

A Lone Ranger, Two Smoking Barrels and the Dirty Dozen: Confirming the Harmonious Nature (Convergence) of the Alternating Harmonic Series and Some Serious Series Stuff

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Abstract: *This study provides a proof of the convergence of the alternating harmonic series. The alternating harmonic series could be termed the more harmonious sibling of the harmonic series, which is divergent. As a byproduct of this proof - and the associated technique - we arrive at the convergence of a few other series and related results. All the main results are stated as propositions and corollaries, complete with proofs that are simple yet formal. In addition, we create a custom program - code - to numerically illustrate the convergence of one of the series. This serves as an example of how to approach series calculations using computational tools. We verify the convergence of all the series using mathematical software. Our findings confirm the convergence of the alternating harmonic series and several related series. To complement the mathematical content, we have given a compilation of several techniques that can aid in disseminating this knowledge across appropriate audiences. We discuss challenges with respect to mathematics education and provide some pedagogical prescriptions - for the classroom, or other educational settings - to approach the topics we have presented. Our suggestions can help ease the anxiety of participants - who tend to get uneasy when engaging with mathematical material - while performing symbolic manipulations. Despite the simplicity of our proofs - and our belief that these results are surely known - we were surprised that we could not find these results easily elsewhere. This indicates that an unintended consequence of the many advances in mathematics - and related knowledge creation - might be that some knowledge could either be getting lost or just becoming harder to find, which is the equivalent of getting lost in many cases. We outline a potential solution to mitigate this unwanted outcome.*

Keywords: Harmonic; Alternating; Series; Harmonious; Convergence; Divergence; Conundrum; π ; Basel; Problem; Numerical; Illustration; Lost; Found; Unintended Consequence; Lone Ranger; Two Smoking Barrels; The Dirty Dozen

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1 Introduction: Numbers and Symbols

Numbers have been associated with many aspects of life and perhaps, the mystery of numbers pervades all corners of the Cosmos waiting to be unraveled (Bell 1933; Schimmel 1993; Dudley 1997; Rowan-Robinson 1999; End-note 1). Somewhat conversely, *numerous symbols* - from various alphabets, or elsewhere, as we will see later in more detail in Section (5) - *have been used to represent numbers*, and to understand numbers better, *maybe because there are just too many numbers*. This - *using numerous symbols to represent numbers* - might have resulted in an *unintended consequence*, which is that *numbers have become harder to comprehend* (Uttal, Scudder & DeLoache 1997; Pimm 2002; Vlassis 2008; MorenoArmella & Sriraman 2010).

The subtle aspect to understand here is that *numbers are nothing but symbols representing numbers; and symbols - which do not seem like numbers, on the surface - can also be used to represent numbers*. The essential idea here is that *the assignment of meaning to symbols - whether they are numbers, or not - happens in our minds, which we can term as comprehension*. The *purpose of education is to ease the burden with respect to comprehension* - by facilitating better understanding - and we turn next to discussing the corresponding issues.

1.1 Permutations of Pedagogical Problems and Prescriptions

One of the goals of this article is to furnish material that can be used in classrooms, or for other educational activities, to promote greater understanding of certain abstract ideas relating to numbers, or mathematics. Most educational curriculums, starting from early school years, emphasize the importance of building a solid foundation in mathematics (Bishop 1991; Sarama & Clements 2009; Cross et al., 2009). There is also a significant amount of research that looks at the challenges involved while teaching mathematics, including numerous suggestions on how to make math more manageable (Inprasitha et al., 2015; Clements & Sarama 2011). Maloney & Beilock (2012); Wang et al. (2014) provide detailed discussions of the anxiety people feel when confronted with mathematical tasks and list techniques to reduce math anxiety that can help to bolster math performance.

Another set of challenges that arise, when teaching or learning mathematics, are related to the use of symbols and the medium of instruction. This introduces another layer the need to relate symbols to vocabulary and to concepts and overcome the difficulties of communication due to deficiencies in language (Hoffert 2009; Whiteford 2009; Chinn 2013; Kashyap 2021a).

Most of us have an innate ability for mathematical thinking, even though the developments in higher math have been in the last 400 years or so, which is a like a nano-second on an evolutionary scale (Devlin 2000).

Hence, we can surmise that we all have the necessary background to grasp mathematical concepts, though recent developments in mathematics are far from becoming second nature to most of us. It is important to realize that when we see a symbol - or hear a sound - the meaning associated with this stimulus happens in our minds (Kashyap 2021b). The real obstacle then is to create familiarity with new shapes, and sounds, that one might encounter. That said, it is simply not possible to become acquainted with all that is new, and hence unknown, until the initial encounter. ***The crux of the challenge is to foster a mind-set that will ensure that any meeting with the unfamiliar will not produce undesired anxiety and unwanted outcomes.***

Human beings have evolved thus far to relate to the stories of others. The bulk of the small amount of time that human beings have been around has been spent trying to survive (Shubin 2008). So, what is interesting to us are the struggles and the tales of survival. What would ensure that we grasp the main lessons from any endeavor is to ensure that we have an interesting enough narrative to transmit it to others.

We would like to make the case that, ***along with every mathematical encoding of knowledge, we need to have artistic means of communicating the message to ensure that the knowledge is easily transmitted to a greater audience.*** What would suffice in many instances would be to ensure that a story, or something engaging, captures the main elements of knowledge.

We need to be realistic in realizing that all mathematical material will not immediately become available as story books. But as more people attempt to create creative methods of mathematics education, better techniques to capture mathematics in amusing ways will develop. Knowledge creation and dissemination involve putting our findings, at-least as many of them as possible, in a mathematical format. Would it be such a stringent requirement to ask that every journal article, which has ample amounts of mathematical proofs, or other thoughts, also be accompanied by a musical encoding? ***The message this holds for all curriculum designers is that we need to find methods of mixing art, music and literature classes with math and science courses.***

1.2 Outline of the Sections Arranged Inline

In this paper, we provide a proof of the convergence of the alternating harmonic series. The alternating harmonic series could be termed the more harmonious sibling of the harmonic series, which is divergent. As a byproduct of this proof - and the associated technique - we arrive at the convergence of a few other series and related results.

Despite the simplicity of our proofs - and our belief that these results are surely known - we were surprised that we could not find these results easily elsewhere. This indicates that an

unintended consequence of the many advances in mathematics - and related knowledge creation - might be that some knowledge could either be getting lost or just becoming harder to find, which is the equivalent of getting lost in many cases. We outline a potential solution to mitigate this unwanted outcome.

All the main results are stated as propositions and corollaries, complete with proofs that are simple yet formal. We have created a custom program - code - to numerically illustrate the convergence of one of the series.

This serves as an example of how to approach series calculations using computational tools. We verify the convergence of all the series using mathematical software. To complement the mathematical content, we have given a compilation of several techniques that can aid in disseminating this knowledge across appropriate audiences.

We discuss challenges with respect to mathematics education and provide some pedagogical prescriptions - for the classroom, or other educational settings - to approach the topics we have presented. Our suggestions can help ease the anxiety of participants - who tend to get uneasy when engaging with mathematical material - while performing symbolic manipulations.

Section (1), which we have already seen, provides an introductory overview - and sets the stage for more detailed explorations in later sections - regarding the context and goals of this article with respect to mathematics education. Section (2) is a discussion of the challenges in mathematics education, and some techniques on how to overcome them, including specific suggestions for using this article in the classroom. Section (4) provides an overview of the main mathematical topics we will discuss including a summary of the main result.

Section (5) gives a detailed proof of our main result and for several related corollaries. Section (6) extends our main result to π and π^2 series and related corollaries. Section (7) illustrates the main result in Section (5) numerically using mathematical software. Sections (9) has the concluding thoughts. The appendices (11; 12; 13; 14) contain supplementary explanations, illustrations and an index of all the series we have used in this article.

2 The Power of Mathematics: Problem Solving with and without Mathematics

The *genesis* of this paper can be traced to a *course* we taught that was titled - "***Problem Solving with and without Mathematics***". This course was designed to illustrate - to undergraduate students the power of proper analytical approaches and the advantages one obtains when trying to solve problems using a mathematical framework (Kashyap 2018). Several problems that we come across in our daily lives were discussed in the class and different ways of solving them - including non-mathematical qualitative methods - were considered, and finally when a rigorous mathematical technique was applied to the problems, the elegance and the precision of the solutions obtained were demonstrated. We will discuss further the motivations for creating such a course.

The bulk of our experience has been mostly restricted to teaching finance and economics related topics to undergraduate, post graduate and Executive Master of Business Administration (MBA) students. Finance and economics courses involve a significant amount of mathematics pertaining to calculus, series summation, optimization and numerical approximation methods (Simon & Blume 1994; Baxter & Rennie 1996). The stumbling block for students to understand financial concepts is their lack of familiarity with - or even enthusiasm for - algebra, calculus and other mathematical notions.

Executive MBA participants can claim *executive immunity* from having to go into the detailed mathematical frameworks being used to study finance. Though we mean the previous sentence in jest, for such individuals - in later stages of their career, with significant responsibilities - the intuition behind financial models is more important. But for individuals that are starting their careers in finance, or business, the appreciation of mathematical subtleties behind financial ideas can be extremely enlightening. A compelling argument could also be made that the intuitions will be stronger when a completely thorough foundational grasp has been obtained.

Such a deeper understanding can help in terms of *uncovering the limitations* of the *models* and being able to *comprehend the corresponding assumptions*. Models will then be rendered as useful instead of merely labelling them and relying on categorizations such as a Normative or a Positive models (McGoun 2003; Kashyap 2018a). This will enable individuals to *decipher the nuances of the scenarios where models can be actually applied and be on the lookout for situations where models will fail*. Once equipped with the suppositions and shortcomings of any model, *practitioners as well as academics again merely titles, which are unnecessary* - are better prepared to both utilize existing models and create new ones.

Series summation is very much applicable when we discuss financial valuation and how to calculate the net present value of a stream of cash flows (Hillier et al., 2009). Infinite series of cash flows occur in many business situations, when suitable assumptions regarding the continuation of operations are made, such as: rents received by property owners from tenants, dividends paid to certain type of equity holders - preferred stock (End-note 2) - from corporations, and so on. The principles of financial valuation can also be useful to find the value of knowledge, which also involves summation of series (Kashyap 2020). We next consider questions and answers - particularly from one of the earlier sessions during a classroom interaction - while discussing topics related to financial valuation and also series summation.

2.1 Q&A (Questions & Answers) with D&A (Definitions & Assumptions) in our DNA

Our main approach to solving problems - and acquiring knowledge, both in mathematics and elsewhere - is through the use of Questions & Answers, or Q&A, which are important. The answers to any question will depend on some Definitions and Assumptions, which we could

term D&A. If we change those D&A, we might get different Q&A. Perhaps, this is pointing us to the possibility that D&A, which are more unfamiliar, are more important. Perhaps, ***D&A are linked to DNA, the biological one*** (Alberts et al., 2002; End-notes 3). We can look at many, - perhaps even all? - aspects of life using the lens of Q&A and D&A, suggesting that ***“Q&A and D&A might be in our very DNA”***.

Below we recount a typical discussion that we have had the privilege of experiencing, regarding mathematics - and series summation - in an introductory finance course.

- Instructor: “How many of you like mathematics?”

A few hands go up, indicating an affinity for mathematics. With them - those that fancy mathematics - finance is an easy conversation. Generally, several students admit that they are not big fans of mathematics. Time permitting, we discuss some aspects from the article “Math anxiety: Who has it, why it develops, and how to guard against it” (Maloney & Beilock 2012). We then display the image in Figure (1) - all the Illustrations are placed in the Appendix (Section: 13) - and inquire as follows.

- Instructor: “How many monsters do you see in this picture”?

The students at this time are more energized, and they play along. A variety of answers are received such as “Four”; “Only on the right are the real monsters”; “Mathematicians create monsters”; and so on.

- Instructor: “There are two monsters in Figure (1). Generally, grown-ups are afraid of the Monster on the right. Children fear the monster on the left. All the monsters on each side are different avatars of the same two creatures.”
- Instructor: “The monster on the right, also known as mathematical equations, is known to induce sleep in grownups at best and can cause all kinds of anxiety in more severe cases. “The monster on the left might act as a stimulant in some people, but is terrifying to many children. This brings up the very important question: ‘What is a Monster?’, which we consider next.”

This discussion also brings up the issue of having to distinguish who are kids and who are adults. Our attempts to address this distinction - between grownups and children - is given in Kashyap (2024). We can define a monster as below and continue the discussion in the classroom.

Definition 1. *A Monster is anything we don't know well, or we are not too familiar with.*

What is in the *“Dean’s Wallet”*? There is a monster there. What is under the *“Editor’s desk”*? There is a monster there. What is in the *“president’s (say, of the United States of America) pants”*? Another monster of course, though it is likely to be a *“tiny one”*. What is in the pants of any person is just another *“body part”*. But given the reaction it invokes in many people, it clearly has become a monster.

There are many such questions, or unknowns to some of us, for which the answer can be related to some monster. Who is *“Kim Jong Kong”*? Where does he live? Must be somewhere close to Hong Kong or next to another monster. What is *“infinity”*? What is a *“call option in finance”*? What is on the *“13th floor”*? What is in the *“first lady’s hand bag”*? What is there at the *“corner of the grand central train station”*? What is *“Walmart”*? What did the *“instructor eat for breakfast”*?

There are monsters everywhere. But not all of them are frightening, though some are highly dreaded.

Here is a list of a few other monsters that are extremely scary: *“Recession, Confusion, Frustration, Examination”*. The first monster is dreaded in business. The rest show up everywhere but are more common in universities. *The main lesson from the story of “Beauty and the Beast” is that we need to “love the beasts to find beauty”* (De Beaumont 1804; End-note 4; Figure 2). Needless to say, *finding beauty is yet another unintended yet welcome consequence*. Beasts are nothing but monsters. Once we understand monsters, we will see that they are simply fantastic creatures. A detailed discussion of these monsters - Recession, Confusion, Frustration, Examination - is given in Kashyap (2024).

When we *meet a monster* we *can get curious* and find out about it. But upon encountering a monster *if we start getting worried, we can simply forget* about the monster. But this also means forgetting the amazing answers it might help us find. It is absolutely okay to get confused and frustrated, but not okay to panic (Figure 3). That being said, does it not seem *inappropriate to be frightened of something that we don’t know much about?* Before we love or hate or judge something, we need to become more aware of it. We might start to love these monsters, as we get familiar with them. This can help us *unlock their awesomeness and find truly stunning solutions*. Many have said that nature has no evil, then the question of hatred, or fear, should not even come up (End-note 5; Nolan & Nolan 2014).

Referring to Figure (3): *“Confusion is the Beginning of Understanding. Necessity is the mother of all invention, (creation or innovation), but the often forgotten father is Frustration”*. Many times the unknown can be intimidating. In other words, when we don’t

know something it can cause confusion or frustration. Confusion and Frustration, though scary and ugly to begin with, can be powerful motivators as long as we don't let them bother us. Preparing for an Examination will significantly boost Learning. Controlled Recessions might hold the key to smoothen the wild turbulence - ups, downs and wild swings - of business cycles.

The *mathematics monster* is like a *swimsuit model wearing a Burka* (End-note 6). We need to see beyond the symbols and the surface to appreciate the beauty. Mathematics has become complex because of different notations, assumptions not made explicit and missing steps. Mathematics without the steps is like a *treasure hunt without the clues*. Mathematics originated with numbers (Dehaene 2011), which are simple but the complexity lies in the abstractions we have created on top of the concept of numbers. Paying enough attention to, and getting familiar with, the tricky ideas, will render them much easier to comprehend.

- Instructor: "Why do we need mathematics in finance?"
- Instructor: "Finance deals extensively with numbers since representing information using numbers gives more precise answers - as compared to answers without numbers - and helps with decision making. Mathematics - which is the study of numbers and their properties - seems like a natural tool to use."
- Instructor: "How many formula do you think there are in finance?"
- Student One: "Many".
- Student Two: "Dozens".
- Instructor: "*There is only one formula you need to understand really well in finance. All the other formulae are different version of the same formula, or they are linked to the one fundamental formula. It is like *one formula wearing Halloween costumes* - or developing tentacles and other such appendages to become distant from the basic form - and appearing like something else.*"

Mathematics is heavily used in many areas of business and specifically in a sub-discipline, which has gained increased prominence of late, finance. The *allure of finance* is because of its *one very simple goal*: "*whatever amount of money we have, we want to make more of it*". Finance also has a reputation of being complex, exotic and full of arcane equations. There is another way to look at finance.

In our perspective, there are only three simple things we can do in finance "*Buy, Sell or Hold*". This has been true about the possible actions in finance through time. The complications are mainly to get to these simple results. To fulfill this one simple objective with three easy actions,

whether we need *better morals or more complicated mathematical models* - which can be extremely elegant - is a *can of worms* better left unopened for now. But we point out the only mathematical equation we really need to be able to comprehend the entire scope and objectives of both finance and accounting below.

$$p = E(mx) \quad (1)$$

Here, p is the price of any financial asset; x is the cash flows we derive from this asset; m is the discount factor; and E is the expectation operator (Cochrane 2009; Hillier et al., 2009). Finance concerns itself with finding appropriate discount factors and accounting is all about estimating accurate cash flows. We are using expectations since the cash flows occur in the future and we can only expect what we might get later. All the formulae we see in Figure (1) are variations of Equation (1). And only mathematics can give us the proof of this statement and convince us of its truth.

Despite being unappealing to many of us on the surface, most of us can be made to agree with - or can be made to be appreciative of - the many wonders of mathematics, by illustrating its many uses in all corners of life (Kashyap 2018). Though this paper has many results, one of the main purposes of the paper is to provide students with several examples to increase their exposure to series summations - and to enable them to get familiar with - and other abstract mathematical objects, by getting them to become comfortable juggling many such items.

3 Concepts for the Classroom

Continuing with the chain of introductory thoughts from Section (1.1), we would like to discuss the sequence of steps, and the corresponding mathematical concepts, that can be taken to adopt this article for classroom usage. The main ideas that are necessary to understand this article are: Addition, Fractions, Algebra and Infinity. Clearly addition is the most fundamental mathematical operation. Many (all?) mathematical operations can be seen as derived from the concept of addition. From there, the next level for the students is to be comfortable with fractions.

The use of symbols instead of actual numbers, which comes with the rules of algebra and the corresponding manipulations, is an essential skill. Once these topics have been mastered, the tricky concept of infinity can be introduced. Blaszczyk (2020) provides ways of measuring infinity - developed at the turn of the 20th and 21st centuries - that provide alternatives to Cantor's theory of cardinal and ordinal numbers (Cantor 1874; Dauben 1990).

The leap from numbers to abstract ideas starts with algebra and infinity. This illustrates that all advanced mathematical abstractions are built from many smaller and simpler operations. When any idea from mathematics becomes difficult to grasp, we need to break it down into simpler parts and focus on the components. Once we are familiar with the components, and we are able to connect the constituent fragments to what we already know, meaning takes shape in our minds. The important role of imagination in teaching mathematics is covered in Degu (2020).

Once a base is set, with the four concepts mentioned earlier, we can approach advanced concepts such as: Limits, Infinite Sequences, Sums of Infinite Series, Convergence, Divergence and lastly Differences of Infinite Series (Roh 2008; González-Martín et al., 2011; Martínez-Planell et al., 2012; Arnal-Palacián & ClarosMellado 2022). These advanced concepts offer some tricky and counter intuitive notions. For someone who has mastered the first four aspects, the latter portions will begin to become much more easy. It is important to know where someone stands, in terms of what they already know and which operations are second nature to them, and then proceed to the next step.

The topics covered here are commonly found in undergraduate, or advanced high school calculus curriculums, perhaps even as a second calculus course after completing an introductory calculus one course. In a formal educational environment, incorporating this article as part of beginner calculus courses would be recommended. That said, we have not explicitly specified which grade, or school year, someone has to be in before they can approach the material here.

An alternate approach can be applied when someone is doing self study, or is being schooled outside a regular school atmosphere. It would be adequate to look at this paper in terms of the concepts mentioned without too much attention to age and level of schooling. It will be seen that the progress from Addition to Limits can take several months - or years - depending on prior experience with, and exposure to, these concepts. Clearly, it is possible to move some of the topics around depending on interest and expertise. This brings in the many permutations that are feasible when we start studying these topics. Suffice it to say - ***as the concept of infinity tells us - the possibilities are endless.***

All of what we have discussed is perhaps, not new. This is because, as someone said, everything has already been said. But maybe not by everyone and to everyone. This scenario arises because many times we forget what we have said and need to be reminded. Our main hope is that this paper will convince everyone - acting as a reminder to those who have forgotten and becoming a stimulus for action to others - that we need to develop creative means to build mathematical aptitude.

4 Alternating Harmoniously

The harmonic series and the alternating harmonic series are well studied. Knopp (1956); Rudin (1964); Schumacher (2008); Bonar & Khoury (2018) are detailed textbooks that cover infinite series and related topics. Kullman (2001) provides an insightful account on how the harmonic series derives its name. Jardine & Shell-Gellasch (2011) is a fascinating historical account of several mathematical marvels including the harmonic series.

The harmonic series is known to be divergent but, its sibling, the alternating harmonic series is known to be convergent, perhaps making it the more harmonious entity of the two kins. The first proof of the divergence of the harmonic series was given by Nicole Oresme in around 1350 (Oresme 1961; End-note 7). Kifowit & Stamps (2006) is a survey of several proofs showing the divergence of the harmonic series. Lubeck & Ponomarenko (2018); Wadhwa (1978) look at the convergence of sub-sums of the harmonic series. Sarmiento (2019) is a review of sub-series of the harmonic series that converge. Boas & Wrench (1971) consider partial sums of harmonic series and look at computing the smallest integer, n , for which the sum of the first n terms of the harmonic series exceeds a given number.

Dunham (1987) is an interesting historical account of the work done by the Bernoulli family, specifically the brothers: Jacob Bernoulli and Johann Bernoulli, on the harmonic series. Bibiloni, Viader & Paradís (2006) look at the nuances of proofs, related to infinite series, by considering tricky aspects with respect to Jakob Bernoulli's proof of the divergence of the harmonic series. Borwein et al., (1989) go on a journey of discovery to find an asymptotic expansion of the error in Euler summation (Apostol 1999; De Bruijn 1981; End-notes 8; 9).

Many well known proofs show the convergence of the alternating harmonic series including the value it converges to (Equation 2; End-note 10). The sum of the alternating harmonic series is equal to the natural logarithm of 2, $\ln 2$. Hudelson (2010) is a graphical proof of the convergence of the alternating harmonic series to $\ln 2$. Gillman (2002) shows a simple proof as opposed to more complicated proofs that rely on integrals. Setting $x = 1$ in the Mercator series (Equation 3) yields the alternating harmonic series, as a special case of the Mercator series, and also the value the alternating harmonic series converges to (Equation 4; Babcock & Dawson 1978; Thomas & Finney 1992; Mungan 2009; Bradley 2017; End-note 11).

$$\sum_{n=1}^{\infty} \left[\frac{(-1)^{n+1}}{n} \right] = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \dots = \ln 2 \quad (2)$$

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \dots \quad (3)$$

$$\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k} = \ln(2) \quad (4)$$

Bényi (2005) gives a recursive formula for the sums of alternating harmonic series of odd order. Cowen, Davidson & Kaufman (1980); Brown et al., (1985); Freniche (2010); An & Edgar (2018); Riddle (2021) are excellent accounts of rearranging the alternating harmonic series. The not so serious arrangements of the material in our paper ensures that Section (6.2) contains more details about series rearrangements.

We provide a proof of the convergence of the alternating harmonic series in Section (5). The techniques used in the proof can be applied to other series and, as additional examples, we consider alternating series for π in Section (6.1). As an unintended consequence of venturing on the path of this proof, we arrive at the convergence of the below series (Equations 5; 6; 7; 8; 9; 10; 11; 12; 13; 14; 15; 16) summarized in Section

(4.1) and a few other related results. We provide numerical illustrations of the convergence of the series $\sum_{n=0}^{\infty} \left[\frac{1}{(n+1)(2n+1)^2} \right]$ to $\left(\frac{\pi^2}{4} - \ln 4 \right)$ in Section (7.1). We also verify the convergence of all the other series using mathematical software in Section (7.1).

We believe that these results might be well known, given their simplicity, despite our initial amusement when we stumbled upon them. That said, perhaps all of mathematics, with a detailed set of steps, will be simple for anyone who can follow the suitable steps accordingly. Though, we were even more amazed that we could not find these results elsewhere even after considerable efforts at locating them. Due to the enormous advances in mathematics and related knowledge creation, lots of information is being generated. Another unintended consequence of all this data being churned out might be that some knowledge could either be getting lost or just becoming harder to find, which is the equivalent of getting lost in many cases. We have outlined a potential solution that might mitigate this unwanted outcome.

We have given detailed steps, at-least in the beginning portions, so that the results are easily accessible to a wide audience. We have marked the statements that contain sort of interesting, and perhaps seemingly novel results, as propositions, lemmas and corollaries so that they stand out from the rest of the working.

There are a total of two propositions and twelve corollaries that depend on one lemma. We would like to call them A Lone ranger, Two Smoking Barrels and The Dirty Dozen (End-notes 12; 13; 14; Allen 1996; Williams 2004; Redhead 2010). As a word of caution this material in this series of pages, though most of us would not find it to be smoking or dirty, could be seriously stimulating to some of us.

4.1 Summary of Some Serious Series Summations

This section summarizes all the new - or seemingly new - series results that we have derived in this paper. Detailed steps to derive these results follow in subsequent sections. This section is only meant to concisely capture the convergence of several series - so the readers can get

somewhat familiar with what series results to expect - before we look at the proofs of convergence, the values the series converge to, and how we approach the proofs.

$$\sum_{n=0}^{\infty} \left[\frac{1}{(n+1)(2n+1)^2} \right] = \sum_{n=1}^{\infty} \left[\frac{1}{(n)(2n-1)^2} \right] = \left(\frac{\pi^2}{4} - \ln 4 \right) \quad (5)$$

$$\sum_{n=0}^{\infty} \left[\frac{1}{(4n+1)^2(4n+3)^2} \right] = \left[\frac{(\pi-1)^2-1}{32} \right] \quad (6)$$

$$4 - (128) \sum_{n=0}^{\infty} \sum_{k=n+1}^{\infty} \left[\frac{1}{(4n+1)(4n+3)(4k+1)(4k+3)} \right] = (\pi-2)^2 \quad (7)$$

$$\sum_{n=1}^{\infty} \left[\frac{1}{(n)^2(2n-1)} \right] = \left(\ln 16 - \frac{\pi^2}{6} \right) \quad (8)$$

$$\sum_{n=1}^{\infty} \left[\frac{(4n-1)}{(2n-1)^2(n)^2} \right] = \frac{\pi^2}{3} \quad (9)$$

$$\sum_{n=1}^{\infty} \left[\frac{1}{(2n-1)(n)} \right] = \ln 4 \quad (10)$$

$$\sum_{n=1}^{\infty} \left[\frac{1}{(2n-1)^2(n)^2} \right] = \left(\frac{2\pi^2}{3} - \ln 256 \right) \quad (11)$$

$$(2 \ln 2)(2 + \ln 2) - \sum_{n=1}^{\infty} \sum_{k=n+1}^{\infty} \left[\frac{1}{(n)(2n-1)(k)(2k-1)} \right] = \frac{\pi^2}{3} \quad (12)$$

$$\sum_{n=1}^{\infty} \left[\frac{1}{(2n-1)^2(n)^3} \right] = \left[\zeta(3) + \left(\frac{5\pi^2}{3} \right) - 12(\ln 4) \right] \quad (13)$$

$$\sum_{n=1}^{\infty} \left[\frac{1}{(n)(2n-1)} \left\{ \frac{(4n-1)}{(2n-1)(n)} + \sum_{k=n+1}^{\infty} \left[\frac{1}{(k)(2k-1)} \right] \right\} \right] = (2 \ln 2)(2 + \ln 2) \quad (14)$$

$$\sum_{n=1}^{\infty} \left[\frac{1}{(2n-1)^2(n)^4} \right] = \left[4 \{ \zeta(3) - 8(\ln 4) \} + \frac{\pi^4}{90} + 4\pi^2 \right] \quad (15)$$

$$\begin{aligned} & \sum_{n=1}^{\infty} \sum_{k=n+1}^{\infty} \left[\frac{1}{(2n-1)(n)^2(2k-1)(k)^2} \right] \\ &= \left[\frac{\pi^4}{120} - 2\pi^2 \left(1 + \frac{\ln 4}{6} \right) - 2 \{ \zeta(3) - (\ln 4)(8 + \ln 4) \} \right] \quad (16) \end{aligned}$$

5 Proofs with Symbols Representing Numbers

In this section we prove the main result about the convergence of the alternating harmonic series - as Proposition (1) - and derive four related results as corollaries. We use a well-known result - stated and proved as a lemma - to establish the convergence of the alternating harmonic series. The four corollaries are obtained as a by-product of the technique we use to prove the main proposition. Each of the corollaries are variations of the summations of terms, similar to the terms, we use in the proof of the central result.

The reason for having the corollaries is to demonstrate how a central result can be varied to get several related results. The secondary results - example corollaries - we have chosen are the ones we could easily establish using the main results. There might be other related results, which we have not been able to discover. Perhaps, other such results, which might get revealed later, can become the material for a later paper.

The related results - in the corollaries - are much easier to understand when seen within the flow of steps leading to the results. If any of these results are taken in isolation, it becomes much harder to prove them, or to come up with a mechanism to understand their validity. We have given more detailed general explanations before the key results, in addition to also outlining the rationale for proceeding in a particular direction. Additional explanations and connections between the results are given before each of the results.

Proposition 1. *The alternating harmonic series is greater than zero and less than $\frac{\pi^2}{8}$. That is, $0 < \sum_{n=1}^{\infty} \left[\frac{(-1)^{n+1}}{n} \right] < \frac{\pi^2}{8}$ and hence it is convergent.*

Proof. We show below one method to confirm the convergence of the alternating harmonic series,

$$\sum_{n=1}^{\infty} \left[\frac{(-1)^{n+1}}{n} \right] = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \dots \quad (17)$$

We combine the positive and negative terms of the series as shown below,

$$= \left(1 - \frac{1}{2}\right) + \left(\frac{1}{3} - \frac{1}{4}\right) + \left(\frac{1}{5} - \frac{1}{6}\right) + \left(\frac{1}{7} - \frac{1}{8}\right) + \dots \quad (18)$$

Summing the consecutive positive and negative terms gives,

$$= \left(\frac{1}{2}\right) + \left(\frac{1}{12}\right) + \left(\frac{1}{30}\right) + \left(\frac{1}{56}\right) + \left(\frac{1}{90}\right) \dots \quad (19)$$

This gives the following alternative series for the alternating harmonic series,

$$\sum_{n=1}^{\infty} \left[\frac{(-1)^{n+1}}{n} \right] = \sum_{n=1}^{\infty} \left[\frac{1}{(2n-1)(2n)} \right] \quad (20)$$

Equation (20) can be further decomposed into,

$$\begin{aligned} \sum_{n=1}^{\infty} \left[\frac{(-1)^{n+1}}{n} \right] &= \sum_{n=1}^{\infty} \left[\frac{1}{(2n-1)(2n)} \right] = \left[\frac{1}{1} - \left(\frac{1}{1} \right) \left(\frac{1}{2} \right) \right] + \left[\frac{1}{9} - \left(\frac{1}{9} \right) \left(\frac{1}{4} \right) \right] + \left[\frac{1}{25} - \left(\frac{1}{25} \right) \left(\frac{1}{6} \right) \right] \\ &\quad + \left[\frac{1}{49} - \left(\frac{1}{49} \right) \left(\frac{1}{8} \right) \right] + \dots \end{aligned} \quad (21)$$

This gives the following decomposition of the alternating harmonic series into two different series,

$$\sum_{n=1}^{\infty} \left[\frac{(-1)^{n+1}}{n} \right] = \sum_{n=1}^{\infty} \left[\frac{1}{(2n-1)^2} - \frac{1}{(2n-1)^2(2n)} \right] \quad (22)$$

On the right hand side of Equation (22) we see that the first terms - which are positive - are always greater than the second terms, which are negative. Hence, we can infer from Equation (22) - and also by directly looking at Equation (20) - that the partial sums of the alternating harmonic series form a monotonic increasing sequence (Definition: 3.13, Rudin 1964; End-note 15). A positive monotonic series has to converge to a positive value or diverge. This implies that,

$$0 < \sum_{n=1}^{\infty} \left[\frac{(-1)^{n+1}}{n} \right] < \sum_{n=1}^{\infty} \left[\frac{1}{(2n-1)^2} \right] \quad (23)$$

We prove next in Lemma (1) that $\sum_{n=1}^{\infty} \left(\frac{1}{(2n-1)^2} \right) = \frac{\pi^2}{8}$. This gives the result that,

$$0 < \sum_{n=1}^{\infty} \left[\frac{(-1)^{n+1}}{n} \right] < \frac{\pi^2}{8} \quad (24)$$

This leads to the conclusion that the alternating harmonic series is convergent since its partial sums are monotonic and bounded (Theorem: 3.24, Rudin 1964; End-note 16).

□

It is well known that $\sum_{n=1}^{\infty} \left(\frac{1}{(2n-1)^2} \right)$ is convergent due to its relation to the Basel problem (End-note 17; Ivan 2008; Benko 2012). We have used this property in the proof of the Proposition (1). This relationship is given in Lemma (1), which follows next. We write the Basel problem below, which is the summation of the reciprocals of the squares of the natural numbers, i.e. the precise sum of the infinite series:

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = 1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \frac{1}{5^2} + \cdots = \frac{\pi^2}{6} \quad (25)$$

Lemma 1. *The series $\sum_{n=1}^{\infty} \left(\frac{1}{(2n-1)^2} \right)$ is convergent and we show that $\sum_{n=1}^{\infty} \left(\frac{1}{(2n-1)^2} \right) = \frac{\pi^2}{8}$.*

Proof. $\sum_{n=1}^{\infty} \left(\frac{1}{(2n-1)^2} \right)$ is convergent due to its relation to the Basel problem. We show this relationship of $\sum_{n=1}^{\infty} \left(\frac{1}{(2n-1)^2} \right)$ with the Basel problem below. We first start with the Basel problem in series form,

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = 1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \frac{1}{5^2} + \cdots = \frac{\pi^2}{6} \quad (26)$$

We multiply both sides of Equation (26) by $\frac{1}{2^2}$,

$$\frac{1}{2^2} \left(1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \frac{1}{5^2} + \cdots \right) = \left(\frac{1}{2^2} \right) \left(\frac{\pi^2}{6} \right) \quad (27)$$

$$\frac{1}{2^2} + \frac{1}{4^2} + \frac{1}{6^2} + \frac{1}{8^2} + \frac{1}{10^2} + \cdots = \frac{\pi^2}{24} \quad (28)$$

Equation (28) can be written as,

$$\sum_{n=1}^{\infty} \frac{1}{(2n)^2} = \frac{1}{2^2} + \frac{1}{4^2} + \frac{1}{6^2} + \frac{1}{8^2} + \frac{1}{10^2} + \cdots = \frac{\pi^2}{24} \quad (29)$$

Subtracting Equation (28) from Equation (26) gives,

$$1 + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \frac{1}{9^2} + \cdots = \left(\frac{\pi^2}{6} - \frac{\pi^2}{24} \right) \quad (30)$$

$$\sum_{n=1}^{\infty} \left[\frac{1}{(2n-1)^2} \right] = \frac{\pi^2}{8} \quad (31)$$

We have proved the desired result. □

We find other results using the convergence of $\sum_{n=1}^{\infty} \left(\frac{1}{(2n-1)^2} \right)$, given in Lemma (1). Combining the result in Lemma (1) with the decomposition of the alternating harmonic series given in Proposition (1) and using the corresponding convergence, we continue further to find the value to which the series $\sum_{n=1}^{\infty} \frac{1}{(n)(2n-1)^2}$ converges to. This result is shown in Corollary (1).

Corollary 1. The series $\sum_{n=1}^{\infty} \frac{1}{(n)(2n-1)^2}$ converges to $\left(\frac{\pi^2}{4} - \ln 4 \right)$.

Proof. We first use the decomposition of the alternating harmonic series given in Equation (22),

$$\sum_{n=1}^{\infty} \left[\frac{(-1)^{n+1}}{n} \right] = \sum_{n=1}^{\infty} \left[\frac{1}{(2n-1)^2} - \frac{1}{(2n-1)^2(2n)} \right] = \ln 2 \quad (32)$$

Rearranging Equation (32) and using Equation (31), from Lemma (1), gives the desired result,

$$\sum_{n=1}^{\infty} \left[\frac{1}{(n)(2n-1)^2} \right] = \sum_{n=0}^{\infty} \left[\frac{1}{(n+1)(2n+1)^2} \right] = \left(\frac{\pi^2}{4} - \ln 4 \right) \quad (33)$$

□

Rewriting Equation (20) - from the main Proposition (1) - in another combination can allow us to find the value to which the series $\sum_{n=1}^{\infty} \left[\frac{1}{(n)^2(2n-1)} \right]$ converges to. We prove this result in Corollary (2). Note that, one of the intermediate results from Lemma (1) is also used here.

Corollary 2. The series $\sum_{n=1}^{\infty} \left[\frac{1}{(n)^2(2n-1)} \right]$ converges to $\left(\ln 16 - \frac{\pi^2}{6} \right)$.

Proof. Another combination of Equation (20) is possible as shown below,

$$\sum_{n=1}^{\infty} \left[\frac{1}{(2n-1)(2n)} \right] = \sum_{n=1}^{\infty} \left[\frac{1}{(2n)^2} + \frac{1}{(2n)^2(2n-1)} \right] \quad (34)$$

Using Equations (28; 20; 34) results in the following formula,

$$\sum_{n=1}^{\infty} \left[\frac{1}{(n)^2(2n-1)} \right] = \left(\ln 16 - \frac{\pi^2}{6} \right) \quad (35)$$

This concludes the proof. □

Using earlier results from Equations (20; 33; 35) - from Proposition (1) and Corollaries (1; 2) - can show the convergence of the series $\sum_{n=1}^{\infty} \left[\frac{1}{(2n-1)(n)} \right]$, $\sum_{n=1}^{\infty} \left[\frac{1}{(2n)(2n-1)^2} \right]$ and $\sum_{n=1}^{\infty} \left[\frac{1}{(2n)^2(2n-1)} \right]$ and also the values the series converge to. It should be emphasized here that the results that follow in Corollary (3) are easy to establish in the flow of results, and by going through the sequence of equations we have considered thus far. But taken in isolation, such results are harder to see and not that trivial to establish.

Corollary 3. *We prove the convergence of the following series and show the values they converge to:*

$$\sum_{n=1}^{\infty} \left[\frac{1}{(2n-1)(n)} \right] = \ln 4, \quad \sum_{n=1}^{\infty} \left[\frac{1}{(2n)(2n-1)^2} \right] = \left(\frac{\pi^2}{8} - \ln 2 \right) \quad \text{and}$$

$$\sum_{n=1}^{\infty} \left[\frac{1}{(2n)^2(2n-1)} \right] = \left(\ln 2 - \frac{\pi^2}{24} \right).$$

Proof. Using Equations (20; 33; 35) gives the following simple, yet potentially useful, possibilities:

$$\sum_{n=1}^{\infty} \left[\frac{1}{(2n-1)(n)} \right] = \ln 4 \quad (36)$$

$$\sum_{n=1}^{\infty} \left[\frac{1}{(2n)(2n-1)^2} \right] = \left(\frac{\pi^2}{8} - \ln 2 \right) \quad (37)$$

$$\sum_{n=1}^{\infty} \left[\frac{1}{(2n)^2 (2n-1)} \right] = \left(\ln 2 - \frac{\pi^2}{24} \right) \quad (38)$$

Note that Equation (37) can also be written as,

$$\sum_{n=0}^{\infty} \left[\frac{1}{(n+1)(2n+1)^2} \right] = \left(\frac{\pi^2}{4} - \ln 4 \right) \quad (39)$$

And that Equation (38) can be written as

$$\sum_{n=1}^{\infty} \left[\frac{1}{(n)^2 (2n-1)} \right] = \left(\ln 16 - \frac{\pi^2}{6} \right) \quad (40)$$

We have shown the desired results.

□

Lastly, to conclude this section, we show how convergence results - using the ones we have shown earlier in Corollary (1) and Lemma (1) - can be used to establish divergence of related series in Corollary (4).

Corollary 4. *The series $\sum_{n=1}^{\infty} \left[\frac{(n-1)}{(2n-1)^2} \right]$ and $\sum_{n=1}^{\infty} \left[\frac{(n)}{(2n-1)^2} \right]$ are divergent.*

Proof. We can decompose $\sum_{n=1}^{\infty} \left[\frac{1}{(n)(2n-1)^2} \right]$ as shown below,

$$\sum_{n=1}^{\infty} \left[\frac{1}{(n)(2n-1)^2} \right] = \sum_{n=1}^{\infty} \frac{1}{n} - \sum_{n=1}^{\infty} \left[\frac{4(n-1)}{(2n-1)^2} \right] \quad (41)$$

We might recognize the first part on the right hand side of Equation (41) as the harmonic series, which is known to be divergent (End-note 7). Hence the series $\sum_{n=1}^{\infty} \left[\frac{(n-1)}{(2n-1)^2} \right]$ and $\sum_{n=1}^{\infty} \left[\frac{(n)}{(2n-1)^2} \right]$ have to be

divergent as well, since Corollary (1) shows that $\sum_{n=1}^{\infty} \left[\frac{1}{(n)(2n-1)^2} \right]$ is convergent, and also $\sum_{n=1}^{\infty} \left(\frac{1}{(2n-1)^2} \right)$ is convergent from Lemma (1).

□

6 Slicing Up The Pi(e)

Clearly, the technique used in Section (5) can be applied to other alternating series as well. Such an approach of creating a series sum - by taking the difference of the consecutive terms of an alternating series and then rewriting each term in the summation as the difference of other series - can result in series formulae that might not be obvious otherwise.

We next consider cutting up a series for π and π^2 , that is, we take appropriate differences of the terms obtained by first adding up successive terms. The following are excellent resources for series, and formulae, involving π : (Berggren, Borwein & Borwein 1997; Arndt & Haenel 2001; Eymard & Lafon 2004; Baruah, Berndt & Chan 2009; Brown 2017; End-note 19).

6.1 Sum The Difference Plus Difference The Sum

We start with known series results involving π and find newer series that converge to terms involving powers of π . The first result - involving a series summation for π - is given as Proposition (2) and it has the bulk of the machinery that is used in the corollaries that follow.

We derive eight simpler results - or corollaries - that are related to Proposition (2). Note that, the result from Lemma (1) is also used in the Proof of Proposition (2).

Proposition 2. The series $\sum_{n=0}^{\infty} \left[\frac{1}{(4n+1)^2(4n+3)^2} \right]$ converges to $\left[\frac{(\pi-1)^2-1}{32} \right]$

Proof. Starting with the well-known alternating series for π (End-note 20), we proceed by taking the sum of the differences and then by taking the difference of the sums as shown below,

$$\sum_{n=0}^{\infty} \left[\frac{(-1)^n}{2n+1} \right] = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \cdots = \frac{\pi}{4} \quad (42)$$

Adding the back-to-back terms gives,

$$\frac{2}{(1)(3)} + \frac{2}{(5)(7)} + \frac{2}{(9)(11)} + \frac{2}{(13)(15)} + \cdots = \frac{\pi}{4} \quad (43)$$

Simplifying gives another well-known formula for π (End-note 19),

$$\sum_{n=0}^{\infty} \left[\frac{1}{(4n+1)(4n+3)} \right] = \frac{1}{(1)(3)} + \frac{1}{(5)(7)} + \frac{1}{(9)(11)} + \frac{1}{(13)(15)} + \cdots = \frac{\pi}{8} \quad (44)$$

Expressing each term in Equation (44) as the difference of two other series in two different ways gives the following two equations (45; 47),

$$\sum_{n=0}^{\infty} \left[\frac{1}{(4n+1)(4n+3)} \right] = \sum_{n=0}^{\infty} \left[\frac{1}{(4n+1)^2} - \frac{2}{(4n+1)^2(4n+3)} \right] = \frac{\pi}{8}$$

$$= \left[\frac{1}{(1)^2} - \frac{2}{(1)^2(3)} \right] + \left[\frac{1}{(5)^2} - \frac{2}{(5)^2(7)} \right] + \left[\frac{1}{(9)^2} - \frac{2}{(9)^2(11)} \right] + \left[\frac{1}{(13)^2} - \frac{2}{(13)^2(15)} \right] + \dots$$

(45), (46)

$$\sum_{n=0}^{\infty} \left[\frac{1}{(4n+1)(4n+3)} \right] = \sum_{n=0}^{\infty} \left[\frac{1}{(4n+3)^2} + \frac{2}{(4n+1)(4n+3)^2} \right] = \frac{\pi}{8} \quad (47)$$

$$= \left[\frac{1}{(3)^2} + \frac{2}{(1)(3)^2} \right] + \left[\frac{1}{(7)^2} + \frac{2}{(5)(7)^2} \right] + \left[\frac{1}{(11)^2} + \frac{2}{(9)(11)^2} \right] + \left[\frac{1}{(15)^2} + \frac{2}{(13)(15)^2} \right] + \dots \quad (48)$$

(47), (48)

Adding Equations (45; 47) and simplifying further gives,

$$\sum_{n=0}^{\infty} \left[\frac{1}{(4n+1)^2} - \frac{2}{(4n+1)^2(4n+3)} \right] + \sum_{n=0}^{\infty} \left[\frac{1}{(4n+3)^2} + \frac{2}{(4n+1)(4n+3)^2} \right] = \frac{\pi}{4} \quad (49)$$

$$\sum_{n=0}^{\infty} \left[\frac{1}{(4n+1)^2} + \frac{1}{(4n+3)^2} \right] - \sum_{n=0}^{\infty} \left[\frac{2}{(4n+1)(4n+3)} \left\{ \frac{1}{(4n+1)} - \frac{1}{(4n+3)} \right\} \right] = \frac{\pi}{4} \quad (50)$$

$$\sum_{n=0}^{\infty} \left[\frac{1}{(2n+1)^2} \right] - \sum_{n=0}^{\infty} \left[\frac{2}{(4n+1)(4n+3)} \left\{ \frac{2}{(4n+1)(4n+3)} \right\} \right] = \frac{\pi}{4} \quad (51)$$

Noticing that the first term on the left hand side of Equation (51) is the well-known sum of the reciprocal of the squares of odd numbers given in Equation (31) and rearranging gives,

$$\sum_{n=0}^{\infty} \left[\frac{1}{(4n+1)^2(4n+3)^2} \right] = \left[\frac{(\pi-1)^2-1}{32} \right] \quad (52)$$

Hence, we have obtained the desired result. □

We now use combinations of Equations (44; 52) from Proposition (2) to arrive at Corollary (5). Notice that we are now arriving at convergence results for double series, which are normally not that simple to calculate (Endou 2004; Habil 2016; End-note 18; Appendix 11). In

this example, the result follows easily because it is embedded within - naturally progression from earlier steps - the results we have derived earlier.

Corollary 5. The expression $4 - (128) \sum_{n=0}^{\infty} \sum_{k=n+1}^{\infty} \left[\frac{1}{(4n+1)(4n+3)(4k+1)(4k+3)} \right]$ converges to $(\pi - 2)^2$.

Proof. Combining Equations (44; 52) we easily arrive at the decreasing double series,

$$(\pi - 2)^2 = 4 - (128) \sum_{n=0}^{\infty} \sum_{k=n+1}^{\infty} \left[\frac{1}{(4n+1)(4n+3)(4k+1)(4k+3)} \right] \quad (53)$$

This completes the required proof. □

We next turn to the π^2 series with alternating reciprocal squares (End-note 19) by using a corresponding well-known convergence result. The series summation with alternating reciprocal squares converges to $\frac{\pi^2}{12}$, and this is used to derive the result in Corollary (6).

Corollary 6. The series $\sum_{n=1}^{\infty} \left[\frac{(4n-1)}{(2n-1)^2 (2n)^2} \right]$ converges to $\frac{\pi^2}{12}$.

Proof. We start with the following well known expression for $\frac{\pi^2}{12}$,

$$\sum_{n=1}^{\infty} \left[\frac{(-1)^{n+1}}{n^2} \right] = \frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \cdots = \frac{\pi^2}{12} \quad (54)$$

This could be denoted as,

$$\frac{3}{(1)^2 (2)^2} + \frac{7}{(3)^2 (4)^2} + \frac{11}{(5)^2 (6)^2} + \frac{15}{(7)^2 (8)^2} + \cdots = \frac{\pi^2}{12} \quad (55)$$

This gives the desired result,

$$\sum_{n=1}^{\infty} \left[\frac{(4n-1)}{(2n-1)^2 (2n)^2} \right] = \frac{\pi^2}{12} \quad (56)$$

□

We next come up with another result - Corollary (7) - based on another combination of the result we started with in Corollary (6). But, in this case, we get a new result because of the separation of the terms we use and also because we connect the dependencies that we have derived in earlier results - Corollary (1) from Section (5). We also see here that slightly different results can be combined to get the same final result in Equation (58).

Corollary 7. The series $\sum_{n=1}^{\infty} \left[\frac{1}{(2n-1)^2 (n)^2} \right]$ converges to $\left(\frac{2\pi^2}{3} - \ln 256 \right)$.

Proof. A simple separation of $\sum_{n=1}^{\infty} \left[\frac{(4n-1)}{(2n-1)^2 (2n)^2} \right] = \frac{\pi^2}{12}$, Equation (56), gives,

$$\sum_{n=1}^{\infty} \left[\frac{1}{(2n-1)^2 (n)} - \frac{1}{(2n-1)^2 (2n)^2} \right] = \frac{\pi^2}{12} \quad (57)$$

Combining Equations (33; 57) or Equations (32; 57) yields simple yet potent results (Equations 58; 59) that we will turn to later,

$$\sum_{n=1}^{\infty} \left[\frac{1}{(2n-1)^2 (2n)^2} \right] = \left(\frac{\pi^2}{6} - \ln 4 \right) \quad (58)$$

We can rewrite Equation (58) to give the desired result,

$$\sum_{n=1}^{\infty} \left[\frac{1}{(2n-1)^2 (n)^2} \right] = \left(\frac{2\pi^2}{3} - \ln 256 \right) \quad (59)$$

□

The result from Corollary (3) - when combined with Equation (59) from Corollary (7) - gives the convergence result for another decreasing double series in Corollary (8).

Corollary 8. The expression $(2 \ln 2) (2 + \ln 2) - \sum_{n=1}^{\infty} \sum_{k=n+1}^{\infty} \left[\frac{1}{(n)(2n-1)(k)(2k-1)} \right]$ converges to $\frac{\pi^2}{3}$.

Proof. Combining Equations (36; 59) we get the decreasing double series,

$$(2 \ln 2) (2 + \ln 2) - \sum_{n=1}^{\infty} \sum_{k=n+1}^{\infty} \left[\frac{1}{(n)(2n-1)(k)(2k-1)} \right] = \frac{\pi^2}{3} \quad (60)$$

This completes the required proof.

□

The result - Equation (56) - from Corollary (6) when combined with Equations (60) from Corollary (8) gives the convergence result for another double series in Corollary (9).

Corollary 9. The expression $\sum_{n=1}^{\infty} \left[\frac{1}{(n)(2n-1)} \left\{ \frac{(4n-1)}{(2n-1)(n)} + \sum_{k=n+1}^{\infty} \left[\frac{1}{(k)(2k-1)} \right] \right\} \right]$ converges to $(2\ln 2)(2 + \ln 2)$.

Proof. Equation (56) also results in,

$$\sum_{n=1}^{\infty} \left[\frac{(4n-1)}{(2n-1)^2 (n)^2} \right] = \frac{\pi^2}{3} \quad (61)$$

Using Equations (60; 61) gives the desired result,

$$\sum_{n=1}^{\infty} \left[\frac{1}{(n)(2n-1)} \left\{ \frac{(4n-1)}{(2n-1)(n)} + \sum_{k=n+1}^{\infty} \left[\frac{1}{(k)(2k-1)} \right] \right\} \right] = (2\ln 2)(2 + \ln 2) \quad (62)$$

□

Moving on to higher powers, we first introduce cubes and derive results for series with terms that have cubes. Using the result - Equation (59) - from Corollary (7) we can find series convergence results for terms with cubes in Corollary (10). Here, we also use the result from Corollary (1).

Corollary 10. The series $\sum_{n=1}^{\infty} \left[\frac{1}{(2n-1)^2 (n)^3} \right]$ converges to $\left[\zeta(3) + \left(\frac{5\pi^2}{3} \right) - 12(\ln 4) \right]$

Proof. We can split $\sum_{n=1}^{\infty} \left[\frac{1}{(2n-1)^2 (n)^2} \right]$ from Equation (59) as,

$$\sum_{n=1}^{\infty} \left[\frac{1}{(2n-1)^2 (n)^2} \right] = \sum_{n=1}^{\infty} \left[\frac{1}{(n)^3} - \frac{4}{(2n-1)^2 (n)} + \frac{5}{(2n-1)^2 (n)^2} - \frac{1}{(2n-1)^2 (n)^3} \right] \quad (63)$$

Using Apéry's constant (End-note 21) for the sum of the inverses of the positive cubes with the results from Equations (33; 59) and rearranging gives,

$$\sum_{n=1}^{\infty} \left[\frac{1}{(2n-1)^2 (n)^3} \right] = \left[\zeta(3) + \left(\frac{5\pi^2}{3} \right) - 12(\ln 4) \right] \quad (64)$$

This concludes the proof. □

Note that ζ is the Zeta function (End-note 22; Ayoub 1974; Titchmarsh, Heath-Brown & Titchmarsh 1986; Steuding & Suriajaya 2020). The Zeta function arises in Apéry's constant (End-note 21) which is explained in the End-notes (Section 12). We also use the Zeta function and Apéry's constant in our next result.

Corollary (11) has the convergence of a double series, which involves Apéry's constant $\zeta(3)$, and another result with a fourth power in its terms. The Basel Problem and other results from Lemma (1) as well as Corollaries (1; 2; 7) are used. Additional explanations for the intermediate result in Equation (65) are given in Appendix (11).

Corollary 11. *We prove the convergence of the following series:*

$$\sum_{n=1}^{\infty} \left[\frac{1}{(2n-1)^2 (n)^4} \right] \text{ and } \sum_{n=1}^{\infty} \sum_{k=n+1}^{\infty} \left[\frac{1}{(2n-1) (n)^2 (2k-1) (k)^2} \right]$$

and show the values they converge to, which are

$$\left[4 \{ \zeta(3) - 8 (\ln 4) \} + \frac{\pi^4}{90} + 4\pi^2 \right] \text{ and } \left[\frac{\pi^4}{120} - 2\pi^2 \left(1 + \frac{\ln 4}{6} \right) - 2 \{ \zeta(3) - (\ln 4) (8 + \ln 4) \} \right] \text{ respectively.}$$

Proof. We next use $\zeta(3)$ and $\zeta(4)$ in the expansion of $\sum_{n=1}^{\infty} \left[\frac{1}{(2n-1)^2 (n)^4} \right]$ as shown below,

$$\begin{aligned} \sum_{n=1}^{\infty} \left[\frac{1}{(2n-1)^2 (n)^4} \right] &= \sum_{n=1}^{\infty} \left[\frac{1}{(n)^4} + \frac{4}{(n)^3} + \frac{1}{(n)^2} + \frac{6}{(2n-1)^2 (n)^2} \right. \\ &\quad \left. - \frac{2}{(2n-1)^2 (n)} - \frac{5}{(2n-1) (n)^2} - \frac{4}{(2n-1)^2} \right] \end{aligned} \quad (65)$$

Using Equations (65; 26; 31; 33; 35; 59) and simplifying gives the below two expressions,

$$\sum_{n=1}^{\infty} \left[\frac{1}{(2n-1)^2 (n)^4} \right] = \left[4 \{ \zeta(3) - 8 (\ln 4) \} + \frac{\pi^4}{90} + 4\pi^2 \right] \quad (66)$$

$$\sum_{n=1}^{\infty} \sum_{k=n+1}^{\infty} \left[\frac{1}{(2n-1)(n)^2(2k-1)(k)^2} \right] = \left[\frac{\pi^4}{120} - 2\pi^2 \left(1 + \frac{\ln 4}{6} \right) - 2 \{ \zeta(3) - (\ln 4)(8 + \ln 4) \} \right] \quad (67)$$

End-note (23) gives the value of $\sum_{n=1}^{\infty} \frac{1}{(n)^4} = \zeta(4)$, the Riemann Zeta function for 4.

□

In our last result, Corollary (12), we link several earlier results to find bounds for Apéry's constant. We use the results from Corollaries (1; 3; 9), Lemma (1) and Proposition (1).

Corollary 12. *We establish the following lower bound and upper bound for Apéry's constant:*

$$\left(\frac{2\pi^2}{3} - 8 \ln 2 \right) < \zeta(3) = \sum_{n=1}^{\infty} \left[\frac{1}{(n)^3} \right] < \left(\frac{\pi^2}{3} \right)$$

Proof. We return to Equation (61) and consider inequalities as follows,

$$\sum_{n=1}^{\infty} \left[\frac{(4n-1)}{(2n-1)^2(n)^2} \right] = \frac{\pi^2}{3} \quad (68)$$

$$\sum_{n=1}^{\infty} \left[\frac{1}{(n)^2} + \frac{4}{(2n-1)^2} - \frac{4}{(2n-1)(n)} \right] = \sum_{n=1}^{\infty} \left[\frac{n}{(2n-1)^2(n)^3} \right] < \sum_{n=1}^{\infty} \left[\frac{(4n-1)}{(2n-1)^2(n)^3} \right] \quad (69)$$

$$\sum_{n=1}^{\infty} \left[\frac{(4n-1)}{(2n-1)^2(n)^3} \right] = \sum_{n=1}^{\infty} \left[\frac{4}{(2n-1)^2(n)} - \frac{1}{(n)^3} \right] < \sum_{n=1}^{\infty} \left[\frac{(4n-1)}{(2n-1)^2(n)^2} \right] \quad (70)$$

Using the results from Equations (20; 26; 31; 33; 36; 61) in Equations (69; 70) gives,

$$\left(\frac{2\pi^2}{3} - 8 \ln 2 \right) < \zeta(3) = \sum_{n=1}^{\infty} \left[\frac{1}{(n)^3} \right] < \left(\frac{\pi^2}{3} \right) \quad (71)$$

We have established the desired bounds on Apéry's constant.

□

6.2 Is Any Series Really Many Series? Any Series Is Really Many Series!

An attempt to find generalizations of the above, sum plus difference (Section 6.1), technique will be pursued later. Though our hunch is that there must be many well known results towards that end. Such a generalization could involve forming partial summations of any series and then taking suitable differences of the resulting summation. Surely, taking differences of the terms first and then doing summations next or varying the number of terms involved in the summation - or the differences, or any combination of these operations - are other possibilities.

This brings up the very general question, “*Is Any Series Really Many Series?*”. The answer in the affirmative could be the challenge to verify the intuition that “*Any Series Is Really Many Series*”. This would also require finding standard ways to separate a given series and arriving at the properties of the component series, while being open to the possibility of presumably non-standard methods since they might become the norm in due course of time. We note that the methods of partial fraction decomposition will be very useful towards disassembling any partial sums or differences formed (Hamilton 1972; Eustice & Klamkin 1979; Straight & Dowds 1984; Scott & Peeples 1988; Kung 2006; Rose 2007; Adkins & Davidson 2008; End-note 24).

The one warning we want to emphasize is that care must be taken not to rearrange the terms when there is a difference involved. This result - known mathematically as the Riemann rearrangement theorem (Rudin 1964; Beigel 1981; Gordon 2001; Dence 2008; Freniche 2010; Chen & Kennedy 2012; End-note 25) and applicable when a series is conditionally convergent - highlights the practical wisdom that any differences (in mathematics or in life) must be handled with caution. Otherwise things could get blown out of proportion, which in the series world could give rise to the *lack of convergence, or the long march towards infinity, or being stuck everywhere in between*.

As an immediate generalization, this can be stated as the need for vigilance when there are conditions present - conditionally convergent series have to be handled with care, since rearrangement of the terms can be done so that the new series converges to an arbitrary real number, or diverges, based on the Riemann rearrangement theorem - and also that when there is absolute involvement there is more flexibility which can aid in obtaining desired goals - when there is absolute convergence the series are much more amenable to diverse kinds of modifications. But, suffice it to say that this is a can of worms best left unopened for now.

7 Numerical Convergence Conundrum: Pro Se Legal, Nothing Illegal Per Se

In this section, we intend the numbers we use to only represent actual numbers. To aid in this effort of letting numbers represent themselves - *somewhat like someone going to court and representing oneself: a seriously complicated affair* - we use computational devices and

software: an approach that could be considered experimental in mathematics, or elsewhere (Borwein et al., 1996; Borwein & Bailey 2008; Swank 2005; Quintanilla, Allen & Hirt 2017; End-notes 26; 27).

7.1 Empirical Intuition with Numbers Representing Themselves

Equation (72) gives the value of $\left(\frac{\pi^2}{4} - \ln 4\right)$ up-to 120 decimal places. Figures (4; 5) show the convergence of the series $\sum_{n=1}^{\infty} \left[\frac{1}{(n)(2n-1)^2}\right]$ to $\left(\frac{\pi^2}{4} - \ln 4\right)$ as we include increasing number of terms of the series. In both Figures (4; 5), the first portion (before the colon), shows the number of terms of the series that have been included in the summation; and the second portion (after the colon) shows the difference between the sum of the series up-to the number of terms shown in portion one and the value the series converges to, which is shown in Equation (72).

It is clearly seen from Figures (4; 5) that as the number of terms included in the series summation increases, the difference - between the partial sums and the value the series converges to - decreases. Though, as we know from limits, the difference will get closer and closer to zero but it will never be zero (Schumacher 2008). The calculations were done on a Macintosh personal computer using the R (Ihaka & Gentleman 1996; Crawley 2012; Team 2013; End-note 28) package **Rmpfr** (Mächler 2015) for greater floating point precision.

$$\left(\frac{\pi^2}{4} - \ln 4\right) = 1.0811067391524490822550947839786769582441827986461892674014 \dots \quad (72)$$

$$\dots 0958997305950100046786910290563810121966525912284851074218750 \dots \quad (73)$$

We have also verified all the other series with single summations using Mathematica (Wolfram 1999; Wellin, Gaylord & Kamin 2005; Hazrat 2010; Tam 2011; End-note 29). Figure (6) shows the convergence results for all the series with single summations for which we have given the proofs in Sections (5; 6.1). The Mathematica code and the corresponding notebook are available upon request.

The double series results - and also all the other results, including single series summations - were verified using another utility within the Wolfram Toolkit, called Wolfram Alpha (Dimiceli et al., 2010; Hindin 2010; End-note 30). Figure (7), and the subsequent text below it, gives the Wolfram Alpha formulae to verify each result and is labelled with the corresponding index - corollary or proposition number.

8 Seeking Subjectively Objective Proofs and Solutions for Objectively Subjective Meanings and Problems

As we discussed in the introduction (Section 1), comprehension ultimately happens in our minds. That is, the meanings to proofs, solutions to problems, and answers to questions are very much dependent on our unique individual perspectives shaped by our experiences. Someone externally - instructors or peers, who become instructors when they help us understanding something - can provide endless amounts of training, and material, to aid our understanding, but the responsibility falls upon us to become involved.

Involvement alone can ensure that our intelligence will shine through to help us attain better awareness. Involvement can be represented - and measured - as *IQ (Involvement Quotient)*: Kashyap 2021b) identical to the *Intelligence Quotient (IQ)* to cause more confusions, which we know is good (Section 2.1). Though with minimal involvement it should become clear that intelligence is omnipresent. This is also because we need intelligence - which comes with involvement - to recognize the intelligence elsewhere.

With involvement from all parties involved, the learning - finding answers or solutions - adventure becomes a smoother venture and perhaps manifest onto the rest of this journey we call as Life. We - whoever is trying to learn something - have to raise the relevant questions to connect the links between the seemingly unrelated elements, which is what we refer to as comprehension, or consciousness.

While there are bound to be subjective evaluations of mathematics - and its usefulness - the fact remains that mathematics is rooted in accuracy as an exact science. Mathematics can be considered to be the language which can express ideas with the most precision, and hence we see its increasing usage across all sciences and other fields as well - to establish new findings. The struggle is mostly to ensure that the untrained eye or individuals unfamiliar with mathematical notation and notions - can get comfortable with this language. Another complementary aspect - to ease the burden of learning - is to ensure that the writers in the language of mathematics provide detailed steps, explicitly specify the notations and related results they use to express their work.

We have tried to provide as many detailed steps as possible, at-least in the first few results, to ensure that the results can be easily understood by a wide audience. While the material presented here might seem trivial to the trained mathematician, we need to remind ourselves that such an approach is essential conditioning to groom future mathematicians, who are just starting out.

Once a budding mathematician is past the initial stages and a certain level of familiarity and fondness for the subject has set in, steps can be skipped building what can be termed mathematical maturity. That being said, we retain the hope that with enough acclimatization to

the notation and the provision of sufficiently granular steps, perhaps, all of mathematics can be rendered elementary to everyone. To assist in building a mathematical foundation, we have provided some suggestions to approach the topics discussed here from a student perspective and to ease the barriers to learning.

We are almost certain that these proofs are well known to many others. Our view is that it has just become harder to find such results in the literature, or other related resources, due to the explosion of information both within and outside the mathematical realm (Kashyap 2017). This highlights a major problem, mathematical and otherwise, that an unintended consequence of the many advances in knowledge creation might be that some knowledge could either be getting lost or just becoming harder to find, which is the equivalent of getting lost in many (most?) cases. Solutions to address this issue by finding better mechanisms of knowledge representation, organization and retrieval might be more necessary than ever.

We have tried our best to make this manuscript somewhat lively and entertaining to the reader. Having said that, even if we are deemed to have failed on that front, we just wish to make our intentions known loud and clear. More authors (and journals) attempting (and welcoming) such an approach, (related not just to mathematics), would be a huge next step in making mathematics (and other knowledge related activities) more endearing to later generations. If more of us get involved in the pursuit, and dissemination, of knowledge then perhaps even if some knowledge is lost it will be found sooner or later.

9 Lost and Found: The Series Conclusion, Serious Not!

We have provided a simple proof of the convergence of the alternating harmonic series, perhaps justifying that the alternating harmonic series could be termed the more harmonious sibling of the harmonic series. As an unintended consequence of exploring this proof, we have uncovered the formulae for other apparently new series. We have shown a few other related results including numerical illustrations of one of the series results, using custom code we have created. All the results have also been verified using computational software.

We have given all the main results as two propositions and twelve corollaries. We would like to call them “*the two smoking barrels*” and “*the dirty dozen*”. The lonely lemma we use becomes “*the lone ranger*”, of course.

Future work can look at ways in which mathematics - and scientific knowledge - can be made to be entertaining, in addition to conveying its usefulness or novelty. Ways to ensure that dictionaries as well as appendices - with terminology, notation, symbols and detailed mathematical steps - are provided as part of the scientific publishing process should assist with

ease of understanding. Mechanisms to increase the involvement of participants - both teachers and students - would be a welcome undertaking.

The humungous amounts of information being generated makes it a challenge to make all of it entertaining, and hence difficult to solicit involvement from the concerned members. ***Information can get buried and stay hidden; Knowledge is shared and spreads; Wisdom is dispersed and treasured*** (Kashyap 2017). So the challenge is to locate “*nuggets of knowledge*” from “*tons of trash*” and to extract “*sprinkles of pearls*” from “*buckets of shells*”.

An appreciation of this distinction - between information, knowledge and wisdom - is likely to convey to us the message that any message conveyed with fewer carefully chosen words will increase its likelihood of being retrieved, should it get misplaced. If a few sections (or chapters) in a paper (or book) are lost, perhaps they will be found if the section names have stayed on in our memories and if entire papers (or books) have gone missing then if the title (or the summary, which should be present in the title) has remained in our thoughts, we will recover - or recreate - whatever has been forgotten.

10 References

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Part I

Appendix of Supplementary Online Material

11 Appendix on Double Series Summation and Partial Fraction Decomposition

To understand the notation used - and conventions followed - in a double series summation can be tricky. Hence, we provide some illustrative examples here. Please see End-notes and references in the main body of the article for additional explanations and resources for double series summation and partial fraction decomposition.

- Below, we show a simple example regarding the use of double series summation notation and how to write an expanded double series summation (Reference: Simple Double Summation, Exchange Link).

$$\sum_{n=0}^2 \sum_{k=0}^3 (2n + 3k) = \sum_{n=0}^2 \left[\sum_{k=0}^3 (2n) + \sum_{k=0}^3 (3k) \right] \quad \text{Math Stack Exchange Link} \quad (74)$$

$$= \sum_{n=0}^2 \left[(2n) \sum_{k=0}^3 (1) + (3) \sum_{k=0}^3 (k) \right] \quad (75)$$

$$= \sum_{n=0}^2 [(2n)(4) + (3)(0 + 1 + 2 + 3)] \quad (76)$$

$$= \sum_{n=0}^2 [(8n) + 18] \quad (77)$$

$$= [(8)(0 + 1 + 2) + 18(3)] = 24 + 54 = 78 \quad (78)$$

A brute force approach will give the following,

$$((0+0)+(0+3)+(0+6)+(0+9))+((2+0)+(2+3)+(2+6)+(2+9))+((4+0)+(4+3)+(4+6)+(4+9)) = 78 \quad (79)$$

2. Applying the above expansion technique (Point 1), we try to understand the results from Corollary (5).

$$\sum_{n=0}^{\infty} \sum_{k=n+1}^{\infty} \left[\frac{1}{(4n+1)(4n+3)(4k+1)(4k+3)} \right] = \frac{1}{(1)(3)(5)(7)} + \frac{1}{(1)(3)(9)(11)} \quad (80)$$

$$+ \frac{1}{(1)(3)(13)(15)} + \dots \quad (81)$$

$$+ \frac{1}{(5)(7)(9)(11)} + \frac{1}{(5)(7)(13)(15)} \quad (82)$$

$$+ \frac{1}{(5)(7)(17)(19)} + \dots \quad (83)$$

Corollary (5) uses the results from Equations (44; 52) which are repeated below,

$$\sum_{n=0}^{\infty} \left[\frac{1}{(4n+1)(4n+3)} \right] = \frac{1}{(1)(3)} + \frac{1}{(5)(7)} + \frac{1}{(9)(11)} + \frac{1}{(13)(15)} + \dots = \frac{\pi}{8} \quad (84)$$

$$\sum_{n=0}^{\infty} \left[\frac{1}{(4n+1)^2(4n+3)^2} \right] = \left[\frac{(\pi-1)^2 - 1}{32} \right] \quad (85)$$

Since the series in Equation (44) is absolutely convergent, we can square both sides and use the result - for the square of a sum of numbers (Reference: Expansion of the Square of the Sum of n Numbers, Math Stack Exchange Link) - to get the desired result in Equation (53), which is the main result in Corollary (5).

$$(\pi-2)^2 = 4 - (128) \sum_{n=0}^{\infty} \sum_{k=n+1}^{\infty} \left[\frac{1}{(4n+1)(4n+3)(4k+1)(4k+3)} \right] \quad (86)$$

3. We next show an example of partial fraction decomposition using Equation (65) in Corollary (11). We are simplifying the example by only using four coefficients, A, B, C, D .

$$\sum_{n=1}^{\infty} \left[\frac{1}{(2n-1)^2 (n)^4} \right] = \sum_{n=1}^{\infty} \left[\frac{1}{(n)^4} + \frac{4}{(n)^3} + \frac{1}{(n)^2} + \frac{A}{(2n-1)^2 (n)^2} - \frac{B}{(2n-1)^2 (n)} + \frac{C}{(2n-1) (n)^2} + \frac{D}{(2n-1)^2} \right] \quad (87)$$

Doing the necessary calculations gives the following simultaneous equations governing the coefficients,

$$D = -4$$

$$A - C = 11$$

$$B + 2C = -12$$

Clearly several solutions are possible and we have chosen one of them for the proceedings in Corollary (11).

12 Appendix of Additional Explanations: End-notes

1. Numerology is any belief in the divine or mystical relationship between a number and one or more coinciding events. It is also the study of the numerical value of the letters in words, names, and ideas. Numerology, Wikipedia Link
2. Preferred stock (also called preferred shares, preference shares, or simply preferreds) is a component of share capital that may have any combination of features not possessed by common stock, including properties of both an equity and a debt instrument, and is generally considered a hybrid instrument. Preferred Stock, Wikipedia Link
3. Deoxyribonucleic acid (DNA) is a molecule composed of two chains (made of nucleotides) that coil around each other to form a double helix carrying the genetic instructions used in the growth, development, functioning and reproduction of all known living organisms and many viruses. DNA, Wikipedia Link

4. Beauty and the Beast (French: La Belle et la Bête) is a fairy tale written by French novelist Gabrielle Suzanne Barbot de Villeneuve and published in 1740 in The Young American and Marine Tales (French: La Jeune Américaine et les contes marins). Her lengthy version was abridged, rewritten, and published first by Jeanne-Marie Leprince de Beaumont in 1756. Beauty and the Beast, Wikipedia Link

5. Cooper: You don't think nature can be evil? Brand: No. Formidable, frightening, but...no, not evil. Is a lion evil because it rips a gazelle to shreds? Quote from the movie Interstellar. Interstellar (Film), Wikipedia Link; Interstellar (Film) Quote from Reddit

(a) It is important to add here that Anne Hathaway - and Hollywood - who plays the role of Brand in the movie Interstellar, is perhaps more convincing than most mathematics instructors. That said, we have to believe only what becomes a part of our experience. Hence, while it is prudent to take suggestions from others, we need to take the onus upon ourselves to verify what is true by trying to understand something better or making it a part of our existential reality.

6. A burqa or a burka (/ˈbʊrːkə/) is an enveloping outer garment worn by some Muslim women which fully covers the body and the face. Burqa or Burkha, Wikipedia Link 7. In mathematics, the harmonic series is the divergent infinite series

$$\sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \cdots$$

There are several well-known proofs of the divergence of the harmonic series. Harmonic Series, Wikipedia

Link; Harmonic Series, Mathworld Wolfram Link; List of Mathematical Series, Wikipedia Link

8. In the mathematics of convergent and divergent series, Euler summation is a summation method. That is, it is a method for assigning a value to a series, different from the conventional method of taking limits of partial sums. Euler Summation, Wikipedia Link

9. In mathematical analysis, asymptotic analysis, also known as asymptotics, is a method of describing limiting behavior. Informally, the term asymptotic means approaching a value or curve arbitrarily closely (i.e., as some sort of limit is taken). Asymptotic Analysis, Wikipedia Link; Asymptotic, Mathworld Wolfram Link

10. In mathematics, the alternating harmonic series is the infinite series

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \dots$$

There are several well-known proofs of the convergence of the alternating harmonic series. Alternating Harmonic Series, [Wikipedia Link](#); Alternating Harmonic Series, [Mathworld Wolfram Link](#)

11. In mathematics, the Mercator series or Newton–Mercator series is the Taylor series for the natural logarithm:

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \dots$$

In summation notation,

$$\ln(1+x) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} x^n$$

The series converges to the natural logarithm (shifted by 1) whenever $-1 < x \leq 1$. Mercator Series,

[Wikipedia Link](#); Mercator Series, [Mathworld Wolfram Link](#)

- (a) In mathematics, the Taylor series or Taylor expansion of a function is an infinite sum of terms that are expressed in terms of the function's derivatives at a single point. Taylor Series, [Wikipedia Link](#)
- (b) The natural logarithm of a number is its logarithm to the base of the mathematical constant e , which is an irrational and transcendental number approximately equal to 2.718281828459. Natural Logarithm, [Wikipedia Link](#)
- (c) The number e , also known as Euler's number, is a mathematical constant approximately equal to 2.718281828 that can be characterized in many ways. e Mathematical Constant, [Wikipedia Link](#)
- (d) Note that 1828 which appears in the mathematical constant e is the birth year of Leo Tolstoy. Count Lev Nikolayevich Tolstoy, usually referred to in English as Leo Tolstoy, was a Russian writer who is regarded as one of the greatest authors of all time. He received nominations for the Nobel Prize in Literature every year from 1902 to 1906 and for the Nobel Peace Prize in 1901, 1902, and 1909; the fact that he never won is a major controversy. Leo Tolstoy, [Wikipedia Link](#)

12. The Lone Ranger is a fictional masked former Texas Ranger who fought outlaws in the American Old West with his Native American friend Tonto. Lone Ranger, Wikipedia Link

(a) There have been several movies made based on the story of the Lone Ranger. A memorable one is a 2013 American western action film, The Lone Ranger, directed by Gore Verbinski and written by Justin Haythe, Ted Elliott and Terry Rossio. The Lone Ranger: 2013 Film, Wikipedia Link

13. Lock, Stock and Two Smoking Barrels is a 1998 British-American black comedy crime film written and directed by Guy Ritchie. Lock, Stock and Two Smoking Barrels, Wikipedia Link

14. The Dirty Dozen is a 1967 American war film directed by Robert Aldrich. The Dirty Dozen, Wikipedia Link

15. If a sequence is increasing or decreasing, then it is called a monotone sequence.

(a) A monotonic function is a function which is either entirely non-increasing or non-decreasing. Monotonic Function, Wikipedia Link; Monotonic Function, Mathworld Wolfram Link

16. A sequence is bounded if it is bounded above and below, that is to say, if there is a number, K , less than or equal to all the terms of sequence and another number, K' greater than or equal to all the terms of the sequence. Therefore, all the terms in the sequence are between K and K' . Bounded Sequence, SuperProf Link

(a) In mathematics, a function f defined on some set X with real or complex values is called bounded if the set of its values is bounded. In other words, there exists a real number M such that

$$|f(x)| \leq M$$

for all x in X . Bounded Function, Wikipedia Link; Convergent Sequence, Mathworld Wolfram Link

17. The Basel problem asks for the precise summation of the reciprocals of the squares of the natural numbers, i.e. the precise sum of the infinite series:

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \cdots$$

The exact sum is equal to $\frac{\pi^2}{6}$. Basel Problem, Wikipedia Link; Riemann Zeta Function zeta(2), Mathworld Wolfram Link

18. A double sum is a series having terms depending on two indices,

$$\sum_{i,j} b_{ij}$$

Double Series, Mathworld Wolfram Link; Double Series, Encyclopedia of Math Link;

19. The following links contain a list of formulae involving the mathematical constant π . List of Formulae involving Pi, Wikipedia Link; Pi Formulas, Mathworld Wolfram Link

20. In mathematics, the Leibniz formula for π , named after Gottfried Leibniz, states that

$$\sum_{n=0}^{\infty} \left[\frac{(-1)^n}{2n+1} \right] = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \cdots = \frac{\pi}{4}$$

It is also called the Madhava–Leibniz series as it is a special case of a more general series expansion for the inverse tangent function, first discovered by the Indian mathematician Madhava of Sangamagrama in the 14th century ([112]; [113]). Leibniz Pi Formula, Wikipedia Link; Leibniz Series, Mathworld Wolfram Link

21. In mathematics, at the intersection of number theory and special functions, Apéry's constant is the sum of the inverses of the positive cubes. That is, it is defined as the number

$$\begin{aligned} \zeta(3) &= \sum_{n=1}^{\infty} \frac{1}{n^3} \\ &= \lim_{n \rightarrow \infty} \left(\frac{1}{1^3} + \frac{1}{2^3} + \cdots + \frac{1}{n^3} \right) \end{aligned}$$

where ζ is the Riemann zeta function. It has an approximate value of

$$\zeta(3) = 1.202056903159594285399738161511449990764986292\dots$$

Apéry's Constant, Wikipedia Link

22. The Riemann zeta function or Euler–Riemann zeta function, denoted by the Greek letter ζ (zeta), is a mathematical function of a complex variable defined as

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} = \frac{1}{1^s} + \frac{1}{2^s} + \frac{1}{3^s} + \dots$$

for $\text{Re}(s) > 1$ and its analytic continuation elsewhere. Riemann Zeta Function, Wikipedia Link

23. The Riemann zeta function of 4 is given by:

$$\zeta(4) = \sum_{n=1}^{\infty} \frac{1}{n^4} = \frac{1}{1^4} + \frac{1}{2^4} + \frac{1}{3^4} + \dots$$

$$\zeta(4) = \frac{\pi^4}{90} = 1.082323\dots$$

Particular Values of the Riemann Zeta Function, Wikipedia Link; Riemann Zeta Function of 4, Proof Wiki Link

24. In algebra, the partial fraction decomposition or partial fraction expansion of a rational function (that is, a fraction such that the numerator and the denominator are both polynomials) is an operation that consists of expressing the fraction as a sum of a polynomial (possibly zero) and one or several fractions with a simpler denominator. Partial Fraction Decomposition, Wikipedia Link; Partial Fraction Decomposition, Mathworld Wolfram Link
25. In mathematics, the Riemann series theorem (also called the Riemann rearrangement theorem), named after 19th-century German mathematician Bernhard Riemann, says that if an infinite series of real numbers is conditionally convergent, then its terms can be arranged in a permutation so that the new series converges to an arbitrary real number, or diverges. Riemann Series Theorem, Wikipedia Link; Riemann Series Theorem, Mathworld Wolfram Link

26. Experimental mathematics is an approach to mathematics in which computation is used to investigate mathematical objects and identify properties and patterns. Experimental Mathematics, Wikipedia Link; Experimental Mathematics, Mathworld Wolfram Link
27. Pro se legal representation (/ˌproʊ ˈsiː/ or /ˌproʊ ˈseɪ/) comes from Latin pro se, meaning "for oneself" or "on behalf of themselves" which, in modern law, means to argue on one's own behalf in a legal proceeding, as a defendant or plaintiff in civil cases, or a defendant in criminal cases, rather than have representation from counsel or an attorney. Pro Se Legal, Wikipedia Link
 - (a) This status is sometimes known as *in propria persona* (abbreviated to "pro per"). In England and Wales the comparable status is that of "litigant in person". In England and Wales, a litigant in person is an individual, company or organization that has rights of audience (this is, the right to address the court) and is not represented in a court of England and Wales by a solicitor or barrister. Litigant in Person, Wikipedia Link
28. R is a programming language and free software environment for statistical computing and graphics supported by the R Foundation for Statistical Computing. R Programming Language, Wikipedia Link
29. Wolfram Mathematica (usually termed Mathematica) is a modern technical computing system spanning most areas of technical computing - including neural networks, machine learning, image processing, geometry, data science, visualizations, and others. Wolfram Mathematica, Wikipedia Link; Wolfram Mathematica, Home Page Link;
30. WolframAlpha (/ˈwʊlf.r@m-/ WUULf-r@m-) is an answer engine developed by Wolfram Research.[3] It is offered as an online service that answers factual queries by computing answers from externally sourced data. Wolfram Alpha, Wikipedia Link; Wolfram Alpha, Home Page Link;
 - (a) WolframAlpha was released on May 18, 2009, and is based on Wolfram's earlier product Wolfram Mathematica, a technical computing platform.

13 Appendix of Figures

- Each of the illustrations below is linked into the main text and additional clarity regarding the context is given there accordingly.

The Two Monsters



Price = Discounted Future Cash Flow

$$P = E(mx)$$

m is the discount factor, x is the cash flow. E is the expectation since we are using future cash flows. We need cash flows and discount factors

Valuation Formulas

$$PV = \frac{FV_n}{(1+r)^n}$$

$$FV_n = PV * (1+r)^n$$

$$PV = \frac{C}{r}$$

$$PV = \frac{C_1}{r-g}$$

$$PV = \frac{C}{r} \left[1 - \frac{1}{(1+r)^T} \right]$$

$$PV = \frac{C_1}{r-g} \left[1 - \left(\frac{1+g}{(1+r)} \right)^T \right]$$

Figure 1: Two Monsters

Beauty and The Beast (The Monster) !!!



Figure 2: Beauty and the Beast

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Monsters are Simply Fantastic Creatures

***Get Confused and Frustrated
But Please Don't Panic.***

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12

Figure 3: Simply Fanstastic Creatures

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1 : 0.08110673915244908225509478397867695824418279864618926740140958997305950100046786910290563810121966526
2 : 0.02555118359689352978349207460411179275427076556513457990140958997305950100046786910290563810121966526
3 : 0.0122178502635601955943618264555362617772135474512725568265958997305950100046786910290563810121966526
4 : 0.007115809447233665389245907552151209560111794255472714667034589973059501000467869102905638101219665259
5 : 0.004646673644764529637105168892219069706630590937830502752972089973059501000467869102905638101219665259
6 : 0.003269263176444970370224210743969877543282393000818784002972089973059501000467869102905638101219665259
7 : 0.002423954638828740497075267680315562787587452314592892889690839973059501000467869102905638101219665259
8 : 0.001868399083273184942001579756975786980479653559423978094768964973059501000467869102905638101219665259
9 : 0.00148393157077799077423623539600114488281007490850980047893964973059501000467869102905638101219665259
10 : 0.001206923260528683272727149135050411163252476237513119940472089973059501000467869102905638101219665259
11 : 0.001000780197242762855744829954040824790286762274600560431194746223059501000467869102905638101219665259
12 : 0.0008432502665559323536803916566026378258591756297126362368588087230595010004678691029056381012196652591
13 : 0.0007201733434790092833692799823450064049446262491717884585384962230595010004678691029056381012196652591
14 : 0.0006221917640159483382425788311917363297887384964004108950619337230595010004678691029056381012196652591
15 : 0.0005429210545430985606084316253291893043986527321711132571224805980595010004678691029056381012196652591
16 : 0.0004778846341476771185401832620517841714225249437466213747982618480595010004678691029056381012196652591
17 : 0.0004238685373508316576253428309858866522752685544863293215756055980595010004678691029056381012196652591
18 : 0.0003785170634279291623655755505858882396985898798182556674008009105595010004678691029056381012196652591
19 : 0.000340071790274263404110966786441921302799357201702887779965040355595010004678691029056381012196652591
20 : 0.0003071986804780766859183394124507775895161210631802628085140821605595010004678691029056381012196652591
21 : 0.0002788708710675783991969176806466284428332181254252741381649610668095010004678691029056381012196652591
22 : 0.0002542875582203391061653594063415800146011059244945595309261915355595010004678691029056381012196652591
23 : 0.0002328168121119118381072755974867054955446399820819447085384962230595010004678691029056381012196652591
24 : 0.0002139545818418047008090802985510952420789288429394314333920118480595010004678691029056381012196652591
25 : 0.0001972948567272690912043844252862716952457156570479223773007032543095010004678691029056381012196652591
26 : 0.0001825076447082231615446353404250073164583335004597852274899122386845010004678691029056381012196652591
27 : 0.0001693225122635677548367157186245895299414876920953342959408399730595010004678691029056381012196652591
28 : 0.0001575161368208286755902346601862445278819103099256703898434278636845010004678691029056381012196652591
29 : 0.000146902791600548695818880650987008837251607312619681124401533324345010004678691029056381012196652591
30 : 0.0001373269992037278592494332627335578560412721548080394312862989574345010004678691029056381012196652591
40 : 7.746799906887819235818418821509161164232120642810559721037122305950100046786910290563810121966525912e-5
50 : 4.96642138184165982098661089758011992678910449929485365872022774700100046786910290563810121966525912e-5
60 : 3.452813427255547737260789828677647038904235375485814065977307852825100046786910290563810121966525912e-5
70 : 2.538808489080688645453919470475971630565875090797862858816358145793850046786910290563810121966525912e-5
80 : 1.944949280171687842207282652286996759374853570153436512991162833293850046786910290563810121966525912e-5
90 : 1.537470717861329234888362303939550445266997700929141492887708243450100046786910290563810121966525912e-5
100 : 1.245817854941208143850010931191145390431467819399631471677991446575100046786910290563810121966525912e-5

Figure 4: Series Difference of Initial Terms

106000000 : 6.018038654224216295828727981248329040753100250327013362032805773860211218106996966525912284851074219e-17
107000000 : 5.997341514636007759328931078634953532137462029527431503077598493045563543546450091525912284851074219e-17
108000000 : 5.977216629766307141847269988268075545367343321456577917154566123580927694547426654025912284851074219e-17
109000000 : 5.957643096067721155993757854109786471025699356507222467365129589533800924527895404025912284851074219e-17
110000000 : 5.938600955810418375775092309761811048249616568683696335022607556974882925260317279025912284851074219e-17
111000000 : 5.920071146208805003253692552185296881069541758468678302045366465544695699918520404025912284851074219e-17
112000000 : 5.902035451671458731157766355406719690416118496212139951233614440506691778409731341525912284851074219e-17
113000000 : 5.884476459023490496364149567015181800112562038928932714036993886774773443448793841525912284851074219e-17
114000000 : 5.867377515469566790299984819021138004372490166227028715177941806113715017545473529025912284851074219e-17
115000000 : 5.850722689076711472801356147561264818524029791057917612300291396176810110318911029025912284851074219e-17
116000000 : 5.834496731663448891258098023831266545272534685462152345274437423294777715909731341525912284851074219e-17
117000000 : 5.818685043873933935039762883648087121083564505263742069467473513877386892545473529025912284851074219e-17
118000000 : 5.803273642361811694844662629405248837558147307402568379068900569772715414273989154025912284851074219e-17
119000000 : 5.788249128805654371791055341833142445681615135322052231452464796102042043912661029025912284851074219e-17
120000000 : 5.773598660843538996730009511964979513692401943905797486871226895099873388517153216525912284851074219e-17
121000000 : 5.759309924547008320006550710183137907243670572998529485624378751462693060148012591525912284851074219e-17
122000000 : 5.74537110863241895887956465504433261743748113159303477908172383403773153531801654025912284851074219e-17
123000000 : 5.73177087993178486621756773385576282027057765710954447961915023124213064420473529025912284851074219e-17
124000000 : 5.718498360456646511197788753377209706757109214991620013273125253117079580533754779025912284851074219e-17
125000000 : 5.705543105564294288961092603660240769955654477951601996204037889337534750211489154025912284851074219e-17
126000000 : 5.692895083443913996882700858397086897963001673239437002842526155954594457853090716525912284851074219e-17
127000000 : 5.680544655670376659969029762527154846652051673527684591129575594758982504117739154025912284851074219e-17
128000000 : 5.668482558837190074596872366365564515949186320031661842215901445096726263273012591525912284851074219e-17
129000000 : 5.656699887194774689866238105018579647951539408123712212721504941081995809781801654025912284851074219e-17
130000000 : 5.645188076215177736044748642074572552112371821075688024387266225612158621061098529025912284851074219e-17
131000000 : 5.63393886973416162096825134257210241207186968759136255351397878921027029264223529025912284851074219e-17
132000000 : 5.622944391465355239891260711463444720003570038213515011187330558395380819547426654025912284851074219e-17
133000000 : 5.612196958596118937846652991401472554421395539436635465784268192446226919400942279025912284851074219e-17
134000000 : 5.601689240960201099056644042139905228810839111275133780503158661967511022794496966525912284851074219e-17
135000000 : 5.591414162238465847297991576488911241281528158919390455587920471525187338102114154025912284851074219e-17
136000000 : 5.581364905348194613162618015827394287450640226244492544531742780899996603238832904025912284851074219e-17
137000000 : 5.571534901050988029579934805791138703481130495853524883380662388598913992154848529025912284851074219e-17
138000000 : 5.561917817301250445160652614881485686207439001538067521317491512244934881437075091525912284851074219e-17
139000000 : 5.552507548976325343486517044532203793145009818654590424551490898674721563565981341525912284851074219e-17
140000000 : 5.543298208203009606747129264067626237853596644133792245150464325642580831754457904025912284851074219e-17
141000000 : 5.534284115197458456494601128622741135586605654761261837841805275028939092862856341525912284851074219e-17
142000000 : 5.525459789431047720435532375059407992115285635559108675800591911619896734464418841525912284851074219e-17
143000000 : 5.516819941321459689919137477925703071318197179884014110933269578462834203946840716525912284851074219e-17

Figure 5: Series Difference of Subsequent Terms

A_Simple_Proof_Conf...20

File Format

In[59]:= **A Simple Proof to Confirm the Harmonious Nature (Convergence) of the Alternating Harmonic Series Kashyap 2020**

In[61]:= **Sum[1/((n+1)(2n+1)^2), {n, 0, ∞}]**

Out[61]= $\frac{1}{4} (\pi^2 - 4 \log[4])$

In[62]:= **Sum[1/((4n+1)^2(4n+3)^2), {n, 0, ∞}]**

Out[62]= $\frac{1}{32} (-2 + \pi) \pi$

In[63]:= **Sum[1/((n^2)(2n-1)), {n, 1, ∞}]**

Out[63]= $\frac{1}{6} (-\pi^2 + 6 \log[16])$

In[64]:= **Sum[(4n-1)/((2n-1)^2(n^2)), {n, 1, ∞}]**

Out[64]= $\frac{\pi^2}{3}$

In[65]:= **Sum[1/((2n-1)(n)), {n, 1, ∞}]**

Out[65]= $\log[4]$

In[66]:= **Sum[1/((2n-1)^2(n^2)), {n, 1, ∞}]**

Out[66]= $\frac{2}{3} (\pi^2 - 6 \log[4])$

In[67]:= **Sum[1/((2n-1)^2(n^3)), {n, 1, ∞}]**

Out[67]= $\frac{1}{3} (5 \pi^2 - 36 \log[4] + 3 \text{Zeta}[3])$

In[68]:= **Sum[1/((2n-1)^2(n^4)), {n, 1, ∞}]**

Out[68]= $\frac{1}{90} (360 \pi^2 + \pi^4 - 2880 \log[4] + 360 \text{Zeta}[3])$

Figure 6: Mathematica Notebook Convergence Verification

+	Result_Verification	Corollary_5	Corollary_8	Corollary_11	Corollary_12				
Corollary	Wolfram Alpha Input					Comments	Comments	Comments	Link + goes missing sometimes, verify before running
1	sum((1/((2n-1)^2*(n))), n=1 to infinity						Verified		https://www.wolframalpha.com/input?i=sum((1/((2n-1)^2*(n))), n=1 to infinity
2	sum((1/((2n-1)^*(n^2))), n=1 to infinity						Verified		https://www.wolframalpha.com/input?i=sum((1/((2n-1)^*(n^2))), n=1 to infinity
3	sum((1/((2n-1)^*(n))), n=1 to infinity						Verified		https://www.wolframalpha.com/input?i=sum((1/((2n-1)^*(n))), n=1 to infinity
4	sum((1/((2n-1)^2*(2n))), n=1 to infinity						Verified		https://www.wolframalpha.com/input?i=sum((1/((2n-1)^2*(2n))), n=1 to infinity
5	sum((1/((2n-1)^2*(n^2))), n=1 to infinity						Verified		https://www.wolframalpha.com/input?i=sum((1/((2n-1)^2*(n^2))), n=1 to infinity
6	sum((n/((2n-1)^2)), n=1 to infinity						Verified		https://www.wolframalpha.com/input?i=sum((n/((2n-1)^2)), n=1 to infinity
7	sum((n-1/((2n-1)^2)), n=1 to infinity						Verified		https://www.wolframalpha.com/input?i=sum((n-1/((2n-1)^2)), n=1 to infinity
8	sum((1/((4n+1)^2*(4n+3)^2)), n=0 to infinity						Verified		https://www.wolframalpha.com/input?i=sum((1/((4n+1)^2*(4n+3)^2)), n=0 to infinity
9	sum(sum(1/((4n+1)(4n+3)(4k+1)(4k+3)), k=n+1 to infinity), n=0 to infinity)					Double Series	Verified		https://www.wolframalpha.com/input?i=sum(sum(1/((4n+1)(4n+3)(4k+1)(4k+3)), k=n+1 to infinity), n=0 to infinity
10	sum((4n-1)/((2n-1)^2*(2n)^2)), n=1 to infinity						Verified		https://www.wolframalpha.com/input?i=sum((4n-1)/((2n-1)^2*(2n)^2)), n=1 to infinity
11	sum((1/((2n-1)^2*(n^2))), n=1 to infinity						Verified		https://www.wolframalpha.com/input?i=sum((1/((2n-1)^2*(n^2))), n=1 to infinity
12	sum(sum(1/((n)(2n-1)(k)(2k-1))), k=n+1 to infinity), n=1 to infinity)					Double Series	Verified		https://www.wolframalpha.com/input?i=sum(sum(1/((n)(2n-1)(k)(2k-1))), k=n+1 to infinity), n=1 to infinity
13	sum(1/(n*(2n-1))*((4n-1)/((2n-1)*n)+sum(1/(k*(2k-1))), k=n+1 to infinity), n=1 to infinity)					Double Series	Verified		https://www.wolframalpha.com/input?i=sum(1/(n*(2n-1))*((4n-1)/((2n-1)*n)+sum(1/(k*(2k-1))), k=n+1 to infinity), n=1 to infinity
14	sum((1/((2n-1)^2*(n^3))), n=1 to infinity						Verified		https://www.wolframalpha.com/input?i=sum((1/((2n-1)^2*(n^3))), n=1 to infinity
15	sum((1/((2n-1)^2*(n^4))), n=1 to infinity					Approximation Wolfram	Verified	LHS Summation	https://www.wolframalpha.com/input?i=sum((1/((2n-1)^2*(n^4))), n=1 to infinity
16	simplify ((1/(n^4)) + (4/(n^3)) + (1/(n^2)) + (5/((2n-1)^2*(n^2))) - (0/((2n-1)^2*(n))) - (6/((2n-1)^*(n^2))) - (4/((2n-1)^2)))					First Equation	Verified	RHS Simplification	https://www.wolframalpha.com/input?i=simplify((1/(n^4)) + (4/(n^3)) + (1/(n^2)) + (5/((2n-1)^2*(n^2))) - (0/((2n-1)^2*(n))) - (6/((2n-1)^*(n^2))) - (4/((2n-1)^2)))
17	simplify ((1/(n^4)) + (4/(n^3)) + (1/(n^2)) + (5/((2n-1)^2*(n^2))) - (2/((2n-1)^2*(n))) - (5/((2n-1)^*(n^2))) - (4/((2n-1)^2)))					First Equation Alternate	Verified	RHS Simplification	https://www.wolframalpha.com/input?i=simplify((1/(n^4)) + (4/(n^3)) + (1/(n^2)) + (5/((2n-1)^2*(n^2))) - (2/((2n-1)^2*(n))) - (5/((2n-1)^*(n^2))) - (4/((2n-1)^2)))
18	sum ((1/(n^4)) + (4/(n^3)) + (1/(n^2)) + (5/((2n-1)^2*(n^2))) - (0/((2n-1)^2*(n))) - (6/((2n-1)^*(n^2))) - (4/((2n-1)^2))), n=1 to infinity						Verified	RHS Summation	https://www.wolframalpha.com/input?i=sum((1/(n^4)) + (4/(n^3)) + (1/(n^2)) + (5/((2n-1)^2*(n^2))) - (0/((2n-1)^2*(n))) - (6/((2n-1)^*(n^2))) - (4/((2n-1)^2))), n=1 to infinity
19	sum ((1/(n^4)) + (4/(n^3)) + (1/(n^2)) + (5/((2n-1)^2*(n^2))) - (2/((2n-1)^2*(n))) - (5/((2n-1)^*(n^2))) - (4/((2n-1)^2))), n=1 to infinity						Verified	RHS Summation	https://www.wolframalpha.com/input?i=sum((1/(n^4)) + (4/(n^3)) + (1/(n^2)) + (5/((2n-1)^2*(n^2))) - (2/((2n-1)^2*(n))) - (5/((2n-1)^*(n^2))) - (4/((2n-1)^2))), n=1 to infinity
20	sum(sum(1/((2n-1)(n^2)(2k-1)(k^2))), k=n+1 to infinity), n=1 to infinity)					Double Series	Verified	Approximation Wolfram	https://www.wolframalpha.com/input?i=sum(sum(1/((2n-1)(n^2)(2k-1)(k^2))), k=n+1 to infinity), n=1 to infinity
21	compute (pi^4/120 - 2 (pi^2) (1+(ln(4)/6) - 2(zeta(3) - ln(4) (8 + ln 4)))					Double Series Result	Verified		https://www.wolframalpha.com/input?i=compute(pi^4/120 - 2 (pi^2) (1+(ln(4)/6) - 2(zeta(3) - ln(4) (8 + ln 4)))
22	sum ((1/n^2)+4/(2n-1)^2-4/((2n-1)(n))), n=1 to infinity					LHS First Inequality	Verified	Issue Fixed	https://www.wolframalpha.com/input?i=sum((1/n^2)+4/(2n-1)^2-4/((2n-1)(n))), n=1 to infinity
23	sum (1/((2n-1)^2*(n^2))), n=1 to infinity					Middle First Inequality	Verified	Upper Bound Changed	https://www.wolframalpha.com/input?i=sum(1/((2n-1)^2*(n^2))), n=1 to infinity
24	sum((4n-1)/((2n-1)^2*(n^3))), n=1 to infinity					RHS First Inequality	Verified		https://www.wolframalpha.com/input?i=sum((4n-1)/((2n-1)^2*(n^3))), n=1 to infinity
25	sum ((4/(2n-1)^2(n))-1/(n^3)), n=1 to infinity					Middle Second Inequality	Verified		https://www.wolframalpha.com/input?i=sum((4/(2n-1)^2(n))-1/(n^3)), n=1 to infinity
26	sum((4n-1)/((2n-1)^2*(n^2))), n=1 to infinity					RHS Second Inequality	Verified		https://www.wolframalpha.com/input?i=sum((4n-1)/((2n-1)^2*(n^2))), n=1 to infinity

Figure 7: Wolfram Alpha Formulae for Series Verification

- Formulae Text Corresponding to Figure (7) are given below. The formulatate can be tried at this link: Wolfram Alpha, Home Page Link.

– Corollary_Result_Label | Wolfram_Alpha_Formulae_Input

– 1 | sum((1/((2n-1)^2*(n))), n=1 to infinity

– 2 | sum((1/((2n-1)^*(n^2))), n=1 to infinity

– 3 | sum((1/((2n-1)^*(n))), n=1 to infinity

– 3 | sum((1/((2n-1)^2*(2n))), n=1 to infinity

– 3 | sum((1/((2n-1)^*(2n)^2)), n=1 to infinity

– 4 | sum((n/((2n-1)^2)), n=1 to infinity

– 4 | sum((n-1/((2n-1)^2)), n=1 to infinity

– Proposition 2 | sum((1/((4n+1)^2*(4n+3)^2)), n=0 to infinity

– 5 | sum(sum(1/((4n+1)(4n+3)(4k+1)(4k+3))), k=n+1 to infinity), n=0 to infinity)

– 6 | sum((4n-1)/((2n-1)^2*(2n)^2)), n=1 to infinity

– 7 | sum((1/((2n-1)^2*(n^2))), n=1 to infinity

– 8 | sum(sum(1/((n)(2n-1)(k)(2k-1))), k=n+1 to infinity), n=1 to infinity)

– 9 | sum(1/(n*(2n-1))*((4n-1)/((2n-1)*n)+sum(1/(k*(2k-1))), k=n+1 to infinity), n=1 to infinity)

– 10 | sum((1/((2n-1)^2*(n^3))), n=1 to infinity

– 11 | sum((1/((2n-1)^2*(n^4))), n=1 to infinity

- 11 | simplify $((1/(n^4)) + (4/(n^3)) + (1/(n^2)) + (5/((2n-1)^2*(n^2))) - (0/((2n-1)^2*(n))) (6/((2n-1)*(n^2))) - (4/((2n-1)^2)))$
- 11 | simplify $((1/(n^4)) + (4/(n^3)) + (1/(n^2)) + (6/((2n-1)^2*(n^2))) - (2/((2n-1)^2*(n))) (5/((2n-1)*(n^2))) - (4/((2n-1)^2)))$
- 11 | sum $((1/(n^4)) + (4/(n^3)) + (1/(n^2)) + (5/((2n-1)^2*(n^2))) - (0/((2n-1)^2*(n))) (6/((2n-1)*(n^2))) - (4/((2n-1)^2)))$, n=1 to infinity
- 11 | sum $((1/(n^4)) + (4/(n^3)) + (1/(n^2)) + (6/((2n-1)^2*(n^2))) - (2/((2n-1)^2*(n))) (5/((2n-1)*(n^2))) - (4/((2n-1)^2)))$, n=1 to infinity
- 11 | sum(sum($1/((2n-1)(n^2)(2k-1)(k^2))$), k=n+1 to infinity), n=1 to infinity
- 11 | compute $(\pi^4/120 - 2(\pi^2)(1+(\ln 4)/6) - 2(\zeta(3) - \ln(4)(8 + \ln 4)))$
- 12 | sum $((1/n^2) + (4/(2n-1)^2) - 4/((2n-1)(n)))$, n=1 to infinity
- 12 | sum $(1/((2n-1)^2(n^2)))$, n=1 to infinity
- 12 | sum $((4n-1)/((2n-1)^2*(n^3)))$, n=1 to infinity
- 12 | sum $((4/((2n-1)^2(n))) - 1/(n^3))$, n=1 to infinity
- 12 | sum $((4n-1)/((2n-1)^2*(n^2)))$, n=1 to infinity

14 Appendix of Index of Series: Summary of Selected Serious Series Summations

1. The following series are well known (End-notes 7; 17; 19).

$$\sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \dots \quad (88)$$

$$\sum_{n=1}^{\infty} \left[\frac{(-1)^{n+1}}{n} \right] = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \dots = \ln 2 \quad (89)$$

$$\zeta(2) = \sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \frac{1}{5^2} + \dots = \frac{\pi^2}{6} \quad (90)$$

$$\sum_{n=1}^{\infty} \left[\frac{1}{(2n-1)^2} \right] = \frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \frac{1}{9^2} + \dots = \frac{\pi^2}{8} \quad (91)$$

$$\sum_{n=1}^{\infty} \left[\frac{1}{(2n)^2} \right] = \frac{1}{2^2} + \frac{1}{4^2} + \frac{1}{6^2} + \frac{1}{8^2} + \frac{1}{10^2} + \dots = \frac{\pi^2}{24} \quad (92)$$

$$\sum_{n=0}^{\infty} \left[\frac{(-1)^n}{2n+1} \right] = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \dots = \frac{\pi}{4} \quad (93)$$

$$\sum_{n=0}^{\infty} \left[\frac{1}{(4n+1)(4n+3)} \right] = \frac{1}{(1)(3)} + \frac{1}{(5)(7)} + \frac{1}{(9)(11)} + \frac{1}{(13)(15)} + \dots = \frac{\pi}{8} \quad (94)$$

$$\sum_{n=1}^{\infty} \left[\frac{(-1)^{n+1}}{n^2} \right] = \frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \dots = \frac{\pi^2}{12} \quad (95)$$

$$\zeta(4) = \sum_{n=1}^{\infty} \frac{1}{n^4} = \frac{1}{1^4} + \frac{1}{2^4} + \frac{1}{3^4} + \frac{1}{4^4} + \frac{1}{5^4} + \dots = \frac{\pi^4}{90}$$

2. The below series are combinations of the series in Point (1) above and are related to the alternating harmonic series (Section 5).

$$\sum_{n=1}^{\infty} \left[\frac{(-1)^{n+1}}{n} \right] = \sum_{n=1}^{\infty} \left[\frac{1}{(2n-1)(2n)} \right] = \frac{1}{(1)(2)} + \frac{1}{(3)(4)} + \frac{1}{(5)(6)} + \frac{1}{(7)(8)} + \frac{1}{(9)(10)} + \dots = \ln 2 \quad (96)$$

$$\sum_{n=1}^{\infty} \left[\frac{1}{(2n-1)(n)} \right] = \frac{1}{(1)(1)} + \frac{1}{(3)(2)} + \frac{1}{(5)(3)} + \frac{1}{(7)(4)} + \frac{1}{(9)(5)} + \dots = \ln 4 \quad (97)$$

$$\sum_{n=1}^{\infty} \left[\frac{1}{(2n-1)(2n)} \right] = \sum_{n=1}^{\infty} \left[\frac{1}{(2n-1)^2} - \frac{1}{(2n-1)^2(2n)} \right] = \ln 2 \quad (99)$$

$$\sum_{n=1}^{\infty} \left[\frac{1}{(2n-1)^2(2n)} \right] = \frac{1}{(1)^2(2)} + \frac{1}{(3)^2(4)} + \frac{1}{(5)^2(6)} + \frac{1}{(7)^2(8)} + \frac{1}{(9)^2(10)} + \dots = \left(\frac{\pi^2}{8} - \ln 2 \right) \quad (100)$$

$$\sum_{n=1}^{\infty} \left[\frac{1}{(2n-1)^2(n)} \right] = \frac{1}{(1)^2(1)} + \frac{1}{(3)^2(2)} + \frac{1}{(5)^2(3)} + \frac{1}{(7)^2(4)} + \frac{1}{(9)^2(5)} + \dots = \left(\frac{\pi^2}{4} - \ln 4 \right) \quad (101)$$

$$\sum_{n=1}^{\infty} \left[\frac{1}{(2n-1)(2n)} \right] = \sum_{n=1}^{\infty} \left[\frac{1}{(2n)^2} + \frac{1}{(2n)^2(2n-1)} \right] = \ln 2 \quad (102)$$

$$\sum_{n=1}^{\infty} \left[\frac{1}{(2n)^2(2n-1)} \right] = \frac{1}{(2)^2(2)} + \frac{1}{(4)^2(3)} + \frac{1}{(6)^2(5)} + \frac{1}{(8)^2(7)} + \frac{1}{(10)^2(9)} + \dots = \left(\ln 2 - \frac{\pi^2}{24} \right) \quad (103)$$

$$\sum_{n=1}^{\infty} \left[\frac{1}{(n)^2(2n-1)} \right] = \frac{1}{(1)^2(1)} + \frac{1}{(2)^2(3)} + \frac{1}{(3)^2(5)} + \frac{1}{(4)^2(7)} + \frac{1}{(5)^2(9)} + \dots = \left(\ln 16 - \frac{\pi^2}{6} \right) \quad (104)$$

$$\sum_{n=1}^{\infty} \left[\frac{1}{(n)(2n-1)^2} \right] = \sum_{n=1}^{\infty} \frac{1}{n} - \sum_{n=1}^{\infty} \left[\frac{4(n-1)}{(2n-1)^2} \right] \quad (105)$$

$$\sum_{n=1}^{\infty} \left[\frac{(n-1)}{(2n-1)^2} \right] = \frac{0}{1^2} + \frac{1}{3^2} + \frac{2}{5^2} + \frac{3}{7^2} + \frac{4}{9^2} + \dots \quad (106)$$

$$\sum_{n=1}^{\infty} \left[\frac{(n)}{(2n-1)^2} \right] = \frac{1}{1^2} + \frac{2}{3^2} + \frac{3}{5^2} + \frac{4}{7^2} + \frac{5}{9^2} + \dots \quad (107)$$

3. The below series are combinations of the series in Point (1) above and are related to various π series (Section 6).

$$\sum_{n=0}^{\infty} \left[\frac{1}{(4n+1)(4n+3)} \right] = \sum_{n=0}^{\infty} \left[\frac{1}{(4n+1)^2} - \frac{2}{(4n+1)^2(4n+3)} \right] = \frac{\pi}{8} \quad (108)$$

$$\sum_{n=0}^{\infty} \left[\frac{1}{(4n+1)^2} \right] = \frac{1}{(1)^2} + \frac{1}{(5)^2} + \frac{1}{(9)^2} + \frac{1}{(13)^2} + \frac{1}{(17)^2} + \dots = TBD \quad (109)$$

$$\sum_{n=0}^{\infty} \left[\frac{1}{(4n+1)^2(4n+3)} \right] = \frac{1}{(1)^2(3)} + \frac{1}{(5)^2(7)} + \frac{1}{(9)^2(11)} + \frac{1}{(13)^2(15)} + \frac{1}{(17)^2(19)} + \dots = TBD \quad (110)$$

$$\sum_{n=0}^{\infty} \left[\frac{1}{(4n+1)(4n+3)} \right] = \sum_{n=0}^{\infty} \left[\frac{1}{(4n+3)^2} + \frac{2}{(4n+1)(4n+3)^2} \right] = \frac{\pi}{8} \quad (111)$$

$$\sum_{n=0}^{\infty} \left[\frac{1}{(4n+3)^2} \right] = \frac{1}{(3)^2} + \frac{1}{(7)^2} + \frac{1}{(11)^2} + \frac{1}{(15)^2} + \frac{1}{(19)^2} + \dots = TBD \quad (112)$$

$$\sum_{n=0}^{\infty} \left[\frac{1}{(4n+1)(4n+3)^2} \right] = \frac{1}{(1)(3)^2} + \frac{1}{(5)(7)^2} + \frac{1}{(9)(11)^2} + \frac{1}{(17)(19)^2} + \frac{1}{(13)(15)^2} + \dots = TBD \quad (113)$$

$$\sum_{n=0}^{\infty} \left[\frac{1}{(4n+1)^2(4n+3)^2} \right] = \frac{1}{(1)^2(3)^2} + \frac{1}{(5)^2(7)^2} + \frac{1}{(9)^2(11)^2} + \frac{1}{(17)^2(19)^2} + \frac{1}{(13)^2(15)^2} + \dots \quad (114)$$

$$\sum_{n=0}^{\infty} \left[\frac{1}{(4n+1)^2(4n+3)^2} \right] = \left[\frac{(\pi-1)^2 - 1}{32} \right] \quad (115)$$

$$\begin{aligned} (\pi-2)^2 = 4 - (128) & \left[\frac{1}{(1)(3)} \left\{ \frac{1}{(5)(7)} + \frac{1}{(9)(11)} + \frac{1}{(13)(15)} + \dots \right\} \right. \\ & + \frac{1}{(5)(7)} \left\{ \frac{1}{(9)(11)} + \frac{1}{(13)(15)} + \frac{1}{(17)(19)} + \dots \right\} \\ & + \frac{1}{(9)(11)} \left\{ \frac{1}{(13)(15)} + \frac{1}{(17)(19)} + \frac{1}{(21)(23)} + \dots \right\} + \dots \left. \right] \\ (\pi-2)^2 = 4 - (128) & \sum_{n=0}^{\infty} \sum_{k=n+1}^{\infty} \left[\frac{1}{(4n+1)(4n+3)(4k+1)(4k+3)} \right] \end{aligned} \quad (116)$$

$$\sum_{n=1}^{\infty} \left[\frac{(4n-1)}{(2n-1)^2(2n)^2} \right] = \frac{3}{(1)^2(2)^2} + \frac{7}{(3)^2(4)^2} + \frac{11}{(5)^2(6)^2} + \frac{15}{(7)^2(8)^2} + \frac{19}{(9)^2(10)^2} + \dots \quad (117)$$

$$\sum_{n=1}^{\infty} \left[\frac{(4n-1)}{(2n-1)^2(2n)^2} \right] = \frac{\pi^2}{12} \quad (118)$$

$$\sum_{n=1}^{\infty} \frac{(4n-1)}{(2n-1)^2(n)^2} = \frac{3}{(1)^2(1)^2} + \frac{7}{(3)^2(2)^2} + \frac{11}{(5)^2(3)^2} + \frac{15}{(7)^2(4)^2} + \frac{19}{(9)^2(5)^2} + \dots = \frac{\pi^2}{3} \quad (119)$$

$$\frac{\pi^2}{3} = (2 \ln 2) (2 + \ln 2) - \left[\frac{1}{(1)(1)} \left\{ \frac{1}{(2)(3)} + \frac{1}{(3)(5)} + \frac{1}{(4)(7)} + \dots \right\} \right. \quad (120)$$

$$\left. + \frac{1}{(2)(3)} \left\{ \frac{1}{(3)(5)} + \frac{1}{(4)(7)} + \frac{1}{(5)(9)} + \dots \right\} \right. \quad (121)$$

$$\left. + \frac{1}{(3)(5)} \left\{ \frac{1}{(4)(7)} + \frac{1}{(5)(9)} + \frac{1}{(6)(11)} + \dots \right\} + \dots \right] \quad (122)$$

$$\frac{\pi^2}{3} = (2 \ln 2) (2 + \ln 2) - \sum_{n=1}^{\infty} \sum_{k=n+1}^{\infty} \left[\frac{1}{(n)(2n-1)(k)(2k-1)} \right]$$

$$\sum_{n=1}^{\infty} \left[\frac{1}{(n)(2n-1)} \left\{ \frac{(4n-1)}{(2n-1)(n)} + \sum_{k=n+1}^{\infty} \left[\frac{1}{(k)(2k-1)} \right] \right\} \right] \quad (123)$$

$$= \left[\frac{1}{(1)(1)} \left\{ \frac{3}{(1)(1)} + \left[\frac{1}{(2)(3)} + \frac{1}{(3)(5)} + \frac{1}{(4)(7)} + \dots \right] \right\} \right. \quad (124)$$

$$+ \frac{1}{(2)(3)} \left\{ \frac{7}{(2)(3)} + \left[\frac{1}{(3)(5)} + \frac{1}{(4)(7)} + \frac{1}{(5)(9)} + \dots \right] \right\} \quad (125)$$

$$+ \frac{1}{(3)(5)} \left\{ \frac{11}{(3)(5)} + \left[\frac{1}{(4)(7)} + \frac{1}{(5)(9)} + \frac{1}{(6)(11)} + \dots \right] \right\} + \dots \quad (126)$$

$$\sum_{n=1}^{\infty} \left[\frac{1}{(n)(2n-1)} \left\{ \frac{(4n-1)}{(2n-1)(n)} + \sum_{k=n+1}^{\infty} \left[\frac{1}{(k)(2k-1)} \right] \right\} \right] = (2 \ln 2) (2 + \ln 2) \quad (127)$$

$$\sum_{n=1}^{\infty} \left[\frac{1}{(2n-1)^2 (2n)^2} \right] = \frac{1}{(1)^2 (2)^2} + \frac{1}{(3)^2 (4)^2} + \frac{1}{(5)^2 (6)^2} + \frac{1}{(7)^2 (8)^2} + \frac{1}{(9)^2 (10)^2} + \dots \quad (128)$$

$$\sum_{n=1}^{\infty} \left[\frac{1}{(2n-1)^2 (2n)^2} \right] = \left(\frac{\pi^2}{6} - \ln 4 \right) \quad (129)$$

$$\sum_{n=1}^{\infty} \left[\frac{1}{(2n-1)^2 (n)^2} \right] = \frac{1}{(1)^2 (1)^2} + \frac{1}{(3)^2 (2)^2} + \frac{1}{(5)^2 (3)^2} + \frac{1}{(7)^2 (4)^2} + \frac{1}{(9)^2 (5)^2} + \dots$$

$$\sum_{n=1}^{\infty} \left[\frac{1}{(2n-1)^2 (n)^2} \right] = \left(\frac{2\pi^2}{3} - \ln 256 \right) \quad (130)$$

$$\sum_{n=1}^{\infty} \left[\frac{1}{(2n-1)^2 (n)^3} \right] = \frac{1}{(1)^2 (1)^3} + \frac{1}{(3)^2 (2)^3} + \frac{1}{(5)^2 (3)^3} + \frac{1}{(7)^2 (4)^3} + \frac{1}{(9)^2 (5)^3} + \dots$$

$$\sum_{n=1}^{\infty} \left[\frac{1}{(2n-1)^2 (n)^3} \right] = \left[\zeta(3) + \left(\frac{5\pi^2}{3} \right) - 12(\ln 4) \right]$$

$$\sum_{n=1}^{\infty} \left[\frac{1}{(2n-1)^2 (n)^4} \right] = \frac{1}{(1)^2 (1)^4} + \frac{1}{(3)^2 (2)^4} + \frac{1}{(5)^2 (3)^4} + \frac{1}{(7)^2 (4)^4} + \frac{1}{(9)^2 (5)^4} + \dots \quad (131)$$

$$\sum_{n=1}^{\infty} \left[\frac{1}{(2n-1)^2 (n)^4} \right] = \left[4 \{ \zeta(3) - 8(\ln 4) \} + \frac{\pi^4}{90} + 4\pi^2 \right] \quad (132)$$

$$\begin{aligned} & \left[\frac{\pi^4}{120} - 2\pi^2 \left(1 + \frac{\ln 4}{6} \right) - 2 \{ \zeta(3) - (\ln 4)(8 + \ln 4) \} \right] \\ &= \left[\frac{1}{(1)(1)^2} \left\{ \frac{1}{(3)(2)^2} + \frac{1}{(5)(3)^2} + \frac{1}{(7)(4)^2} + \dots \right\} \right. \\ & \quad + \frac{1}{(3)(2)^2} \left\{ \frac{1}{(5)(3)^2} + \frac{1}{(7)(4)^2} + \frac{1}{(9)(5)^2} + \dots \right\} \\ & \quad \left. + \frac{1}{(5)(3)^2} \left\{ \frac{1}{(7)(4)^2} + \frac{1}{(9)(5)^2} + \frac{1}{(11)(6)^2} + \dots \right\} + \dots \right] \quad (133) \end{aligned}$$

$$\begin{aligned} & \left[\frac{\pi^4}{120} - 2\pi^2 \left(1 + \frac{\ln 4}{6} \right) - 2 \{ \zeta(3) - (\ln 4)(8 + \ln 4) \} \right] \\ &= \sum_{n=1}^{\infty} \sum_{k=n+1}^{\infty} \left[\frac{1}{(2n-1)(n)^2(2k-1)(k)^2} \right] \quad (134) \end{aligned}$$

4. The following series are a summary of the new results.

$$\sum_{n=0}^{\infty} \left[\frac{1}{(n+1)(2n+1)^2} \right] = \sum_{n=1}^{\infty} \left[\frac{1}{(n)(2n-1)^2} \right] = \left(\frac{\pi^2}{4} - \ln 4 \right)$$

$$\sum_{n=0}^{\infty} \left[\frac{1}{(4n+1)^2(4n+3)^2} \right] = \left[\frac{(\pi-1)^2-1}{32} \right]$$

$$4 - (128) \sum_{n=0}^{\infty} \sum_{k=n+1}^{\infty} \left[\frac{1}{(4n+1)(4n+3)(4k+1)(4k+3)} \right] = (\pi-2)^2$$

$$\sum_{n=1}^{\infty} \left[\frac{1}{(n)^2(2n-1)} \right] = \left(\ln 16 - \frac{\pi^2}{6} \right)$$

$$\sum_{n=1}^{\infty} \left[\frac{(4n-1)}{(2n-1)^2(n)^2} \right] = \frac{\pi^2}{3}$$

$$\sum_{n=1}^{\infty} \left[\frac{1}{(2n-1)(n)} \right] = \ln 4$$

$$\sum_{n=1}^{\infty} \left[\frac{1}{(2n-1)^2(n)^2} \right] = \left(\frac{2\pi^2}{3} - \ln 256 \right)$$

$$(2 \ln 2)(2 + \ln 2) - \sum_{n=1}^{\infty} \sum_{k=n+1}^{\infty} \left[\frac{1}{(n)(2n-1)(k)(2k-1)} \right] = \frac{\pi^2}{3}$$

$$\sum_{n=1}^{\infty} \left[\frac{1}{(2n-1)^2(n)^3} \right] = \left[\zeta(3) + \left(\frac{5\pi^2}{3} \right) - 12(\ln 4) \right]$$

$$\sum_{n=1}^{\infty} \left[\frac{1}{(n)(2n-1)} \left\{ \frac{(4n-1)}{(2n-1)(n)} + \sum_{k=n+1}^{\infty} \left[\frac{1}{(k)(2k-1)} \right] \right\} \right] = (2 \ln 2)(2 + \ln 2)$$

$$\sum_{n=1}^{\infty} \left[\frac{1}{(2n-1)^2(n)^4} \right] = \left[4 \{ \zeta(3) - 8(\ln 4) \} + \frac{\pi^4}{90} + 4\pi^2 \right]$$

$$\sum_{n=1}^{\infty} \sum_{k=n+1}^{\infty} \left[\frac{1}{(2n-1)(n)^2(2k-1)(k)^2} \right]$$

$$= \left[\frac{\pi^4}{120} - 2\pi^2 \left(1 + \frac{\ln 4}{6} \right) - 2 \{ \zeta(3) - (\ln 4)(8 + \ln 4) \} \right]$$