Corrigendum: A conjecture of De Koninck regarding particular values of the sum of divisors function

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The proof of Lemma 7 of [1] is made complete by giving the proof of a missing case (4). This omission was pointed out to the authors by Min Tang, to whom we are most grateful. The same definitions and notation are employed as in op. cit., and one should replace the first paragraph of the proof by the following argument.

To obtain a contradiction, let us assume that there is no odd prime p such that $p^4 \mid n$. We can also assume $e \geq 4$, otherwise the result follows easily as n will be divisible by the fourth power of a prime by a result of [2]. In the same notation as Lemma 6 of [1], since $p_1 \equiv 1 \bmod 4$ we must have $p_2 \equiv 1 \bmod 4$ and $a_2 \equiv 1 \bmod 4$. Therefore $a_2 = 1$, and we can write

$$(2^{e+1}-1) \cdot \left(\frac{p_1+1}{2}\right) \cdot \left(\frac{p_2+1}{2}\right) \cdot \prod_{j=3}^{m} \left(p_j^2 + p_j + 1\right) \quad = \quad p_1^2 p_2^2 \cdot \prod_{j=3}^{m} p_j^2. \qquad (1)$$

Furthermore, since $3 \nmid n$ it follows that $Q \mid p_1^2 p_2^2$, hence Q has at most four quadratic factors. However, if $i \neq j$ we have $p_i^2 + p_i + 1 \neq p_j^2 + p_j + 1$, so in fact Q has at most three quadratic factors.

Each of the resulting possibilities was then covered in op. cit., except for the missing case (4) below.

Case (4): Here one considers $Q = p_1^2 p_2^2$, and

$$\begin{aligned} p_1 &= p_3^2 + p_3 + 1, \\ p_2 &= p_4^2 + p_4 + 1, \\ p_1 p_2 &= p_5^2 + p_5 + 1. \end{aligned}$$

Now $p_1 \equiv p_2 \equiv 1 \bmod 4$ implies $p_j \equiv 3 \bmod 4$ for $j \geq 3$, and $3 \nmid n$ implies $p_j \equiv 2 \bmod 3$ for $j \geq 3$, so that $p_1 \equiv p_2 \equiv 1 \bmod 3$. Moreover, note that as $2^{e+1} - 1 \equiv 3 \bmod 4$, we must have $2^{e+1} - 1 \neq \square$.

Cancelling Q from Equation (1) allows us to write

$$(2^{e+1} - 1) \cdot \frac{p_1 + 1}{2} \cdot \frac{p_2 + 1}{2} \quad = \quad p_3^2 \cdot p_4^2 \cdot p_5^2.$$

The symmetry of these constraints on p_1, p_2 and on p_3, p_4, p_5 enables us to reduce this expression to the following six potential situations:

$$\begin{array}{lllcll}
(4.1) & 2^{e+1} - 1 = p_3 & \Longrightarrow & \frac{p_1+1}{2} \cdot \frac{p_2+1}{2} & = & p_3 p_4^2 p_5^2, \\
(4.2) & 2^{e+1} - 1 = p_3 p_4 & \Longrightarrow & \frac{p_1+1}{2} \cdot \frac{p_2+1}{2} & = & p_3 p_4 p_5^2, \\
(4.3) & 2^{e+1} - 1 = p_3 p_4 p_5 & \Longrightarrow & \frac{p_1+1}{2} \cdot \frac{p_2+1}{2} & = & p_3 p_4 p_5, \\
(4.4) & 2^{e+1} - 1 = p_3^2 p_4 & \Longrightarrow & \frac{p_1+1}{2} \cdot \frac{p_2+1}{2} & = & p_4 p_5^2, \\
(4.5) & 2^{e+1} - 1 = p_3^2 p_4^2 p_5 & \Longrightarrow & \frac{p_1+1}{2} \cdot \frac{p_2+1}{2} & = & p_5, \\
(4.6) & 2^{e+1} - 1 = p_3^2 p_4 p_5 & \Longrightarrow & \frac{p_1+1}{2} \cdot \frac{p_2+1}{2} & = & p_4 p_5.
\end{array}$$

In situations (4.1), (4.3), (4.4) and (4.5) the left-hand side of the implied expression is 1 modulo 3 but the right hand side, having an odd number of prime factors, is 2 modulo 3.

In the situation (4.2), one knows that $2^{e+1} - 1$ is 3 modulo 4 while $p_3 p_4$ is 1 modulo 4.

Finally, in situation (4.6) we deduce that

$$\frac{p_1 + 1}{2} = p_4 \quad \text{and} \quad \frac{p_2 + 1}{2} = p_5.$$

This latter case also cannot occur, since the left-hand side of each of these equations is 1 modulo 3 but the right-hand side is 2 modulo 3. $\square$

Remark: At the start of Case (1) on page 58 of the article [1], we claimed that $(p_2+1)/2$ has at most 3 prime divisors. To exclude the possible scenario where

$$\frac{p_2+1}{2} \;=\; p_1 p_3^2 p_4 \quad \text{and} \quad (2^{e+1}-1)\cdot\frac{p_1+1}{2} \;=\; p_2 p_4,$$

one first notes that as $p_2 \equiv 1 \bmod 4$ and $2^{e+1}-1 \equiv 3 \bmod 4$, consequently $p_2 = (p_1+1)/2$ and $p_4 = 2^{e+1}-1$. It follows that $p_4 \geq 19$ and $p_3 \geq 7$, whence

$$\frac{\frac{p_1+1}{2}+1}{2} \;\geq\; p_1 \times 19 \times 7^2$$

which is clearly impossible!

References

[1] K. Broughan, D. Delbourgo and Q. Zhou, *A conjecture of De Koninck regarding particular values of the sum of divisors function*, Journal of Number Theory **137** (2014), 50–66.

[2] K. Broughan, J.-M. De Koninck, I. Kátai and F. Luca, *On integers for which the sum of divisors is the square of the squarefree core*, Journal of Integer Sequences **15** (2012), 1–12.