Lie theory and separation of variables. 9. Orthogonal R-separable coordinate systems for the wave equation $\psi_{tt}-\Delta_2\psi=0$

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A list of orthogonal coordinate systems which permit R-separation of the wave equation $\psi_{tt}-\Delta_2\psi=0$ is presented. All such coordinate systems whose coordinate curves are cyclides or their degenerate forms are given. In each case the coordinates and separation equations are computed. The two basis operators associated with each coordinate system are also presented as symmetric second order operators in the enveloping algebra of the conformal group O(3,2).

INTRODUCTION

In this article we complement the contents of our previous article[1] (hereafter referred to as I) by giving a detailed treatment of the orthogonal coordinate systems for which the two-dimensional wave equation

$$\partial_{tt}\psi=\Delta_2\psi \tag{*}$$

admits an R-separable solution.[2] We recall that an R-separable solution of (*) can be written in the form $\exp[Q(\mu,\rho,\nu)]A(\mu)B(\rho)C(\nu)$. Here μ,ρ,ν are curvilinear coordinates and Q is a function such that either

$$\frac{\partial^2 Q}{\partial\lambda\partial\lambda'}\neq 0,\quad \lambda\neq\lambda',\quad \lambda,\lambda'=\mu,\rho,\nu,$$

for at least two distinct pairs λ,λ' or $Q=0$. The latter case is the familiar one of separation of variables. In searching for R-separable solutions of (*) we restrict our attention in this article to orthogonal curvilinear coordinate systems. These are systems of coordinates μ,ρ,ν such that the differential form

$$ds^2=dt^2-dx^2-dy^2 \tag{0.1}$$

can be written

$$ds^2=F\,d\mu^2+G\,d\rho^2+H\,d\nu^2, \tag{0.2}$$

with F, G, and H real functions of μ,ρ,ν. In a subsequent article we shall give a systematic treatment of the nonorthogonal systems for which (*) admits a separation of variables.

The methods necessary for systematically finding all such orthogonal R-separable coordinate systems have been developed in some detail in the book by Bôcher.[3] These methods can be readily adapted to the problem of interest in this article. There are however a number of new developments occurring in the case of (*). These developments stem from the fact that (*) is inherently more complicated than Laplace's equation

$$(\partial_{xx}+\partial_{yy}+\partial_{zz})\psi=0, \tag{0.3}$$

which Bôcher treated in detail. The contents of the article are arranged as follows.

In Sec. I we give the basic ideas necessary to construct the coordinates which allow an R-separation of (*). This involves a treatment of pentaspherical space, relevant properties of cyclides, and the method of finding the pentaspherical coordinates (and hence the coordinates t,x,y) in terms of the various curvilinear coordinates. Enough detail is presented in this section so as to make the article reasonably self-contained. In Sec. II the connection between the wave equation (*) and pentaspherical coordinates is discussed.

Section III contains the classification of orthogonal R-separable coordinates of (*). In addition the separation equations are given and identified as much as possible. We also give the two symmetric second order operators whose eigenvalues are the separation constants. These operators are expressed in terms of the symmetry group of (*) discussed in detail in I.

The best-known coordinate systems which permit separation of variables in the wave, Laplace, and Helmholtz equations have the property that the coordinate surfaces are orthogonal families of confocal quadrics

$$\frac{x^2}{\lambda-a_1}+\frac{y^2}{\lambda-a_2}+\frac{z^2}{\lambda-a_3}=1,\quad a_i \text{ const} \tag{0.4}$$

or their limits.[4] Thus the coordinate surfaces are ellipsoids, hyperboloids, spheres, planes, etc. The Helmholtz equation separates only in coordinate systems of this type, but the wave and Laplace equations admit more general separable systems. This fact is related to the greater symmetry of the latter differential equations. Indeed, the wave equation admits an inversion symmetry which transforms the coordinates x,y,t to $x/\mathbf{x}\cdot\mathbf{x}$, $y/\mathbf{x}\cdot\mathbf{x}$, $t/\mathbf{x}\cdot\mathbf{x}$, where $\mathbf{x}\cdot\mathbf{x}=t^2-x^2-y^2$. Under inversion and space–time translations the orthogonal coordinate surfaces (0.4) are transformed into orthogonal surfaces, each of the form

$$\begin{aligned}&\alpha(t^2-x^2-y^2)^2+ax^2+by^2+ct^2\\&+dx+ey+ft+h=0.\end{aligned} \tag{0.5}$$

The fourth-order surfaces (0.5) are cyclides[3,5,6] and the coordinate surfaces are orthogonal families of confocal cyclides. The set of all cyclides is invariant under the conformal symmetry group of the wave equation. Moreover, one can show by explicit construction that certain confocal families of cyclides define orthogonal coordinate systems which permit separation of variables in the wave equation. No separable systems other than these are known. Two families of confocal cyclides define equivalent coordinate systems if one can be obtained from the other by a transformation belonging to the conformal symmetry group SO(3, 2) of the wave equation. Certain special families of cyclides can be mapped to the form (0.5) with $\alpha=0$ by a conformal symmetry, and these families lead to the special coordinate surfaces (0.4) and their limits.

To determine all distinct cyclidic separable coordinate systems, we clearly need to classify the distinct equivalence classes of cyclides under the action of the conformal group.

However, as shown explicitly in I, the action of this group on x, y, t (Minkowski) space is rather complicated. To simplify the computation of equivalence classes, one sets up a correspondence between three-dimensional Minkowski space and five-dimensional pentaspherical space as defined in Sec. I. In pentaspherical space the general cyclide takes the simple form (1.9) and the action of the conformal group SO(3, 2) reduces to matrix multiplication. Thus the classification of cyclides into SO(3, 2) symmetry classes can be carried out in a straightforward manner, and the results mapped back to Minkowski space to yield R-separable coordinate systems for the wave equation.

I. PENTASPHERICAL COORDINATES AND ORTHOGONAL FAMILIES OF CONFOCAL CYCLIDES

In this section we will outline the use of pentaspherical coordinates in classifying orthogonal families of confocal cyclides. Such orthogonal families, each provide an R-separable coordinate system for (*). The results presented here summarize those aspects of the work of Bőcher that are relevant for this article. Further details can be found in Bőcher's book and also the book by Coolidge.[5]

Any set of objects that can be put into one to one correspondence with sets of five homogeneous coordinates $x_1 : x_2 : x_3 : x_4 : x_5$ not all simultaneously zero but connected by the relation

$$x_1^2+x_2^2+x_3^2+x_4^2+x_5^2=0 \tag{1.1}$$

are called points in pentaspherical space. It is clear that in general the quantities x_i are complex numbers. For our purposes the subset of pentaspherical coordinates of interest for the wave equation (*) can be obtained from the coordinates t, x, y as follows. Instead of considering the usual Cartesian coordinates t, x, y in three-dimensional Minkowski space, consider the Cartesian coordinates defined by

$$Z=t, \quad X=ix, \quad Y=iy. \tag{1.2}$$

The correspondence between a point (t, x, y) in Minkowski space and a point in five-dimensional space is then achieved as follows. The stereographic projection of the Cartesian coordinates with respect to the four-dimensional unit sphere embeds the point (Z, X, Y) in a four-dimensional space. The homogeneous or projective coordinates of the corresponding four-vector are

$$\begin{aligned} &y_1=r^2-p^2-q^2+s^2, \quad y_2=r^2-p^2-q^2-s^2, \\ &y_3=2ips, \quad y_4=2iqs, \quad y_5=2rs, \end{aligned} \tag{1.3}$$

where the coordinates t, x, y are given by

$$t=r/s, \quad x=p/s, \quad y=q/s. \tag{1.4}$$

If we adopt entirely real coordinates by writing $z_i=y_i$, $i=1,2,5$, and $z_i=-iy_i$, $i=3,4$, we see that these coordinates satisfy

$$z_1^2-z_2^2+z_3^2+z_4^2-z_5^2=0 \quad (z_i \text{ all real}). \tag{1.5}$$

The subset of pentaspherical space of interest then consists of those points whose pentaspherical coordinates are

$$\begin{aligned} &x_1=i(r^2-p^2-q^2+s^2), \quad x_2=r^2-p^2-q^2-s^2, \\ &x_3=2ips, \quad x_4=2iqs, \quad x_5=2rs. \end{aligned} \tag{1.6}$$

In this work we are concerned only with these points in pentaspherical space which correspond to the real coordinates z_i satisfying (1.5) (i.e., having the same signature as this equation). An alternative equation to (1.6) can be obtained via the substitutions $p\to -ip$, $q\to -iq$, $r\to -ir$. From the form of (1.1) it can be seen that to transform one set of pentaspherical coordinates x_i into another set x_i' via a linear transformation

$$x_i'=V_{ij}x_j, \tag{1.7}$$

which preserves

$$\Omega=x_1^2+x_2^2+x_3^2+x_4^2+x_5^2$$

is only possible if $V=(V_{ij})$ is an orthogonal matrix:

$$VV^T=1 \quad [V^T=(V_{ji}), \ V_{ij}\in C]. \tag{1.8}$$

In particular for the case of interest here the orthogonal transformations V corresponding to points in pentaspherical space of the form (1.6) are isomorphic to elements of the group O(3, 2). This is the symmetry group of (*).

A cyclide is defined to be the locus of points x_i in pentaspherical space lying on the quadric surface

$$\Phi=\sum_{i,j=1}^{5} a_{ij}x_ix_i=0 \tag{1.9}$$

with $a_{ij}=a_{ji}$ and $\det(a_{ij})\neq 0$. The problem of classifying types of cyclides under the group of orthogonal transformations V as in (1.7) and (1.8) is then the problem of classifying the intersections of two quadric forms in five-dimensional projective space, where one form is required to be equivalent to Ω, (1.5). This is performed by the method of elementary divisors applied to the two quadratic forms.[6] If we take the quadratic forms to be Φ as in (1.9) and $\Omega=\sum_{i,j=1}^{5} b_{ij}x_ix_j$, each class of quadratic forms Φ, Ω is then specified by the corresponding invariant factors. The invariant factors form a complete set of invariants for each class of pairs Ω, Φ. This means that if Ω', Φ' have the same invariant factors as Ω, Φ, the two systems are related by a linear substitution

$$x_i'=c_{ij}x_j, \quad \det(c_{ij})\neq 0.$$

The invariant factors of a given pair of quadratic forms are obtained as follows. Suppose $D = \det|\lambda a_{ij} - b_{ij}|$ contains the factor $(\lambda - u)^{l_0}$. A second index l_1 is defined to be the highest power of $(\lambda - u)$ which divides all the first minors of D. Proceeding in this manner we obtain the terminating set of indices $e_1 = l_0 - l_1$, $e_2 = l_1 - l_2, \ldots, e_r = l_{r-1}$. The powers $(\lambda - u)^{e_1}$, $(\lambda - u)^{e_2}, \ldots, (\lambda - u)^{e_r}$ are called the invariant factors to the base $\lambda - u$ of the determinant D of the family of forms. All possible invariant factors of D then determine a complete set of invariants. The standard notation for the inequivalent classes of pairs Ω, Φ of quadratic forms is to display the indices e_i for each of the roots of $D = 0$ within a square bracket. Those indices belonging to the same base or root of $D = 0$ are enclosed in conventional brackets. As an example consider the invariant factors $(\lambda - a)^2$, $(\lambda - b)$, $(\lambda - c)$, $(\lambda - d)$ the corresponding notation is [2111]. If the invariant factors are $(\lambda - a)^2$, $(\lambda - a)$, $(\lambda - c)$, $(\lambda - d)$, then there is more than one invariant factor to the base a. Such a cyclide is then called a degenerate form of the corresponding cyclide in which there is only one invariant factor to each different base. For this second example we have a degenerate case of the cyclide [2111] and write this as [(21)11]. If the set of invariant factors are $(\lambda - a)^2$, $(\lambda - b)$, $(\lambda - b)$, $(\lambda - c)$, then the notation would be [2(11)1] and so on. The list of pairs of quadratic forms in five variables which are inequivalent are (this does not include the singular cases, which we do not need here, see, for instance, Bromwich[6]):

1. [11111] $$\Omega = x_1^2 + x_2^2 + x_3^2 + x_4^2 + x_5^2, \quad \Phi = \lambda_1 x_1^2 + \lambda_2 x_2^2 + \lambda_3 x_3^2 + \lambda_4 x_4^2 + \lambda_5 x_5^2; \tag{1.10}$$

2. [2111] $$\Omega = 2x_1x_2 + x_3^2 + x_4^2 + x_5^2, \quad \Phi = 2\lambda_1 x_1 x_2 + x_1^2 + \lambda_3 x_3^2 + \lambda_4 x_4^2 + \lambda_5 x_5^2; \tag{1.11}$$

3. [311] $$\Omega = 2x_1x_3 + x_2^2 + x_4^2 + x_5^2, \quad \Phi = \lambda_1(2x_1x_3 + x_2^2) + 2x_1x_2 + \lambda_4 x_4^2 + \lambda_5 x_5^2; \tag{1.12}$$

4. [221] $$\Omega = 2x_1x_2 + 2x_3x_4 + x_5^2, \quad \Phi = 2\lambda_1 x_1 x_2 + x_1^2 + 2\lambda_3 x_3 x_4 + x_3^2 + \lambda_5 x_5^2; \tag{1.13}$$

5. [41] $$\Omega = 2x_1x_4 + 2x_2x_3 + x_5^2, \quad \Phi = 2\lambda_1(x_1x_4 + x_2x_3) + 2x_1x_3 + x_2^2 + \lambda_5 x_5^2; \tag{1.14}$$

6. [32] $$\Omega = 2x_1x_3 + 2x_4x_5 + x_2^2, \quad \Phi = \lambda_1(2x_1x_3 + x_2^2) + 2x_1x_2 + 2\lambda_4 x_4 x_5 + x_4^2; \tag{1.15}$$

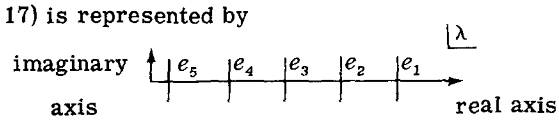
7. [5] $$\Omega = 2x_1x_5 + 2x_2x_4 + x_3^2, \quad \Phi = \lambda_1(2x_1x_5 + 2x_2x_4 + x_3^2) + 2x_1x_4 + 2x_2x_3. \tag{1.16}$$

The pairs of forms for a degenerate cyclide can be obtained from these formulas, e.g., the quadratic forms Ω, Φ corresponding to the configuration [(11)111] are obtained from (1.10) by putting $\lambda_1 = \lambda_2$, and so on. Each type of cyclide is then associated with one of the seven types listed or one of the corresponding degenerate forms. The corresponding equations defining the cyclide are $\Omega = 0$ and $\Phi = 0$. The types of cyclides of particular interest here are those belonging to a confocal family. The most general such family is associated with the configuration [11111] and is given by the pair of equations,

$$\Omega = x_1^2 + x_2^2 + x_3^2 + x_4^2 + x_5^2 = 0, \quad \Phi = \frac{x_1^2}{\lambda - e_1} + \frac{x_2^2}{\lambda - e_2} + \frac{x_3^2}{\lambda - e_3} + \frac{x_4^2}{\lambda - e_4} + \frac{x_5^2}{\lambda - e_5} = 0, \tag{1.17}$$

where λ is the parameter specifying the family. For the subset of pentaspherical space of interest to us Eqs. (1.17) may correspond to a number of different real nondegenerate coordinate curves in t, x, y space. These possibilities are,

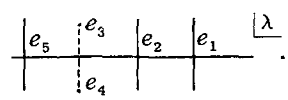
(i) The coordinates x_i are in fact the pentaspherical coordinates and are given by (1.6) [or the substitution $p \to -ip$, $q \to -iq$, $r \to -ir$ applied to (1.6)]. To give a real curve all the e_i must then be real. Bôcher introduces a diagramatic notation for such a confocal family of cyclides as follows. In the complex λ plane (1.17) is represented by



(ii) Two of the quantities e_i are mutually complex conjugate, say e_3, e_4. The corresponding choice of variables for x_i is

$$x_1 = i(r^2 - p^2 - q^2 + s^2), \quad x_2 = r^2 - p^2 - q^2 - s^2, \quad x_3 = \sqrt{2}(r + ip)s, \quad x_4 = \sqrt{2}(r - ip)s, \quad x_5 = 2iqs. \tag{1.18}$$

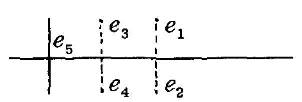
Another associated choice is obtained by taking $p \to -ip$, $q \to -iq$, $r \to -ir$ in these formulas. The notation for such a family of cyclides is $[\widehat{11}111]$ and the corresponding diagrammatic representation is



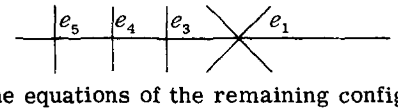
(iii) Two pairs of the quantities e_i are mutually complex conjugate, say e_1, e_2 and e_3, e_4. The corresponding choice of variables for x_i is

$$x_1 = \sqrt{i}\,(r^2 - p^2 - q^2 + is^2), \quad x_2 = \sqrt{-i}\,(r^2 - p^2 - q^2 - is^2) \quad x_3 = \sqrt{2}(r + ip)s, \quad x_4 = \sqrt{2}(r - ip)s, \quad x_4 = 2iqs. \tag{1.19}$$

Another associated choice is obtained by taking $p \to -ip$, $q \to -iq$, $r \to -ir$. The notation for such a family of cyclides is $[\widehat{11}\widehat{11}1]$ with the corresponding diagrammatic representation



The equations for a family of cyclides corresponding to the configuration [(11)111] are readily obtained from Eqs. (1.17) by putting $e_1 = e_2$. The corresponding diagrammatic representation of this configuration is



The equations of the remaining configurations 2–7 are obtained as limiting cases of the general configuration (1.17). This leads to equations which are more convenient than those found in Eqs. (1.11)–(1.16). The method is illustrated here for the [2111] configuration and is explained in detail in Bôcher's book. As an illustration of the procedure we subject (1.17) to the transformation $x_i \to \sqrt{a_i}\, x_i$ (a_i real) and take

$$e_2 = e_1 + \epsilon, \quad x_2 = x_1 + \epsilon x_2, \tag{1.20}$$

where ϵ is a first order quantity. Then by choosing a_i such that

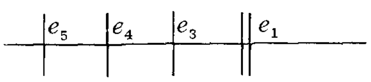
$$a_1 + a_2 = 0, \quad a_2\epsilon = 1, \quad a_3 = a_4 = a_5 = 1, \tag{1.21}$$

Eqs. (1.17) become

$$\begin{aligned} &\Omega = 2x_1x_2 + x_3^2 + x_4^2 + x_5^2 = 0, \\ &\Phi = \frac{x_1^2}{(\lambda - e_1)^2} + \frac{2x_1x_2}{\lambda - e_1} + \frac{x_3^2}{\lambda - e_3} + \frac{x_4^2}{\lambda - e_4} + \frac{x_5^2}{\lambda - e_5} = 0. \end{aligned} \tag{1.22}$$

These are then the equations of cyclides of type [2111]. The coordinates x_i in (1.22) have two interpretations:

(i) The e_i are all real. The corresponding diagrammatic representation is

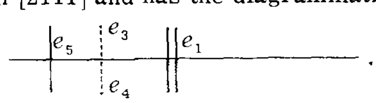


Here the two close parallel lines at e_1 signify the invariant factor index 2 in the [2111] configuration. The choice of variables x_i in this case is

$$\begin{aligned} &x_1 = -2s^2, \quad x_2 = r^2 - p^2 - q^2, \\ &x_3 = 2ips, \quad x_4 = 2iqs, \quad x_5 = 2rs. \end{aligned} \tag{1.23}$$

The variables x_i are in this case a complex linear combination of the pentaspherical coordinates given in (1.6). An associated set of variables is given by the transformation $p \to -ip,\ q \to -iq,\ r \to -ir$.

(ii) Two of the quantities e_i, say e_3, e_4, are mutually complex conjugate. This corresponds to the configuration $[2\widehat{11}1]$ and has the diagrammatic representation



The choice of variables x_i is given by

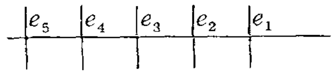
$$\begin{aligned} &x_1 = -2s^2, \quad x_2 = r^2 - p^2 - q^2, \\ &x_3 = \sqrt{2}(r + ip)s, \quad x_4 = \sqrt{2}(r - ip)s, \quad x_5 = 2iqs. \end{aligned} \tag{1.24}$$

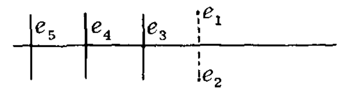
An associated set of variables is given by the transformation $p \to -ip,\ q \to -iq,\ r \to -ir$.

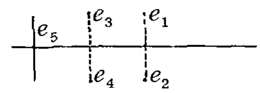
As we have mentioned, the expressions for all confocal families of cyclides can be derived from the general system (1.17) by methods similar to those illustrated here to pass to the configuration [2111]. We now list the equations for these families of curves and their associated diagrams. In the case of the configuration [221] we give the coordinates x_i in terms of the homogeneous coordinates p, q, r and s.

1. [11111], $[\widehat{11}111]$, and $[\widehat{11}\,\widehat{11}1]$

$$\begin{aligned} &\Omega = x_1^2 + x_2^2 + x_3^2 + x_4^2 + x_5^2 = 0, \\ &\Phi = \frac{x_1^2}{\lambda - e_1} + \frac{x_2^2}{\lambda - e_2} + \frac{x_3^2}{\lambda - e_3} + \frac{x_4^2}{\lambda - e_4} + \frac{x_5^2}{\lambda - e_5} = 0: \end{aligned} \tag{1.25}$$

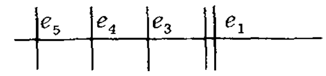
(i) [11111] 

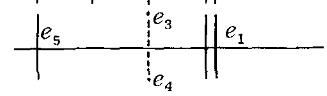
(ii) $[\widehat{11}111]$ 

(iii) $[\widehat{11}\,\widehat{11}1]$ 

2. [2111] and $[2\widehat{11}1]$:

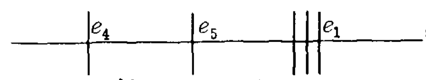
$$\begin{aligned} &\Omega = 2x_1x_2 + x_3^2 + x_4^2 + x_5^2 = 0, \\ &\Phi = \frac{x_1^2}{(\lambda - e_1)^2} + \frac{2x_1x_2}{\lambda - e_1} + \frac{x_3^2}{\lambda - e_3} + \frac{x_4^2}{\lambda - e_4} + \frac{x_5^2}{\lambda - e_5} = 0: \end{aligned} \tag{1.26}$$

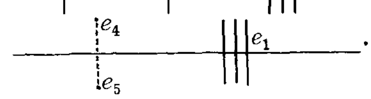
(i) [2111] 

(ii) $[2\widehat{11}1]$ 

3. [311] and $[3\widehat{11}]$

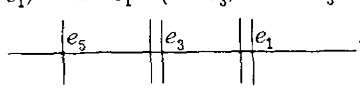
$$\begin{aligned} &\Omega = 2x_1x_3 + x_2^2 + x_4^2 + x_5^2 = 0, \\ &\Phi = \frac{x_1^2}{(\lambda - e_1)^3} + \frac{2x_1x_2}{(\lambda - e_1)^2} + \frac{2x_1x_3 + x_2^2}{\lambda - e_1} + \frac{x_4^2}{\lambda - e_4} + \frac{x_5^2}{\lambda - e_5} = 0: \end{aligned} \tag{1.27}$$

(i) [311] 

(ii) $[3\widehat{11}]$ 

4. [221]

$$\begin{aligned} &\Omega = 2x_1x_2 + 2x_3x_4 + x_5^2 = 0, \\ &\Phi = \frac{x_1^2}{(\lambda - e_1)^2} + \frac{2x_1x_2}{\lambda - e_1} + \frac{x_3^2}{(\lambda - e_3)^2} + \frac{2x_3x_4}{\lambda - e_3} + \frac{x_5^2}{\lambda - e_5} = 0: \end{aligned} \tag{1.28}$$

(i) [221] 

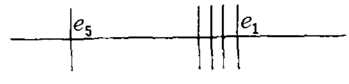
The corresponding expressions for the coordinates x_i in this case are

$$\begin{aligned} &x_1 = -2s^2, \quad x_2 = r^2 - p^2 - q^2, \\ &x_3 = \sqrt{2}\,(r - p)s, \quad x_4 = \sqrt{2}\,(r + p)s, \quad x_5 = 2iqs. \end{aligned} \tag{1.29}$$

The associated set of coordinates being given as usual by $p \to -ip,\ q \to -iq,\ r \to -ir$. From Eqs. (1.10)–(1.16) it is seen that Ω is always one of the types found in systems corresponding to the configurations [11111], [2111], or [221]. The correspondence between the x_i's in this list with p, q, r, and s has now been determined in all cases.

5. [41]

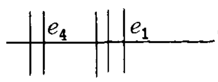
$$\begin{aligned} &\Omega = 2x_1x_4 + 2x_2x_3 + x_5^2 = 0, \\ &\Phi = \frac{x_1^2}{(\lambda - e_1)^4} + \frac{2x_1x_2}{(\lambda - e_1)^3} + \frac{2x_1x_3 + x_2^2}{(\lambda - e_1)^2} + \frac{2x_1x_4 + 2x_2x_3}{\lambda - e_1} \\ &\qquad + \frac{x_5^2}{\lambda - e_5} = 0: \end{aligned} \tag{1.30}$$

(i) [41] 

6. [32]

$$\Omega = 2x_1x_3 + 2x_4x_5 + x_2^2 = 0, \tag{1.31}$$

$$\Phi = \frac{x_1^2}{(\lambda - e_1)^3} + \frac{2x_1x_2}{(\lambda - e_1)^2} + \frac{2x_1x_3 + x_2^2}{\lambda - e_1} + \frac{x_4^2}{(\lambda - e_4)^2} + \frac{2x_4x_5}{\lambda - e_4} = 0:$$

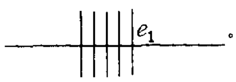
(i) [32] 

.

7. [5]

$$\Omega = 2x_1x_5 + 2x_2x_4 + x_3^2 = 0, \tag{1.32}$$

$$\Phi = \frac{x_1^2}{(\lambda - e_1)^5} + \frac{2x_1x_2}{(\lambda - e_1)^4} + \frac{x_2^2 + 2x_1x_3}{(\lambda - e_1)^3} + \frac{2x_2x_3 + 2x_1x_4}{(\lambda - e_1)^2} + \frac{2x_1x_5 + 2x_2x_4 + x_3^2}{\lambda - e_1} = 0:$$

(i) [5] 

.

In the expression for Φ in this last case the final term is identically zero as it is proportional to Ω.

As was mentioned earlier, the coordinate curves for the cases in which brackets are inserted inside the square brackets can be obtained from this list by the appropriate substitution, e.g., [(32)] corresponds to curves (1.31) with $e_1 = e_4$.

Any two confocal families of the same type and configuration are equivalent under the action of linear transformations of the x_i which preserve the form Ω if their parameters e_i', λ' and e_i, λ are related by the equations

$$e_i = \frac{\alpha e_i' + \beta}{\gamma e_i' + \delta}, \quad \lambda = \frac{\alpha\lambda' + \beta}{\gamma\lambda' + \delta}, \quad \alpha, \beta, \gamma, \delta \in \mathbf{R}, \tag{1.33}$$

with $\alpha\delta - \beta\gamma \neq 0$. This equivalence is with respect to transformations which are isomorphic to the orthogonal transformations V which in our case are elements of O(3, 2).

We now turn our attention to the problem of relating the coordinates x_i in Eqs. (1.25)–(1.28), (1.30)–(1.32) to the parameters which specify an orthogonal family of such surfaces. These latter quantities are the curvilinear coordinates whose coordinate curves are mutually orthogonal at the common point of intersection. The problem of the ranges of variation of the parameters and the number of inequivalent types of parametrization for the real subset (1.6) are the subject of Sec. III. Here we just give the form of the coordinates x_i corresponding to each of the cases 1–7 outlined above when the coordinate curves are all of this type. The corresponding curvilinear coordinates are denoted by $\lambda = \mu, \rho, \nu$. For a coordinate system generated by cyclides of the type [11111] the coordinate curves are given by the equations

$$\Omega = x_1^2 + x_2^2 + x_3^2 + x_4^2 + x_5^2 = 0,$$
$$\Phi = \frac{x_1^2}{\lambda - e_1} + \frac{x_2^2}{\lambda - e_2} + \frac{x_3^2}{\lambda - e_3} + \frac{x_4^2}{\lambda - e_4} + \frac{x_5^2}{\lambda - e_5} = 0, \tag{1.34}$$

with $\lambda = \mu, \rho$ or ν. The corresponding expression for the coordinates x_i is:

1. [11111]

$$\sigma x_i^2 = \phi(e_i)/f'(e_i), \quad i = 1, \ldots, 5, \tag{1.35}$$

where

$$-1/\sigma = e_1x_1^2 + e_2x_2^2 + e_3x_3^2 + e_4x_4^2 + e_5x_5^2$$

and

$$f(\lambda) = (\lambda - e_1)(\lambda - e_2)(\lambda - e_3)(\lambda - e_4)(\lambda - e_5),$$
$$\phi(\lambda) = (\mu - \lambda)(\nu - \lambda)(\rho - \lambda).$$

The coordinates in Minkowski space can be found from these expressions via the relations

$$t = \frac{-x_5}{x_2 + ix_1}, \quad x = \frac{ix_3}{x_2 + ix_1}, \quad y = \frac{ix_4}{x_2 + ix_1} \tag{1.36}$$

if the x_i are given as in (1.6). We will see in Sec. III that this need not always be the case, and we may be required to permute the expressions on the right-hand side of equations (1.6) so as to correspond to the correct signature as in (1.5). We now give the expressions for the coordinates x_i for the remaining six types of families of cyclides. These can be deduced by the same methods as used to deduce the form of the cyclides [2111] from the general case [11111]. We again refer to Bócher's book for details. (Bócher has given the formulas required to pass the configurations [2111] and [311]. The authors have extended this to include all remaining cases. Only the results are presented here.)

2. [2111]

$$\begin{aligned}
\sigma x_1^2 &= \frac{(\mu - e_1)(\nu - e_1)(\rho - e_1)}{(e_3 - e_1)(e_4 - e_1)(e_5 - e_1)},\\
2\sigma x_1x_2 &= \frac{\partial}{\partial e_1}\left[\frac{(\mu - e_1)(\nu - e_1)(\rho - e_1)}{(e_3 - e_1)(e_4 - e_1)(e_5 - e_1)}\right],\\
\sigma x_3^2 &= \frac{(\mu - e_3)(\nu - e_3)(\rho - e_3)}{(e_1 - e_3)^2(e_4 - e_3)(e_5 - e_3)},\\
\sigma x_4^2 &= \frac{(\mu - e_4)(\nu - e_4)(\rho - e_4)}{(e_1 - e_4)^2(e_3 - e_4)(e_5 - e_4)},\\
\sigma x_5^2 &= \frac{(\mu - e_5)(\nu - e_5)(\rho - e_5)}{(e_1 - e_5)^2(e_3 - e_5)(e_4 - e_5)},
\end{aligned} \tag{1.37}$$

where

$$-1/\sigma = 2e_1x_1x_2 + x_1^2 + e_3x_3^2 + e_4x_4^2 + e_5x_5^2,$$

$$\begin{aligned}
\sigma x_1^2 &= \frac{(\mu - e_1)(\nu - e_1)(\rho - e_1)}{(e_4 - e_1)(e_5 - e_1)},\\
2\sigma x_1x_2 &= \frac{\partial}{\partial e_1}\left[\frac{(\mu - e_1)(\nu - e_1)(\rho - e_1)}{(e_4 - e_1)(e_5 - e_1)}\right],\\
\sigma(2x_1x_3 + x_2^2) &= \frac{1}{2}\frac{\partial^2}{\partial e_1^2}\left[\frac{(\mu - e_1)(\nu - e_1)(\rho - e_1)}{(e_4 - e_1)(e_5 - e_1)}\right],\\
\sigma x_4^2 &= \frac{(\mu - e_4)(\nu - e_4)(\rho - e_4)}{(e_1 - e_4)^3(e_5 - e_4)},
\end{aligned} \tag{1.38}$$

$$\sigma x_5^2=\frac{(\mu-e_5)(\nu-e_5)(\rho-e_5)}{(e_1-e_5)^3(e_4-e_5)},$$

where

$$\frac{-1}{\sigma}=e_1(2x_1x_3+x_2^2)+2x_1x_2+e_4x_4^2+e_5x_5^2.$$

4. [221]

$$\begin{aligned}
\sigma x_1^2&=\frac{(\mu-e_1)(\nu-e_1)(\rho-e_1)}{(e_3-e_1)^2(e_1-e_5)},\\
2\sigma x_1x_2&=-\frac{\partial}{\partial e_1}\left[\frac{(\mu-e_1)(\nu-e_1)(\rho-e_1)}{(e_3-e_1)^2(e_1-e_5)}\right],\\
\sigma x_3^2&=\frac{(\mu-e_3)(\nu-e_3)(\rho-e_3)}{(e_3-e_1)^2(e_3-e_5)},\\
2\sigma x_3x_4&=-\frac{\partial}{\partial e_3}\left[\frac{(\mu-e_3)(\nu-e_3)(\rho-e_3)}{(e_3-e_1)^2(e_5-e_4)}\right],\\
\sigma x_5^2&=\frac{(\mu-e_5)(\nu-e_5)(\rho-e_5)}{(e_1-e_5)^2(e_3-e_5)^2},
\end{aligned}\tag{1.39}$$

where

$$-1/\sigma=2e_1x_1x_2+x_1^2+2e_3x_3x_4+x_3^2+e_5x_5^2.$$

5. [41]

$$\begin{aligned}
\sigma x_1^2&=\frac{(\mu-e_1)(\nu-e_1)(\rho-e_1)}{(e_1-e_5)},\\
2\sigma x_1x_2&=\frac{\partial}{\partial e_1}\left[\frac{(\mu-e_1)(\nu-e_1)(\rho-e_1)}{(e_1-e_5)}\right],\\
\sigma(2x_1x_3+x_2^2)&=\frac{1}{2}\frac{\partial^2}{\partial e_1^2}\left[\frac{(\mu-e_1)(\nu-e_1)(\rho-e_1)}{(e_1-e_5)}\right],\\
\sigma(2x_1x_4+2x_2x_3)&=\frac{1}{6}\frac{\partial^3}{\partial e_1^3}\left[\frac{(\mu-e_1)(\nu-e_1)(\rho-e_1)}{(e_1-e_5)}\right],\\
\sigma x_5^2&=\frac{(\mu-e_5)(\nu-e_5)(\rho-e_5)}{(e_5-e_1)^4},
\end{aligned}\tag{1.40}$$

where

$$-1/\sigma=2e_1(x_2x_3+x_1x_4)+2x_1x_3+x_2^2+e_5x_5^2.$$

6. [32]

$$\begin{aligned}
\sigma x_1^2&=\frac{(\mu-e_1)(\nu-e_1)(\rho-e_1)}{(e_4-e_1)^2},\\
2\sigma x_1x_2&=\frac{\partial}{\partial e_1}\left[\frac{(\mu-e_1)(\nu-e_1)(\rho-e_1)}{(e_4-e_1)^2}\right],\\
\sigma(2x_1x_3+x_2^2)&=\frac{1}{2}\frac{\partial^2}{\partial e_1^2}\left[\frac{(\mu-e_1)(\nu-e_1)(\rho-e_1)}{(e_4-e_1)^2}\right],\\
\sigma x_4^2&=\frac{(\mu-e_4)(\nu-e_4)(\rho-e_4)}{(e_4-e_1)^3},\\
2\sigma x_4x_5&=\frac{\partial}{\partial e_4}\left[\frac{(\mu-e_4)(\nu-e_4)(\rho-e_4)}{(e_4-e_1)^3}\right],
\end{aligned}\tag{1.41}$$

where

$$-1/\sigma=e_1(2x_1x_3+x_2^2)+2x_1x_2+2e_4x_4x_5+x_4^2.$$

7. [5]

$$\begin{aligned}
\sigma x_1^2&=(\mu-e_1)(\nu-e_1)(\rho-e_1),\\
2\sigma x_1x_2&=\frac{\partial}{\partial e_1}[(\mu-e_1)(\nu-e_1)(\rho-e_1)],\\
\sigma(2x_1x_3+x_2^2)&=\frac{1}{2}\frac{\partial^2}{\partial e_1^2}[(\mu-e_1)(\nu-e_1)(\rho-e_1)],\\
2\sigma(x_2x_3+x_1x_4)&=-1,
\end{aligned}\tag{1.42}$$

where σ is given by the last equation in this case.

These are the basic formulas for the pentaspherical coordinates expressed in terms of the curvilinear coordinates $\mu,\rho,$ and ν for all nondegenerate cyclides. The expressions for the coordinates in the case of a degenerate configuration are directly derivable from these formulas. The explicit methods for doing this will be discussed in Sec. III, where we evaluate all the possible inequivalent systems 1—7 and the associated degenerate forms. Finally in this section we give the formula expressing the line element ds^2 in terms of the curvilinear coordinates μ,ρ,ν and the pentaspherical coordinates x_i,

$$ds^2=\frac{1}{4\sigma s^2}\left(\frac{(\mu-\nu)(\mu-\rho)}{f(\mu)}d\mu^2+\frac{(\nu-\mu)(\nu-\rho)}{f(\nu)}d\nu^2+\frac{(\rho-\mu)(\rho-\nu)}{f(\rho)}d\rho^2\right)\tag{1.43}$$

with $f(\lambda)=\Pi_{i=1}^5(\lambda-e_i)$ as in (1.35). In each case σ is the quantity in the above list given for each configuration. The quantity s is the homogeneous coordinate that was introduced in (1.3) and can be expressed in terms of the x_i depending on the configuration in question. This formula is basic to the classification of coordinate systems which are inequivalent under the action of the underlying transformation group O(3, 2).

We summarize what has been done to this point. We have given the equations required to pass to a subspace of pentaspherical space having definite real signature as in (1.5). The associated group of transformations which preserve this subspace is isomorphic to O(3, 2) the local symmetry group of (*). The corresponding second order curves or cyclides in these coordinates can then be classified into equivalence classes under the action of this group of transformations. Those curves of special interest are the families of confocal cyclides and the coordinate systems to which they correspond. An important feature here is that all families of confocal orthogonal cyclides can be obtained as specified limits of the most general case corresponding to the configuration [11111].

II. THE TWO-DIMENSIONAL WAVE EQUATION AND *R*-SEPARABLE COORDINATES GENERATED FROM ORTHOGONAL FAMILIES OF CONFOCAL CYCLIDES

In this section we summarize the results that enable (*) to have an R-separable solution. For more details we refer to Bocher's book[3] and also to Morse and Feshbach.[4] The central result with which we will be concerned is the form of the equation (*) when written in terms of the cyclidic coordinates discussed in the previous section. Of central interest is the case of cyclidic coordinates corresponding to the configuration [11111]. The result is the following. If ψ is a solution of $\partial_{tt}\psi=\Delta_2\psi$ and if we write

$$\psi = \sqrt{2}\sigma^{1/4} s\phi(\mu,\nu,\rho), \tag{2.1}$$

where μ, ν, ρ are cyclidic coordinates of the type [11111] and

$$-1/\sigma = e_1x_1^2 + e_2x_2^2 + e_3x_3^2 + e_4x_4^2 + e_5x_5^2$$

with S as in (1.3), then ψ satisfies the differential equation

$$(\rho-\nu)\frac{\partial^2\phi}{\partial u^2} + (\mu-\rho)\frac{\partial^2\phi}{\partial v^2} + (\nu-\mu)\frac{\partial^2\phi}{\partial w^2} + (\mu-\nu)(\nu-\rho)(\rho-\mu)\left(\tfrac{5}{4}(\mu+\nu+\rho) - \tfrac{3}{4}\sum_{i=1}^{5} e_i\right)\phi = 0, \tag{2.2}$$

where

$$\frac{\partial}{\partial u} = 2\sqrt{f(\mu)}\frac{\partial}{\partial\mu}, \quad \frac{\partial}{\partial v} = 2\sqrt{f(\nu)}\frac{\partial}{\partial\nu},$$

and

$$\frac{\partial}{\partial w} = \sqrt{f(\rho)}\frac{\partial}{\partial\rho}.$$

Here $f(\lambda)$ is as usual given by

$$f(\lambda) = (\lambda-e_1)(\lambda-e_2)(\lambda-e_3)(\lambda-e_4)(\lambda-e_5).$$

Equation (2.2) admits a separable solution

$$\phi = E_1(\mu)E_2(\nu)E_3(\rho) \tag{2.3}$$

with each of the separated functions satisfying an equation of the form

$$\sqrt{f(\lambda)}\frac{d}{d\lambda}\sqrt{f(\lambda)}\frac{dE_i}{d\lambda} + \left[\frac{5}{16}\lambda^3 - \frac{3}{16}\left(\sum_{i=1}^{5} e_i\right)\lambda^2 - \frac{A}{4}\lambda - \frac{B}{4}\right]E_i = 0. \tag{2.4}$$

With this result all the separation equations for the coordinate systems given in the previous section can be obtained by taking appropriate limits in the above equations. Equation (2.4) is an equation of the Lamé type with six elementary singularities.[7] The quantities A and B are separation constants.

III. CLASSIFICATION OF ORTHOGONAL R-SEPARABLE COORDINATE SYSTEMS FOR THE WAVE EQUATION

In this section a systematic treatment is given of the orthogonal R-separable coordinates of (*), which can be constructed as limiting cases of general cyclidic coordinates with configurations [11111], $[\widehat{11}111]$, and $[\widehat{11}\widehat{11}1]$. For each coordinate system we give the expression for the corresponding pentaspherical coordinates x_i and the Cartesian coordinates t, x, and y. The operators whose eigenvalues are the separation constants are also given in each case and expressed in terms of the generators of the symmetry group of (*) which were derived in I. We also say what we can about the solutions of the separated equations. Our procedure is the following. For the completely cyclidic coordinates listed in Eqs. (1.35)–(1.42) we must choose ranges for the curvilinear coordinates in such a way that the differential form (1.43) when expressed as in (0.2) must satisfy

$$\mathrm{sgn}F = \mathrm{sgn}G = -\mathrm{sgn}H. \tag{3.1}$$

This ensures that the space is three-dimensional Minkowski space. The classification of all such parametrizations into equivalence classes under the relation (1.33) then gives the inequivalent coordinate systems we need. For the general configuration [11111] the equivalence relation (1.33) allows us to interchange all the e_i in the way specified by these formulas. However, for the remaining configurations such as [2111] only the three unit indices can change under the relation (1.33) when classifying equivalence classes of this type. In addition for each class of coordinate systems we choose a standardized representative which has a simple form. In most cases this will involve taking one of the indices e_i to be ∞.

The method of connecting the operators whose eigenvalues are the separation constants with the generators of the symmetry group $G = \mathrm{O}(3,2)$ is achieved by noting that the generators Γ_{ij} as defined in I are related to the generators of the underlying O(3, 2) group which preserves the pentaspherical space identity (1.1) with the choice of coordinates (1.6). The relations are

$$\begin{aligned} &\Gamma_{13} = L_{14}, \quad \Gamma_{12} = L_{15}, \quad \Gamma_{23} = L_{34}, \\ &\Gamma_{45} = L_{52}, \quad \Gamma_{15} = -iL_{12}, \quad \Gamma_{14} = -iL_{15}, \\ &\Gamma_{52} = iL_{23}, \quad \Gamma_{53} = iL_{24}, \quad \Gamma_{24} = -iL_{35}, \end{aligned} \tag{3.2}$$

where

$$L_{ij} = x_j\partial_i - x_i\partial_j, \quad i,j = 1,2,\ldots,5,$$

with the x_j as in (1.6). By means of the relations (see I)

$$\begin{aligned} &M_{12} = \Gamma_{23}, \quad M_{01} = \Gamma_{42}, \quad M_{02} = \Gamma_{43}, \quad D = \Gamma_{15}, \\ &P_0 = \Gamma_{14} + \Gamma_{45}, \quad K_0 = \Gamma_{14} - \Gamma_{45}, \quad P_1 = \Gamma_{12} + \Gamma_{25}, \\ &K_1 = \Gamma_{12} - \Gamma_{25}, P_2 = \Gamma_{13} + \Gamma_{35}, \quad K_2 = \Gamma_{13} - \Gamma_{35}, \end{aligned} \tag{3.3}$$

the operators whose eigenvalues are the separation constants in a given R-separable coordinate system can be expressed as second order symmetric operators in the generators of the O(3, 2) symmetry group of (*). In the subsequent classification of R-separable orthogonal solutions of (*) we will have occasion to introduce a number of modifications of Bôcher's diagramatic notation as well as some of the limiting procedures of interest for the various degenerate configurations being considered.

A further comment is in order here. In order to give all the coordinate systems that are potentially of interest, we give in the subsequent listing, with the exception of systems of the type [11111], all the separable systems of (*) which are inequivalent under the underlying $E(2,1)$ group. This gives a more thorough treatment of these coordinate systems already considered in an earlier article.[8] In the concluding remarks we indicate which of these systems, which are not equivalent under $E(2,1)$, are equivalent under the symmetry group O(3, 2) of (*).

We now proceed to the classification of the coordinate systems of interest.

A. The configurations [11111], $[\widehat{11}111]$, $[\widehat{11}\widehat{11}1]$ and their degenerate forms

1. The configurations [11111], $[\widehat{11}111]$ and $[\widehat{11}\widehat{11}1]$

Here we give those configurations of the form [11111] which are inequivalent under the procedure outlined in the introductory paragraphs of this section. For configurations of this type we can transform the quantities e_i via (1.33) to be

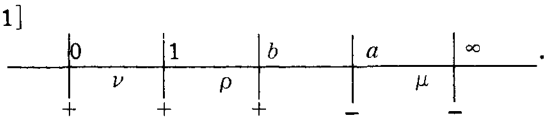
$$e_1=\infty,\quad e_2=a,\quad e_3=b,\quad e_4=1,\quad e_5=0.$$

In addition to Bôcher's diagrammatic notation for such a configuration, as given in Sec. I, we put the sign of the expression σx_i^2 at the bottom of the vertical line in the diagram of the [11111] configuration. From the formulas (1.6) the arrangement of these signs indicates how the choice of pentaspherical coordinates should be made. This involves a permutation of the quantities on the right-hand side of (1.6). In each of the inequivalent parametrizations for the configuration [11111] a specific choice of the x_i is made to within a permutation of those x_i whose squares have the same sign. This is sufficient for our purposes as all coordinate systems that are related by such permutations will be equivalent and related by a group transformation. The two additional operators $\hat{A}$, $\hat{B}$ whose eigenvalues are A and B, respectively, as in (2.4), have the form

$$\hat{A}=\frac{(\nu+\rho)}{(\mu-\rho)(\mu-\nu)}\frac{\partial^2}{\partial u^2}+\frac{(\mu+\rho)}{(\nu-\rho)(\nu-\mu)}\frac{\partial^2}{\partial v^2}+\frac{(\mu+\nu)}{(\rho-\nu)(\rho-\mu)}\frac{\partial^2}{\partial w^2},$$
$$\hat{B}=\frac{\nu\rho}{(\mu-\rho)(\mu-\nu)}\frac{\partial^2}{\partial u^2}+\frac{\mu\rho}{(\nu-\rho)(\nu-\mu)}\frac{\partial^2}{\partial v^2}+\frac{\mu\nu}{(\rho-\nu)(\rho-\mu)}\frac{\partial^2}{\partial w^2}\tag{3.4}$$

when acting on the functions $\phi(\mu,\nu,\rho)$ as in (2.1). The part of the solution of (*) that gives the R-separation (called hereafter the modulation factor following Morse and Feshbach) is from (2.1), $\sqrt{2}\,\sigma^{1/4}s$. Corresponding to the configuration [11111] being considered in this subsection we have the following inequivalent possibilities.

(a) [11111]



For such a configuration the pentaspherical coordinates are

$$\sigma x_1^2=-1,\quad \sigma x_2^2=-\frac{(\mu-a)(\nu-a)(\rho-a)}{(a-b)(a-1)a},$$
$$\sigma x_3^2=-\frac{(\mu-b)(\nu-b)(\rho-b)}{(b-a)(b-1)b},\tag{3.5}$$
$$\sigma x_4^2=-\frac{(\mu-1)(\nu-1)(\rho-1)}{(a-1)(b-1)},\quad \sigma x_5^2=\frac{\mu\nu\rho}{ab}.$$

The coordinates in three-dimensional Minkowski space are given by the formulas

$$t=\frac{-x_2}{(x_1+ix_5)}=\frac{1}{R}\left(\frac{(\mu-a)(\nu-a)(\rho-a)}{(a-b)(a-1)a}\right)^{1/2},$$
$$x=\frac{ix_4}{(x_1+ix_5)}=\frac{1}{R}\left(\frac{(\mu-1)(\rho-1)(1-\nu)}{(a-1)(b-1)}\right)^{1/2},\tag{3.6}$$
$$y=\frac{ix_3}{(x_1+ix_5)}=\frac{1}{R}\left(\frac{(\mu-b)(\rho-b)(\nu-b)}{(a-b)(b-1)b}\right)^{1/2},$$

where $R=i(1+\sqrt{\mu\nu\rho/ab})$. The modulation factor is

$$\sqrt{2}\sigma^{1/4}s=(1+\sqrt{\mu\nu\rho/ab})^{1/2}.$$

The operators $\hat{A}$, $\hat{B}$ defining the eigenvalues of the separation constants (we refer to these as basis operators subsequently) are

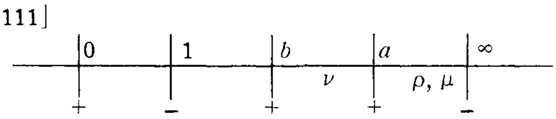
$$\hat{A}=bM_{01}^2-M_{02}^2-aM_{12}^2-\tfrac14(a+1)(P_2+K_2)^2+\tfrac14(b+1)(P_0+K_0)^2+\tfrac14(a+b)(P_1+K_1)^2,$$
$$4\hat{B}=b(P_0+K_0)^2-a(P_2+K_2)^2+ab(P_1+K_1)^2\tag{3.7}$$

and the separation equations have the form

$$\sqrt{h(\lambda)}\frac{d}{d\lambda}\sqrt{h(\lambda)}\frac{dE_i}{d\lambda}-(\tfrac{3}{16}\lambda^2+A\lambda+B)E_i=0\tag{3.8}$$

with $h(\lambda)=\lambda(\lambda-1)(\lambda-b)(\lambda-a)$ and $\lambda=\mu,\nu$ or ρ for $i=1,2,$ or 3 respectively just as in (2.3). Equation (3.10) is a standard form of an equation with five elementary singularities (see, for instance, Ince, Ref. 7, p. 500). It should be noted here that the form of the pentaspherical coordinates (1.6) when subjected to the transformation $p\to ip$, $q\to -iq$, $r\to -ir$ gives no new information, i.e., exactly the same coordinate system results.

(b) [11111]



The pentaspherical coordinates are as in (3.6) with the three-dimensional Minkowski space coordinates given by

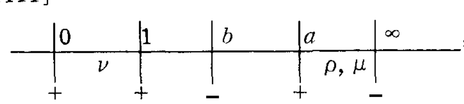
$$t=-x_4/(x_1+ix_5),\quad x=ix_2/(x_1+ix_5),$$
$$y=ix_3/(x_1+ix_5).\tag{3.9}$$

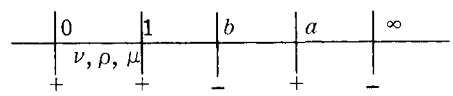
The modulation factor $\sqrt{2}\,\sigma^{1/4}S$ is the same as in (a). The basis operators are

$$\hat{A}=aM_{02}^2+bM_{01}^2-M_{12}^2+\tfrac14(a+1)(P_2+K_2)^2+\tfrac14(b+1)(P_0-K_0)^2+\tfrac14(a+b)(P_1-K_1)^2,\tag{3.10}$$
$$-4\hat{B}=b(P_1+K_1)^2+a(P_2-K_2)^2+ab(P_0+K_0)^2,$$

and the separation equations have the form (3.10).

(c) [11111]

(i) 

(ii) 

The pentaspherical coordinates are as in (3.6) with the three space coordinates given by

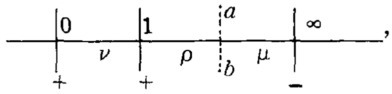
$$t=-x_3/(x_1+ix_5),\quad x=ix_4/(x_1+ix_5),$$
$$y=ix_2/(x_1+ix_5).\tag{3.11}$$

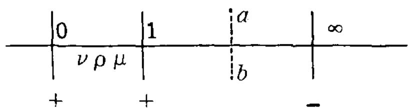
The modulation factor is the same as in (a). The basis operators are

$$\hat{A}=aM_{01}^2-bM_{12}^2-M_{02}^2-\tfrac14(a+1)(P_0+K_0)^2+\tfrac14(b+1)(P_2+K_2)^2-\tfrac14(a+b)(P_1+K_1)^2,\tag{3.12}$$
$$4\hat{B}=b(P_2+K_2)^2-a(P_0+K_0)^2-ab(P_1+K_1)^2,$$

and the separation equations have the form (3.10).

(*d*) $[\widehat{11}111]$

(i) 

(ii) 

The pentaspherical coordinates are as in (3.6) with $a = \alpha + i\beta,\ b = \alpha - i\beta,\ \alpha, \beta \in \mathbb{R}$. The three space coordinates are given by

$$t = \frac{(x_2 + x_3)}{\sqrt{2}(x_1 + ix_5)}, \quad x = \frac{i(x_2 - x_3)}{\sqrt{2}(x_1 + ix_5)}, \quad y = \frac{ix_4}{(x_1 + ix_5)} \tag{3.13}$$

The basis operators are

$$\begin{aligned} 2\hat{A} &= \alpha(M_{01}^2 - M_{12}^2) + \beta(M_{01}M_{12} + M_{12}M_{01}) \\ &\quad + (\alpha + 1)[(P_2 + K_2)^2 - (P_0 + K_0)^2] + \alpha(P_1 + K_1)^2 \\ &\quad - (\beta + 1)[(P_2 + K_2)(P_0 + K_0) + (P_0 + K_0)(P_2 + K_2)], \end{aligned} \tag{3.14}$$

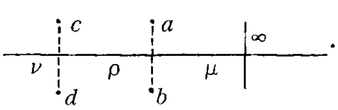
$$\begin{aligned} 4\hat{B} &= \alpha[(P_0 + K_0)^2 - (P_2 + K_2)^2] + (\alpha^2 + \beta^2)(P_1 + K_1)^2 \\ &\quad + \beta[(P_0 + K_0)(P_2 + K_2) + (P_2 + K_2)(P_0 + K_0)]. \end{aligned}$$

The modulation factor is

$$\sqrt{2}\sigma^{1/4}s = \{1 + [\mu\nu\rho/(\alpha^2 + \beta^2)]^{1/2}\}^{1/2},$$

and the separation equations have the form (3.10).

(e) $[\widehat{11}\widehat{11}1]$



The pentaspherical coordinates are given by

$$\sigma x_1^2 = -1, \quad \sigma x_2^2 = -\frac{(\mu - a)(\nu - a)(\rho - a)}{(a - b)(a - c)(a - d)},$$

$$\sigma x_3^2 = -\frac{(\mu - b)(\nu - b)(\rho - b)}{(b - a)(b - c)(b - d)}, \tag{3.15}$$

$$\sigma x_4^2 = -\frac{(\mu - c)(\nu - c)(\rho - c)}{(c - a)(c - b)(c - d)}, \quad \sigma x_5^2 = -\frac{(\mu - d)(\nu - d)(\rho - d)}{(d - c)(d - a)(d - b)},$$

where $a = \alpha + i\beta,\ b = \alpha - i\beta,\ c = \gamma + i\delta,\ d = \gamma - i\delta$ with $\alpha, \beta, \gamma, \delta \in \mathbb{R}$. The three space coordinates are given by

$$\begin{aligned} t &= -(x_4 + x_5)/[(x_2 + x_3) + i\sqrt{2}x_1], \\ x &= (x_3 - x_2)/[i(x_2 + x_3) - \sqrt{2}x_1], \\ y &= (x_5 - x_4)/[i(x_2 + x_3) - \sqrt{2}x_1], \end{aligned} \tag{3.16}$$

and the modulation factor is

$$\sqrt{2}\sigma^{1/4}s = [(-x_2 - x_3)/\sqrt{2} - ix_1]^{1/2}.$$

The basis operators are

$$\begin{aligned} \hat{A} &= 2\beta[(P_0 - K_0)M_{01} + M_{01}(P_0 - K)] + \beta[(P_0 - K_0)(P_2 - K_2) \\ &\quad + (P_2 - K_2)(P_0 - K_0)] - 4\gamma(M_{12}M_{01} + M_{01}M_{12}) \\ &\quad - 2\gamma[M_{12}(P_2 - K_2) + (P_2 - K_2)M_{12}] \\ &\quad + 2(\gamma - \alpha\ [\tfrac{1}{4}(P_1 - K_1)^2 - M_{02}^2], \\ \hat{B} &= 2[(\alpha + \delta)^2 + (\beta + \delta)^2][M_{12}^2 + \tfrac{1}{4}(P_0 - K_0)^2 \\ &\quad + M_{01}^2 + \tfrac{1}{4}(P_2 - K_2)^2 + \tfrac{1}{2}M_{01}(P_2 - K_2) + \tfrac{1}{2}(P_2 - K_2)M_{01}] \\ &\quad - 2\beta\delta[(P_0 - K_0)M_{12} + M_{12}(P_0 - K_0)] \\ &\quad + 4\alpha\gamma[(P_1 - K_1)^2 - 4M_{02}^2] + [1/(\alpha + \gamma)] \\ &\quad [\{2(\alpha + \gamma)(\delta\gamma - \alpha\beta) + (\beta - \delta)(\alpha^2 + \gamma^2 - \beta^2 - \delta^2)\} \\ &\quad \times\{[(M_{12} + \tfrac{1}{2}(P_0 - K_0)][\tfrac{1}{2}(K_2 - P_2) - M_{01}] \\ &\quad + [\tfrac{1}{2}(K_2 - P_2) - M_{01}][M_{12} + \tfrac{1}{2}(P_0 - K_0)]\} \\ &\quad + \{2(\alpha + \gamma)(\delta\gamma + \alpha\beta) - (\beta + \delta)(\alpha^2 + \gamma^2 - \beta^2 - \delta^2)\} \\ &\quad \times\{[\tfrac{1}{2}(K_0 - P_0) - M_{12}][\tfrac{1}{2}(P_2 - K_2) + M_{01}] + [\tfrac{1}{2}(P_2 - K_2) + M_{01}] \\ &\quad \times[\tfrac{1}{2}(K_0 - P_0) - M_{12}]\} + 2\alpha(\gamma^2 - \delta^2)[\tfrac{1}{4}(P_1 - K_1)^2 - M_{02}^2]. \end{aligned} \tag{3.17}$$

2. The configurations [(11)111], [11(11)1]

Here we must digress briefly to explain how the pentaspherical coordinates for the configuration [(11)111] can be obtained from the formulas (1.35) for the general configuration. To find the pentaspherical coordinates for the configuration [(11)111] for which say $e_1 = e_2$, we proceed as follows, putting

$$e_1 = e_2 + \epsilon, \quad \lambda = e_2 + \epsilon\lambda',$$

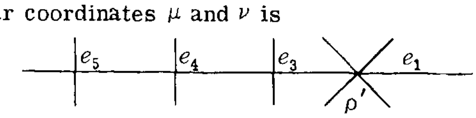
where for definiteness we take $\lambda = \rho$. The resulting expression for the pentaspherical coordinates is

$$\begin{aligned} \sigma x_1^2 &= \frac{(\mu - e_1)(\nu - e_1)}{(e_3 - e_1)(e_4 - e_1)(e_5 - e_1)}(1 - \rho'), \\ \sigma x_2^2 &= \frac{(\mu - e_1)(\nu - e_1)}{(e_3 - e_1)(e_4 - e_1)(e_5 - e_1)}\rho', \\ \sigma x_3^2 &= \frac{(\mu - e_3)(\nu - e_3)}{(e_1 - e_3)(e_4 - e_3)(e_5 - e_3)}, \\ \sigma x_4^2 &= \frac{(\mu - e_4)(\nu - e_4)}{(e_1 - e_4)(e_3 - e_4)(e_5 - e_4)}, \\ \sigma x_5^2 &= \frac{(\mu - e_5)(\nu - e_5)}{(e_1 - e_5)(e_3 - e_5)(e_4 - e_5)}. \end{aligned} \tag{3.18}$$

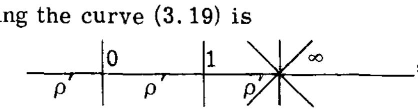
The coordinate curves corresponding to the new curvilinear coordinate ρ' are

$$x_1^2/\rho' + x^2(\rho' - 1) = 0. \tag{3.19}$$

This defines a family of real curves for $0 < \rho' < 1$ if $\mathrm{sgn}(x_1^2/x_2^2) = -1$. Otherwise for a real curve we must have $\mathrm{sgn}(x_1^2/x_2^2) = 1$. The diagrammatic notation for the family of degenerate cyclides specified by the curvilinear coordinates μ and ν is



The method of obtaining other degenerate forms corresponding to a configuration [(11)111] is to generalize the procedure outlined here to the case of two adjacent parameters $e_i,\ e_{i+1}$ becoming equal. The diagram representing the curve (3.19) is



where ρ' may be in one of the regions indicated according as the relative sign of x_1^2 and x_2^2 is ± 1, as we have discussed above. The separation equations for the function $\psi(\mu, \nu, \rho')$ are given by (2.4) with $e_1 = e_2$ and $\lambda = \mu, \nu$. For ρ' we obtain

$$(e_2 - e_3)(e_2 - e_4)(e_2 - e_5)\sqrt{|\rho'(\rho'-1)|}\,\frac{d}{d\rho'}\sqrt{|\rho'(\rho'-1)|}\,\frac{dE_3}{d\rho'}$$

$$= [\tfrac{1}{16} e_2^2(e_2 + 3e_3 + 3e_4 + 3e_5) + \tfrac{1}{4}Ae_2 + \tfrac{1}{4}B]E_3.$$

For all the classes of inequivalent coordinate systems of the type $[(11)111]$ the quantities e_i will be standardized to be $0, 1, a$, and ∞. This greatly simplifies all the calculations. For instance, in the example we have presented here this standardization can be achieved by taking

$$e_1 = \infty, \quad e_3 = a, \quad e_4 = 1, \quad e_5 = 0.$$

The resulting standardized form then gives the following expressions for the pentaspherical coordinates:

$$\sigma x_1^2 = \rho' - 1, \quad \sigma x_2^2 = -\rho', \quad \sigma x_3^2 = \frac{(\mu - a)(\nu - a)}{a(a-1)},$$
$$\sigma x_4^2 = \frac{(\mu - 1)(\nu - 1)}{(1 - a)}, \quad \sigma x_5^2 = \frac{\mu\nu}{a}. \tag{3.20}$$

The separation equations are

$$\sqrt{p(\lambda)}\frac{d}{d\lambda}\sqrt{p(\lambda)}\frac{dE_i}{d\lambda} - (A\lambda + B)E_i = 0. \tag{3.21}$$

$\lambda = \mu, \nu$ and $i = 1, 2$ respectively, and

$$\sqrt{|\rho'(\rho'-1)|}\,\frac{d}{d\rho}\sqrt{|\rho'(\rho'-1)|}\,\frac{dE_3}{d\rho'} - (\tfrac{1}{4} + A)E_3 = 0.$$

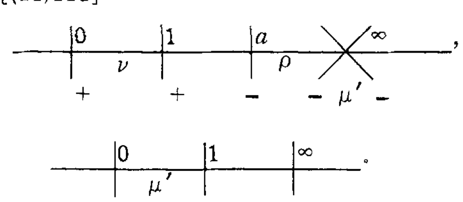
Here $p(\lambda) = \lambda(\lambda - 1)(\lambda - a)$. Equation (3.21) is a form of Lamé's equation (see, for instance, Ince, Ref. 7, p. 502). The basis operators $\hat{A}$, $\hat{B}$ whose eigenvalues are the separation constants A and B respectively are in this case

$$\hat{A} = -\tfrac{1}{4} - \sqrt{|\rho'(\rho'-1)|}\,\frac{d}{d\rho'}\sqrt{|\rho'(\rho'-1)|}\,\frac{d}{d\rho'} \tag{3.22}$$
$$\hat{B} = \frac{1}{(\nu - \mu)}\left(\nu\sqrt{p(\mu)}\frac{\partial}{\partial\mu}\sqrt{p(\mu)}\frac{\partial}{\partial\mu} - \mu\sqrt{p(\nu)}\frac{\partial}{\partial\nu}\sqrt{p(\nu)}\frac{\partial}{\partial\nu}\right).$$

acting on ϕ.

We now proceed to the evaluation of the inequivalent types of coordinate systems of type $[(11)111]$ and $[\widehat{11}(11)1]$.

(a) $[(11)111]$



The pentaspherical coordinates are obtained from (3.19) subjected to the transformation

$$\rho' \to \mu', \quad \nu \to \nu, \quad \mu \to \rho.$$

The three space Minkowski coordinates are given by

$$t = -x_5/(ix_3 + x_4) = (1/R)\sqrt{\nu\rho/a},$$
$$x = ix_2/(ix_3 + x_4) = (1/R)\cos\phi, \tag{3.23}$$
$$y = ix_1/(ix_3 + x_4) = (1/R)\sin\phi,$$

where

$$R = \left[\left(\frac{(\mu - a)(\nu - a)}{a(1-a)}\right)^{1/2} + \left(\frac{(\mu - 1)(\nu - 1)}{(a - 1)}\right)^{1/2}\right]$$

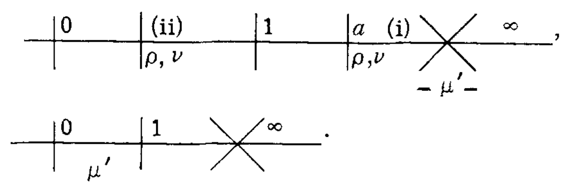
and we have put $\nu' = \sin^2\phi$. The modulation factor is

$$\sqrt{2}\,\sigma^{1/4}s = R^{1/2} \tag{3.24}$$

and the basis operators are

$$4\hat{B} = (P_0 + K_0)^2 - a(P_0 - K_0)^2, \quad \hat{A} = -\tfrac{1}{4} - M_{12}^2. \tag{3.25}$$

(c)[†] $[(11)111]$



The pentaspherical coordinates are as in (a). The three space coordinates are given by

$$t = x_3/(ix_4 + x_5), \quad x = ix_1/(ix_4 + x_5),$$
$$y = ix_2/(ix_4 + x_5) \tag{3.26}$$

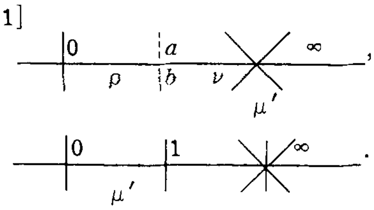
The modulation factor is

$$\sqrt{2}\,\sigma^{1/4}s = \left[\left(\frac{\mu\nu}{a}\right)^{1/2} + \left(\frac{(\mu - 1)(\nu - 1)}{(a - 1)}\right)^{1/2}\right]^{1/2} \tag{3.27}$$

and the basis operators are,

$$4\hat{B} = -(P_0 - K_0)^2 + 4aD^2, \quad \hat{A} = -\tfrac{1}{4} - M_{12}^2. \tag{3.28}$$

(d) $[\widehat{11}(11)1]$



The pentaspherical coordinates are given by

$$\sigma x_1^2 = \mu' - 1, \quad \sigma x_2^2 = -\mu', \quad \sigma x_3^2 = \frac{(\rho - a)(\nu - a)}{a(a - b)},$$
$$\sigma x_4^2 = \frac{(\rho - b)(\nu - b)}{b(b - a)}, \quad \sigma x_5^2 = \frac{\rho\nu}{ab}. \tag{3.29}$$

The three space coordinates are given by

$$t = \sqrt{2}\,x_5/R, \quad x = i\sqrt{2}\,x_1/R, \quad y = i\sqrt{2}\,x_2/R, \tag{3.30}$$

where $R = i(x_3 - x_4) - (x_3 + x_4)$. The modulation factor is

$$\sqrt{2}\,\sigma^{1/4}s = \sqrt{i}\left\{2\mathrm{Re}\left[\left(\frac{i(\rho - a)(\nu - a)}{a(a - b)}\right)^{1/2}\right]\right\}^{1/2} \tag{3.31}$$

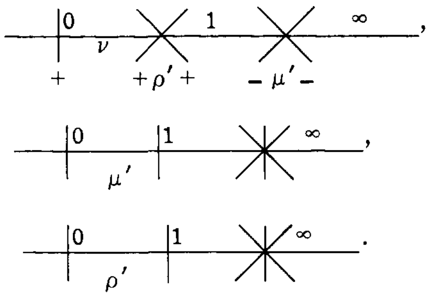
and the basis operators are

$$\hat{B} = \alpha(P_0K_0 + K_0P_0) + 2\beta(P_0^2 - K_0^2),$$
$$\hat{A} = -\tfrac{1}{4} - M_{12}^2. \tag{3.32}$$

Here as usual $a = \alpha + i\beta$, $b = \alpha - i\beta$, $\alpha, \beta \in \mathbb{R}$. The separation equations have the form (3.20) and (3.21) with $p(\lambda) = \lambda(\lambda - a)(\lambda - b)$ and a is replaced in (3.21) by ab.

3. The configuration [(11)(11)1]

There is only one such coordinate system of interest here. The diagrams of this system are



The pentaspherical coordinates are given by

$$\sigma x_1^2=\mu'-1,\quad \sigma x_2^2=-\mu',\quad \sigma x_3^2=(1-\nu)(1-\rho'),$$
$$\sigma x_4^2=(1-\nu)\rho',\quad \sigma x_5^2=\nu. \tag{3.33}$$

The three space coordinates are given by

$$t=ix_2/(ix_1+x_5),\quad x=x_4/(ix_1+x_5),$$
$$y=x_3/(ix_1+x_5), \tag{3.34}$$

and the modulation factor is

$$\sqrt{2}\,\sigma^{1/4}s=(\sqrt{1-\mu'}+\sqrt{\nu})^{1/2}. \tag{3.35}$$

If we write $\mu'=\cos^2\phi$, $\nu=\cos^2\psi$, $\rho'=\cos^2\theta$, then

$$t=\cos\phi/(\sin\phi+\cos\psi),$$
$$x=\sin\psi\cos\theta/(\sin\phi+\cos\psi), \tag{3.36}$$
$$y=\sin\psi\sin\theta/(\sin\phi+\cos\psi).$$

The separation equations for this system of coordinates are given by

$$\sqrt{p(\mu)}\frac{d}{d\mu}\sqrt{p(\mu)}\frac{dE_1}{d\mu}-(A\mu+B)E_1=0, \tag{3.37}$$

where $\sqrt{p(\mu)}=\mu(\mu-1)$.

Equation (3.37) is a form of the Legendres equation with spherical harmonic solution,

$$\frac{d^2E_2}{d\theta^2}+BE_2=0,\quad \frac{d^2E_3}{d\phi^2}+AE_3=0. \tag{3.38}$$

The basis defining operators are

$$4\hat{A}=(P_0-K_0)^2,\quad \hat{B}=M_{12}^2. \tag{3.39}$$

This completes the classification of inequivalent coordinate systems of type [11111] and its degenerate forms. These are the only coordinate systems which will prove to be strictly R separable in the classification presented in this article.

B. THE CONFIGURATIONS [2111], $[2\hat{1}11]$ and their degenerate forms

1. The configurations [2111] and $[2\hat{1}11]$

Here we give the configurations of the form [2111] and $[2\hat{1}11]$, which are inequivalent under the equivalence relation discussed in the begining of this section. By applying a transformation of the type (1.33) to the indices e_i for the configuration [2111] it is always possible to choose these numbers in the standard form

$$e_1=\infty,\quad e_3=a,\quad e_4=1,\quad e_5=0 \tag{3.40}$$

with e_1 the number associated with the invariant factor index 2. The two operators whose eigenvalues are the separation constants are given as in (3.5) with $f(\lambda)=\lambda(\lambda-1)(\lambda-a)$.

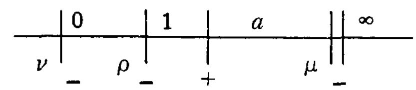
The separation equations with the choice of e_i given above are

$$\sqrt{f(\lambda)}\frac{d}{d\lambda}\sqrt{f(\lambda)}\frac{dE_i}{d\lambda}-(A\lambda+B)E_i=0 \tag{3.41}$$

with $f(\lambda)=\lambda(\lambda-1)(\lambda-a)$ and $\lambda=\mu,\nu$, or ρ for $i=1,2$, or 3 respectively. This is Lamé's equation with four elementary singularities. For the configuration $[2\hat{1}11]$ the separation equations are as in (3.41) with $f(\lambda)=\lambda(\lambda-a)(\lambda-b)$, where $a=\alpha+i\beta$ and $b=\alpha-i\beta$, $\alpha,\beta\in\mathbb{R}$. We now give the inequivalent coordinate systems.

For the choice of e_i given in (3.40) there is no modulation factor in the R-separated solutions. The solutions of (*) of the type [2111] are therefore separable.

(a) [2111]



The pentaspherical coordinates for this configuration are given by

$$\sigma x_1^2=-1,\quad 2\sigma x_1x_2=\mu+\nu+\rho+a+1,$$
$$\sigma x_3^2=\frac{(\mu-a)(\nu-a)(\rho-a)}{a(a-1)}, \tag{3.42}$$
$$\sigma x_4^2=\frac{(\mu-1)(\nu-1)(\rho-1)}{(1-a)},\quad \sigma x_5^2=\frac{\mu\nu\rho}{a}.$$

The three space variables are given by

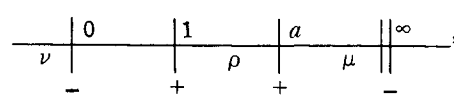
$$t^2=-x_3^2/x_1^2=(\mu-a)(\nu-a)(\rho-a)/a(a-1),$$
$$x^2=x_4^2/x_1^2=(\mu-1)(\nu-1)(\rho-1)/(a-1), \tag{3.43}$$
$$y^2=x_5^2/x_1^2=-\mu\nu\rho/a,$$

and the basis operators are

$$\hat{A}=P_0^2-(a+1)P_2^2-aP_1^2+M_{12}^2-M_{01}^2-M_{02}^2,$$
$$\hat{B}=aP_2^2+aM_{12}^2-M_{02}^2. \tag{3.44}$$

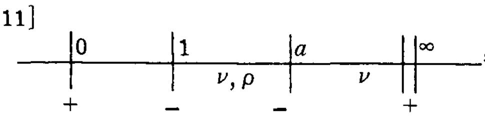
For the remaining inequivalent systems of type [2111] we give the corresponding diagrams and the transformation which relates the three space coordinates given in (3.43) to each system. The expressions for the operators $\hat{A}$, $\hat{B}$ can be obtained from (3.44) via this substitution. In each case the pentaspherical coordinates are given by (3.42).

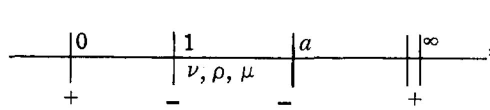
(b) [2111]



(a) → (b), $t\to x$, $x\to iy$, $y\to t$.

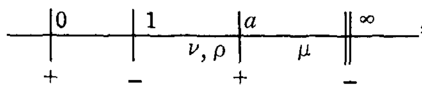
(c) [2111]

(i) 

(ii) 

(a) → (c), $t \to ix$, $x \to y$, $y \to it$.

(d) [2111]

(i) 

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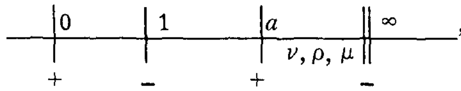
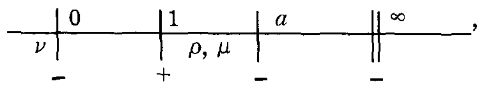
(iii) 

,

(a) → (d), $t \to x$, $x \to t$, $y \to iy$.

(e) [2111]



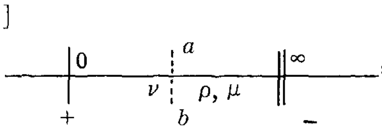
,

(a) → (e), $t \to ix$, $x \to it$, $y \to y$.

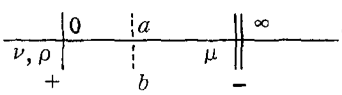
In all the above systems the choice of pentaspherical coordinates is made in two ways. If the net signature of the terms σx_i^2 from (1.37) for $i = 1, 2, 3, 4, 5$ is plus, then the form of the x_i's is as in (1.23). If the net signature is minus, then the required form of the x_i is obtained from (1.23) via the transformation

$$p \to -ip, \quad q \to -iq, \quad r \to -ir.$$

(f) $[2\widehat{11}1]$

(i) 

,

(ii) 

.

The pentaspherical coordinates are given by

$$\sigma x_1^2 = -1, \quad 2\sigma x_1 x_2 = \mu + \nu + \rho + a + b$$

$$\sigma x_3^2 = \frac{(\mu - a)(\nu - a)(\rho - a)}{a(a-b)},$$

$$\sigma x_4^2 = \frac{(\mu - b)(\nu - b)(\rho - b)}{b(b-a)}, \quad \sigma x_5^2 = \frac{\mu\nu\rho}{ab}, \tag{3.45}$$

where $a = \alpha + i\beta$, $b = \alpha - i\beta$, $\alpha\beta \in \mathbb{R}$. The three space coordinates are given by

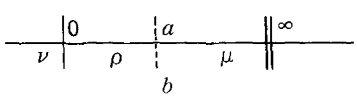
$$t + ix = i\sqrt{2}\, x_3/x_1, \quad y = ix_5/x_1; \tag{3.46}$$

this follows from the use of formula (1.24) relating the x_i's to p, q, r, and s. [More exactly the coordinates obtained from (1.24) via the transformation $p \to -ip$, $q \to -iq$, $r \to -ir$.] The basis operators are

$$\hat{A} = \alpha(P_0^2 - P_1^2 - P_2^2) + 2\beta P_0 P_1 + M_{12}^2 - M_{01}^2 - M_{02}^2,$$
$$\hat{B} = -(\alpha^2 + \beta^2)P_2^2 + \alpha(M_{12}^2 - M_{02}^2) - \beta(M_{12}M_{02} + M_{02}M_{12}). \tag{3.47}$$

The term $\alpha(P_0^2 - P_1^2 - P_2^2)$ is included in the above expression for $\hat{A}$ so as to correspond to the correct operator derived from equations (3.5).

(g) $[2\widehat{11}1]$

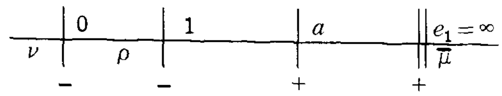


(f) → (g), $t \to x$, $x \to t$, $y \to iy$.

2. Configurations having a radial coordinate in three space derivable from configurations of the form [2111] and [$2\widehat{11}1$]

Such coordinates can be derived in a straightforward manner, which we illustrate in detail for the first system of this section.

(a) (i)



For this diagram write $\mu = e_1 + \bar{\mu}$ in formulas (1.37) and then take $e_1 \to \infty$. The resulting pentaspherical coordinates have the form

$$\sigma x_1^2 = \bar{\mu}, \quad 2\sigma x_1 x_2 = 1, \quad \sigma x_3^2 = \frac{(\nu - a)(\rho - a)}{a(a-1)},$$
$$\sigma x_4^2 = -\frac{(\nu - 1)(\rho - 1)}{(a-1)}, \quad \sigma x_5^2 = \frac{\nu\rho}{a}, \tag{3.48}$$

and the three space coordinates are given by

$$t = \frac{x_3}{x_1} = r\left(\frac{(\nu - a)(\rho - a)}{a(a-1)}\right)^{1/2},$$
$$x = \frac{ix_4}{x_1} = r\left(\frac{(\nu - 1)(\rho - 1)}{(a-1)}\right)^{1/2}, \quad y = \frac{ix_5}{x_1} = r\left(\frac{-\nu\rho}{a}\right)^{1/2}, \tag{3.49}$$

where $r^2 = 1/\bar{\mu}$. The basis operators are given by

$$\hat{A} = \frac{4}{(\rho - \nu)}\left(\sqrt{P(\nu)}\frac{d}{d\nu}\sqrt{P(\nu)}\frac{d}{d\nu} - \sqrt{P(\rho)}\frac{d}{d\rho}\sqrt{P(\rho)}\frac{d}{d\rho}\right)$$
$$= \tfrac{1}{4} - D^2 \tag{3.50}$$

and

$$\hat{B} = \frac{4}{(\rho - \nu)}\left(\rho\sqrt{P(\nu)}\frac{d}{d\nu}\sqrt{P(\nu)}\frac{d}{d\nu} - \nu\sqrt{P(\rho)}\frac{d}{d\rho}\sqrt{P(\rho)}\frac{d}{d\rho}\right)$$
$$= M_{02}^2 - aM_{12}^2,$$

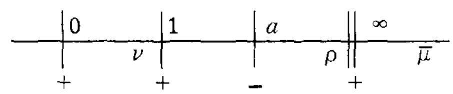
where $P(\lambda) = \lambda(\lambda - 1)(\lambda - a)$. The separation equations have the form

$$\sqrt{P(\lambda)}\frac{d}{d\lambda}\sqrt{P(\lambda)}\frac{dE_i}{d\lambda} + (A\lambda + B)E_i = 0, \tag{3.51}$$

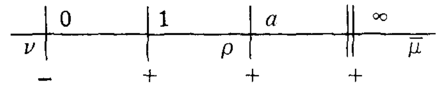
$\lambda = \nu, \rho$ and $i = 1, 2$, respectively, and $A = j(j+1)$,

$$r^2\frac{d^2E_3}{dr^2} + 2r\frac{dE_3}{dr} - j(j+1)E_3 = 0. \tag{3.52}$$

Equation (3.51) is a form of the Lamé equation and the solutions of (3.52) are r^j and r^{-j-1}.

(ii) 

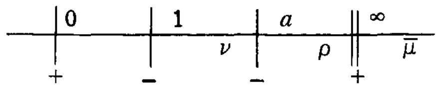
(i) → (ii), $t \to it$, $x \to ix$, $y \to iy$.

(b) (i) 

(a) (i) → (b) (i), $t \to x$, $x \to iy$, $y \to t$,

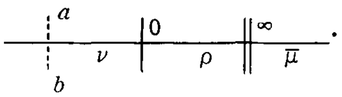
$\hat{B} = M_{01}^2 + aM_{02}^2$.

(ii)



(b) (i) → (b) (ii), $t \to it$, $x \to ix$, $y \to iy$.

(c) (i)



Here as usual $a=\alpha+i\beta$, $b=\alpha-i\beta$, $\alpha,\beta\in\mathbb{R}$. The pentaspherical coordinates are given by

$$\sigma x_1^2=\bar{\mu},\quad 2\sigma x_1x_2=1, \tag{3.53}$$

$$\sigma x_3^2=\frac{(\nu-a)(\rho-a)}{a(a-b)},\quad \sigma x_4^2=\frac{(\nu-b)(\rho-b)}{b(b-a)},\quad \sigma x_5^2=\frac{\nu\rho}{ab},$$

and the three space coordinates are given by

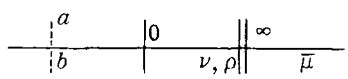
$$(t+ix)=\frac{x_3}{x_1}=\sqrt{2}\,r\frac{(\nu-a)(\rho-a)}{a(a-b)},\quad y=\frac{ix_5}{x_1}=r\frac{\nu\rho}{ab} \tag{3.54}$$

with $r^2=1/\bar{\mu}$. The basis operators are as in (3.50) with $P(\lambda)=\lambda(\lambda-a)(\lambda-b)$. In particular

$$\hat{B}=\alpha(M_{12}^2-M_{02}^2)-\beta(M_{02}M_{12}+M_{12}M_{02}), \tag{3.55}$$

and the separation equations are as in (3.51), (3.52) with appropriate changes in $P(\lambda)$ as above.

(ii)



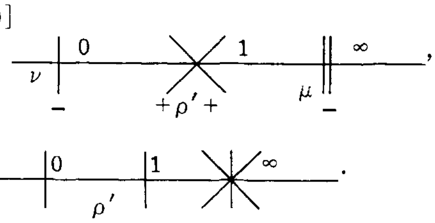
(c) (i) → (c) (ii), $t \to it$, $x \to ix$, $y \to iy$.

The three space parametrizations corresponding to (a), (b), and (c) in this subsection are recognized as the three possible Lamé bases for the group O(2, 1). These bases have been discussed by the authors[9] and Macfadyen and Winternitz.[10] The results presented in this subsection give the parametrization of these bases inside and outside the cone $t^2-x^2-y^2=0$.

3. Degenerate systems of the type [21(11)]

The coordinate systems of this type are chosen in such a way that the parameters e_i are $e_1=\infty$, with the remaining free parameters 1 and 0

(a) [21(11)]



The pentaspherical coordinates are given by

$$\sigma x_1^2=-1,\quad 2\sigma x_1x_2=-\mu-\nu+1, \tag{3.56}$$

$$\sigma x_3^2=-(\mu-1)(\nu-1)(1-\rho'),$$

$$\sigma x_4^2=-(\mu-1)(\nu-1)\rho',\quad \sigma x_5^2=\mu\nu.$$

The three space coordinates are given by

$$t=ix_5/x_1,\quad x=x_3/x_1,\quad y=x_4/x_1. \tag{3.57}$$

With $\mu=\cosh^2A$, $\nu=-\sinh^2B$, $\rho'=\sin^2\phi$, the three space coordinates assume the form

$$t=\cosh A\sinh B,\quad x=\sinh A\cosh B\cos\phi,$$
$$y=\sinh A\cosh B\sin\phi. \tag{3.58}$$

The separation equations have the form

$$\frac{1}{\cosh A}\frac{d}{dA}\cosh A\frac{dE_1}{dA}+\left(\frac{m^2}{\cosh^2A}+K\right)E_1=0, \tag{3.59}$$

$$\frac{1}{\sinh B}\frac{d}{dB}\frac{dE_2}{dB}+\left(\frac{-m^2}{\sinh^2B}+K\right)E_2=0, \tag{3.60}$$

and

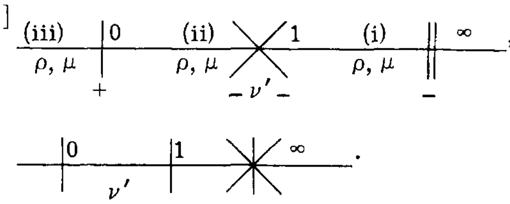
$$\frac{d^2E_3}{d\phi^2}+m^2E_3=0, \tag{3.61}$$

where (*) has the solution $E_1(A)E_2(B)E_3(\phi)$. The basis operators $\hat{A}$ and $\hat{B}$ whose eigenvalues are $-m^2-K$ and $-m^2$ respectively are

$$\hat{A}=M_{01}^2+M_{02}^2+p_1^2+p_2^2$$
$$=P_0^2-\tfrac{1}{2}(P_0K_0+K_0P_0+1), \tag{3.62}$$
$$\hat{B}=M_{12}^2.$$

The separation equations (3.59), and (3.60) can be identified with Legendre's equation. The linearly independent solutions of (3.60) are $P_j^m(\cosh B)$, $Q_j^m(\cosh B)$, where $K=-j(j+1)$. The solutions of (3.59) can be obtained from those of (3.60) by putting $B\to A+i\pi/2$.

(b) [21(11)]



There are three cases to consider here as indicated in the above diagram. We put

(i) $\mu=\cosh^2A$, $\rho=\cosh^2B$,

(ii) $\mu=\cos^2\alpha$, $\rho=\cos^2\beta$,

(iii) $\mu=-\sinh^2A$, $\rho=-\sinh^2\beta$,

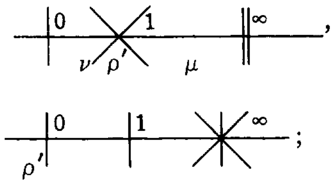
with $\nu'=\cos^2\phi$ in all cases. The resulting three space variables are in these cases:

$$\begin{aligned}
&\text{(i)}\quad t=\cosh A\cosh B,\quad x=\sinh A\sinh B\cos\phi,\\
&\qquad y=\sinh A\sinh B\sin\phi,\\
&\text{(ii)}\quad t=\cos\alpha\cos\beta,\quad x=\sin\alpha\sin\beta\cos\phi,\\
&\qquad y=\sin\alpha\sin\beta\sin\phi,\\
&\text{(iii)}\quad t=\sinh A\sinh B,\quad x=\cosh A\cosh B\cos\phi,\\
&\qquad y=\cosh A\cosh B\sin\phi.
\end{aligned} \tag{3.63}$$

The basis defining operators are

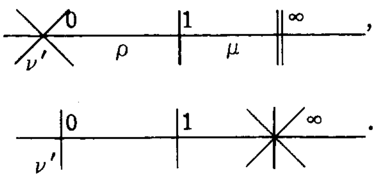
$$\hat{A}=M_{01}^2+M_{02}^2-P_1^2-P_2^2$$
$$=-P_0^2-\tfrac{1}{2}(P_0K_0+K_0P_0+1), \tag{3.64}$$
$$\hat{B}=M_{12}^2.$$

(c) [21(11)]

(i) 

by putting $\mu=\cosh^2 A$, $\nu=\cos^2\alpha$, $\rho'=-\sinh^2 B$, this system gives the three space coordinates

$$t=\sin\alpha\sinh A\sinh B,\quad x=\cos\alpha\cosh A,\quad y=\sin\alpha\sinh A\cosh B. \tag{3.65}$$

(ii) 

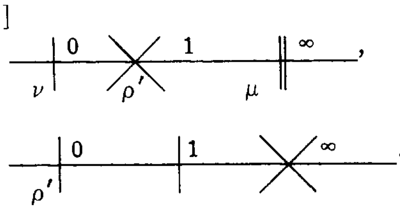
With $\mu=\cosh^2 A$, $\rho=\cos^2\alpha$, and $\nu'=-\sinh^2 B$ the three space coordinates are

$$t=\sin\alpha\cosh A\sinh B,\quad x=\cos\alpha\sinh A,\quad y=\sin\alpha\cosh A\cosh B. \tag{3.66}$$

The basis defining operators for these coordinate systems are

$$\hat{A}=M_{01}^2-M_{12}^2+P_0^2-P_2^2,\quad \hat{B}=M_{02}^2. \tag{3.67}$$

(d) [21(11)]



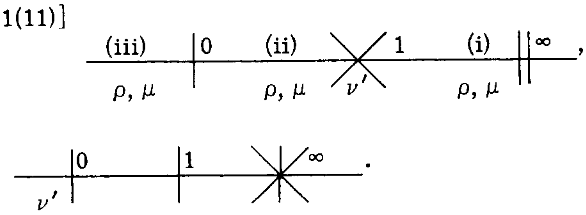
With $\mu=\cosh^2 A$, $\nu=-\sinh^2 B$ and $\rho'=-\sinh^2 C$ the three space coordinates become

$$t=\sinh A\cosh B\cosh C,\quad x=\cosh A\sinh B,\quad y=\sinh A\cosh B\sinh C. \tag{3.68}$$

The basis defining operators of this system are

$$\hat{A}=M_{01}^2-M_{12}^2+P_0^2-P_2^2,\quad \hat{B}=M_{02}^2. \tag{3.69}$$

(e) [21(11)]



There are three possible cases to consider here:

(i) $\mu=\cosh^2 A$, $\rho=\cosh^2 B$,

(ii) $\mu=\cos^2\alpha$, $\rho=\cos^2\beta$,

(iii) $\mu=-\sinh^2 A$, $\rho=-\sinh^2 B$,

where in all cases $\nu'=-\sinh^2 C$. The resulting coordinate systems in three space are

(i) $t=\sinh A\sinh B\cosh C$, $x=\cosh A\cosh B$, $y=\sinh A\sinh B\sinh C$,

(ii) $t=\cos\alpha\cos\beta\cosh C$, $x=\sin\alpha\sin\beta$, $y=\cos\alpha\cos\beta\sinh C$, (3.70)

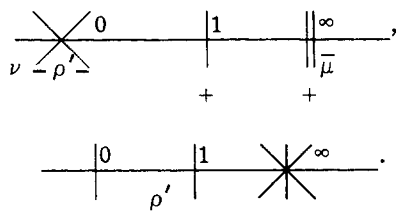
(iii) $t=\cosh A\cosh B\cosh C$, $x=\sinh A\sinh B$, $y=\cosh A\cosh B\sinh C$

and the basis operators are

$$\hat{A}=M_{01}^2-M_{12}^2-P_0^2+P_2^2,\quad \hat{B}=M_{02}^2. \tag{3.71}$$

4. Coordinate systems containing a radial coordinate in three space and derivable from the configuration [21(11)]

These systems are derivable in exactly the same manner as those of subsection 2.

(a) (i) 

The pentaspherical coordinates are given by

$$\sigma x_1^2=\bar{\mu},\quad 2\sigma x_1x_2=-1,\quad \sigma x_3^2=(1-\nu),\quad \sigma x_4^2=\nu(1-\rho'),\quad \sigma x_5^2=\nu\rho', \tag{3.72}$$

With $\nu=-\sinh^2 A$ and $\rho'=\sin^2\alpha$, $\bar{\mu}=1/r^2$ these formulas give the three space coordinates

$$t=x_3/x_1=r\cosh A,\quad x=ix_4/x_1=r\sinh A\cos\alpha,\quad y=-ix_5/x_1=r\sinh A\sin\alpha. \tag{3.73}$$

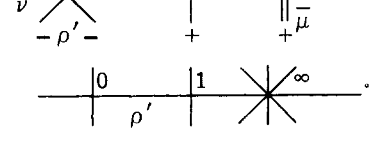
These are just the familiar polar coordinates inside the cone $t^2-x^2-y^2=0$. The basis operators are

$$\hat{A}=\tfrac{1}{4}-D^2,\quad \hat{B}=M_{12}^2. \tag{3.74}$$

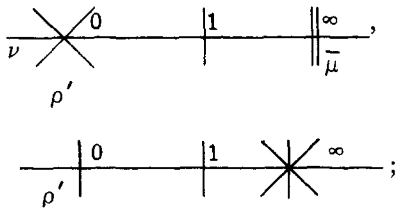
The separation equations are

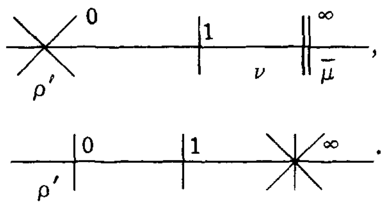
$$\frac{d^2E_1}{dr^2}+\frac{1}{r}\frac{dE_1}{dr}-\frac{j(j+1)}{r^2}E_1=0,$$
$$\frac{1}{\sinh A}\frac{d}{dA}\sinh A\frac{dE_2}{dA}-\left(\frac{m^2}{\sinh^2 A}+j(j+1)\right)E_2=0, \tag{3.75}$$
$$\frac{d^2E_3}{d\phi^2}+m^2E_3=0,$$

where $E_1(r)E_2(A)E_3(\phi)$ is a solution of (*). The second of these equations is just a form of the Legendre equation with solutions $P_j^m(\cosh A)$, $Q_j^m(\cosh A)$. The other two equations have the elementary solutions $E_1=r^j$, r^{-j-1}, and $E_3=\exp(\pm im\phi)$

(ii) 

The three space coordinates for this second configuration are obtained from (3.73) via the transformation $\cosh A \rightleftharpoons \sinh A$.

(b) (i) 

(ii) 

The system given by diagram (i) yields the three space coordinates

$$t = r\sinh A\cosh B, \quad x = r\sinh A\sinh B, \quad y = r\cosh A, \tag{3.76}$$

where

$$\nu = -\sinh^2 A, \quad \rho' = \sinh^2 B, \text{ and } \bar{\mu} = 1/r^2.$$

The defining operators are

$$\hat{A} = \tfrac{1}{4} - D^2, \quad \hat{B} = M_{01}^2. \tag{3.77}$$

The coordinates (3.76) are the familiar hyperbolic coordinates inside the cone $t^2 - x^2 - y^2 = 0$. For diagram (ii) the only change is in the three space variables subjected to the transformation $\sinh A \rightleftharpoons \cosh A$.

5. Coordinate systems corresponding to the configuration [(21) 11]

To obtain the expression for the pentaspherical coordinates corresponding to [(21)11] requires the substitution

$$e_3 = e_1 + \epsilon, \quad \lambda = e_1 + \epsilon + \epsilon^2\lambda' \tag{3.78}$$

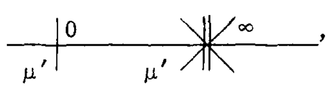
into (1.37). Here ϵ is a first order quantity and for definiteness we may take $\lambda = \mu$. The resulting expression for the pentaspherical coordinates is

$$\begin{aligned}
\sigma x_1^2 &= -\frac{(\rho - e_1)(\nu - e_1)}{(e_4 - e_1)(e_5 - e_1)}, \\
2\sigma x_1 x_2 &= -\frac{\partial}{\partial e_1}\left[\frac{(\rho - e_1)(\nu - e_1)}{(e_4 - e_5)(e_5 - e_1)}\right] - \frac{(\rho - e_1)(\nu - e_1)}{(e_4 - e_1)(e_5 - e_1)}\mu', \\
\sigma x_3^2 &= \frac{(\rho - e_1)(\nu - e_1)}{(e_4 - e_1)(e_5 - e_1)}\mu', \quad \sigma x_4^2 = \frac{(\rho - e_4)(\nu - e_4)}{(e_1 - e_4)^2(e_5 - e_4)} \\
\sigma x_5^2 &= \frac{(\rho - e_5)(\nu - e_5)}{(e_1 - e_5)^2(e_4 - e_5)}.
\end{aligned} \tag{3.79}$$

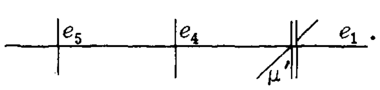
The resulting coordinate curve for the coordinate μ' is

$$x_1^2 + x_3^2/\mu' = 0. \tag{3.80}$$

The diagram corresponding to such a curve is



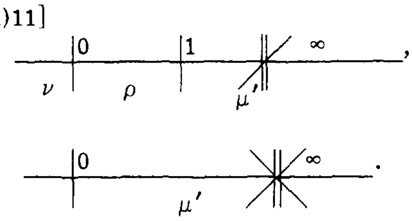
while the diagram representing the coordinate curves of the curvilinear coordinates ρ and ν is



The inequivalent classes of coordinates of this type are now given. In each case the e_i can be standardized as usual to be

$$e_1 = \infty, \quad e_4 = 1, \quad e_5 = 0. \tag{3.81}$$

(a) [(21)11]



From formulas (3.79) the pentaspherical coordinates for the coordinate system are given by

$$\begin{aligned}
&\sigma x_1^2 = 1, \quad 2\sigma x_1 x_2 = 1 - \nu - \rho - \mu', \\
&\sigma x_3^2 = \mu', \quad \sigma x_4^2 = (1 - \rho)(\nu - 1), \quad \sigma x_5^2 = \nu\rho.
\end{aligned} \tag{3.82}$$

With $\nu = -\sinh^2 A$ and $\rho = \sin^2\alpha$ this gives the three space coordinates

$$\begin{aligned}
&t = ix_3/x_1 = K, \quad x = x_4/x_1 = \cosh A\cos\alpha, \\
&y = x_5/x_1 = \sinh A\sin\alpha.
\end{aligned} \tag{3.83}$$

Here $K = \sqrt{\mu'}$. The separation equations have the form

$$\frac{d^2E_1}{dA^2} + (-\tau^2\sinh^2 A + V)E_1 = 0, \tag{3.84}$$

$$\frac{d^2E_2}{d\alpha^2} + (-\tau^2\sin^2\alpha - V)E_2 = 0, \tag{3.85}$$

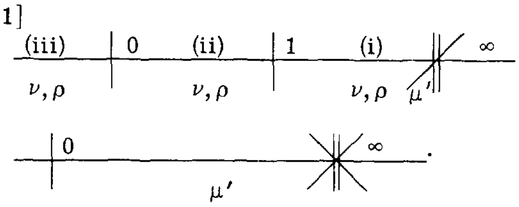
$$\frac{d^2E_3}{dK^2} + \tau^2 E_3 = 0, \tag{3.86}$$

and the basis operators $\hat{A}$, $\hat{B}$ whose eigenvalues are the separation constants V, $-\tau^2$ respectively are

$$\hat{A} = M_{12}^2 - P_2^2, \quad \hat{B} = P_0^2. \tag{3.87}$$

Equations (3.84) and (3.85) are easily seen to be forms of Mathieu's equation. Here as usual $\psi = E_1(A)E_2(\alpha)E_3(K)$ is a solution of (*).

(b) [(21)11]



There are three cases to consider here. If we choose

(i) $\rho = \cosh^2 A$, $\nu = \cosh^2 B$,

(ii) $\rho = \cos^2\alpha$, $\nu = \cos^2\beta$,

(iii) $\rho = \sinh^2 A$, $\nu = -\sinh^2 B$,

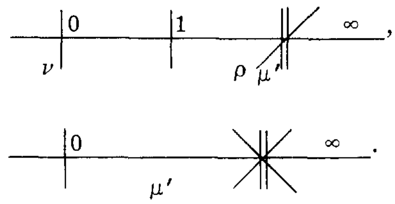
with $\mu' = K^2$ in all cases, then the resulting three space coordinates are

$$\begin{aligned}
&\text{(i)} \quad t = \sinh A\sinh B, \quad x = K, \quad y = \cosh A\cosh B, \\
&\text{(ii)} \quad t = \sin\alpha\sin\beta, \quad x = K, \quad y = \cos\alpha\cos\beta, \\
&\text{(iii)} \quad t = \cosh A\cosh B, \quad x = K, \quad y = \sinh A\sinh B.
\end{aligned} \tag{3.88}$$

The basis operators in this case are

$$\hat{A} = M_{02}^2 - P_0^2, \quad \hat{B} = P_1^2. \tag{3.89}$$

(c) [(21)11]



If we put $\rho=\cosh^2 A$, $\nu=-\sinh^2 B$, and $\mu'=K^2$, then the three space coordinates are

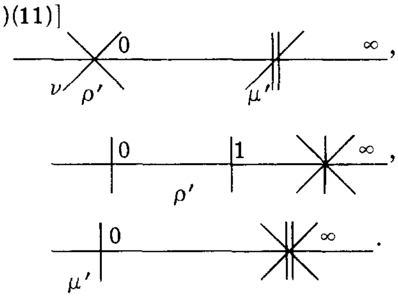
$$t=\sinh A\cosh B,\quad x=K,\quad y=\cosh A\sinh B \tag{3.90}$$

with basis operators

$$\hat{A}=M_{02}^2+P_0^2,\quad \hat{B}=P_1^2. \tag{3.91}$$

6. Coordinate systems corresponding to the configuration [(21)(11)] and [(21) 11]

(a) [(21)(11)]



The pentaspherical coordinates are given by

$$\begin{aligned}&\sigma x_1^2=-1,\quad 2\sigma x_1x_2=\mu'-\nu,\\ &\sigma x_3^2=-\mu',\quad \sigma x_4^2=\nu(1-\rho'),\quad \sigma x_5^2=\nu\rho'.\end{aligned} \tag{3.92}$$

Set

$$\nu=-r^2,\ \rho'=\sin^2\phi,\ \text{and}\ \mu'=-K^2.$$

The corresponding three space coordinates are given by

$$\begin{aligned}&t=x_3/x_1=K,\quad x=ix_4/x_1=r\cos\phi,\\ &\qquad\qquad y=ix_5/x_1=r\sin\phi.\end{aligned} \tag{3.93}$$

The separation equations are

$$\frac{1}{r}\frac{d}{dr}\left(r\frac{dE_1}{dr}\right)-\left(\frac{m^2}{r^2}+S^2\right)E_1=0, \tag{3.94}$$

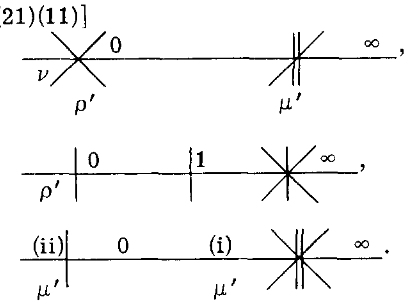
$$\frac{d^2E_2}{d\phi^2}+m^2E_2=0,\quad \frac{d^2E_3}{dK^2}+S^2E_3=0. \tag{3.95}$$

The corresponding basis defining operators $\hat{A}$, $\hat{B}$ with eigenvalues $-m^2$, $-S^2$ respectively are

$$\hat{A}=M_{12}^2,\quad \hat{B}=P_0^2. \tag{3.96}$$

Equation (3.94) is a form of Bessel's equation.

(b) [(21)(11)]



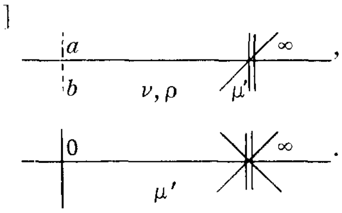
For the case (i) with $\mu'=K^2$, $\rho'=-\sinh^2 A$, and $\nu=-r^2$, the three space coordinates are

$$t=r\sinh A,\quad x=K,\quad y=r\cosh A. \tag{3.97}$$

For (ii) the three space coordinates are as in (3.97) with $\cosh A \leftrightharpoons \sinh A$. The basis operators for this system are

$$\hat{A}=M_{02}^2,\quad \hat{B}=P_1^2 \tag{3.98}$$

(c) [(21)$\hat{1}\hat{1}$]



This is the only coordinate system of this type. The pentaspherical coordinates are given by

$$\begin{aligned}&\sigma x_1^2=-1,\quad 2\sigma x_1x_2=b+a-\nu-\rho-\mu',\\ &\sigma x_3^2=\mu',\\ &\sigma x_4^2=\frac{(\nu-a)(\rho-a)}{(b-a)},\quad \sigma x_5^2=\frac{(\nu-b)(\rho-b)}{(a-b)}.\end{aligned} \tag{3.99}$$

This corresponds to a choice of three space coordinates:

$$(x+it)=\sqrt{2}\,i\,x_4/x_1,\quad y=ix_3/x_1=K, \tag{3.100}$$

where $K=\sqrt{\mu'}$. The separation equations have the form (3.51) with $P(\lambda)=(\lambda-a)(\lambda-b)$ and

$$\frac{d^2E_3}{dK^2}+BE_3=0. \tag{3.101}$$

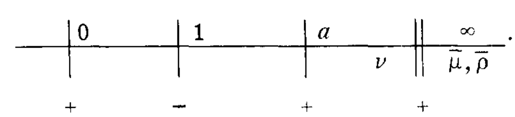
The basis operators $\hat{A}$, $\hat{B}$ are

$$\hat{A}=-M_{01}^2+\alpha(P_1^2-P_0^2)-\beta P_0P_1,\quad \hat{B}=P_2^2. \tag{3.102}$$

7. Coordinate systems on the cone $t^2-x^2-y^2=0$ that can be obtained from the configuration [2111] and its degenerate forms

The method for obtaining coordinate systems on the cone is similar to that for obtaining the coordinate systems with a radial coordinate in three space. The method is illustrated for the first coordinate system of this subsection.

(a)



The pentaspherical coordinates for such a diagram are obtained from (1.37) by putting $\mu=e_1+\bar{\mu}$, $\rho=e_1+\bar{\rho}$ and making the substitutions $x_i\to e_ix_i$ $(i=3,4,5)$ and $x_2\to e_1^2x_2$, $x_1\to x_1$. Then the pentaspherical coordinates assume the form

$$\begin{aligned}&\sigma x_1^2=\bar{\mu}\bar{\rho},\quad 2\sigma x_1x_2=0,\quad \sigma x_3^2=(\nu-a)/a(a-1),\\ &\sigma x_4^2=(1-\nu)/(a-1),\quad \sigma x_5^2=\nu/a.\end{aligned} \tag{3.103}$$

The corresponding choice of three space variables is

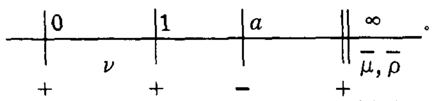
$$\begin{aligned}&t=r\sqrt{(\nu-1)/(a-1)},\quad x=r\sqrt{\nu/a},\\ &y=r\sqrt{(\nu-a)/a(a-1)},\end{aligned} \tag{3.104}$$

where $r^2=1/\bar{\mu}\bar{\rho}$. The separation equations here are given by (3.41) for the variable ν and

$$r^2\frac{d^2E_2}{dr^2}+r\frac{dE_2}{dr}-j(j+1)E_2=0. \tag{3.105}$$

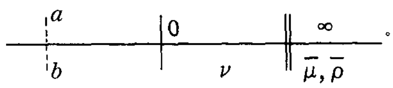
The basis defining operators $\hat{A}$, $\hat{B}$ corresponding to the separation constants $j(j+1)$ and B are

$$\hat{A}=\tfrac{1}{4}-D^2,\quad \hat{B}=M_{02}^2-aM_{12}^2. \tag{3.106}$$

(b) 

The properties of this system are obtained from those of (a) via

(a) → (b), $t\to ix$, $x\to y$, $y\to it$.

(c) 

The pentaspherical coordinates are

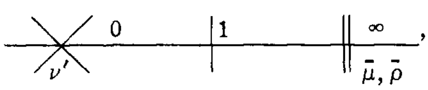
$$\sigma x_1^2=\bar{\mu}\bar{\rho},\quad 2\sigma x_1x_2=0,\quad \sigma x_3^2=(\nu-a)/a(a-b),$$
$$\sigma x_4^2=(\nu-b)/b(b-a),\quad \sigma x_5^2=\nu/ab. \tag{3.107}$$

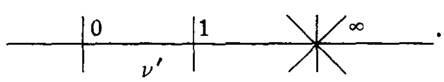
The corresponding three space coordinates are

$$(x+it)=i\sqrt{2}x_4/x_1,\quad y=ix_5/x_1. \tag{3.108}$$

The separation equations are given by (3.41) with $P(\lambda)=\lambda(\lambda-a)(\lambda-b)$ and (3.105), where as usual $r^2=1/\bar{\mu}\bar{\rho}$. The basis defining operators $\hat{A}$, $\hat{B}$ are

$$\hat{A}=\tfrac{1}{4}-D^2,\quad \hat{B}=\alpha(M_{12}^2-M_{02}^2)-\beta(M_{02}M_{12}+M_{12}M_{02}). \tag{3.109}$$

(d) 



The pentaspherical coordinates are

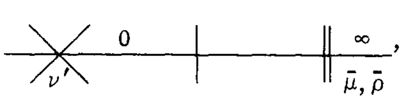
$$\sigma x_1^2=\bar{\mu}\bar{\rho},\quad 2\sigma x_1x_2=0,\quad \sigma x_3^2=-1,$$
$$\sigma x_4^2=(1-\nu')\quad \sigma x_5^2=\nu'. \tag{3.110}$$

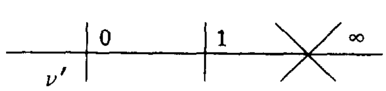
The three space coordinates on the cone are

$$t=r,\quad x=r\cos\phi,\quad y=r\sin\phi,$$

where $1/\bar{\mu}\bar{\rho}=r^2$, $\nu'=\cos^2\phi$. The separation equations are (3.105) and the third equation of (3.75). The basic defining operators are clearly

$$\hat{A}=\tfrac{1}{4}-D^2,\quad \hat{B}=M_{12}^2. \tag{3.111}$$

(e) 



The three space coordinates are

$$t=r\cosh A,\quad x=r\sinh A,\quad y=\pm r$$

with the basis defining operators

$$\hat{A}=\tfrac{1}{4}-D^2,\quad \hat{B}=M_{01}^2. \tag{3.112}$$

C. The configurations [311], [3$\widehat{11}$], and their degenerate forms

Here we give the configurations of the form [311] and [3$\widehat{11}$] which are inequivalent with respect to the by now familiar equivalence relation. It is possible to standardize the parameters e_i such that

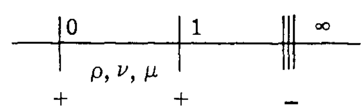
$$e_1=\infty,\quad e_4=1,\quad e_5=0, \tag{3.113}$$

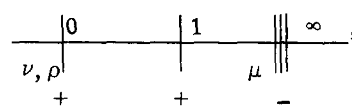
where, of course, e_1 is the parameter associated with the invariant factor index 3. The two operators whose eigenvalues are the separation constants are given as in (3.5) with $f(\lambda)=\lambda(\lambda-1)$. The separation equations with the above choice of the e_i are

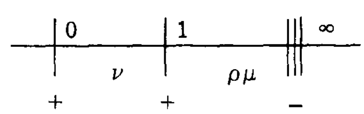
$$\sqrt{f(\lambda)}\frac{d}{d\lambda}\left(\sqrt{f(\lambda)}\frac{dE_i}{d\lambda}\right)-(A\lambda+B)E_i=0 \tag{3.114}$$

with $f(\lambda)$ as above and $E_1(\mu)E_2(\nu)E_3(\rho)$ as the separable solution of (*). For the configuration [311] the separation equations are as in (3.114) with $f(\lambda)=(\lambda-a)(\lambda-b)$, where as usual $a=b^*=\alpha+i\beta$, $\alpha,\beta\in\mathbf{R}$. We now list the inequivalent systems of this type.

(a) [311]

(i) 

(ii) 

(iii) 

The pentaspherical coordinates are given by

$$\sigma x_1^2=-1,\quad 2\sigma x_1x_2=\mu+\nu+\rho,$$
$$\sigma(2x_1x_3+x_2^2)=\nu+\mu+\rho-(\mu\nu+\mu\rho+\nu\rho)-1, \tag{3.115}$$
$$\sigma x_4^2=-(\mu-1)(\nu-1)(\rho-1),\quad \sigma x_5^2=\mu\nu\rho.$$

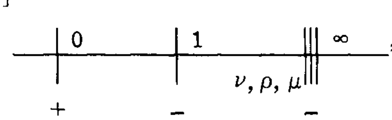
The three space coordinates are given by

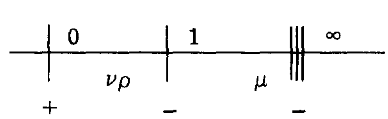
$$2t=2x_2/x_1=-\mu-\nu-\rho, \tag{3.116}$$
$$x=ix_4/x_1=\sqrt{(\mu-1)(\nu-1)(1-\rho)},\quad y=x_5/x_1=\sqrt{\mu\nu\rho}.$$

The basis operators are

$$\hat{A}=P_0M_{20}+M_{20}P_0-P_1M_{02}-M_{02}P_1+P_0^2+P_1^2,$$
$$\hat{B}=-M_{12}^2+P_2^2+P_2M_{02}+M_{02}P_2. \tag{3.117}$$

(b) [311]

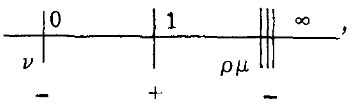
(i) 

(ii) 

The properties of this system can be deduced from those of (a) via the transformation

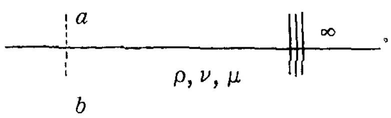
(a) → (b), $t \to y$, $x \to ix$, $y \to t$.

(c) [311]



(a) → (b), $t \to it$, $x \to ix$, $y \to iy$.

(d) $[3\widehat{11}]$



The pentaspherical coordinates are given by

$$\begin{aligned}
&\sigma x_1^2 = -1, \quad 2\sigma x_1 x_2 = \mu + \nu + \rho,\\
&\sigma(2x_1x_3 + x_2^2) = \mu\rho + \nu\rho + \nu\mu - (a+b)(\mu+\nu+\rho)\\
&\qquad + a^2 + ab + b^2, \qquad (3.118)\\
&\sigma x_4^2 = (\mu - a)(\rho - a)(\nu - a)/(b - a),\\
&\sigma x_5^2 = (\mu - b)(\rho - b)(\nu - b)/(a - b),
\end{aligned}$$

where $a = \alpha + i\beta$, $b = \alpha - i\beta$, $\alpha, \beta \in \mathbf{R}$. The three space coordinates are given by

$$(x + it) = i\sqrt{2}\, x_4/x_1, \quad y = -x_2/x_1 \qquad (3.119)$$

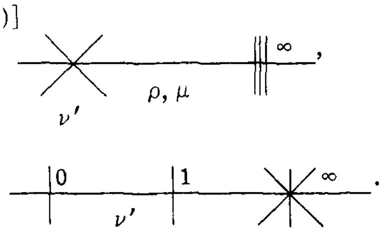
with the basis operators given by

$$\begin{aligned}
\hat{A} &= M_{12}P_1 + P_1M_{12} + M_{02}P_0 + P_0M_{02} + 2\alpha P_2^2\\
&\quad + \alpha(P_0^2 - P_1^2) - 2\beta P_0P_1, \qquad (3.120)\\
\hat{B} &= \alpha(M_{12}P_1 + P_1M_{12} + M_{02}P_0 + P_0M_{02}) + \beta(M_{02}P_1 + P_1M_{02}\\
&\quad - M_{12}P_0 - P_0M_{12}) + (\alpha^2 + \beta^2)P_2^2 - M_{01}^2\\
&\quad + (\alpha^2 - \beta^2)(P_1^2 - P_0^2) - 4\alpha\beta P_0P_1.
\end{aligned}$$

1. Degenerate systems of type [3(11)]

The coordinate systems of this type are chosen such that the free parameters e_1 and e_5 are ∞ and 0 respectively.

(a) [3(11)]



The pentaspherical coordinates are given by

$$\begin{aligned}
&\sigma x_1^2 = -1, \quad 2\sigma x_1x_2 = -\mu - \rho - 1, \qquad (3.121)\\
&\sigma(2x_1x_3 + x_2^2) = -\mu\rho, \quad \sigma x_4^2 = \mu\rho(1 - \nu'),\\
&\sigma x_5^2 = \mu\rho\nu'.
\end{aligned}$$

The three space coordinates are given by the formulas

$$t = \pm x_2/x_1, \quad x = ix_4/x_1, \quad y = ix_5/x_1. \qquad (3.122)$$

Translating t by $\pm\frac{1}{2}$ and putting $\mu = \xi^2$, $\rho = \eta^2$, and $\nu' = \sin^2\alpha$. We obtain the more familiar form,

$$t = \pm\tfrac{1}{2}(\xi^2 + \eta^2), \quad x = \xi\eta\cos\alpha, \quad y = \xi\eta\sin\alpha. \qquad (3.123)$$

The separation equations for this system assume the form

$$\frac{d^2E_i}{d\lambda^2} + \frac{1}{\lambda}\frac{dE_i}{d\lambda} + \left(q^2 - \frac{m^2}{\lambda^2}\right)E_i = 0, \qquad (3.124)$$

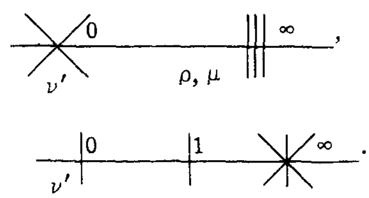
where $\lambda = \xi, \eta$ and $i = 1, 2$ respectively, and

$$\frac{d^2E_3}{d\alpha^2} + m^2E_3 = 0.$$

Equation (3.124) is Bessel's equations with linearly independent solutions $J_m(q\lambda)$, $Y_m(q\lambda)$. The solutions of the third equation are $E_3 = \exp(\pm im\alpha)$. The basis operators $\hat{A}$, $\hat{B}$ whose eigenvalues are q^2 and $-m^2$, respectively, are

$$\begin{aligned}
&\hat{A} = M_{01}P_1 + P_1M_{01} + M_{02}P_2 + P_2M_{02} = P_0D + DP_0,\\
&\hat{B} = M_{12}^2. \qquad (3.125)
\end{aligned}$$

(b) [3(11)]



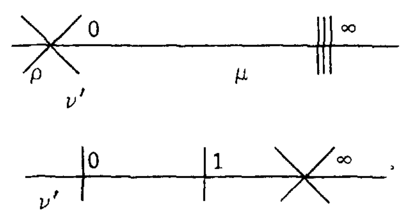
The three space coordinates in this case are

$$\begin{aligned}
&y = (x_2/x_1 - \tfrac{1}{2}) = \pm\tfrac{1}{2}(\xi^2 + \eta^2), \quad t = ix_4/x_1 = \xi\eta\cosh A,\\
&x = x_5/x_1 = \xi\eta\sinh A. \qquad (3.126)
\end{aligned}$$

where $\mu = \xi^2$, $\rho = \eta^2$, and $\nu' = -\sinh^2 A$. The resulting basis operators are

$$\begin{aligned}
&\hat{A} = M_{02}P_0 + P_0M_{02} + M_{12}P_1 + P_1M_{12},\\
&\hat{B} = M_{01}^2. \qquad (3.127)
\end{aligned}$$

(c)



This system corresponds to the choice of three space coordinates

$$\begin{aligned}
&t = ix_5/x_1 = \xi\eta\sinh A, \quad x = (x_2/x_1 - \tfrac{1}{2}) = \tfrac{1}{2}(\xi^2 - \eta^2)\\
&y = x_4/x_1 = \xi\eta\cosh A, \qquad (3.128)
\end{aligned}$$

$\mu = \xi^2$, $\rho = -\eta^2$, and $\nu' = -\sinh A$. The basis operators are

$$\hat{A} = M_{12}P_2 + P_2M_{12} + M_{01}P_0 + P_0M_{01}, \quad \hat{B} = M_{02}^2. \qquad (3.129)$$

2. Degenerate systems of type [(31)1]

The formulas for the pentaspherical coordinates corresponding to the degenerate configuration [(31)1] are obtained from these of (1.38) via the substitution

$$e_4 = e_1 + \epsilon, \quad \lambda = e_1 + \epsilon + \epsilon^3\lambda', \qquad (3.130)$$

where for definiteness we can take $\lambda = \nu$. The resulting expression for the pentaspherical coordinates is

$$\sigma x_1^2=\frac{(\mu-e_1)(\rho-e_1)}{(e_5-e_1)},\quad 2\sigma x_1x_2=\frac{\partial}{\partial e_1}\left(\frac{(\mu-e_1)(\rho-e_1)}{(e_5-e_1)}\right), \tag{3.131}$$

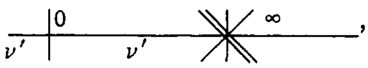
$$\sigma(2x_1x_3+x_2^2)=\frac{1}{2}\frac{\partial^2}{\partial e_1^2}\left(\frac{(\mu-e_1)(\rho-e_1)}{(e_5-e_1)}\right)+\frac{(\mu-e_1)(\rho-e_1)}{(e_5-e_1)}\nu',$$

$$\sigma x_4^2=-\frac{(\mu-e_1)(\rho-e_1)}{(e_5-e_1)}\nu',\quad \sigma x_5^2=\frac{(\mu-e_5)(\rho-e_5)}{(e_1-e_5)^3}.$$

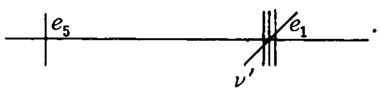
The coordinate curve for thecoordinate ν' is

$$x_1^2+x_4^2/\nu'=0, \tag{3.132}$$

and the diagram corresponding to such a curve is

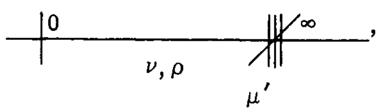


while the diagram representing the coordinate curves of the curvilinear coordinates μ,ρ is

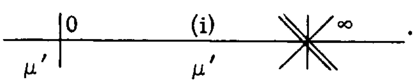


The inequivalent classes of this type are now given. In each case e_1 and e_5 can be taken to be ∞ and 0, respectively.

(a) [(31)1]



(ii)



The pentaspherical coordinates are given by

$$\begin{aligned}&\sigma x_1^2=-1,\quad 2\sigma x_1x_2=-\nu-\rho-1,\\&\sigma(2x_1x_3+x_2^2)=-\mu'-\nu\rho,\quad \sigma x_4^2=\mu',\quad \sigma x_5^2=\nu\rho.\end{aligned} \tag{3.133}$$

The corresponding three space coordinates are

$$\begin{aligned}&\text{(i) } t=ix_5/x_1=\xi\eta,\quad x=ix_4/x_1=k,\\&\quad y=(x_2/x_1-\tfrac{1}{2})=\pm\tfrac{1}{2}(\xi^2+\eta^2),\\&\text{(ii) } t=\pm\tfrac{1}{2}(x_2/x_1-\tfrac{1}{2})=\pm\tfrac{1}{2}(\xi^2+\eta^2),\\&\quad x=x_4/x_1=k,\quad y=ix_5/x_1=\xi\eta,\end{aligned} \tag{3.134}$$

where $\rho=\xi^2$, $\nu=\eta^2$, and $k=(|\mu'|)^{1/2}$. The separation equations are

$$\frac{d^2E_1}{d\lambda^2}+(Q-\tau^2\lambda^2)E_i=0 \tag{3.135a}$$

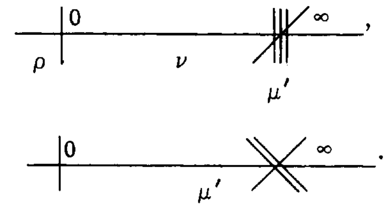
for $\lambda=\xi,\eta$ and $i=1,2$ respectively, and

$$\frac{d^2E_3}{dk^2}+\tau^2E_3=0. \tag{3.135b}$$

Equation (3.135a) is well known to have solutions expressed as parabolic cylinder functions. The basis operators $\hat{A}$, $\hat{B}$ whose eigenvalues are the separation constants Q and $-\tau^2$, respectively, are

$$\hat{A}=M_{02}P_0+P_0M_{02},\quad \hat{B}=P_1^2. \tag{3.136}$$

(b)



In this case the three space coordinates are given by

$$\begin{aligned}&t=ix_4/x_1=k,\quad x=(x_2/x_1-\tfrac{1}{2})=\tfrac{1}{2}(\xi^2-\eta^2),\\&\qquad y=ix_5/x_1=\xi\eta\end{aligned} \tag{3.137}$$

with $\nu=\xi^2$, $\rho=-\eta^2$, and $k=(\mu')^{1/2}$. The basis operators are

$$\hat{A}=M_{12}P_2+P_2M_{12},\quad \hat{B}=P_0^2. \tag{3.138}$$

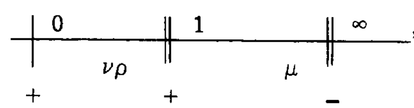
D. The configuration [221] and its degenerate forms

1. Systems of the type [221]

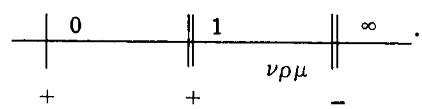
The inequivalent coordinate systems of type [221] are given in the following list. In each case the three free parameters e_1, e_3, and e_5 are standardized to be ∞, 1, and 0, respectively.

(a) [221]

(i)



(ii)



The pentaspherical coordinates are given by

$$\begin{aligned}&\sigma x_1^2=-1,\quad 2\sigma x_1x_2=-\mu-\nu-\rho+2,\\&\sigma x_3^2=(\mu-1)(\nu-1)(\rho-1),\\&2\sigma x_3x_5=\mu+\nu+\rho-\mu\nu\rho-2,\quad \sigma x_5^2=\mu\nu\rho.\end{aligned} \tag{3.139}$$

A suitable choice of three space coordinates is

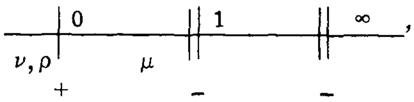
$$\begin{aligned}&(t+x)^2=-x_3^2/x_1^2=(\mu-1)(\nu-1)(\rho-1),\\&x^2-t^2=-2x_3x_5/x_1^2=\mu+\nu+\rho-\mu\nu\rho-2,\\&y=\pm ix_5/x_1=\pm\sqrt{\mu\nu\rho}.\end{aligned} \tag{3.140}$$

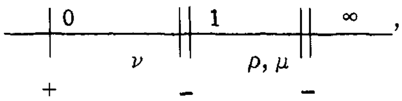
The separation equations have the form (3.51) with $p(\lambda)=\lambda(\lambda-1)^2$, i.e., the associated Legendre equation. The defining operators are

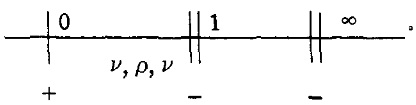
$$\hat{A}=2(P_2^2-P_0^2-P_1P_0)+M_{12}^2-M_{01}^2-M_{02}^2,$$

$$\hat{B}=P_2^2-2M_{02}^2-M_{12}M_{02}-M_{02}M_{12}. \tag{3.141}$$

(b) [221]

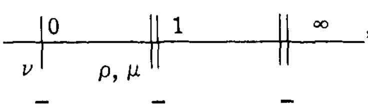
(i) 

(ii) 

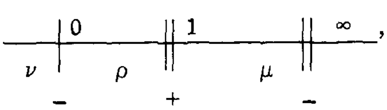
(iii) 

This system is related to (a) via the transformation

(a) → (b), $t \to ix$, $x \to it$, $y \to y$.

(c) [221] 

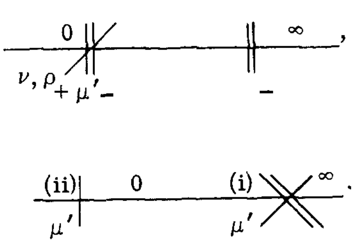
(a) → (c), $t \to it$, $x \to ix$, $y \to iy$.

(d) [221] 

(a) → (d), $t \to x$, $x \to t$, $y \to iy$.

2. Coordinate systems corresponding to the configuration [2(21)]

Here the two free parameters e_1 and e_3 may be taken as 0 and ∞ (not necessarily respectively, as will be seen).

(a) [2(21)] 

The pentaspherical coordinates are given by

$$\sigma x_1^2 = -1, \quad 2\sigma x_1 x_2 = -\nu - \rho, \quad \sigma x_3^2 = -\nu\rho,$$

$$2\sigma x_3 x_4 = \nu + \rho - \mu'\nu\rho, \quad \sigma x_5^2 = 2\nu\rho\mu'. \tag{3.142}$$

A suitable choice of three space coordinates is

$$(t-x)^2 = x_3^2/x_1^2 = \nu\rho,$$
$$(t^2 - x^2) = 2x_3x_4/x_1^2 = \mu'\nu\rho - \nu - \rho, \tag{3.143}$$
$$y = \pm ix_5/\sqrt{2}\,x_1 = \pm\sqrt{\mu'\nu\rho}\,.$$

The separation equations for this system are of the form (3.51) with $p(\lambda) = \lambda^3$ for the variables ν and ρ. This equation can be related to Bessel's equation. The separation equation in the variable μ' is

$$\mu'^{1/2}\frac{d}{d\mu'}\left(\mu'^{1/2}\frac{dE_3}{d\mu'}\right) + BE_3 = 0. \tag{3.144}$$

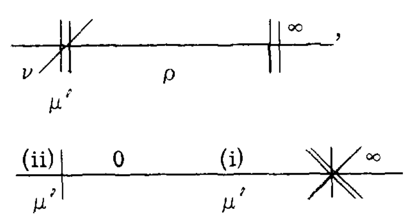
The basis operators $\hat{A}$, $\hat{B}$ are

$$\hat{A} = M_{01}^2 - (P_0 + P_1)^2, \quad \hat{B} = (M_{12} - M_{20})^2. \tag{3.145}$$

For (ii) the results follow from (i) via the transformation

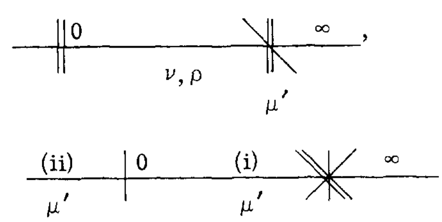
(i) → (ii), $t \to x$, $x \to t$, $y \to iy$.

This does not change the operators $\hat{A}$ and $\hat{B}$ but gives new expressions for the three space coordinates.

(b) [2(21)] 

(b) (i) and (b) (ii) are obtained from (a) (i) and (a) (ii), respectively, via the transformation

$t \to it$, $x \to ix$, $y \to iy$.

(c) [2(21)] 

The pentaspherical coordinates in this case are given by

$$\sigma x_1^2 = \nu\rho, \quad 2\sigma x_1 x_2 = \nu + \rho, \quad \sigma x_3^2 = -1,$$
$$2\sigma x_3 x_4 = \mu' - \nu - \rho, \quad \sigma x_5^2 = \mu'. \tag{3.146}$$

For (i) a suitable choice of three space coordinates is

$$(t-x)^2 = -x_1^2/x_3^2 = \nu\rho, \tag{3.147}$$
$$(x^2 - t^2) = -2x_1x_2/x_3^2 = \nu + \rho, \quad y = \pm ix_5/x_3 = \pm(\mu')^{1/2}.$$

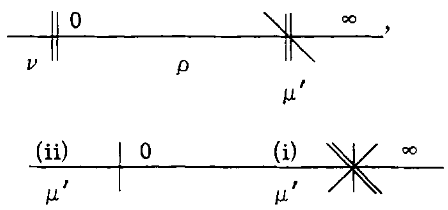
The separation equations for this system are of the form (3.51) with $p(\lambda) = \lambda^2$, $\lambda = \nu, \rho$. This is a form of Bessel's equation. For the variable μ' the equation has the form (3.144) and the basis defining operators $\hat{A}$, $\hat{B}$ are

$$\hat{A} = M_{01}^2 - (P_0 + P_1)^2, \quad \hat{B} = P_2^2. \tag{3.148}$$

The corresponding properties for (ii) are obtained from those of (i) via the transformation

$$t \to x, \quad x \to t, \quad y \to iy.$$

(d) [2(21)]

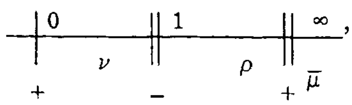


(d) (i) and (d) (ii) are obtained from (c) (i) and (c) (ii), respectively, via the transformation

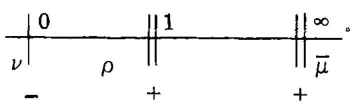
$$t \to it, \quad x \to ix, \quad y \to iy.$$

3. Coordinate systems having a radial coordinate in three space and derivable from the configurations [221] and [2(21)]

(a) (i)



(ii)



The pentaspherical coordinates are given by

$$\sigma x_1^2 = \bar{\mu}, \quad 2\sigma x_1 x_2 = 1, \quad \sigma x_3^2 = (\nu - 1)(\rho - 1),$$
$$2\sigma x_3 x_4 = -1 - \nu\rho, \quad \sigma x_5^2 = \nu\rho. \tag{3.149}$$

For the case (i) a suitable choice of three space coordinates is given by the equations

$$(t + x)^2 = x_3^2/x_1^2 = r^2(\nu - 1)(1 - \rho),$$
$$y^2 = x_5^2/x_1^2 = r^2\nu\rho, \quad x^2 + y^2 - t^2 = 2x_2/x_1 = r^2, \tag{3.150}$$

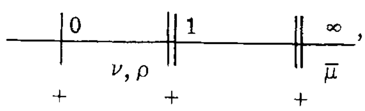
where $r^2 = 1/\bar{\mu}$. The separation equations have the form (3.51) with $p(\lambda) = (\lambda - 1)^2\lambda$ and $\lambda = \rho, \nu$ and the first of equations (3.75) for the radial coordinate. The equations in ρ and ν are associated Legendre function equations. The basis defining operators $\hat{A}$, $\hat{B}$ are

$$\hat{A} = \tfrac{1}{4} - D^2, \quad \hat{B} = 2M_{12}^2 + M_{12}M_{20} + M_{20}M_{12}. \tag{3.151}$$

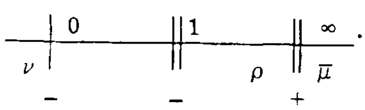
System (ii) is related to (i) via the transformation

$$t \to it, \quad x \to ix, \quad y \to iy.$$

(b) (i)



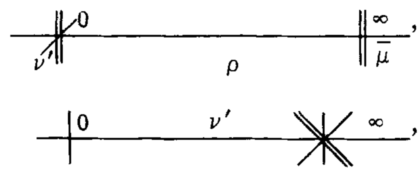
(ii)



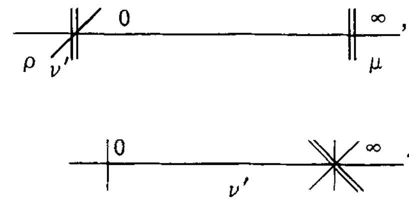
The coordinate systems (b) (i) and (b) (ii) are related to (a) (i) and (a) (ii), respectively, via the transformation

$$t \to ix, \quad x \to it, \quad y \to y.$$

(c) (i)



(ii)



The pentaspherical coordinates are given by

$$\sigma x_1^2 = \bar{\mu}, \quad 2\sigma x_1 x_2 = 1, \quad \sigma x_3^2 = \rho,$$
$$2\sigma x_3 x_4 = -\nu'\rho - 1, \quad \sigma x_5^2 = \nu'\rho. \tag{3.152}$$

A suitable choice of three space coordinates for system (i) is

$$x - t = x_3/x_1 = re^a, \qquad x^2 - t^2 = -2x_3x_4/x_1^2,$$
$$= r^2(1 + s^2e^{2a}), \quad y = x_5/x_1 = rse^a, \tag{3.153}$$

where $s = \sqrt{\nu'}$, $\sqrt{\rho} = e^a$, and $1/\bar{\mu} = r^2$. This system corresponds to horospherical coordinates on the unit hyperboloid. The separation equations are

$$\frac{d^2E_1}{da^2} + \frac{dE_1}{da} - (e^{-2a}B + A)E_1 = 0,$$
$$\frac{d^2E_3}{ds^2} + BE_3 = 0, \tag{3.154}$$

the equation for $E_2(r)$ being identical with (3.105). The basis operators $\hat{A}$, $\hat{B}$ are

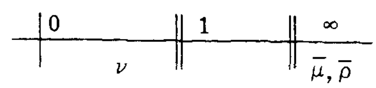
$$\hat{A} = \tfrac{1}{4} - D^2, \quad \hat{B} = (M_{01} - M_{12})^2. \tag{3.155}$$

System (ii) is related to (i) via the transformation

$$t \to it, \quad x \to ix, \quad y \to iy.$$

4. Coordinate systems on the cone $t^2 - x^2 - y^2 = 0$ obtainable from [221] and its degenerate forms

The method of obtaining such coordinates follows along the lines outlined previously. Consider the diagram



The pentaspherical coordinates are given by

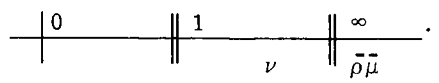
$$\sigma x_1^2 = \bar{\mu}\bar{\rho}, \quad 2\sigma x_1 x_2 = 0, \quad \sigma x_3^2 = \nu - 1,$$
$$2\sigma x_3 x_4 = -\nu, \quad \sigma x_5^2 = \nu. \tag{3.156}$$

A suitable choice of three space coordinates is

$$(t + x)^2 = -x_3^2/x_1^2 = r^2(1 - \nu), \quad y^2 = x_5^2/x_1^2 = r^2\nu,$$
$$x^2 + y^2 - t^2 = 2x_2/x_1 = 0, \tag{3.157}$$

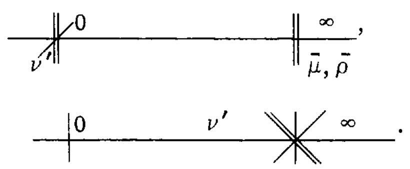
where $1/\bar{\mu}\bar{\rho} = r^2$. The separation equations are given by (3.51) with $P(\lambda) = (\lambda - 1)^2\lambda$ for the coordinate $\lambda = \nu$ and for the radial coordinate the equation is (3.105). The basis operators are

$$\hat{A}=\tfrac{1}{4}-D^2, \quad \hat{B}=2M_{12}^2+M_{12}M_{20}+M_{20}M_{12}. \tag{3.158}$$

(b) 

This system is related to (a) via the transformation

(a) → (b), $t\to ix$, $x\to it$, $y\to y$.

(c) 

The pentaspherical coordinates are given by

$$\sigma x_1^2=\bar{\mu}\bar{\rho}, \quad 2\sigma x_1x_2=0, \quad \sigma x_3^2=-1,$$
$$2\sigma x_3x_4=-\nu', \quad \sigma x_5^2=\nu'. \tag{3.159}$$

A suitable choice of three space coordinates is

$$t-x=ix_3/x_1=r, \quad t^2-x^2=-2x_3x_4/x_1^2=r^2\nu',$$
$$y=x_5/x_1=r(\nu')^{1/2}. \tag{3.160}$$

The separation equations have the form (3.51) with $P(\lambda)=\lambda^3$. The equation in the radial coordinate is (3.105), and the basis defining operators $\hat{A}$, $\hat{B}$ are

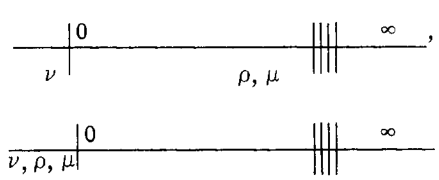
$$\hat{A}=\tfrac{1}{4}-D^2, \quad \hat{B}=(M_{01}-M_{12})^2. \tag{3.161}$$

E. The configuration [41] and the degenerate form [(41)]

1. The configuration [41]

The inequivalent coordinate systems of type [41] are given in the following list. In each case the two free parameters e_1 and e_5 can be chosen to be ∞ and 0, respectively.

(a) [41]

(i), (ii) 

The corresponding pentaspherical coordinates are

$$\sigma x_1^2=-1, \quad 2\sigma x_1x_2=-(\mu+\nu+\rho+2),$$
$$\sigma(2x_1x_3+x_2^2)=-(\nu\rho+\mu\nu+\mu\rho+\rho+\mu+\nu+1), \tag{3.162}$$
$$2\sigma(x_1x_4+x_2x_3)=\mu\nu\rho, \quad \sigma x_5^2=-\mu\nu\rho.$$

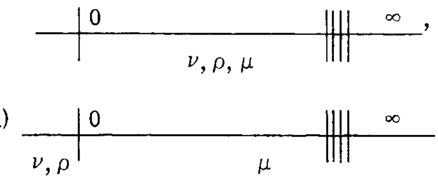
A suitable choice of three space coordinates is

$$2(t+x)=4x_3/x_1=\mu\nu+\mu\rho+\nu\rho-\tfrac{1}{2}(\mu^2+\nu^2+\rho^2),$$
$$2(x-t)=2(x_2/x_1-1)=\mu+\nu+\rho, \tag{3.163}$$
$$y^2=-x_5^2/x_1^2=-\mu\nu\rho.$$

Here the second equation has been subjected to a translation. This is merely a convenience. The separation equations for this coordinate system have the form (3.51) with $p(\lambda)=\lambda$. The solutions are expressible in terms of Bessel functions. The corresponding basis operators are

$$\hat{A}=(P_0+P_1)M_{10}+M_{10}(P_1+P_0)-2P_2(M_{12}-M_{20})$$
$$-2(M_{12}-M_{20})P_2-(P_1-P_0)^2, \tag{3.164}$$
$$\hat{B}=(M_{12}-M_{20})^2-P_2(M_{12}+M_{20})-(M_{12}+M_{20})P_2.$$

(b)

(i), (ii) 

This system is related to (a) via the transformation

(a) → (b), $t\to x$, $x\to t$, $y\to iy$.

2. The degenerate case with configuration [(41)]

The pentaspherical coordinates corresponding to such a system are obtained by making substitutions

$$e_5=e_1+\epsilon, \quad \lambda=e_1+\epsilon+\epsilon^4\lambda', \tag{3.165}$$

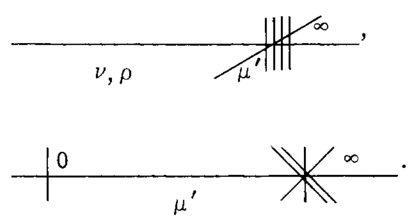
where $\lambda=\mu$, say. The resulting expression for the pentaspherical coordinates is

$$\sigma x_1^2=-(\nu-e_1)(\rho-e_1), \quad 2\sigma x_1x_2=-(\nu-e_1)-(\rho-e_1),$$
$$\sigma(2x_1x_3+x_2^2)=1, \quad 2\sigma(x_1x_4+x_2x_3)=-\mu'(\nu-e_1)(\rho-e_1),$$
$$\sigma x_5^2=\mu'(\nu-e_1)(\rho-e_1). \tag{3.166}$$

If we further specialize to the case $e_1=\infty$, these equations simplify to

$$\sigma x_1^2=-1, \quad 2\sigma x_1x_2=\rho+\nu, \quad \sigma(2x_1x_3+x_2^2)=\rho\nu,$$
$$2\sigma(x_1x_4+x_2x_3)=-\mu', \quad \sigma x_5^2=\mu'. \tag{3.167}$$

The one coordinate system of this type corresponds to the diagrams



A suitable choice of three space coordinates is

$$(y-t)=-4x_2/x_1=2(\rho+\nu), \quad (y+t)=2ix_3/x_1,$$
$$x=ix_5/x_1=k. \tag{3.168}$$

The separation equations have the form (3.51) for the variables ν, ρ with $p(\lambda)=$ const and (3.135b) in the variable k. The basis operators $\hat{A}$, $\hat{B}$ are

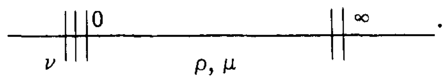
$$\hat{A}=M_{02}(P_0+P_2)+(P_0+P_2)M_{02}+(P_0-P_2)^2,$$
$$\hat{B}=P_1^2. \tag{3.169}$$

F. The configuration [32] and associated coordinate systems

1. The configuration [32]

As usual in the classification of inequivalent coordinate systems the two free parameters e_1 and e_4 can be standardized to be 0 and ∞ (not necessarily respectively).

(a) [32]



The pentaspherical coordinates are given by

$$\sigma x_1^2=\mu\nu\rho,\quad 2\sigma x_1x_2=-(\mu\nu+\mu\rho+\nu\rho),\tag{3.170}$$
$$\sigma(2x_1x_3+x_2^2)=-\mu-\nu-\rho,\quad \sigma x_4^2=-1,\quad 2\sigma x_4x_5=\mu+\nu+\rho.$$

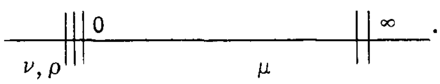
The suitable choice of three space coordinates is

$$(t-x)^2=x_1^2/x_4^2=-\mu\nu\rho,\quad 2y(x-t)=2x_1x_2/x_4^2=\mu\nu+\nu\rho+\mu\rho,$$
$$t^2-x^2-y^2=-x_5/x_4=\mu+\nu+\rho.\tag{3.171}$$

The separation equations are given by (3.51) with $p(\lambda)=\lambda^3$. The corresponding basis defining operators are

$$\hat{A}=M_{12}^2-M_{01}^2-M_{02}^2-2P_2(P_0+P_1),$$
$$\hat{B}=M_{01}(M_{12}-M_{02})+(M_{12}-M_{02})M_{01}-(P_0+P_1)^2.\tag{3.172}$$

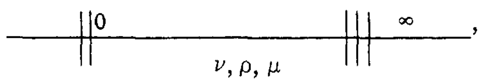
(b) [32]



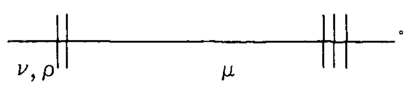
This system is related to (a) via the transformation (a) → (b), $t\to it$, $x\to ix$, $y\to iy$.

(c) [32]

(i)



(ii)



The pentaspherical coordinates are given by

$$\begin{aligned}&\sigma x_1^2=-1,\quad 2\sigma x_1x_2=-(\mu+\nu+\rho),\\&\sigma(2x_1x_3+x_2^2)=-\mu\nu-\mu\rho-\nu\rho,\\&\sigma x_4^2=-\mu\nu\rho,\quad 2\sigma x_4x_5=\mu\nu+\mu\rho+\nu\rho.\end{aligned}\tag{3.173}$$

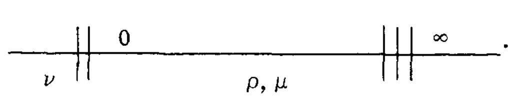
The suitable choice of three space coordinates is

$$(t-x)^2=x_4^2/x_1^2=\mu\nu\rho,\quad t^2-x^2=-2x_4x_5/x_1^2=\mu\nu+\mu\rho+\nu\rho,$$
$$2y=2x_2/x_1=\mu+\nu+\rho\tag{3.174}$$

with the pentaspherical coordinates chosen as in (3.173). The separation equations are (3.41) with $f(\lambda)=\lambda^2$ and the basis operators are

$$\hat{A}=(P_0+P_1)(M_{12}+M_{02})+(M_{12}+M_{02})(P_0+P_1)$$
$$+3(P_0+P_1)^2\tag{3.175}$$
$$\hat{B}=2(P_1+P_0)(M_{12}-M_{02})+2(M_{12}-M_{02})(P_0+P_1)+M_{01}^2.$$

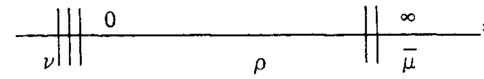
(d)



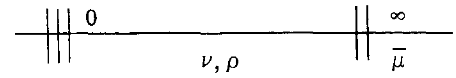
(c) → (d), $t\to ix$, $x\to it$, $y\to y$.

2. Coordinate systems of type [32] corresponding to a radial coordinate in three space

(a) (i)



(ii)



The pentaspherical coordinates are given by

$$\begin{aligned}&\sigma x_1^2=\nu\rho,\quad 2\sigma x_1x_2=-\nu-\rho,\\&\sigma(2x_1x_3+x_2^2)=1,\quad \sigma x_4^2=\bar{\mu},\quad 2\sigma x_4x_5=2.\end{aligned}\tag{3.176}$$

For (i) this corresponds to a choice of three space coordinates

$$(t-x)^2=-x_1^2/x_4^2=-\nu\rho r^2,\quad 2y(x-t)=2x_1x_2/x_4^2=(\nu+\rho)r^2,$$
$$x^2+y^2-t^2=-x_5/x_4.\tag{3.177}$$

The separation equations are given by (3.51) with $p(\lambda)=\lambda^3$ for $\lambda=\nu,\rho$ and the equation in the variable r is (3.105). The basis operators $\hat{A}$, $\hat{B}$ are

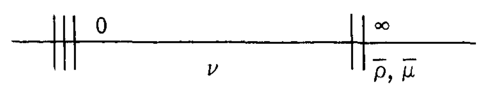
$$\hat{A}=\tfrac14-D^2,$$
$$\hat{B}=M_{01}(M_{12}-M_{02})+(M_{12}-M_{02})M_{01}.\tag{3.178}$$

The corresponding results for (ii) follow via the transformation $t\to it$, $x\to ix$, $y\to iy$.

3. Coordinates on the cone arising from the configuration [32]

There is one case to consider here.

(a)



The pentaspherical coordinates are given by

$$\begin{aligned}&\sigma x_1^2=\nu,\quad 2\sigma x_1x_2=-1,\quad \sigma(2x_1x_3+x_2^2)=0,\\&\sigma x_4^2=\bar{\mu}\bar{\rho},\quad 2\sigma x_4x_5=0.\end{aligned}\tag{3.179}$$

The associated choice of three space variables is

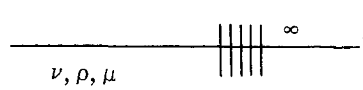
$$\begin{aligned}&(t-x)^2=x_1^2/x_4^2=\nu r^2,\quad 2y(x-t)=2x_1x_2/x_4^2=-r^2,\\&t^2-x^2-y^2=x_5/x_4=0.\end{aligned}\tag{3.180}$$

The separation equations are (3.51) with $p(\lambda)=\lambda^3$ for $\lambda=\nu$ and (3.105) for the variable r. The basis operators are

$$\hat{A}=\tfrac14-D^2,$$
$$\hat{B}=M_{01}(M_{12}-M_{02})+(M_{12}-M_{02})M_{01}.\tag{3.181}$$

G. The configuration [5]

There is only one coordinate system for such a configuration and it has the diagram



The pentaspherical coordinates are given by

$$\sigma x_1^2 = 1, \quad 2\sigma x_1 x_2 = -(\mu+\nu+\rho), \tag{3.182}$$
$$\sigma(x_2^2 + 2x_1x_3) = \mu\nu + \mu\rho + \nu\rho, \quad \sigma(2x_2x_3 + 2x_1x_4) = \mu\nu\rho.$$

This gives the three space variables

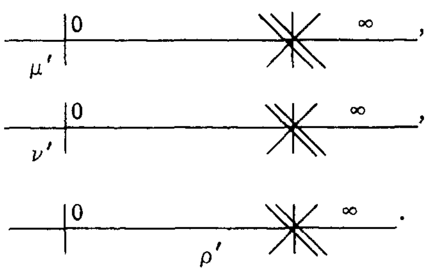
$$\begin{aligned} 2(t-x) &= -2x_2/x_1 = \mu+\nu+\rho, \\ 2(t+x) &= -2x_4/x_1 = -\tfrac{1}{2}\mu\nu\rho + \tfrac{1}{4}[\nu^2(\rho+\mu) + \rho^2(\mu+\nu) \\ &\quad + \mu^2(\nu+\rho) - (\mu^3+\nu^3+\rho^3)], \\ 4y &= 4x_3/x_1 = \mu\nu + \mu\rho + \nu\rho - \tfrac{1}{2}(\mu^2+\lambda^2+\rho^2). \end{aligned} \tag{3.183}$$

The separation equations are (3.41) with $f(\lambda)=1$. This gives the product of three solutions of Airy's equation. The resulting basis operators are

$$\begin{aligned} \hat{A} &= 8[2(P_0-P_1)^2 + (P_0+P_1)(M_{12}+M_{20}) \\ &\quad + (M_{12}+M_{20})(P_0+P_1) - P_2M_{10} - M_{10}P_2], \\ \hat{B} &= M_{01}(P_0+P_1) + (P_0+P_1)M_{01} + 4P_2(M_{12}-M_{20}-P_1+P_0) \\ &\quad + 4(M_{12}-M_{20}-P_1+P_0)P_2. \end{aligned} \tag{3.184}$$

H. Cartesian coordinates

The defining coordinates t, x, y can be incorporated into the scheme we have used here in the same way that Bôcher has done for the Laplace equation in three space. The diagrams for such a coordinate system are



The expressions for the pentaspherical coordinates are

$$\begin{aligned} &\sigma x_1^2 = -1, \quad 2\sigma x_1x_2 = -\mu' - \nu' - \rho', \\ &\sigma x_3^2 = \rho', \quad \sigma x_4^2 = \nu', \quad \sigma x_5^2 = \mu', \end{aligned} \tag{3.185}$$

where the x_i are as in (1.26). The separation equations are obviously of the form

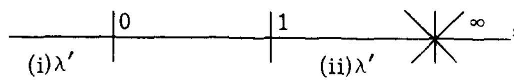
$$\frac{d^2E_i}{d\lambda'^2} + K_iE_i = 0 \tag{3.186}$$

with basis defining operators any two of the operators P_i^2 $(i=0,1,2)$.

IV. CONCLUDING REMARKS

In this paper we have made a detailed study of the orthogonal coordinate systems in three-dimensional Minkowski space for which the two-dimensional wave equation (*) admits an R-separable solution. The method for doing this is due to Bôcher and involves the use of pentaspherical coordinates. The direct relation between pentaspherical coordinates and the symmetry group of (*) was clearly demonstrated. The utility of the method over alternate ways of finding separable solutions of differential equations such as the classification of differential forms[11] is clear. Not only can the coordinates be found, but the separation equations and modulation factor can be determined from the key formulas in Sec. II.

As mentioned in the introductory comments of Sec. III, we have given a list of coordinate systems, some of which are equivalent under the action of the O(3, 2) symmetry group of (*) but not under the action of the $E(2,1)$ subgroup. The coordinate systems corresponding to the configurations [21(11)], [(21)(11)], and [3(11)], in which the coordinate curve for a contracted variable λ' has a diagram



are equivalent under the O(3, 2) group action to one of the nine classes of coordinate systems which have a radial coordinate. This reflects the fact that the operator $\hat{B}$ as in (3.21) with $\lambda' = \rho'$ can always be chosen to be $-\frac{1}{4} + D^2$. Further, for the systems corresponding to the configuration [32], those which have $e_1 = \infty$, $e_4 = 0$ and $e_1 = 0$, $e_4 = \infty$ are equivalent under the action of O(3, 2) but not under the E(2, 1) subgroup. Similar comments apply to the systems with configuration [2(21)].

No attempt has been made to firmly establish that all inequivalent classes of orthogonal R separable solutions of (*) have been found. This topic will be the subject of subsequent work. Taking into account the equivalent systems as indicated in the preceding comments, we have presented 53 coordinate systems inequivalent under the O(3, 2) group action. In addition all the coordinate systems except those belonging to the configuration [11111] give separable solutions of the Helmholtz equation $\partial_{tt}\psi - \Delta_2\psi = K^2\psi$. There are 53 such systems. All the coordinate systems in Secs. 1–4, 6 of I are represented here. In particular the nine coordinate systems of the Euler–Poincaré–Darboux (EPD) equation. In subsequent articles it is our intention to look at the EPD equation in detail and to examine solutions of (*) which are R-separable but not orthogonal.

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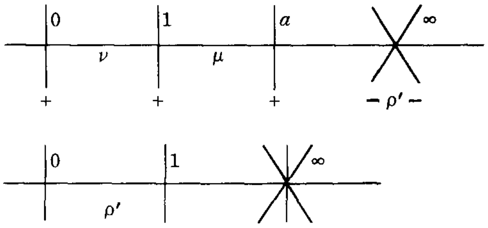
†(b) [(11)111]



The pentaspherical coordinates are given as in (3.20). The three space coordinates are given by

$$t = ix_2/(x_5 + ix_1), \quad x = x_4/(x_5 + ix_1),$$
$$y = x_3/(x_5 + ix_1). \tag{P1}$$

The modulation factor is

$$\sqrt{2}\sigma^{1/4} s = (\sqrt{\nu\mu/a} + \sqrt{1-\rho'})^{1/2}.$$

The basis operators are

$$4\hat{A} = a(P_2 + K_2)^2 + (P_1 + K_1)^2,$$
$$4\hat{B} = -1 - (P_0 - K_0)^2. \tag{P2}$$