

## Reply

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### Introduction

In their preceding comment on our paper [Matthaeus *et al.*, 1994] (hereinafter referred to as paper 1), Tu and Marsch [this issue] object vehemently to what they perceive as undeserved criticism of the class of solar wind turbulence models developed by Tu and coworkers [Tu *et al.*, 1984; Tu, 1987, 1988]. Our intent was not to slight the valuable contributions made in those groundbreaking theories for incorporating turbulence into models for spatial transport. Rather, we wished to point out certain shortcomings in existing theories and suggest how they might be improved.

Spectral evolution is modeled by Tu using a locally defined cascade function; that is, the energy flux at a wave number  $k$  (or frequency  $f$ ) depends only on  $k$  and the energy spectrum at  $k$ . This method assumes that each part of the fluctuation spectrum evolves without influence from any other part. The hypothesis arose originally from considerations of inertial-range dynamics. Indeed, the distinctive feature of an inertial range is that the energy spectrum remains unchanged by linear rescalings of  $k$ , i.e., it is self-similar. Because of this, there are no privileged wave numbers (equivalently, length scales), so it is natural that only local quantities appear in a model for spectral evolution. Obviously, such a condition cannot hold for indefinitely large or small scales, and self-similarity can only persist over a limited region of the spectrum. As discussed in paper 1, our principal reservation with Tu's model is that the local cascade hypothesis is used regardless of the spatial scale in question. We propose that other models of turbulence should be used when dealing with turbulence where particular spatial scales assume special roles.

The termination of the self-similar character of the inertial range at sufficiently low wave numbers naturally leads to theories in which distinguished length scales appear. For example, von Karman and Howarth [1938] and others postulated that the correlation function would preserve its shape, remaining geometrically similar with time but not self-similar. The spectrum is

then fully specified at any moment by a single characteristic length scale and another scalar function. These are conveniently chosen to be the correlation length and the total energy. Modeling turbulence then reduces to the problem of finding appropriate governing equations for these two parameters as functions of time. Although considerations of observed spectra can influence the equations chosen, the approach focuses on the evolution of the total energy, which is dominated by fluctuations near the peak of the turbulent spectrum, the energy-containing eddies. Kolmogoroff [1941] later argued that a subsection of the spectrum could be modeled as self-similar and proposed a spectral theory for that subrange, i.e., the inertial range. The nomenclature "energy-containing range" was used to contrast the more comprehensive theories that actually encompassed the entire spectrum.

In the solar wind there are several characteristic spatial scales that can be identified. The first is the well-known correlation length. The second is the extent of the largest parcel of plasma that can reasonably be treated as homogeneous in the expanding solar wind. A third is the causality cutoff, i.e., the maximum distance over which two parcels of plasma can communicate via turbulent MHD fluctuations in the time elapsed since they left the Sun [Matthaeus and Goldstein, 1986]. These scales might be difficult to calculate precisely, but they obviously represent the influence of important effects that depend on specific spatial scales. Models for solar wind turbulence must take them into consideration or use physical arguments to show they are not important.

The presence of important physical length scales motivated our development of a non-self-similar theory that treats them explicitly. Paper 1 proposed one example of such a model that concentrates on the evolution of the correlation length. The continuation of this line of research to include other scales is an ongoing endeavor. Our goal is to advance solar wind turbulence theory beyond inertial-range models, which are limited to the self-similar range, to a more comprehensive description encompassing a wider range of the turbulence. It is interesting to note that hydrodynamic turbulence theory proceeded in the opposite direction, both conceptually and historically. Kolmogoroff started from models for the decay of the total energy and introduced the similarity assumption for a subset of the turbulence.

In Tu's model one particular scale does assume a distinguished role, i.e., the break frequency  $f_c$  where the low-frequency fluctuations with  $f^{-1}$  slope gradually meld into the inertial range. Within that formulation,

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as *Tu and Marsch* [this issue] state in their recapitulation of the model's properties, "the heating rate is ultimately controlled only by the rate of decrease of the frequency  $f_c$ ." It is immediately questionable whether a hypothesis explicitly derived from considerations of scale-free fluctuations continues to be valid for manifestly scale-dependent turbulence. The local cascade hypothesis can certainly be used as an ad hoc assumption, but it should be recognized as lacking conceptual justification outside of the inertial range. Furthermore, if the dynamics are most crucial near the break frequency, which is the least self-similar part of the spectrum, then perhaps that region in particular should be treated with a turbulence model that does not originate in assumptions of self-similarity.

It might be argued that the *Tu* models for the turbulent spectrum are an improvement over nonspectral models that restrict consideration to a few length scales. However, the *Tu et al.* [1984] model is just such an example of this class of model! Indeed, the power spectrum they derive (equation (54)) can be written as the product of a function of radial distance (the Lagrangian time coordinate in the solar wind) with another function that depends only on  $f/f_c$ . The evolution of the power spectrum is mathematically similar and can be characterized solely by  $f_c$  and the magnitude of the spectrum at  $f_c$ . However, there are two major conceptual differences between their approach and ours (beyond the nontrivial distinction between working in frequency and wave number). The first is that *Tu et al.*'s power spectrum is not bounded at low frequencies, so the total energy cannot be defined by integrating over the entire spectrum. The dissipation rate is therefore calculated from the cascade rate at high frequencies rather than being given by the decay of total energy. The second concerns the equations used to evolve the characteristic scales. Our approach relies on relatively straightforward, physically motivated phenomenologies that can and have been evaluated against numerical simulations of MHD turbulence [*Pontius et al.*, 1993; *W. H. Matthaeus et al.*, Phenomenology for the decay of energy-containing eddies in homogeneous MHD turbulence, submitted to *J. Geophys. Res.*, 1995]. From equation (1) in the comment and equation (39) in *Tu et al.* [1984], the time derivative of  $f_c$  is equal to  $f_c^3$  times the spectral power at  $f_c$  and a function strictly of radius. The physical significance of such a relation is not clear, although the local cascade hypothesis is ultimately responsible for the expression.

In later papers [*Tu*, 1987, 1988] the problem of indefinite total energy was addressed by introducing another solution at low wave number and joining the two solutions at some matching frequency. Although this procedure ensures that no more energy is dissipated than is present, it suffers from more than simply being ad hoc. In paper 1 we pointed out that the local cascade hypothesis is inappropriate at low wave number. Furthermore, the introduction into the formalism of yet another distinguished length scale, the matching frequency, makes the use of a scale-free cascade function

increasingly questionable. Indeed, the solar wind turbulence spectra presented in Figure 1 of *Tu and Marsch's* comment seem to have several distinguished frequencies. For such a situation, contemporary turbulence theory strongly compels discarding the local cascade hypothesis.

*Tu and Marsch's* brief description of the theory of energy-containing eddies reveals some fundamental misconceptions about turbulence theory. As a guiding perspective, it is not the energy-containing eddies that "should be treated as a group clearly separated from the inertial range"; rather, if an inertial range exists (which need not be true), it must be at much higher wave numbers than the dominant energy-containing eddies. The distinction is not a reciprocal one because the energy-containing structures are described by a coarser theory than the inertial range. A portion of the spectrum containing most of the turbulent energy can almost always be identified, but an inertial range is not necessarily present. The latter depends on the conditions of self-similar interactions and statistical equilibrium, which need not be obtained. A theory for the energy-containing eddies models the decay of all the turbulent energy at once, including whatever might be in an inertial range if one exists, but that region typically contains only a small fraction of the total energy.

In the final paragraph of their comment, *Tu and Marsch* reemphasize that the *Tu* spectral model is intended to describe the low-frequency part of the spectrum with slope  $-1$ . However, it is a mistake to conclude that a local cascade model is appropriate simply because the fluctuation spectrum exhibits a power law. Fluctuations at such low frequencies correspond to spatial extents approaching or larger than the causality cutoff, and the observed power law there cannot be the result of any in situ turbulent process. These structures are instead the result of conditions in the inner heliosphere [*Matthaeus and Goldstein*, 1986]. They cannot play any dynamic role in the evolution of the turbulence until sufficient time (radial distance) has elapsed to allow MHD fluctuations to propagate across them. These structures behave as a reservoir of energy that gradually becomes available to the turbulent dynamics, which we propose to model via a driving term at the causality cutoff.

*Tu and Marsh's* continued emphasis on a spectral description of the turbulence leads them to search the observations for a sharply delineated range that could be identified as the energy-containing range. Indeed, their arguments in section 2 presume that the supposed boundaries of such a range must necessarily remain fixed. Again we emphasize that the model we propose deliberately avoids the very great difficulties associated with precisely describing the spectral evolution throughout the spectrum. Their statement that the "partial self-preservation of the shape... is a principal feature of the energy-containing eddies" is similarly misguided. In the cited material, *Batchelor*, [1953] points out some intriguing features observed for energy-containing eddies in freely decaying turbulence, but

those are not stipulated as prerequisites by which they are identified. Moreover, the cited material is principally a caveat about tentative theories for the spectrum in the energy-containing range and specifically states [Batchelor, 1953, p. 148]: “To find [statistically similar] solutions has been one task; to determine the conditions under which they can and do provide a correct description of the turbulence is another.” The phenomenology we adopted for those eddies is a conservative one that avoids unjustified and inappropriate hypotheses. This should be contrasted with Tu’s approach that attempts to model the phenomenon to a more exact degree than can be reasonably expected to hold.

## Observational Considerations for the Energy-Containing Range

We now turn to Tu and Marsch’s attempt to exclude any energy-containing range from observations of the solar wind. For brevity, the present discussion oversimplifies the full analysis carried out in an expanding, evolving wind (see paper 1 for a thorough discussion). The effect of mixing between fluctuations of opposite sense is ignored, the efficiency of which varies with radial distance in our model (but not in Tu’s), and the influence of varying cross helicity is not present. We offer the following order-of-magnitude analysis as a quantitative refutation of their criticism with the understanding that a truly meaningful evaluation of our model cannot be carried out using the simple WKB scaling Tu and Marsch employ. Indeed, the WKB formalism is constructed to be an asymptotic, short-wavelength theory, so it is not altogether appropriate for the long-wavelength fluctuations of the energy-containing eddies. It should also be recognized that isolated observations necessarily reflect conditions that may not be representative of the solar wind at other times. This is particularly important in the poorly sampled, low-frequency part of the spectrum where the statistical weight of the data is low. The analysis involves comparing spectra taken some months apart, so the possibility of confusion with changing conditions at the Sun cannot be ruled out. Finally, driving or forcing would conflict with Tu and Marsch’s presumption of freely decaying turbulence. Although they argue against driving in the high-speed solar wind, we do not agree that the situation is so clear.

The claims of Tu and Marsch rely on dividing the fluctuation spectrum into separate regions based on the slope. As discussed above, our model for the energy-containing eddies is not a spectral one so there is no justification for the strict partitioning they adopt. (Indeed, it makes more sense to examine the spectrum to discern where one might expect a local cascade theory to be appropriate.) Tu and Marsch conclude that the range between  $10^{-4}$  and  $6 \times 10^{-3}$  Hz could not be part of the energy-containing range because it gradually becomes part of the inertial range. They then argue that slower fluctuations could not form the energy-containing range

because they do not decay sufficiently. But these conclusions rest on their unfounded presumption that the partitioning must be fixed. In contrast, the evolution of the characteristic scale describing the energy-containing eddies is a central part of our model, so one would expect the spectrum to exhibit changes in consequence. It is clear that a large amount of energy exists where the spectrum does not have an inertial-range slope and that this energy decays over the distance examined. Fretting over the precise demarcation of this region misses the point that most of the turbulent energy lies at frequencies below the inertial range, a necessary consequence of the steep decline of spectral energy in an inertial range. The dynamics of the energy-containing eddies must ultimately control the heating rate.

Given the above caveats on comparisons between data and WKB-adjusted spectra, we now show that the decay of fluctuations in the neighborhood of several times  $10^{-4}$  Hz is indeed consistent with our predictions for the energy-containing range. The MHD nonlinear decay timescale for  $Z^+$  fluctuations of outward sense is [Dobrowolny *et al.*, 1980]

$$\tau^+ = \lambda_+/Z^-, \quad (1)$$

where  $Z^-$  is the root-mean-square magnitude for fluctuations of inward sense. (The  $+/-$  convention of paper 1 is reversed here.) For the characteristic length scale of the energy-containing eddies  $\lambda_+$ , we take the correlation length  $L_C^+ = 7.6 \times 10^6$  km given in Table 1a of Marsch and Tu [1990] (hereinafter referred to as MT90).  $Z^-$  is given by  $\sqrt{e^-(t)}$  at zero time separation, which Figure 1 of MT90 gives as  $600 \text{ km}^2/\text{s}^2$  at 0.29 AU. The characteristic nonlinear time for the energy-containing eddies is therefore

$$\tau_{nl} = \frac{7 \times 10^6 \text{ km}}{\sqrt{600 \text{ km}^2/\text{s}^2}} \cong 3 \times 10^5 \text{ s}. \quad (2)$$

Assuming the geometry (rotational symmetry of the fluctuations) is such that the MHD turbulence experiences decorrelation due to Alfvénic propagation, the decay rate is estimated as follows. Decorrelation depends on the Alfvén timescale  $\tau_A = L_C^+/V_A$ , where  $V_A$  is the Alfvén speed, 136 km/s at 0.29 AU (Table 1a of MT90). The resulting decay timescale at 0.29 is [Dobrowolny *et al.*, 1980]

$$\tau_s = \frac{(\tau_{nl})^2}{\tau_A} = \frac{(3 \times 10^5 \text{ s})^2}{6 \times 10^4 \text{ s}} \cong 1.5 \times 10^6 \text{ s}. \quad (3)$$

Both the correlation length and the Alfvén speed decrease by roughly a factor of 2 by 0.87 AU (see Table 1a of MT90), so the actual value may be lower by as much as a factor of 4, and we have neglected additional decorrelation due to the nonlinear turnover itself. The solar wind takes  $t = 10^5$  s to travel from 0.29 to 0.87 AU, so the energy-containing eddies should only decay about  $1 - \exp[-t/\tau_s] \cong 10\%$  over this distance. The WKB-adjusted spectra in Marsch and Tu’s [1980] Figure 1 show a decay of at most a few tens of percent

between 1 and  $3 \times 10^{-4}$  Hz, increasing to roughly 50% by  $5 \times 10^{-4}$  Hz. The data are too jittery to allow a more quantitative evaluation, but these values are certainly consistent with our hypothesis. For other turbulence geometries (such as two dimensional) the Alfvén speed may not enter, and a reasonable estimate could be as short as  $\tau_{nl}$  itself. This would imply a decay of 50%, which is consistent with the observed decay of eddies having length scales corresponding to approximately  $5 \times 10^{-4}$  Hz.

## Observational Considerations for the Inertial Range

We have shown that part of the fluctuation spectrum can reasonably be associated with the energy-containing eddies. A similar analysis will allow us to evaluate whether there is also a plausible inertial range in which the local cascade hypothesis may be appropriate. In section 3 of their comment Tu and Marsch concede that the local cascade hypothesis does not always have rigorous justification. However, their discussion is somewhat disingenuous in suggesting that the conditions under which the hypothesis obviously fails cannot be determined.

The local cascade hypothesis requires that turbulent eddies of the scale in question reorganize many times before their energy decays substantially. We now calculate the decorrelation time for solar wind eddies and compare that to the observed decay timescale for the same eddies. The decorrelation time is roughly the Alfvén timescale, which is the measured period times the solar wind Mach number. The spectrum at  $5 \times 10^{-4}$  Hz has a decorrelation time of  $3 \times 10^4$  s, so it undergoes roughly three decorrelations during the  $10^5$  s transit time. That part of the spectrum is observed to decay by 50%, which means that the local cascade hypothesis is dubious for this and lower frequencies. Fluctuations at  $10^{-3}$  Hz decorrelate twice as fast, while the spectrum decays to roughly  $1/e$ , so the local cascade hypothesis is probably a viable approximation for frequencies  $\gtrsim 10^{-3}$  Hz.

The data therefore place a stringent condition on where one can set a high-frequency limit above which the decay time is longer than many characteristic periods of the fluctuations. The points emphasized by Tu and Marsch in section 3 dwell on the ability of Tu's model to produce spectral profiles with trends similar to those observed, but this begs the question of whether the physics is properly represented. A model that mimics data in the absence of proper physical justification leads one to question whether the coincidence might not be fortuitous. We also take exception to their second claim, i.e., that the origin of the  $e^-(f)$  fluctuations is unknown. We have discussed previously how mixing leads naturally to inward fluctuations [Zhou and Matthaeus, 1989], an effect not included in any of the Tu models. Regardless, the nonlinear evolution time can certainly be evaluated from the observed magnitude of the  $e^-(f)$  spectrum.

## Final Comments

Tu and Marsch feel that our discussion of their low-frequency boundary condition and their neglect of energy-containing eddies is a second basic criticism of the Tu model. In reality, that criticism follows immediately from our contention that their governing equations are not appropriate there. The extended discussions in sections 4 and 5 of how the Tu models go about describing spectral evolution do not address the question of whether the models themselves are valid. It should be pointed out that the so-called break frequency was introduced by Tu *et al.* [1984] as a fit to solar wind observations. If, as Tu and Marsh claim, it is the behavior of the spectrum near that region that controls the heating rate, then it is certainly important to avoid modeling it with unphysical hypotheses. Note that they are unconcerned whether the fluctuations at lower frequencies evolve according to WKB theory or actually experience a local cascade process. The difference may be negligible in some circumstances but the physical distinction is apparent. Their statement, "this method does not violate any principles of traditional turbulence theory," suggests an incautious attitude toward the careful attention to physical justification underlying even the simplest theories of turbulence.

We have shown here that the characteristic timescales support our interpretation, so our criticism of their use of the local cascade hypothesis stands. The defense of the mathematical solution is irrelevant because the governing equations lack physical justification. It does not matter that the mathematical model is in some sense complete or whether the parameters can be chosen to make the solutions mimic the observations. Because the inertial-range model must fail at sufficiently low wave numbers, an energy-containing range is required for physical completeness and logical consistency.

Our model for the energy-containing eddies should be viewed as complementary to inertial-range models. The various Tu models are flawed because they apply the idea of a locally determined energy cascade throughout the spectrum. They may produce some spectra in reasonable agreement with observations, but the logical motivation for the model is inadequate and does not correctly represent crucial physics contained in an energy-containing model. We claim that the transfer of energy within the larger eddies, the energy-containing range, is poorly described by the model and is not robust. In the inner heliosphere the spectrum adjusts from its initial state to one with a spectral index of approximately  $-5/3$  at high frequencies, and this supplies energy to the dissipation scale, initially with less influence from the low-frequency fluctuations. However, once this reservoir of energy is depleted, the ultimate supply of energy must move to lower frequencies. Because the spectrum tends to relax toward a statistical Kolmogorov state faster at higher frequencies, the conditions required for the Tu model are first achieved at higher wave numbers. However, there will always be a lower frequency limit below which the model is inappropriate. We stand by our original criticism.

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