Multiply perfect numbers of low abundancy

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Abstract

The purpose of this thesis is to investigate the properties of multiperfect numbers with low abundancy, and to include the structure, bounds, and density of certain multiperfect numbers.

As a significant result of this thesis, an exploration of the structure of an odd 4-perfect number has been made. An extension of Euler's theorem on the structure of any odd perfect number to odd 2^k-perfect numbers has also been obtained.

In order to study multiperfect numbers, it is necessary to discuss the factorization of the sum of divisors, in particular for $\sigma(q^e)$, for prime q. This concept is applied to investigate multiperfect numbers with a so-called flat shape $N = 2^a p_1 \cdots p_m$. If some prime divisors of N are fixed then there are finitely many flat even 3-perfect numbers. If N is a flat 4-perfect number and the exponent of 2 is not congruent to 1 (mod 12), then the exponent is even. If all odd prime divisors of N are Mersenne primes, where N is even, flat and multiperfect, then N is a perfect number. In more general cases, some necessary conditions for the divisibility by 3 of an even 4-perfect number $N = 2^a b$ are obtained, where b is an odd positive integer.

Two new ideas, namely flat primes and thin primes, are introduced since these appear often in multiperfect numbers. The relative density of flat primes to all primes is given by 2 times Artin's constant. An upper bound of the number of thin primes is $T(x) \ll \frac{x}{\log^2 x}$. The sum of the reciprocals of the thin primes is finite.

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Contents

Chapter 1

Summary of the Literature

1.1 Mathematical notations and symbols

The lowercase letters p and q denote prime numbers.

The following notations are standard:

$(a_1, \cdots, a_n)$ denotes the greatest common divisor of the integers $a_1, \cdots, a_n$.

$[x]$ denotes the integer part of the real number x.

$\{x\}$ denotes the fractional part of the real number x.

$\pi(x)$ denotes the number of prime numbers less than or equal to x.

$\mathbb{Z}$ denotes the set of all integers.

$\mathbb{N}$ denotes the set of all positive integers.

$\mathbb{P}$ denotes the set of all prime numbers.

$\#$ denotes the number of elements in a set.

We use Landau's O, o, and $\ll$ notation [71]:

$$f(x) = O(g(x)) \text{ or } \quad f(x) \ll g(x)$$

for a range of a real x, there is a constant A such that the inequality

$$|f(x)| \leq Ag(x)$$

holds over the range.

$$f(x) = o(g(x)) \text{ as } x \to \infty,$$

means

$$\lim_{x\to\infty} \frac{f(x)}{g(x)} = 0$$

($g(x) \neq 0$ for x sufficiently large.) The same meaning is used when $x \to \infty$ is replaced by $x \to \alpha$, for any fixed α.

In this thesis, $f(x) = O_\epsilon(x^\epsilon)$ means there is a constant C depending on $\epsilon > 0$, such that

$$|f(x)| \leq Cx^\epsilon,$$

holds over the range of a real x.

A subset A of positive integers has asymptotic density $d(A) = \alpha$, where $0 \leq \alpha \leq 1$, if the proportion of elements of A, among all positive integers from 1 to n, has a limit α as n tends to infinity. That is,

$$d(A) = \lim_{n\to\infty} \frac{|A(n)|}{n} = \alpha,$$

where $A(n) = A \cap \{1, 2, \cdots, n\}$.

A function $f(x)$ of a real variable $x > 0$ is written as $o(x)$ if $\frac{f(x)}{x} \to 0$, as $x \to \infty$. If $|A(x)| = \alpha x + o(x)$, then $d(A) = \alpha$.

A function f is said to have an upper bound C if $f(x) \leq C$ for all x in its domain.

Let k be a positive integer. Any collection of positive integers whose sum is equal to k is said to form a partition of k [54]. For example,

$$\begin{aligned} 5 &= 4+1 = 3+2 = 3+1+1 = 2+2+1 \\ &= 2+1+1+1 = 1+1+1+1+1, \end{aligned}$$

so that there are 7 partitions of 5.

The factorization of a positive integer N as a product of powers of distinct primes p_i, $i = 1, \cdots, k$ can be given by

$$N = \prod_{i=1}^{k} p_i^{\alpha_i}.$$

The notation $v_p(N) = e$ means $p^e || N$, that is, $p^e \mid N$ and $p^{e+1} \nmid N$. $v_p(N) = 0$ if p does not divide N.

The number of distinct prime divisors of N is denoted by

$$\omega(N) = k$$

and the total number of prime divisors of N is denoted by

$$\Omega(N) = \sum_{i=1}^{k} \alpha_i.$$

The sum of divisors function is denoted by

$$\sigma(N) = \sum_{d|N} d.$$

For prime p and positive integer α, we have

$$\sigma(p^\alpha) = 1 + p + p^2 + \cdots + p^\alpha = \frac{p^{\alpha+1} - 1}{p - 1}.$$

It is well-known that σ is a multiplicative function. Given the prime factorization of N we have

$$\sigma(N) = \prod_{i=1}^{k} \sigma(p_i^{\alpha_i}) = \prod_{i=1}^{k} \frac{p_i^{\alpha_i+1} - 1}{p_i - 1}.$$

Given integers a, b and m with $m > 0$, we say that a is congruent to b modulo m and write

$$a \equiv b \pmod{m}$$

if m divides the difference $a - b$.

A powerful number is a positive integer m such that for every prime number p dividing m, p^2 also divides m.

By convention positive integers of the form

$$M_n = 2^n - 1, (n \geq 1)$$

are called Mersenne numbers. If a prime p is a Mersenne number, we say p is a Mersenne prime.

A multiperfect (or multiply perfect) number (MPN) N is any positive integer which satisfies the equation $\sigma(N) = kN$ with $k \geq 2$ an integer called the abundancy of N. A MPN of abundancy k is also called a k-perfect number. If $k = 2$, we call this MPN a perfect number.

Definition 1.1 *We say p is a* **super flat prime** *if either p is a Mersenne prime, $p + 1 = 2^a$, or $p + 1 = 2^a p_1 \cdots p_m$, where $a \geq 1$, $m \geq 1$ and each p_i is a super flat prime.*

Example 1.1 *Since $19 + 1 = 2^2 \cdot 5$, $5 + 1 = 2^1 \cdot 3$, $3 + 1 = 2^2$, then 19 is a super flat prime.*

Definition 1.2 *We say a positive integer N is an* **upper flat number** *if $N + 1 = 2^e$ or $N + 1 = 2^e q_1 \cdots q_m$ where $e \geq 1$ and the q_i's are distinct odd primes. If a prime p is an upper flat number we say p is an* **upper flat prime**. *Let*

$$F(x) := \#\{p \leq x : p \text{ is an upper flat prime}\}.$$

There are corresponding definitions of the terms **lower flat number** and **lower flat prime** obtained by replacing a shift by $+1$ with a shift by -1:

Definition 1.3 *We say a positive integer N is a* **lower flat number** *if $N - 1 = 2^e$ or $N - 1 = 2^e q_1 \cdots q_m$ where $e \geq 1$ and the q_i's are distinct odd primes. If a prime p is a lower flat number we say p is a* **lower flat prime**.

Definition 1.4 *We say a positive integer N is an* **upper thin number** *if $N+1 = 2^e q$ or $N+1 = 2^e$ where $e \geq 1$ and q is an odd prime. If a prime p is an upper thin number we say p is an* **upper thin prime**. *Let*

$$T(x) := \#\{p \leq x : p \text{ is an upper thin prime}\}.$$

Definition 1.5 *We say a positive integer N is a* **lower thin number** *if $N-1 = 2^e q$ or $N-1 = 2^e$ where $e \geq 1$ and q is an odd prime. If a prime p is a lower thin number we say p is a* **lower thin prime**.

Definition 1.6 *Let p and q be distinct primes. The* **exponent of q modulo p,** $\exp_p q$, *is the minimum positive integer k such that $p \mid q^k - 1$, [2, Chapter 10].*

Definition 1.7 *The* **discrete power of p to base q,** $[p|q]$, *is the maximum positive integer l such that $p^l \mid q^{\exp_p q} - 1$. If $p = q$ we set $[p|q] = 0$.*

Definition 1.8 *We say a prime p is a* **super thin prime**, *if $p = p_i$, and $p_i + 1 = 2^{a_i} p_{i-1}$, where $i = 2, \cdots, m$, $a_i \geq 1$ and $p_1 + 1 = 2^{a_1}$ with $a_1 \geq 2$, the p_i's are distinct odd primes.*

Definition 1.9 *We say a positive integer N is* **flat** *if its odd part is squarefree, i.e. if N can be written in the form $N = 2^a \cdot p_1 \cdots p_m$ where $a \geq 0$, $m \geq 0$ and $p_1 < p_2 < \cdots < p_m$, where the p_i are odd primes. If N is flat then the value of a is called its* **exponent** *and the value of m its* **length**.

1.2 A historical background

In the history of mathematics, many famous mathematicians such as Pythagoras (500 B. C.), Euclid (275 B. C.), Fermat (1636-1643), Mersenne (1639-1643), Euler (1772), Lucas (1876), and Lehmer (1901) have investigated the properties of perfect numbers. However the history of the study of MPNs with

abundancy $k > 2$ is only about 400 years old. In 1918, Dickson [28] traced the development of the mathematics of MPNs. According to Schroeppel (1995) [90], the frequency of finding multiperfect numbers (including perfect numbers) has increased dramatically in the last 20 years. From antiquity to 1910, only 47 MPNs had been discovered, but to date 5190 MPNs are known (of which 5145 have abundancy $k > 2$, and the rest 45 are even perfect numbers).

1.2.1 Perfect numbers

Euclid produced the first significant mathematical result on perfect numbers. In Proposition 36 of Book IX of the *Elements*, he provided a form for a set of even perfect numbers using the formula for the sum of a geometric progression. That is, if the sum $1 + 2 + 2^2 + 2^3 + \cdots + 2^{k-1} = 2^k - 1$ is a prime number ($k > 1$), then $N = 2^{k-1}(2^k - 1)$ is a perfect number, [14, p. 220].

From the literature we see ([78] and [29]) that Nicomachus (about A. D. 100) classified numbers into three types: abundant numbers which satisfy $\sigma(N) > 2N$; perfect numbers which satisfy $\sigma(N) = 2N$; and deficient numbers which satisfy $\sigma(N) < 2N$. Nicomachus also stated that perfect numbers will be arranged in regular order; that is, only one among the units, one among the tens, one among the hundreds, and one among the thousands; for example, 6, 28, 496, 8128 are the only perfect numbers in the corresponding intervals between 1, 10, 100, 1000, 10000; and the last digit of the successive perfect numbers is alternately 6 and 8. These statements of Nicomachus imply that (1) all perfect numbers are even; (2) the nth perfect number has n digits; (3) all perfect numbers end in 6 and 8 alternately; (4) Euclid's formula provides all perfect numbers; (5) there are infinitely many perfect numbers.

With the test of time, it has been discovered that some of Nicomachus's assertions are correct, some are incorrect and some are still open questions. In

1536, Hudalrichus Regius gave the factorization $2^{11} - 1 = 2047 = 23 \cdot 89$, in which 11 is the first prime such that $2^p - 1$ is not a prime number. He also found that $2^{13} - 1 = 8191$ is prime, so he discovered the 5th perfect number $2^{12}(2^{13} - 1) = 33550336$. This showed that Nicomachus' assertion (2) is false because the 5th perfect number has 8 digits, (See [78]).

In 1548-1626, Cataldi [28] proved that all perfect numbers given by Euclid's form end in 6 or 8. In 1603, Cataldi [78] found and listed all primes ≤ 750, then proved that $2^{17} - 1 = 131071$ is a prime because $131071 < 562500 = 750^2$, and he could check the number with his list of primes (≤ 750) to show 131071 does not contain any other prime divisor. We now know, following Euler (see below), this means that the 6th perfect number $2^{16}(2^{17} - 1) = 8589869056$. From this result we can see that Nicomachus's assertion (3) is false, because the last digit of both the 5th and 6th perfect numbers is 6. The two perfect numbers do not end in 6 and 8 alternately. Cataldi also found the 7th perfect number $2^{18}(2^{19} - 1) = 137438691328$ by the same method.

In 1652, Broscius [28] pointed out that perfect numbers could be expressed using sums of arithmetical progressions: $6 = 1 + 2 + 3$, $28 = 1 + 2 + 3 + 4 + 5 + 6 + 7$, $496 = 1 + 2 + 3 + \cdots + 31$. He also speculated that perfect numbers end in 6 or 28.

In 1647, Mersenne [28] stated that $M_p = 2^p - 1$ is prime for p=2, 3, 5, 7, 13, 17, 19, 31, 67, 127, 257 and composite for all other primes $p < 257$. But Mersenne could not test the prime factors for all of the numbers he had announced without advanced calculation tools. Traditionally, a number of the form $M_n = 2^n - 1$ ($n \geq 1$) is called a Mersenne number, and if M_n is prime, then this number is said to be a Mersenne prime.

In 1849, one of Euler's posthumous manuscripts revealed the relation between Mersenne primes and even perfect numbers. Euler provided a proof of Euclid's type, that is, every even perfect number must be of the form

$2^{k-1}(2^k-1)$. Therefore, if one could find primes of the form 2^p-1 (where necessarily p is prime, so M_p is a Mersenne prime), then one would find a corresponding even perfect number, (see [28]).

By observing the form of Mersenne numbers, we can see that a higher Mersenne prime can come from a lower Mersenne prime. For example, 7 is a Mersenne prime, and a new Mersenne prime 127 can be obtained from 2^7-1. It was hoped that if the number M_n was prime, then M_{M_n} would also be a prime. However, in 1953 a high-speed computer found an exception $M_{M_{13}}=2^{M_{13}}-1=2^{8191}-1$, which is a composite number with 2466 digits, (see [102]). A useful way to search for Mersenne primes on a computer was provided by Lucas's (1876) criterion, (see [33]). The criterion is that for a given sequence by $u_1=4$, $u_{n+1}=u_n^2-2$ $(n=1,2,\cdots)$, given a prime $p\geq 3$, the Mersenne number M_p is prime if and only if M_p divides u_{p-1}. This can be tested by computing the residue modulo M_p of the u_n. Today this method is still used. For instance, on September 4, 2006, Curtis Cooper and Steven Boone discovered the 44th known Mersenne prime, $2^{32,582,657}-1$, just nearly nine months from their last discovery, the 43rd Mersenne prime. The new prime has 9,808,358 digits, (see [105]). We still do not know whether there are infinitely many Mersenne primes. Therefore the question regarding the number of even perfect numbers is still unsolved.

In order to decide whether a certain Mersenne number is prime or composite, Fermat (1640) stated and Euler (1747) proved that, if p is a prime, and $M_p=2^p-1$ is composite, then there are no prime factors other than those of the form $kp+1$ with k an integer. For example, $2^{11}-1=23\cdot 89$ has the factor 23; $2^{37}-1$ has the factor 223; and $2^{23}-1$ has the factor 47. From this result, Cunningham (1894) stated that the corollary proved by Lucas (1878) is that if p and $2p+1$ are both odd primes with $p\equiv 3 \pmod 4$, then $M_p=2^p-1$ has the factor $2p+1$. (See [102]).

Bachet de Mëzirac (1581-1638) [28] proved Euclid's theorem that $N = 2^{p-1}(2^p - 1)$ is perfect if $2^p - 1$ is a prime, but if $2^p - 1$ is composite, then N is abundant. He also asserted that every multiple of a perfect or abundant number is abundant; every divisor of a perfect number is deficient; and the product of two odd prime powers is deficient. These results follow from the following well known definitions: if $N_1, N_2, \cdots N_s$ are the divisors of a perfect or abundant number N, and a is an integer > 1, then $\sigma(aN) \geq aN_1 + aN_2 + \cdots + aN_s + 1 = a\sigma(N) + 1$, implies $\frac{\sigma(aN)}{aN} \geq \frac{\sigma(N)}{N} + \frac{1}{aN} > 2$; if p is a prime, then $\frac{\sigma(p^k)}{p^k} = \frac{p^{k+1}-1}{p^k(p-1)} < \frac{p^{k+1}}{p^k(p-1)} = \frac{p}{p-1} = 1 + \frac{1}{p-1} \leq 2$; if the two primes are p and q with $2 < p < q$, then $\frac{\sigma(p^j q^k)}{p^j q^k} = \frac{p^{j+1}-1}{p^j(p-1)} \cdot \frac{q^{k+1}-1}{q^k(q-1)} < \frac{p}{p-1} \cdot \frac{q}{q-1} \leq (1 + \frac{1}{2})(1 + \frac{1}{4}) = \frac{15}{8} < 2$.

In the history of mathematics, interest has also focused on discovering whether odd perfect numbers exist or show their non-existence, but mathematicians could only find some necessary conditions for the existence of odd perfect numbers. Euler was the first to provide a significant result on odd perfect numbers. Euler not only proved that every even perfect number is of Euclid's type, but he also proved that any odd perfect number N if it exists must be of the form $N = p^k m^2$, where p is a prime, the common factor of both p and m is only 1, and $p \equiv k \equiv 1 \pmod 4$; and in particular, $N \equiv 1 \pmod 4$. For more details see [28].

After obtaining the basic structure which is Euler's form, mathematicians investigated the properties of the size, the factors, the number of prime factors, and the exponents of prime factors of odd perfect numbers. During the 19th century, some significant results were discovered. Lebesgue (1844) [28] proved that if N is an odd perfect number then $\omega(N) \geq 4$. Sylvester (1887) [36] proved that no odd perfect number has factor 105 and there is no odd perfect number N with $\omega(N) < 6$. In 1888, Sylvester and Servais [36] proved that no odd perfect number exists with $\omega(N) = 3$. Sylvester (1888) [36] also proved that any odd perfect number N with $\omega(N) < 8$ must be divisible by 3 and

that there is no odd perfect number N with $\omega(N) = 4$. Catalan (1888) [17] proved that if 3, 5, or 7 are not factors of an odd perfect number N, then $\omega(N) \geq 26$ and thus $N > 10^{44}$. In 1951, Webber [101] proved that $\omega(N) \geq 6$, Robbins (1972) [87] and Pomerance (1974) [80] improved this to $\omega(N) \geq 7$. Chein (1979) [18] and Hagis (1980) [40] independently proved that $\omega(N) \geq 8$. Cohen (1991) [22] and Sorli (1999) [25] used different methods to prove that $\omega(N) \geq 6$ and then 7. In 2007 Nielsen [77] improved this to $\omega(N) \geq 9$.

Sylvester (1888) [36] proved that if N, an odd perfect number, is not divisible by 3, then $\omega(N) \geq 8$. This was improved to $\omega(N) \geq 10$ by Kishore (1977) [61]; to $\omega(N) \geq 11$ by Hagis [42] and Kishore [63] in 1983; and to $\omega(N) \geq 12$ by Nielsen [77] in 2007. Another area of research interest was to investigate the finiteness of odd perfect numbers. In 1913, Dickson [27] stated that "for a given s, there are at most finitely many odd perfect numbers N such that $\omega(N) = s$". This was proved by Shapiro (1949) [91] and Rózsa Péter (1959) [33].

Some results on the number of total prime factors of an odd perfect number N are $\Omega(N) \geq 23$ by Cohen (1982) [20], $\Omega(N) \geq 29$ by Sayers (1986) [89], $\Omega(N) \geq 37$ by Iannucci and Sorli (2003) [57]. Hare improved the result that $\Omega(N) \geq 47$ (2004) [49] and $\Omega(N) \geq 75$ (2007) [48] using an idea from Pomerance (1974) [80].

On the upper bound of an odd multiperfect number N with $\omega(N) = k$, Pomerance (1977) [82] proved that $N < (4k)^{(4k)^{2^{k^2}}}$. This result was improved to 4^{4^k} by Heath-Brown in 1994 [51] and to D^{4^k} with $D = (195)^{1/7}$ by Cook in 1999 [26], and further to 2^{4^k} by Nielsen in 2003 [76].

On the lower bound of an odd perfect number N, the best result we have today is very close to 10^{500} by William Lipp [66], which is an application of a method developed by Brent, Cohen, and te Riele (1991) [10], whose result was $N > 10^{300}$. Early results were $N > 10^{50}$ by Hagis (1973) [39] and $N > 10^{160}$ by

Brent and Cohen (1989) [9]. Bernhard (1949) [6], Kühnel (1949) [65], Kanold (1957) [60], Tuckerman (1973) [100], and Kishore (1977) [61] contributed to the lower bound of odd perfect numbers. Some related results are that $N > q^{3k/2}$, if N is an odd perfect number with q a prime, k an even integer, and $q^k \mid N$ but $q^{k+1} \nmid N$ (by Brent, Cohen, and te Riele [10]), and that some prime power ($> 10^{20}$) is a factor of any odd perfect number (by Cohen, 1987 [21]).

For the lower bound of the large prime factors of an odd perfect number N, the largest prime $p_k \geq 100129$ was proved by Hagis and McDaniel (1973) [45], improved to $p_k \geq 10^6$ by Cohen and Hagis (1998) [24], and further increased to $p_k \geq 10^7$ by Jenkins (2003) [58]. The second largest prime p_{k-1} was shown to be greater than or equal to 139 by Pomerance (1975) [81] and extended to be greater than or equal to 1009 by Hagis (1981) [41]. Iannucci (1999 [55], 2000 [56]) proved $p_{k-1} > 10^4$ and the third largest prime factor $p_{k-2} > 10^2$. Grün (1952) [38] proved the smallest prime factor $p_1 < \frac{2}{3}k + 2$. Kishore (1981 [62]) showed that for $2 \leq i \leq 6$, $p_i < 2^{2^{i-1}}(k - i + 1)$. This result was improved by Cohen and Sorli (2003) [25].

1.2.2 Multiperfect numbers

The early study of multiperfect numbers was done in the 17th century by mathematicians Mersenne, Descartes, Fermat, and Frenicle among others. Descartes provided 5 rules for finding multiperfect numbers: (1), if N is 3-perfect, and $3 \nmid N$, then $3N$ is 4-perfect; (2), if N is 3-perfect, $3 \mid N$, but $5 \nmid N$, $9 \nmid N$, then $45N$ is 4-perfect; (3), if N is 3-perfect, $3 \mid N$, but $7 \nmid N$, $9 \nmid N$, $13 \nmid N$, then $3 \cdot 7 \cdot 13N$ is 4-perfect; (4), if $\sigma(N) = kN$ with k a natural number, $2^9 \mid N$, but $2^{10} \nmid N$, $31 \nmid N$, $43 \nmid N$, $127 \nmid N$, then $31N$ and $16 \cdot 43 \cdot 127N$ are $(k+1)$-perfect numbers; (5), if $3 \nmid N$, and if $3N$ is 4-perfect, then N is 3-perfect.

Descartes derived his six 4-perfect numbers and tested the first four 3-perfect numbers by his rules. Frenicle and Mersenne doubted that it would be possible to find all multiperfect numbers by Descartes' rules. However Descartes responded that infinitely many multiperfect numbers could be generated by his rules. In 1638, Descartes obtained the 3rd 5-perfect number $2^7 3^5 5 \cdot 7^2 \cdot 13 \cdot 17 \cdot 19$. In 1644, Mersenne asserted that he could find all k-perfect numbers with k an integer through his general method. For details see [28].

With the appearance of high-speed computers since the 1950s the search for multiperfect numbers has significantly improved. Between 1955 and 2006, 4634 new multiperfect numbers had been found including new perfect numbers, as well as 9-perfect, 10-perfect and 11-perfect numbers. It should be mentioned that all known six 3-perfect numbers had been discovered by the end of 1643, in the time of Mersenne, Fermat and Descartes (1639-1643). It seems that all 3-perfect numbers have been discovered. See [78].

From the late 19th century to the early 20th century, there were some discoveries of multiperfect numbers of lower abundancy. Desboves (1878) [28] observed that 3-perfect numbers of the form $2^n \cdot 3 \cdot p$ with p a prime are only 120 and 672. This result was proved by Westlund (1900) [103], that is, the only 3-perfect numbers of the form $p_1^a p_2^b p_3$ with p_i's primes are $2^3 3 \cdot 5 = 120$ and $2^5 3 \cdot 7 = 672$. Carmichael (1906) [15] proved that (1) $2^3 3 \cdot 5 = 120$ and $2^5 3 \cdot 7 = 672$ are the only two multiperfect numbers with $\omega(N) = 3$; (2) multiperfect numbers with $\omega(N) = 4$ are only the third 3-perfect number $2^9 3 \cdot 11 \cdot 31$ (discovered by Jumeau and Croix [28]) and the first 4-perfect number $2^5 3^3 5 \cdot 7 = 30240$ (discovered by Descartes) [28]; (3) those with $\omega(N) = 5$ are only the fourth 3-perfect number $2^{13} 3 \cdot 11 \cdot 43 \cdot 127$, the second 4-perfect number $2^3 3^2 5 \cdot 7 \cdot 13$, the fourth 4-perfect number $2^9 3^3 5 \cdot 11 \cdot 31$ (discovered by Descartes) and the eighth 4-perfect number $2^7 3^3 5^2 17 \cdot 31$ (discovered by Mersenne) [28]. Westlund (1901) [104] also proved that the only 3-perfect number of the form

$p_1^a p_2 p_3 p_4$ with $p_1 < p_2 < p_3 < p_4$ is the third 3-perfect number $2^9 3 \cdot 11 \cdot 31$. Lehmer (1900) [28] proved that for a 3-perfect number N, $\omega(N) \geq 3$; for a 4-perfect number, $\omega(N) \geq 4$; for a 5-perfect number, $\omega(N) \geq 6$; for a 6-perfect number, $\omega(N) \geq 9$; for a 7-perfect number, $\omega(N) \geq 14$. In 1902, Cunningham [28] stated that the number of multiperfect numbers N with a special form $2^{q-1}(2^q - 1)F$, where F is a factor of N, is at least one, for $1 \leq q \leq 39$ (except 33, 35, 36) or $q = 45, 51, 62$. For example, the third 4-perfect number $2^2 \cdot 3^2 \cdot 5 \cdot 7^2 \cdot 13 \cdot 19$. Tables 4.1, 6.1 and 7.1 provide some examples of k-perfect numbers for every value of k up to and including $k = 11$.

The properties of odd multiperfect numbers are very similar to odd perfect numbers. It was shown by Carmichael [16] in 1907 that $\omega(N) \geq 4$, where N is an odd multiperfect number. Artuhov (1973) [3] proved that there exist finitely many odd multiperfect numbers with $\omega(N) = s$, where s is an arbitrary positive integer, (comparing with Dickson's theorem for odd perfect numbers in 1913 [27]). $\omega(N) \geq 11$ and $N > 10^{70}$ were shown by Cohen and Hagis (1985) [23], which improved on $N > 10^{50}$ by Beck and Najar (1982) [5]. Iannucci (1999) [55] showed that for odd 3-perfect numbers the largest prime factor must be greater than 10^7. Hagis (1986) [43] proved that the third largest prime factor of an odd 3-perfect number has to be at least 100. Kanold (1957) [60] proved that for an odd 3-perfect number N with $\omega(N) \geq 9$, then N is a square, and $N > 10^{20}$. In 1987, Kishore [64] showed that $\omega(N) \geq 12$ for an odd 3-perfect number N. In 1993 Hagis [44] gave a simple proof of Kishore's result.

1.3 An outline of this thesis

Here is an outline of the thesis: In Chapter 2 restricted forms for an odd multiperfect number of abundancy 4 are developed. In Chapter 3 the factorization

of the sum of divisors of an integer is studied. Chapter 4 treats counting multiperfect numbers up to x. Chapter 5 treats even 3-perfect numbers of a so-called flat shape; Chapter 6 deals with even perfect numbers of abundancy 4. In Chapter 7 other properties of multiply perfect numbers are developed, and unsolved problems are presented in the form of conjectures.

1.4 Summary of the main findings

Here are some of the main findings in this thesis:

Chapter 2:

Theorem 2.3 gives the structure of an odd 4-perfect number: Let N be an odd 4-perfect number. Then N has one of the following forms, where the α_i are positive integers and the p_i odd primes:

(A) $N = q_1^{e_1} q_2^{e_2} p_1^{2\alpha_1} \cdots p_m^{2\alpha_m}$ for primes q_i and positive integers e_i with $q_i \equiv e_i \equiv 1 \pmod 4$.

In the remaining types $N = q^e p_1^{2\alpha_1} \cdots p_m^{2\alpha_m}$ with q prime where:

(B) $q \equiv 1 \pmod 4$ and $e \equiv 3 \pmod 8$ or

(C) $q \equiv 3 \pmod 8$ and $e \equiv 1 \pmod 4$.

Corollary 2.4 shows that no square or squarefree number is odd and 4-perfect.

Theorem 2.11 shows that if 9 is the maximum power of 3 dividing N then N is not an odd 4-perfect number.

Chapter 3:

Theorem 3.9 investigates the factorization of $\sigma(q^e)$:

(1) Let q be an odd prime and e a positive integer. Then

$$\sigma(q^e) = 2^{(v_2((q+1)(e+1))-1)(2\{\frac{e}{2}\})} \times \prod_{p|(e+1,q-1)} p^{v_p(e+1)} \times \prod_{1 \neq \exp_p q | e+1} p^{[p|q]+v_p(\frac{e+1}{\exp_p q})}$$

where p is an odd prime.

(2) Let $q = 2$ and e a positive integer, then

$$\sigma(2^e) = \prod_{1<\exp_p 2|e+1} p^{[p|2]+v_p(e+1)}$$

where p is an odd prime.

And Theorem 3.15 is a consequence of Theorem 3.9:

(1) Let q be prime, p an odd prime and $e \geq 1$. If $\sigma(q^e) = p^j$, then $j = [p|q]$ and $e + 1 = \exp_p q$ is prime. For example, $\sigma(3^4) = 11^2$, and $\sigma(3^2) = 13$.

(2) Conversely, let e be even and the prime p be odd. If $e + 1 = \exp_p q$ is prime and the equation $e+1 = \exp_p q$, for given e and q, has a unique solution $p \in \mathbb{P}$, then $\sigma(q^e) = p^{[p|q]}$.

Chapter 4:

Theorem 4.35 shows that there are infinitely many groups of 8 consecutive odd upper flat numbers.

Theorem 4.15 shows that asymptotically the number of thin numbers is the same as that of the primes.

Theorem 4.23 shows that the relative density of upper flat primes to all primes is given by 2 times Artin's constant.

Theorem 4.25 proves that the relative density of primes which are both lower and upper flat to all primes is $\prod_{p \text{ odd}}(1 - \frac{2}{p^2-p}) = 0.53511\cdots$, where the product is taken over odd prime values of p.

Corollary 4.26 shows that those primes which are both lower and upper flat are about 54% of all primes, those either lower or upper flat but not both - each about 21%, and those neither upper nor lower flat - 4%.

Corollary 4.29 proves that the sum of the reciprocals of the upper thin primes is finite.

Chapter 5:

Theorem 5.9 proves that $N = 2^a p_1 \cdots p_m$, with some restrictions, is not a 3-perfect number:

Let $N = 2^a p_1 \cdots p_m$, with $a \geq 1$, $p_i + 1 = 2^{a_i} p_{i-1}$, $a_i \geq 1$, where p_i is an odd prime for $1 \leq i \leq m$ and p_0 is a prime. Then N is not a 3-perfect number.

Proposition 5.13 proves that, for a 3-perfect number $N = 2^a p_1 \cdots p_m$ with a even, not all prime factors of $\sigma(2^a)$ are Mersenne primes.

Some examples are given of particular classes of number with a flat shape $N = 2^a p_1 \cdots p_m$ which are not 3-perfect numbers:

Example 5.3 shows that $N = 2^e \cdot 3 \cdot 23 \cdot 7 \cdot 31$ is not a 3-perfect number for any $e \geq 1$.

Example 5.9 shows that if $N = 2^e \cdot 5 \cdot 19 \cdot 37 \cdot 73 \cdot 9343 \cdot p_1 \cdots p_m$, where $p_i + 1 = 2^{f_i} \cdot q_i$, $(i = 1, 2, \cdots, m)$, p_i's are distinct, with p_i's, q_i's odd primes, and $p_i \neq 5, 19, 37, 73, 9343$, and $e \geq 1$, then N is not a 3-perfect number.

Chapter 6:

Theorem 6.3 shows that all primes appearing in a flat 4-perfect number $N = 2^a p_1 p_2 \cdots p_m$ are super flat primes (which are defined in Definition 1.1).

Theorem 6.4 gives some necessary conditions for the divisibility by 3 of an even 4-perfect number $N = 2^a b$, where b is an odd positive integer. For example, if a is odd, then N is divisible by 3.

Theorem 6.9 proves that if $N = 2^a p_1 \cdots p_m$ is a 4-perfect number with the p_i's distinct primes and $a \not\equiv 1 \pmod{12}$ then a is even.

Chapter 2

Restricted forms for an odd multiperfect number of abundancy 4

2.1 Introduction

This chapter is based on the paper by Broughan and Zhou [12]. No odd k-perfect numbers are known for any $k \geq 2$, and it is believed that none exist. For a survey of known results see [93] or [35] and the references given there. For example, if N is odd and 4-perfect then N has at least 22 distinct prime factors. If it is also not divisible by 3 then it has at least 142 prime factors.

In this chapter we consider the properties of classes of odd numbers which must be satisfied if they are to be 4-perfect. Conversely, we also consider the properties of classes which can never be 4-perfect. In a number of cases theorems follow, with some changes, in the pattern of corresponding results for 2-perfect numbers. However, mostly because of the number of primes involved, some of those techniques, from the theory of 2-perfect numbers, are not so readily available.

We show that Euler's structure theorem, that every odd 2-perfect number has the shape $N = q^e p_1^{2\alpha_1} \cdots p_m^{2\alpha_m}$, where $q \equiv e \equiv 1 \pmod 4$, has an extension to odd 4-perfect numbers, and then to odd 2^k-perfect numbers. For 4-perfect numbers there are three possible shapes like Euler's form, (A) with 2 q's instead of 1, (B) with $q \equiv 3 \pmod 8$ and $e \equiv 1 \pmod 4$, and (C) with $q \equiv 1 \pmod 4$ and $e \equiv 3 \pmod 8$, (Theorem 2.3). An immediate corollary is that no square or squarefree number is 4-perfect, (Corollary 2.4).

For 2^k-perfect numbers we need to derive a fact, which could be of independent interest. For $j \geq 1$, odd primes p and odd e, we have $2^j || \sigma(p^e)$ if and only if $2^{j+1} || (p+1)(e+1)$, (Theorem 2.5).

We include negative results (i.e. shapes which no odd 4-perfect number can have) for odd 4-perfect cubes (Theorem 2.10), numbers with 9 being the maximum power of 3 dividing N (Theorem 2.11), numbers with each of the p_i occurring to the power 2 (Theorem 2.12), and a positive result on the power of 3 dividing any odd 2^k-perfect number (Theorem 2.13).

2.2 Lemmas

We begin with two lemmas, summarizing well known results.

Lemma 2.1 *Let d and n be positive integers and p a prime number. If $d+1 \mid n+1$ then $\sigma(p^d) \mid \sigma(p^n)$.*

The converse of this lemma is also true - see Lemma 3.10.

Lemma 2.2 *(Congruences modulo 3)*
Let $p > 3$ be a prime number and let e be a positive integer.

$$\text{If } p \equiv 1 \pmod 3, \text{ then } \sigma(p^e) \equiv \begin{cases} 1 \pmod 3 & \text{if } e \equiv 0 \pmod 3, \\ 2 \pmod 3 & \text{if } e \equiv 1 \pmod 3, \\ 0 \pmod 3 & \text{if } e \equiv 2 \pmod 3. \end{cases}$$

$$\textit{If } p \equiv 2 \pmod 3, \textit{ then } \sigma(p^e) \equiv \begin{cases} 1 \pmod 3 & \textit{if } e \equiv 0 \pmod 2, \\ 0 \pmod 3 & \textit{if } e \equiv 1 \pmod 2. \end{cases}$$

2.3 Results

Theorem 2.3 *(Euler equivalent)*

Let N be an odd 4-perfect number. Then N has one of the following forms, where the α_i are positive integers, the p_i are odd primes and $m \geq 2$:

(A) $N = q_1^{e_1} q_2^{e_2} p_1^{2\alpha_1} \cdots p_m^{2\alpha_m}$ for primes q_i and positive integers e_i with $q_i \equiv e_i \equiv 1 \pmod 4$.

In the remaining types $N = q^e p_1^{2\alpha_1} \cdots p_m^{2\alpha_m}$ with q prime where:

(B) $q \equiv 1 \pmod 4$ and $e \equiv 3 \pmod 8$ or

(C) $q \equiv 3 \pmod 8$ and $e \equiv 1 \pmod 4$.

Proof. (1) Let $N = p_1^{\beta_1} \cdots p_m^{\beta_m}$ where the p_i are odd primes and the β_i whole numbers. Then $\sigma(N) = 4N$ implies $2^2 || \sigma(p_1^{\beta_1}) \cdots \sigma(p_m^{\beta_m})$ so either 2^1 is the maximum power of two dividing two distinct terms in the product and the remaining terms are odd, or 2^2 is the maximum power dividing one term and the remaining terms are odd. So type (A) is the former shape and (B) and (C) the latter. In the treatment for type (A), we need to know that if $2 || \sigma(q^e)$ and q is prime, then $q \equiv e \equiv 1 \pmod 4$. This has not yet been proved; the case $j = 1$ of Theorem 2.5 can be used for this. Therefore we need only consider primes q and powers α such that $2^2 || \sigma(q^\alpha)$.

(2) Claim: If $q \equiv 1 \pmod 4$ and $\alpha \equiv 3 \pmod 8$ then $4 \mid \sigma(q^\alpha)$.

$$\begin{aligned} \sigma(q^\alpha) &= 1 + q + \cdots + q^\alpha \\ &\equiv 1 + \alpha \pmod 4 \\ &\equiv 0 \pmod 4. \end{aligned}$$

so $4 \mid \sigma(q^\alpha)$.

(3) In the same situation as in (2), $8 \nmid \sigma(q^\alpha)$: Write

$$\sigma(q^\alpha) = 1 + q + q^2 + q^3 + \cdots + q^{3+8e},$$

group the $4 + 8e$ terms in $1 + 2e$ sets of 4 terms, so that

$$\sigma(q^\alpha) \equiv (1 + q + q^2 + q^3)(1 + 2e) \pmod 8,$$

where we have used $q^4 \equiv 1 \pmod 8$. Replacing q by $1+4x$, for some integer x, and reducing modulo 8 we get $\sigma(q^\alpha) \equiv 4 \cdot (1+2e) \pmod 8$, which is non-zero, so $8 \nmid \sigma(q^\alpha)$.

(4) Claim: If $q \equiv 3 \pmod 8$ and $\alpha \equiv 1 \pmod 4$ then $4 \mid \sigma(q^\alpha)$. Let $\alpha = 1 + 4e$ and $q = 3 + 8x$ then (where f, x, y, z and w are integers)

$$\begin{aligned}
\sigma(q^\alpha) &= \frac{(3+8x)^{2f} - 1}{2 + 8x} \text{ where } f \text{ is odd} \\
&= \frac{(1+2y)^{2f} - 1}{2y} \text{ where } y \text{ is odd} \\
&= \frac{1}{2y}\left(\binom{2f}{1}(2y)^1 + \binom{2f}{2}(2y)^2 + \cdots\right) \\
&= 2f + 2f(2f-1)y + 4z \\
&= 4w
\end{aligned}$$

so $4 \mid \sigma(q^\alpha)$.

(5) In the same situation as in (4) $8 \nmid \sigma(q^\alpha)$: write

$$\sigma(q^\alpha) = \frac{q^{2+4e} - 1}{2 + 8x} \equiv \frac{q^2 - 1}{2} \equiv \frac{(3+8x)^2 - 1}{2} \equiv 4 \pmod 8.$$

so, again $8 \nmid \sigma(q^\alpha)$.

(6) The remainder of the proof consists in showing the above cases constitute the only possibilities by examining in turn the 14 possible additional values of $\{q, e\}$ modulo 8. In summary, using the notation q^e for the values of q and e modulo 8, and using the same techniques as used in parts (2), (3) and (4) of the proof, the cases $1^1, 1^5, 5^1, 5^5$ give $4 \nmid \sigma(q^e)$. The cases $1^7, 3^3, 3^7, 5^7, 7^1, 7^3, 7^5, 7^7$ give $8 \mid \sigma(q^e)$, so cannot occur. The remaining cases

$5^3, 3^5$ are covered by (B) and (C). We have already proved 1^3 in (B), and 3^1 in (C).

Finally, since a 4-perfect number must have at least 4 different prime factors [28, Lehmer (1900)], we get $m \geq 2$. □

Corollary 2.4 *No square or squarefree number is odd and 4-perfect.*

Proof. Since the exponents of the leading primes are odd, and one of the three forms is always present, the first part of the claim is immediate. For the second part we need only consider the special forms $N = q_1 q_2$ and $N = q_1$, where the q_i are odd primes which are not possible, so no odd 4-perfect number is squarefree. □

It might be of interest to speculate, on the basis of Euler's theorem and the above, on the general form for division of $\sigma(p^\alpha)$ by powers of 2. However for powers 2^3, and beyond, the situation appears to be well structured but mysterious.

For example, in the following each pair corresponds to the classes modulo 2^4 of an odd prime and odd exponent (p, e) such that $2^3 \| \sigma(p^e)$. The list appears to be complete for this power of 2:

$$(1,7), (3,3), (3,11), (5,7), (7,1), (7,5),$$
$$(7,9), (7,13), (9,7), (11,3), (11,11), (13,7).$$

Note that in each case $2^4 \| (p+1)(e+1)$. It is a beautiful fact that this is true in general for all powers of 2.

Theorem 2.5 *For all odd primes p, powers $j \geq 1$ and odd exponents $e > 0$ we have*

$$2^j \| \sigma(p^e) \iff 2^{j+1} \| (p+1)(e+1).$$

Proof. (1) Let $2^j \| \sigma(p^e)$. First expand p to base 2:

$$p = 1 + e_1 2^1 + e_2 2^2 + \cdots + 2^{j+1}\eta,$$

where $\eta \in \{0\} \cup \mathbb{N}$ and $e_i \in \{0, 1\}$. There exists a minimum i with $1 \leq i \leq j$ so that

$$p = 1 + 2^1 + 2^2 + \cdots + 2^{i-1} + 0 \cdot 2^i + \cdots + 2^{j+1}\eta, \eta \in \{0\} \cup \mathbb{N}$$

since otherwise

$$p = 1 + 2^1 + \cdots + 2^j + 2^{j+1}\eta \equiv -1 \pmod{2^{j+1}}$$

so

$$\begin{aligned} \sigma(p^e) &= 1 + p + p^2 + \cdots + p^e \\ &\equiv 1 - 1 + 1 \cdots - 1 \equiv 0 \pmod{2^{j+1}} \end{aligned}$$

so $2^{j+1} | \sigma(p^e)$ which is impossible. Hence we can write

$$p = 2^i - 1 + 2^{i+1}\beta, \beta \in \{0\} \cup \mathbb{N}.$$

Therefore $p + 1 = 2^i \cdot o$ where here, and in what follows, "o" represents a generic odd integer, with not necessarily the same value in a given expression.

Since $e + 1$ is even, there exists a positive integer l such that $e + 1 = 2^l \cdot o$. Since $2^j \| \sigma(p^e)$ we have

$$\frac{p^{2^l \cdot o} - 1}{p - 1} = 2^j \cdot o$$

and therefore

$$(2^i \cdot o - 1)^{2^l \cdot o} - 1 = 2^j \cdot o \cdot (2^i \cdot o - 2). \tag{2.1}$$

(1a) If $i > 1$ examine both sides of equation (2.1) in base 2 and equate the lowest powers of 2. This leads to $i + l = j + 1$ since $2^i \cdot o - 2 = 2 \cdot o$. Therefore $l = j - i + 1$.

(1b) If $i = 1$ write $p + 1 = 2 \cdot o$ so $p - 1 = 2^k \cdot o$ for some $k \geq 2$. Hence, because $2^j || \sigma(p^e)$,

$$\begin{aligned} p^{2^l \cdot o} - 1 &= 2^j \cdot 2^k \cdot o \\ (1 + 2^k \cdot o)^{2^l \cdot o} - 1 &= 2^{j+k} \cdot o \end{aligned}$$

so, again comparing the lowest powers of 2 on both sides, $k + l = j + k$ so $l = j = j - 1 + 1$. Hence, for all $i \geq 1$, $l = j - i + 1$ and we can write

$$\begin{aligned} p &= 2^i - 1 + 2^{i+1} \cdot x \\ e &= 2^{j-i+1} - 1 + 2^{j+1-i+1} \cdot y \end{aligned}$$

where x, y are integers. Hence $(p + 1)(e + 1) = 2^{j+1}(1 + 2x)(1 + 2y)$ so $2^{j+1} || (p + 1)(e + 1)$.

(2) Conversely, let $2^{j+1} || (p + 1)(e + 1)$ so for some $i > 0$, $2^i || p + 1$ and $2^{j+1-i} || e + 1$. We now consider two cases, depending on the values of i and j.

(2a) Let $i = 1$ and $j = 1$. (This is really Euler's theorem). In this case $p + 1 = 2 \cdot o = 2(2x + 1)$ so $p = 4x + 1$ and $e + 1 = 2 \cdot o$. Therefore

$$\begin{aligned} \sigma(p^e) &= \frac{p^{2 \cdot o} - 1}{p - 1} = \frac{p^o - 1}{p - 1}(p^o + 1) \\ &= (1 + p + \cdots + p^{o-1})((4x + 1)^o + 1) \\ &= o \cdot (4y + 2) = 2 \cdot o \end{aligned}$$

so $2^1 || \sigma(p^e)$.

(2b) Let $i = 1$ and $j > 1$. Again $p = 4x + 1$. The inductive hypothesis is that for all $j' < j$, $2^{j'} || (p^{2^{j'} \cdot o} - 1)/(p - 1)$. Then

$$\begin{aligned} \sigma(p^e) &= \frac{p^{2^{j-1} \cdot o} - 1}{p - 1}(p^{2^{j-1} \cdot o} + 1) \\ &= 2^{j-1} \cdot o((4x + 1)^{2^{j-1} \cdot o} + 1) \\ &= 2^{j-1} \cdot o(4y + 2) \\ &= 2^j \cdot o \end{aligned}$$

so in this case also $2^j \| \sigma(p^e)$.

(2c) Let $i > 1$. First we make some preliminary polynomial constructions where all polynomials are in $\mathbb{Z}[x]$. For $n \in \mathbb{N}$ define f_n, q_n, s_n, r_n by

$$\begin{aligned} f_n(x) &= (1+x)^n - 1 = x q_n(x) \\ s_n(x) &= (1+x)^n + 1 = (x+2) r_n(x) \text{ for } n \text{ odd.} \end{aligned}$$

Then

$$f_{2\cdot o}(x) = ((1+x)^o - 1)((1+x)^o + 1) = x \cdot r_o(x) \cdot (x+2) \cdot q_o(x),$$

and for $l \geq 1$

$$\begin{aligned} f_{2^l \cdot o}(x) &= f_{2^{l-1} \cdot o}(x) \cdot s_{2^{l-1} \cdot o}(x) \\ &= s_{2^{l-1} \cdot o}(x) \cdot s_{2^{l-2} \cdot o}(x) \cdots s_{2 \cdot o}(x) x(x+2) \cdot r_o(x) \cdot q_o(x) \\ s_{2^l \cdot o}(x) &= (((1+x)^{2^l})^o - (-1)) \\ &= ((1+x)^{2^l} - (-1))(((1+x)^{2^l})^{o-1} + \cdots + 1) \\ &= ((1+x)^{2^l} + 1)(\cdots) \end{aligned}$$

Since $i > 1$, if $x = 2^i \cdot o - 2 = 2 \cdot o$, then $x + 1 = 2 \cdot o + 1 = o$ and

$$s_{2^l \cdot o}(x) = (o^{2^l} + 1) \text{ (an even number of odd terms } + 1) = 2 \cdot o \cdot o = 2 \cdot o,$$

and $x + 2 = 2^i \cdot o$. Note also that $q_o(x) = ((1+2y)^o - 1)/(2y) = o + 2z = o$ and $r_o(x) = ((1+x)^o + 1)/(x+2) = (o^{o-1} - o^{o-2} \cdots + 1) = o$, where x, y and z are integers. Therefore, with this value of x

$$\frac{f_{2^l \cdot o}(x)}{x} = 2^{l-1} \cdot o \cdot 2^i \cdot o \cdot o \cdot o = 2^{l+i-1} \cdot o.$$

Now, at last, we can complete the proof. Let $x = p - 1 = 2^i \cdot o - 2$ and $l = j + 1 - i$. Then

$$\begin{aligned} \sigma(p^e) &= \frac{p^{e+1} - 1}{p - 1} = \frac{(1+x)^{2^l \cdot o} - 1}{x} \\ &= \frac{f_{2^l \cdot o}(x)}{x} = 2^{l+i-1} \cdot o = 2^j \cdot o \end{aligned}$$

so $2^j || \sigma(p^e)$. $\square$

Remark: Theorem 2.5 can be recovered from early papers on Lucas sequences, ([7] and [106]). In a private communication, Florian Luca stated that the general version in this current setting is the following:

Theorem 2.6 (Luca) *(A) Let $q > 2$ and p be distinct primes and put f for the exponent of p modulo q. Then*

$$v_q\left(\frac{p^{e+1}-1}{p-1}\right) = v_q\left(\frac{p^f-1}{p-1}\right) + v_q\left(\frac{e+1}{f}\right). \tag{2.2}$$

In the above, it is understood that the right hand side is zero if f does not divide $e+1$.

(B) Let p be an odd prime. Then

$$v_2\left(\frac{p^{e+1}-1}{p-1}\right) = v_2\left(p+1\right) + v_2\left(\frac{e+1}{2}\right).$$

In the above, it is understood that the right hand side is zero if e is even.

Actually Theorem 2.5 is a corollary of Theorem 2.6 (B). Furthermore, if $q = 3$, we can obtain another corollary as follows:

Corollary 2.7 *Let p be a prime, $e \geq 1$ and suppose $3 \mid \sigma(p^e)$. Then*

$$v_3(\sigma(p^e)) = v_3((p+1)(e+1)).$$

Proof. Assume $3 \mid \sigma(p^e)$. By Theorem 2.6, taking $q = 3$, then either $p \equiv 1 \pmod 3$ or $p \equiv 2 \pmod 3$. In the first case, $f = 1$ and the right hand side of equation (2.2) becomes $v_3(e+1) = v_3((p+1)(e+1))$, since $p+1$ is not a multiple of 3. In the second case, $f = 2$ and if $e+1$ is odd, then the right hand side of equation (2.2) is not zero, but the left hand side is zero, which is a contradiction, so $e+1$ is even in which case it is $v_3(p+1) + v_3((e+1)/2) = v_3((p+1)(e+1))$, again as desired. □

The following corollary is a consequence of Theorem 2.6 (B), by taking $p = M_q$:

Corollary 2.8 *Let M_q be a Mersenne prime and let e be an odd positive integer. If $2^j||\sigma(M_q^e)$ then $j \geq q$.*

From Theorem 2.6 we also get the following corollary, which is an extension of Euler's theorem to perfect numbers of abundancy 2^k.

Corollary 2.9 *Let N be odd and 2^k-perfect. Then there exists a partition of k, $k = k_1 + \cdots + k_n$, with $k_i \geq 1$, such that*

$$N = \prod_{i=1}^{n} p_i^{e_i} \prod_{j=1}^{m} q_j^{2f_j}$$

where the e_i are odd, the p_i, q_j odd primes, and for each i with $1 \leq i \leq n$ there exist positive integers l_i and m_i such that $2^{l_i}||p_i + 1$, $2^{m_i}||e_i + 1$ and $l_i + m_i = k_i + 1$.

Proof. Let $\sigma(N) = 2^k N$. Since σ is a multiplicative function, then

$$\sigma\left(\prod_{i=1}^{n} p_i^{e_i} \prod_{j=1}^{m} q_j^{2f_j}\right) = \prod_{i=1}^{n} \sigma(p_i^{e_i}) \prod_{j=1}^{m} \sigma(q_j^{2f_j}) = 2^k \prod_{i=1}^{n} p_i^{e_i} \prod_{j=1}^{m} q_j^{2f_j}.$$

Since $2 \nmid \sigma(q_j^{2f_j})$, then

$$2^k || \prod_{i=1}^{n} \sigma(p_i^{e_i}), \text{ and } 2^{k_i}||\sigma(p_i^{e_i}),$$

so,

$$k = \sum_{i=1}^{n} k_i.$$

Also, by Theorem 2.6 (B),

$$2^{k_i+1}||(p_i + 1)(e_i + 1),$$

so, the result follows. □

Theorem 2.10 *(Cubes)*

Let N be an odd cube with $3 \nmid N$.

(A) If N has shape $N = q_1^{e_1} \cdot q_2^{e_2} \cdot p_1^{2\alpha_1} \cdots p_m^{2\alpha_m}$ with $q_1 \equiv 5 \pmod{12}$, $q_2 \equiv 1 \pmod 4$ and $e_1 \equiv e_2 \equiv 1 \pmod 4$, then N is not a 4-perfect number.

(B) If N has shape $N = q^e p_1^{2\alpha_1} \cdots p_m^{2\alpha_m}$ with $q \equiv 5 \pmod{12}$ and $e \equiv 3 \pmod 8$, then N is not a 4-perfect number.

(C) If N has shape $N = q^e p_1^{2\alpha_1} \cdots p_m^{2\alpha_m}$ with $q \equiv 11 \pmod{24}$ and $e \equiv 1 \pmod 4$, then N is not a 4-perfect number.

Proof. Let N be an odd cube with $3 \nmid N$ and $\sigma(N) = 4N$.

(A) Since N is a cube, then $e_1 \equiv 0 \pmod 3$, but $e_1 \equiv 1 \pmod 4$, so $e_1 \equiv 9 \pmod{12}$. Since $q_1 \equiv 5 \pmod{12}$, which implies $q_1 \equiv 2 \pmod 3$, by Lemma 2.2, we have $\sigma(q_1^{e_1}) \equiv 0 \pmod 3$.

Since σ function is multiplicative, then we can write

$$\sigma(N) = \sigma(q_1^{e_1})\sigma(q_2^{e_2})\sigma(p_1^{2\alpha_1}) \cdots \sigma(p_m^{2\alpha_m}),$$

the first factor on the right hand side is congruent to 0 modulo 3, so $\sigma(N) \equiv 0 \pmod 3$. Since $\sigma(N) = 4N$,

$$0 \equiv q_1^{e_1} q_2^{e_2} p_1^{2\alpha_1} \cdots p_m^{2\alpha_m} \pmod 3,$$

but each factor on the right hand side is non-zero modulo 3. Hence N is not 4-perfect.

In part (B), $q \equiv 5 \pmod{12}$ implies $q \equiv 2 \pmod 3$. Since $e \equiv 3 \pmod 8$ and $e \equiv 0 \pmod 3$, then

$$\sigma(q^e) = \sigma(q^{3+24x}) \equiv 0 \pmod 3, \text{ where } x \text{ is some integer.}$$

In part (C), $q \equiv 11 \pmod{24}$ implies $q \equiv 2 \pmod 3$. Since $e \equiv 1 \pmod 4$ and $e \equiv 0 \pmod 3$, then

$$\sigma(q^e) = \sigma(q^{9+12y}) \equiv 0 \pmod 3, \text{ where } y \text{ is some integer.}$$

By the same argument as part (A), the results in parts (B) and (C) also follow. □

Theorem 2.11 *Let N be an odd positive integer with $3^2 \| N$ and if any one of* 13, 61 *or* 97 *appears in the prime factorization of N, then it does so to a power congruent to* 2 *modulo* 6. *Then N is not an odd* 4*-perfect number.*

Proof. Let the hypotheses of the theorem hold for N, but let it also be odd and $\sigma(N) = 4N$.

Now if 13, 61 or 97 appear, even though each is congruent to 1 modulo 4, their powers, being congruent to 2 modulo 6, are even, so must appear amongst the p_i in each of the three shapes given in Theorem 2.3.

Then $3^2 \| N$ implies $13 = \sigma(3^2) \mid N$. So $13^{2+6e_1} || N$ with $e_1 \geq 0$, and by the argument given below, $61^{2+6e_2} || N$ and $97^{2+6e_3} || N$ with $e_2, e_3 \geq 0$.

Now, by Lemma 2.1, for all primes p and positive numbers e, $\sigma(p^2) \mid \sigma(p^{2+6e})$. So $3 \cdot 61 = \sigma(13^2) \mid \sigma(13^{2+6e_1}) \mid \sigma(N) = 4N$, which implies $3 \cdot 61 \mid N$. Again $3 \cdot 13 \cdot 97 = \sigma(61^2) \mid \sigma(61^{2+6e_2}) \mid \sigma(N)$, which implies $3 \cdot 13 \cdot 97 \mid N$. Finally $3 \cdot 3169 = \sigma(97^2) \mid \sigma(97^{2+6e_3}) \mid \sigma(N)$, so $3^3 \mid \sigma(13^{2+6e_1})\sigma(61^{2+6e_2})\sigma(97^{2+6e_3}) \mid \sigma(N)$. Therefore $3^3 \mid N$, which is a contradiction to $3^2 \| N$. Therefore N is not 4-perfect. □

The following result uses techniques similar to those developed for 2-perfect numbers by Steuerwald in [95].

Theorem 2.12 *(Small powers)*

(1) If N is odd, $3 \mid N$ and N has the shape either (1a) $N = q_1^{1+4e_1} \cdot q_2^{1+4e_2} \cdot 3^2 \cdot p_1^2 \cdots p_m^2$ or (1b) $N = q_1^{3+8e_1} \cdot 3^2 \cdot p_1^2 \cdots p_m^2$ where, in either case, $q_i \equiv 1 \pmod 4$, or (1c) $q_1^{1+4e} \cdot 3^2 \cdot p_1^2 \cdots p_m^2$, where $q_1 \equiv 3 \pmod 8$, where the primes are distinct, then N is not an odd 4*-perfect number.*

(2) If N is odd, $3 \nmid N$ and N has the shape either (2a) $N = q^{3+8e} \cdot p_1^2 \cdots p_m^2$ with $q \equiv 1 \pmod 4$ or (2b) $N = q^{1+4e} \cdot p_1^2 \cdots p_m^2$ with $q \equiv 3 \pmod 8$, or (2c) $N = q_1^{1+4e_1} \cdot q_2^{1+4e_2} \cdot p_1^2 \cdots p_m^2$, with $q_i \equiv 1 \pmod 4$, then N is not a 4*-perfect number.*

Proof. (1) Let N satisfy $\sigma(N) = 4N$. Then $\sigma(3^2) = 13 \mid N$. In case (1c), q_1 is *not* in the set $\{13, 61, 97\}$. Assume first that the q_i are not in this set in cases (1a) and (1b). (Below we consider the situation which arises when a q_i *is* in this set.)

Under this assumption we obtain the chain:

$$\sigma(13^2) = 3 \cdot 61, \quad \sigma(61^2) = 3 \cdot 13 \cdot 97, \quad \sigma(97^2) = 3 \cdot 3169,$$

so $3^3 \mid N$, which is false. Hence N is not 4-perfect.

Since the exponent of each q_i is odd, for $q = q_1$ or q_2, $e = e_1$ or e_2, $q + 1 \mid \sigma(q^e)$.

If $q = 13$, since $q + 1 \mid N$ we obtain the chain:

$$\sigma(7^2) = 3 \cdot 19 \mid N, \sigma(19^2) = 3 \cdot 127 \mid N, \sigma(127^2) = 3 \cdot 5419 \mid N,$$

giving $3^3 \mid N$, which is false.

If $q = 61$ we can assume also $\sigma(13^2) = 3 \cdot 61 \mid N$. Again, since $q + 1 \mid N$ We obtain the chain:

$$\sigma(31^2) = 3 \cdot 331 \mid N, \sigma(331^2) = 3 \cdot 7 \cdot 5233 \mid N, \sigma(127^2) = 3 \cdot 5419 \mid N,$$

again giving $3^3 \mid N$, which is false.

If $q = 97$ then $(q+1)/2 = 7^2 \mid N$ and the same chain as in the $q = 13$ case can be derived with the same conclusion. Thus our assumption that no q_i is in the set $\{13, 61, 97\}$ is valid and the proof is complete.

(2a) and (2b): Let N satisfy $\sigma(N) = 4N$ and $3 \nmid N$, with shape

$$N = q^f \cdot p_1^2 \cdots p_m^2,$$

where $3 < p_1 < \cdots < p_m$ and f is odd.

Since, for each i, $\sigma(p_i^2) = 1 + p_i + p_i^2$ and $3 \nmid N$, we must have $p_i \equiv 2 \pmod 3$.

By Theorem 2.3, q is congruent to 1 modulo 4 or 3 modulo 8. Because f is odd, $q+1 \mid \sigma(q^f) \mid N$ and since also $3 \nmid N$ we cannot have $q \equiv 2 \pmod 3$, so must have $q \equiv 1 \pmod 3$.

Since $\sigma(p_1^2) < (p_1+1)^2 < p_2^2$, $\sigma(p_1^2)$ is divisible by at most one p_i. Therefore either (a) $\sigma(p_1^2) = q^g$ with $1 \leq g$ or (b) $\sigma(p_1^2) = q^g \cdot p_i$ for some i. Case (b) is impossible, since it is invalid modulo 3. In case (a), [8, Lemma 1] shows the only possibility is $g = 1$.

Let $x = (q+1)/2$. Then $x \equiv 1 \pmod 3$. Since x is too small to include a power of at least two q's, it must be a product of the p_i. We cannot have $x = p_i$ since $p_i \equiv 2 \pmod 3$, so it must have at least 2 prime factors, with the smallest factor being less than or equal to $\sqrt{x}$, and therefore $p_i \leq \sqrt{x}$ for some i. But then

$$q = 1 + p_1 + p_1^2 \leq 1 + p_i + p_i^2 \leq 1 + \sqrt{x} + x \leq \frac{q+3}{2} + \sqrt{\frac{q+1}{2}}$$

so $q = 5$ or $q = 7$. Each of these is impossible since $q \equiv 1 \pmod 4$ and $1 \pmod 3$ or $q \equiv 3 \pmod 8$.

(2c): Now let $N = q_1^{1+4e_1} q_2^{1+4e_2} p_1^2 \cdots p_m^2$, be odd and 4-perfect with $3 \nmid N$. Since $\sigma(N) = 4N$ we can write:

$$\sigma(q_1^{1+4e_1})\sigma(q_2^{1+4e_2})(1+p_1+p_1^2)(\cdots)(1+p_m+p_m^2) = 4q_1^{1+4e_1} q_2^{1+4e_2} p_1^2 \cdots p_m^2.$$

Considering this equation modulo 3 shows each $p_i \equiv 2 \pmod 3$ and then $\sigma(q_1^{1+4e_1})\sigma(q_2^{1+4e_2}) \equiv q_1^{1+4e_1} q_2^{1+4e_2} \pmod 3$. But $q_i \equiv 2 \bmod 3$ implies, by Lemma 2.2, $3 \mid \sigma(q_i^{1+4e_i})$, which is impossible. This means $q_1 \equiv 1 \pmod 3$, $q_2 \equiv 1 \pmod 3$.

(Now we modify the argument of Steuerwald, and find that the Lemma of Brauer [8, Lemma 1] is not needed.) Since $\sigma(p_1^2) < p_2^2$, $\sigma(p_1^2)$ is divisible by at most one of the p_i, so we can write

$$\sigma(p_1^2) = q_1^{g_1} q_2^{g_2} p_i \text{ or } \sigma(p_1^2) = q_1^{g_1} q_2^{g_2} \text{ or } \sigma(p_1^2) = q_1^{g_1} \text{ or } \sigma(p_1^2) = q_2^{g_2},$$

where $q_1 < q_2$, $g_i \geq 1$ except in the first case where $g_i \geq 0$. Consideration of these possibilities modulo 3 shows that the first case cannot occur.

Since e_1 is odd, by Lemma 2.1, $x = \frac{q_1+1}{2} \mid N$ and $x \equiv 1 \pmod 3$. Now x is too small to include a q_i in its prime factorization, so must be a product of the p_i. We cannot have $x = p_i$ (consider modulo 3 again), so there must be two or more of the p_i in the factorization of x, so there exists an i with $p_i \leq \sqrt{x}$. But then, in all remaining cases,

$$q_1 \leq 1 + p_1 + p_1^2 \leq 1 + p_i + p_i^2 \leq 1 + \sqrt{x} + x = 1 + \sqrt{\frac{q_1+1}{2}} + \frac{q_1+1}{2},$$

so $q_1 \leq 1 + \sqrt{\frac{q_1+1}{2}} + \frac{q_1+1}{2}$. But this means q_1 must be $2, 3, 5$ or 7. Each of these is impossible, since $q \equiv 1 \pmod 4$ and $1 \pmod 3$. This contradiction verifies our conclusion (that no such 4-perfect number exists) in this final case. □

If we call the leading prime(s) to odd power(s) with special shape the "Euler part" and the rest the "squared part", then the previous result says that "no odd 4-perfect number exists with squared part a square of a squarefree number".

The following result is based on the technique of Starni [94] whose theorem, for 2-perfect numbers, had uniform powers for the p_i. This, in turn depended on a result of McDaniel [69] (incorrectly cited), where the powers are not uniform.

Theorem 2.13 *Let $N = \Pi 3^{2\beta} \prod_{i=1}^{M} p_i^{2\alpha_i}$ be odd and 2^k-perfect, where the p_i's are distinct odd primes with $p_i > 3$, $\beta > 0$, the Euler part Π has any of the forms given by Theorem 2.3, and, for all i $\alpha_i \not\equiv 1 \pmod 3$. Then $3^{2\beta} \mid \sigma(\Pi)$.*

Proof. Firstly $(\sigma(3^{2\beta}), 3^{2\beta}) = 1$. Since $\alpha_i \not\equiv 1 \pmod 3$, $1 + 2\alpha_i \equiv 1, 5 \pmod 6$. Since $p_i \equiv 1, -1 \pmod 6$, $\sigma(p_i^{2\alpha_i}) \equiv 1 \pmod 6$ if $p_i \equiv -1 \pmod 6$, or $\sigma(p_i^{2\alpha_i}) \equiv 1 + 2\alpha_i \pmod 6$ if $p_i \equiv 1 \pmod 6$. But then, subject maybe to

some reordering, there exists an $m \geq 0$ with

$$\begin{aligned} P := \prod_{i=1}^{M} \sigma(p_i^{2\alpha_i}) &\equiv \prod_{i=1}^{m} (1 + 2\alpha_i) \pmod 6 \\ &\equiv \prod_{i=1}^{m} (1 + 2\alpha_i) \pmod 3. \end{aligned}$$

By the given assumption, $1 + 2\alpha_i \not\equiv 0 \pmod 3$, so $P \not\equiv 0 \pmod 3$, and thus $(P, 3^{2\beta}) = 1$.

But for some positive integer k, $\sigma(N) = 2^k \cdot N$ so therefore

$$\sigma(\Pi)\sigma(3^{2\beta})P = 2^k \Pi 3^{2\beta} \prod_{i=1}^{M} p_i^{2\alpha_i}.$$

Therefore $3^{2\beta} \mid \sigma(\Pi)$. □

Chapter 3

The factorization of the sum of divisors

3.1 Introduction

In this chapter we examine the factorization of $N = \sigma(q^e)$. Of course there is the well known factorization using cyclotomic polynomials, but we take a different point of view, examining which primes and the power to which each prime divides N.

The purpose of this study is to lay some groundwork to continue the study of multiperfect numbers begun in Chapter 2. Some well known concepts are applied, in case p and q are primes, the exponent of q modulo p, $\exp_p q$, and a positive integer which is called here the discrete power of p to base q, $[p|q]$.

For example $\exp_2 q = 1$ for all odd primes q. If $q \neq 3$, $\exp_3 q \equiv q \pmod 3 = \frac{3-(q|3)}{2}$, where we have used the least positive residue and $(a|b)$ is the Legendre symbol. If $p > q$ then $\exp_p q > 1$. If $\exp_p q > 1$ and $\alpha > 1$ then $\exp_p q \mid \alpha$ if and only if $p \mid q^\alpha - 1$.

As a consequence of the above concepts we can write

$$p^{[p|q]} || q^{\exp_p q} - 1.$$

It is obvious that $1 \leq \exp_p q \mid p-1$ (by Fermat's Theorem [47, p.63]) and $\exp_p q = 1$ if and only if $p \mid q-1$ (by the definition of the exponent of q modulo p). Also $1 \leq [p|q] \leq \exp_p q \cdot \log q / \log p$ by taking the logarithm of each side of the inequality $p^{[p|q]} \leq q^{\exp_p q}$. For each $a \geq 1$ and $q \geq 3$, there exists at most a finite number of odd primes p with $\exp_p q = a$, (since the values of p depend on $q^a - 1$).

In Section 3.2 we set out a number of lemmas and corollaries, using the language of discrete powers and exponents. These add some insight into the question of why most prime divisors of $\sigma(q^e)$ occur to the first power. Section 3.3 has the main theorem of this chapter, namely Theorem 3.9 which gives a complete breakdown of the prime factorization of $\sigma(q^e)$. For example when $q = 2$:

$$\sigma(2^e) = \prod_{1<\exp_p 2 \mid e+1} p^{[p|2]+v_p(e+1)}.$$

The corollaries of this theorem include necessary condition $e+1 \mid f+1$ for $\sigma(q^e)$ to divide $\sigma(q^f)$.

In Section 3.4 a set of conditions is found under which $\sigma(q^e)$ is a prime power. This is closely related to a classical question studied by many people including Suryanarayana [97], Edgar [31], and Estes [34] et al. Although in our case by limiting the scope to primes q, we are able to use Theorem 3.9 to obtain a converse result which is Theorem 3.15.

3.2 Exponent of q modulo p

The following two lemmas and their proofs are derived from standard results [84, Part P] which date back to Euler.

Lemma 3.1 *If p is an odd prime and x an integer such that $p \mid x-1$, then*

for every $i \geq 0$

$$v_p(x^{p^i} - 1) = v_p(x-1) + i.$$

Proof. Let $v_p(x-1) = e \geq 1$. Then for some integer y, $x = 1 + yp^e$. Therefore, using the Binomial Theorem,

$$x^p - 1 = \binom{p}{1} yp^e + \binom{p}{2} y^2 p^{2e} + \cdots + \binom{p}{p} y^p p^{pe}.$$

But $v_p(\binom{p}{1} yp^e) = e + 1$ and for $2 \leq j < p$, $v_p(\binom{p}{j} y^j p^{je}) \geq 1 + je$. Using the property $v_p(u) < v_p(v)$ implies $v_p(u+v) = v_p(u)$ gives

$$\begin{aligned} v_p(x^p - 1) &= v_p\left(\binom{p}{1} yp^e + \binom{p}{2} y^2 p^{2e} + \cdots + \binom{p}{p} y^p p^{pe}\right) \\ &= v_p\left(\binom{p}{1} yp^e\right) = e + 1. \end{aligned}$$

The proof of the lemma is completed using induction, replacing x by x^p. □

Lemma 3.2 *If p is an odd prime and $x > 1$ an integer with $p \mid x - 1$ then for every $e \geq 1$*

$$v_p\left(\frac{x^e - 1}{x - 1}\right) = v_p(e).$$

Proof. Let $e = mp^f$ with $(m, p) = 1$, $f \geq 0$ and $f = v_p(e)$. Since $x \equiv 1 \pmod{p}$, $1 + x + \cdots + x^{m-1} \equiv m \pmod{p}$ and $p \nmid m$, so $v_p(\frac{x^m - 1}{x-1}) = 0$. Therefore $v_p(x^m - 1) = v_p(x - 1) \geq 1$, where the last inequality follows from the hypothesis $p \mid x - 1$.

Then, by Lemma 3.1 applied to x^m, $v_p(x^{mp^f} - 1) = v_p(x^m - 1) + f$ so

$$v_p\left(\frac{x^e - 1}{x^m - 1}\right) = v_p\left(\frac{x^{mp^f} - 1}{x^m - 1}\right) = f = v_p(e).$$

□

Theorem 3.3 *(Prime factorization of $\sigma(q^e)$)*

Let $i \geq 1$ and p be any odd prime, q a prime with $q \geq 2$ such that $p \neq q$. Then

(1) if $\exp_p q = 1$ *then* $p \mid \sigma(q^e)$ *if and only if* $p \mid e+1$*, and*

(2) if $\exp_p q > 1$ *then* $p \mid \sigma(q^e)$ *if and only if* $\exp_p q \mid e+1$.

In case (1) $p^i || \sigma(q^e)$ *if and only if for some* $h \geq 1$ *with* $(h,p) = 1$ *we have* $e+1 = p^i \cdot h$.

In case (2) $p^i || \sigma(q^e)$ *if and only if for some* $h \geq 1$ *with* $(h,p) = 1$ *we have* $e+1 = p^{i-[p|q]} \cdot \exp_p q \cdot h$.

Proof. (1) Now $\exp_p q = 1$ if and only if $p \mid q-1$. By Lemma 3.2

$$v_p(\sigma(q^e)) = v_p\left(\frac{q^{e+1}-1}{q-1}\right) = v_p(e+1)$$

and both implications of this part follow directly.

(2) First the divisibility criteria: If $\exp_p q > 1$ we have $p \nmid q-1$, so $q-1 \not\equiv 0 \pmod{p}$. Hence $p \mid \sigma(q^e)$ if and only if $p \mid q^{e+1}-1$ which is true if and only if $\exp_p q \mid e+1$.

Now consider the order of p when it does divide $\sigma(q^e)$: note first that $\exp_p q \geq 2$ if and only if $p \nmid q-1$ if and only if $v_p(q-1) = 0$. First assume that $h = 1$ and $e+1 = p^{i-[p|q]} \cdot \exp_p q$ for some $i \geq [p|q]$. Then $p^{[p|q]} || q^{\exp_p q} - 1$ implies $v_p(q^{\exp_p q} - 1) = [p|q] \geq 1$. By Lemma 3.1, for all $j \geq 0$, $v_p(q^{\exp_p q \cdot p^j} - 1) = [p|q] + j$, so if we write $i = v_p(q^{\exp_p q \cdot p^j} - 1)$, it follows that $j = i - [p|q]$. Therefore

$$\begin{aligned} v_p(\sigma(q^e)) &= v_p\left(\frac{q^{e+1}-1}{q-1}\right) = v_p(q^{e+1}-1) \\ &= v_p(q^{\exp_p q \cdot p^{i-[p|q]}} - 1) = i. \end{aligned}$$

The result for the form $e+1 = p^{i-[p|q]} \cdot \exp_p q \cdot h$ with $h > 1, (h,p) = 1$ follows by replacing q by q^h in the above argument. □

For example when $p = 3$ and $q = 2$, if $h \geq 1$ has $(h,3) = 1$ and $i \geq 1$, then

$$3^i || \sigma(2^{2 \cdot 3^{i-1} h - 1})$$

and these are the only possibilities for $3^i||\sigma(2^e)$.

It is a matter of observation that numbers of the form $\sigma(q^e)$, apparently, are never powerful. Indeed, most prime factors of such numbers are to power 1, and it is only the 'small' primes which occur to powers higher than 1. The next result is an attempt to quantify this observed phenomena: It is really just a restatement of Theorem 3.3.

Corollary 3.4 *Let $p^i\|\sigma(q^e)$ where, as before, p is an odd prime, q a prime, and $e, i \geq 1$. If $p \mid (q-1, e+1)$ then $i = v_p(e+1)$. If $p \nmid q-1$ and $\exp_p q \mid e+1$ then $i = v_p(e+1) + [p|q]$.*

So for a prime to divide $\sigma(q^e)$ to the power 2 or greater, that prime must either divide $e+1$ to a power higher than 1 if it divides $q-1$, or have its exponent of q modulo p divide $e+1$ and itself divide $e+1$, or satisfy this exponent condition and have a discrete power to base q which is 2 or more. So even though there are three possible situations for the square of a prime to divide $\sigma(q^e)$, they all restrict the size of the prime in terms of e or are quite hard to satisfy.

Note that for each q there are only a finite number of primes with $\exp_p q = 1$, namely the prime divisors of $q-1$. Also prime pairs with $[p|q] > 1$ are somewhat rare - discrete powers are usually 1.

Among the primes $p < 10^6$ and for $q \in \{2, 3, 5, 7, 11, 13, 17, 19\}$, the only discrete powers which are greater than 1 are:

(1) $q = 2$, $1093^2\|2^{364} - 1$; $3511^2\|2^{1755} - 1$;

(2) $q = 3$, $11^2\|3^5 - 1$;

(3) $q = 5$, $2^2\|5^1 - 1$; $20771^2\|5^{10385} - 1$; $40487^2\|5^{40486} - 1$;

(4) $q = 7$, $5^2\|7^4 - 1$;

(5) $q = 11$, $71^2\|11^{70} - 1$;

(6) $q = 13$, $2^2\|13^1 - 1$; $863^2\|13^{862} - 1$;

(7) $q = 17$, $2^4 \| 17^1 - 1$; $3^2 \| 17^2 - 1$; $46021^2 \| 17^{7670} - 1$; $48947^2 \| 17^{24473} - 1$;

(8) $q = 19$, $3^2 \| 19^1 - 1$; $7^3 \| 19^6 - 1$; $13^2 \| 19^{12} - 1$; $43^2 \| 19^{42} - 1$; $137^2 \| 19^{68} - 1$.

Corollary 3.5 *If $e+1$ is prime or a power of 2, then the power of any prime p dividing $\sigma(2^e)$ is given by $[p|2]$.*

By considering the equation $\sigma(N) = kN$ modulo 2 it is easy to see that, if N is even with $N = 2^a \cdot p_1^{2\alpha_1} \cdots p_m^{2\alpha_m}$, $a \geq 1$ and $k \geq 2$, then N is not k-perfect. It follows that no even square is k-perfect.

Corollary 3.6 *Let p be an odd prime and q a distinct prime with $\exp_p q > 1$. If $p \mid e+1$ then $p^i || \sigma(q^e)$ with $i \geq 2$. If $p \nmid e+1$ and $[p|q] = 1$ then $p^i || \sigma(q^e)$ with $i = 1$.*

Corollary 3.7 *If $p^2 \mid \sigma(q^e)$ and $\exp_p q > 1$ and $[p|q] = 1$ then $p \mid e+1$.*

Example 3.1 *Consider the factors of $\sigma(2^{209})$:*

$$\begin{aligned} \sigma(2^{(2\cdot 3\cdot 5\cdot 7-1)}) &= 3^2 \cdot 7^2 \cdot 11 \cdot 31 \cdot 43 \cdot 71 \cdot 127 \cdot 151 \cdot 211 \cdot \\ &\quad 281 \cdot 331 \cdot 337 \cdot 5419 \cdot 29191 \cdot 86171 \cdot 106681 \cdot \\ &\quad 122921 \cdot 152041 \cdot 664441 \cdot 1564921. \end{aligned}$$

Here the only prime factors which appear on the right hand side, to other than the first power are divisors of $209+1$ in accordance with the corollaries.

Corollary 3.8 *Let p and q be odd primes with $\exp_p q > 1$ and such that $\exp_p q \mid e+1$. If $p^j \| \sigma(q^e)$, then*

$$[p|q] \leq j \leq [p|q] + \left(\frac{\log(\frac{e+1}{\exp_p q})}{\log p} \right).$$

Hence if $p > (e+1)/\exp_p q$, then $j = [p|q]$.

This again explains why most large primes which appear in the factorization of $\sigma(q^e)$ do so to the first power. Moreover, to visualize this we use the following graph.

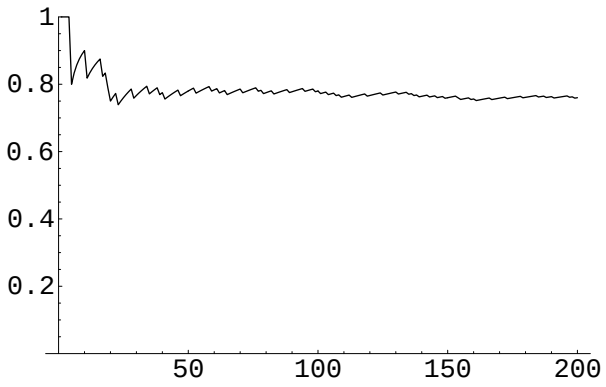

Figure 3.1: The proportion of squarefree $\sigma(2^n)$, more than 70% of n where $1 \leq n \leq 200$.

This graph is a plot of b_n versus n where

$$b_n := \#\{e | e \leq n, \sigma(2^e) \text{ is squarefree}\}.$$

It indicates that not only large primes appear to power 1, but in a significant proportion of cases, all primes appear to power 1 in $\sigma(2^e)$. I was not able to prove this but have the conjecture: there exists a constant $\alpha > 0$ such that

$$\liminf_{n\to\infty} \left(\frac{b_n}{n}\right) \geq \alpha.$$

3.3 The factorization theorem

Theorem 3.9 *Let q be an odd prime and e a positive integer. Then*

$$\begin{aligned}\sigma(q^e) &= 2^{(v_2((q+1)(e+1))-1)(2\{\frac{e}{2}\})} \times \prod_{p|(e+1,q-1)} p^{v_p(e+1)} \\ &\quad \times \prod_{1\neq \exp_p q | e+1} p^{[p|q]+v_p(\frac{e+1}{\exp_p q})}\end{aligned}$$

where p is an odd prime.

Let $q = 2$ and e a positive integer, then

$$\sigma(2^e) = \prod_{1<\exp_p 2|e+1} p^{[p|2]+v_p(e+1)}$$

where p is an odd prime.

Proof. Let $\sigma(q^e) = \prod_{i=1}^{m} p_i^{\alpha_i}$, then we can separate the p_i's into three types.

Type 1: If $p = 2$. If e is even, then $2 \nmid \sigma(q^e)$, because $\sigma(q^e)$ is an odd number. If e is odd, then by Theorem 2.5, the highest power of base 2 in $\sigma(q^e)$ is $v_2((q+1)(e+1)) - 1$, so, the component of the prime 2 in $\sigma(q^e)$ is $2^{v_2((q+1)(e+1))-1}$.

Type 2: If p is an odd prime and $\exp_p q = 1$. Then, $p \mid q - 1$. And, by Theorem 3.3 part (1), we have $p^j \| \sigma(q^e) \Leftrightarrow e + 1 = p^j \cdot h$, for some $h \geq 1$, with $(h, p) = 1$, so, $j = v_p(e+1)$. So, the component of type 2 in $\sigma(q^e)$ is $\prod_{p|(e+1,q-1)} p^{v_p(e+1)}$.

Type 3: If p is an odd prime and $\exp_p q > 1$. Then, $\exp_p q | e + 1$. Also, by Theorem 3.3 part (2), we have

$$p^j \| \sigma(q^e) \Leftrightarrow e + 1 = p^{j-[p|q]} \cdot \exp_p q \cdot h,$$

for some $h \geq 1$, with $(h, p) = 1$, so

$$j = [p|q] + v_p\left(\frac{e+1}{\exp_p q}\right).$$

Thus the component of type 3 in $\sigma(q^e)$ is

$$\prod_{1 \neq \exp_p q | e+1} p^{[p|q]+v_p(\frac{e+1}{\exp_p q})}.$$

Therefore the factorization of $\sigma(q^e)$ is:

$$\begin{aligned} \sigma(q^e) &= 2^{(v_2((q+1)(e+1))-1)(2\{\frac{e}{2}\})} \times \prod_{p|(e+1,q-1)} p^{v_p(e+1)} \\ &\quad \times \prod_{1 \neq \exp_p q | e+1} p^{[p|q]+v_p(\frac{e+1}{\exp_p q})} \end{aligned}$$

where the products extend over odd primes p that satisfy the given conditions.

□

Example 3.2 *We use the factorization theorem of the σ function to explain the 29th 4-perfect number d_{29}, (see Table 6.1).*

$$d_{29} = 2^{25} \cdot 3^4 \cdot 7 \cdot 11^2 \cdot 19^4 \cdot 151 \cdot 683 \cdot 911 \cdot 2731 \cdot 8191.$$

Since σ function is multiplicative, then $\sigma(d_{29})$ can be expressed as a product of divisor sums that involve prime factors of d_{29}, and this is why $\sigma(151)$ is being considered.

$$\begin{aligned}\sigma(151) &= 2^{(v_2((151+1)(1+1))-1)(2\{\frac{1}{2}\})} \times \prod_{p|(1+1,151-1)} p^{v_p(1+1)} \\ &\quad \times \prod_{1\neq \exp_p 151 | 1+1} p^{[p|151]+v_p(\frac{1+1}{\exp_p 151})} \\ &= 2^3 \cdot 19.\end{aligned}$$

Since $(e+1, q-1) = (2, 150) = 2$, $p \nmid 2$ for any odd prime p, so the term $\prod_{p|(e+1,q-1)} p^{v_p(e+1)}$ does not exist.

Since $1 \neq \exp_p 151 \mid 2$, so $\exp_p 151 = 2$, then $p \mid 151^2 - 1 = 2^4 \cdot 3 \cdot 5^2 \cdot 19$, since $3 \mid 151^1 - 1$ and $5 \mid 151^1 - 1$, so we only choose $p = 19$. Since $[19|151] = 1$ and $v_{19}(\frac{2}{\exp_{19} 151}) = 0$, so the term $\prod_{1\neq \exp_p q | e+1} p^{[p|q]+v_p(\frac{e+1}{\exp_p q})} = 19$.

The same method can be applied for the other prime divisors on $\sigma(d_{29})$.

The next result is the converse of Lemma 2.1. It is well known but useful.

Lemma 3.10 *If $\sigma(q^e) \mid \sigma(q^f)$, then $e + 1 \mid f + 1$.*

Proof. Let $\sigma(q^e) \mid \sigma(q^f)$ and $e < f$. Suppose $e + 1 \nmid f + 1$, then

$$(f+1) = (r+1)(e+1) + s, \text{ where } r \geq 0,\ 1 \leq s < e+1.$$

Then there exists $n \in \mathbb{N}$, such that

$$\begin{aligned}(1 + q + \cdots + q^e)n &= (1 + q + q^2 + q^3 + \cdots + q^{r(e+1)+e}) + \\ &\quad q^{r(e+1)+e+1} + q^{r(e+1)+e+2} + \cdots + q^f.\end{aligned}$$

But by

$$(1+q+\cdots+q^e)(1+q^{e+1}+q^{2(e+1)}+\cdots+q^{r(e+1)}) = (1+q+q^2+q^3+\cdots+q^{r(e+1)+e}),$$

there exists $m \in \mathbb{N}$, such that

$$(1+q+\cdots+q^e)m = q^{(r+1)(e+1)}(1+q+q^2+\cdots+q^{f-(r+1)(e+1)}).$$

But $q \nmid (1+q+\cdots+q^e)$ implies

$$1+q+\cdots+q^e \mid 1+q+\cdots+q^{f-(r+1)(e+1)}$$

so

$$f-(r+1)(e+1) \geq e,$$

implies

$$s \geq e+1,$$

which is a contradiction, so

$$e+1 \mid f+1.$$

□

Note this is also a corollary of Theorem 3.17(B) given below.

Corollary 3.11 *If n, m are positive integers, and for all odd p if $\exp_p 2 \mid n$ implies $\exp_p 2 \mid m$, then $n \mid m$.*

Example 3.3 *We can also use Theorem 3.9 to examine cases where $\sigma(q^e)$ is not squarefree, or investigate the forms directly.*

(1) $\sigma(2^e)$ is not squarefree if and only if there exists an odd prime p with $\exp_p 2 \mid e+1$ and either $[p|2] \geq 2$ or $p \mid e+1$. This is the case if for example, $e \equiv 5 \pmod 6$ when $p = 3$ or $e \equiv 19 \pmod{20}$ when $p = 5$.

(2) $\sigma(3^e)$ is not squarefree if and only if either e is odd (when $4 \mid \sigma(3^e)$) or there exists a prime $p > 3$ with $\exp_p 3 \mid e+1$ and ($[p|3] > 1$ or $p \mid e+1$).

(3) $\sigma(5^e)$ *is not squarefree if* $e \equiv 3 \pmod 4$ *or* $e \equiv 2 \pmod 3$ *or* $e \equiv 41 \pmod{42}$ *when* $2^2 \mid \sigma(5^e)$, $3^2 \mid \sigma(5^e)$ *and* $7^2 \mid \sigma(5^e)$ *respectively. Note that if* $7^2 \mid \sigma(5^e)$ *we have also* $3^2 \mid \sigma(5^e)$.

(4) Let q *be an odd prime with* $q \equiv 3 \pmod 4$ *and let* e *be odd. Then* $2^2 \mid \sigma(q^e)$ *so* $\sigma(q^e)$ *is not squarefree.*

Corollary 3.12 *Let* q *be an odd prime and* $e \geq 1$. *Then* $\sigma(q^e)$ *is not squarefree if and only if*

either (1) $q \equiv 3 \pmod 4$ *or* $e \equiv 3 \pmod 4$*;*

or (2) there exists an odd prime p *with either subcase (a) (*$\exp_p q = 1$ *and* $p^2 \mid e+1$*); or subcase (b) (*$\exp_p q > 1$ *and* $\exp_p q \mid e+1$*) together with (either* $[p|q] > 1$ *or* $p \mid e+1$*).*

Proof. ($\Rightarrow$): If $\sigma(q^e)$ is not squarefree, then, $\exists\, p \in \mathbb{P}$, such that $p^i \| \sigma(q^e)$, with $i \geq 2$.

Case 1: If $2^i \| \sigma(q^e)$, with $i \geq 2$, then, both q and e are odd. By Theorem 2.5, we have, for all odd primes q, powers $i \geq 1$ and odd exponents $e > 0$, then,

$$2^i \| \sigma(q^e) \Leftrightarrow 2^{i+1} \| (q+1)(e+1).$$

So $2^3 \mid (q+1)(e+1)$ therefore $2 \mid (\frac{q+1}{2})(\frac{e+1}{2})$, so either $2 \mid \frac{q+1}{2} \Rightarrow q \equiv 3 \pmod 4$; or $2 \mid \frac{e+1}{2} \Rightarrow e \equiv 3 \pmod 4$.

Case 2: If $p^i \| \sigma(q^e)$, with $i \geq 2$ and p is an odd prime and q a prime, then, $p \neq q$. Then, by Theorem 3.3, we have possibility subcase (a), if $\exp_p q = 1$, then, $p^i \| \sigma(q^e) \Leftrightarrow e+1 = p^i \cdot h$, for some $h \geq 1$ with $(h,p) = 1$, so $p^2 \mid e+1$; or possibility subcase (b), if $\exp_p q > 1$, then, $p^i \| \sigma(q^e) \Leftrightarrow e+1 = p^{i-[p|q]} \cdot \exp_p q \cdot h$, for some $h \geq 1$ with $(h,p) = 1$, so, $i - [p|q] \geq 0 \Rightarrow i \geq [p|q]$, since $i \geq 2$, so either $[p|q] \geq 2$, or $i - [p|q] \geq 1 \Rightarrow p \mid e+1$.

($\Leftarrow$): Case 1: If $q \equiv 3 \pmod 4$ and e is odd, then,

$$\begin{aligned}\sigma(q^e) &= 1 + q + q^2 + \cdots + q^e \\ &\equiv 1 + (-1) + (-1)^2 + \cdots + (-1)^e \\ &\equiv 0 \pmod 4.\end{aligned}$$

So, $2^2 \mid \sigma(q^e)$, so, $\sigma(q^e)$ is not squarefree.

Case 2: If $e \equiv 3 \pmod 4$ and q is odd, then, by Theorem 2.5,

$$1 + v_2(\sigma(q^e)) = v_2((e+1)(q+1)) \geq 3.$$

So, $2^2 \mid \sigma(q^e)$, so, $\sigma(q^e)$ is not squarefree.

Case 3: If $\exp_p q = 1$ and $p \mid e+1$, then, $p \mid q-1 \Rightarrow q \equiv 1 \pmod p$. Also, $p \mid e+1 \Rightarrow e+1 \equiv 0 \pmod p$. So,

$$\begin{aligned}\sigma(q^e) &= 1 + q + q^2 + \cdots + q^e \\ &\equiv 1 + e \pmod p \\ &\equiv 0 \pmod{p^2}.\end{aligned}$$

So, $p^2 \mid \sigma(q^e)$, so, $\sigma(q^e)$ is not squarefree.

Case 4a: If $\exp_p q > 1$, and $\exp_p q \mid e+1$, and $[p|q] \geq 2$, then, by Theorem 3.3, we have, $p^i \| \sigma(q^e) \Leftrightarrow e+1 = p^{i-[p|q]} \cdot \exp_p q \cdot h$, for some $h \geq 1$ with $(h,p) = 1$. So, $i - [p|q] \geq 0 \Leftrightarrow i \geq [p|q] \geq 2$, so, $\sigma(q^e)$ is not squarefree.

Case 4b: If $\exp_p q > 1$, and $\exp_p q \mid e+1$, and $p \mid e+1$, then, by Theorem 3.3 again, $e+1 = p^{i-[p|q]} \cdot \exp_p q \cdot h$, for some $h \geq 1$ with $(h,p) = 1$. Since, $p \mid e+1$, so, $i - [p|q] \geq 1$, so, $i \geq [p|q] + 1 \geq 2$, so, $p^2 \mid \sigma(q^e)$, so, $\sigma(q^e)$ is not squarefree. $\square$

Corollary 3.13 *For odd primes $p \geq 2$, if $(e+1, f+1) = 1$, then*

$$(\sigma(p^e), \sigma(p^f)) = 1$$

Corollary 3.14 *If q_1 and q_2 are distinct odd primes and $(e+1, f+1) = 1$, then*

$$(\sigma(q_1^e), \sigma(q_2^f)) = \prod_{p|(q_1^{e+1}-1, q_2^{f+1}-1)} p^{min\{[p|q_1],[p|q_2]\}}.$$

3.4 Prime power values of $\sigma(q^e)$

Since $\sigma(2^e) = 2^{e+1} - 1$, $\sigma(2^e)$ will be prime if and only if $2^{e+1} - 1$ is a Mersenne prime. Are there any examples of $\sigma(q^e) = p^j$ with p an odd prime, q a prime and $j \geq 2$ other than $\sigma(3^4) = 11^2$? A computer search ($q \leq 10^6$ and $e \leq 100$) did not reveal any additional solutions. By a theorem of Ljunggren [67], $\frac{x^n-1}{x-1} = y^j$ with $n > 2$ and $j = 2$, has only two solutions $(3, 5, 11, 2)$ and $(7, 4, 20, 2)$ for (x, n, y, j). Thus, $\sigma(3^4) = 11^2$ and $\sigma(7^3) = 20^2 = 2^4 5^2$, and only the first case is a solution for $j = 2$ of the equation $\sigma(q^e) = p^j$. Theorem 3.15 will address this question, and we will see that the solutions are rare.

First, we will give some well-known identities (see [73, p160] and [84, p22]) about cyclotomic polynomials before considering general cases:

(A) for p an odd prime, q a prime,

$$\Phi_p(q) = \frac{q^p - 1}{q - 1} = \sigma(q^{p-1}) \tag{3.1}$$

(B) for p a prime and $(p, n) = 1$, then for all real x

$$\Phi_{np}(x) = \frac{\Phi_n(x^p)}{\Phi_n(x)} \tag{3.2}$$

(C) for p a prime and $p \mid n$, then for all real x

$$\Phi_{np}(x) = \Phi_n(x^p) \tag{3.3}$$

(D) if a real number $x \geq 2$, then each number in the sequence

$$\Phi_3(x), \Phi_4(x), \Phi_5(x), \Phi_6(x), \Phi_7(x) \cdots$$

(with $\Phi_6(x)$ excluded when $x = 2$) has a prime factor which is not a factor of any of the preceding numbers.

Theorem 3.15 *Let q be prime, p an odd prime and $e \geq 1$. If $\sigma(q^e) = p^j$, then $j = [p|q]$ and $e+1 = \exp_p q$ is prime.*

Conversely, let e be even and the prime p be odd. If $e+1 = \exp_p q$ is prime and the equation $e+1 = \exp_p q$, for given e and q, has a unique solution $p \in \mathbb{P}$, then $\sigma(q^e) = p^{[p|q]}$.

Proof. Case 1. Suppose $\exp_p q = 1$, then, (by Theorem 3.3), $p^j \| \sigma(q^e)$ if and only if for some $h \geq 1$ with $(h,p) = 1$, we have $e+1 = p^j \cdot h$. By the definitions of $\exp_p q$ and $[p|q]$, we have $p^{[p|q]} \| q^{\exp_p q} - 1$, so therefore $q - 1 = p^{[p|q]} \cdot w$, with $(p,w) = 1$, and so $q = p^{[p|q]}w + 1$. Since $\sigma(q^e) = p^j$, we can write

$$\begin{aligned} p^j &= q^e + q^{e-1} + \cdots + q + 1 \\ &= (p^{[p|q]w} + 1)^e + \cdots + (p^{[p|q]w} + 1) + 1. \end{aligned}$$

We want to obtain a contradiction, and it is enough to consider the highest power of p in the right hand side of above equation. Then

$$\begin{aligned} p^j \cdot h > j+1 &\Leftrightarrow p^j \cdot h - 1 > j \\ &\Rightarrow [p|q]w(p^j \cdot h - 1) > j \\ &\Leftrightarrow [p|q]we > j \text{ (since } e+1 = p^j \cdot h) \\ &\Leftrightarrow p^{[p|q]we} > p^j. \end{aligned}$$

Since

$$p^j = \sigma(q^e) > p^{[p|q]we} > p^j,$$

so, it is a contradiction. Therefore $\exp_p q = 1$ is false.

Case 2. Suppose $\exp_p q > 1$, then, again by Theorem 3.3, $p^j \| \sigma(q^e)$ if and only if for some $h \geq 1$ with $(h,p) = 1$, we have

$$e + 1 = p^{j-[p|q]} \cdot \exp_p q \cdot h.$$

Since $\exp_p q > 1$, so, $p \nmid q-1$, and $p^{[p|q]} \| q^{\exp_p q} - 1$, and therefore $q^{\exp_p q} - 1 = p^{[p|q]} \cdot w$ with $(p, w) = 1$. Since,

$$\begin{aligned} p^j &= \sigma(q^e) \\ &= \frac{q^{e+1}-1}{q-1}, \end{aligned}$$

therefore

$$\begin{aligned} p^j(q-1) &= q^{e+1} - 1 \\ &= q^{\exp_p q \cdot p^{j-[p|q]} \cdot h} - 1. \end{aligned}$$

Let $k = j - [p|q]$, and suppose $k \geq 1$. Then

$$\begin{aligned} p^j(q-1) &= q^{\exp_p q \cdot h \cdot p^k} - 1 \\ &= (q^{\exp_p q} - 1)((q^{\exp_p q})^{h \cdot p^k - 1} + \cdots + (q^{\exp_p q}) + 1). \end{aligned}$$

Therefore

$$p^k(q-1) = w((p^{[p|q]}w + 1)^{(h \cdot p^k - 1)} + \cdots + (p^{[p|q]}w + 1) + 1) \qquad (3.4)$$

Case 3(a). If $p > q$, then we can find a contradiction, because the value of the right hand side in above equation (3.4) is greater than the value of the left hand side. This follows since the LHS is less than p^{k+1} and the RHS is at least as great as p^{p^k}.

Case 3(b). If $p < q$, then

$$\begin{aligned} LHS &= p^k((p^{[p|q]}w + 1)^{\frac{1}{\exp_p q}} - 1) \\ &\leq p^k(p^{[p|q]}w + 1 - 1) \\ &= p^{k+[p|q]} \cdot w. \end{aligned}$$

Since p is an odd prime, $p \geq 3$, and $k \leq 3^k - 2$ for $k \geq 1$, and therefore, for

equation (3.4)

$$\begin{aligned} k &\leq [p|q] \cdot (h \cdot p^k - 2) \\ &\Leftrightarrow k + [p|q] \leq [p|q] \cdot (h \cdot p^k - 1) \\ &\Leftrightarrow p^{k+[p|q]} \leq p^{[p|q] \cdot (h \cdot p^k - 1)} \\ &\Leftrightarrow p^{k+[p|q]} \cdot w \leq p^{[p|q] \cdot (h \cdot p^k - 1)} \cdot w \\ &\Rightarrow LHS \leq p^{k+[p|q]} \cdot w \leq p^{[p|q] \cdot (h \cdot p^k - 1)} \cdot w < RHS. \end{aligned}$$

So, this is a contradiction. Therefore $k = j - [p|q]$ must be 0, so, $j = [p|q]$.

Case 4. Now, we can write the simplified relation $e + 1 = \exp_p q \cdot h$. Next, we will show that $h = 1$. Suppose, $h > 1$, then, $e + 1 > \exp_p q$. Since $p \nmid q - 1$, and

$$p^j = \sigma(q^e) = q^e + \cdots + q + 1, \text{ for some } j \geq 1,$$

and since $j = [p|q]$,

$$p^{[p|q]} \| q^{\exp_p q} - 1 \Rightarrow p^j \| q^{\exp_p q} - 1 \Rightarrow p^j \| (q^{\exp_p q - 1} + \cdots + q + 1),$$

so,

$$\begin{aligned} p^j &\leq q^{\exp_p q - 1} + \cdots + q + 1 \\ &< q^e + \cdots + q + 1 \\ &= p^j. \end{aligned}$$

So, it is a contradiction and therefore $h = 1$. Therefore $e + 1 = \exp_p q$.

Case 5. Now, we want to show that $e + 1 = \exp_p q$ is prime. Firstly, we consider a special case: suppose $e + 1 = p_1 \cdot p_2$, the product of two primes with $3 \leq p_1 < p_2$. (Note that e is even since $2 \nmid \sigma(q^e)$.) By the factorization property of cyclotomic polynomials, we have,

$$\begin{aligned} \sigma(q^e) &= \Phi_{p_1}(q) \cdot \Phi_{p_2}(q) \cdot \Phi_{p_1 p_2}(q) \\ &= \sigma(q^{(p_1 - 1)}) \cdot \Phi_{p_2}(q^{p_1}) \\ &= \sigma(q^{(p_2 - 1)}) \cdot \Phi_{p_1}(q^{p_2}). \end{aligned}$$

Since $\sigma(q^{(p_1-1)}) \mid \sigma(q^e)$ and $\sigma(q^{(p_2-1)}) \mid \sigma(q^e)$, then $\sigma(q^{(p_1-1)}) = p^a$ and $\sigma(q^{(p_2-1)}) = p^b$ for some positive integers a and b, where we may assume $a < b$, so $\sigma(q^{(p_1-1)}) \mid \sigma(q^{(p_2-1)})$, so $p_1 \mid p_2$, but this is impossible. Hence $e+1$ is not the product of two distinct primes.

Now we consider two cases, the first when $e+1$ is the product of two or more prime powers and the second when $e+1$ is a power of a single prime.

Subcase 1. Assume that $e+1$ is not a prime and let $e+1 = hk$, where h, $k > 1$. Then

$$\sigma(q^e) = p^j = \frac{q^{e+1}-1}{q-1} = \left(\frac{q^{hk}-1}{q^k-1}\right)\left(\frac{q^k-1}{q-1}\right).$$

Since $\frac{q^k-1}{q-1} > 1$ then p divides $\frac{q^k-1}{q-1}$ implies $q^k \equiv 1 \bmod p$. Hence

$$\frac{q^{hk}-1}{q^k-1} = 1 + q^k + \cdots + q^{k(h-1)} \equiv h \bmod p.$$

But p divides $\frac{q^{hk}-1}{q^k-1}$, hence $p \mid h$. Since this is true for every divisor h of $e+1$, then $e+1$ is a power of p.

Subcase 2. Let $e+1 = p^k$, with $k > 1$ and p a prime. Then by (3.3)

$$\begin{aligned} \sigma(q^e) &= \frac{q^{e+1}-1}{q-1} \\ &= \Phi_p(q) \cdot \Phi_{p^2}(q) \cdots \Phi_{p^k}(q). \end{aligned}$$

Each number in the sequence

$$\Phi_p(q), \Phi_{p^2}(q), \Phi_{p^3}(q), \cdots, \Phi_{p^k}(q)$$

has a prime factor which is not a factor of any of the preceding numbers (except $\Phi_6(2)$, which is not the case here). Since $\sigma(q^e) = p^j$, we get a contradiction. Therefore if $e+1 = p^k$, $k = 1$.

Combing Subcase 1 and Subcase 2, we have $e+1 = p$, with p a prime.

For the converse, we simply apply Theorem 3.9. □

Note that in Theorem 3.15 above, in order to show that $e+1 = \exp_p q$ is prime, in case $p \mid q-1$, a proof can be found in some papers by Suryanarayana

(1967, 1970, [96], [97]), Edgar (1971, 1985, [30], [31]), and Estes et al. (1985, [34]). In the situation here, $p \nmid q-1$, but we are able to use the first part of these proofs.

Corollary 3.16 *If for fixed $q \geq 2$ and prime a the equation $a = \exp_p q$ has a unique prime solution p and $\sigma(q^{a-1}) = p^j$, then $\sigma(q^{a-1})$ is prime if and only if $[p|q] = 1$.*

Proof. Case 1. We first want to show that if $q > 2$, the hypotheses are not satisfied.

By the definition of the exponent of q modulo p, $p \mid q^a - 1$, where $a = \exp_p q$, p is a prime.

Since, p is the unique prime solution of the equation $a = \exp_p q$, so, $p = 2$, (since $q^a - 1$ is an even number). But then $2 \mid q-1$, so $a = 1$, because by the definition, a is the minimum natural number such that $p \mid q^a - 1$ holds. But, a is a prime, so, this is a contradiction. Therefore $q > 2$ is impossible.

Case 2. Now we fix $q = 2$.

By the definition of $[p|q]$, we have $p^{[p|2]} \| 2^a - 1$. Let $[p \mid 2] = j$.

Since, $\sigma(q^{a-1}) = \sigma(2^{a-1}) = 2^a - 1 = p^j$, by Theorem 3.15, $j = [p|q]$, so, therefore $\sigma(2^{a-1})$ is a prime if and only if $[p|q] = 1$. □

The following is a well known result [2, p.23] which is quite useful:

Theorem 3.17 *If $a > 1$, $m \geq 1$ and $n \geq 1$. (A) If $d \mid m$ then $a^d - 1 \mid a^m - 1$; (B) If $a^d - 1 \mid a^m - 1$, then $d \mid m$; (C) $(a^m - 1, a^n - 1) = a^{(m,n)} - 1$.*

Proof. (A) If $d \mid m$ then $a^d - 1 \mid a^m - 1$:

Since if $de = m$, then

$$\begin{aligned} a^m - 1 &= (a^d)^e - 1 \\ &= (a^d - 1)(a^{d(e-1)} + a^{d(e-2)} + \cdots + 1). \end{aligned}$$

(B) If $a^d - 1 \mid a^m - 1$, then $d \mid m$:

Suppose $d \nmid m$. Let $m = dq + r$ with $1 \leq r < d$. Then $\exists b$, such that

$$\begin{aligned}(a^d - 1)b &= a^m - 1 \\ &= a^{dq+r} - 1 \\ &= a^{dq}a^r - 1 \\ &= (a^{dq} - 1)a^r + a^r - 1 \\ &= (a^d - 1)ca^r + (a^r - 1),\end{aligned}$$

where $c = a^{d(q-1)} + a^{d(q-2)} + \cdots + a + 1$.

So

$$(a^d - 1)(b - ca^r) = a^r - 1,$$

so

$$a^d - 1 \mid a^r - 1,$$

so $d \leq r$, but $r < d$, which is a contradiction. Hence Lemma 3.10 follows immediately.

(C) Let $m > n$. Since $(m, n) \mid m$ and $(m, n) \mid n$, by (A), we get

$$a^{(m,n)} - 1 \mid (a^m - 1, a^n - 1).$$

$$\begin{aligned}(a^m - 1, a^n - 1) &= (a^m - 1 - (a^n - 1), a^n - 1) \\ &= (a^m - a^n, a^n - 1) \\ &= (a^n(a^{m-n} - 1), a^n - 1), \text{ (but } (a^n, a^n - 1) = 1 \text{)} \\ &= (a^{m-n} - 1, a^n - 1), \text{ (let } m = nq + r \text{)} \\ &= (a^r - 1, a^n - 1) \\ &= (a^n - 1, a^r - 1).\end{aligned}$$

By the Euclidean algorithm, $n \to (m,n)$, as $r \to 0$, so

$$\begin{aligned}(a^m - 1, a^n - 1) &= (a^{(m,n)} - 1, a^0 - 1) \\ &= (a^{(m,n)} - 1, 0) \\ &= a^{(m,n)} - 1.\end{aligned}$$

□

Corollary 3.18 *If p is a prime, $e \geq 1$, and $f \geq 1$, then*

$$(\sigma(p^e), \sigma(p^f)) = \frac{p^{(e+1,f+1)} - 1}{p - 1} = \sigma(p^{(e+1,f+1)-1}).$$

Chapter 4

Counting multiperfect numbers up to x

4.1 Introduction

In this chapter there are some results about counting multiperfect numbers. In Section 4.2 we provide some estimates of counting perfect numbers developed by Hornfeck [52], Kanold [59], and Wirsing [83], (Theorems 4.8, 4.9, and 4.10). In Section 4.3 we discuss two classes of primes which are flat primes and thin primes [11]. Some properties are described: the asymptotic density of thin numbers (Theorem 4.15); the density of flat numbers (Theorem 4.17); the relative density of flat primes (Theorem 4.23); the density of primes which are both lower and upper flat (Theorem 4.25); an upper bound of the number of thin primes (Theorem 4.28 and Corollary 4.29). In Subsection 4.3.4 we introduce the Hardy-Littlewood-Bateman-Horn conjectures. In Section 4.4 we investigate flat numbers: the maximum number of successive odd numbers which are flat (Theorem 4.30); and show there are infinitely many groups of 8 consecutive flat numbers (Theorem 4.35).

4.2 Counting perfect numbers

In this section we consider the perfect numbers N, corresponding to N we define $N(x)$ as the number of perfect numbers less than or equal to x, that is, $\#\{N \leq x : \sigma(N) = 2N\}$.

Hornfeck [52](1955) showed that $N(x) = O(x^{1/2})$, and [53] (1956) improved his result in this form $\overline{\lim}_{N\to\infty} \frac{N(x)}{\sqrt{x}} \leq \frac{1}{2}$, where $\overline{\lim}$ denotes the limit supremum. Kanold [59](1956) improved Hornfeck's result to $N(x) = o(\sqrt{x})$.

I first give the details of the works by Hornfeck and Kanold, and then describe the much stronger and more general theorem of Wirsing. Before these theorems we provide some lemmas. In this chapter, absolute constants $c_0, c_1, \cdots$ and $x_0, x_1, \cdots$ in different theorems or lemmas are not necessarily the same. No work in this section is original, but proofs of theorems from the literature have been elaborated.

Lemma 4.1 *[72, Lemma7.7, p208] Let $m, n \geq 1$ be natural numbers and let $A(m, n)$ denote the number of solutions of the inequality $a_1 + a_2 + \cdots + a_m \leq n$ with integers $a_i \geq 0$, $i = 1, 2, \cdots, m$. Then $A(m, n) = \binom{m+n}{m}$.*

Proof. This proof is taken from [72]. It is included to assist the reader. Suppose there are n balls and m walls. Let a_1 be the number of balls between the first and second wall, let a_2 be the number of balls between the second and third wall, and so on, so that a_{m-1} is the number of balls between the last two walls. Let a_0 be the number of balls to the left of the first wall, and let a_m be the number of balls to the right of the mth wall. Then $a_0 = n - \sum_{i=1}^{m} a_i \geq 0$. Thus an arrangement of n balls and m walls determines a choice of non-negative a_i with $a_1 + a_2 + \cdots + a_m \leq n$ and vice versa. So the number of solutions, $A(m, n)$, is $\binom{m+n}{n}$ (or $\binom{m+n}{m}$) from $n+m$ possible positions choosing n balls (or m walls). □

Lemma 4.2 *The number of solutions* $(b_1, b_2, \cdots, b_r)$*, with* $b_1+b_2+\cdots+b_r \leq k$ *and* $b_i \geq 1$*,* $(i = 1, 2, \cdots, r)$*, is* $\binom{k}{r}$.

Proof. Since $b_1 + b_2 + \cdots + b_r \leq k$, $(b_i \geq 1,\ i = 1, \cdots, r)$, so

$$(b_1 - 1) + (b_2 - 1) + \cdots + (b_r - 1) \leq k - r.$$

By Lemma 4.1, we have the number of solutions $(b_1, b_2, \cdots, b_r)$ is

$$A(r, k - r) = \binom{k}{r}.$$

□

Lemma 4.3 *Let* $p_1, p_2, \cdots, p_k$ *be primes without necessarily being distinct, then*

$$\prod_{i=1}^{k}\left(1 + \frac{1}{p_i - 1}\right) < \exp\left(\sum_{i=1}^{k}\frac{1}{p_i - 1}\right).$$

Proof. Firstly we have

$$1 + \frac{1}{n} < 1 + \frac{1}{n} + (\frac{1}{n})^2\frac{1}{2!} + \cdots = \exp(\frac{1}{n}),$$

for any $n \in \mathbb{N}$.

Thus, for any prime p_i,

$$\prod_{i=1}^{k}\left(1 + \frac{1}{p_i - 1}\right) < \exp\left(\sum_{i=1}^{k}\frac{1}{p_i - 1}\right).$$

□

Note that in the following lemmas and theorems all constants x_i are absolute.

Lemma 4.4 *There is an absolute constant* x_2 *such that if* $x \geq x_2$*, then*

$$\exp\left(\frac{\log x/\log\log x}{(\log x) - 1}\right) < 2.$$

Proof. Observe that

$$\lim_{x\to\infty} \frac{\log x}{(\log x - 1)\log\log x} = 0$$

Therefore, for any positive number ϵ, there exists a real number N (with $N > \epsilon$) such that if $x \geq N$ then

$$0 \leq \frac{\log x}{(\log x - 1)\log\log x} < \epsilon.$$

The absolute constant x_2 is the value of N that corresponds to $\epsilon = \log 2$. The result follows. □

Lemma 4.5 *Let x be a real number sufficiently large, then*

$$\left(\frac{\log x}{\log 2}\right)^{\log^{3/4} x} = O\left(2^{\frac{\log x}{\log\log x}}\right).$$

Proof. There exists a real number x_3 such that if $x \geq x_3$ then

$$\left(\frac{\log\log x - \log\log 2}{\log 2}\right)\log^{3/4} x \leq \frac{\log x}{\log\log x}$$

we have

$$\begin{aligned}\left(\frac{\log x}{\log 2}\right)^{\log^{3/4} x} &= \left(2^{\frac{\log\log x - \log\log 2}{\log 2}}\right)^{\log^{3/4} x} \\ &\leq 2^{\frac{\log x}{\log\log x}}.\end{aligned}$$

Thus, the result follows. □

We can obtain the following Lemma 4.6 from [88, eqn.(3.6)]:

Lemma 4.6 *For $x > 1$, we have*

$$\pi(x) < 2x/\log x.$$

Lemma 4.7 *Let $A(x)$ denote the number of $a \leq x$, where $a = 2^{p-1}$, with p prime, then $A(x) \leq c_1\sqrt{x}$, where c_1 is an absolute constant.*

Proof. Since $a = 2^{p-1} \leq x$, so $p \leq \frac{\log x}{\log 2} + 1 = y$, (say). Then by Lemma 4.6,

$$\begin{aligned} A(x) &\leq \pi(y) < \frac{2y}{\log y} \\ &\leq c_0 \frac{\log x}{\log\log x} \leq c_1\sqrt{x}, \end{aligned}$$

for x sufficiently large, where c_0, c_1 are some absolute constants. □

Theorem 4.8 *(Hornfeck [52]) $N(x) < c\sqrt{x}$, where $c > 0$ is a constant.*

Proof. This proof is the same as [52], but with more details.

Case 1. For odd perfect numbers. Let N_1, N_2 be odd perfect numbers, and $N_1 = p_1^{\alpha_1} m^2$, $(p_1, m) = 1$; $N_2 = p_2^{\alpha_2} m^2$, $(p_2, m) = 1$. Since

$$2 = \frac{\sigma(N_1)}{N_1} = \frac{\sigma(N_2)}{N_2}$$

which implies

$$\frac{1 + p_1 + \cdots + p_1^{\alpha_1}}{p_1^{\alpha_1}} = \frac{1 + p_2 + \cdots + p_2^{\alpha_2}}{p_2^{\alpha_2}}$$

so

$$(1 + p_1 + \cdots + p_1^{\alpha_1}) p_2^{\alpha_2} = (1 + p_2 + \cdots + p_2^{\alpha_2}) p_1^{\alpha_1}$$

since $p_1^{\alpha_1} \nmid (1 + p_1 + \cdots + p_1^{\alpha_1})$, and $p_2^{\alpha_2} \nmid (1 + p_2 + \cdots + p_2^{\alpha_2})$, so $p_1^{\alpha_1} \mid p_2^{\alpha_2} \mid p_1^{\alpha_1}$, and therefore $p_1 = p_2$, and $\alpha_1 = \alpha_2$. Hence, once we fix m, then p^α is determined so $p^\alpha m^2$ is perfect. Therefore there exists a one to one correspondence between a subset of squares m^2 and its corresponding perfect numbers $p^\alpha m^2$.

Let $N_o(x) = \#\{N \leq x : \sigma(N) = 2N,\ N\ odd\}$. Since $m^2 \leq x$ implies $m \leq \sqrt{x}$, thus we have $N_o(x) \leq \sqrt{x}$.

Case 2. For even perfect numbers. Let $N_e(x) = \#\{N \leq x : N = 2^{p-1}(2^p - 1), \sigma(N) = 2N\}$ and $A(x) = \#\{a \leq x : a = 2^{p-1}\}$. Since N is an even perfect number, we have $N = 2^{p-1}(2^p - 1)$, with $2^p - 1$ primes, so $N_e(x) \leq A(x)$. By Lemma 4.7, $N_e(x) < c_1\sqrt{x}$.

Therefore, $N(x) < c\sqrt{x}$, where $c > 0$ is an absolute constant. □

Theorem 4.9 *(Kanold [59])*

$$N(x) = o(\sqrt{x})$$

Proof. Case 1. For even perfect numbers. Let $N_1(x) = \#\{N \leq x : N = 2^{p-1}(2^p - 1), \sigma(N) = 2N\}$, then only for even perfect numbers, we have $N = 2^{p-1}(2^p - 1)$, where p and $2^p - 1$ are primes. Since $2^{p-1} < 2^p - 1$ and $2^{p-1}(2^p - 1) \leq x$, then $4^{p-1} < x$ implies $p < \frac{\log x}{\log 4} + 1$.

By Lemma 4.6 we have

$$N_1(x) \leq \pi\left(\frac{\log x}{\log 4} + 1\right) < 2\frac{\frac{\log x}{\log 4} + 1}{\log(\frac{\log x}{\log 4} + 1)}, \tag{4.1}$$

for all sufficiently large x.

From inequality (4.1), we can get $N_1(x) \ll \frac{\log x}{\log\log x} = o(\sqrt{x})$, for all sufficiently large x.

Case 2. For odd perfect numbers. If N is an odd perfect number, then it has the shape $N = p^\alpha \cdot q_1^{2\beta_1} \cdots q_r^{2\beta_r}$, with $p \equiv \alpha \equiv 1 \pmod 4$. Suppose $q_1 < \cdots < q_r$. If $k > 1$ is a given integer, then according to a theorem of Dickson [27], there are at most finitely many odd perfect numbers N when $\omega(N) \leq k$. If $\omega(N) > k$, then $r \geq k$.

Let $N_2(x) = \#\{N \leq x : \sigma(N) = 2N,\ N\ odd,\ \omega(N) > k,\ p^\alpha \geq k\}$, then by the theorem of Hornfeck (Theorem 4.8), there is a one to one correspondence $N \leftrightarrow m^2 = \frac{N}{p^\alpha} \leq \frac{N}{k}$, so we can get the estimate $N_2(x) \leq \sqrt{\frac{x}{k}}$.

Let $N_3(x) = \#\{N \leq x : \sigma(N) = 2N, N\ odd,\ \omega(N) > k,\ p^\alpha < k\}$. Now, we want to show that for $N_3(x)$, there also exists a one to one correspondence between m and N, where

$$m^2 = \frac{N}{p^\alpha q_r^{2\beta_r}}, (m \neq N).$$

Suppose the same m corresponds to N and $\overline{N}$, $N \neq \overline{N}$, then

$$\begin{aligned} N &= p^{\alpha} \cdot q_r^{2\beta_r} \cdot m^2 && (4.2)\\ \overline{N} &= \overline{p}^{\overline{\alpha}} \cdot \overline{q}_r^{2\overline{\beta}_r} \cdot m^2 && (4.3)\\ 2 = \frac{\sigma(N)}{N} &= \frac{\sigma(m^2)}{m^2} \cdot \frac{\sigma(p^{\alpha})}{p^{\alpha}} \cdot \frac{\sigma(q_r^{2\beta_r})}{q_r^{2\beta_r}} && (4.4)\\ &= \frac{\sigma(\overline{N})}{\overline{N}} = \frac{\sigma(m^2)}{m^2} \cdot \frac{\sigma(\overline{p}^{\overline{\alpha}})}{\overline{p}^{\overline{\alpha}}} \cdot \frac{\sigma(\overline{q}_r^{2\overline{\beta}_r})}{\overline{q}_r^{2\overline{\beta_r}}}. && (4.5) \end{aligned}$$

From $N = p^{\alpha} \cdot q_1^{2\beta_1} \cdots q_r^{2\beta_r}$(4.2) and $p^{\alpha} < k$, since $q_1 \geq 3, q_2 \geq 5, q_3 \geq 7, \cdots$, by observation the following inequalities hold:

$$q_r, \overline{q}_r \geq 2k+1 > 2p^{\alpha}, 2\overline{p}^{\overline{\alpha}} \tag{4.6}$$

From (4.4) and (4.5) we obtain by a simple manipulation:

$$\begin{aligned} &\overline{p}^{\overline{\alpha}} \cdot \overline{q}_r^{2\overline{\beta}_r}(1 + p + \cdots + p^{\alpha})(1 + q_r + \cdots + q_r^{2\beta_r}) \\ &= p^{\alpha} \cdot q_r^{2\beta_r}(1 + \overline{p} + \cdots + \overline{p}^{\overline{\alpha}})(1 + \overline{q}_r + \cdots + \overline{q}_r^{2\overline{\beta}_r}) \end{aligned} \tag{4.7}$$

Now from the inequalities (4.6) we get $q_r^{2\beta_r} \mid \overline{q}_r^{2\overline{\beta}_r} \mid q_r^{2\beta_r}$ therefore $q_r = \overline{q}_r$; $\beta_r = \overline{\beta}_r$.

After canceling equal terms in (4.7) we obtain $p = \overline{p};\ \alpha = \overline{\alpha}$. Since

$$\begin{aligned} m^2 &= \frac{N}{p^{\alpha} q_r^{2\beta_r}} \leq \frac{N}{p^{\alpha} q_r^2} \\ &\leq \frac{N}{p^{\alpha}(2k+1)^2} \leq \frac{x}{5(2k+1)^2}, \end{aligned}$$

where $p \equiv 1 \pmod 4$ and $\alpha \geq 1$, so $m \leq \sqrt{\frac{x}{5(2k+1)^2}}$, therefore, $N_3(x)$ is the same as the number of m.

From $N_1(x) \ll \frac{\log x}{\log\log x}$, $N_2(x) \leq \sqrt{\frac{x}{k}}$, and $N_3(x) \leq \sqrt{\frac{x}{5(2k+1)^2}}$, for all sufficiently large x and fixed $k > 1$ we get

$$N(x) \ll \frac{\log x}{\log\log x} + \sqrt{\frac{x}{k}} + \sqrt{\frac{x}{5(2k+1)^2}} < 3\sqrt{\frac{x}{k}}, x > x_0. \tag{4.8}$$

Now let $\epsilon > 0$ be given and choose $k > (\frac{3}{\epsilon})^2$, then $N(x) < \epsilon\sqrt{x}$. Therefore $N(x) = o(\sqrt{x})$ has been proved. □

We now provide an expanded version of the theorem of Wirsing, which is given by [83, Theorem 7.8. pp.1008-1010].

Theorem 4.10 *There are absolute constants c_0, x_0 such that if $x \geq x_0$ and α is any rational number, then the number of $n \leq x$ with $\sigma(n) = \alpha n$ is at most $x^{c_0/\log\log x}$.*

Proof. Suppose α is given in the reduced form $\alpha = u/v$. Suppose x is large, $n \leq x$, and $\sigma(n) = \alpha n$. Let $n = ab$, where b is the largest divisor of n, all of whose prime factors p satisfy $p \leq \log x$ or $p \mid v$ (if $v \neq 1$). Let the prime factorization of a be $p_1^{\beta_1} p_2^{\beta_2} \cdots p_k^{\beta_k}$, with each $p_i > \log x$. Then $(a, b) = 1$. Since $\sigma(n)$ is a multiplicative function, we have

$$\sigma(n) = \sigma(ab) = \sigma(a)\sigma(b) = a \cdot \alpha b. \tag{4.9}$$

From $v\sigma(n) = aub$, if $p^e \mid v$, then $p \mid v$ and $p^e \mid aub$. But $p \nmid u$, since $(u, v) = 1$ and $p \nmid a$, since $(a, b) = 1$, so $p^e \mid ub$. But this is true for every prime power divisor of v, hence $v \mid ub$, and $\alpha b = ub/v$ is an integer.

We want to show that $\sigma(b) \nmid \alpha b$, and apply this idea to show the number a depends only on b, and thus determine n.

Let l be the least integer $\geq \log x/\log\log x$. Since $a \leq n \leq x$ and $\log x < p_i$, we have

$$\begin{aligned} (\log x)^{\beta_1+\cdots+\beta_k} < p_1^{\beta_1} \cdots p_k^{\beta_k} = a \leq x, \\ \beta_1 + \beta_2 + \cdots + \beta_k < \frac{\log x}{\log\log x} \leq l, \end{aligned} \tag{4.10}$$

so that $k \leq l$. Then by Lemmas 4.3 and 4.4

$$\begin{aligned} 1 \leq \frac{\sigma(a)}{a} = \prod_{i=1}^{k} (1 + p_i^{-1} + \cdots + p_i^{-\beta_i}) < \prod_{i=1}^{k} \left(1 + \frac{1}{p_i - 1}\right) \\ < \exp\left(\sum_{i=1}^{k} \frac{1}{p_i - 1}\right) < \exp\left(\frac{l}{(\log x) - 1}\right) < 2 \end{aligned} \tag{4.11}$$

for $x \geq x_1$. Thus for $x \geq x_1$ we have $a \mid \sigma(a)$ if and only if $a = 1$. So by equation (4.9) we see that for $x \geq x_1$ either $a = 1$ or $\sigma(b) \nmid \alpha b$. Furthermore,

if $a' \mid a$, $(a', a/a') = 1$, and $a' < a$, then applying inequalities (4.11) to a/a' we have $\sigma(a'b) \nmid a' \cdot \alpha b$.

Now let us use b to construct the number a, which is an ordered k-tuple (with $k \geq 0$) of positive integers $\beta_1, \beta_2, \cdots, \beta_k$ satisfying inequality (4.10). First, if $k = 0$, then $a = 1$, and we get $n = b$. So assume $k > 0$, then $\sigma(b) \nmid \alpha b$ (otherwise, there does not exist solutions for a, since $\sigma(a) \nmid a$ and $\sigma(b) \mid \alpha b$, so a contradiction for equation (4.9)). Let p_1 be the least prime that divides $\sigma(b)$ to a higher power than it divides αb. If there is a solution for the following equation (4.12),

$$\frac{\sigma(b)}{p_1^{\beta_1}} = \frac{\alpha b}{\sigma(p_1^{\beta_1})}, \tag{4.12}$$

we say $a = p_1^{\beta_1}$.

If there is no solution for the equation (4.12), then let p_2 be the least prime that divides $\sigma(bp_1^{\beta_1})$ to a higher power than it divides $\alpha b p_1^{\beta_1}$. Now try to find the solution for the equation (4.13),

$$\frac{\sigma(b)\sigma(p_1^{\beta_1})}{p_2^{\beta_2}} = \frac{\alpha b p_1^{\beta_1}}{\sigma(p_2^{\beta_2})}. \tag{4.13}$$

If there is a solution for equation (4.13), then $a = p_1^{\beta_1} p_2^{\beta_2}$.

This procedure either ends in an integer $a = p_1^{\beta_1} p_2^{\beta_2} \cdots p_k^{\beta_k}$ or there does not exist a satisfying equation (4.9). If a is constructed, it does not necessarily to satisfy equation (4.9). But if some a satisfying equation (4.9) does exist, this procedure will find it.

Thus for $x \geq x_1$ the number of $n \leq x$ satisfying $\sigma(n) = \alpha n$ (or $v\sigma(n) = uab$) is at most BC, where B is the number of $b \leq x$ such that $v \mid b$ and for every prime p in b we have $p \leq \log x$ or $p \mid v$ and C is the number of ordered tuples of natural numbers satisfying inequality (4.10).

By Lemma 4.2 we have $C \leq \binom{l}{k} \leq 2^l$. Now we have $B \leq B_1 B_2 B_3$, where B_1 is the number of $b_1 \leq x$ of the form $q_1^{\gamma_1} q_2^{\gamma_2} \cdots q_t^{\gamma_t}$ where $q_1, q_2, \cdots, q_t$ are all of the primes in v exceeding $\log x$ and $\gamma_1, \gamma_2, \cdots, \gamma_t$ are natural numbers, B_2 is the

number of $b_2 \leq x$ such that every prime in b_2 is in the interval $(\log^{3/4} x, \log x]$, and B_3 is the number of $b_3 \leq x$ divisible by no prime exceeding $\log^{3/4} x$.

An upper bound for B_1 is the number of sequences $\gamma_1, \gamma_2, \cdots, \gamma_t$ of natural numbers such that

$$\gamma_1 + \gamma_2 + \cdots + \gamma_t \leq l.$$

Thus by Lemma 4.2 $B_1 \leq \binom{l}{t} \leq 2^l$.

The total number of prime factors in a choice for b_2 is at most

$$(\log x)/\log(\log^{3/4} x) \leq 2l.$$

Say the primes in $(\log^{3/4} x, \log x]$ are $r_1, r_2, \cdots, r_m$. Then B_2 is at most the number of sequences $\delta_1, \delta_2, \cdots, \delta_m$ of non-negative integers with

$$\delta_1 + \delta_2 + \cdots + \delta_m \leq 2l.$$

Thus by Lemma 4.1, we have

$$B_2 \leq \binom{m+2l}{m} \leq 2^{m+2l}.$$

Note that $m = \pi(\log x) - \pi(\log^{3/4} x) < \pi(\log x)$. By Lemma 4.6,

$$\pi(\log x) < 2\log x/\log\log x$$

holds for all $\log x > 1$. Thus $m < 2l$, so that $B_2 \leq 2^{4l}$.

If p is a prime and p^β divides some choice for b_3, then $p^\beta \leq x$ so that $\beta \leq (\log x)/\log 2$. Thus B_3 is at most the number of ordered $\pi(\log^{3/4} x)$-tuples with each coordinate a non-negative integer at most $(\log x)/\log 2$. Thus by

Lemma 4.5,

$$\begin{aligned} B_3 &\leq (1+(\log x)/\log 2)^{\pi(\log^{3/4} x)} \\ &\leq (1+(\log x)/\log 2)^{\log^{3/4} x} \\ &\ll (\log x/\log 2)^{\log^{3/4} x} \\ &= (2^{(\log\log x - \log\log 2)/\log 2})^{\log^{3/4} x} \\ &\ll 2^{\log x/\log\log x} \\ &= 2^l, \end{aligned}$$

for $x \geq x_2$.

From the above, if $x \geq x_2$, then $B \leq B_1 B_2 B_3 \leq 2^{6l}$. Since $C \leq 2^l$, if we choose $x \geq x_0 = max\{x_1, x_2\}$, then the number of $n \leq x$ with $\sigma(n) = \alpha n$ is at most $2^{7l} < x^{7/\log\log x}$, completing the proof of the theorem. □

Note: The number "b" appearing in this proof are all "smooth", i.e. all of the prime factors are small. We can shorten the above proof by using the theorem estimating the number of smooth numbers up to x with prime factors $\leq \log^a x, a \geq 1$, namely [72, p.203, eqn.(7.16)], $\phi(x, \log^a x) = x^{1-1/a+o(1)}$, in the case $a = 1$.

Pollack [79] applied the distribution of $gcd(N, \sigma(N))$ on the natural numbers $N \leq x$ to obtain another proof of Wirsing's theorem.

Theorem 4.11 *[79, Theorem 1.3.] For each $x \geq 3$, we have*

$$\sum_{N \leq x} gcd(N, \sigma(N)) \leq x^{1+c/\sqrt{\log\log x}},$$

where c is an absolute positive constant.

Theorem 4.12 *[79, Theorem 1.4.] Fix $\epsilon > 0$. The number of $N \leq x$ with $gcd(N, \sigma(N)) > A$ is at least $x/A^{1+o(1)}$ as $x \to \infty$, uniformly for $2 \leq A \leq x^{1-\epsilon}$.*

Theorems 4.11 and 4.12 immediately have the following consequence:

Theorem 4.13 *[79, Corollary 1.5.] Fix $\alpha \in (0,1)$. The number of $N \leq x$ for which $gcd(N, \sigma(N)) > x^{\alpha}$ equals $x^{1-\alpha+o(1)}$ as $x \to \infty$.*

To get the multiperfect number bound from Theorem 4.13, we first want to count the multiperfect numbers in $(x/2, x]$, then $(x/4, x/2]$, etc. More explicitly, suppose that j is the smallest positive integer with $x/2^j \leq \log x$. If N is a multiperfect number in $(x/2^j, x/2^{j-1}]$, then

$$N = gcd(N, \sigma(N)) > x/2^j > (x/2^{j-1})^{1-\epsilon},$$

for any fixed $\epsilon > 0$ and all large x, (see the remark below). So by the upper bound half of Theorem 4.13 with $\alpha = 1 - \epsilon$, the number of such N is at most $(x/2^{j-1})^{\epsilon+o(1)} < (x/2^{j-1})^{2\epsilon}$, for large x. Summing over j we get an upper bound of $O_\epsilon(x^{2\epsilon})$ for the number of multiperfect numbers in $(\log x, x]$. Since there are only $O(\log x)$ multiperfect numbers below $\log x$, which is negligible.

Remark: Given $\epsilon > 0$, choose x, so $2^{\frac{1}{\epsilon}-1} < \log x/2$, then choose $j \in \mathbb{N}$ such that $\log x/2 < x/2^j \leq \log x$. With these choices it follows that $2^{\frac{1}{\epsilon}-1} < x/2^j$, which implies $x/2^j > (x/2^{j-1})^{1-\epsilon}$.

So we get the following corollary:

Corollary 4.14 *If*

$$M(x) = \{N \leq x : N \mid \sigma(N)\},$$

then $\forall \epsilon > 0$,

$$\#M(x) = O_\epsilon(x^\epsilon).$$

4.3 Flat primes and thin primes

Some interesting subclasses of primes have been identified and actively considered. These include Mersenne primes (of the form $2^p - 1$), Sophie Germain primes (of the form $2p + 1$), Fermat primes (of the form $2^{2^n} + 1$), Cullen's

primes (of the form $p \cdot 2^p + 1$), Wieferich primes (which are primes p such that $p^2 \mid 2^{p-1} - 1$), primes of the form $N^2 + 1$, of the form $N! \pm 1$, etc. See for example [85, Chapter 5] and the references in that text. For any one of these classes, determining whether or not it is infinite has proved to be a very difficult problem.

In this section we explore two classes of primes, the so-called lower or upper flat primes and the lower or upper thin primes. They have simple representations, and we are able to get an idea of their densities relative to the full set of primes.

These primes are similar to primes of the form $k \cdot 2^e + 1$ considered by Erdős and Odlyzko, Chen and Sierpiński among others ([32],[19],[92]). There the focus is mainly on the admissible values of odd integers k with $k \leq x$, rather on the density of primes themselves having that structure. Erdős showed [32, Theorem 1] that the number $N(x)$ of odd numbers less than or equal to x of the form $(p+1)/2^e$ satisfies

$$c_1 x \leq N(x) \leq c_2 x,$$

where c_1 and c_2 are positive absolute constants. In the opposite direction, a simple modification of the derivation of Sierpinski [92] gives an infinite number of integers N (including an infinite set of primes) such that $N \cdot 2^e - 1$ is composite for every $e = 1, 2, 3, \cdots$.

In Theorem 4.17, we will show that the number of upper flat (or lower flat) numbers is asymptotically the same as that of the odd squarefree numbers, i.e. the number of upper (or lower) flat numbers is given by $4x/\pi^2 + O(\sqrt{x})$.

For example, among the first 100 primes, 75 primes are either upper flat or lower flat and among the first 1000 primes, 742 are either upper flat or lower flat. For upper thin or lower thin primes the corresponding numbers are 38 and 213 respectively. The first 10 upper flat primes are 3, 5, 7, 11, 13, 19, 23,

29, 31, and 37. The first 10 upper thin primes are 3, 5, 7, 11, 13, 19, 23, 31, 37 and 43.

If $M(x)$ is the number of Mersenne primes up to x, then clearly, for all $x \geq 1$:

$$M(x) \leq T(x) \leq F(x) \leq \pi(x), \tag{4.14}$$

where for each $x > 0$, $\pi(x)$ is the number of primes up to x.

Figure 4.1 shows the ratio of $F(x)/\pi(x)$ over a small range. This gives some indication of the strength of Theorem 4.23 below - in the given range over 70% of all primes are upper flat or lower flat.

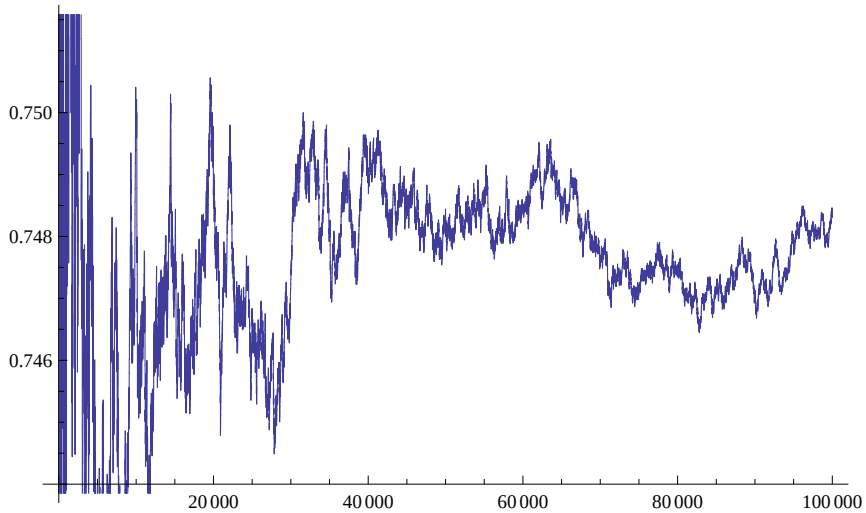

Figure 4.1: The ratio $F(x)/\pi(x)$ for $1 \leq x \leq 10^5$

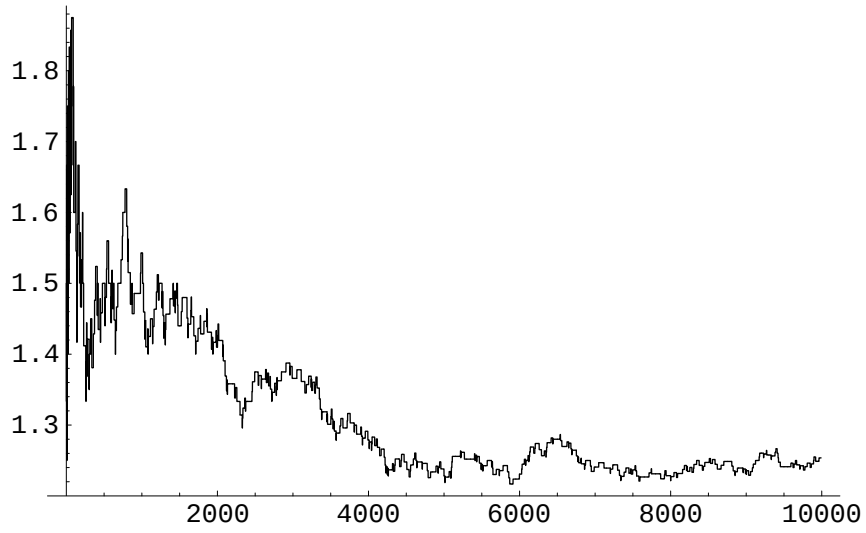

Figure 4.2: The ratio of thin primes to twin primes up to x for $1 \leq x \leq 10^4$

$$\begin{aligned} c_1 &= 2^3 \cdot 3 \cdot 5, \\ c_2 &= 2^5 \cdot 3 \cdot 7, \\ c_3 &= 2^9 \cdot 3 \cdot 11 \cdot 31, \\ c_4 &= 2^8 \cdot 5 \cdot 7 \cdot 19 \cdot 37 \cdot 73, \\ c_5 &= 2^{13} \cdot 3 \cdot 11 \cdot 43 \cdot 127, \\ c_6 &= 2^{14} \cdot 5 \cdot 7 \cdot 19 \cdot 31 \cdot 151. \end{aligned}$$

Table 4.1: Known 3-perfect numbers

Figure 4.2 shows the ratio of the number of thin primes up to x to the number of twin primes up to x. The relationship between thin and twin primes comes from the method of proof of Theorem 4.28 below.

These types of number arise frequently in the context of multiperfect numbers. For example, when $k = 3$ all of the known examples of so-called 3-perfect numbers $\{c_i : 1 \leq i \leq 6\}$ are included in Table 4.1 [90]. Each $c_i - 1$ is an upper flat number and each odd prime appearing on the right hand side is an upper thin number.

This section is organized as follows: in Subsection 4.3.1 we first show that the number of upper thin numbers up to x, is asymptotically equal to that of the primes up to x. In Subsection 4.3.2 we show that the relative density of upper flat primes is given by $2A$ where A is Artin's constant. A corollary to this is that there is an upper flat prime in every interval $[x, (1+\epsilon)x]$, for any $\epsilon > 0$ and sufficiently large x. This is followed by a demonstration that primes which are both lower and upper flat have an asymptotic density and constitute more than half of all primes. In Subsection 4.3.3 we then show that the upper

thin primes are sufficiently sparse that the sum of their reciprocals converges. The final subsection is a numerical validation of what might be expected for the density of thin primes under the Bateman-Horn conjectures.

4.3.1 Upper or lower thin numbers

Theorem 4.15 *As $x \to \infty$, the number of upper (or lower) thin numbers up to x is the same as that of the primes up to x.*

Proof. Firstly, the number of upper thin numbers less than or equal to x, as $x \to \infty$, namely $N(x)$, is given by

$$N(x) = \sum_{n=1}^{\lfloor \frac{\log x}{\log 2} \rfloor} \pi\Big(\frac{x}{2^n}\Big) + O(1).$$

Next we will show that $\lim_{x\to\infty} N(x)/\pi(x) = 1$. To this end first consider a single term in the sum. By [88, Theorem 2, p.69], there is a positive real absolute constant α such that for x sufficiently large,

$$\frac{x}{\log x + \alpha} < \pi(x) < \frac{x}{\log x - \alpha}.$$

Therefore, for all $n \in \mathbb{N}$ such that $1 \le n \le \lfloor \frac{\log x}{\log 2} \rfloor$ (which makes the numerators and denominators of the ratios below positive for sufficiently large x),

$$lb := \frac{1 - \frac{\alpha}{\log x}}{1 + \frac{\alpha}{\log x} - \frac{n \log 2}{\log x}} < \frac{2^n \pi(\frac{x}{2^n})}{\pi(x)} < \frac{1 + \frac{\alpha}{\log x}}{1 - \frac{\alpha}{\log x} - \frac{n \log 2}{\log x}} =: ub.$$

Clearly lb and ub tend to 1 as $x \to \infty$ uniformly for n in the range

$$1 \le n \le \log x / \log\log x =: b.$$

The difference between the upper and lower bounds is

$$\begin{aligned} ub - lb &= \frac{1}{d}\Big(\frac{4\alpha}{\log x} - \frac{2\alpha n \log 2}{\log^2 x}\Big) \le \frac{4\alpha}{d \log x}, \\ \text{where } d &= \Big(1 - \frac{\alpha}{\log x} - \frac{n\log 2}{\log x}\Big)\Big(1 + \frac{\alpha}{\log x} - \frac{n \log 2}{\log x}\Big) \\ &= 1 - \frac{\alpha^2}{\log^2 x} + \frac{n^2 \log^2 2}{\log^2 x} - \frac{2n \log 2}{\log x} \ge \frac{1}{4}, \\ \text{so } ub - lb &\le \frac{16\alpha}{\log x}, \end{aligned}$$

for all n in the given range and x sufficiently large.

Now ensure that x is sufficiently large so

$$\begin{aligned} |1-lb| &\leq \frac{\frac{2\alpha}{\log x}-\frac{n\log 2}{\log x}}{1+\frac{\alpha}{\log x}-\frac{n\log 2}{\log x}} \\ &\leq \frac{\frac{2\alpha}{\log x}}{\frac{1}{2}+\frac{\alpha}{\log x}} \\ &\leq \frac{4\alpha}{\log x}. \end{aligned}$$

This implies, for x sufficiently large and $1 \leq n \leq b$,

$$\begin{aligned} |\frac{2^n\pi(\frac{x}{2^n})}{\pi(x)}-1| &\leq |\frac{2^n\pi(\frac{x}{2^n})}{\pi(x)}-lb|+|lb-1| \\ &\leq |ub-lb|+|lb-1| < \frac{20\alpha}{\log x}. \end{aligned}$$

Using this bound we derive

$$\begin{aligned} &\Big|\sum_{n\leq b}\frac{\pi(\frac{x}{2^n})}{\pi(x)}-1\Big| \\ &\qquad \leq \Big|\frac{\sum_{n\leq b}\pi(\frac{x}{2^n})}{\pi(x)}-\frac{\sum_{n\leq b}\frac{\pi(x)}{2^n}}{\pi(x)}\Big|+\sum_{n>b}\frac{1}{2^n} \\ &\qquad \leq \sum_{n\leq b}\frac{1}{2^n}\Big|\frac{2^n\pi(\frac{x}{2^n})}{\pi(x)}-1\Big|+o(1) \\ &\qquad \leq \sum_{1\leq n}\frac{20\alpha}{2^n\log x}+o(1)=o(1) \end{aligned}$$

as $x \to \infty$.

For the remaining part of the summation range for $N(x)$, namely for $b < n \leq \lfloor \log x/\log 2 \rfloor$, note that this corresponds to values of x and n which satisfy

$$\frac{x}{2^n} \leq x^{1-\frac{\log 2}{\log\log x}}.$$

Using $\pi(x) \leq x$ and defining

$$S(x) := \sum_{n=b}^{\lfloor\frac{\log x}{\log 2}\rfloor}\pi\Big(\frac{x}{2^n}\Big) \ll \log x \cdot x^{1-\frac{\log 2}{\log\log x}},$$

it follows (using say l'Hôpital's rule) that $S(x)/\pi(x) \to 0$ as $x \to \infty$. Hence $N(x)/\pi(x) \to 1$. The proof for lower thin numbers is similar. □

From this we have consequences such as that an infinite number of successive primes are separated by a thin number and vice versa.

4.3.2 Upper or lower flat primes

Define the so-called logarithmic integral for $x \geq 2$:

$$\mathrm{Li}(x) := \int_2^x \frac{dt}{\log t}.$$

Lemma 4.16 *([73, p.130]and [47, pp.269-270]) The asymptotic number $Q(x)$ of squarefree numbers less than or equal to x is given by*

$$Q(x) = \frac{6x}{\pi^2} + O(\sqrt{x}).$$

Theorem 4.17 *The number of upper flat numbers up to x, namely $F(x)$, is given by for all $x \geq 2$:*

$$F(x) = \frac{4x}{\pi^2} + O(\sqrt{x}).$$

Proof. Let $D(x)$ be the number of odd squarefree numbers. Then

$$F(x) = D(x/2) + D(x/4) + \cdots,$$

where there are at most $\lfloor \frac{\log x}{\log 2} \rfloor$ non-zero terms in this sum. Also the number of all squarefree numbers is given by

$$Q(x) = D(x) + D(x/2).$$

Let $E(x)$ be the number of the even squarefree numbers so $E(2x) = D(x)$. Counting the squarefree numbers in $[x, 2x]$ gives

$$(D(2x) - D(x)) + (E(2x) - E(x)) = Q(2x) - Q(x)$$

and therefore

$$D(2x) - D(\frac{x}{2}) = Q(2x) - Q(x).$$

Hence

$$D(x) = \sum_{n=0}^{\infty} \left(Q\left(\frac{x}{2^{2n}}\right) - Q\left(\frac{x}{2^{2n+1}}\right)\right).$$

By Lemma 4.16,

$$Q(x) = \frac{6x}{\pi^2} + O(\sqrt{x}),$$

so

$$\begin{aligned} F(x) &= D(x/2) + D(x/4) + \cdots \\ &= \sum_{n=0}^{\infty} \left(Q\left(\frac{x}{2^{2n+1}}\right) - Q\left(\frac{x}{2^{2n+2}}\right)\right) + \sum_{n=0}^{\infty} \left(Q\left(\frac{x}{2^{2n+2}}\right) - Q\left(\frac{x}{2^{2n+3}}\right)\right) + \cdots \\ &= \sum_{n=0}^{\infty} Q\left(\frac{x}{2^{2n+1}}\right) \\ &= \frac{6x}{\pi^2}\left(\frac{1/2}{1-1/4}\right) + O(\sqrt{x}) \\ &= \frac{4x}{\pi^2} + O(\sqrt{x}). \end{aligned}$$

□

Lemma 4.18 *(Bombieri-Vinogradov theorem)[71] Let* $\pi(x;q,a) = \sum_{\substack{p \le x \\ p \equiv a \bmod q}} 1$, *where* $q > 0$, $(q,a) = 1$ *(integers). Then for fixed* $A > 0$, *there exists* $B = B(A) > 0$ *such that*

$$\sum_{q \le Q, (q,a)=1} \left| \pi(x;q,a) - \frac{\pi(x)}{\phi(q)} \right| = O\Big(\frac{x}{\log^A x}\Big),$$

where $Q = \frac{\sqrt{x}}{\log^B x}$.

Lemma 4.19 *Let* $n = 2^e a^2$, *with* e, a *positive integers. Then*

$$\frac{1}{\phi(2^e a^2)} = O\Big(\frac{1}{2^e \phi(a^2)}\Big),$$

where ϕ *is Euler's function.*

Proof.

$$\begin{aligned}
\phi(n) &= \phi(2^e a^2) \\
&= n \prod_{p|2^e a^2} \left(1 - \frac{1}{p}\right) \\
&= 2^e a^2 \left(1 - \frac{1}{2}\right) \prod_{p|a^2;\ p \text{ odd}} \left(1 - \frac{1}{p}\right) \\
&= 2^{e-1} a^2 \prod_{p|a^2;\ p \text{ odd}} \left(1 - \frac{1}{p}\right) \\
&= \begin{cases} 2^e \phi(a^2) & \text{if } a \text{ is even} \\ 2^{e-1}\phi(a^2) & \text{if } a \text{ is odd} \end{cases} \\
&\geq 2^{e-1}\phi(a^2),
\end{aligned}$$

so

$$\frac{1}{\phi(2^e a^2)} \leq \frac{1}{2^{e-1}\phi(a^2)} = \frac{2}{2^e \phi(a^2)},$$

so

$$\frac{1}{\phi(2^e a^2)} = O\Big(\frac{1}{2^e \phi(a^2)}\Big).$$

□

Lemma 4.20 *Let p be a prime and a, b, e, x, y be integers, then*

$$\sum_{p \leq x} \sum_{\substack{a > y \\ p+1 = 2^e a^2 b}} 1 = O\Big(\frac{x}{2^e y}\Big).$$

Proof. Here we first replace $p + 1$ by n, and then allow n to range over all

positive integers up to x.

$$\begin{aligned}\sum_{p\le x}\sum_{\substack{a>y\\ p+1=2^e a^2 b}} 1 &\ll \sum_{n\le x}\sum_{\substack{a>y\\ 2^e a^2\mid n}} 1\\ &= \sum_{y<a\le\sqrt{\frac{x}{2^e}}} \lfloor \frac{x}{2^e a^2}\rfloor\\ &\le \frac{x}{2^e}\sum_{y<a\le\sqrt{\frac{x}{2^e}}}\frac{1}{a^2}\\ &< \frac{x}{2^e}\sum_{y<a}\frac{1}{a^2}\\ &\le \frac{x}{2^e y}.\end{aligned}$$

Thus

$$\sum_{p\le x}\sum_{\substack{a>y\\ p+1=2^e a^2 b}} 1 = O\Big(\frac{x}{2^e y}\Big).$$

□

Lemma 4.21 *[47, Theorem 328, p.267]*

$$\frac{n}{\phi(n)} = O(\log\log n).$$

Lemma 4.22 *The function $g(n) = 2^{1-e}\phi(2^e n^2)$ is multiplicative, where e, n are positive integers.*

Proof. Let $(a,b)=1$. Case 1. Suppose both a and b are odd. Then,

$$\begin{aligned}g(a)g(b) &= 2^{1-e}\phi(2^e a^2)2^{1-e}\phi(2^e b^2)\\ &= 2^{1-e}\phi(2^e)\phi(a^2)2^{1-e}\phi(2^e)\phi(b^2)\\ &= \phi(a^2b^2),\end{aligned}$$

$$\begin{aligned}g(ab) &= 2^{1-e}\phi(2^e a^2 b^2)\\ &= 2^{1-e}\phi(2^e)\phi(a^2b^2)\\ &= 2^{1-e}2^{e-1}\phi(a^2b^2)\\ &= \phi(a^2b^2).\end{aligned}$$

Case 2. Suppose a is even and b is odd, then $a = 2^\alpha c$, where c is odd.

$$\begin{aligned} g(a)g(b) &= 2^{1-e}\phi(2^e 2^{2\alpha} c^2) 2^{1-e}\phi(2^e b^2) \\ &= 2^{2-2e} 2^{e+2\alpha-1} 2^{e-1} \phi(c^2 b^2) \\ &= 2^{2\alpha}\phi(c^2 b^2), \end{aligned}$$

$$\begin{aligned} g(ab) &= 2^{1-e}\phi(2^e a^2 b^2) \\ &= 2^{1-e}\phi(2^{e+2\alpha} c^2 b^2) \\ &= 2^{1-e} 2^{e+2\alpha-1}\phi(c^2 b^2) \\ &= 2^{2\alpha}\phi(c^2 b^2). \end{aligned}$$

So, $g(ab) = g(a)g(b)$, that is, $g(n)$ is a multiplicative function. □

Theorem 4.23 *For all $H > 0$*

$$F(x) = 2\prod_p \left(1 - \frac{1}{p(p-1)}\right) \mathrm{Li}(x) + O\left(\frac{x}{\log^H x}\right),$$

i.e., the relative density of upper (or lower) flat primes p is $2A = 0.7480\cdots$ where A is Artin's constant,

$$A = \prod_p \left(1 - \frac{1}{p(p-1)}\right).$$

Proof. We begin with following the method of Mirsky [70]. Fix $e \geq 1$ and let x and y satisfy $1 < y < x$ and be sufficiently large. Let $H > 0$ be the given positive constant. Define

$$F_e(x) := \#\{p \leq x : p \text{ is prime and } m \text{ squarefree such that } 2^e m = p + 1\}.$$

If $\mu(n)$ is the Möbius function and $\mu_2(n)$ the characteristic function of the squarefree numbers, we can write

$$\mu_2(n) = \sum_{d^2|n} \mu(d).$$

Then

$$\begin{aligned} F_e(x) &= \sum_{\substack{p\le x\\ p+1=2^e m}} \mu_2(m) = \sum_{p\le x} \sum_{\substack{a:a\ge 1\\ a^2b2^e=p+1}} \mu(a) \\ &= \Sigma_1 + \Sigma_2, \\ \text{where } \Sigma_1 &:= \sum_{p\le x} \sum_{\substack{a:1\le a\le y\\ a^2b2^e=p+1}} \mu(a), \\ \text{and } \Sigma_2 &:= \sum_{p\le x} \sum_{\substack{a>y\\ a^2b2^e=p+1}} \mu(a). \end{aligned}$$

Now using the Bombieri-Vinogradov theorem (Lemma 4.18) for the number of primes in an arithmetic progression, which is valid with a uniform error bound for the values of e which will be needed:

$$\begin{aligned} \Sigma_1 &= \sum_{a\le y} \mu(a) \sum_{\substack{p:p\le x\\ p\equiv -1 \bmod 2^e a^2}} 1 \\ &= \sum_{a\le y} \mu(a) \left(\frac{\mathrm{Li}(x)}{\phi(2^e a^2)} + O\left(\frac{x}{\log^{2H+1} x} \right) \right) \\ &= \left(\sum_{a\ge 1} \frac{\mu(a)}{\phi(2^e a^2)} \right) \mathrm{Li}(x) + O\left(\frac{x}{\log x} \sum_{a>y} \frac{1}{\phi(2^e a^2)} \right) + O\left(\frac{xy}{\log^{2H+1} x} \right). \end{aligned}$$

By Lemma 4.22 the function $g(n) := 2^{1-e}\phi(2^e n^2)$ is multiplicative, and the series in the sum below is absolutely convergent, so the coefficient of $\mathrm{Li}(x)$ may be rewritten

$$\begin{aligned} \frac{1}{2^{e-1}} \sum_{a\ge 1} \frac{2^{e-1}\mu(a)}{\phi(2^e a^2)} &= \frac{1}{2^{e-1}} \prod_p \left(1 - \frac{2^{e-1}}{\phi(2^e p^2)} \right) \\ &= \frac{1}{2^{e-1}} \frac{3}{4} \prod_{p \text{ odd}} \left(1 - \frac{1}{p^2 - p} \right) = \frac{3A}{2^e}. \end{aligned}$$

Now consider the sum in the first error term for Σ_1. By Lemmas 4.19 and 4.21,

$$\sum_{a>y} \frac{1}{\phi(2^e a^2)} \ll \sum_{a>y} \frac{1}{2^e \phi(a^2)} \ll \frac{1}{2^e} \sum_{a>y} \frac{\log\log a}{a^2}.$$

Therefore

$$O\left(\frac{x}{\log x} \sum_{a>y} \frac{1}{\phi(2^e a^2)} \right) = O\left(\frac{x \log\log y}{2^e y \log x} \right).$$

For the second sum we use Lemma 4.20:

$$|\Sigma_2| \le \sum_{p<x} \sum_{\substack{a>y \\ p+1=2^e a^2 b}} 1 \le \sum_{\substack{a>y \\ 2^e a^2 b \le x}} 1 = O\Big(\frac{x}{2^e y}\Big),$$

and therefore

$$F_e(x) = \frac{3A}{2^e}\mathrm{Li}(x) + O\Big(\frac{x \log\log y}{2^e y \log x}\Big) + O\Big(\frac{x}{2^e y}\Big) + O\Big(\frac{xy}{\log^{2H+1} x}\Big).$$

If we choose $y = \log^H x$, then

$$F_e(x) = \frac{3A}{2^e}\mathrm{Li}(x) + O\left(\frac{x}{\log^{H+1} x}\right).$$

Now let

$$D_e(x) := \#\{p \le x : p \text{ is prime, } p+1 = 2^e m, \text{ with } m \text{ squarefree and odd}\}.$$

By, [70, Theorem 1], $D_1(x) = A \cdot \mathrm{Li}(x) + O\big(\frac{x}{\log^{H+1} x}\big)$. Considering the even and odd cases, for all $e \ge 1$, we have $F_e(x) = D_e(x) + D_{e+1}(x)$ so

$$F_1(x) + F_2(x) + \cdots = D_1(x) + 2(D_2(x) + D_3(x) + \cdots)$$

and therefore

$$\begin{aligned} F(x) &= \sum_{e=1}^{\lfloor \frac{\log x}{\log 2} \rfloor} D_e(x) + O(\log x) \\ &= \frac{1}{2}(D_1(x) + F_1(x) + F_2(x) + \cdots) + O(\log x) \\ &= \frac{A}{2}\left(1 + \frac{3}{2^1} + \frac{3}{2^2} + \cdots\right)\mathrm{Li}(x) + O\left(\frac{x}{\log^{H+1} x}\right) \\ &= 2A\mathrm{Li}(x) + O\left(\frac{x}{\log^H x}\right) \end{aligned}$$

and this completes the proof for upper flat primes. The proof for lower flat primes is similar. □

Since $2A > 0.74$, the relative density of either lower flat or upper flat primes is greater than 74%. Thus, in the worst possible case the density of primes

which are neither lower nor upper flat would be less than $26\% + 26\% = 52\%$, leading to a lower bound of 48% for the density of the set of primes which are both upper flat and lower flat. However, this figure underestimates the proportion of such primes - see Theorem 4.25 and its corollary below.

Corollary 4.24 *For all $\epsilon > 0$ and $x \geq x_\epsilon$ there exist an upper flat prime and a lower flat prime in the interval $[x, (1+\epsilon)x]$.*

Proof. Since $F(x) = 2Ax/\log x + O(x/\log^2 x)$, for fixed $\epsilon > 0$ we have

$$F(x+\epsilon x) - F(x) = \frac{2A\epsilon x}{\log x} + O\left(\frac{x}{\log^2 x}\right),$$

which is strictly positive for all x sufficiently large. □

Note also that it would be possible to adapt the method of Adleman, Pomerance and Rumely [1, Proposition 9] to count lower or upper flat primes in arithmetic progressions.

Theorem 4.25 *Let the constant $H > 0$ and the real variable x be sufficiently large. Let the set of primes p which are both lower and upper flat which are less than x be given by*

$$\begin{aligned} B(x) &= \{p \leq x : \exists e \geq 1,\ f \geq 1 \text{ and odd squarefree } u, v \text{ so } p-1 = 2^e v, \\ &\quad p+1 = 2^f u\}. \end{aligned}$$

Then

$$B(x) = A_2 \mathrm{Li}(x) + O\left(\frac{x}{\log^H x}\right)$$

where the constant

$$A_2 = \prod_{p \text{ odd}} \left(1 - \frac{2}{p^2 - p}\right) = 0.53511....$$

Proof. Let $e, f \geq 1$ and define the sets:

$$\begin{aligned} L_e &:= \{p \leq x : \exists \text{ odd squarefree } v \text{ so } p-1 = 2^e v\} \\ U_f &:= \{p \leq x : \exists \text{ odd squarefree } u \text{ so } p+1 = 2^f u\}. \end{aligned}$$

Then $L_1 \cap U_1 = \emptyset$ and $L_e \cap U_f = \emptyset$ for all $e \geq 2, f \geq 2$ so we can write

$$B(x) = \{\cup_{f\geq 2} L_1 \cap U_f\} \cup \{\cup_{e \geq 2} U_1 \cap L_e\}$$

where all of the unions are disjoint.

Now fix $e \geq 2$. We will first estimate the size of $U_1 \cap L_e$,where

$$U_1 \cap L_e = \{p \leq x : \exists \text{ odd squarefree } u, v \text{ so } p+1 = 2u, p-1 = 2^e v\}.$$

Then

$$\begin{aligned} \#U_1 \cap L_e &= \sum_{p \leq x} \sum_{\substack{p+1=2u, \\ p-1=2^e v, \\ u,v \text{ odd and squarefree}}} 1 \\ &= \sum_{p \leq x} \sum_{\substack{a,b \text{ odd},(a,b)=1, \\ p \equiv -1 \bmod a^2,\ a^2 \leq x/2 \\ p \equiv 1 \bmod b^2,\ b^2 \leq x/2^e \\ p \equiv 1+2^e \bmod 2^{e+1}}} \mu(a)\mu(b) \\ &= \sum_{p \leq x} \sum_{a,b \text{ odd},(a,b)=1, a^2b^2 \leq x^2 2^{-e-1}, \atop p \equiv u \bmod 2^{e+1}a^2b^2} \mu(a)\mu(b), \\ &= \sum_{p \leq x} \sum_{d \text{ odd}, d^2 \leq x^2 2^{-e-1}, \atop p \equiv u \bmod 2^{e+1}d^2} \tau^*(d)\mu(d) \end{aligned}$$

where u, the residue obtained through an application of the Chinese Remainder Algorithm, is dependent on d and e, and $\tau^*(d)$ is the number of unitary divisors of d, a multiplicative function with $\tau^*(p) = 2$. This function arises because for fixed $d \geq 1$, the number of decompositions $d = ab$ with $(a, b) = 1$ is $\tau^*(d)$.

We then split and reverse the sum in a similar manner as in the proof of Theorem 4.23.

$$\begin{aligned} \#U_1 \cap L_e &= \sum_{p \leq x} \sum_{\substack{d \text{ odd}, \ d \leq y \\ p \equiv u \bmod 2^{e+1}d^2}} \tau^*(d)\mu(d) + \sum_{p \leq x} \sum_{\substack{y < d \leq x/\sqrt{2^{e+1}} \\ p \equiv u \bmod 2^{e+1}d^2}} \tau^*(d)\mu(d) \\ &= \Sigma_1 + \Sigma_2. \end{aligned}$$

For Σ_1:

$$\begin{aligned}\Sigma_1 &= \sum_{d \text{ odd},\ d\le y} \tau^*(d)\mu(d)\left(\frac{\mathrm{Li}(x)}{\phi(2^{e+1}d^2)} + O\left(\frac{x}{\log^{3H+3} x}\right)\right) \\ &= \left(\sum_{d\ge 1,\ d \text{ odd}} \frac{\tau^*(d)\mu(d)}{\phi(2^{e+1}d^2)}\right)\mathrm{Li}(x) + O\left(\frac{x}{\log x}\sum_{d>y}\frac{\tau^*(d)}{\phi(2^{e+1}d^2)}\right) \\ &\quad + O\left(\frac{xy}{\log^{3H+3} x}\right).\end{aligned}$$

Let $\epsilon > 0$ be given. Indeed $\epsilon = \frac{1}{2}$ is sufficient for our proof. To bound the sum in the second error term, we use the estimate $\tau^*(d) \ll \tau(d) \ll d^\epsilon$ and Lemma 4.22. Then,

$$\begin{aligned}\sum_{d>y}\frac{\tau^*(d)}{\phi(2^{e+1}d^2)} &\le \frac{1}{2^e}\sum_{d>y}\frac{\tau^*(d)}{\phi(d^2)} \\ &\ll \frac{1}{2^e}\sum_{d>y}\frac{\tau^*(d)\log\log d}{d^2} \\ &\ll \frac{1}{2^e}\sum_{d>y}\frac{\log\log d}{d^{2-\epsilon}},\end{aligned}$$

so

$$O\left(\frac{x}{\log x}\sum_{d>y}\frac{\tau^*(d)}{\phi(2^{e+1}d^2)}\right) = O\left(\frac{x}{\log x}\frac{1}{2^e}\frac{\log\log y}{y^{1-\epsilon}}\right)$$

For Σ_2:

$$\begin{aligned}\Sigma_2 &= \sum_{p\le x}\sum_{\substack{y<d\le x/\sqrt{2^{e+1}} \\ p\equiv u \bmod 2^{e+1}d^2}} \tau^*(d)\mu(d) \\ &\ll \sum_{n\le x}\sum_{\substack{2^{e+1}d^2\mid n \\ y<d}} d^\epsilon \\ &= \sum_{y<d}\lfloor\frac{x}{2^{e+1}d^2}\rfloor d^\epsilon \\ &\le \frac{x}{2^{e+1}}\sum_{y<d}\frac{1}{d^{2-\epsilon}} \\ &\ll \frac{x}{y^{1-\epsilon}}.\end{aligned}$$

Putting these bounds together and choosing $y = \log^{2H+2} x$,

$$\begin{aligned} \#U_1 \cap L_e &= \left(\sum_{d \geq 1,\ d\ odd} \frac{\tau^*(d)\mu(d)}{\phi(2^{e+1}d^2)} \right) \mathrm{Li}(x) + O\left(\frac{x}{\log x} \frac{\log\log x}{\log^{H+1} x} \right) \\ &\quad + O\left(\frac{x \log^{2H+2} x}{\log^{3H+3} x} \right) + O\left(\frac{x}{\log^{(2H+2)/2} x} \right) \\ &= \frac{1}{2^e} \prod_{p\ odd} \left(1 - \frac{2}{p^2 - p} \right) \mathrm{Li}(x) + O\left(\frac{x}{\log^{H+1} x} \right). \end{aligned}$$

Summing over $e \geq 2$ and, noticing that the sizes for each corresponding $L_1 \cap U_e$ are the same, we obtain the stated value of $B(x)$. □

Figure 4.3 compares the number of primes up to $80,000$ with the number of primes up to $80,000$ which are both lower and upper flat.

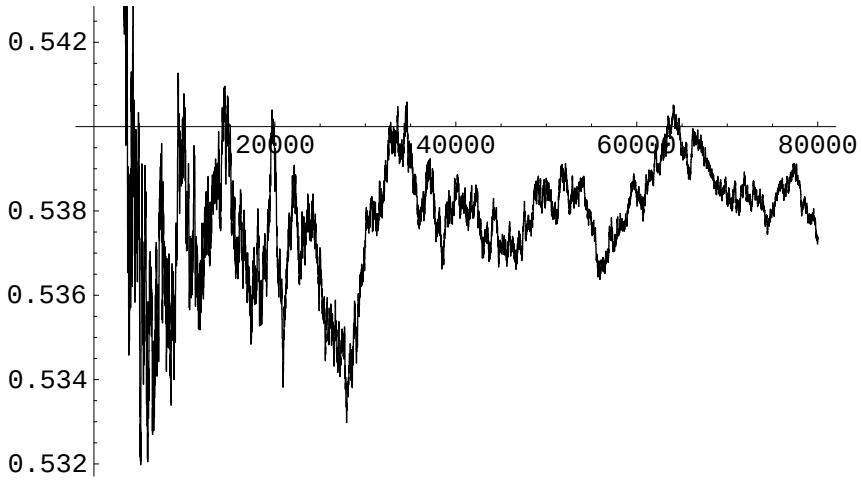

Figure 4.3: The ratio $B(x)/\pi(x)$ for $1 \leq x \leq 8 \cdot 10^4$

Corollary 4.26 *It follows from Theorems 4.23 and 4.25 that the set of rational primes may be divided into* 4 *disjoint classes: those both lower and upper flat - about* 54%*, those either lower or upper flat but not both - each about* 21%*, and those neither upper nor lower flat -* 4%*.*

Corollary 4.27 *It follows that primes which are both upper and lower flat and congruent to* 1 *modulo* 4 *(respectively* 3 *modulo* 4*) are a positive relative*

proportion of all primes. These must therefore be of the form $p = 2s - 1$ *(respectively* $p = 2s + 1$*) where* s *is odd and squarefree.*

Comment: Note that not both $p + 1$ and $p - 1$ can be squarefree for odd primes p, and that the same applies to $p \pm h$ for odd shifts h. It appears, numerically, that for fixed odd h, the proportion of primes p with both $p+h$ and $p-h$ flat is always over 50%, with smaller proportions for $h \equiv \pm 1 \pmod 6$ and larger for $h \equiv 3 \pmod 6$. The larger proportions appear to be all significantly larger than the smaller. This warrants further investigation.

4.3.3 Upper and lower thin primes

In the paper [99, Theorem 3] a proof is set out for a result given below on the number of primes up to x giving a lower bound for the number primes with fixed consecutive values of the number of distinct prime divisors of shifts of the primes by a, with the parameter a having the explicit value 2. It is remarked that a similar proof will work for all integer (non-zero) a. Here is the statement taken from the review of [99] in Mathematical Reviews (MR1347377) (although the lower bound for m is not given):

Let a be a non-zero integer and (for $m \geq 1$) define

$$\mathcal{P}(m, x, \omega) := \#\{p : p \leq x, \omega(p + a) = m\}.$$

Then there exist positive absolute constants b and c such that as $x \to \infty$

$$\mathcal{P}(m, x, \omega) + \mathcal{P}(m + 1, x, \omega) \geq c\frac{x(\log\log x)^{m-1}}{(m - 1)! \log^2 x}$$

holds for $1 \leq m \leq b \log\log x$.

If we use the result in case $a = 1$, we are able to show the number of thin primes is infinite.

To see this let $a = 1$, $m = 1$ and x be sufficiently large. Then

$$\begin{aligned} T(x) + M(x) &= \mathcal{P}(1, x, \omega) + \mathcal{P}(2, x, \omega) \\ &= \#\{p \le x : p + 1 = 2^e \text{ or } p + 1 = 2^e q^f, e \ge 1, f \ge 1, \text{ or } p = 2\} \end{aligned}$$

where

$$M(x) := \#\{p \le x : p + 1 = 2^e q^f, e \ge 1, f \ge 2\}.$$

(Note this is not the same as the $M(x)$ in equation (4.14).) Then

$$\begin{aligned} M(x) &\le \sum_{e=1}^{\log x} \sum_{f=2}^{\log x} \pi\left(\left(\frac{x}{2^e}\right)^{\frac{1}{f}}\right) + O(\log x) \\ &\ll \log x \sum_{e=1}^{\log x} \pi\left(\sqrt{\frac{x}{2^e}}\right) \\ &\ll \log^2 x \; \pi\left(\sqrt{x}\right) \ll \sqrt{x} \log x \end{aligned}$$

Therefore, by the quoted result above, the number of thin primes less than or equal to x is bounded below by a constant times $x/\log^2 x$, so must be infinite.

However there are parts of the proof of [99, Theorem 3] that do not appear to work, even for the given case $a = 2$, and, in addition, the implied lower bound should be $m \ge 2$. Apparently the best available safe result, using the method of Chen [19], appears to be that of Heath-Brown [50, Lemma 1] from which we can easily show that if $H(x)$ is the number of primes such that $p \le x$ and either $p + 1 = 2p_1$ or $p + 1 = 2p_1p_2$, with the p_i's odd primes, then $H(x) \gg x/\log^2 x$.

Based on this evidence, the Bateman-Horn conjecture set out in Section 5 below, and numerical evidence, we are led to the conjecture on lower bound for $T(x)$:

Conjecture: The number of upper thin primes up to x satisfies

$$T(x) \gg \frac{x}{\log^2 x}$$

and has the same asymptotic density as the number of twin primes up to x and the same is true for the lower thin primes.

The order of difficulty of this conjecture appears to be similar to showing that there are an infinite number of twin primes or Sophie-Germain primes. As usual upper bounds are much easier to obtain (see Theorem 4.28 below):

Theorem 4.28 *As* $x \to \infty$

$$T(x) \ll \frac{x}{\log^2 x}.$$

Proof. First let $e \geq 1$ be fixed and apply the sieve of Brun in the same manner as for the classical twin primes problem (for example [98, Theorem 4]) or [4, Theorem 13.1]) to count

$$J_e(x) := \#\{p \leq x : 2^e p - 1 \text{ is prime}\}.$$

Note that if

$$\mathcal{A} = \{m(2^e m - 1) : m \leq x\}$$

and $\rho(d)$ is the number of solutions modulo d which satisfy

$$m(2^e m - 1) \equiv 0 \pmod{d},$$

then ρ is a multiplicative function. Also $\rho(2) = 1$ and $\rho(p) = 2$ for odd primes p, leading to the same bound as in the twin primes problem, namely

$$J_e(x) \ll \frac{x}{\log^2 x}.$$

Now we use the fact, proved using induction for $m \geq 4$, that, for all $m \geq 1$,

$$\sum_{n=1}^{m} \frac{2^n}{n^2} < 5\frac{2^m}{m^2}. \tag{4.15}$$

Finally, let x be large and choose $m \in \mathbb{N}$ so $2^m \leq x < 2^{m+1}$. Then

$$\begin{aligned}
T(x) &= \sum_{e=1}^{\lfloor \frac{\log x}{\log 2} \rfloor} \left(J_e\left(\frac{x}{2^e}\right) + O(1) \right) \\
&\ll \sum_{e=1}^{\lfloor \frac{\log x}{\log 2} \rfloor - 1} \frac{x}{2^e} \frac{1}{\log^2 \frac{x}{2^e}} + O(\log x) \\
&\leq \sum_{e=0}^{\lfloor \frac{\log 2^{m+1}}{\log 2} \rfloor - 1} \frac{2^{m+1}}{2^e} \frac{1}{\log^2 \frac{2^{m+1}}{2^e}} + O(\log x) \\
&= \frac{1}{\log^2 2} \sum_{n=1}^{m+1} \frac{2^n}{n^2} + O(\log x) \\
&< 5 \frac{1}{\log^2 2} \frac{2^{m+1}}{(m+1)^2} + O(\log x) \text{ by (4.15)} \\
&\ll \frac{x}{\log^2 x},
\end{aligned}$$

completing the proof of the theorem. □

So the asymptotic bound is the same as that for twin primes. In the same manner as originally derived by Brun for the sum of reciprocals of the twin primes (for example [74, Theorem 6.12]) we obtain:

Corollary 4.29 *The sum of the reciprocals of the thin primes is finite.*

Proof. If p_n is the n'th thin prime then, by Theorem 4.28,

$$\begin{aligned}
n &= T(p_n) \ll \frac{p_n}{\log^2 p_n} \\
&\ll \frac{p_n}{(\log n)^2} \text{ so} \\
\frac{1}{p_n} &\ll \frac{1}{n \log^2 n}.
\end{aligned}$$

□

4.3.4 Hardy-Littlewood-Bateman-Horn conjectures

The well known Hardy-Littlewood-Bateman-Horn conjectures ([46], [4]) give an asymptotic formula for the number of simultaneous prime values of sets of

polynomials in $\mathbb{Z}[x]$, with some restrictions on the polynomials. In the case of twin primes the polynomials are $f_o(x) = x,\ f_1(x) = x + 2$ and if

$$\pi_2(x) := \#\{p \le x : p + 2 \text{ is prime}\}$$

then the formula predicted is

$$\pi_2(x) \sim 2C_2 \int_2^x \frac{du}{\log^2 u}$$

where C_2 is the so-called twin prime constant [75] defined by

$$C_2 := \prod_{p>2} \Big(1 - \frac{1}{(p-1)^2}\Big).$$

In the case of thin primes the conjectures only apply to forms with fixed $e \ge 1$ with polynomials $f_0(x) = x,\ f_e(x) = 2^e x - 1$. If

$$T_e(x) := \#\{p \le x : p + 1 = 2^e q\}$$

Then the formulas predict

$$T_e(x) \sim \frac{2C_2}{2^e} \int_2^x \frac{du}{\log^2 u}.$$

The factor $1/2^e$ occurs simply because $p \le x+1$ if and only if $q \le x/2^e$. Hence

$$\frac{T(x)}{\pi_2(x)} \sim \frac{\sum_{e=1}^{\log x/\log 2} T_e(x)}{\pi_2(x)} \sim 1.$$

To test this numerically we evaluated the ratio of the number of thin primes up to x to the number of twin primes up to x for x up to 4×10^6 in steps of 10^5 and obtained the following values:

```
{1., 1.20343, 1.16852, 1.17134, 1.16036, 1.15882, 1.14819, 1.1447,
1.14499,  1.1428, 1.13515, 1.12896, 1.12543, 1.1234, 1.11715,
1.1184, 1.11729, 1.11438,  1.11168, 1.1099, 1.11169, 1.1106,
1.11125, 1.11095, 1.11221, 1.11317, 1.1134,  1.11251, 1.1118,
1.11306, 1.11179, 1.11015, 1.10986, 1.1096, 1.10876,  1.10924,
1.10912, 1.10676, 1.10623, 1.10536}.
```

demonstrating some convergence towards the predicted value 1. If the relationship between the thin and twin primes could be made explicit this would assist in a proof of the twin primes conjecture.

Igor Shparlinski, in a private communication, alerted us to the theorem of Heath-Brown. The computations were produced using Mathematica.

4.4 Theorems on flat numbers

The following Theorems 4.30, 4.31, 4.32, 4.33, 4.34 and 4.35 investigate the properties of the upper flat numbers.

Theorem 4.30 *The maximum number of successive odd numbers which are upper flat is* 8. *That is,*

$$N+1 = 2^{e_1} b_1$$

$$N+3 = 2^{e_3} b_3$$

$$N+5 = 2^{e_5} b_5$$

$$N+7 = 2^{e_7} b_7$$

$$N+9 = 2^{e_9} b_9$$

$$N+11 = 2^{e_{11}} b_{11}$$

$$N+13 = 2^{e_{13}} b_{13}$$

$$N+15 = 2^{e_{15}} b_{15}$$

where N is an upper flat number, b_j's are squarefree numbers.

Proof. Let N be an upper flat number. By the definition of an upper flat number,

$$N+1 = 2^e p_1 \cdots p_m,$$

where $e \geq 1$, and $p_i \in \mathbb{P}$, $i = 1, 2, \cdots m$. Let $b = p_1 \cdots p_m$, then b is a squarefree odd number. So, b only can exist between the power of 3, 3^α, with $\alpha \geq 2$. Since $N \cdot 3^2 - (N-1) \cdot 3^2 = 9$, $\forall N \in \mathbb{Z}$, so, an upper flat number N in intervals $[3^2(N-1), 3^2N]$, where there are 9 consecutive odd numbers, could not be $3^2N - 1$, so the number of members in a group of upper flat numbers having above restriction is no more than 8. □

Theorem 4.31 *Restrictions on the values of the (e_j), where e_j's are the exponents in Theorem 4.30, ($j = 1, 3, 5, 7, 9, 11, 13, 15$). For $e_1e_3e_5e_7e_9e_{11}e_{13}e_{15}$, we have the following patterns:*

(A) $1, 2, 1, *, 1, 2, 1, *$

(B) $*, 1, 2, 1, *, 1, 2, 1$

(C) $1, *, 1, 2, 1, *, 1, 2$

(D) $2, 1, *, 1, 2, 1, *, 1$

where the symbol $*$ denotes any positive integer ≥ 3.

Proof. Let N be an upper flat number having restriction in Theorem 4.30. We will separately consider the value of e_j with j=1, 3, 5, 7, 9, 11, 13, 15. First, let $e_1 = 1$. Then, $N + 1 = 2b_1$, $N + 3 = 2^{e_3}b_3$, so $1 + b_1 = 2^{e_3-1}b_3$. So, there are 2 possibilities: (1), if $1 + b_1 = 2c$, with c odd, then $e_3 = 2$, since $c = 2^{e_3-2}b_3$, c and b_3 are odd, force $2^{e_3-2} = 1$; (2), if $1 + b_1 = 2^m c$, with $m \geq 2$, and c odd, then $1 + b_1 = 2^m c = 2^{e_3-1}b_3$, then $c = 2^{e_3-1-m}b_3$, since b_3 is odd, so $2^{e_3-1-m} = 1$, so $e_3 = 1 + m \geq 3$, and we use symbol $*$ to express $e_j \geq 3$. Since, $N + 5 = 2^{e_5}b_5$, so $4 + 2b_1 = 2^{e_5}b_5$, so $2 + b_1 = 2^{e_5-1}b_5$, since $2 + b_1$ and b_5 are odd numbers, so force $e_5 = 1$. Since $N + 7 = 2^{e_7}b_7$, so $6 + 2b_1 = 2^{e_7}b_7$, so $3 + b_1 = 2^{e_7-1}b_7$. (1), if $1 + b_1 = 2c$, with c odd, then $3 + b_1 = 2 + (1 + b_1) = 2(1 + c) = 2^{e_7-1}b_7$, then $1 + c = 2^{e_7-2}b_7$, since $1 + c$ is

even and b_7 is odd, so, $e_7 - 2 \geq 1$, so $e_7 \geq 3$, so $e_7 = *$; (2), if $1 + b_1 = 2^m c$, with $m \geq 2$ and c odd, then $3 + b_1 = 2 + 2^m c = 2(1 + 2^{m-1}c) = 2^{e_7-1}b_7$, since $m \geq 2$, so $1 + 2^{m-1}c$ is odd, so $2^{e_7-2}b_7 = 1$, so $e_7 = 2$. Since $N + 9 = 2^{e_9}b_9$, and $N + 13 = 2^{e_{13}}b_{13}$, so $4 + b_1 = 2^{e_9-1}b_9$, and $6 + b_1 = 2^{e_{13}-1}b_{13}$, since b_j's are odd, so $4 + b_1$ and $6 + b_1$ are odd, so $e_9 = e_{13} = 1$. Since, $N + 11 = 2^{e_{11}}b_{11}$, so $10 = 2^{e_{11}}b_{11} - 2b_1$, so $5 + b_1 = 2^{e_{11}-1}b_{11}$. (1), if $1 + b_1 = 2c$, with c odd, then $5 + b_1 = 4 + (1 + b_1) = 2(2 + c) = 2^{e_{11}-1}b_{11}$, then $2 + c = 2^{e_{11}-2}b_{11}$, since $2 + c$ and b_{11} are odd, so $e_{11} = 2$; (2), if $1 + b_1 = 2^m c$, with $m \geq 2$ and c odd, then $5 + b_1 = 4 + 2^m c = 2(2 + 2^{m-1}c) = 2^{e_{11}-1}b_{11}$, then $2 + 2^{m-1}c = 2^{e_{11}-2}b_{11}$, since $2 + 2^{m-1}c$ is even and b_{11} is odd, so $e_{11} - 2 \geq 1$, so $e_{11} \geq 3$, so $e_{11} = *$. Since, $N + 15 = 2^{e_{15}}b_{15}$, so $7 + b_1 = 2^{e_{15}-1}b_{15}$, (1), if $1 + b_1 = 2c$, with c odd, then $7 + b_1 = 6 + (1 + b_1) = 2(3 + c) = 2^{e_{15}-1}b_{15}$, so $3 + c = 2^{e_{15}-2}b_{15}$, since $3 + c$ is even and b_{15} is odd, so $e_{15} - 2 \geq 1$, $e_{15} \geq 3$, so $e_{15} = *$; (2), if $1 + b_1 = 2^m c$, with $m \geq 2$ and c odd, then $7 + b_1 = 6 + 2^m c = 2(3 + 2^{m-1}c) = 2^{e_{15}-1}b_{15}$, so $3 + 2^{m-1}c = 2^{e_{15}-2}b_{15}$, since $3 + 2^{m-1}c$ and b_{15} are odd, so $e_{15} = 2$. Now, we have proved that when $e_1 = 1$ there are 2 patterns: (A) $121 * 121*$, if $1 + b_1 = 2c$ with c odd; and (C) $1 * 121 * 12$, if $1 + b_1 = 2^m c$ with $m \geq 2$ and c odd.

Next, we fix $e_1 = 2$. Since $N + 1 = 2^2 b_1$, $N + 3 = 2^{e_3}b_3$, $N + 7 = 2^{e_7}b_7$, $N + 11 = 2^{e_{11}}b_{11}$, $N + 15 = 2^{e_{15}}b_{15}$, so $1 + 2b_1 = 2^{e_3-1}b_3 \Rightarrow e_3 = 1$; $3 + 2b_1 = 2^{e_7-1}b_7 \Rightarrow e_7 = 1$; $5 + 2b_1 = 2^{e_{11}-1}b_{11} \Rightarrow e_{11} = 1$; and $7 + 2b_1 = 2^{e_{15}-1}b_{15} \Rightarrow e_{15} = 1$, since b_j's are odd. Since, $N + 5 = 2^{e_5}b_5$, so $1 + b_1 = 2^{e_5-2}b_5$, so either (1), if $1 + b_1 = 2c$, with $c > 0$ odd, then $e_5 = 3 = *$, or (2), if $1 + b_1 = 2^m c$, with $m \geq 2$ and c odd, then $e_5 \geq 4$, so $e_5 = *$. Since $N + 9 = 2^{e_9}b_9$, so $2 + b_1 = 2^{e_9-2}b_9 \Rightarrow e_9 = 2$. Since, $N + 13 = 2^{e_{13}}b_{13}$, so $3 + b_1 = 2^{e_{13}-2}b_{13}$ so either (1), if $1 + b_1 = 2c$, with $c > 0$ odd, then $3 + b_1 = 2(c + 1) = 2^{e_{13}-2}b_{13} \Rightarrow e_{13} \geq 4$, so $e_{13} = *$, or (2), if $1 + b_1 = 2^m c$, with $m \geq 2$ and c odd, then $3 + b_1 = 2(2^{m-1}c + 1) = 2^{e_{13}-2}b_{13} \Rightarrow e_{13} = 3 = *$.

Therefore, when $e_1 = 2$, the pattern is: (D) $21 * 121 * 1$.

Finally, let $e_1 = * \geq 3$. Since $N + 1 = 2^* b_1$, $N + 3 = 2^{e_3} b_3$, $N + 7 = 2^{e_7} b_7$, $N + 11 = 2^{e_{11}} b_{11}$, and $N + 15 = 2^{e_{15}} b_{15}$, so $1 + 2^{*-1} b_1 = 2^{e_3 - 1} b_3 \Rightarrow e_3 = 1$, $3 + 2^{*-1} b_1 = 2^{e_7 - 1} b_7 \Rightarrow e_7 = 1$, $5 + 2^{*-1} b_1 = 2^{e_{11} - 1} b_{11} \Rightarrow e_{11} = 1$, and $7 + 2^{*-1} b_1 = 2^{e_{15} - 1} b_{15} \Rightarrow e_{15} = 1$, since the left hand side of the equations are odd. Since, $N + 5 = 2^{e_5} b_5$, and $N + 13 = 2^{e_{13}} b_{13}$, so $1 + 2^{*-2} b_1 = 2^{e_5 - 2} b_5 \Rightarrow e_5 = 2$, and $3 + 2^{*-2} b_1 = 2^{e_{13} - 2} b_{13} \Rightarrow e_{13} = 2$. Since, $N + 9 = 2^{e_9} b_9$, so $1 + 2^{*-3} b_1 = 2^{e_9 - 3} b_9$, if $1 + 2^{*-3} b_1$ is odd, then $e_9 = 3 = *$, if $1 + 2^{*-3} b_1 = 2^m c$, with $m \geq 1$ and c odd, then $e_9 \geq 4$, so $e_9 = *$. Therefore, when $e_1 \geq 3$, then the pattern is: (B) $*121 * 121$. □

Theorem 4.32 *Restrictions on the values of the (b_j) for (A) and (C):*

$$b_5 = b_1 + 2$$

$$b_9 = b_5 + 2$$

$$b_{13} = b_9 + 2;$$

for (B) and (D):

$$b_7 = b_3 + 2$$

$$b_{11} = b_7 + 2$$

$$b_{15} = b_{11} + 2,$$

where (A), (B), (C) and (D) are the patterns in Theorem 4.31 and b_j's are squarefree numbers in Theorem 4.30.

Proof. For patterns (A) and (C), $e_1 = e_5 = e_9 = e_{13} = 1$, so $N + 1 = 2b_1$; $N+5 = 2b_5$; $N+9 = 2b_9$; and $N+13 = 2b_{13}$. Therefore, $b_5 = b_1+2$; $b_9 = b_5+2$; $b_{13} = b_9 + 2$.

For patterns (B) and (D), $e_3 = e_7 = e_{11} = e_{15} = 1$, so $N + 3 = 2b_3$; $N+7 = 2b_7$; $N+11 = 2b_{11}$; $N+15 = 2b_{15}$. Therefore, $b_7 = b_3+2$; $b_{11} = b_7+2$; $b_{15} = b_{11} + 2$. □

Theorem 4.33 $3 \mid b_5$; $3 \mid b_{11}$, *where* b_5 *and* b_{11} *are the squarefree numbers in Theorem 4.30.*

Proof. In a group of 8 odd consecutive integers, each number lies between powers of 3, 3^α with $\alpha > 1$, we will show that only the first number N_1 of such a group possibly satisfies the following form:

$$N_1 + 1 = 2^{e_1} b_1$$

$$N_1 + 3 = 2^{e_3} b_3$$

$$N_1 + 5 = 2^{e_5} b_5$$

$$N_1 + 7 = 2^{e_7} b_7$$

$$N_1 + 9 = 2^{e_9} b_9$$

$$N_1 + 11 = 2^{e_{11}} b_{11}$$

$$N_1 + 13 = 2^{e_{13}} b_{13}$$

$$N_1 + 15 = 2^{e_{15}} b_{15}$$

where b_j's are odd squarefree numbers. Since for other numbers N_i of a group having above restriction with $2 \leq i \leq 8$, $N_i + k$ with $k = 1, 3, 5, \cdots 15$ will be broken by some 3^α with $\alpha \geq 2$. Therefore, N_1 can be expressed as $18N + 1$ for some integer $N \geq 0$. So, $N_1 + 5 = 2^{e_5} b_5$ becomes $18N + 6 = 3(6N + 2) = 2^{e_5} b_5$, so $3 \mid b_5$. Similarly, $N_1 + 11 = 18N + 12 = 3(6N + 4) = 2^{e_{11}} b_{11}$, so $3 \mid b_{11}$. $\square$

Theorem 4.34 *One or two* b_j *have* $5 \mid b_j$, *where* b_j*'s are the squarefree numbers in Theorem 4.30.*

Proof. From Theorem 4.30, we can deduce that only the first number $18N+1$ with $N \in \mathbb{Z}$ of a group of 8 successive odd numbers possibly satisfies the structure in Theorem 4.30. Hence, $(18N+1)+1 = 2^{e_1} b_1 \Rightarrow 2^{e_1 - 1} b_1 = 9N+1 \equiv 4N+1 \pmod 5$; $(18N+1)+3 = 2^{e_3} b_3 \Rightarrow 2^{e_3 - 1} b_3 = 9N+2 \equiv 4N+2 \pmod 5$;

$(18N+1)+5 = 2^{e_5}b_5 \Rightarrow 2^{e_5-1}b_5 = 9N+3 \equiv 4N+3 \pmod 5$; $(18N+1)+7 = 2^{e_7}b_7 \Rightarrow 2^{e_7-1}b_7 = 9N+4 \equiv 4N+4 \pmod 5$; $(18N+1)+9 = 2^{e_9}b_9 \Rightarrow 2^{e_9-1}b_9 = 9N+5 \equiv 4N \pmod 5$; $(18N+1)+11 = 2^{e_{11}}b_{11} \Rightarrow 2^{e_{11}-1}b_{11} = 9N+6 \equiv 4N+1 \pmod 5$; $(18N+1)+13 = 2^{e_{13}}b_{13} \Rightarrow 2^{e_{13}-1}b_{13} = 9N+7 \equiv 4N+2 \pmod 5$; $(18N+1)+15 = 2^{e_{15}}b_{15} \Rightarrow 2^{e_{15}-1}b_{15} = 9N+8 \equiv 4N+3 \pmod 5$. There are 5 possibilities about the relation between 5 and b_j's: (1), if $N \equiv 0 \pmod 5$, then $5 \mid b_9$; (2), if $N \equiv 1 \pmod 5$, then $5 \mid b_1$ and $5 \mid b_{11}$; (3), if $N \equiv 2 \pmod 5$, then $5 \mid b_3$ and $5 \mid b_{13} \pmod 5$; (4), if $N \equiv 3 \pmod 5$, then $5 \mid b_5$ and $5 \mid b_{15}$; (5), if $N \equiv 4 \pmod 5$, then $5 \mid b_7$. Therefore, this theorem is true. □

Theorem 4.35 *There are infinitely many groups of* 8 *consecutive odd upper flat numbers.*

Proof. Consider n groups of 8 odd consecutive integers, each lying between powers of 3, 3^α with $\alpha > 1$. Since a group of upper flat integers can be written in the form

$$N+1 = 2^{e_1}b_1$$

$$N+3 = 2^{e_3}b_3$$

$$N+5 = 2^{e_5}b_5$$

$$N+7 = 2^{e_7}b_7$$

$$N+9 = 2^{e_1}b_9$$

$$N+11 = 2^{e_{11}}b_{11}$$

$$N+13 = 2^{e_{13}}b_{13}$$

$$N+15 = 2^{e_{15}}b_{15}$$

where the b_j's are odd. The largest number in these groups is $18n-1$. A group of 8 numbers will have all of the b_j's squarefree unless one of them is divisible by a square of an odd prime. If p is a prime and $p^2 \leq 18n-1$, then

the number of odd numbers less than or equal to $18n-1$ all divisible by p^2 is $\lfloor \frac{18n-1}{2p^2} \rfloor$. Therefore, $S(n)$, the number of groups of 8 which have all of the b_j's squarefree, satisfies

$$\begin{aligned} S(n) &\geq n - \sum_{p=5}^{\sqrt{18n}} \frac{18n}{2p^2} \\ &> n - 9n \sum_{p=5}^{\infty} \frac{1}{p^2} \\ &> n - 9n \frac{1}{10} \\ &= \frac{n}{10}. \end{aligned}$$

So,

$$\lim_{n \to \infty} S(n) = \infty$$

Therefore, there exist an infinite number of groups of 8 consecutive odd integers, where b_j's are squarefree, i.e. upper flat integers. $\square$

Chapter 5

Even 3-perfect numbers of a flat shape

5.1 Introduction

In this chapter we study even 3-perfect numbers with a flat shape

$$N = 2^a p_1 \cdots p_m,$$

where $a \geq 1$ and $p_1 < p_2 < \cdots < p_m$. All known even 3-perfect numbers $\{c_i : 1 \leq i \leq 6\}$ have a flat shape, (see Table 4.1). If some prime divisors of N are fixed then there are finitely many even 3-perfect numbers, (Theorem 5.10). If all odd prime divisors of N are super thin primes, and p_0 is a prime, $(p_1 + 1 = 2^{a_1} p_0)$, then N is not a 3-perfect number, (Theorem 5.9). If the sum of the reciprocals of odd primes of N is greater than $\frac{3}{5}$, then N is not a 3-perfect number, (Theorem 5.8). If $N = 2^a p_1 \cdots p_m M_{q_1} \cdots M_{q_l}$, where p_i's are super thin primes, $p_i + 1 = 2^{a_i} p_{i-1}$, $(1 \leq i \leq m)$, M_{q_j}'s are Mersenne primes, and p_0 is one of the M_{q_j}'s, then the number of such N's is finite, (Theorem 5.11). We also provide some special cases and examples.

5.2 Special cases

In this section, we discuss six special flat patterns. Some of them are not multiperfect numbers, some of them are only even 3-perfect numbers, and some are only even perfect numbers. It is convenient to investigate the numbers which have a flat shape, and we include a case of odd squarefree numbers.

Theorem 5.1 *If $N = p_1 \cdots p_m$, $p_1 < p_2 < \cdots < p_m$ are odd primes, then N is not a multiperfect number.*

Proof. Suppose $N = p_1 \cdots p_m$ is a multiperfect number, then $N \mid \sigma(N)$, implies $p_m \mid (p_1 + 1) \cdots (p_m + 1)$. Since $p_1 < \cdots < p_m$, so all prime factors of $(p_i + 1)$ are no more than $\frac{p_i+1}{2} < p_m$, for all p_i's. Therefore, N is not a multiperfect number. □

Theorem 5.2 *If $N = 2p_1 \cdots p_m$, and $N \neq 6$, where $m \geq 1$ and $2 < p_1 < \cdots < p_m$, then N is not a multiperfect number.*

Proof. Suppose $N = 2p_1 \cdots p_m$ is a multiperfect number, then $N \mid \sigma(N)$, so $p_m \mid 3(p_1 + 1) \cdots (p_m + 1)$, since $p_1 < \cdots < p_m$, so $p_m \nmid (p_1 + 1) \cdots (p_m + 1)$, so $p_m = 3$, but $N \neq 6$, so N is not a multiperfect number. □

Theorem 5.3 *If $N = 4p_1 \cdots p_m$ is a multiperfect number, where $m \geq 1$ and $2 < p_1 < \cdots < p_m$, then $N = 28$ is the only one solution.*

Proof. Suppose $N = 4p_1 \cdots p_m$ is a k-perfect number, $k \geq 2$, then $N \mid \sigma(N)$, so $p_m \mid 7(p_1+1) \cdots (p_m+1)$, since $2 < p_1 < \cdots < p_m$, so $p_m \nmid (p_1+1) \cdots (p_m+1)$, so $p_m = 7$. Now there are four possibilities as follows: Case (1), $N = 4 \cdot 7$; Case (2), $N = 4 \cdot 3 \cdot 5 \cdot 7$; Case (3), $N = 4 \cdot 3 \cdot 7$; Case (4), $N = 4 \cdot 5 \cdot 7$. By checking $\frac{\sigma(N)}{N}$, $N = 4 \cdot 7 = 28$ is the only one solution. □

Theorem 5.4 *If $N = 2^a$, then N is not a multiperfect number.*

Proof. Since $\sigma(N) = \sigma(2^a) = 2^{a+1} - 1$ is an odd number, but $N = 2^a$ is an even number, so $\frac{\sigma(N)}{N}$ is not an integer. Therefore, N is not a multiperfect number. □

Theorem 5.5 *If $N = 2^a p_1$ is a k-perfect number, with $a \geq 1$ and p_1 is an odd prime, ($k \geq 2$), then $k = 2$.*

Proof. Let $N = 2^a p_1$ be a k-perfect number and $k \neq 2$. Then

$$\sigma(N) = (2^{a+1} - 1)(p_1 + 1) = k \cdot 2^a p_1$$

implies

$$\frac{(2^{a+1} - 1)}{p_1} \cdot \frac{(p_1 + 1)}{2^a} = k,$$

so $2^a \leq p_1 + 1 \leq 2^{a+1}$, so $1 \leq \frac{p_1+1}{2^a} \leq 2$. Since $\frac{p_1+1}{2^a}$ is an integer, so we have the following two cases:

Case (1). If $\frac{p_1+1}{2^a} = 1$. Then $p_1 + 1 = 2^a$ implies $2p_1 + 2 = 2^{a+1}$, so $2p_1 + 1 = 2^{a+1} - 1 = h_1 p_1$, where $h_1 \geq 1$, so $p_1(h_1 - 2) = 1$, so false.

Case (2). If $\frac{p_1+1}{2^a} = 2$. Then $p_1 + 1 = 2^{a+1}$, so $p_1 = 2^{a+1} - 1$, hence $k = 2$. □

The following theorem is a particular case of the result of Carmichael [15]. Carmichael proved that if $N = 2^{a_1} p_2^{a_2} p_3^{a_3}$ is a multiperfect number, then N is not a perfect number and there are only two such values of N (120 and 672), and these are 3-perfect numbers.

Theorem 5.6 *If $N = 2^a p_1 p_2$ is a k-perfect number, with $k \geq 2$, $a \geq 1$, and $p_1 < p_2$ odd primes, then $k = 3$ and the only two solutions are $c_1 = 2^3 \cdot 3 \cdot 5 = 120$ and $c_2 = 2^5 \cdot 3 \cdot 7 = 672$.*

Proof.

$$\begin{aligned} k &= \frac{\sigma(N)}{N} \\ &< 2\left(\frac{1+p_1}{p_1}\right)\left(\frac{1+p_2}{p_2}\right) \\ &\leq 2\cdot\frac{4}{3}\cdot\frac{6}{5} \\ &< 4. \end{aligned}$$

We get $k = 3$ or 2. But $k \neq 2$, by Euler's Theorem. Therefore, $k = 3$.

Now $\sigma(N) = 3N$ and $N = 2^a p_1 p_2$, so

$$(2^{a+1}-1)(p_1+1)(p_2+1) = 3\cdot 2^a p_1 p_2.$$

If $3 \nmid N$, then

$$3 = \frac{\sigma(N)}{N} < 2\left(\frac{6}{5}\right)\left(\frac{8}{7}\right) < 3,$$

which is false.

Hence $p_1 = 3 \mid N$, so

$$(2^{a+1}-1)(3+1)(p_2+1) = 3^2\cdot 2^a p_2,$$

implies the following three cases:

Case (1). $2^{a+1}-1 = 3^2 p_2$ and $p_2+1 = 2^{a-2}$. Then, $p_2 = 7$, so $a = 5$, so $N = c_2 = 2^5\cdot 3\cdot 7$.

Case (2). $2^{a+1}-1 = p_2$ and $p_2+1 = 3^2\cdot 2^{a-2}$. Then, $2^{a+1} = 3^2\cdot 2^{a-2}$, which is false.

Case (3). $2^{a+1}-1 = 3p_2$ and $p_2+1 = 3\cdot 2^{a-2}$. Then, $a = 3$ and $p_2 = 5$, so $N = c_1 = 2^3\cdot 3\cdot 5$. □

5.3 Lemmas

Lemma 5.7 *Let $q_1 < q_2 < \cdots < q_l$, where q_i's are odd primes, and $M_{q_i} = (2^{q_i}-1)$ are primes, $i = 1, 2, \cdots, l$, $l \geq 3$. Then*

$$q_1 \cdots q_l > 6 + q_1 + \cdots + q_l.$$

Proof. Suppose $q_1 \cdots q_l \leq 6 + q_1 + \cdots + q_l$.

If $l = 3$, since Mersenne primes $M_3 = 7$ and $M_5 = 31$, so we have

$$\begin{aligned} 3 \cdot 5 \cdot q_3 &\leq q_1 \cdot q_2 \cdot q_3 \\ &\leq 6 + q_1 + q_2 + q_3 \\ &\leq 6 + 3q_3. \end{aligned}$$

So $15q_3 \leq 6 + 3q_3$, which implies $q_3 \leq \frac{1}{2}$, a contradiction.

If $l \geq 4$, then

$$\begin{aligned} 2^{l-1} \cdot q_l &< q_1 \cdots q_l \\ &\leq 6 + q_1 + \cdots + q_l \\ &\leq 6 + l \cdot q_l. \end{aligned}$$

So

$$2 \cdot (2^{l-1} - l) < q_l \cdot (2^{l-1} - l) < 6$$

so

$$2 \cdot (2^{l-1} - l) < 6$$

so

$$2^l - 2l < 6$$

a contradiction for $l \geq 4$.

Therefore,

$$q_1 \cdots q_l > 6 + q_1 + \cdots + q_l.$$

□

5.4 Results

Theorem 5.8 *If $\sum_{i=1}^{m} \frac{1}{p_i} \geq \frac{3}{5}$, with distinct odd primes $p_1 < p_2 < \cdots < p_m$, then any $N = 2^a \cdot p_1 \cdots p_m$ with $a \geq 1$ is not a 3-perfect number.*

Proof. Let $N = 2^a \cdot p_1 \cdots p_m$ be a 3-perfect number, then $\sigma(N) = 3N$. Using Theorems 5.2 and 5.3, we can assume $a \geq 3$. We have

$$2^a \cdot 3 \cdot p_1 \cdots p_m = (2^{a+1} - 1) \cdot (p_1 + 1) \cdots (p_m + 1)$$

then

$$\left(2 - \frac{1}{2^a}\right)\left(1 + \frac{1}{p_1}\right) \cdots \left(1 + \frac{1}{p_m}\right) = 3$$

so

$$\frac{8}{5} \geq \frac{3}{2 - \frac{1}{2^a}} = \prod_{i=1}^{m}\left(1 + \frac{1}{p_i}\right) > 1 + \sum_{i=1}^{m}\frac{1}{p_i} \geq \frac{8}{5},$$

so a contradiction.

Therefore, $N = 2^a \cdot p_1 \cdots p_m$ is not a 3-perfect number. □

Theorem 5.9 *If $N = 2^a p_1 \cdots p_m$, with $a \geq 1$, $p_i + 1 = 2^{a_i} p_{i-1}$, $a_i \geq 1$, where p_i is an odd prime for $1 \leq i \leq m$ and p_0 is a prime. Then N is not a 3-perfect number.*

Proof. Suppose N is a 3-perfect number. Then, $\sigma(N) = 3N$. We get

$$\begin{aligned} \sigma(N) &= (2^{a+1} - 1)(p_1 + 1) \cdots (p_m + 1) \\ &= (2^{a+1} - 1)2^{(\sum_{i=1}^{m} a_i)} p_0 \cdots p_{m-1} \\ &= 3 \cdot 2^a p_1 \cdots p_m \end{aligned}$$

implies

$$a = \sum_{i=1}^{m} a_i$$

and

$$(2^{a+1} - 1)p_0 = 3p_m.$$

If $p_0 = 3$, then $p_m = 2^{a+1} - 1$, so $p_m + 1 = 2^{a+1}$, but $p_m + 1 = 2^{a_m} p_{m-1}$, where p_{m-1} is an odd prime, so a contradiction.

If $p_0 = p_m$. From $p_i + 1 = 2^{a_i} p_{i-1}$, we know $p_0 < p_1 < \cdots < p_m$. So $p_m = p_0 < p_m$, which is again a contradiction.

Therefore, $N = 2^a p_1 \cdots p_m$, with the above conditions is not a 3-perfect number. □

Theorem 5.10 *If $N = 2^e p_0 p_1 \cdots p_m$ is 3-perfect, where the p_i's are odd, distinct primes, then any $N' = 2^f p_0 p_1 \cdots p_m p_{m+1}$, and $N'' = 2^g p_0 p_1 \cdots p_m p_{m+1} \cdots p_l$, where $e \geq 1$; $f \geq 1$; $g \geq 1$; $m \geq 0$; $p_i + 1 = 2^{a_i} p_{i-1}$, $a_i \geq 1$, $i = 1, 2, \cdots$, and p_0 is a Mersenne prime, then N' and N'' are not 3-perfect numbers.*

Proof. Since $N = 2^e p_0 p_1 \cdots p_m$ is a 3-perfect number, then

$$\begin{aligned}
\sigma(N) &= \sigma(2^e)\sigma(p_0)\sigma(p_1)\cdots\sigma(p_m) \\
&= (2^{e+1}-1)(p_0+1)(p_1+1)\cdots(p_m+1) \\
&= (2^{e+1}-1)2^{a_0}2^{a_1}p_0\cdots 2^{a_m}p_{m-1} \\
&= 3N \\
&= 3\cdot 2^e p_0 p_1 \cdots p_m
\end{aligned}$$

implies

$$e = a_0 + a_1 + \cdots + a_m,$$

and

$$2^{e+1} - 1 = 3p_m.$$

Suppose $N' = 2^f p_0 p_1 \cdots p_m p_{m+1}$ is a 3-perfect number, then

$$\begin{aligned}
\sigma(N') &= \sigma(2^f)\sigma(p_0)\sigma(p_1)\cdots\sigma(p_m)\sigma(p_{m+1}) \\
&= (2^{f+1}-1)(p_0+1)(p_1+1)\cdots(p_m+1)(p_{m+1}+1) \\
&= (2^{f+1}-1)2^{(a_0+a_1+\cdots+a_m+a_{m+1})}p_0 p_1 \cdots p_{m-1}p_m \\
&= 3N' \\
&= 3\cdot 2^f p_0 p_1 \cdots p_m p_{m+1}
\end{aligned}$$

implies

$$\begin{aligned}
f &= a_0 + a_1 + \cdots + a_m + a_{m+1} \\
&= e + a_{m+1},
\end{aligned}$$

and

$$\begin{aligned} 2^{f+1}-1 &= 2^{(e+1+a_{m+1})}-1 \\ &= 3p_{m+1} \\ &= 3(2^{a_{m+1}}p_m-1) \\ &= 3p_m 2^{a_{m+1}}-3 \\ &= (2^{e+1}-1)2^{a_{m+1}}-3 \\ &= 2^{(e+1+a_{m+1})}-2^{a_{m+1}}-3 \end{aligned}$$

implies

$$2^{a_{m+1}}+2=0,$$

so a contradiction. Therefore, N' is not a 3-perfect number.

Suppose $N''=2^g p_0 p_1 \cdots p_m p_{m+1} \cdots p_l$ is a 3-perfect number, then

$$\begin{aligned} \sigma(N'') &= (2^{g+1}-1)2^{(a_0+a_1+\cdots+a_l)}p_0 p_1 \cdots p_{l-1} \\ &= 3N'' \\ &= 3 \cdot 2^g p_0 p_1 \cdots p_m p_{m+1} \cdots p_{l-1} p_l \end{aligned}$$

implies

$$\begin{aligned} g &= (a_0+a_1+\cdots+a_m)+a_{m+1}+\cdots+a_l \\ &= e+a_{m+1}+\cdots+a_l, \end{aligned}$$

and

$$2^{g+1}-1=3p_l.$$

So

$$\begin{aligned} 2^{g+1}-1 &= 3(2^{a_l}p_{l-1}-1) \\ &= 3\cdot 2^{a_l}p_{l-1}-3 \\ &= 3\cdot 2^{a_l}(2^{a_{l-1}}p_{l-2}-1)-3 \\ &= 3\cdot 2^{(a_l+a_{l-1}+\cdots+a_{m+1})}p_m-3y-3 \text{ (for some } y\in\mathbb{Z}^+) \\ &= (2^{e+1}-1)2^{(a_l+a_{l-1}+\cdots+a_{m+1})}-3y-3 \\ &= 2^{g+1}-2^{(a_l+a_{l-1}+\cdots+a_{m+1})}-3y-3, \end{aligned}$$

implies

$$3y+2+2^{(a_l+a_{l-1}+\cdots+a_{m+1})}=0,$$

so a contradiction. Therefore, N'' is not 3-perfect. □

Theorem 5.11 *Let $N=2^a p_1\cdots p_m M_{q_1}\cdots M_{q_l}$, where $p_i+1=2^{a_i}p_{i-1}$, $a_i\geq 1$, $i=1,2,\cdots,m$; M_{q_j}'s are Mersenne primes; p_0 is one of M_{q_j}'s.*
For given a, then there are only finitely many 3-perfect numbers of the above shape N.

Proof. Let $N=2^a p_1\cdots p_m M_{q_1}\cdots M_{q_l}$ be a 3-perfect number. We have

$$\begin{aligned} \sigma(N) &= \sigma(2^a)\sigma(p_1)\cdots\sigma(p_m)\sigma(M_{q_1})\cdots\sigma(M_{q_l}) \\ &= (2^{a+1}-1)(p_1+1)\cdots(p_m+1)(M_{q_1}+1)\cdots(M_{q_l}+1) \\ &= (2^{a+1}-1)2^{(\sum_{i=1}^m a_i+\sum_{j=1}^l q_j)}p_0\cdots p_{m-1} \\ &= 3N \\ &= 3\cdot 2^a p_1\cdots p_m\prod_{j=1}^l M_{q_j}. \end{aligned}$$

So

$$a+1=\sum_{i=1}^m a_i+\sum_{j=1}^l q_j+1$$

and

$$(2^{a+1}-1)p_0=3p_m\prod_{j=1}^l M_{q_j}.$$

Say $p_0 = M_{q_1}$, then

$$2^{a+1} - 1 = 3p_m \prod_{j=2}^{l} M_{q_j}.$$

Since $1 \leq a < +\infty$, so $2^{a+1} - 1 < +\infty$, and thus the numbers of p_m and M_{q_j}'s are limited, $(j = 1, 2, \cdots, l)$. So there are finitely many 3-perfect numbers N of the shape given above. □

Theorem 5.12 *Let $N = 2^a p_1 p_2 \cdots p_m$ be a 3-perfect number, with a even, and $p_1 < p_2 < \cdots < p_m$ odd primes. If $3 \mid N$, then $9 \nmid p_m + 1$.*

Proof. Let N be 3-perfect number, and $a = 2b$, then $\sigma(N) = 3N$. Because $3 \mid N$, we have $p_1 = 3$. Then

$$\begin{aligned} \sigma(N) &= (2^{2b+1} - 1)(3 + 1)(p_2 + 1) \cdots (p_m + 1) \\ &= 2^{2b} 3^2 p_2 \cdots p_m, \end{aligned}$$

so

$$(2^{2b+1} - 1)4(p_2 + 1) \cdots (p_m + 1) = 2^{2b} 3^2 p_2 \cdots p_m.$$

Suppose $9 \mid p_m + 1$, then $3 \nmid p_j + 1$, $(2 \leq j \leq m - 1)$, so $p_m \equiv 2 \pmod 3$, and for $2 \leq j \leq m - 1$, $p_j \equiv 1 \pmod 3$.

Since $p_m \nmid p_i + 1$, $1 \leq i \leq m$, so $p_m \mid 2^{2b+1} - 1$. So

$$2^{2b+1} - 1 = p_m \prod_{j \in S} p_j,$$

for some subset S, where $S \subset \{2, \cdots, m - 1\}$.

Since $2^{2b+1} - 1 \equiv 1 \pmod 3$, and $p_m \prod_j p_j \equiv 2 \pmod 3$, we obtain a contradiction. Therefore $9 \nmid p_m + 1$. □

Proposition 5.13 *Let $N = 2^a p_1 p_2 \cdots p_m$ be a 3-perfect number, with a even, and $p_1 < p_2 < \cdots < p_m$ odd primes. Then not all prime factors of $\sigma(2^a)$ are Mersenne primes.*

Proof. Let N be 3-perfect number and $a = 2b$. Then $\sigma(N) = 3N$ so

$$\begin{aligned} \sigma(N) &= (2^{2b+1} - 1)(p_1 + 1)(p_2 + 1) \cdots (p_m + 1) \\ &= 3 \cdot 2^{2b} p_1 p_2 \cdots p_m. \end{aligned}$$

Suppose all prime factors of $2^{2b+1} - 1$ are Mersenne primes.

Since $p_1 < p_2 < \cdots < p_m$, so $p_m \nmid p_i + 1$, for all $1 \leq i \leq m$, so $p_m \mid 2^{2b+1} - 1$, so p_m is a Mersenne prime, then $p_m + 1 = 2^{a_m}$. Similarly, $p_{m-1} \nmid p_i + 1$, for $1 \leq i \leq m - 1$, and $p_{m-1} \mid 2^{2b+1} - 1$, so p_{m-1} is a Mersenne prime. By induction, $p_1, \cdots, p_{m-2}$ are prime factors of $2^{2b+1} - 1$, then they are Mersenne primes, so 3 does not divide the left hand side of the above equation. But 3 divides the right hand side of the equation, which is a contradiction.

Therefore, not all prime factors of $2^{2b+1} - 1$ are Mersenne primes. □

Similarly we can show the following property:

Proposition 5.14 *Let $N = 2^a p_1 p_2 \cdots p_m$ be a 3-perfect number, with a even, and $p_1 < p_2 < \cdots < p_m$ odd primes. Then the maximum non-Mersenne prime factor of N divides $\sigma(2^a)$.*

Lemma 5.15 *If N is a flat 3-perfect number with odd exponent and $3 \nmid N$ then every odd prime divisor of N is congruent to 1 modulo 3.*

Proof. Let N be flat and 3-perfect with $N = 2^a \cdot p_1 \cdots p_m$, where the exponent a is odd, and suppose that $3 \nmid N$. Then

$$3 \cdot 2^a \cdot p_1 \cdots p_m = \sigma(2^a) \cdot (p_1 + 1) \cdots (p_m + 1). \tag{5.1}$$

Since a is odd and $2^2 \equiv 1 \pmod 3$, so $\sigma(2^a) = 2^{a+1} - 1 \equiv 0 \pmod 3$. Therefore

$$2^a \cdot p_1 \cdots p_m = \frac{\sigma(2^a)}{3}(p_1 + 1) \cdots (p_m + 1). \tag{5.2}$$

Since $3 \nmid N$, so all of the prime factors p_i's are not 3, $i = 1, 2, \cdots, m$.

If there exists a prime factor p_i of N with $p_i \equiv 2 \pmod 3$, for some $i \in \{1, 2, \cdots, m\}$, then in the right hand side of equation (5.2), $(p_i + 1) \equiv 0 \pmod 3$, giving $3 \mid N$, a contradiction. □

Theorem 5.16 *Let N be flat and 3-perfect with exponent a and length m and with $3 \nmid N$. If $a \not\equiv 1 \pmod{12}$ then a is even. If $a \equiv 1 \pmod{12}$ then m is odd and every odd prime divisor of N is congruent to 1 modulo 3.*

Proof. See Chapter 6 (Section 6.3, after Theorem 6.11). □

5.5 Examples

Example 5.1 *If $N = 2^a \cdot 23$, then N is not a 3-perfect number.*

Proof. Suppose N is 3-perfect, then $\sigma(N) = 3N$, so

$$\begin{aligned} \sigma(N) &= (2^{a+1} - 1) \cdot 2^3 \cdot 3 \\ &= 3N \\ &= 3 \cdot 2^a \cdot 23. \end{aligned}$$

So

$$2^{a+1} - 1 = 23 \cdot 2^{a-3}. \tag{5.3}$$

Since the left hand side of equation (5.3) is odd, so $a = 3$, so $23 = 2^4 - 1$ is a contradiction. Therefore, $N = 2^a \cdot 23$ is not 3-perfect. □

Example 5.2 *If $N = 2^a \cdot 3 \cdot 23$, then N is not a 3-perfect number.*

Proof. Suppose N is 3-perfect, then $\sigma(N) = 3N$, so

$$\begin{aligned} \sigma(N) &= (2^{a+1} - 1) \cdot 2^2 \cdot 2^3 \cdot 3 \\ &= 3N \\ &= 3 \cdot 2^a \cdot 3 \cdot 23. \end{aligned}$$

So

$$2^{a+1} - 1 = 2^{a-5} \cdot 3 \cdot 23. \tag{5.4}$$

Since the left hand side of equation (5.4) is odd, so $a = 5$, so $2^6 - 1 = 3 \cdot 23$ is a contradiction. Therefore, $N = 2^a \cdot 3 \cdot 23$ is not a 3-perfect number. □

Example 5.3 *If $N = 2^a \cdot 3 \cdot 23 \cdot 7 \cdot 31$, then N is not a 3-perfect number, for any $a \geq 1$.*

Proof. Suppose N is a 3-perfect number, then $\sigma(N) = 3N$, so

$$\begin{aligned} \sigma(N) &= (2^{a+1} - 1) \cdot 2^2 \cdot 2^3 \cdot 3 \cdot 2^3 \cdot 2^5 \\ &= 3N \\ &= 3 \cdot 2^a \cdot 3 \cdot 23 \cdot 7 \cdot 31. \end{aligned}$$

So

$$2^{a+1} - 1 = 2^{a-13} \cdot 3 \cdot 23 \cdot 7 \cdot 31 \tag{5.5}$$

implies $a = 13$ and $2^{14} - 1 = 3 \cdot 43 \cdot 127 \neq 3 \cdot 23 \cdot 7 \cdot 31$, so a contradiction. Therefore, $N = 2^a \cdot 3 \cdot 23 \cdot 7 \cdot 31$ is not a 3-perfect number. □

Example 5.4 *If $N = 2^a \cdot 13 \cdot 7$, then N is not a 3-perfect number.*

Proof. Suppose N is 3-perfect, then $\sigma(N) = 3N$, so

$$\begin{aligned} \sigma(N) &= (2^{a+1} - 1) \cdot 2 \cdot 7 \cdot 2^3 \\ &= 3N \\ &= 3 \cdot 2^a \cdot 13 \cdot 7. \end{aligned}$$

So

$$2^{a+1} - 1 = 2^{a-4} \cdot 3 \cdot 13 \tag{5.6}$$

implies $a = 4$ and $2^5 - 1 = 31 \neq 3 \cdot 13$, so a contradiction. Therefore, $N = 2^a \cdot 13 \cdot 7$ is not a 3-perfect number. □

Example 5.5 *If $N = 2^a \cdot 13 \cdot 7 \cdot M_p$, where M_p is a Mersenne prime, then N is not a 3-perfect number.*

Proof. Suppose N is a 3-perfect number, then $\sigma(N) = 3N$, so

$$\begin{aligned} \sigma(N) &= (2^{a+1} - 1) \cdot 2 \cdot 7 \cdot 2^3 \cdot 2^p \\ &= 3N \\ &= 3 \cdot 2^a \cdot 13 \cdot 7 \cdot M_p. \end{aligned}$$

So

$$2^{a+1} - 1 = 2^{a-4-p} \cdot 3 \cdot 13 \cdot M_p \tag{5.7}$$

implies $a = 4 + p$, and

$$\begin{aligned} 2^{5+p} - 1 &= 3 \cdot 13 \cdot M_p \\ &= 3 \cdot 13 \cdot (2^p - 1). \end{aligned}$$

So

$$2^p = \frac{38}{7}$$

a contradiction. Therefore, $N = 2^a \cdot 13 \cdot 7 \cdot M_p$ is not a 3-perfect number. □

Example 5.6 *If $N = 2^a \cdot 79 \cdot 5 \cdot M_p \cdot M_q$, where M_p, M_q are distinct Mersenne primes, then N is not a 3-perfect number.*

Proof. Suppose N is a 3-perfect number, then $\sigma(N) = 3N$, so

$$\begin{aligned} \sigma(N) &= (2^{a+1} - 1)2^4 \cdot 5 \cdot 2 \cdot 3 \cdot 2^p \cdot 2^q \\ &= 3N \\ &= 3 \cdot 2^a \cdot 79 \cdot 5 \cdot M_p \cdot M_q. \end{aligned}$$

So

$$2^{a+1} - 1 = 2^{a-5-p-q} \cdot 79 \cdot M_p \cdot M_q \tag{5.8}$$

implies $a = 5 + p + q$ and

$$\begin{aligned} 2^{a+1} - 1 &= 2^{6+p+q} - 1 \\ &= 79 \cdot M_p \cdot M_q \\ &= 79 \cdot (2^p - 1)(2^q - 1). \end{aligned}$$

Since p, q are odd, so $3 \mid (2^{6+p+q} - 1)$ implies $3 \mid 79 \cdot (2^p - 1)(2^q - 1)$, so a contradiction. Therefore, $N = 2^a \cdot 79 \cdot 5 \cdot M_p \cdot M_q$ is not a 3-perfect number. □

Example 5.7 *If $N = 2^a \cdot 79 \cdot 5 \cdot M_{q_1} \cdots M_{q_l}$, where l is odd, and M_{q_i} are Mersenne primes, $i = 1, \cdots, l$. Then, N is not a 3-perfect number.*

Proof. Suppose N is a 3-perfect number, then we can get

$$2^{6+q_1+\cdots+q_l} - 1 = 79 \cdot \prod_{i=1}^{l} (2^{q_i} - 1)$$

Since

$$2^r - 1 \mid 2^s - 1 \Leftrightarrow r \mid s$$

so

$$q_j \mid 6 + \sum_{i=1}^{l} q_i$$

where $j = 1, \cdots, l$. So

$$\prod_{j=1}^{l} q_j \cdot x = 6 + \sum_{i=1}^{l} q_i$$

where x is a positive integer. So

$$q_1 \cdots q_l \leq 6 + q_1 + \cdots + q_l.$$

By Lemma 5.7, this is a contradiction. Therefore, $N = 2^a \cdot 79 \cdot 5 \cdot M_{q_1} \cdots M_{q_l}$ is not a 3-perfect number. □

Example 5.8 *If $N = 2^a \cdot 5 \cdot 19 \cdot 37 \cdot 73 \cdot 9343$, then N is not a 3-perfect number.*

Proof. Suppose N is a 3-perfect number, then

$$\begin{aligned}\sigma(N) &= (2^{a+1}-1)\cdot 2\cdot 3\cdot 2^2\cdot 5\cdot 2\cdot 19\cdot 2\cdot 37\cdot 2^7\cdot 73\\ &= 3N\\ &= 3\cdot 2^a\cdot 5\cdot 19\cdot 37\cdot 73\cdot 9343,\end{aligned}$$

implies

$$(2^{a+1}-1)\cdot 2^{12} = 2^a\cdot 9343,$$

implies $a = 12$, and $2^{a+1}-1 = 2^{13}-1 = 8191 \neq 9343$, so a contradiction. Therefore, N is not a 3-perfect number. □

Example 5.9 *If $N = 2^a\cdot 5\cdot 19\cdot 37\cdot 73\cdot 9343\cdot p_1\cdots p_m$, where $p_i+1 = 2^{a_i}\cdot q_i$, $(i = 1,2,\cdots,m)$, p_i's are distinct, with p_i's, q_i's odd primes, and $p_i \neq 5, 19, 37, 73, 9343$, then N is not a 3-perfect number, for any $a \geq 1$.*

Proof. Let $p_1 < p_2 < \cdots < p_m$. Suppose N is a 3-perfect number, then

$$\begin{aligned}\sigma(N) &= (2^{a+1}-1)\cdot 2\cdot 3\cdot 2^2\cdot 5\cdot 2\cdot 19\cdot 2\cdot 37\cdot 2^7\cdot 73\cdot 2^{a_1}\cdot q_1\cdots 2^{a_m}\cdot q_m\\ &= (2^{a+1}-1)\cdot 2^{12+\sum_{i=1}^m a_i}\cdot 3\cdot 5\cdot 19\cdot 37\cdot 73\cdot\prod_{i=1}^m q_i\\ &= 3N\\ &= 3\cdot 2^a\cdot 5\cdot 19\cdot 37\cdot 73\cdot 9343\cdot\prod_{i=1}^m p_i,\end{aligned}$$

implies

$$(2^{a+1}-1)\cdot 2^{12+\sum_{i=1}^m a_i}\cdot\prod_{i=1}^m q_i = 2^a\cdot 9343\cdot\prod_{i=1}^m p_i$$

implies

$$a = 12+\sum_{i=1}^m a_i$$

and

$$\omega((2^{a+1}-1)\prod_{i=1}^m q_i = \omega(9343\cdot\prod_{i=1}^m p_i) = m+1,$$

so

$$\omega(2^{a+1}-1) = 1.$$

Since

$$p_m \nmid \prod_{i=1}^{m} q_i,$$

so

$$p_m = 2^{a+1} - 1,$$

but

$$p_m = 2^{a_m} q_m - 1,$$

implies $q_m = 1$, so a contradiction. Therefore, N is not a 3-perfect number. □

Example 5.10 *Let* $N = 2^a \cdot (3 \cdot 5 \cdot 19 \cdot 37 \cdot 73 \cdot 9343) \cdot (7 \cdot 13 \cdot 103 \cdot 823) \cdot (31 \cdot 61 \cdot 487) \cdot (127 \cdot 4261412863) \cdot (8191 \cdot 16381) \cdot (131071 \cdot 4194271) \cdot (524287 \cdot 1048573)$, *where* $a \geq 1$, *then* N *is not a* 3*-perfect number.*

Proof. Suppose N is a 3-perfect number, then $\sigma(N) = 3N$.

$$\begin{aligned}
\sigma(N) &= (2^{a+1} - 1)(2^{14} \cdot 3 \cdot 5 \cdot 19 \cdot 37 \cdot 73)(2^{10} \cdot 7 \cdot 13 \cdot 103)(2^9 \cdot 31 \cdot 61) \\
&\quad (2^{32} \cdot 127)(2^{14} \cdot 8191)(2^{22} \cdot 131071)(2^{20} \cdot 524287) \\
&= 3 \cdot 2^a \cdot (3 \cdot 5 \cdot 19 \cdot 37 \cdot 73 \cdot 9343)(7 \cdot 13 \cdot 103 \cdot 823)(31 \cdot 61 \cdot 487) \\
&\quad (127 \cdot 4261412863)(8191 \cdot 16381)(131071 \cdot 4194271)(524287 \cdot 1048573),
\end{aligned}$$

then

$$\begin{aligned}
(2^{a+1} - 1) \cdot 2^{121} &= 3 \cdot 2^a \cdot 9343 \cdot 823 \cdot 487 \cdot 4261412863 \cdot \\
&\quad 16381 \cdot 4194271 \cdot 1048573
\end{aligned}$$

implies $a = 121$, then

$$\begin{aligned}
2^{122} - 1 &= 3 \cdot 9343 \cdot 823 \cdot 487 \cdot 4261412863 \cdot \\
&\quad 16381 \cdot 4194271 \cdot 1048573,
\end{aligned}$$

so a contradiction. So N is not a 3-perfect number. □

Chapter 6

Even perfect numbers of abundancy 4

6.1 Introduction

In this chapter the known set of 4-perfect numbers, in particular those having a flat shape, will be discussed [13]. We begin with observations based on the structure of the known thirty-six 4-perfect numbers. Eight phenomena are presented in Section 6.2, but we are not able to prove all of these properties. The structure of 4-perfect numbers with a flat shape $N = 2^a p_1 \cdots p_m$ is considered. Only the 7th 4-perfect number $d_7 = 2^8 \cdot 3 \cdot 5 \cdot 7 \cdot 19 \cdot 37 \cdot 73$ and the 10th 4-perfect number $d_{10} = 2^{14} \cdot 3 \cdot 5 \cdot 7 \cdot 19 \cdot 31 \cdot 151$ have a flat shape. We can show that the exponent of 2 in a 4-perfect number is not congruent to 3 (mod 4), not congruent to 5 (mod 6), not congruent to 9 (mod 10), (Lemma 6.2), not congruent to 9 (mod 12) (Lemma 6.6). Therefore, we can show that if the exponent of 2 is not congruent to 1 (mod 12), then it is even (Theorem 6.9). The length m of a flat 4-perfect number is also discussed (in Lemmas 6.1, 6.7 and 6.8, Theorem 6.9). Furthermore, we provide a more general result for 4-perfect numbers with a shape $N = 2^a p_1^{a_1} \cdots p_m^{a_m}$. Under some conditions, N

is divisible by 3 (Theorem 6.4). The structure of $\sigma(2^a)$ is described (in Theorems 6.3, 6.10 and 6.11). Theorem 6.12 shows that if all odd prime divisors of N are Mersenne primes, where N is even, flat and multiperfect, then N is perfect.

6.2 Observations

At the end of this chapter the list of all of the 4-perfect numbers is given, d_1 through d_{36}, which have been discovered up until the date of writing this thesis [90]. Below are some observations based on this list. These are easy to make, but most appear to be quite difficult in general to resolve. Here N is a generic 4-perfect number.

Observations:

(1) $4 \mid N$.

(2) $3 \mid N$.

(3) The power of the largest prime divisor of N is 1.

(4) There are only two numbers N with shape $2^a \cdot p_1 \cdots p_m$.

(5) The largest prime always occurs in the factorization of $\sigma(2^e)$.

(6) Primes with odd discrete powers to the base 2 always appear in the factorization of $\sigma(2^e)$ which consists exactly of those primes.

(7) The number of Mersenne primes in N is exactly the number of distinct primes in the factorization of $\sigma(2^e)$.

(8) Each odd prime which appears in N is super thin.

In Theorem 6.11 of this chapter we prove that for any number $N = 2^a b$ with b an odd positive integer, then not all of prime factors of $\sigma(2^a)$ are Mersenne primes, provided $\sigma(2^a)$ is not a prime and $a \neq 5$. In Chapter 5, we obtained a similar result in Proposition 5.13, but in that case where $N = 2^a p_1 \cdots p_m$ is a 3-perfect number with a even and the p_i's are distinct odd primes.

6.3 Lemmas and theorems

First we show that if the number N has a flat shape, then the power of 2 "determines" the number.

Lemma 6.1 *If $a \geq 1$ is fixed, then there exists at most one 4-perfect number of shape $2^a p_1 \cdots p_m$, where $p_1 < p_2 < \cdots < p_m$ are distinct odd primes, and then $3 \leq m \leq a+2$.*

Proof. Let $N_1 = 2^a \cdot b_1$ and $N_2 = 2^a \cdot b_2$, where the b_i are odd and squarefree. Then

$$\sigma(N_1) = \sigma(2^a)\sigma(b_1) = 2^{a+2} b_1$$

implies $\sigma(2^a) \mid b_1$. Similarly $\sigma(2^a) \mid b_2$ and $b_1/b_2 = \sigma(b_1)/\sigma(b_2)$, therefore $b_1\sigma(b_2) = b_2\sigma(b_1)$.

If $b_1 \neq b_2$, let $b_1 = c \cdot p_1 \cdots p_m$, $b_2 = c \cdot q_1 \cdots q_l$ where none of the primes p_i equal any of the primes q_j. Then

$$p_1 \cdots p_m (q_1 + 1) \cdots (q_l + 1) = q_1 \cdots q_l (p_1 + 1) \cdots (p_m + 1).$$

If p is the maximum prime in the set $\{p_i, q_j\}$, then p divides one side of this equation but not the other. Hence $b_1 = b_2$.

Derivation of the bounds on m:

Let $N = 2^a p_1 \cdots p_m$ be a 4-perfect number, and $\sigma(N) = 4N$.

Suppose $m = 1$, then

$$\begin{aligned} \sigma(N) &= (2^{a+1} - 1)(p_1 + 1) \\ &= 2^{a+2} p_1. \end{aligned}$$

Since $(p_1 + 1) \nmid p_1$, p_1 is a Mersenne prime, so $p_1 + 1 = 2^{a+2}$, and $p_1 = 2^{a+1} - 1$, so $a + 2 = a + 1$, which is a contradiction. So, $m \neq 1$.

Suppose $m = 2$, then

$$\begin{aligned}\sigma(N) &= (2^{a+1}-1)(p_1+1)(p_2+1)\\ &= 2^{a+2}p_1p_2.\end{aligned}$$

Assume $p_1 + 1 = 2^{a_1}q_1 \cdots q_l$, where the q_j's are primes, $1 \leq j \leq l$.

So $q_j < p_1 < p_2$, so $q_j = 2$, so p_1 is a Mersenne prime. Let $p_1 + 1 = 2^{a_1}$.

Since $(p_2 + 1) \nmid p_1$ and $(p_2 + 1) \nmid p_2$, so $(p_2 + 1) \mid 2^{a+2}$, so p_2 is also a Mersenne prime. Let $p_2 + 1 = 2^{a_2}$. So

$$(2^{a+1}-1)2^{(a_1+a_2)} = 2^{a+2}(2^{a_1}-1)(2^{a_2}-1)$$

So, $a_1 + a_2 = a + 2$, $a_1 < a_2$, $a_1 \mid a + 1$, and $a_2 \mid a + 1$ imply

$$a_1a_2 \leq a + 1 < a + 2 = a_1 + a_2,$$

So, $a_1 = 1$, and thus $p_1 = 1$, which is a contradiction. So $m \geq 3$.

Since $N = 2^a p_1 \cdots p_m$ is a 4-perfect number, so

$$\sigma(2^a p_1 \cdots p_m) = 4 \cdot 2^a p_1 \cdots p_m,$$

so

$$(2^{a+1}-1)\left(\frac{p_1+1}{2}\right)\cdots\left(\frac{p_m+1}{2}\right) = 2^{a+2-m}p_1 \cdots p_m,$$

so

$$a + 2 - m \geq 0$$

Therefore, $3 \leq m \leq a + 2$. □

Now we show that there are restrictions on a for $N = 2^a p_1 \cdots p_m$ to be 4-perfect.

Lemma 6.2 *Let the 4-perfect number N have shape $2^a p_1 \cdots p_m$ with $p_1 < p_2 < \cdots < p_m$ distinct odd primes, then $a \not\equiv 3 \pmod 4$; $a \not\equiv 5 \pmod 6$; and $a \not\equiv 9 \pmod{10}$.*

Proof. (1) Let $a \equiv 3 \pmod 4$ and $N = 2^a p_1 \cdots p_m$. Since $4 \mid a+1$ we have $15 = \sigma(2^3) \mid \sigma(2^a)$ so we can write

$$\begin{aligned}
(2^{a+1}-1)(p_1+1)\cdots(p_m+1) &= 2^{a+2}p_1\cdots p_m, \\
15 \cdot \frac{2^{a+1}-1}{15}(p_1+1)\cdots(p_m+1) &= 2^{a+2}3\cdot 5\cdot p_3\cdots p_m, \\
15 \cdot \frac{2^{a+1}-1}{15}\cdot 2^2\cdot 2\cdot 3\cdot(p_3+1)\cdots(p_m+1) &= 2^{a+2}3\cdot 5\cdot p_3\cdots p_m,
\end{aligned}$$

and therefore 3^2 divides the right hand side, a contradiction, showing that $a \not\equiv 3 \pmod 4$.

(2) Let $a \equiv 5 \pmod 6$ and $N = 2^a p_1 \cdots p_m$ be a 4-perfect number. So

$$(2^{a+1}-1)(p_1+1)\cdots(p_m+1) = 2^{a+2}p_1\cdots p_m.$$

Since $a+1 \equiv 0 \pmod 6$, so $5+1 \mid a+1$, so

$$63 = 3^2\cdot 7 = \sigma(2^5) \mid \sigma(2^a) = 2^{a+1}-1.$$

So, $3^2 \mid p_1\cdots p_m$, which is a contradiction. Therefore, $a \not\equiv 5 \pmod 6$.

(3) Let $a \equiv 9 \pmod{10}$ and $N = 2^a p_1 \cdots p_m$ be a 4-perfect number. So $10 \mid a+1$, then $\sigma(2^9) \mid \sigma(2^a)$, so $3\cdot 11 \mid \sigma(2^a)$, and thus $p_1 = 3$, $p_j = 11$. So

$$\sigma(2^a)(p_1+1)\cdots(p_j+1)\cdots(p_m+1) = 2^{a+2}3\cdots 11\cdots p_m$$

implies

$$\frac{\sigma(2^a)}{3\cdot 11}\cdot 2^2\cdots 2^2\cdot 3\cdots(p_m+1) = 2^{a+2}\cdot p_2\cdots p_{j-1}\cdot p_{j+1}\cdots p_m$$

so $3 \mid p_2p_3\cdots p_{j-1}p_{j+1}\cdots p_m$, which is a contradiction. Therefore, $a \not\equiv 9 \pmod{10}$. □

Now we show that all primes appearing in a flat 4-perfect number are in fact super flat.

Theorem 6.3 *Let $a \geq 1$. If there exist odd primes $p_1, \cdots, p_m$ such that $N = 2^a p_1 \cdots p_m$ is 4-perfect then $\sigma(2^a)$ is squarefree and each of its prime factors is a super flat prime, such that $(q_i + 1, q_j + 1) = 2^b$, where q_i and q_j are distinct prime factors of $\sigma(2^a)$, $b \geq 1$.*

Proof. Let $N = 2^a p_1 \cdots p_m$ be a 4-perfect number, where the p_i's are odd primes, $(i = 1, \cdots m)$. Then $\sigma(N) = 4N$, so

$$\sigma(2^a)\sigma(p_1 \cdots p_m) = 2^{a+2} p_1 \cdots p_m,$$

since $p_1 \cdots p_m$ is squarefree and $\sigma(2^a)$ is odd, we have $\sigma(2^a) \mid p_1 \cdots p_m$, so $\sigma(2^a)$ is squarefree.

Since $\sigma(p_i) = p_i + 1$, and $\sigma(p_i) \mid 2^a p_1 \cdots p_m$, so $p_i + 1 = 2^{a_i} \prod_{j \in I} p_j$, where $I \subseteq \{1, \cdots, m\}$, and $j \neq i$, so p_i is a super flat prime.

Suppose there are at least two distinct prime factors q_i and q_j of $\sigma(2^a)$, such that $(q_i + 1, q_j + 1) = 2^b c$ and $(2, c) = 1$ with $c \geq 3$.

Since $c \mid q_i + 1$ and $c \mid q_j + 1$, so $c^2 \mid (q_i + 1)(q_j + 1) \mid \sigma(N)$, so $c^2 \mid N$, which is a contradiction. Therefore, $(q_i + 1, q_j + 1) = 2^b$, $b \geq 1$. □

Although the table of examples suggests that all even multiperfect numbers of abundancy 4 are divisible by 3, we are not able to show this completely, but have the following conditions:

Theorem 6.4 *Let N be 4-perfect and even and let $N = 2^a p_1^{\alpha_1} \cdots p_m^{\alpha_m}$ be its standard prime factorization. Then in the following three cases N is divisible by 3:*

(A) If a is odd,

(B) If there exists an i with α_i odd and $p_i \equiv 2 \pmod 3$,

(C) If there exists an i with $\alpha_i \equiv 2 \pmod 3$ and $p_i \equiv 1 \pmod 3$.

If $3 \mid N$ with a even then necessarily at least one of (B) or (C) hold.

Proof. (A) If $N = 2^a \cdot p_1^{\alpha_1} \cdots p_m^{\alpha_m}$ and a is odd, then since

$$(2^{a+1} - 1)\sigma(p_1^{\alpha_1}) \cdots \sigma(p_m^{\alpha_m}) = 2^{a+2} p_1^{\alpha_1} \cdots p_m^{\alpha_m}$$

and $3 \mid 2^{a+1} - 1$, one of the p_i must be 3, so $3 \mid N$.

(B) If one of the α_i is odd and the corresponding $p_i \equiv 2 \pmod 3$, then, since $2 \mid \alpha_i + 1$, by Lemma 2.1, $1 + p_i \mid \sigma(p_i^{\alpha_i})$, so again $3 \mid N$.

(C) Let us suppose that 3 does not divide N. Let $b = p_1^{\alpha_1} \cdots p_m^{\alpha_m}$ so $3 \nmid b$. And, because of point (A) we may assume that a is even. Then the hypothesis $\sigma(N) = 4N$ gives

$$(2^{a+1} - 1)\sigma(b) = 2^{a+2} b$$

which implies

$$\sigma(b) = 2b + \frac{2b}{2^{a+1} - 1}.$$

Suppose $b \equiv 2 \pmod 3$. Then since each divisor d of b satisfies $3 \nmid d$, each sum $b/d + d \equiv 0 \pmod 3$. But from the equation above, $\sigma(b) \equiv 0 \pmod 2$, so, since each divisor of b is odd, b has an even number of divisors. Arrange them in pairs $\{b/d, d\}$ and add to show that $3 \mid \sigma(b)$ leading to $3 \mid b$, a contradiction.

This means $b \equiv 1 \pmod 3$. Then by the given hypothesis and definition of b, there is a $p_i \equiv 1 \pmod 3$ and, by (B) if any of the $p_i \equiv 2 \pmod 3$, then its corresponding α_i is even (otherwise $3 \mid b$).

Now consider the equation $\sigma(N) = 4N$:

$$(2^{a+1} - 1)\sigma(p_1^{\alpha_1}) \cdots \sigma(p_m^{\alpha_m}) = 2^{a+2} \cdot b$$

with a even, and take this equation modulo 3. This leads to

$$(1 + \alpha_1) \cdots (1 + \alpha_l) \equiv 1 \pmod 3,$$

where, if needed, we have reordered the α_i to place the non-empty set of those with $p_i \equiv 1 \pmod 3$ first. But given an $\alpha_i \equiv 2 \pmod 3$ we obtain $0 \equiv 1 \pmod 3$, a contradiction which implies therefore $3 \mid b$, so finally $3 \mid N$.

For the necessary condition assume $N = 2^{2a} \cdot p_1^{\alpha_1} \cdots p_m^{\alpha_m}$ and $3 \mid N$. Because $2^{2a+1} - 1 \equiv 1 \pmod 3$ we must have an i with $3 \mid \sigma(p_i^{\alpha_i})$. If $\exp_3 p_i = 1$ if and only if $p_i \equiv 1 \pmod 3$, we must have, by Theorem 3.3, $3 \mid \alpha_i + 1$ so $\alpha_i \equiv 2 \pmod 3$ which is (C). If however $\exp_3 p_i = 2$ then $3 \nmid p_i - 1$ and $3 \mid p_i^2 - 1$, so we must have $2 \mid \alpha_i + 1$ so α_i is odd and $p_i \equiv 2 \pmod 3$, which is (B). □

Note that, in greater generality than Lemma 6.1, we have the following Theorem 6.5:

Theorem 6.5 *If a is powerful, then there exists at most one squarefree number b such that $N = ab$ is multiperfect with $(a, b) = 1$ of any given fixed abundancy k, ($k \geq 2$).*

Proof. Assume both $N_1 = ab_1$ and $N_2 = ab_2$ are k-perfect numbers, where b_1 and b_2 are two distinct squarefree numbers.

Let $b_1 = cp_1p_2 \cdots p_m$, $b_2 = cq_1q_2 \cdots q_l$, where $c \geq 1$ is squarefree, and $p_1 < p_2 < \cdots < p_m$, $q_1 < q_2 < \cdots < q_l$, p_i's and q_j's are odd distinct primes.

Since N_1 and N_2 are k-perfect, so $\sigma(N_1) = kN_1$ and $\sigma(N_2) = kN_2$, which implies,

$$\frac{\sigma(acp_1 \cdots p_m)}{acp_1 \cdots p_m} = \frac{\sigma(acq_1 \cdots q_l)}{acq_1 \cdots q_l}.$$

Since $(a, b_1) = 1$, $(a, b_2) = 1$, $(c, \prod_{i=1}^{m} p_i) = 1$, $(c, \prod_{j=1}^{l} q_j) = 1$, so we have

$$\frac{\sigma(p_1) \cdots \sigma(p_m)}{p_1 \cdots p_m} = \frac{\sigma(q_1) \cdots \sigma(q_l)}{q_1 \cdots q_l},$$

so,

$$(p_1 + 1) \cdots (p_m + 1)q_1 \cdots q_l = (q_1 + 1) \cdots (q_l + 1)p_1 \cdots p_m. \tag{6.1}$$

Since $p_1 < \cdots < p_m$, then $p_m \nmid p_i + 1$, for all i, $(i = 1, 2, \cdots, m)$.

Since $p_i \neq q_j$, for all i and j, then $p_m \nmid q_j$, for all j, $(j = 1, 2, \cdots, l)$.

So the left hand side of equation (6.1) does not divide by p_m, which is a contradiction. Therefore, $b_1 = b_2$.

This completes the proof of this Theorem. □

Note: Using the same method as in Lemma 6.2 above we can show that all indices a with $a \equiv j \pmod{j+1}$ for $15 \leq j \leq 50$ do not give rise to a 4-perfect number of the flat shape.

A superficially more straight forward approach to this problem involves the simple relationship between perfect numbers of abundancy 3 and 4: If $3 \nmid m$ then m is 3-perfect if and only if $3m$ is 4-perfect. However, to succeed with this approach, we need to assume the table of the six known 3-perfect numbers is complete, and we are not able to do this.

We now develop another restriction on the exponent a.

Lemma 6.6 *Let N be flat and 4-perfect with exponent a and length m. Then $a \not\equiv 9 \pmod{12}$.*

Proof. Let $\sigma(N) = 4N$ and $a = 12b + 9$ with $b \geq 0$, then

$$\begin{aligned}
\sigma(N) &= \sigma(2^a)\sigma(p_1)\cdots\sigma(p_m) \\
&= (2^{a+1}-1)(p_1+1)\cdots(p_m+1) \\
&= (2^{6b+5}+1)(2^{6b+5}-1)(p_1+1)\cdots(p_m+1) \\
&= 2^{12b+11}p_1p_2\cdots p_m.
\end{aligned}$$

If for any i, $(2 \leq i \leq m)$, $p_i \equiv 2 \pmod{3}$, then $3 \mid p_i + 1$, implies N has too many 3's. So we can say $p_i \equiv 1 \pmod{3}$, for all i, $(2 \leq i \leq m)$.

Since

$$2^{6b+5} + 1 = 3(21x + 11) = 3(3y + 2),$$

then $p_1 = 3$, and $3y + 2$ is the product of some prime factors of N. So $3y + 2 = \prod_{i \in I} p_i$, where $I = \{2, 3, \cdots, m\}$. Since $3y + 2 \equiv 2 \pmod{3}$, but $\prod_{i \in I} p_i \equiv 1 \pmod{3}$, so a contradiction.

Therefore, N is not a 4-perfect number, if $a \equiv 9 \pmod{12}$. □

Lemma 6.7 *Let N be flat, 4-perfect with exponent a, $N = 2^a p_1 \cdots p_m$. If $a \equiv 1 \pmod{12}$ then $3 \mid N$, for $2 \le i \le m$, $p_i \equiv 1 \pmod 3$, and m is even.*

Proof. Suppose $a = 12b + 1$, $b \ge 0$. We can assume $b \ge 1$ by Theorem 5.2. Because $\sigma(N) = 4N$ we have

$$\begin{aligned}
\sigma(N) &= (2^{12b+2} - 1)(p_1 + 1)(p_2 + 1) \cdots (p_m + 1) \\
&= 2^{12b+3} p_1 p_2 \cdots p_m, \text{ and} \\
\sigma(2^a) &= 2^{12b+2} - 1 \\
&= 3(21x + 1) \\
&= 3(3y + 1),
\end{aligned}$$

where $x = 2^{12b-4} + 2^{12b-10} + \cdots + 2^2$ and $y = 7x$.

So, $p_1 = 3$, and

$$\frac{(2^{12b+2} - 1)}{3}(p_2 + 1) \cdots (p_m + 1) = 2^{12b+1} p_2 \cdots p_m \tag{6.2}$$

If any $p_i \equiv 2 \pmod 3$, with $2 \le i \le m$, then $p_i + 1 \equiv 0 \pmod 3$, implies there would be too many 3's, so for all i with $2 \le i \le m$, we must have $p_i \equiv 1 \pmod 3$. Now taking the equation (6.2) modulo 3, we get

$$\prod_{i=2}^{m} (p_i + 1) \equiv 2^{m-1} \equiv 2^a \equiv 2 \pmod 3$$

and therefore $2^m \equiv 1 \pmod 3$ so m must be even. □

Lemma 6.8 *Let N be flat and 4-perfect with even exponent and suppose also $3 \nmid N$. Then the length of N is even.*

Proof. Let $N = 2^a p_1 \cdots p_m$ and $a = 2b$. Since $3 \nmid N$, for $1 \le i \le m$ each $p_i \equiv 1 \pmod 3$, and if $2^{\beta_i} \| p_i + 1$, since $(p_i + 1)2^{-\beta_i}$ is a product of primes congruent to 1 modulo 3, it must also be congruent to 1 modulo 3. Thus each β_i is odd. Since $\beta_1 + \cdots + \beta_m = 2b + 2$, m must be even. □

Theorem 6.9 *Let N be flat and 4-perfect with exponent a and length m. If $a \not\equiv 1 \pmod{12}$, then a is even. If a is even and $3 \nmid N$ then m is also even. If $a \equiv 1 \pmod{12}$ then $3 \mid N$ and m is even.*

Proof. Suppose $N = 2^a p_1 p_2 \cdots p_m$ is a 4-perfect number. By Lemma 6.2, we know $a \not\equiv 5 \pmod 6$, which implies $a \not\equiv 5 \pmod{12}$ and $a \not\equiv 11 \pmod{12}$. By Lemma 6.6 $a \not\equiv 9 \pmod{12}$. By Lemma 6.2 again, since $a \not\equiv 3 \pmod 4$, we have $a \not\equiv 3 \pmod{12}$ and $a \not\equiv (7 \bmod 12)$. Therefore, since by hypothesis $a \not\equiv 1 \pmod{12}$, a must be even. By Lemma 6.8 if a is even and $3 \nmid N$, then m is even. By Lemma 6.7 if $a \equiv 1 \pmod{12}$ then $3 \mid N$ and m is even. □

Theorem 6.10 *Let $N = 2^a p_1 p_2 \cdots p_m$ be 4-perfect, where $p_1 < p_2 < \cdots < p_m$ are odd primes, $a = 2b+1$, and $b \geq 1$. If $2^{b+1}-1$ is prime, then $2^{b+1}-1 = p_m$.*

Proof. Case 1. Suppose $2^{b+1}-1 = p_{m-1}$, then $p_m \mid 2^{b+1}+1 = p_{m-1}+2$. Since $p_{m-1}+2 \leq p_m$, so $p_m = p_{m-1}+2$, so $p_m = 2^{b+1}+1$, but this is impossible, since $3 \mid 2^{b+1}+1$.

Case 2. Suppose $2^{b+1}-1 = p_j$, for some j with $1 \leq j \leq m-2$, then $p_m \mid 2^{b+1}+1$, so $p_m \leq p_j + 2$, but $p_j + 2 < p_m$, so a contradiction.

Combining Case 1 and Case 2, if $2^{b+1}-1$ is a prime, then $2^{b+1}-1 = p_m$. □

Theorem 6.11 *If $2^{a+1}-1$ is not a prime, where $a \neq 5$ is a positive integer, then not all of its prime factors are Mersenne primes.*

Proof. Suppose all prime factors of $2^{a+1}-1$ are Mersenne primes. Then for $m \geq 2$,

$$2^{a+1}-1 = (2^{r_1}-1)(2^{r_2}-1)\cdots(2^{r_m}-1). \tag{6.3}$$

By the primitive divisor theorem [86, (3.17), p20.], since $a \neq 5$, so $a+1 \neq 6$, then there is a prime factor $p \mid 2^{a+1}-1$, but $p \nmid 2^{r_i}-1$ for some i, $1 \leq i \leq m$, which is a contradiction.

Therefore, not all prime factors of $2^{a+1}-1$ are Mersenne primes, if $a \geq 1$ and $a \neq 5$. □

Theorem 5.16 is a corollary of Theorem 6.9. Now we give the proof of Theorem 5.16 as follows:

Proof. Let $M = 3N$. Then M is a flat 4-perfect number with the same exponent a as N. By Theorem 6.9, when $a \not\equiv 1 \pmod{12}$, a is even. When $a \equiv 1 \pmod{12}$, again by Theorem 6.9 the length of M is even so the length of N is odd and, by Lemma 5.15, every odd prime divisor of N is congruent to 1 modulo 3. □

It is also of some interest to observe the existence of Mersenne primes in the factorizations of the multiperfect numbers. Of course every 2-perfect number must be divisible by a Mersenne prime. We are able to show this persists for flat multiperfect numbers of multiplicities 3 and 4, but that non-Mersenne primes must always be present:

Theorem 6.12 *Let N be even, flat and multiperfect. (A) If the multiplicity is not greater than 4 then N is divisible by at least one Mersenne prime. (B) If all odd prime divisors of N are Mersenne primes then N is perfect.*

Proof. Let $N = 2^a p_1 \cdots p_m$ with $m \geq 1$.

(A) We can assume that $3 \nmid N$. If the multiplicity $k = 2$ then $N = 2^{p-1}M_p$ where p is prime and M_p is a Mersenne prime.

Let $k = 4$. Write

$$(2^{a+1}-1)(p_1+1)\cdots(p_m+1) = 2^{a+2}p_1\cdots p_m.$$

If p_1 is not Mersenne, the least odd divisor of p_1+1 is an odd prime $q < p_1$ which divides $p_1 \cdots p_m$ and, therefore, divides N. This contradicts the fact that p_1 is the least odd divisor of N. Thus p_1 is Mersenne.

Now let $k = 3$. If a is odd, write

$$\left(\frac{2^{a+1}-1}{3}\right)(p_1+1)\cdots(p_m+1) = 2^a p_1 \cdots p_m.$$

Like in the case $k = 4$, we deduce from this equation that p_1 is Mersenne.

Assume none of the p_i are Mersenne. Since $3 \nmid N$, then $p_i \neq 3$ and

$$(2^{a+1}-1)(p_1+1)\cdots(p_m+1) = 3 \cdot 2^a p_1 \cdots p_m$$

Also, the $(2^{a+1}-1, p_i+1) = 1$ and $(p_i+1, p_j+1) = 2^{\alpha_{ij}}$, $\alpha_{ij} \geq 1$, $\forall i \neq j$.

Hence

$$\omega(2^{a+1}-1) + 1 + m \leq \omega(3 \cdot 2^a p_1 \cdots p_m) = m + 2$$

which implies

$$\omega(2^{a+1}-1) = 1.$$

Since $2^{a+1}-1$ is odd, it must be squarefree, so $\omega(2^{a+1}-1) = 1$ implies $2^{a+1}-1$ is a Mersenne prime.

(B) Let $\sigma(N) = kN$ for some $k \geq 2$ and suppose that all of the p_i are Mersenne. Then

$$\sigma(2^a)(p_1+1)\cdots(p_m+1) = k \cdot 2^a \cdot p_1 \cdots p_m.$$

There exist primes q_i such that $p_i = 2^{q_i} - 1$. Therefore

$$(2^{a+1}-1) \cdot 2^{q_1} \cdots 2^{q_m} = k \cdot 2^a \cdot (2^{q_1}-1)\cdots(2^{q_m}-1)$$

so $a \leq q_1 + \cdots + q_m$ and for each i, $2^{q_i}-1 \mid 2^{a+1}-1$. But then Lemma 2.1 implies $q_i \mid a+1$, and, since necessarily these q_i's are distinct primes, $q_1 \cdots q_m \mid a+1$ giving

$$q_1 \cdots q_m \leq a+1 \leq q_1 + q_2 + \cdots + q_m + 1.$$

It follows (say by induction on m) that $m = 1$, therefore $N = 2^a \cdot p_1$. Then

$$(2^{a+1}-1)(p_1+1) = k 2^a p_1$$

implies $p_1 \mid 2^{a+1} - 1$ and $2^a \mid p_1 + 1$, so $2^a \leq p_1 + 1 \leq 2^{a+1}$ or

$$1 \leq \frac{p_1 + 1}{2^a} \leq 2.$$

If $(p_1 + 1)/2^a = 1$ then $p_1 + 1 = 2^a$, so $p_1 = 2^a - 1$ and $2^a - 1 \mid 2^{a+1} - 1$, which implies $a = 1$. It leads to the perfect number 6. If $(p_1 + 1)/2^a = 2$, then $p_1 = 2^{a+1} - 1$ giving $k = 2$, so N is perfect. □

Comment: The six flat 3-perfect numbers given in Chapter 5 have been known for over 100 years. There are no flat multiperfect numbers known of abundancy 5 or more, so in addition to the conjecture that all even 4-perfect numbers, flat or otherwise, are divisible by 3, an additional problem in this area is to find an upper bound for the possible multiplicities of flat multiperfect numbers. We have not been able to do this.

$$
\begin{aligned}
d_1 &= 2^5 \cdot 3^3 \cdot 5 \cdot 7 \\
d_2 &= 2^3 \cdot 3^2 \cdot 5 \cdot 7 \cdot 13 \\
d_3 &= 2^2 \cdot 3^2 \cdot 5 \cdot 7^2 \cdot 13 \cdot 19 \\
d_4 &= 2^9 \cdot 3^3 \cdot 5 \cdot 11 \cdot 31 \\
d_5 &= 2^7 \cdot 3^3 \cdot 5^2 \cdot 17 \cdot 31 \\
d_6 &= 2^9 \cdot 3^2 \cdot 7 \cdot 11 \cdot 13 \cdot 31 \\
d_7 &= 2^8 \cdot 3 \cdot 5 \cdot 7 \cdot 19 \cdot 37 \cdot 73 \\
d_8 &= 2^{10} \cdot 3^3 \cdot 5^2 \cdot 23 \cdot 31 \cdot 89 \\
d_9 &= 2^{13} \cdot 3^3 \cdot 5 \cdot 11 \cdot 43 \cdot 127 \\
d_{10} &= 2^{14} \cdot 3 \cdot 5 \cdot 7 \cdot 19 \cdot 31 \cdot 151 \\
d_{11} &= 2^{13} \cdot 3^2 \cdot 7 \cdot 11 \cdot 13 \cdot 43 \cdot 127 \\
d_{12} &= 2^5 \cdot 3^4 \cdot 7^2 \cdot 11^2 \cdot 19^2 \cdot 127 \\
d_{13} &= 2^8 \cdot 3^2 \cdot 7^2 \cdot 13 \cdot 19^2 \cdot 37 \cdot 73 \cdot 127 \\
d_{14} &= 2^7 \cdot 3^{10} \cdot 5 \cdot 17 \cdot 23 \cdot 107 \cdot 3851 \\
d_{15} &= 2^7 \cdot 3^6 \cdot 5 \cdot 17 \cdot 23 \cdot 137 \cdot 547 \cdot 1093 \\
d_{16} &= 2^{14} \cdot 3^2 \cdot 7^2 \cdot 13 \cdot 19^2 \cdot 31 \cdot 127 \cdot 151 \\
d_{17} &= 2^5 \cdot 3^4 \cdot 7^2 \cdot 11^2 \cdot 19^4 \cdot 151 \cdot 911 \\
d_{18} &= 2^9 \cdot 3^4 \cdot 7 \cdot 11^3 \cdot 31^2 \cdot 61 \cdot 83 \cdot 331
\end{aligned}
$$

$$
\begin{aligned}
d_{19} &= 2^8 \cdot 3^2 \cdot 7^2 \cdot 13 \cdot 19^4 \cdot 37 \cdot 73 \cdot 151 \cdot 911 \\
d_{20} &= 2^{25} \cdot 3^3 \cdot 5^2 \cdot 19 \cdot 31 \cdot 683 \cdot 2731 \cdot 8191 \\
d_{21} &= 2^{17} \cdot 3^{10} \cdot 7 \cdot 19^2 \cdot 23 \cdot 37 \cdot 73 \cdot 107 \cdot 127 \cdot 3851 \\
d_{22} &= 2^{17} \cdot 3^6 \cdot 7 \cdot 19^2 \cdot 23 \cdot 37 \cdot 73 \cdot 127 \cdot 137 \cdot 547 \cdot 1093 \\
d_{23} &= 2^{25} \cdot 3^4 \cdot 7 \cdot 11^2 \cdot 19^2 \cdot 127 \cdot 683 \cdot 2731 \cdot 8191 \\
d_{24} &= 2^{25} \cdot 3^5 \cdot 7^2 \cdot 13 \cdot 19^2 \cdot 127 \cdot 683 \cdot 2731 \cdot 8191 \\
d_{25} &= 2^{17} \cdot 3^{10} \cdot 7 \cdot 19^4 \cdot 23 \cdot 37 \cdot 73 \cdot 107 \cdot 151 \cdot 911 \cdot 3851 \\
d_{26} &= 2^{25} \cdot 3^{10} \cdot 5 \cdot 19 \cdot 23 \cdot 107 \cdot 683 \cdot 2731 \cdot 3851 \cdot 8191 \\
d_{27} &= 2^{17} \cdot 3^6 \cdot 7 \cdot 19^4 \cdot 23 \cdot 37 \cdot 73 \cdot 137 \cdot 151 \cdot 547 \cdot 911 \cdot 1093 \\
d_{28} &= 2^{25} \cdot 3^6 \cdot 5 \cdot 19 \cdot 23 \cdot 137 \cdot 547 \cdot 683 \cdot 1093 \cdot 2731 \cdot 8191 \\
d_{29} &= 2^{25} \cdot 3^4 \cdot 7 \cdot 11^2 \cdot 19^4 \cdot 151 \cdot 683 \cdot 911 \cdot 2731 \cdot 8191 \\
d_{30} &= 2^{25} \cdot 3^5 \cdot 7^2 \cdot 13 \cdot 19^4 \cdot 151 \cdot 683 \cdot 911 \cdot 2731 \cdot 8191 \\
d_{31} &= 2^{33} \cdot 3^4 \cdot 7 \cdot 11^3 \cdot 31 \cdot 61 \cdot 83 \cdot 331 \cdot 43691 \cdot 131071 \\
d_{32} &= 2^{33} \cdot 3^{10} \cdot 7 \cdot 11 \cdot 23 \cdot 83 \cdot 107 \cdot 331 \cdot 3851 \cdot 43691 \cdot 131071 \\
d_{33} &= 2^{33} \cdot 3^6 \cdot 7 \cdot 11 \cdot 23 \cdot 83 \cdot 137 \cdot 331 \cdot 547 \cdot 1093 \cdot 43691 \cdot 131071 \\
d_{34} &= 2^{37} \cdot 3^4 \cdot 7 \cdot 11^3 \cdot 31 \cdot 61 \cdot 83 \cdot 331 \cdot 43691 \cdot 174763 \cdot 524287 \\
d_{35} &= 2^{37} \cdot 3^{10} \cdot 7 \cdot 11 \cdot 23 \cdot 83 \cdot 107 \cdot 331 \cdot 3851 \cdot 43691 \cdot 174763 \cdot 524287 \\
d_{36} &= 2^{37} \cdot 3^6 \cdot 7 \cdot 11 \cdot 23 \cdot 83 \cdot 137 \cdot 331 \cdot 547 \cdot 1093 \cdot 43691 \cdot 174763 \cdot 524287
\end{aligned}
$$

Table 6.1: Known 4-perfect numbers

Chapter 7

Other properties of multiply perfect numbers and unsolved problems

7.1 Introduction

In Section 7.2 of this chapter we apply the method of Goto [37] (Proposition 7.1) to obtain an upper bound for 5-perfect numbers with a flat shape $N = 2^a p_1 p_2 \cdots p_m$ (Proposition 7.2). We also discuss an example to show that multiply perfect Fermat numbers do not exist ([68], Example 7.1). Finally we provide some conjectures about multiperfect numbers, which come from Chapters 4, 5 and 6, (Section 7.3).

7.2 Upper bound for 5-perfect numbers of a flat shape

Let $N = N_1 N_2 \cdots N_k$, where $N_1, \cdots, N_k$ denote prime powers satisfying $N_i < N_j$, $(N_i, N_j) = 1$ for $i < j$.

Proposition 7.1 *[37, Takeshi Goto]*

Let a, b, k be positive integers. Suppose that N is a positive integer with k distinct prime factors. If

$$\prod_{i=1}^{k}\left(1+\frac{1}{N_i}\right)=\frac{a}{b},$$

then

$$N\leq(b+1)^{2^{k-1}-1}\left((b+1)^{2^{k-1}}-1\right).$$

Proposition 7.2 *Let e be a positive integer. If $N=2^{e}p_1p_2\cdots p_m$ with the p_i's odd primes ($i=1,\cdots,m$), is a 5-perfect number, and we define a and b by*

$$\prod_{i=1}^{m}\left(1+\frac{1}{p_i}\right)=\frac{a}{b},$$

with $(a,b)=1$, then we have the following upper bound for N:

If $(a+1,b)=1$ then

$$N\leq(b+1)^{2^{m}-1}\left((b+1)^{2^{m}}-1\right);$$

If $(a+1,b)\geq 3$ then

$$N\leq\left(\frac{1}{3}b+1\right)^{2^{m}-1}\left(\left(\frac{1}{3}b+1\right)^{2^{m}}-1\right).$$

Proof. Since $N=2^{e}p_1\cdots p_m$ is a 5-perfect number, we have $\sigma(N)=5N$. Therefore

$$\frac{\sigma(N)}{N}=\left(2-\frac{1}{2^{e}}\right)\prod_{i=1}^{m}\left(1+\frac{1}{p_i}\right)=5.$$

Since

$$\prod_{i=1}^{m}\left(1+\frac{1}{p_i}\right)=\frac{a}{b},$$

so

$$2^{e}=\frac{a}{2a-5b}.$$

It follows that

$$2^{e}\cdot 5b=a(2^{e+1}-1),$$

implies $a \mid 2^e \cdot 5b$, since $(a,b)=1$, so $a \mid 2^e \cdot 5$. Now we need only consider the following 3 cases:

Case 1. If $a = 2^\alpha$, $(\alpha \le e)$, then

$$5b = 2^{\alpha+1} - 2^{\alpha-e},$$

implies $\alpha = e$, so

$$1 + \frac{1}{2^e} = \frac{a+1}{a}.$$

So

$$\begin{aligned}\left(1+\frac{1}{2^e}\right)\prod_{i=1}^{m}\left(1+\frac{1}{p_i}\right) &= \left(1+\frac{1}{2^e}\right)\frac{a}{b} \\ &= \frac{a+1}{a}\frac{a}{b} \\ &= \frac{a+1}{b}.\end{aligned}$$

We have either $(a+1,b)=1$ or $(a+1,b)=c_1 \ge 3$ (since a is even and b is odd).

If $(a+1,b)=1$, then by Proposition 7.1, we get an upper bound for N:

$$N = 2^e \prod_{i=1}^{m} p_i \le (b+1)^{2^m-1}\left((b+1)^{2^m}-1\right).$$

If $(a+1,b)=c_1 \ge 3$. Let $\frac{a+1}{b} = \frac{a_1}{b_1}$, $(a_1,b_1)=1$, then $b_1 \le \frac{1}{3}b$. By Proposition 7.1 again, we get an upper bound for N:

$$N = 2^e \prod_{i=1}^{m} p_i \le \left(\frac{1}{3}b+1\right)^{2^m-1}\left(\left(\frac{1}{3}b+1\right)^{2^m}-1\right).$$

Case 2. If $a = 2^\alpha \cdot 5$, $(\alpha \le e)$, then $b = 2^{\alpha-e}(2^{e+1}-1)$, implies $\alpha = e$, then $a = 2^e \cdot 5$, $b = 2^{e+1}-1$. So

$$\begin{aligned}\left(1+\frac{1}{2^e}\right)\prod_{i=1}^{m}\left(1+\frac{1}{p_i}\right) &= \frac{a+5}{a}\frac{a}{b} \\ &= \frac{a+5}{b}.\end{aligned}$$

Implies either $(a+5,b)=1$ or $(a+5,b)=c_2\geq 3$. Therefore, we get an upper bound for N by Proposition 7.1:

$$N=2^e\prod_{i=1}^{m}p_i\leq (b+1)^{2^m-1}\left((b+1)^{2^m}-1\right),$$

where $(a+5,b)=1$;

$$N=2^e\prod_{i=1}^{m}p_i\leq \left(\frac{1}{3}b+1\right)^{2^m-1}\left(\left(\frac{1}{3}b+1\right)^{2^m}-1\right),$$

where $(a+5,b)=c_2$.

Case 3. If $a=5$, then $2^eb=2^{e+1}-1$, implies $1=2^e(2-b)$, so $e=0$ and $b=1$, which is a contradiction. □

Here I give an example of an infinite set of numbers which are not multiperfect.

Example 7.1 *[68] There are no multiperfect Fermat numbers.*

Proof. Let $F_n=\prod_{i=1}^{k}p_i^{\alpha_i}$. Then

$$\frac{\sigma(F_n)}{F_n}=\prod_{i=1}^{k}\left(\frac{p_i-p_i^{-\alpha_i}}{p_i-1}\right)<\prod_{i=1}^{k}\left(\frac{p_i}{p_i-1}\right)=\prod_{i=1}^{k}\left(1+\frac{1}{p_i-1}\right),$$

then

$$\log\left(\frac{\sigma(F_n)}{F_n}\right)\leq\sum_{i=1}^{k}\log\left(1+\frac{1}{p_i-1}\right)<\sum_{i=1}^{k}\left(\frac{1}{p_i-1}\right).$$

Since $F_n=2^{2^n}+1$, so $p_i\equiv 1\pmod{2^{n+2}}$, [85] it follows that

$$p_i-1=2^{n+2}\cdot n_i\geq 2^{n+2}\cdot i,$$

then $p_i\geq 2^{n+2}\cdot i+1$ for all $i=1,\cdots,k$. Hence,

$$\log F_n=\sum_{i=1}^{k}\alpha_i\log p_i\geq k\log p_1\geq k\log(2^{n+2}+1).$$

So,

$$k\log\left(2^{n+2}+1\right)\leq\log F_n=\log\left(2^{2^n}+1\right).$$

Since

$$\frac{\log(y+1)}{\log(z+1)}\leq\frac{\log y}{\log z}\text{ whenever }y\geq z>2,$$

it follows that

$$k \leq \frac{\log(2^{2^n}+1)}{\log(2^{n+2}+1)} \leq \frac{\log(2^{2^n})}{\log(2^{n+2})} = \frac{2^n}{n+2}.$$

Thus

$$\begin{aligned}
\sum_{i=1}^{k} \frac{1}{p_i - 1} &\leq \frac{1}{2^{n+2}} \sum_{i=1}^{k} \frac{1}{i} \\
&\leq \frac{1}{2^{n+2}}(1 + \log k) \\
&\leq \frac{1}{2^{n+2}} \left(1 + \log\left(\frac{2^n}{n+2}\right)\right) \\
&< \frac{n \log 2}{2^{n+2}} \\
&< \log 2.
\end{aligned}$$

Hence, $\frac{\sigma(F_n)}{F_n} < 2$, therefore F_n cannot be multiply perfect. □

7.3 Conjectures

If $p_i + 1 = 2^{a_i} p_{i-1}$, with p_i's odd primes, and $a_i \geq 1$, $(i = 1, 2, \cdots, m)$, where p_0 is the n'th Mersenne prime, we say the corresponding sequence p_i's are in the n'th tree. For example, $5 + 1 = 2^1 \cdot 3$; $11 + 1 = 2^2 \cdot 3$; $23 + 1 = 2^3 \cdot 3$; $19 + 1 = 2^2 \cdot 5$; $37 + 1 = 2^1 \cdot 19$, so we say the primes 5, 11, 23, 19, 37 are in the 1st tree. The first seven trees are given in figures below.

By observing the form of each of the 6 known 3-perfect numbers I am led to Conjectures 7.3 and 7.4:

Conjecture 7.3 *Any 3-perfect number has all of its non-Mersenne primes from the 1st tree, and any primes from other trees are Mersenne primes.*

Conjecture 7.4 *There are at most a finite number of integers with the flat shape, $N = 2^a \cdot p_1 \cdots p_m$, which are 3-perfect.*

There are only 8 known flat multiperfect numbers, six with abundancy 3 and two with abundancy 4. Thus I have Conjectures 7.5, 7.6 and 7.7:

Conjecture 7.5 *There are only a finite number of multiperfect numbers which have the flat shape* $N = 2^a \cdot p_1 \cdots p_m$.

Conjecture 7.6 *If* N *is a flat and multiperfect number then its abundancy is less than or equal to* 4.

Conjecture 7.7 *All* 3*-perfect numbers have the flat shape* $N = 2^a \cdot p_1 \cdots p_m$.

I also have the following Conjectures. Unfortunately, I was not able to make progress with these problems. They seem to be very difficult.

Conjecture 7.8 *If* N *is an even* 4*-perfect number, then* $4 \mid N$.

Conjecture 7.9 *If* N *is an even* 4*-perfect number, then* $3 \mid N$.

Conjecture 7.10 *If* N *is a* 4*-perfect number, then the power of the largest prime divisor of* N *is* 1.

Conjecture 7.11 *There are only two* 4*-perfect numbers with a flat shape* $N = 2^a p_1 \cdots p_m$.

Conjecture 7.12 *The largest prime factor of an even* 4*-perfect number* N *with* $2^a \| N$ *always occurs in the factorization of* $\sigma(2^a)$.

Conjecture 7.13 *Prime factors of an even* 4*-perfect number* N *with* $2^a \| N$*, with odd discrete logarithms to the base* 2*, always appear in the factorization of* $\sigma(2^a)$ *which consists exactly of those primes.*

Conjecture 7.14 *The number of Mersenne primes in an even* 4*-perfect number* N *is exactly the number of distinct primes in the factorization of* $\sigma(2^a)$.

Conjecture 7.15 *Each odd prime which appears in a flat* 4*-perfect number* N *is a super thin prime.*

$$
\begin{aligned}
e_1 &= 2^7 \cdot 3^4 \cdot 5 \cdot 7 \cdot 11^2 \cdot 17 \cdot 19 \\
f_1 &= 2^{15} \cdot 3^5 \cdot 5^2 \cdot 7^2 \cdot 11 \cdot 13 \cdot 17 \cdot 19 \cdot 31 \cdot 43 \cdot 257 \\
g_1 &= 2^{32} \cdot 3^{11} \cdot 5^4 \cdot 7^5 \cdot 11^2 \cdot 13^2 \cdot 17 \cdot 19^3 \cdot 23 \cdot 31 \cdot 37 \cdot \\
&\quad 43 \cdot 61 \cdot 71 \cdot 73 \cdot 89 \cdot 181 \cdot 2141 \cdot 599479 \\
h_1 &= 2^{62} \cdot 3^{15} \cdot 5^9 \cdot 7^7 \cdot 11^3 \cdot 13^3 \cdot 17^2 \cdot 19 \cdot 23 \cdot 29 \cdot 31^2 \cdot 37 \cdot 41 \cdot \\
&\quad 43 \cdot 53 \cdot 61^2 \cdot 71^2 \cdot 73 \cdot 83 \cdot 89 \cdot 97^2 \cdot 127 \cdot 193 \cdot 283 \cdot 307 \cdot ... \\
i_1 &= 2^{104} \cdot 3^{43} \cdot 5^9 \cdot 7^{12} \cdot 11^6 \cdot 13^4 \cdot 17 \cdot 19^4 \cdot 23^2 \cdot 29 \cdot 31^4 \cdot \\
&\quad 37^3 \cdot 41^2 \cdot 43^2 \cdot 47^2 \cdot 53 \cdot 59 \cdot 61 \cdot 67 \cdot 71^3 \cdot 73 \cdot 79^2 \cdot ... \\
j_1 &= 2^{209} \cdot 3^{77} \cdot 5^{23} \cdot 7^{26} \cdot 11^{14} \cdot 13^{11} \cdot 17^9 \cdot 19^{12} \cdot 23^4 \cdot \\
&\quad 29^3 \cdot 31^9 \cdot 37^4 \cdot 41^5 \cdot 43^7 \cdot 47 \cdot 53 \cdot 59 \cdot 61^3 \cdot 67 \cdot ... \\
k_1 &= 2^{468} \cdot 3^{140} \cdot 5^{66} \cdot 7^{49} \cdot 11^{40} \cdot 13^{31} \cdot 17^{11} \cdot 19^{12} \cdot \\
&\quad 23^9 \cdot 29^7 \cdot 31^{11} \cdot 37^8 \cdot 41^5 \cdot 43^3 \cdot 47^3 \cdot 53^4 \cdot 59^3 \cdot ...
\end{aligned}
$$

Table 7.1: Examples of multiperfect numbers (abundancy from 5 to 11)

Conjecture 7.16 *If N is a 3-perfect number with the shape $2^a p_1 \cdots p_m$, where a is an even positive integer, then $3 \nmid N$.*

Conjecture 7.17 *If N is a flat even 4-perfect number, then the exponent of 2 is even.*

Conjecture 7.18 *The number of thin primes up to x satisfies*

$$T(x) \gg \frac{x}{\log^2 x}.$$

Table 7.1 includes the smallest multiperfect numbers of abundancy from 5 to 11 [90].

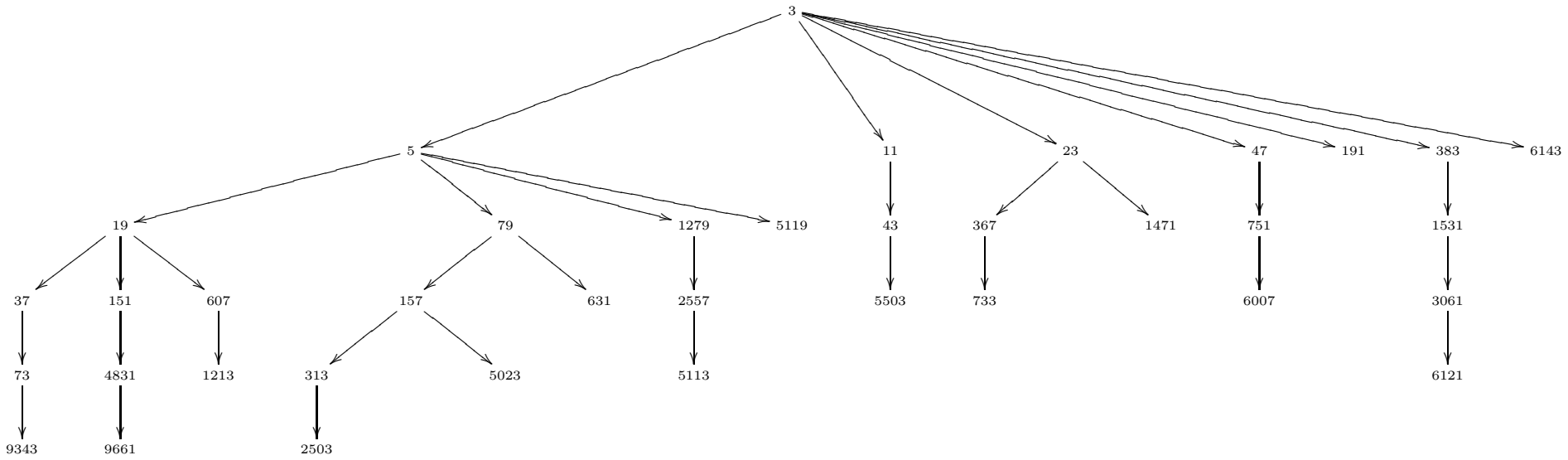

Figure 7.1: The 1st tree of primes up to 10^4

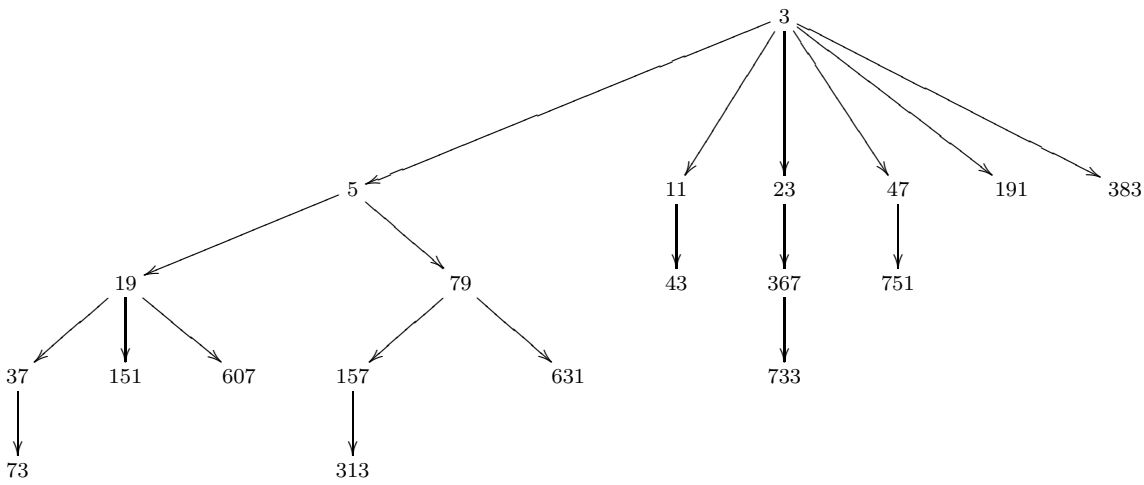

Figure 7.2: The 1st tree of primes up to 10^3

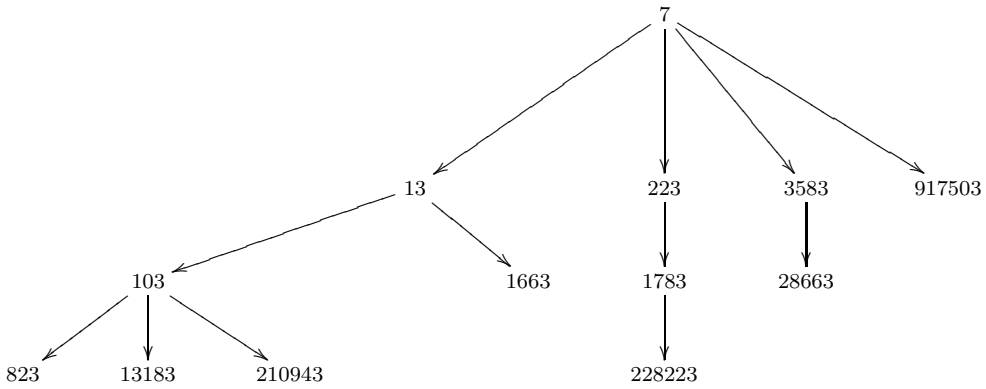

Figure 7.3: The 2nd tree of primes up to 10^6

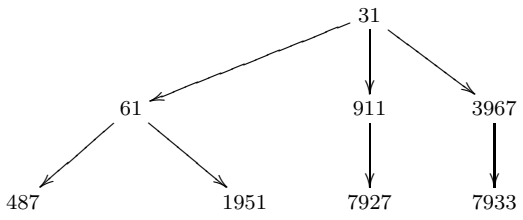

Figure 7.4: The 3rd tree of primes up to 10^4

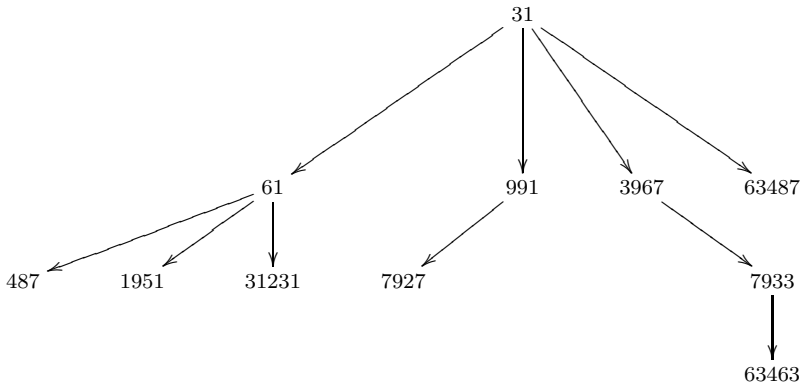

Figure 7.5: The 3rd tree of primes up to 10^5

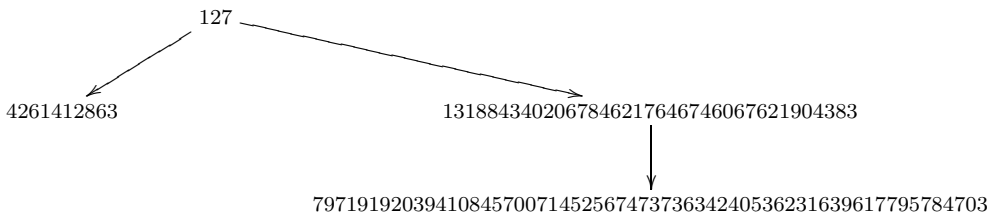

Figure 7.6: The 4th tree of primes up to 10^{60}

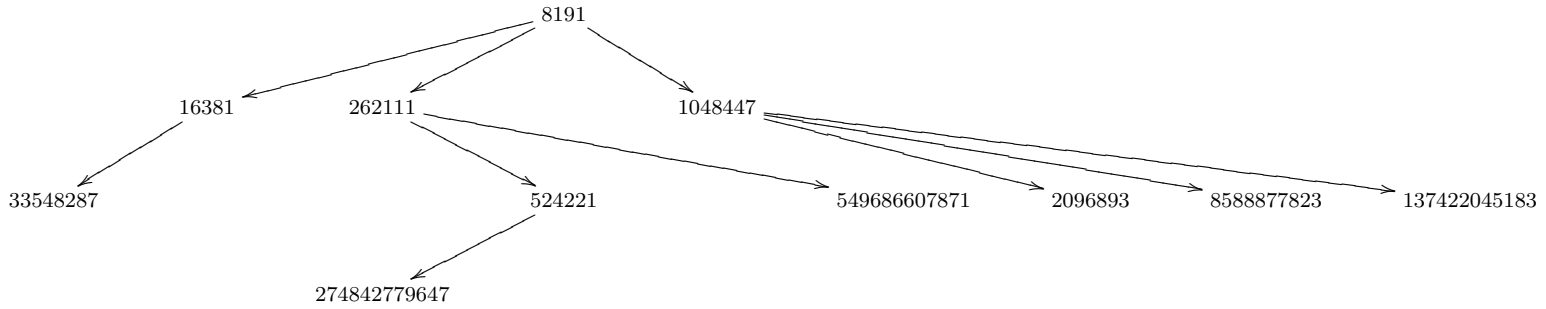

Figure 7.7: The 5th tree of primes up to 10^{12}

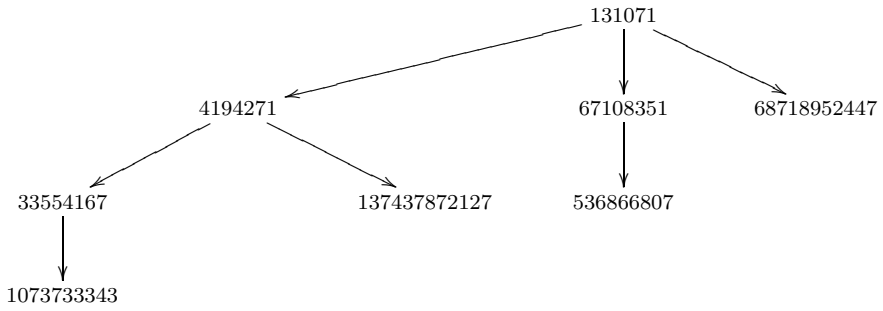

Figure 7.8: The 6th tree of primes up to 10^{12}

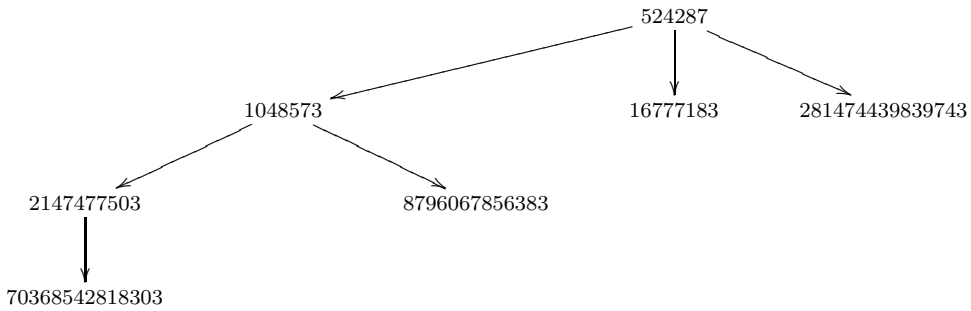

Figure 7.9: The 7th tree of primes up to 10^{15}

References

[1] L. M. Adleman, C. Pomerance, and R. S. Rumely. On distinguishing prime numbers from composite numbers. *Ann. Math.*, 117(1):173–206, 1983.

[2] T. M. Apostol. *Introduction to Analytic Number Theory.* Springer-Verlag, New York, 1976.

[3] M. M. Artuhov. On the problem of odd h-fold perfect numbers. *Acta Arith.*, 23(3):249–255, 1973.

[4] P. T. Bateman and H. Diamond. *Analytic Number Theory: An Introductory Course.* World Scientific, New Jersey, 2004.

[5] W. Beck and R. Najar. A lower bound for odd triperfects. *Math. Comp.*, 38(157):249–251, 1982.

[6] H. A. Bernhard. On the least possible odd perfect number. *Amer. Math. Monthly*, 56:628–629, 1949.

[7] G. D. Birkhoff and H. S. Vandiver. On the integral divisors of $a^n - b^n$. *Ann. Math.*, 5:173–180, 1904.

[8] A. Brauer. On the non-existence of odd perfect numbers of the form $p^\alpha q_1^2 q_2^2 \cdots q_{t-1}^2 q_t^4$. *Bull. Amer. Math. Soc.*, 49:712–718, 1943.

[9] R. P. Brent and G. L. Cohen. A new lower bound for odd perfect numbers. *Math. Comp.*, 53(187):431–437, 1989.

[10] R. P. Brent, G. L. Cohen, and H. J. J. te Riele. Improved techniques for lower bounds for odd perfect numbers. *Math. Comp.*, 57(196):857–868, 1991.

[11] K. A. Broughan and Q. Zhou. Flat primes and thin primes. *Bull. Aust. Math. Soc.* (to appear).

[12] K. A. Broughan and Q. Zhou. Odd multiperfect numbers of abundancy 4. *J. Number Theory*, 128:1566–1575, 2008.

[13] K. A. Broughan and Q. Zhou. Divisibility by 3 of even multiperfect numbers of abundancy 3 and 4. *J. Integer Sequences*, 13, 2010. Article 10.1.5.

[14] D. M. Burton. *Elementary Number Theory*. Allyn and Bacon, Inc., Boston, revised printing edition, 1980.

[15] R. D. Carmichael. Multiply perfect numbers of three different primes. *Ann. Math.*, 8(1):49–56, 1906.

[16] R. D. Carmichael. Multiply perfect numbers of four different primes. *Ann. Math.*, 8(4):149–158, 1907.

[17] E. Catalan. Propositions et questions diverses. *Bull. Soc. Math. France*, 16:128–129, 1888.

[18] J. E. Z. Chein. *An Odd Perfect Number has at Least 8 Prime Factors*. PhD thesis, Pennsylvania State University, 1979.

[19] Y. G. Chen. On integers of the form $k \cdot 2^n + 1$. *Proc. Amer. Math. Soc.*, 129:355–361, 2000.

[20] G. L. Cohen. *Generalised Quasiperfect Numbers with an Appendix on Odd Perfect Numbers*. PhD thesis, University of New South Wales, Sydney, 1982.

[21] G. L. Cohen. On the largest component of an odd perfect number. *J. Aust. Math. Soc. (Series A)*, 42:280–286, 1987.

[22] G. L. Cohen. On the number of distinct prime factors of an odd perfect number. Internal Report 24, University of Technology, Sydney, School of Mathematical Sciences, Sydney, 1991.

[23] G. L. Cohen and P. Hagis, Jr. Results concerning odd multiperfect numbers. *Bull. Malaysian Math. Soc.*, 8(1):23–26, 1985.

[24] G. L. Cohen and P. Hagis, Jr. Every odd perfect number has a prime factor which exceeds 10^6. *Math. Comp.*, 67:1323–1330, 1998.

[25] G. L. Cohen and R. M. Sorli. On the number of distinct prime factors of an odd perfect number. *J. Discrete Alg.*, 1:21–35, 2003.

[26] R. J. Cook. Bounds for odd perfect numbers. In *Fifth Conference of the Canadian Number Theory Association*, number 19 in Centre de Recherches Mathématiques Proceedings and Lecture Notes, pages 67–71, Carleton University, Ottawa, Ontario, Canada, 1999.

[27] L. E. Dickson. Finiteness of the odd perfect and primitive abundant numbers with n distinct prime factors. *Amer. J. Math.*, 35:413–422, 1913.

[28] L. E. Dickson. *History of the Theory of Numbers*, volume 1. Chelsea, New York, 1952.

[29] M. L. D'Ooge. *Nicomachus, Introduction to Arithmatic*. The Macmillan Company, New York, 1926.

[30] H. Edgar. On a theorem of Suryanarayana. *Proc. Washington State Univ. Conf. Number Theory Pullman*, pages 52–54, 1971.

[31] H. Edgar. Problems and some results concerning the Diophantine equation $1 + a + a^2 + \cdots + a^{x-1} = p^y$. *Rocky Mountain J. Math.*, 15:325–327, 1985.

[32] P. Erdős and A. M. Odlyzko. On the density of odd integers of the form $(p-1)2^{-n}$ and related questions. *J. Number Theory*, 11:257–263, 1979.

[33] P. Erdős and J. Surányi. *Topics in the Theory of Numbers*. Springer, New York, 2003.

[34] D. Estes, R. Guralnick, M. Schacher, and E. Straus. Equations in prime powers. *Pacific J. Math.*, 118:359–367, 1985.

[35] A. Flammenkamp. Multiperfect number web site. `http://www.uni-bielefeld.de/achim/mpn.html`, 2007.

[36] S. Gimbel and J. H. Jaroma. Sylvester: ushering in the modern era of research on odd perfect numbers. *Integers: Electronic J. Comb. Number Theory*, 3. A16.

[37] T. Goto. Upper bounds for unitary perfect numbers and unitary harmonic numbers. *Rocky Mountain J. Math.*, 37(5):1557–1576, 2007.

[38] O. Grün. Über ungerade vollkommenen Zahlen. *Math. Zeit.*, 55(3):353–354, 1952.

[39] P. Hagis, Jr. A lower bound for the set of odd perfect numbers. *Math. Comp.*, 27(124):951–953, 1973.

[40] P. Hagis, Jr. Outline of a proof that every odd perfect number has at least eight prime factors. *Math. Comp.*, 35(151):1027–1032, 1980.

[41] P. Hagis, Jr. On the second largest prime divisor of an odd perfect number. In *Analytic Number Theory*, number 899 in Lecture Notes in Mathematics, pages 254–263. Springer-Verlag, Berlin, 1981.

[42] P. Hagis, Jr. Sketch of a proof that an odd perfect number relatively prime to 3 has at least eleven prime factors. *Math. Comp.*, 40(161):399–404, 1983.

[43] P. Hagis, Jr. The third largest prime factor of an odd multiperfect number exceeds 100. *Bull. Malaysian Math. Soc.*, 9(2):43–49, 1986.

[44] P. Hagis, Jr. A new proof that every odd triperfect number has at least twelve prime factors. *Contemp. Math.*, 143:445–450, 1993.

[45] P. Hagis, Jr. and W. L. McDaniel. On the largest prime divisor of an odd perfect number. *Math. Comp.*, 27(124):955–957, 1973.

[46] G. H. Hardy and J. E. Littlewood. Some problems of 'partitio numerorum';III: on the expression of a number as a sum of primes. *Acta Math.*, 44:1–70, 1923.

[47] G. H. Hardy and E.M. Wright. *An Introduction to the Theory of Numbers*. Oxford Univervisty Press, Oxford, fifth edition, 1979.

[48] K. G. Hare. New techniques for bounds on the total number of prime factors of an odd perfect number. *Math. Comp.* (to appear).

[49] K. G. Hare. More on the total number of prime factors of an odd perfect number. *Math. Comp.*, 74(250):1003–1008, 2004.

[50] D. R. Heath-Brown. Artin's conjecture for primitive roots. *Quart. J. Math. Oxford Ser. (2)*, 37:27–38, 1986.

[51] D. R. Heath-Brown. Odd perfect numbers. *Math. Proc. Cambridge Phil. Soc.*, 115(2):191–196, 1994.

[52] B. Hornfeck. Zur dichte der menge der follkommenen zahlen. *Arch. Math.*, 6:442–443, 1955.

[53] B. Hornfeck. Bemerkung zu meiner Note Uber vollkommene Zahlen. *Arch. Math.*, 7:273, 1956.

[54] L. K. Hua. *Introduction to number theory*. Springer-verlag, Berlin, 1982.

[55] D. E. Iannucci. The second largest prime divisor of an odd perfect number exceeds ten thousand. *Math. Comp.*, 68(228):1749–1760, 1999.

[56] D. E. Iannucci. The third largest prime divisor of an odd perfect number exceeds one hundred. *Math. Comp.*, 69(230):867–879, 2000.

[57] D. E. Iannucci and R. M. Sorli. On the total number of prime factors of an odd perfect number. *Math. Comp.*, 72(244):2077–2084, 2003.

[58] P. M. Jenkins. Odd perfect numbers have a prime factor exceeding 10^7. *Math. Comp.*, 72(243):1549–1554, 2003.

[59] H. J. Kanold. Eine Bemerkung über die Menge der vollkommenen zahlen. *Math. Ann.*, 131:390–392, 1956.

[60] H. J. Kanold. Über mehrfach vollkommenen Zahlen II. *J. Reine Angew. Math.*, 197(1):82–96, 1957.

[61] M. Kishore. Odd perfect numbers not divisible by three are divisible by at least ten distinct primes. *Math. Comp.*, 31(137):274–279, 1977.

[62] M. Kishore. On odd perfect, quasiperfect, and odd almost perfect numbers. *Math. Comp.*, 36(154):583–586, 1981.

[63] M. Kishore. Odd perfect numbers not divisible by 3 II. *Math. Comp.*, 40(161):405–411, 1983.

[64] M. Kishore. Odd triperfect numbers are divisible by twelve distinct prime factors. *J. Aust. Math. Soc. (Series A)*, 42:183–195, 1987.

[65] U. Kühnel. Verschärfung der notwendingen Bedingungen für die Existenz von ungeraden vollkommenen Zahlen. *Math. Zeit.*, 52:202–211, 1949.

[66] W. Lipp. Odd perfect number search. http://www.oddperfect.org, 2006.

[67] W. Ljunggren. Some theorems on indeterminate equations of the form $\frac{x^n-1}{x-1} = y^q$. *Norsk Mat. Tidsskr.*, 25:17–20, 1943.

[68] F. Luca. The anti-social Fermat number. *Amer. Math. Monthly*, 107(2):171–173, 2000.

[69] W. L. McDaniel. The non-existence of odd perfect numbers of a certain form. *Arch. Math.*, 21(1):52–53, 1970.

[70] L. Mirsky. The number of representations of an integer as the sum of a prime and a k-free integer. *Amer. Math. Monthly*, 56:17–19, 1949.

[71] D. S. Mitrinović, J. Sándor, and B. Crstici. *Handbook of Number Theory.* Kluwer Academic Publishers, Dordrecht, 1996.

[72] H. L. Montgomery and R. C. Vaughan. *Multiplicative Number Theory I. Classical Theory.* Cambridge University Press, Cambridge London, 2007.

[73] T. Nagell. *Introduction to Number Theory.* Chelsea, New York, 1964.

[74] M. B. Nathanson. *Additive Number Theory: The Classical Bases.* Springer, New York, 1996.

[75] T. R. Nicely. http://www.trnicely.net/twins/twins.html, 2008.

[76] P. P. Nielsen. An upper bound for odd perfect numbers. *Integers: Electronic J. Comb. Number Theory*, 3, 2003. A14.

[77] P. P. Nielsen. Odd perfect numbers have at least nine distinct prime factors. *Math. Comp.*, 76:2109–2126, 2007.

[78] J. J. O'Connor and E. F. Robertson. History topic: Perfect numbers. http://www-history.mcs.standrews.ac.uk/histtopics/perfectnumbers.html, 2006.

[79] P. Pollack. On the greatest common divisor of a number and its sum of divisors. http://www.math.uiuc.edu/~pppollac/combined2.pdf, 2009.

[80] C. Pomerance. Odd perfect numbers are divisible by at least seven distinct primes. *Acta Arith.*, 25:265–300, 1974.

[81] C. Pomerance. The second largest prime factor of an odd perfect number. *Math. Comp.*, 29(131):914–921, 1975.

[82] C. Pomerance. Multiply perfect numbers, Mersenne primes and effective computability. *Math. Ann.*, 226(3):195–206, 1977.

[83] C. Pomerance and A. Sárközy. *Combinatorial Number Theory*. Elsevier Science, New York, 1995.

[84] P. Ribenboim. *Catalan's Conjecture*. Academic Press, Boston, 1994.

[85] P. Ribenboim. *The New Book of Prime Number Records*. Springer, New York, third edition, 1996.

[86] P. Ribenboim. *My numbers, my friends*. Springer-Verlag, New York, 2000.

[87] N. Robbins. *The non-existence of odd perfect numbers with less than seven distinct prime factors*. PhD thesis, Polytechnic Institute of Brooklyn, New York, June 1972.

[88] J. B. Rosser and L. Schoenfeld. Approximate formulas for some functions of prime numbers. *Illinois J. Math.*, 6:64–94, 1962.

[89] M. D. Sayers. An improved lower bound for the total number of prime factors of an odd perfect number. Master's thesis, New South Wales Institute of Technology, Sydney, 1986.

[90] R. Schroeppel. Multiperfect number list, December 95, 2094 numbers. `ftp://ftp.cs.arizona.edu/~/xkernel/rcs/mpfn.htm`. Also the mpfn mailing list., 2006.

[91] H. N. Shapiro. Note on a theorem of Dickson. *Bull. Amer. Math. Soc.*, 55:450–452, 1949.

[92] W. Sierpiński. Sur un problém concernant les nombres $k \cdot 2^n + 1$. *Elem. Math.*, 15:73–74, 1960.

[93] R. Sorli. *Algorithms in the study of multiperfect and odd perfect numbers*. PhD thesis, University of Technology, Sydney, 2003. `http://hdl.handle.net/2100/275`.

[94] P. Starni. On some properties of the Euler's factor of certain odd perfect numbers. *J. of Number Theory*, 116:483–486, 2006.

[95] R. Steuerwald. Verschaeffung einer notwendigen Bedingung fuer die Existenz einer ungeraden vollkommenen. *Zahl. S.-B Math.-Nat. Kl.Bayer. Akad. Wiss.*, pages 68–73, 1937.

[96] D. Suryanarayana. Certain Diophantine Equations. *Math. Student*, 35:197–199, 1967.

[97] D. Suryanarayana and P. Hagis, Jr. A theorem concerning odd perfect numbers. *Fibonacci Q.*, 8(4):337–346, 374, 1970.

[98] G. Tenenbaum. *Introduction to Analytic and Probabilistic Number Theory*. Cambridge University Press, Cambridge London, 1995.

[99] N. M. Timofeev. The Hardy-Ramanujan and Halasz inequalities for shifted primes. *Math. Notes*, 57:522–535, 1995.

[100] B. Tuckerman. A search procedure and lower bound for odd perfect numbers. *Math. Comp.*, 27(124):943–949, 1973.

[101] G. C. Webber. Non-existence of odd perfect numbers of the form $3^{2\beta} \cdot p^{\alpha} \cdot s_1^{2\beta_1} s_2^{2\beta_2} s_3^{2\beta_3}$. *Duke Math. J.*, 18:741–749, 1951.

[102] D. Wells. *Prime Numbers: The Most Mysterious Figures in Math*. John Wiley & Sons, New Jersey, 2005.

[103] J. Westlund. Note on multiply perfect numbers. *Ann. Math., 2nd Ser.*, 2(1):172–174, 1900.

[104] J. Westlund. Note on multiply perfect numbers. *Ann. Math., 2nd Ser.*, 3(1):161–163, 1901.

[105] G. Woltman. GIMP: Great Internet Mersenne Prime search. `http://www.mersenne.org`, 2007.

[106] K. Zsigmondy. Zur theorie der potenzreste. *Monatsh. Math*, 3:265–284, 1892.