S^n and E^n that correspond to case 1 of Theorem 1, i.e., such that the new time coordinate f, (1.13), is ignorable: $f=y^\delta$. (Here we will treat only the zero potential equations. The nonzero potential treatment is similar.) Since the time-independent Hamilton–Jacobi equations on S^n and E^n separate only in orthogonal coordinates,[12] we can assume $n_1=n$, $n_2=0$, $n_3=2$ so that the transformed equation (1.13) takes the form

$$2\lambda Z_{y^\delta}+\sum_{a=1}^{n}H_a^{-2}(y^b)Z^2_{y^a}+\lambda^2U(y^b)=0.\tag{4.1}$$

For S^n, $\xi^\delta=\widetilde{Q}=1$ so $y^\delta=t$. Furthermore, the argument leading up to Theorem 5 shows that the separable $\{y^a\}$ coordinates are expressible entirely in terms of the $\{x^i\}$, i.e., $\partial_t y^a=0$. Combining this fact with Theorem 5 we have the following.

Theorem 6: Every R-separable coordinate system for the (zero potential) time-dependent Hamilton–Jacobi equation on S^n is purely separable and of the form $\{t,y^a\}$ where $\{y^a\}$ is an orthogonal separable system for the time-independent Hamilton–Jacobi equation on S^n.

For E^n and case 1 the results are a bit more complicated. It is easy to see that case 1 for E^n corresponds to $Q=1$ in (3.9) where now we must allow for the possibility that $h^2(t)\equiv h^2_1(t)=1$. Thus expressions (3.15)–(3.24) are correct with $Q=J=1$, $N_J=n$, $\mathscr{F}_J(y^r)\equiv 0$. The relation between the time coordinates is

$$\frac{dy^\delta}{dt}=h^2(t)\tag{4.2}$$

and the original equation

$$2\lambda W_t+\sum_{l=1}^{n}W^2_{x^l}=0\tag{4.3}$$

maps to

$$2\lambda Z_{y^\delta}+\sum_{i=1}^{n}\left(Z^2_{z^i}+\frac{\alpha}{2}(z^i)^2-2\beta z^1\right)=0.\tag{4.4}$$

Using time translation and dilation invariance for simplification, we obtain the following distinct possibilities:

(a) $\alpha>0,\ \beta=0,$
$h^2(t)=(1+t^2)^{-1},\quad h^{-1}(t)z^i(y^a)=x^i;$

(b) $\alpha<0,\ \beta=0,$
$h^2(t)=(1-t^2)^{-1},\quad h^{-1}(t)z^i(y^a)=x^i;$

(c) $\alpha<0,\ \beta=0,$
$h^2(t)=t^{-1},\quad \sqrt{|t|}\,z^i(y^a)=x^i;$

(d) $\alpha=0,\ \beta\neq 0,$
$h^2(t)=t^{-2},\quad tz^i(y^a)=x^i+(\delta^{i1}/2t)\beta;$

(e) $\alpha=0,\ \beta\neq 0,$
$h^2(t)=1,\quad z^i(y^a)=x^i+\delta^{i1}(\beta/2)t^2;$

(f) $\alpha=\beta=0,$
$h^2(t)=t^{-2},\quad tz^i(y^a)=x^i;$

(g) $\alpha=\beta=0,$
$h^2(t)=1,\quad z^i(y^a)=x^i.$

In each case the y^a are orthogonal separable coordinates for the Hamilton–Jacobi equation

$$\sum_{i=1}^{n}\left(Z_{z^i}^2+\frac{\alpha}{2}(z^i)^2-2\beta z^1\right)=E \tag{4.5}$$

and

$$R=\sum_{i=1}^{n}\left[\frac{1}{4}(\dot h^{-2})(z^i)^2\right]-\frac{\dot c^1}{h}z^1. \tag{4.6}$$

Basically, all case 1 R-separable systems originate from separable systems for the zero-potential equation (4.5), $\alpha=\beta=0$. For each of the types (a)–(g) one need merely determine which of the zero-potential separable systems remains separable for an added linear or quadratic equation.

For example, if $n=2$ there are four separable systems in the zero-potential types (f) and (g): Cartesian, polar, parabolic, and elliptic. For types (a)–(c), Cartesian, polar, and elliptic coordinates remain separable. Thus there are a total of $2(4)+3(3)+2(2)=21$ R-separable systems corresponding to case 1. See Ref. 11, Chap. 2, for more details.

A classification of case 2 coordinates for E^n with general n has recently been worked out by Reid.[15] Reid shows that the Hamiltonian can always be written in the form (3.10) where the functions $h_J^2(t)$ can be selected from

$$h_J^2(t)=\begin{cases}[(t+A_J)^2+B_J^2]^{-1},\\ |(t+A_J)^2-B_J^2|^{-1},\\ (t+A_J)^{-2},\\ (t+A_J)^{-1}.\end{cases}$$

He has also worked out the case 1 separable systems for general n.

V. THE TIME-DEPENDENT SCHRÖDINGER EQUATION

Our results extend rather easily to the time-dependent Schrödinger (or heat) equation

$$2\lambda\psi_t+\Delta_n\psi+2\lambda\sum_{l=1}^{n}A^l(\mathbf{x},t)\psi_{x^l}+\lambda^2V(\mathbf{x},t)\psi=0, \tag{5.1}$$

where Δ_n is the Laplace–Beltrami operator on the Riemannian manifold V^n,

$$\Delta_n=\frac{1}{\sqrt{g}}\sum_{l,m=1}^{n}\partial_{x^l}(\sqrt{g}\,g^{lm}\partial_{x^m}),\quad g^{-1}=\det(g^{lm}).$$

In analogy with Sec. I, one writes

$$\vartheta(x,t,\tau)=e^{\lambda\tau}\psi(x,t),$$

where ϑ satisfies the Laplace equation

$$\Delta_{n+2}\,\vartheta=0,$$
$$\Delta_{n+1}=\frac{1}{\sqrt{K}}\sum_{i,j=1}^{n+2}\partial_{z^i}(\sqrt{K}\,K^{ij}\partial_{z^j}), \tag{5.2}$$
$$K^{-1}=\det(K^{ij})=-g^{-1},$$

and (K^{ij}), z^i are defined by (1.2). Equations (1.3)–(1.11) continue to hold, but, from the theory of R-separation for Laplace equations,[4,16] the R-separable solutions for (5.1) take the form

$$\psi(x,t)=\exp[-\lambda R(\mathbf{y})+S(\mathbf{y})]\prod_{i=1}^{n+1}\psi^{(i)}(y^i), \tag{5.3}$$

where R and S do not depend on the separation parameters. Theorem 1 holds with only minor modifications. The analysis of Secs. I–III for the Hamilton–Jacobi equation applies without change for the second-derivative terms in Eq. (5.2). The only complication is that the Laplace–Beltrami operator also contains first-derivative terms and, if the coefficients of these terms do not have the proper form, they could invalidate the variable separation. We can use our freedom in choosing S to partially offset this difficulty.

To be more specific we consider the case $n_2=n_3=1$ and adopt the notation of Eq. (3.3). Then (5.2) in the separable coordinates $\{y^r,y^a,\mu\}$ takes the form

$$2\vartheta_{\mu y^r}+\partial_{y^r}(\ln h)\,\vartheta_\mu+\frac{1}{h}\sum_a\partial_{y^a}(hH_a^{-2}\partial_{y^a})\vartheta$$
$$+\left(\sum_l\mathscr{F}_l(y^l)H_l^{-2}\right)\vartheta_{\mu\mu}=0,\quad h=\prod_a H_a, \tag{5.4}$$

where the conditions for R-separation of the second-derivative terms are as in (2.2). Now set $\vartheta=e^{S(y^a,y^r)}\Phi$ in (5.4) to obtain

$$2\Phi_{\mu y^r}+\partial_{y^r}(2S+\ln h)\Phi_\mu+\frac{1}{h}\sum_a\partial_{y^a}(hH_a^{-2}\partial_{y^a})\Phi$$
$$+2\sum_a H_a^{-2}S_a\Phi_a+\left(\sum_l\mathscr{F}_lH_l^{-2}\right)\Phi_{\mu\mu}$$
$$+\sum_a H_a^{-2}(S_{aa}+S_a^2+S_a\,\partial_{y^a}\ln(hH_a^{-2}))\Phi=0.$$

(Here $S_a=\partial_{y^a}S$, $\Phi_\mu=\partial_\mu\Phi$ but $\mathscr{F}_l$, H_a^{-2} are merely subscripted.) The coefficients of Φ_a, Φ_μ, and Φ, respectively, will be compatible with R-separation in the coordinates $\{y^a,y^r,\mu\}$ if and only if there exist functions $g_i(y^i)$, $k_i(y^i)$, each depending on the variable y^i alone, such that

$$\partial_{y^ay^b}[2S+\ln(hH_a^{-2})]=0,\quad 1\leqslant a<b\leqslant n, \tag{5.5a}$$

$$\partial_{y^r}[2S+\ln h]=\sum_{c=1}^{n_1}g_c(y^c)H_c^{-2}+g_r(y^r), \tag{5.5b}$$

$$\sum_{a=1}^{n_1}H_a^{-2}(S_{aa}-2S_a^2)=\sum_{c=1}^{n_1}k_c(y^c)H_c^{-2}+k_r(y^r). \tag{5.5c}$$

[Note that (5.5b) and (5.5c) imply that the left-hand sides of these expressions must be Stäckel multipliers.] The case where the new time variable is ignorable ($n_2=0$, $n_3=2$) leads to conditions (5.5a) and (5.5c) with $k_r\equiv 0$. However, (5.5b) is omitted in this case.

For $V^n=S^n$ we know from Theorem 3 that $\partial_{y^a y^b}\ln(hH_a^{-2})=0, a\neq b$ and $\partial_{y^r}h=0$ so $S=0$ satisfies the above equations. For $V^n=E^n$, Theorem 4 implies that

$$\partial_{y^a y^b}\ln(hH_a^{-2})=0,\quad a\neq b,\quad \partial_{y^r}h=g_r(y^r),$$

so again $S=0$ satisfies Eqs. (5.5). Thus we have the following.

Theorem 7: For any potential $(A^i(\mathbf{x},t),\ V(\mathbf{x},t))$ on the spaces S^n or E^n the time-dependent Hamilton–Jacobi equation additively R-separates in a given coordinate system if and only if the corresponding time-dependent Schrödinger equation multiplicatively R-separates in the same coordinates.

In general, R-separation of the Schrödinger equation implies R-separation of the Hamilton–Jacobi equation. However, it is not difficult to find examples where Eq. (5.5) cannot be satisfied, so the converse is false.

See Ref. 17 and references contained therein for applications of R-separation to time-dependent Schrödinger equations.

VI. INTRINSIC CHARACTERIZATION OF THE EQUATIONS

As was pointed out in Sec. I, the time-dependent Hamilton–Jacobi equation (1.1) can be considered as a special case of the conformal Hamilton–Jacobi equation (1.2). This suggests the interest in characterizing those pseudo-Riemannian spaces V^{n+2} for which the infinitesimal distance can be written in the form

$$ds^2=Q\left(2\,dt\,d\tau+\sum_{a,b=1}^{n}g_{ab}\,dx^a\,dx^b\right),\qquad(6.1)$$

where

$$\partial_t g_{ab}=\partial_\tau g_{ab}=0.$$

Here Q is a nonzero function on V^{n+2}. We will employ the root structure of conformal Killing tensors to provide this characterization.

Let V^m be a pseudo-Riemannian manifold with metric $ds^2=\Sigma G_{ij}\,dz^i\,dz^j$ in local coordinates $\{z^i\}$, and let V^m be its associated $2m$-dimensional symplectic manifold (with local canonical coordinates $\{z^i,p_j\}$). The Hamiltonian on V^m is $\mathscr{H}=\Sigma G^{ij}p_ip_j$. A (conformal) Killing tensor $\mathscr{P}(z,p)$ on V^m is a function on V^m, a polynomial in the p's with z-dependent coefficients, such that $\{\mathscr{H},\mathscr{P}\}=\mathscr{R}\mathscr{H}$, where $\mathscr{R}$ is a function on V^m which is also a polynomial in the p's and $\{\cdot,\cdot\}$ is the Poisson bracket. If $\mathscr{R}\equiv 0$ then $\mathscr{P}$ is a Killing tensor. If $\mathscr{P}$ is linear in the p's it is a conformal Killing vector, a Killing vector if $\mathscr{R}\equiv 0$.

Let $\mathscr{A}=\Sigma A^{ij}(z)p_ip_j, A^{ij}=A^{ji}$, be a second-order Killing tensor on V^m. A root $\rho(z)$ of $\mathscr{A}$ is an analytic solution of the characteristic equation

$$\det(A^{ij}-\rho G^{ij})=0\qquad(6.2)$$

and an eigenform $\omega=\Sigma q_k dz^k$ corresponding to ρ is a nonzero one-form such that

$$\sum_{j=1}^{m}(A^{ij}-\rho G^{ij})q_j=0,\quad i=1,\ldots,m.\qquad(6.3)$$

We denote by W^ρ the vector space (over the reals) generated by the eigenforms corresponding to ρ. Roots and eigenforms are defined independent of local coordinates.

Theorem 8: Necessary and sufficient conditions that the infinitesimal distance $ds^2=\Sigma G_{ij}\,dz^i\,dz^j$ on the pseudo-Riemannian space V^{n+2} can be expressed in the form

$$ds^2=Q\left(2\,dt\,d\tau+\sum_{a,b=1}^{n}g_{ab}(x)dx^a\,dx^b\right)\qquad(6.4)$$

are the following.

(1) There is a second-order conformal Killing tensor $\mathscr{A}=\Sigma A^{ij}p_{z^i}p_{z^j}$ $(A^{ij}=A^{ji})$ on V^{n+2} with roots 0 (multiplicity n) and $\rho\neq 0$ (multiplicity 2); $\dim W^0=n$, $\dim W^\rho=2$.

(2) There are two conformal Killing vectors

$$\mathscr{L}_\alpha=\sum\xi^i_\alpha\,p_{z^i},\ \alpha=1,2,\qquad(6.5)$$

on V^{n+2} such that $\mathscr{A}=2\mathscr{L}_1\mathscr{L}_2$. Furthermore $\mathscr{L}_1$, $\mathscr{L}_2$ are in involution: $\{\mathscr{L}_1,\mathscr{L}_2\}=0$.

(3) The first covariant derivatives of $\mathscr{A}$ vanish: $A^{ij}{}_{,k}=0$, $1\leqslant i,j,k\leqslant n+2$. Here the covariant derivatives are taken with respect to the metric $d\bar{s}^2=\rho^{-1}\,ds^2$.

Proof: Suppose conditions (1)–(3) are satisfied. It follows immediately from conditions (1) and (3), and the principal result of Eisenhart's paper on symmetric second-order tensor whose covariant derivatives are zero[18] (p. 303), that there is a coordinate system $\{y^1,y^2,x^1,\ldots,x^n\}$ on V^{n+2} with respect to which

$$\mathscr{A}=\sum_{c,d=1}^{2}\varphi^{c,d}(y)p_{y^c}p_{y^d},\qquad(6.6a)$$

$$\mathscr{H}=\sum_{c,d=1}^{2}\varphi^{c,d}(y)p_{y^c}p_{y^d}+\sum_{a,b=1}^{n}\gamma^{a,b}(x)p_{x^a}p_{x^b}.\qquad(6.6b)$$

(Although Eisenhart's result is stated only for Riemannian spaces, his proof remains valid for pseudo-Riemannian spaces.) Condition (2) and (6.6a) imply that $\mathscr{L}_\alpha=\Sigma_{c=1}^2\xi^c_\alpha p_{y^c}$, $\alpha=1,2$. Since obviously $\{\mathscr{A},\rho,\mathscr{H}\}=\{2\mathscr{L}_1\mathscr{L}_2,\rho\mathscr{H}\}=0$ and the $\mathscr{L}_\alpha$ are conformal Killing vectors for $\rho\mathscr{H}$, it follows that $\{\mathscr{L}_\alpha,\rho\,\mathscr{H}\}=0$. Thus $\Sigma_b\gamma^{a,b}\partial_{x^b}\xi^c_\alpha=0$ and by the nondegeneracy of $\mathscr{H}$ we have $\partial_{x^b}\xi^c_\alpha=0$. It follows that there is a coordinate system $\{t,\tau,x^1,\ldots,x^n\}$ on V^{n+2}, such that $t=t(y^1,y^2)$, $\tau=\tau(y^1,y^2)$ and

$$\mathscr{A}=2p_t\,p_\tau,\quad \rho\mathscr{H}=2p_t\,p_\tau+\sum_{a,b=1}^{n}\gamma^{a,b}(x)p_{x^a}\,p_{x^b}.$$

Setting $\rho=Q$ we obtain (6.4), where $\Sigma_l\,g_{al}\gamma^{lb}=\delta^b_a$.

Conversely, if the metric on V^{n+2} can be expressed in the form (6.4), it is straightforward to verify that conditions (1)–(3) are satisfied where $\mathscr{L}_1=p_t$, $\mathscr{L}_2=p_\tau$. Q.E.D.

With this result one can use existing classifications of separable coordinate systems for Hamilton–Jacobi equations $\Sigma g^{ij}p_i\,p_j=0$ to classify separable coordinates for the time-dependent equation (1.1), e.g., Ref. 19.

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