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THE NATURAL FREQUENCY ESTIMATION FOR A SYSTEM COMPOSED OF A BEAM AND A SPRING-MASS SYSTEM

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ABSTRACT

This study presents an approximate method for determining the natural frequency of a system composed of a beam and a two-degree-of-freedom spring-mass system. The expression for estimating the natural frequency is derived following the standard procedures of the forced vibration analysis of a beam. The results obtained using the current method are in good agreement with those obtained through the dynamic stiffness method, especially when the spring stiffness is large. The innovativeness of the current method is that it reveals the relevance between the natural frequency of the system, the natural frequency of the beam, and the mode shape data at the positions where the spring-mass system is attached to the beam. The method is potentially useful in the dynamic wheel-track interaction analysis because the train wheel is normally simplified as a spring-mass system with high spring stiffness. It may also be applied to natural frequency-based damage detection where an auxiliary spring-mass system is used. When the spring-mass system roves on the beam, the curve of natural frequency versus spring-mass system location would be relevant to the mode shape square which is sensitive to local damage.

1 THEORETICAL DERIVATION

An Euler-Bernoulli beam carrying a two-degree-of-freedom (2dof) spring-mass system is shown in Figure 1. The centroid of the mass is located at the centre.

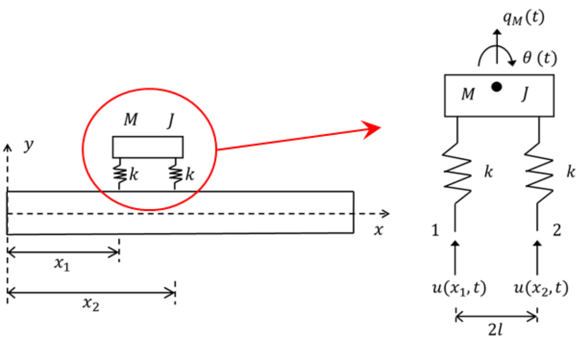

Figure 1: A beam carrying a 2dof spring-mass system.

The governing equation of motion of the beam can be expressed as

$$\bar{m}\ddot{u}(x,t) + EIu''''(x,t) = f_1(t)\delta(x - x_1) + f_2(t)\delta(x - x_2) \tag{1}$$

where $\bar{m}$ is mass per unit length and $u(x,t)$ is the deflection of the beam. The overdot represents the derivative with respect to time and the prime represents the derivative with respect to x. $f_1(t)$ and $f_2(t)$ are the forces that the spring-mass system exerts on the beam at x_1 and x_2, respectively.

For the 2dof spring-mass system in Figure 1, $q_M(t)$ and $\theta(t)$ are the vertical displacement and rotation of the mass, respectively. The governing equations of motion of the spring-mass system can be written as

$$\begin{cases} [u(x_1,t) - q_M(t) - l\theta(t)]k + [u(x_2,t) - q_M(t) + l\theta(t)]k - M\ddot{q}_M(t) = 0 \\ [u(x_1,t) - q_M(t) - l\theta(t)]kl - [u(x_2,t) - q_M(t) + l\theta(t)]kl - J\ddot{\theta}(t) = 0 \end{cases} \tag{2}$$

Thus

$$f_1(t) = -[u(x_1,t) - q_M(t) - l\theta(t)]k = -\frac{Ml\ddot{q}_M(t) + J\ddot{\theta}(t)}{2l} \tag{3}$$

$$f_2(t) = -[u(x_2,t) - q_M(t) + l\theta(t)]k = -\frac{Ml\ddot{q}_M(t) - J\ddot{\theta}(t)}{2l} \tag{4}$$

Using the modal superposition method, the beam deflection can be expressed as

$$u(x,t) = \sum_n \phi_n(x) q_n(t) \tag{5}$$

where $\phi_n(x)$ is the nth mode shape of the beam found by solving the equation

$$EI{\phi_n}''''(x) - \ \bar{m}\omega_{bn}^2\phi_n(x) = 0 \tag{6}$$

and $q_n(t)$ is the generalized coordinate or modal participation coefficient. In Equation (6), ω_{bn} is the nth natural frequency of the beam without carrying a spring-mass system.

Substituting Equation (5) into Equation (1) and multiplying $\phi_m(x)$ on both sides and integrating over the whole beam

$$\begin{gathered} \int_0^L \bar{m}\phi_m(x)\sum_n \phi_n(x)\ddot{q}_n(t)dx + \int_0^L EI\phi_m(x)\sum_n \phi_n''''(x)q_n(t)dx = \\ \int_0^L \phi_m(x)f_1(t)\delta(x - x_1)dx + \int_0^L \phi_m(x)f_2(t)\delta(x - x_2)dx \end{gathered} \tag{7}$$

Considering the orthogonal property of mode shapes

$$\int_0^L \phi_m(x)\phi_n(x)dx = \begin{cases} 0 & (m \neq n) \\ \psi_m & (m = n) \end{cases} \tag{8}$$

the first term on the left-hand side of Equation (7) can be simplified as

$$\int_0^L \bar{m}\phi_m(x)\sum_n \phi_n(x)\ddot{q}_n(t)dx = \bar{m}\psi_m\ddot{q}_m(t) \tag{9}$$

Considering Equation (6) and Equation (8), the second term on the left-hand side of Equation (7) can be written as

$$\int_0^L EI\phi_m(x)\sum_n \phi_n''''(x)q_n(t)dx = \bar{m}\omega_{bm}^2\psi_m q_m(t) \tag{10}$$

When the spring stiffness k is large enough, the displacement and rotation of the mass can be approximated as

$$q_M(t) \approx \frac{1}{2}\left[\sum_i \phi_i(x_1)q_i(t) + \sum_i \phi_i(x_2)q_i(t)\right] \tag{11}$$

$$\theta(t) \approx \frac{1}{2l}\left[\sum_i \phi_i(x_1)q_i(t) - \sum_i \phi_i(x_2)q_i(t)\right] \tag{12}$$

Considering Equations (3), (11), and (12), the first term on the right-hand side of Equation (7) can be written as

$$\int_0^L \phi_m(x)\, f_1(t)\delta(x-x_1)dx = -\frac{M}{4}\phi_m^2(x_1)\ddot{q}_m(t) - \frac{J}{4l^2}\phi_m^2(x_1)\ddot{q}_m(t) + R_1 \tag{13}$$

where

$$\begin{aligned} R_1 = &-\frac{M}{4}\phi_m(x_1)\sum_{i\neq m}\phi_i(x_1)\ddot{q}_i(t) - \frac{M}{4}\phi_m(x_1)\sum_i \phi_i(x_2)\ddot{q}_i(t) \\ &-\frac{J}{4l^2}\phi_m(x_1)\sum_{i\neq m}\phi_i(x_1)\ddot{q}_i(t) + \frac{J}{4l^2}\phi_m(x_1)\sum_i \phi_i(x_2)\ddot{q}_i(t) \end{aligned} \tag{14}$$

Considering Equations (4), (11), and (12), the second term on the right-hand side of Equation (7) can be written as

$$\int_0^L \phi_m(x)\, f_2(t)\delta(x-x_2)dx = -\frac{M}{4}\phi_m^2(x_2)\ddot{q}_m(t) - \frac{J}{4l^2}\phi_m^2(x_2)\ddot{q}_m(t) + R_2 \tag{15}$$

where

$$\begin{aligned} R_2 = &-\frac{M}{4}\phi_m(x_2)\sum_{i\neq m}\phi_i(x_2)\ddot{q}_i(t) - \frac{M}{4}\phi_m(x_2)\sum_i \phi_i(x_1)\ddot{q}_i(t) \\ &-\frac{J}{4l^2}\phi_m(x_2)\sum_{i\neq m}\phi_i(x_2)\ddot{q}_i(t) + \frac{J}{4l^2}\phi_m(x_2)\sum_i \phi_i(x_1)\ddot{q}_i(t) \end{aligned} \tag{16}$$

Therefore, rearranging and rewriting Equation (7) gives

$$\left\{\bar{m}\psi_m + \left(\frac{M}{4} + \frac{J}{4l^2}\right)[\phi_m^2(x_1) + \phi_m^2(x_2)]\right\}\ddot{q}_m(t) + \bar{m}\omega_{bm}^2\psi_m q_m(t) = R_1 + R_2 \tag{17}$$

The mth natural frequency of the whole system which includes the beam and the spring-mass system can be approximated as

$$\omega_m^2 = \frac{\omega_{bm}^2}{1 + \left(\frac{M}{4} + \frac{J}{4l^2}\right)\frac{\phi_m^2(x_1) + \phi_m^2(x_2)}{\bar{m}\psi_m}} \tag{18}$$

From Equation (18), ω_m is directly relevant to the natural frequency of the beam and the mode shape data at the two connecting points.

2 NUMERICAL RESULTS

To verify the accuracy of Equation (18), the natural frequency of a simply supported steel beam carrying a spring-mass system is calculated using the dynamic stiffness method (DSM). The dimension of the beam is $1.2m \times 0.05m \times 0.02m$. ρ=7850 kg/m^3, E=200 GPa, and ν=0.3. For the spring-mass system, l=0.05m, $\tau = M/M_{beam}$, and $\varphi = J/J_{beam}$. $J_{beam} = 1.1307 kg \cdot m^2$ is the rotary inertia about the central axis of the beam.
When τ=0.1 and φ=0.001, assuming the spring stiffness is large, the natural frequency results are listed in Table 1.

Table 1: Natural frequency results when the spring stiffness $k = 10^6 \cdot EI/L^3$.

Mode	DSM results (rad/s)	Equation (18) results (rad/s)	Percentage error (%)
1	192.6584169	194.4136025	-0.91%
2	739.6493993	751.7465197	-1.64%
3	1700.760988	1705.455221	-0.28%
4	3128.143133	3110.617641	0.56%
5	4901.413572	4860.340063	0.84%
6	6982.350624	6938.343671	0.63%
7	9554.649659	9526.266524	0.30%
8	12502.05514	12442.47056	0.48%

3 CONCLUSION

Equation (18) shows that when the spring-mass system roves on a beam, the curve of natural frequency versus spring-mass system location would depend on the mode shape square, which is sensitive to local damage. This is a useful feature for natural frequency-based crack detection where an auxiliary mass or spring-mass system is involved [1, 2]. Natural frequency is known to be less sensitive to local damage than mode shape, while its measurement is more straightforward. Equation (18) shows it is possible to combine the advantage of natural frequency and mode shape.

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