

# A projection-based solution to the $Sp(2N)$ state labeling problem

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A projection-based solution to the symplectic group state labeling problem is presented. The approach yields a nonorthogonal Gel'fand–Tsetlin basis for the irreducible representations of  $Sp(2n)$ . A method for evaluating the corresponding overlap coefficients is discussed. The action of the  $Sp(2n)$  generators, in the basis obtained, is determined and some matrix element formulas are derived. The results obtained are comparable to the matrix element formulas for  $O(n)$  and  $U(n)$ .

## I. INTRODUCTION

The theory of Lie groups has been established as an invaluable tool in physical applications where they usually appear as the symmetry group of the system. The states of a physical system are then to comprise irreducible representations of the symmetry group. Lie groups afford not only convenient analytic methods but in practice are essential to the numerical solution of the equations of motion of the system by allowing the Hamiltonian to be broken into a convenient block form. Lie groups also provide suitable labels (i.e., quantum numbers) for physical states, even though such Lie groups need not be symmetry groups.

Thus, from the point of view of physical applications the principle problem to be solved in the representation theory of a semisimple (compact) Lie group is the complete determination of the basis states of an irreducible representation. The first major step in this direction was made by Gel'fand and Tsetlin,<sup>1</sup> who constructed, with a full set of labels, a complete set of basis vectors for the irreducible representations of the orthogonal and unitary groups. The work of Gel'fand and Tsetlin on  $U(n)$  was extended by Baird and Biedenharn,<sup>2</sup> who revealed the group theoretic nature of the Gel'fand–Tsetlin results. The work of Baird and Biedenharn has recently been extended to  $O(n)$  by Gould.<sup>3</sup>

The solution to the  $O(n)$  and  $U(n)$  state labeling problem, as proposed by Gel'fand and Tsetlin,<sup>1</sup> relies on the fact that  $O(n)$  and  $U(n)$  admit a so-called canonical<sup>4</sup> chain of subgroups whose Casimir invariants provide a complete set of labels for the irreducible representations. For general subgroup chains, however, this method of labeling is incomplete and it is necessary to supplement the Casimir invariants of the subgroup chain one is considering with additional labeling operators. In such a case there still remains the problem of finding the eigenvalues of these extra invariant operators, which are known to be irrational in general (and thus the action of the group generators in such a basis is likely to be complicated). This behavior is typical of the general state labeling problem. Examples are afforded by the subgroup embeddings  $U(2n) \supset Sp(2n)$ ,  $U(n) \supset O(n)$ ,  $U(n+m) \supset U(n) \times U(m)$ ,  $Sp(2n) \supset Sp(2n-2)$ , etc.

An alternative approach to the state labeling problem is to use the method of projection, which has proved in the past to be a powerful tool for handling the multiplicity problem

numerically and, in some cases, analytically. The methods of projection have been successfully employed by Elliot<sup>5</sup> to the  $U(3) \supset O(3)$  state labeling problem. More recently the methods of projection have been applied to give a solution to the Clebsch–Gordan multiplicity problem for a semisimple Lie group  $G$  (i.e., the  $G \times G \supset G$  state labeling problem). A detailed account of the various methods of projection can be found in Moshinsky *et al.*,<sup>6</sup> Asherova and Smirnov,<sup>7</sup> and Edwards and Gould.<sup>8</sup>

In this paper we present a projection-based solution to the symplectic group state labeling problem. Our method consists of embedding an irreducible representation of the symplectic group  $Sp(2n)$  in a suitable representation of the unitary group  $U(2n)$ . A basis for the irreducible representations of  $Sp(2n)$  is then obtained by (central) projection from a suitable set of Gel'fand–Tsetlin (GT) basis states for  $U(2n)$ . This leads to a GT-type labeling scheme for  $Sp(2n)$  in analogy with the solutions to the  $O(n)$  and  $U(n)$  state labeling problems. The principle feature of our approach is that the action of the  $Sp(2n)$  generators in the basis obtained is simple and comparable to the  $O(n)$  and  $U(n)$  cases. By expanding the  $Sp(2n)$  generators in terms of  $U(2n)$  generators we are able to deduce the action of the  $Sp(2n)$  generators in the basis obtained. We give explicit matrix element formulas for certain generators but only give the general form of action for the remaining generators. However, as it turns out, the matrix elements of the elementary generators (which generate the symplectic group Lie algebra) are not difficult to obtain and will be evaluated in a forthcoming publication. It is evident that the approach of this paper may be extended to other subgroup embeddings such as, for example, the problem of obtaining a weight basis for the irreducible representations of  $O(n)$ .

The solution to the symplectic group state labeling problem, as proposed in this paper, suffers from the disadvantage that the basis obtained is nonorthogonal. To this end we have found it convenient to introduce a dual Gel'fand–Tsetlin pattern labeling. One (upper) pattern, which has direct group theoretical significance, refers to the representation labels of the group  $Sp(2n)$  and its subgroups  $Sp(2n-2)$ , ...,  $Sp(2)$ . The other (lower) pattern carries the labels of the  $U(2n)$  GT states from which we are projecting. Thus our basis is orthogonal with respect to the upper patterns but not with respect to the lower ones. However, the projection operators used may be

constructed as a polynomial in the universal Casimir elements of the subgroups  $\text{Sp}(2n)$ ,  $\text{Sp}(2n-2)$ , ...,  $\text{Sp}(2)$  and thus the overlap coefficients may, in principle, be evaluated using the known matrix element formulas of the  $\text{U}(2n)$  generators. The problem of evaluating the overlap coefficients and various simplifications is discussed in the final section of the paper.

The dual pattern labeling of this paper may be compared to the approach of Louck *et al.*<sup>9,10</sup> to the  $\text{U}(N)$  tensor operator problem. In this case two GT patterns appear. One, which has direct group theoretical significance, refers to the components of the tensor operator. The other is an operator pattern which was shown in Ref. 8 to correspond to a projection-type labeling in analogy with the labeling scheme proposed in this paper. We remark that this dual pattern labeling for the symplectic groups also appears in the work of Zhelobenko<sup>11</sup> but without any group theoretical significance.

Other developments in connection with the symplectic group have been made by Lohe and Hurst,<sup>12</sup> who have advocated the use of modified boson operators as a method of constructing basis states for the irreducible representations of  $\text{Sp}(2n)$  in analogy with the boson polynomials used in the theory of  $\text{U}(n)$ . Explicit matrix element formulas, in certain degenerate representations of  $\text{Sp}(2n)$ , have recently been obtained by Klimyk.<sup>13</sup> The method of raising and lowering operators to construct a basis for the irreducible representations of  $\text{Sp}(2n)$  has been advocated by Bincer<sup>14</sup> and Mickelson.<sup>15</sup> The symplectic groups also figure prominently in Cartan's classification of homogeneous spaces, which afford certain degenerate representations of  $\text{Sp}(2n)$  which have been studied by Pajas and Raczka<sup>16</sup> and Kalnins and Gould.<sup>17</sup>

## II. FUNDAMENTALS

We begin by introducing the symplectic group  $\text{Sp}(2n)$  as a subgroup of the unitary group  $\text{U}(2n)$ . The  $(2n)^2$  generators  $a_{ij}$  ( $i, j = 1, \dots, 2n$ ) of the Lie group  $\text{U}(2n)$  satisfy the commutation relations

$$[a_{ij}, a_{kl}] = \delta_{kj}a_{il} - \delta_{il}a_{kj}$$

and are, moreover, required to satisfy the Hermiticity condition

$$a_{ij}^\dagger = a_{ji}$$

on finite-dimensional (i.e., unitary) representations of the group. In order to define the symplectic subgroup of  $\text{U}(2n)$  we introduce a nonsingular antisymmetric metric  $g_{ij} = -g_{ji}$  ( $i, j = 1, \dots, 2n$ ). One may then take for the infinitesimal generators of the Lie group  $\text{Sp}(2n)$  the  $n(2n+1)$  independent operators

$$\alpha_{ij} = g_{ip}a_{pj} + g_{jp}a_{pi} = \alpha_{ji},$$

where we have summed over repeated index  $p$  from 1 to  $2n$ . These generators satisfy the commutation relations

$$[\alpha_{ij}, \alpha_{kl}] = g_{kj}\alpha_{il} - g_{il}\alpha_{kj} + g_{ki}\alpha_{jl} - g_{jl}\alpha_{ki}.$$

Without loss of generality we choose the symplectic group metric  $g_{ij}$  to be given by

$$g_{ij} = \begin{cases} \delta_{j,i+1}, & i \text{ odd}, \\ -\delta_{j,i-1}, & i \text{ even}, \end{cases} \quad i, j = 1, \dots, 2n.$$

We remark that as far as the representation theory is concerned the actual choice of symplectic group metric is immaterial and there exist several other standard choices, all of which should lead to equivalent formalisms.

It is useful to introduce the inverse metric  $g^{ij}$  defined by

$$g^{ij}g_{jk} = \delta^i_k.$$

This equation is to be understood in the sense of the summation convention, which we employ throughout the paper, where any repeated index is to be summed from 1 to  $2n$  (unless otherwise stated). Note that the metric  $g_{ij}$  satisfies the property  $g_{ij}g_{jk} = -\delta_{ik}$  (i.e.,  $g^2 = -I$ ) from which it follows that  $g^{ij} = -g_{ji} = g_{ji}$ . Using the inverse metric we define generators

$$\alpha_j^i = g^{ik}\alpha_{kj}, \quad (1)$$

which satisfy the commutation relations

$$[\alpha_j^i, \alpha_l^k] = \delta^k_j \alpha_l^i - \delta^i_l \alpha_j^k + g^{ik}\alpha_{jl} - g_{jl}\alpha^{ki}, \quad (2)$$

where we define

$$\alpha^{ki} = g^{ij}\alpha_j^k.$$

We note moreover that the generators (1) satisfy the Hermiticity condition

$$(\alpha_j^i)^\dagger = \alpha_i^j.$$

The advantage of working with the generators (1) is that they are automatically in Cartan-Weyl form. We choose as a Cartan subalgebra (CSA) the diagonal generators

$$\alpha_i^i = a_{ii} - (g_{ip})^2 a_{pp}, \quad i = 1, \dots, 2n$$

(where the repeated index  $p$  is to be summed from 1 to  $2n$ ). However, only  $n$  of these operators are linearly independent so we only need consider the Cartan generators

$$\begin{aligned} h_i &= \alpha^{2i-1}_{2i-1} = a_{2i-1, 2i-1} - a_{2i, 2i} \\ &= -\alpha^{2i}_{2i}, \quad i = 1, \dots, n, \end{aligned}$$

whose eigenvalues provide a unique labeling for the system of weights. Note, with our choice of metric, that the CSA for  $\text{Sp}(2n)$  is embedded in the CSA for  $\text{U}(2n)$ .

To see that the generators (1) are in Cartan-Weyl form we note that the commutation relations (2) imply the result

$$[h_i, \alpha_l^k] = (\delta^k_{2i-1} - \delta'_{2i-1} + \delta'_{2i} - \delta^k_{2i})\alpha_l^k. \quad (3)$$

If we introduce the fundamental weights  $\Delta_r$  ( $r = 1, \dots, n$ ) consisting of 1 in the  $r$ th position and zeros elsewhere one sees immediately from Eq. (3) that the roots for the symplectic group Lie algebra are given by the weights

$$\pm(\Delta_i + \Delta_j), \quad i \leq j \quad \text{and} \quad \pm(\Delta_i - \Delta_j), \quad i < j.$$

We take as a system of positive roots the weights

$$\Delta_i + \Delta_j \quad (i \leq j), \quad \Delta_i - \Delta_j \quad (i < j).$$

The corresponding generators are given by

$$\alpha^{2i-1}_{2j} \quad (i \leq j) \quad \text{and} \quad \alpha^{2i-1}_{2j-1} \quad (i < j), \quad (4)$$

respectively, which we henceforth refer to as raising generators. Note that the raising generators (4) are given in terms of  $\text{U}(2n)$  generators by

$$\alpha^{2i-1}_{2j} = a_{2i-1,2j} + a_{2j-1,2i} = \alpha^{2j-1}_{2i}, \quad (5)$$

$$\alpha^{2i-1}_{2j-1} = a_{2i-1,2j-1} - a_{2j,2i} = -\alpha^{2j}_{2i}.$$

By taking the Hermitian conjugate of Eqs. (4) and (5) we obtain the set of lowering generators.

We draw particular attention to the generators  $\alpha^{2i-1}_{2i}, \alpha^{2i}_{2i-1}$  which may be expressed in terms of  $U(2n)$  generators according to

$$\alpha^{2i-1}_{2i} = 2a_{2i-1,2i}, \quad \alpha^{2i}_{2i-1} = 2a_{2i,2i-1}. \quad (6)$$

The operators

$$h_n = \alpha^{2n-1}_{2n-1}, \alpha^{2n-1}_{2n}, \alpha^{2n}_{2n-1} \quad (7)$$

form the generators of the subgroup  $Sp(2)$  of  $Sp(2n)$ . The generators (7) together with the  $Sp(2n-2)$  generators  $\alpha^i_j (i, j = 1, \dots, 2n-2)$  form the generators of the subgroup  $Sp(2n-2) \times Sp(2)$  of  $Sp(2n)$ .

We note that the symplectic group Lie algebra is generated (as a Lie algebra) by the elementary generators

$$\alpha^{2i-1}_{2i+1}, \alpha^{2i+1}_{2i-1}, \alpha^{2n-1}_{2n}, \alpha^{2n}_{2n-1}, \quad (8)$$

$$i = 1, \dots, n-1.$$

Every symplectic group generator  $\alpha^i_j$  may be obtained by repeated commutation with generators of the form (8).

With regard to the group  $U(2n)$  we follow the notation of Gould.<sup>18</sup> We choose as a CSA for the Lie algebra of  $U(2n)$  the Abelian Lie algebra spanned by the diagonal generators  $a_{ii} (i = 1, \dots, 2n)$  whose eigenvalues uniquely label the weights of  $U(2n)$ . With respect to the usual lexicographical ordering imposed on the weights we see that the  $U(2n)$  generators  $a_{ij}$  with  $i < j$  (resp.  $i > j$ ) are raising (resp. lowering) generators.

We let  $L$  (resp.  $L_0$ ) denote the Lie algebra of  $U(2n)$  [resp.  $Sp(2n)$ ] and we let  $H$  (resp.  $H_0$ ) denote the CSA of  $L$  (resp.  $L_0$ ). The weights for  $L$  (resp.  $L_0$ ) may be identified with the CSA dual  $H^*$  (resp.  $H_0^*$ ) in an obvious manner. We let  $B$  (resp.  $B_0$ ) denote the nilpotent Lie subalgebra of  $L$  (resp.  $L_0$ ) generated by the raising generators and we let  $N$  (resp.  $N_0$ ) denote the nilpotent Lie subalgebra of  $L$  (resp.  $L_0$ ) generated by the lowering generators. We furthermore set

$$\bar{N} = N \oplus H, \quad \bar{B} = B \oplus H,$$

$$\bar{N}_0 = N_0 \oplus H_0, \quad \bar{B}_0 = B_0 \oplus H_0.$$

Note that the subalgebras  $\bar{N}, \bar{B}$  (resp.  $\bar{N}_0, \bar{B}_0$ ) are reductive Lie algebras. In this notation the Lie algebras  $L, L_0$  may be written

$$L = H \oplus N \oplus B = \bar{N} \oplus B = N \oplus \bar{B},$$

$$L_0 = H_0 \oplus N_0 \oplus B_0 = \bar{N}_0 \oplus B_0 = N_0 \oplus \bar{B}_0.$$

We let  $U$  (resp.  $U_0$ ) denote the universal enveloping algebra of  $L$  (resp.  $L_0$ ) and we denote the universal enveloping algebras of  $H, N, B, H_0, N_0, B_0$  etc., by  $U(H), U(N), U(B), U(H_0), U(N_0), U(B_0)$ , respectively.

We now recall some basic facts on the structure of universal enveloping algebras (see, e.g., Humphreys<sup>19</sup>). According to the PBW theorem the universal enveloping algebra  $U$  of  $L$  may be written

$$U = U(N)U(H)U(B). \quad (9)$$

Now the Lie algebra  $L$  is generated (as a Lie algebra) by the CSA  $H$  together with the elementary generators

$$x_i = a_{i,i+1}, \quad y_i = a_{i+1,i}, \quad i = 1, \dots, 2n-1.$$

The nilpotent subalgebra  $B$  (resp.  $N$ ) is generated as an algebra by the set  $\{x_i\}_{i=1}^{2n-1}$  (resp.  $\{y_i\}_{i=1}^{2n-1}$ ). Again, by the PBW theorem, we may choose as a basis for the universal enveloping algebra  $U(B)$  the identity  $1 \in \mathbb{C}$  together with the set of all basis monomials of degree  $k$  ( $k = 1, 2, 3, \dots$ )

$$x_{i_1} x_{i_2} \cdots x_{i_k}, \quad 1 \leq i_r \leq 2n-1.$$

Letting  $k$  range over all positive integers and letting the integers  $i_r$  ( $r = 1, \dots, k$ ) take all possible values in the range  $1, \dots, 2n-1$  we thereby get a basis for  $U(B)$ . A similar analysis may be applied to the universal enveloping algebras  $U(H)$  and  $U(N)$ . In view of Eq. (9) we may choose as a basis for  $U$  the set of all monomials of the form

$$u = nhb,$$

where  $n, h$ , and  $b$  are basis monomials for  $U(N), U(H)$ , and  $U(B)$ , respectively.

We remark that we may impose a total ordering on the basis monomials with respect to their degrees. One may also impose a partial ordering on the basis monomials according to their weights under the adjoint action of  $H$  in  $U$ . A similar analysis may be applied to the algebras  $U_0, U(H_0), U(N_0)$ , and  $U(B_0)$ .

We conclude this section by setting up an (associative) algebra homomorphism of  $U(B)$  into  $U_0$  which will be needed in the following section. From Eq. (5) we may write

$$\alpha^{2i}_{2i+1} = a_{2i,2i+1} + a_{2i+2,2i-1}, \quad (10)$$

$$\alpha^{2i-1}_{2i} = 2a_{2i-1,2i}.$$

We then define a mapping

$$\theta: U(B) \rightarrow U_0, \quad (11)$$

defined by

$$\theta(a_{2i,2i+1}) = \alpha^{2i}_{2i+1},$$

$$\theta(a_{2i-1,2i}) = \frac{1}{2}\alpha^{2i-1}_{2i},$$

$$\theta(1) = 1, \quad i = 1, \dots, n-1,$$

which we extend to an algebra homomorphism to all of  $U(B)$ ; that is, if  $b = x_{i_1} \cdots x_{i_k}$  is a basis monomial of  $U(B)$  we define  $\theta(b) = \theta(x_{i_1}) \cdots \theta(x_{i_k})$  and extend linearly. It is easily verified that the mapping  $\theta$ , as defined above, satisfies the algebra homomorphism requirements

$$\theta(\alpha b_1 + \beta b_2) = \alpha \theta(b_1) + \beta \theta(b_2),$$

$$\theta(b_1 b_2) = \theta(b_1) \theta(b_2), \quad \text{for all } b_1, b_2 \in U(B),$$

and is well defined. One may check moreover that  $\theta$  is one-to-one (although this fact will not be required).

In view of Eq. (10) we may write  $\theta(x_i)$  in the form  $\theta(x_i) = x_i + n_i$ , where  $n_i \in N$ . We see from this that if  $b$  is a basis monomial of  $U(B)$  then we may write

$$\theta(b) = b + w, \quad (12)$$

where  $w$  is a sum of basis monomials in  $U$  with  $U(2n)$  weight strictly less than  $b$ .

### III. PROJECTED GEL'FAND BASIS FOR $\text{Sp}(2n)$

In this section we obtain a (nonsymmetry-adapted) basis for the irreducible representations of  $\text{Sp}(2n)$  by a method of projection from the unitary group Gel'fand-Tsetlin (GT) basis. We begin by recalling the solution to the unitary group state labeling problem obtained by Gel'fand and Tsetlin.<sup>1</sup>

The  $U(2n)$  generators  $a_{ij}$ , where  $i$  and  $j$  are restricted to values  $1, \dots, m$  (for some positive integer  $m$  less than  $2n$ ) form the generators of the unitary subgroup  $U(m)$  of  $U(2n)$ . We see therefore that  $U(2n)$  admits the canonical chain of subgroups

$$U(2n) \supset U(2n-1) \supset U(2n-2) \supset \dots \supset U(1). \quad (13)$$

Following Baird and Biedenharn,<sup>2</sup> the Casimir invariants, for each subgroup occurring in the chain (13), provide a complete set of commuting (Hermitian) operators whose normalized eigenstates form an orthonormal basis (ONB) for the irreducible representations of  $U(2n)$ . Now the eigenvalues of the Casimir invariants for the subgroup  $U(m)$  uniquely label the irreducible representations of  $U(m)$ . An alternative characterization of the irreducible representations of  $U(m)$  is in terms of their highest weights  $(\lambda_{1m}, \lambda_{2m}, \dots, \lambda_{mm})$ , where the  $\lambda_{im}$  are integers satisfying the inequalities  $\lambda_{1m} \geq \lambda_{2m} \geq \dots \geq \lambda_{mm}$ .

By virtue of Weyl's subgroup branching laws the highest weights of two groups  $U(m+1)$  and  $U(m)$  occurring in the chain (13) are related by the inequalities

$$\lambda_{1,m+1} \geq \lambda_{1m} \geq \lambda_{2,m+1} \geq \lambda_{2m} \geq \dots \geq \lambda_{mm} \geq \lambda_{m+1,m+1}.$$

The set of partitions for the chain (13) is most conveniently arranged into a GT pattern which labels the GT basis states for the irreducible representations of  $U(2n)$ . More details are given in the paper by Baird and Biedenharn.<sup>2</sup>

The crucial property that makes the GT scheme work for  $U(2n)$  is that in the reduction of an irreducible representation of  $U(m+1)$  into irreducible representations of  $U(m)$  all irreducible representations occur with unit multiplicity. This property is also shared by the orthogonal groups for which a GT scheme exists (see, e.g., Gould<sup>3</sup>).

One would ideally like to obtain a similar solution to the symplectic group state labeling problem. One method is to consider the subgroup chain

$$\text{Sp}(2n) \supset \text{Sp}(2n-2) \supset \dots \supset \text{Sp}(2) \supset U(1), \quad (14)$$

but it is well known (see Zhelobenko<sup>11</sup>) that the Casimir invariants for the subgroups occurring in the chain (14) do not give a complete labeling. The situation may be improved by considering the refinement

$$\begin{aligned} \text{Sp}(2n) \supset \text{Sp}(2) \times \text{Sp}(2n-2) \supset \text{Sp}(2n-2) \\ \supset \dots \supset \text{Sp}(2) \times \text{Sp}(2) \supset \text{Sp}(2) \supset U(1). \end{aligned} \quad (15)$$

The subgroup chain (15) in fact works for the cases  $n \leq 2$ , where we have the local isomorphisms  $\text{Sp}(4) \cong \text{O}(5)$ ,  $\text{Sp}(2) \times \text{Sp}(2) \cong \text{O}(4)$ , and  $\text{Sp}(2) \cong \text{O}(3)$ . However, for  $n > 2$  the chain (15) fails in general to provide a complete set of labels.

This failure is due to the fact that in the reduction of an

irreducible representation of  $\text{Sp}(2n)$  into irreducible representations of its subgroups  $\text{Sp}(2n-2)$  or  $\text{Sp}(2) \times \text{Sp}(2n-2)$  multiplicities may occur and extra invariants are required to completely specify the irreducible representations.

One method of obtaining a solution to the symplectic group state labeling problem is to supplement the Casimir invariants of the chain (15) with an additional set of labeling invariants. However, there still remains the problem of obtaining the eigenvalues of these additional invariants which are known to be irrational in general (and thus the action of the generators in such a basis is likely to be complicated). We propose here an alternative solution based on projection.

For our purposes it suffices to consider irreducible representations  $V(\lambda)$  of  $U(2n)$  with highest weights  $\lambda$  of the special form

$$\lambda = (\lambda_1, \lambda_2, \dots, \lambda_n, 0, 0, \dots, 0).$$

The space  $V(\lambda)$  constitutes a reducible representation of the subgroup  $\text{Sp}(2n)$ . The branching rules for the reduction of  $V(\lambda)$  into irreducible representations of  $\text{Sp}(2n)$  are given by Hamermesh<sup>20</sup> and Zhelobenko.<sup>11</sup> In general the irreducible representations of  $\text{Sp}(2n)$  occurring in the space  $V(\lambda)$  occur with multiplicities [the  $U(2n) \supset \text{Sp}(2n)$  state labeling problem].

We recall however that the space  $V(\lambda)$  contains exactly one copy of the irreducible representation of  $\text{Sp}(2n)$  with highest weight  $\lambda = (\lambda_1, \dots, \lambda_n)$ , which we denote by  $V_0(\lambda)$ . Thus the space  $V_0(\lambda)$  may be obtained by central projection from  $V(\lambda)$ . To this end let  $\Pi(\lambda)$  denote the set of all  $\text{Sp}(2n)$  highest weights occurring in  $V(\lambda)$  but excluding  $\lambda = (\lambda_1, \dots, \lambda_n)$ . Then set

$$P_n^\lambda = \prod_{\nu \in \Pi(\lambda)} \left[ \frac{\sigma_2 - \chi_\nu(\sigma_2)}{\chi_\lambda(\sigma_2) - \chi_\nu(\sigma_2)} \right], \quad (16)$$

where  $\sigma_2 = \alpha^i \alpha^j$  is the second-order invariant of  $\text{Sp}(2n)$  and

$$\chi_\nu(\sigma_2) = 2 \sum_{r=1}^n \nu_r (\nu_r + 2n - 2r)$$

is the eigenvalue of  $\sigma_2$  in the irreducible representation  $V_0(\nu)$  of  $\text{Sp}(2n)$  (see, e.g., Green<sup>21</sup>). We have explicitly included the subscript  $n$  on the left-hand side of (16) to indicate we are considering the group  $\text{Sp}(2n)$ . We have

$$V_0(\lambda) = P_n^\lambda V(\lambda). \quad (17)$$

We remark that the proof of Eq. (17) follows from noticing that the universal Casimir element separates  $\lambda$  from the weights in  $\Pi(\lambda)$ , a fact which is easily deduced from the known form of the weights in  $\Pi(\lambda)$  (see, e.g., Hamermesh<sup>20</sup>).

Thus we may obtain a set of vectors spanning the irreducible representation  $V_0(\lambda)$  by considering the central projector (16) applied to the  $U(2n)$  GT basis states of the space  $V(\lambda)$ . However, such a basis will be overcomplete and we need to consider a certain restricted set of GT vectors to yield a complete basis for the space  $V_0(\lambda)$ . To this end we restrict ourselves to Gel'fand vectors of the form

$$\begin{array}{cccccccccccc}
 \lambda_{1n} & \lambda_{2n} & & \dots & & \lambda_{nn} & 0 & 0 & & \dots & 0 \\
 & \mu_{1n} & & & \mu_{2n} & & & \mu_{nn} & 0 & 0 & \dots & 0 \\
 & & \lambda_{1,n-1} & & \lambda_{2,n-1} & & \dots & \lambda_{n-1,n-1} & 0 & & \dots & 0 \\
 & & & \mu_{1,n-1} & & \mu_{2,n-2} & & \dots & \mu_{n-1,n-1} & 0 & & 0 \\
 & & & & \ddots & & & & \ddots & & & \\
 & & & & & \lambda_{1,2} & & \lambda_{2,2} & 0 & & 0 & \\
 & & & & & & \mu_{1,2} & \mu_{2,2} & & 0 & & \\
 & & & & & & & & \lambda_{1,1} & 0 & & \\
 & & & & & & & & & \mu_{1,1} & & 
 \end{array} \quad (18)$$

From the Gel'fand betweenness conditions the integers  $\mu_{ij}$  and  $\lambda_{ij}$  in the above pattern must satisfy

$$\lambda_{1,m} \geq \mu_{1,m} \geq \lambda_{2,m} \geq \mu_{2,m} \geq \dots \geq \lambda_{m,m} \geq \mu_{m,m} \geq 0, \quad (19)$$

$$\mu_{1,m} \geq \lambda_{1,m-1} \geq \mu_{2,m} \geq \lambda_{2,m-1} \geq \dots \geq \lambda_{m-1,m-1} \geq \mu_{m,m} \geq 0.$$

For simplicity we denote the GT state (18) by

$$\begin{pmatrix}
 \lambda_{1n} & \lambda_{2n} & \dots & \lambda_{nn} \\
 \mu_{1n} & \mu_{2n} & \dots & \mu_{nn} \\
 \lambda_{1,n-1} & \lambda_{2,n-1} & \dots & \lambda_{n-1,n-1} \\
 \mu_{1,n-1} & \mu_{2,n-1} & \dots & \mu_{n-1,n-1} \\
 \vdots & & & \\
 \mu_{12} & \mu_{22} & & \\
 \lambda_{11} & & & \\
 \mu_{11} & & & 
 \end{pmatrix}. \quad (20)$$

We remark that these are the patterns appearing in the work of Zhelobenko<sup>11</sup> (although no group theoretic meaning is attached to such patterns in his work).

It is our aim now to show that we may obtain a complete set of basis states for the space  $V_0(\lambda)$  by central projection from the GT states (20). For simplicity we denote the space spanned by the GT basis states of the special form (18) by  $A_0(\lambda)$  which we refer to as the space of allowed GT states.

Now from the work of Zhelobenko,<sup>11</sup> the number of allowed Gel'fand states (20) is precisely equal to  $\dim V_0(\lambda)$ ; that is,

$$\dim A_0(\lambda) = \dim V_0(\lambda). \quad (21)$$

Thus in order to prove our result it suffices to prove that the central projector  $P^\lambda_n$  is one-to-one on  $A_0(\lambda)$ , that is,

$$(\ker P^\lambda_n) \cap A_0(\lambda) = (0),$$

where

$$\ker P^\lambda_n = \{v \in V(\lambda) | P^\lambda_n v = 0\}.$$

Equation (21) then guarantees that  $P^\lambda_n A_0(\lambda) = V_0(\lambda)$ .

Note that the GT state (20) has  $U(2n)$  weight  $(\rho_1, \rho_2, \dots, \rho_{2n})$ , where

$$\rho_{2i-1} = \sum_{j=1}^i \mu_{j,i} - \sum_{j=1}^{i-1} \lambda_{j,i-1},$$

$$\rho_{2i} = \sum_{j=1}^i \lambda_{j,i} - \sum_{j=1}^i \mu_{j,i}, \quad i = 1, \dots, n.$$

Now, since the CSA  $H_0$  of  $Sp(2n)$  is contained in the CSA  $H$  of  $U(2n)$  we see that the state (20) is also a weight state of  $Sp(2n)$  with weight  $(\nu_1, \dots, \nu_n)$ , where

$$\nu_i = \rho_{2i-1} - \rho_{2i}, \quad i = 1, \dots, n.$$

In particular the maximal allowable GT state

$$\Omega^\lambda_0 = \begin{pmatrix}
 \lambda_{1n} & \lambda_{2n} & \dots & \lambda_{nn} \\
 \lambda_{1n} & \lambda_{2n} & \dots & \lambda_{nn} \\
 \lambda_{1n} & \lambda_{2n} & \dots & \lambda_{n-1,n} \\
 \lambda_{1n} & \lambda_{2n} & \dots & \lambda_{n-1,n} \\
 \vdots & & & \\
 \lambda_{1n} & & & \\
 \lambda_{1n} & & & 
 \end{pmatrix} \quad (22)$$

has  $Sp(2n)$  weight  $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_n)$  which is the highest weight of  $V_0(\lambda)$ . It is easily shown that the GT state (22) in fact constitutes a highest weight state for  $Sp(2n)$  by verifying that the elementary raising generators [see Eq. (8)] vanish on the state  $\Omega^\lambda_0$ . Thus we have immediately  $\Omega^\lambda_0 \in V_0(\lambda)$  whence

$$P^\lambda_n \Omega^\lambda_0 = \Omega^\lambda_0.$$

We note moreover that  $\Omega^\lambda_0$  has  $U(2n)$  weight  $(\lambda_1, 0, \lambda_2, 0, \dots, \lambda_n, 0)$  which is conjugate under the Weyl group to the  $U(2n)$  highest weight  $(\lambda_1, \lambda_2, \dots, \lambda_n, 0, \dots, 0)$  and hence occurs with unit multiplicity in  $V(\lambda)$ .

Some of the properties of the space  $A_0(\lambda)$  are summarized in the following (notation as in Sec. II).

**Lemma 1:** (a) If  $\Omega \in A_0(\lambda)$  is a  $U(2n)$  weight vector then there exists a basis monomial  $b \in U(B)$  such that  $b\Omega = \alpha \Omega^\lambda_0$ ,  $0 \neq \alpha \in \mathbb{C}$ .

(b) If  $\Omega \in A_0(\lambda)$  is arbitrary then there exists  $b \in U(\bar{B})$  such that  $b\Omega = \Omega^\lambda_0$ .

(c)  $A_0(\lambda)$  is a module over the algebras  $U(N)$  and  $U(\bar{N})$  and is cyclically generated by  $\Omega^\lambda_0$ ; that is,

$$A_0(\lambda) = U(\bar{N})\Omega^\lambda_0 = U(N)\Omega^\lambda_0.$$

**Proof:** (a) Our proof of this result is based on a lengthy induction argument (which is not relevant to the remainder of this paper) and is presented in Appendix A for clarity of presentation.

(b) The proof of (b) is an immediate consequence of (a) since one may project out weight states from a general state with elements from  $U(H)$ .

(c) From the known action of the  $U(2n)$  generators on GT states (see Baird and Biedenharn<sup>2</sup> and Gould<sup>22</sup>) it is clear

that the space  $A_0(\lambda)$  is stable under the action of  $N$  and  $\bar{N}$ . Hence  $A_0(\lambda)$  constitutes a module over  $U(N)$  and  $U(\bar{N})$ . Now set

$$A'_0(\lambda) = U(\bar{N})\Omega^{\lambda_0} \subseteq A_0(\lambda).$$

We prove  $A'_0(\lambda) = A_0(\lambda)$  by a contradiction argument. For suppose on the contrary  $A'_0(\lambda) \neq A_0(\lambda)$  and choose  $\Omega \in A_0(\lambda)$  orthogonal to  $A'_0(\lambda)$ . Then we have

$$0 = \langle \Omega | U(\bar{N})\Omega^{\lambda_0} \rangle = \langle U(\bar{B})\Omega | \Omega^{\lambda_0} \rangle. \quad (23)$$

But by (b) there exists  $b \in U(\bar{B})$  such that  $b\Omega = \Omega^{\lambda_0}$  whence (23) implies  $0 = \langle b\Omega | \Omega^{\lambda_0} \rangle = \langle \Omega^{\lambda_0} | \Omega^{\lambda_0} \rangle = 1$  and a contradiction has been reached. Thus our assumption was false and we must have  $A'_0(\lambda) = U(\bar{N})\Omega^{\lambda_0} = A_0(\lambda)$ . From the PBW theorem we may write

$$U(\bar{N}) = U(N)U(H),$$

whence

$$A_0(\lambda) = U(\bar{N})\Omega^{\lambda_0} = U(N)U(H)\Omega^{\lambda_0} = U(N)\Omega^{\lambda_0}. \quad \text{Q.E.D.}$$

Now from Sec. II there exists an algebra homomorphism  $\theta: U(\bar{B}) \rightarrow U_0$  [see Eq. (11)]. This result together with the previous lemma implies the following.

**Lemma 2:** If  $\Omega \in A_0(\lambda)$  there exists  $u \in U_0$  such that

$$\langle \Omega^{\lambda_0} | u\Omega \rangle = 1.$$

**Proof:** Let  $\Omega \in A_0(\lambda)$  be arbitrary. Then  $\Omega$  may be decomposed into a sum of weight vectors

$$\Omega = \sum_{i=1}^k \Omega_{\mu_i},$$

where  $\Omega_{\mu_i}$  has  $U(2n)$  weight  $\mu_i$ . Assume the weights  $\mu_1, \dots, \mu_k$  are ordered in decreasing order with respect to the partial ordering induced by the positive roots (i.e., lexical ordering). Thus  $\mu_1$  is maximal in the set of weights  $\{\mu_i\}_{i=1}^k$ . Then there exists  $h \in U(H)$  such that  $h\Omega = \Omega_{\mu_1}$ . Then by Lemma 1(a) there exists a basis monomial  $b \in U(\bar{B})$  such that

$$bh\Omega = b\Omega_{\mu_1} = \alpha\Omega^{\lambda_0}, \quad \alpha \neq 0.$$

If we denote the  $U(2n)$  weight of the state  $\Omega^{\lambda_0}$  by  $\lambda_0$  then we see that the basis monomial  $b$  has weight  $\lambda_0 - \mu_1$  and we may write  $bh = h'b$  for suitable  $h' \in U(H)$ . Thus we have

$$\alpha = \langle \Omega^{\lambda_0} | h'b\Omega \rangle = \langle h'\Omega^{\lambda_0} | b\Omega \rangle = \beta \langle \Omega^{\lambda_0} | b\Omega \rangle, \quad (24)$$

for some  $\beta \neq 0$ . Now we let  $u = \theta(b) \in U_0$ . From Eq. (12) we have  $u = b + w$  where  $w$  is a sum of basis monomials in  $U$  with  $U(2n)$  weight strictly less than  $\lambda_0 - \mu_1$ . We may thus write

$$u\Omega = b\Omega + w\Omega = b\Omega + \sum_{i=1}^k w\Omega_{\mu_i}.$$

Since  $\mu_1$  is maximal in the set of  $U(2n)$  weights  $\{\mu_j\}_{j=1}^k$  and since  $w$  has  $U(2n)$  weight strictly less than  $\lambda_0 - \mu_1$  it follows that  $w\Omega_{\mu_i}$  has weight, strictly less than  $\lambda_0 - \mu_1 + \mu_i$ , which cannot equal  $\lambda_0$ . Thus each state  $w\Omega_{\mu_i}$  must be orthogonal to  $\Omega^{\lambda_0}$ , from which we obtain, in view of (24),

$$\langle \Omega^{\lambda_0} | u\Omega \rangle = \langle \Omega^{\lambda_0} | b\Omega \rangle = \alpha/\beta \neq 0.$$

Replacing  $u \in U_0$  by  $(\beta/\alpha)u$  the result is seen to follow. Q.E.D.

We are now in a position to prove our main result.

**Theorem 1:**  $[\ker P^{\lambda_n}] \cap A_0(\lambda) = (0)$  and  $P^{\lambda_n}A_0(\lambda) = V_0(\lambda)$ . In particular if  $\{\Omega_i\}$  is a basis for  $A_0(\lambda)$  then the projected states  $\tilde{\Omega}_i = P^{\lambda_n}\Omega_i$  constitute a basis  $\{\tilde{\Omega}_i\}$  for  $V_0(\lambda)$ .

**Proof:** Clearly  $P^{\lambda_n}A_0(\lambda) \subseteq V_0(\lambda)$ . We prove  $P^{\lambda_n}$  is one-to-one on  $A_0(\lambda)$  using a contradiction argument. Suppose on the contrary there exists  $\Omega \in A_0(\lambda)$  such that  $P^{\lambda_n}\Omega = 0$ . But Lemma 2 implies there exists  $u \in U_0$  such that  $\langle u\Omega | \Omega^{\lambda_0} \rangle = 1$ . Now since  $P^{\lambda_n}$  commutes with the action of  $Sp(2n)$  we have

$$0 = P^{\lambda_n}\Omega = uP^{\lambda_n}\Omega = P^{\lambda_n}u\Omega.$$

Thus

$$0 = \langle P^{\lambda_n}u\Omega | \Omega^{\lambda_0} \rangle = \langle u\Omega | P^{\lambda_n}\Omega^{\lambda_0} \rangle = \langle u\Omega | \Omega^{\lambda_0} \rangle = 1$$

and a contradiction has been reached. Thus our assumption was false and  $P^{\lambda_n}$  must be one-to-one on  $A_0(\lambda)$ ; that is,  $[\ker P^{\lambda_n}] \cap A_0(\lambda) = (0)$ . Since  $P^{\lambda_n}A_0(\lambda) \subseteq V_0(\lambda)$  Eq. (21) implies that  $P^{\lambda_n}A_0(\lambda) = V_0(\lambda)$ .

Thus if  $\{\Omega_i | i = 1, \dots, d_\lambda = \dim V_0(\lambda)\}$  denotes a basis for  $A_0(\lambda)$  then the projected states  $P^{\lambda_n}\Omega_i$  constitute a basis for  $V_0(\lambda)$ . Q.E.D.

As a particular case of the above result we may choose the GT states (20) as a basis for  $A_0(\lambda)$  and the projected states constitute a basis for  $V_0(\lambda)$  which we denote by

$$\left( \begin{array}{cccc} \lambda_{1,n} & \lambda_{2,n} & \dots & \lambda_{nn} \\ \mu_{1,n} & \mu_{2,n} & \dots & \mu_{nn} \\ \lambda_{1,n-1} & \lambda_{2,n-1} & \dots & \lambda_{n-1,n-1} \\ \mu_{1,n-1} & \mu_{2,n-1} & \dots & \mu_{n-1,n-1} \\ \vdots & & & \\ \lambda_{12} & \lambda_{22} & & \\ \mu_{12} & \mu_{22} & & \\ \lambda_{11} & & & \\ \mu_{11} & & & \end{array} \right) = P^{\lambda_n} \left( \begin{array}{cccc} \lambda_{1,n} & \lambda_{2,n} & \dots & \lambda_{nn} \\ \mu_{1,n} & \mu_{2,n} & \dots & \mu_{nn} \\ \lambda_{1,n-1} & \lambda_{2,n-1} & \dots & \lambda_{n-1,n-1} \\ \mu_{1,n-1} & \mu_{2,n-1} & \dots & \mu_{n-1,n-1} \\ \vdots & & & \\ \lambda_{12} & \lambda_{22} & & \\ \mu_{12} & \mu_{22} & & \\ \lambda_{11} & & & \\ \mu_{11} & & & \end{array} \right). \quad (25)$$

These states form a complete basis (of weight vectors) for the space  $V_0(\lambda)$  which we refer to as the projected Gel'fand-Tsetlin (PGT) basis. We remark however that the

PGT basis (25) is not orthonormal and moreover is not symmetry adapted to either of the subgroup chains (14) or (15). We shall consider the problem of constructing a symmetry-

adapted basis for the space  $V_0(\lambda)$  in the next section.

The labels appearing in the PGT basis for the space  $V_0(\lambda)$  are those of the GT states from which we project. This method of labeling may be compared with a recently announced solution (see Edwards and Gould<sup>8</sup>) to the Clebsch-Gordan multiplicity problem (for arbitrary semisimple Lie algebras) where the highest weight states occurring in the tensor product representation  $V(\lambda) \otimes V(\mu)$  are obtained by central projection from states of the form  $e_i \otimes e^\mu_+$ , where  $\{e_i\}$  denotes a basis for the irreducible representation  $V(\lambda)$  and  $e^\mu_+$  is the maximal weight vector of  $V(\mu)$ . Thus the maximal weight states (and hence the irreducible representations they generate) may be labeled by the vector  $e_i \otimes e^\mu_+$  (or equivalently  $e_i$ ) from which these states are projected. In the case of  $U(N)$  where  $\{e_i\}$  is the GT basis, this leads to a GT pattern labeling for the irreducible representations occurring in the tensor product representation  $V(\lambda) \otimes V(\mu)$ . In terms of the equivalent problem of determining all tensor operators for a semisimple Lie algebra this leads to a GT pattern labeling for tensor operators which is closely related to the operator patterns of Biedenharn *et al.*<sup>9,10</sup> More details are given in Ref. 8.

Although the basis (25) is nonorthogonal one may obtain, at least in principle, the overlap coefficients for this basis from the known matrix elements of the  $U(2n)$  generators together with the explicit form (16) for the projection operators  $P^\lambda_n$ . Whether this leads to an efficient algorithm for adaption to computers (or ideally analytic manipulation) remains to be seen. However having obtained the overlap coefficients the action of the symplectic group generators in the basis (25) may be obtained from the known matrix elements of the  $U(2n)$  generators, viz.,

$$\begin{aligned}\alpha_{ij} |(\lambda)\rangle &= \alpha_{ij} P^\lambda_n |(\lambda)\rangle \\ &= P^\lambda_n \alpha_{ij} |(\lambda)\rangle \\ &= P^\lambda_n [g_{ip} a_{pj} + g_{jp} a_{pi}] |(\lambda)\rangle,\end{aligned}$$

where  $(\lambda)$  denotes an allowed GT pattern for  $Sp(2n)$ . The trouble with this method, however, is that the  $Sp(2n)$  generators  $\alpha_{ij}$  do not leave the space  $A_0(\lambda)$  invariant and hence it is necessary to obtain the matrix elements of  $P^\lambda_n$  in the GT basis for  $V(\lambda)$  [and not just for  $A_0(\lambda)$ ]. This deficiency will be removed, by considering a symmetry-adapted basis, in the following section.

#### IV. SYMMETRY-ADAPTED BASIS FOR $Sp(2n)$

As mentioned in Sec. III the trouble with the PGT basis (25) is that it is not symmetry adapted to the subgroup chain (14). In order to obtain a symmetry-adapted basis we need to apply the above projective scheme recursively for each of the subgroups  $Sp(2n)$ ,  $Sp(2n-2)$ , ...,  $Sp(2)$ .

We denote the PGT state (25) by the simpler notation

$$\left| \begin{matrix} (\lambda) \\ (\mu) \end{matrix} \right\rangle = P^\lambda_n \left| \begin{matrix} (\lambda) \\ (\mu) \end{matrix} \right\rangle,$$

where  $(\lambda)$  and  $(\mu)$  denote the patterns

$$\begin{aligned}(\lambda) &= \begin{pmatrix} \lambda_{1n} & \cdots & \lambda_{nn} \\ \lambda_{1n-1} & \cdots & \lambda_{n-1n-1} \\ \vdots & & \\ \lambda_{12} & \lambda_{22} & \\ \lambda_{11} & & \end{pmatrix}, \\ (\mu) &= \begin{pmatrix} \mu_{1n} & \cdots & \mu_{nn} \\ \mu_{1n-1} & \cdots & \mu_{n-1n-1} \\ \vdots & & \\ \mu_{12} & \mu_{22} & \\ \mu_{11} & & \end{pmatrix}.\end{aligned}$$

The numbers  $\lambda_{ij}$  and  $\mu_{ij}$  satisfy the betweenness conditions of Eq. (19). Corresponding to each row in the  $(\lambda)$  (i.e., upper) pattern

$$\lambda_m = (\lambda_{1m}, \lambda_{2m}, \dots, \lambda_{mm}),$$

we construct the associated  $Sp(2m)$  projector  $P^{\lambda_m}_m$  [cf. Eq. (16) for  $Sp(2n)$ ]. Clearly these subgroup projectors satisfy the rules

$$(P^{\lambda_m}_m)^2 = P^{\lambda_m}_m, \quad P^{\lambda_m}_m P^{\lambda_k}_k = P^{\lambda_k}_k P^{\lambda_m}_m. \quad (26)$$

We now consider the compound projector

$$P_{(\lambda)} = \prod_{m=1}^n P^{\lambda_m}_m. \quad (27)$$

In view of Eq. (26) it is clear that the projection operators (27) obey the rule

$$P_{(\lambda)} P_{(\lambda')} = P_{(\lambda \cap \lambda')}, \quad P_{(\lambda)}^2 = P_{(\lambda)}.$$

By repeated application of Theorem 1 it is easily deduced that the states  $P_{(\lambda)} \left| \begin{matrix} (\lambda) \\ (\mu) \end{matrix} \right\rangle$  form a basis for the irreducible representation  $V_0(\lambda)$  which, by our construction, is symmetry adapted to the subgroup chain (14). For ease of notation we denote these symmetry-adapted states by

$$\left| \begin{matrix} (\lambda) \\ (\mu) \end{matrix} \right\rangle_0 = P_{(\lambda)} \left| \begin{matrix} (\lambda) \\ (\mu) \end{matrix} \right\rangle. \quad (28)$$

The group theoretical interpretation of the (upper)  $(\lambda)$  pattern is now obvious and refers to the highest weights (or equivalently the eigenvalues of Casimir invariants) of the groups in the subgroup chain (14) in analogy with the GT states for  $U(n)$  and  $O(n)$ . This implies that two symmetry-adapted states (28) are orthogonal unless they have the same upper pattern, i.e.,

$$\left\langle \begin{matrix} (\lambda') \\ (\mu') \end{matrix} \middle| \begin{matrix} (\lambda) \\ (\mu) \end{matrix} \right\rangle_0 = 0, \quad \text{unless } (\lambda') = (\lambda).$$

The (lower)  $(\mu)$  pattern is a multiplicity label which, in our approach, refers to the representation labels of the groups  $U(2n-1), U(2n-3), \dots, U(3), U(1)$  of the  $U(2n)$  GT vectors (20) from which we are projecting. This dual-pattern labeling may be compared to the pattern calculus of Biedenharn *et al.*<sup>9,10</sup> developed for the tensor operator problem of  $U(N)$ . In this latter approach two GT patterns appear, one (which has group theoretical significance) for labeling the components of the tensor operator and the other a multiplicity label (i.e., operator pattern). This dual-pattern idea also appears in the work of Zhelobenko<sup>11</sup> but without any group theoretical significance.

We note that the basis (28) is not symmetry adapted to the subgroup chain (15). However the generators of the subgroup

$$G = \text{Sp}(2) \times \text{Sp}(2) \times \cdots \times \text{Sp}(2) \quad (n \text{ times})$$

of  $\text{Sp}(2n)$  have a simple action on the basis states (28). To see this consider the infinitesimal generators of the subgroup  $G$  [see Eqs. (6) and (7)]

$$\alpha^{2m-1}_{2m} = 2a_{2m-1,2m}, \quad \alpha^{2m}_{2m-1} = 2a_{2m,2m-1}, \quad (29)$$

$$h_m = a_{2m-1,2m-1} - a_{2m,2m}, \quad m = 1, \dots, n.$$

The Cartan generators  $h_m$  are diagonal in the basis (28) with eigenvalue given by

$$h_m \left| \begin{smallmatrix} \lambda \\ \mu \end{smallmatrix} \right\rangle_0 = \left[ 2 \sum_{j=1}^m \mu_{j,m} - \sum_{j=1}^m \lambda_{j,m} - \sum_{j=1}^{m-1} \lambda_{j,m-1} \right] \left| \begin{smallmatrix} \lambda \\ \mu \end{smallmatrix} \right\rangle_0. \quad (30)$$

The generators (29) of the group  $G$  moreover commute with the Casimir invariants of the subgroup chain (14) so that we may write

$$N^m_r = (\mu_{r,m} + m + 1 - r)^{1/2} \times \left( \frac{(-1)^{m+1} \prod_{p=1}^m (\lambda_{p,m} - \mu_{r,m} + r - p) \prod_{l=1}^{m-1} (\mu_{r,m} - \lambda_{l,m-1} + l - r + 1)}{\prod_{\substack{l=1 \\ l \neq r}}^m (\mu_{r,m} - \mu_{l,m} + l - r)(\mu_{r,m} - \mu_{l,m} + l - r - 1)} \right)^{1/2}. \quad (31)$$

Thus we obtain the result

$$\alpha^{2m-1}_{2m} \left| \begin{smallmatrix} \lambda \\ \mu \end{smallmatrix} \right\rangle_0 = 2 \sum_{r=1}^m N^m_r \left| \begin{smallmatrix} \lambda \\ (\mu) + \Delta^m_r \end{smallmatrix} \right\rangle_0,$$

with  $N^m_r$  as in Eq. (31). Similarly for the lowering generator  $\alpha^{2m}_{2m-1}$  we obtain

$$\alpha^{2m}_{2m-1} \left| \begin{smallmatrix} \lambda \\ \mu \end{smallmatrix} \right\rangle_0 = 2 \sum_{r=1}^m \bar{N}^m_r \left| \begin{smallmatrix} \lambda \\ (\mu) - \Delta^m_r \end{smallmatrix} \right\rangle_0,$$

where

$$\bar{N}^m_r = (\mu_{r,m} + 2m - r - 1)^{1/2} \left( \frac{(-1)^{m+1} \prod_{p=1}^m (\lambda_{p,m} - \mu_{r,m} + r - p + 1) \prod_{l=1}^{m-1} (\mu_{r,m} - \lambda_{l,m-1} + l - r)}{\prod_{\substack{l=1 \\ l \neq r}}^m (\mu_{r,m} - \mu_{l,m} + l - r)(\mu_{r,m} - \mu_{l,m} + l - r - 1)} \right)^{1/2}. \quad (32)$$

Thus using the matrix element formulas (31) and (32) one may obtain, in principle, a basis symmetry adapted to the subgroup chain (15). This can be done using either a lowering operator method or by construction of projection operators for the subgroup chain (15) in analogy with the projection operators of Eq. (27) for the subgroup chain (14). We aim to consider this aspect of the problem in a future publication.

Although the generators (29) of the subgroup  $G$  have a simple action on the basis states (28) the action of the remaining generators  $\alpha_{ij}$  is not so clear. It suffices, in principle, to determine the action of the elementary generators (8). The matrix elements of the remaining generators can then be obtained using repeated commutation. However, unlike the generators (29), the elementary generators (8) do not commute with the projection operators  $P_{(\lambda)}$  so it is necessary to proceed indirectly. From Eq. (5) we may expand the generators  $\alpha^{2m-1}_{2m+1}$  and  $\alpha^{2m+1}_{2m-1}$  in terms of  $\text{U}(2n)$  generators according to

$$\begin{aligned} \alpha^{2m-1}_{2m} \left| \begin{smallmatrix} \lambda \\ \mu \end{smallmatrix} \right\rangle_0 &= \alpha^{2m-1}_{2m} P_{(\lambda)} \left| \begin{smallmatrix} \lambda \\ \mu \end{smallmatrix} \right\rangle \\ &= P_{(\lambda)} \alpha^{2m-1}_{2m} \left| \begin{smallmatrix} \lambda \\ \mu \end{smallmatrix} \right\rangle. \end{aligned}$$

Note that the generators (29) leave the space  $A_0(\lambda)$  (to which the GT state  $\left| \begin{smallmatrix} \lambda \\ \mu \end{smallmatrix} \right\rangle_0$  belongs) invariant, and moreover the generators  $\alpha^{2m-1}_{2m}$  (and  $\alpha^{2m}_{2m-1}$ ) can only effect the  $m$ th row  $\mu_m = (\mu_{1m}, \dots, \mu_{mm})$  of the (lower)  $(\mu)$  pattern. Using the known matrix element formulas of the  $\text{U}(2n)$  generators (see, e.g., Gould<sup>22</sup>) we have

$$\begin{aligned} \alpha^{2m-1}_{2m} \left| \begin{smallmatrix} \lambda \\ \mu \end{smallmatrix} \right\rangle_0 &= 2a_{2m-1,2m} \left| \begin{smallmatrix} \lambda \\ \mu \end{smallmatrix} \right\rangle_0 \\ &= 2 \sum_{r=1}^m N^m_r \left| \begin{smallmatrix} \lambda \\ (\mu) + \Delta^m_r \end{smallmatrix} \right\rangle, \end{aligned}$$

where  $(\mu) + \Delta^m_r$  is the pattern obtained from  $(\mu)$  by the shifts

$$\mu_{k,l} \rightarrow \mu_{k,l}, \quad \mu_{r,m} \rightarrow \mu_{r,m} + 1, \quad \text{for } (k,l) \neq (r,m).$$

From the known matrix elements of the  $\text{U}(2n)$  generators we deduce the result (see Appendix B)

$$\alpha^{2m-1}_{2m+1} = a_{2m-1,2m+1} - a_{2m+2,2m}, \quad (33)$$

$$\alpha^{2m+1}_{2m-1} = a_{2m+1,2m-1} - a_{2m,2m+2}.$$

Using the known action of the  $\text{U}(2n)$  generators on the GT basis states (20) together with the shift properties of the  $\text{Sp}(2n)$  generators (33) on the representation labels of the subgroup  $\text{Sp}(2m)$ , we deduce that the action of the generators (33) on the basis states (28) is of the form

$$\begin{aligned} \alpha^{2m-1}_{2m+1} \left| \begin{smallmatrix} \lambda \\ \mu \end{smallmatrix} \right\rangle_0 &= \sum_{r,l=1}^m N^{m,+}_{r,l} \left| \begin{smallmatrix} \lambda \\ (\mu) + \Delta^{m,+}_{r,l} \end{smallmatrix} \right\rangle_0 \\ &+ \sum_{r=1}^m \sum_{l=1}^{m+1} N^{m,-}_{r,l} \left| \begin{smallmatrix} \lambda \\ (\mu) - \Delta^{m,-}_{r,l} \end{smallmatrix} \right\rangle_0, \end{aligned} \quad (34)$$



$$\begin{aligned} & \alpha^{2m+1} {}_{2m-1} \left| \begin{matrix} (\lambda) \\ (\mu) \end{matrix} \right\rangle_0 \\ &= \sum_{r,l=1}^m \bar{N}^{m,+}_{r,l} \left| \begin{matrix} (\lambda) - \Delta^m_r \\ (\mu) - \Delta^m_l \end{matrix} \right\rangle_0 \\ &+ \sum_{r=1}^m \sum_{l=1}^{m+1} \bar{N}^{m,-}_{r,l} \left| \begin{matrix} (\lambda) + \Delta^m_r \\ (\mu) + \Delta^{m+1}_l \end{matrix} \right\rangle_0. \end{aligned} \quad (35)$$

We hope to be permitted to evaluate the matrix elements in Eqs. (34) and (35) [which follow directly from the  $U(2n)$  matrix element formulas] in a future publication. Since the basis states (28) are nonorthogonal there still remains the problem of evaluating overlap coefficients, to which we now turn.

## V. OVERLAP COEFFICIENTS

The overlap coefficients for the basis states (28) are given by the matrix elements of the projection operator (27) between GT basis states in the space  $A_0(\lambda)$ , viz.,

$${}_0 \left\langle \begin{matrix} (\lambda') \\ (\mu') \end{matrix} \middle| \begin{matrix} (\lambda) \\ (\mu) \end{matrix} \right\rangle_0 = \left( \begin{matrix} (\lambda') \\ (\mu') \end{matrix} \middle| P_{(\lambda)} \middle| \begin{matrix} (\lambda) \\ (\mu) \end{matrix} \right). \quad (36)$$

Since the projection operator  $P_{(\lambda)}$  may be expressed as a polynomial in the second-order Casimir invariants for the groups in the chain (14) one may in principle calculate these overlap coefficients using the matrix element formulas for the  $U(2n)$  generators. Whether this yields an effective algorithm for adaption to computers (or ideally analytic manipulation) remains to be seen. Nevertheless we may deduce some elementary properties of the coefficients (36) from general considerations.

We have already noted that the overlap coefficient (36) vanishes unless the (upper)  $(\lambda)$  patterns coincide. Thus we have

$${}_0 \left\langle \begin{matrix} (\lambda') \\ (\mu') \end{matrix} \middle| \begin{matrix} (\lambda) \\ (\mu) \end{matrix} \right\rangle_0 = \delta_{(\lambda'),(\lambda)} {}_0 \left\langle \begin{matrix} (\lambda) \\ (\mu') \end{matrix} \middle| \begin{matrix} (\lambda) \\ (\mu) \end{matrix} \right\rangle_0. \quad (37)$$

Next, since the states (28) are eigenstates of the Cartan generators  $h_m$  Eq. (30) implies that the overlap coefficient (37) vanishes unless  $N_m(\mu) = N_m(\mu')$  where we define

$$N_m(\mu) = \sum_{i=1}^m \mu_{i,m}.$$

Thus we obtain

$${}_0 \left\langle \begin{matrix} (\lambda) \\ (\mu') \end{matrix} \middle| \begin{matrix} (\lambda) \\ (\mu) \end{matrix} \right\rangle_0 = \delta_{N(\mu),N(\mu')} {}_0 \left\langle \begin{matrix} (\lambda) \\ (\mu') \end{matrix} \middle| \begin{matrix} (\lambda) \\ (\mu) \end{matrix} \right\rangle_0, \quad (38)$$

where  $N(\mu) = (N_1(\mu), N_2(\mu), \dots, N_n(\mu))$ . Note that the maximal state

$$\left| \begin{matrix} (\lambda) \\ (\lambda) \end{matrix} \right\rangle_0 = \left| \begin{matrix} (\lambda) \\ (\lambda) \end{matrix} \right\rangle_0$$

[see Eq. (22)] satisfies

$${}_0 \left\langle \begin{matrix} (\lambda) \\ (\mu') \end{matrix} \middle| \begin{matrix} (\lambda) \\ (\lambda) \end{matrix} \right\rangle_0 = \delta_{(\mu'),(\lambda)}. \quad (39)$$

Now in view of Schur's lemma we may write

$$\begin{aligned} & \left( \begin{matrix} (\lambda) \\ (\mu') \end{matrix} \middle| P^{\lambda_n} P^{\lambda_{n-1}} \middle| \begin{matrix} (\lambda) \\ (\mu) \end{matrix} \right) \\ &= \alpha \left( \begin{matrix} (\lambda)_{n-1} \\ (\mu')_{n-1} \end{matrix} \middle| P^{\lambda_{n-1}} \middle| \begin{matrix} (\lambda)_{n-1} \\ (\mu)_{n-1} \end{matrix} \right), \end{aligned}$$

for some constant  $\alpha$ , where the patterns  $(\lambda)_{n-1}, (\mu)_{n-1}$  are obtained from the patterns  $(\lambda), (\mu)$ , respectively, by omission of the top rows. We call the constant  $\alpha$  the reduced  $\text{Sp}(2n):\text{Sp}(2n-2)$  overlap coefficient and write it in the form

$$\left\langle \begin{matrix} \lambda_n \\ \lambda_{n-1} \\ \mu'_n : \mu_n \end{matrix} \right\rangle \quad (40)$$

to indicate that it depends only on the labels  $\lambda_n, \lambda_{n-1}, \mu_{n-1}$ , and  $\mu'_{n-1}$ . Thus we may write

$${}_0 \left\langle \begin{matrix} (\lambda) \\ (\mu') \end{matrix} \middle| \begin{matrix} (\lambda) \\ (\mu) \end{matrix} \right\rangle_0 = \left\langle \begin{matrix} \lambda_n \\ \lambda_{n-1} \\ \mu'_n : \mu_n \end{matrix} \right\rangle {}_0 \left\langle \begin{matrix} (\lambda)_{n-1} \\ (\mu')_{n-1} \end{matrix} \middle| \begin{matrix} (\lambda)_{n-1} \\ (\mu)_{n-1} \end{matrix} \right\rangle_0,$$

showing that the  $\text{Sp}(2n)$  overlap coefficients may be written as an  $\text{Sp}(2n):\text{Sp}(2n-2)$  reduced overlap coefficient times an  $\text{Sp}(2n-2)$  overlap coefficient. Thus it suffices to evaluate only the reduced overlap coefficients (40) [for each group in the chain (14)]. If we choose the representation labels of the subgroup  $\text{Sp}(2n-2)$  to be maximal, Eq. (39) implies

$$\left\langle \begin{matrix} \lambda_n \\ \lambda_{n-1} \\ \mu'_n : \mu_n \end{matrix} \right\rangle = \left( \begin{matrix} \lambda_n \\ \lambda_{n-1} \\ (\lambda)_{\max} \\ \mu'_n \\ (\lambda)_{\max} \end{matrix} \middle| P^{\lambda_n} \middle| \begin{matrix} \lambda_n \\ \lambda_{n-1} \\ (\lambda)_{\max} \\ \mu_n \\ (\lambda)_{\max} \end{matrix} \right).$$

In other words it suffices to evaluate the matrix elements of the projector  $P^{\lambda_n}$  between the GT states of the space  $A_0(\lambda)$  which are maximal in the subgroup  $\text{Sp}(2n-2)$  (i.e., semi-maximal states). This observation clearly reduces the problem of evaluating the overlap coefficient (37).

From Eqs. (37) and (39) we deduce that the reduced overlap coefficients satisfy

$$\left\langle \begin{matrix} \lambda_n \\ \lambda_{n-1} \\ \mu'_n : \mu_n \end{matrix} \right\rangle = 0, \quad \text{unless } N_n(\mu) = N_n(\mu').$$

Moreover for the maximal reduced overlap coefficients (i.e.,  $\lambda_{i,n-1} = \lambda_{i,n}, i = 1, \dots, n-1$  and  $\mu_{i,n} = \lambda_{i,n}$ , for  $i = 1, \dots, n$ ) we have

$$\left\langle \begin{matrix} \lambda_n \\ \lambda_n \\ \mu'_n : \lambda_n \end{matrix} \right\rangle = \delta_{\mu'_n, \lambda_n}.$$

Since the matrix elements of the  $U(2n)$  generators in the GT basis may be chosen real we deduce that the reduced overlap coefficients are real and satisfy

$$\left\langle \begin{matrix} \lambda_n \\ \lambda_{n-1} \\ \mu'_n : \mu_n \end{matrix} \right\rangle = \left\langle \begin{matrix} \lambda_n \\ \lambda_{n-1} \\ \mu_n : \mu'_n \end{matrix} \right\rangle.$$

Also, due to the properties of projectors, we must have

$$1 \geq \left\langle \begin{matrix} \lambda_n \\ \lambda_{n-1} \\ \mu'_n : \mu_n \end{matrix} \right\rangle^2 \geq 0.$$

We do not consider the problem of overlap coefficients any further here. We remark however that there still remain other general features of the labeling scheme we have advocated which may be studied from general considerations. One of these is the property of asymptotic orthogonality where the basis (28) becomes orthogonal in a certain limit of large quantum numbers. This property is satisfied by the solution to the Clebsch-Gordan multiplicity problem (for semisimple Lie algebras) given by Edwards and Gould.<sup>8</sup> Another example of this asymptotic behavior is afforded by Elliot's<sup>5,23</sup> well-known solution to the  $U(3) \supset O(3)$  state labeling problem where the projected  $O(3)$  states of Elliot rapidly approach orthogonality when the  $U(3)$  quantum numbers get sufficiently large. It would be of interest to determine whether such an asymptotic orthogonality is satisfied by the states (28). In terms of reduced overlap coefficients this would require that the coefficients (40) approach unity as the quantum numbers  $\lambda_{i,n}, \lambda_{j,n-1}, \mu_{i,n} = \mu'_{i,n}$  become suitably large. This problem has been considered by Biedenharn *et al.*,<sup>9</sup> for the  $U(N)$  tensor operator problem, who investigate the behavior of certain coupling coefficients in various limits. It would be of interest to see whether analogous results could be obtained for the reduced overlap coefficients (40).

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## APPENDIX A

We prove here part (a) of Lemma 1. We adopt the notation of Secs. II and III of the paper. Let  $\Omega \in A_0(\lambda)$  be a  $U(2n)$  (allowable) weight state. We prove that there exists a basis monomial  $b \in U(B)$  such that  $b\Omega = \alpha\Omega^\lambda$ ,  $0 \neq \alpha \in \mathbb{C}$ , by induction on  $n$ . Since we shall be proceeding down the subgroup chain we adopt the notation of Secs. II and III, except that we add a subscript  $n$  to everything to indicate precisely which unitary group we are considering. The notation we adopt is obvious in the present context.

The result holds for  $n = 1$  since  $U(2)$  is trivial. Proceeding inductively assume the result holds for  $U(2n-2)$ ; that is, if  $\Omega$  is any allowable  $U(2n-2)$  GT state of  $U(2n-2)$  weight  $\nu$ , in the irreducible representation  $V(\lambda_{n-1})$  of  $U(2n-2)$ , then there exists a basis monomial  $b \in U_{n-1}(B)$  such that

$$b\Omega = \alpha\Omega^{\lambda_{n-1}}, \quad 0 \neq \alpha \in \mathbb{C},$$

where

$$\Omega^{\lambda_{n-1}} = \left| \begin{array}{c} \lambda_{1,n-1}, \lambda_{2,n-1}, \dots, \lambda_{n-1,n-1} \\ \text{[max]} \end{array} \right| \quad (\text{A1})$$

is the  $\text{Sp}(2n-2)$  maximal weight state of  $\text{Sp}(2n-2)$  weight  $(\lambda_{1,n-1}, \dots, \lambda_{n-1,n-1})$  [cf. Eq. (22)]. We recall, from the remarks of Sec. III, that  $\Omega^{\lambda_{n-1}}$  is the unique vector in  $V(\lambda_{n-1})$  with  $U(2n-2)$  weight

$$\lambda_{0,n-1} = (\lambda_{1,n-1}, 0, \lambda_{2,n-1}, 0, \dots, \lambda_{n-1,n-1}, 0). \quad (\text{A2})$$

Now let  $\Omega$  be any allowable  $U(2n)$  basis state of  $U(2n)$  weight  $\nu$ . Then  $\Omega$  may be expressed as a sum of Gel'fand states

$$\Omega = \sum_{\mu_n, \lambda_{n-1}, (\tau)} \xi(\mu_n, \lambda_{n-1}, (\tau)) \left| \begin{array}{c} \lambda_n \\ \mu_n \\ \lambda_{n-1} \\ (\tau) \end{array} \right|, \quad (\text{A3})$$

where the sum is over all allowable GT states of weight  $\nu$ . Choose  $\lambda_{n-1}$  to be maximal (under the lexical ordering) such that  $\xi(\mu_n, \lambda_{n-1}, (\tau)) \neq 0$  [for some  $(\tau)$  and  $\mu_n$ ]. Then for  $\mu_n$  fixed we see that

$$\Omega' = \sum_{(\tau)} \xi(\mu_n, \lambda_{n-1}, (\tau)) \left| \begin{array}{c} \lambda_n \\ \mu_n \\ \lambda_{n-1} \\ (\tau) \end{array} \right|$$

is a  $U(2n-2)$  weight state of weight  $(\nu_1, \nu_2, \dots, \nu_{2n-2})$ . Hence, by our inductive hypothesis, there exists a basis monomial  $b \in U_{n-1}(B)$ , of  $U(2n-2)$  weight  $\lambda_{0,n-1} - \nu$ , such that

$$b\Omega' = \alpha \left| \begin{array}{c} \lambda_n \\ \mu_n \\ \lambda_{n-1} \\ (\text{max}) \end{array} \right|, \quad 0 \neq \alpha \in \mathbb{C}.$$

For other labels  $\lambda'_{n-1}$ , occurring in the sum (A3), we necessarily have

$$b \left| \begin{array}{c} \lambda_n \\ \mu_n \\ \lambda'_{n-1} \\ (\tau) \end{array} \right| = 0, \quad \lambda_{n-1} \neq \lambda'_{n-1},$$

since this state has  $U(2n)$  weight  $\lambda_{0,n-1}$  [see Eq. (A2)] which is conjugate under the Weyl group to the maximal weight  $(\lambda_{n-1}, 0)$  and hence, in view of the maximal nature of  $\lambda_{n-1}$ , cannot occur unless  $\lambda'_{n-1} = \lambda_{n-1}$  [recall<sup>19</sup> that the weights in  $V(\lambda'_{n-1})$  consist of all integral weights  $\nu_{n-1} < \lambda'_{n-1}$  together with their Weyl group conjugates].

For all labels  $\mu'_n$  occurring in the sum (A3) such that  $\xi(\mu'_n, \lambda_{n-1}, (\tau)) \neq 0$ , for some  $(\tau)$ , we have, since the state (A1) is the unique  $U(2n-2)$  weight state of weight  $\lambda_{0,n-1}$ , that

$$b \sum_{(\tau)} \xi(\mu'_n, \lambda_{n-1}, (\tau)) \left| \begin{array}{c} \lambda_n \\ \mu'_n \\ \lambda_{n-1} \\ (\tau) \end{array} \right| = \alpha' \left| \begin{array}{c} \lambda_n \\ \mu'_n \\ \lambda_{n-1} \\ (\text{max}) \end{array} \right|, \quad \alpha' \in \mathbb{C}.$$

Thus we must have

$$b\Omega = \sum_{\mu_n} \alpha(\mu_n) \left| \begin{array}{c} \lambda_n \\ \mu_n \\ \lambda_{n-1} \\ (\text{max}) \end{array} \right|, \quad (\text{A4})$$

for suitable scalars  $\alpha(\mu_n) \in \mathbb{C}$ .

Now choose  $\mu_n$  to be maximal (under the lexical ordering) such that  $\alpha(\mu_n) \neq 0$  (by the above such a  $\mu_n$  exists) and set

$$b' = (a_{2n-3, 2n-1})^{\mu_{n-1, n} - \lambda_{n-1, n-1}} \cdots (a_{3, 2n-1})^{\mu_{2, n} - \lambda_{2, n-1}} (a_{1, 2n-1})^{\mu_{1, n} - \lambda_{1, n-1}}.$$

From the known action of the  $U(2n)$  generators we deduce

$$b' \begin{pmatrix} \lambda_n \\ \mu_n \\ \lambda_{n-1} \\ (\max) \end{pmatrix} \neq 0.$$

Moreover this state has weight  $(\mu_{1,n}, 0, \mu_{2,n}, \dots, \mu_{n-1,n}, 0, \mu_{n,n})$  which is conjugate under the Weyl group to the  $U(2n-1)$  maximal weight  $(\mu_{1,n}, \mu_{2,n}, \dots, \mu_{n,n}, 0)$  and thus occurs with unit multiplicity. We thus deduce

$$b' \begin{pmatrix} \lambda_n \\ \mu_n \\ \lambda_{n-1} \\ (\max) \end{pmatrix} = \alpha \begin{pmatrix} \lambda_n \\ \mu_n \\ (\max) \end{pmatrix}, \quad 0 \neq \alpha \in \mathbb{C}.$$

For other labels  $\mu'_n \neq \mu_n$  occurring in the sum (A4) we deduce, in view of the maximality of  $\mu_n$ , that

$$b' \begin{pmatrix} \lambda_n \\ \mu'_n \\ \lambda_{n-1} \\ (\max) \end{pmatrix} = 0, \quad \text{for } \mu'_n \neq \mu_n.$$

We thus obtain

$$b' b \Omega = \beta \begin{pmatrix} \lambda_n \\ \mu_n \\ (\max) \end{pmatrix}, \quad 0 \neq \beta \in \mathbb{C}.$$

Now set

$$b'' = (a_{2n-1,2n})^{\lambda_{n,n}-\mu_{n,n}} \times (a_{2n-3,2n})^{\lambda_{n-1,n}-\mu_{n-1,n}} \dots (a_{1,2n})^{\lambda_{1,n}-\mu_{1,n}}.$$

Again, in view of the known action of the  $U(2n)$  generators

on GT basis states, we deduce, as before

$$b'' \begin{pmatrix} \lambda_n \\ \mu_n \\ (\max) \end{pmatrix} = \beta' \begin{pmatrix} \lambda_n \\ (\max) \end{pmatrix}, \quad \beta' \neq 0.$$

Thus we have

$$b'' b' b \Omega = \gamma \begin{pmatrix} \lambda_n \\ (\max) \end{pmatrix} = \gamma \Omega^{\lambda_0}, \quad 0 \neq \gamma \in \mathbb{C}.$$

Thus we have established the result that given a  $U(2n)$  weight vector  $\Omega \in A_0(\lambda)$ , of weight  $\nu$ , there exists  $b \in U(B)$ , of weight  $\lambda_0 - \nu$  [where  $\lambda_0$  is the  $U(2n)$  weight of  $\Omega^{\lambda_0}$ ] such that

$$b \Omega = \gamma \Omega^{\lambda_0}, \quad 0 \neq \gamma \in \mathbb{C}.$$

Now  $b$  may be expressed as a sum of basis monomials of weight  $\lambda_0 - \nu$ :

$$b = \sum_i \alpha_i b_i, \quad \alpha_i \in \mathbb{C}.$$

Then for some  $i$  we must have  $b_i \Omega \neq 0$ . Since  $\Omega^{\lambda_0}$  is the unique vector in  $V(\lambda)$  of weight  $\lambda_0$  we must have

$$b_i \Omega = \alpha \Omega^{\lambda_0}, \quad 0 \neq \alpha \in \mathbb{C}.$$

Our argument is now complete and the result is proved. We remark that, in view of the simplicity of the final result, there probably exists a simpler proof of this result.

## APPENDIX B

Let  $|\lambda_{ij}\rangle$  denote a GT basis state for  $U(2n)$ . Then from the known matrix element formulas of the  $U(2n)$  generators (see, e.g., Gould<sup>22</sup>) we have

$$a_{2m-1,2m} |\lambda_{ij}\rangle = \sum_{r=1}^{2m-1} N^{2m-1}_r |\lambda_{ij} + \Delta^{2m-1}_r\rangle,$$

where

$$N^{2m-1}_r = \left( \frac{(-1)^{2m-1} \prod_{p=1}^{2m} (\lambda_{p,2m} - \lambda_{r,2m-1} + r - p) \prod_{l=1}^{2m-2} (\lambda_{r,2m-1} - \lambda_{l,2m-2} + l - r + 1)}{\prod_{l=1}^{2m-1} (\lambda_{r,2m-1} - \lambda_{l,2m-1} + l - r) (\lambda_{r,2m-1} - \lambda_{l,2m-1} + l - r + 1)} \right)^{1/2}. \quad (B1)$$

Now for the special representations of  $U(2n)$  we are considering, we have

$$\lambda_{i,2m} = \lambda_{i,2m-1} = 0, \quad \text{for } i > m, \quad \lambda_{i,2m-2} = 0, \quad \text{for } i > m-1. \quad (B2)$$

For this case only the matrix elements  $N^{2m-1}_r$ , for  $r \leq m$  are nonzero. Substituting Eq. (B2) into Eq. (B1) we obtain (for  $r \leq m$ )

$$\begin{aligned} N^{2m-1}_r &= \left( \frac{(-1) \prod_{p=1}^m (\lambda_{p,2m} - \lambda_{r,2m-1} + r - p) \prod_{l=1}^{m-1} (\lambda_{r,2m-1} - \lambda_{l,2m-2} + l - r + 1)}{\prod_{l=1}^m (\lambda_{r,2m-1} - \lambda_{l,2m-1} + l - r) (\lambda_{r,2m-1} - \lambda_{l,2m-1} + l - r + 1)} \right)^{1/2}, \\ &= \frac{\prod_{p=m+1}^{2m} (r - p - \lambda_{r,2m-1}) \prod_{l=m}^{2m-2} (\lambda_{r,2m-1} + l - r + 1)^{1/2}}{\prod_{l=m+1}^{2m-1} (\lambda_{r,2m-1} + l - r) (\lambda_{r,2m-1} + l - r + 1)} \\ &= \left( \frac{(-1)^{m+1} \prod_{p=1}^m (\lambda_{p,2m} - \lambda_{r,2m-1} + r - p) \prod_{l=1}^{m-1} (\lambda_{r,2m-1} - \lambda_{l,2m-2} + l - r + 1)}{\prod_{l=1}^m (\lambda_{r,2m-1} - \lambda_{l,2m-1} + l - r) (\lambda_{r,2m-1} - \lambda_{l,2m-1} + l - r + 1)} \right)^{1/2} (\lambda_{r,2m-1} + m + 1 - r)^{1/2}. \end{aligned} \quad (B3)$$

In the dual pattern notation of Sec. IV we set  $\lambda_{r,m} = \lambda_{r,2m}$ ,  $\mu_{r,m} = \lambda_{r,2m-1}$  ( $r = 1, \dots, m$ ), and  $\lambda_{r,m-1} = \lambda_{r,2m-2}$  ( $r = 1, \dots, m-1$ ) and we denote the matrix element (B3) by  $N^m_r$ . Thus we obtain, in the notation of Sec. IV

$$a_{2m-1,2m} \begin{pmatrix} \lambda \\ \mu \end{pmatrix} = \sum_{r=1}^m N^m_r \begin{pmatrix} \lambda \\ (\mu) + \Delta^m_r \end{pmatrix},$$

where

$$N^m_r = [\mu_{r,m} + m + 1 - r]^{1/2} \left( \frac{(-1)^{m+1} \prod_{p=1}^m (\lambda_{p,m} - \mu_{r,m} + r - p) \prod_{l=1}^{m-1} (\mu_{r,m} - \lambda_{l,m-1} + l - r + 1)}{\prod_{\substack{l=1 \\ l \neq r}}^m (\mu_{r,m} - \mu_{l,m} + l - r) (\mu_{r,m} - \mu_{l,m} + l - r + 1)} \right)^{1/2},$$

which gives the matrix element formula of Eq. (31) as required. A similar analysis may be applied to the matrix element formula (32).

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