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Processing mathematical thinking through digital pedagogical media: the spreadsheet.

A thesis

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Abstract

This study is concerned with the ways mathematical understanding emerges when mathematical phenomena are encountered through digital pedagogical media, the spreadsheet, in particular. Central to this, was an examination of the affordances digital technologies offer, and how the affordances associated with investigating mathematical tasks in the spreadsheet environment, shaped the learning trajectories of the participants. Two categories of participating students were involved, ten-year-old primary school pupils, and pre-service teachers.

An eclectic approach to data collection, including qualitative and quantitative methods, was initially undertaken, but as my research perspective evolved, a moderate hermeneutic frame emerged as the most productive way in which to examine the research questions. A hermeneutic process transformed the research methodology, as well as the manner in which the data were interpreted. The initial analysis and evolving methodology not only informed this transition to a moderate hermeneutic lens, they were constitutive of the ongoing research perspectives and their associated interpretations. The data, and some that was subsequently collected, were then reconsidered from this modified position.

The findings indicated that engaging mathematical tasks through the pedagogical medium of the spreadsheet, influenced the nature of the investigative process in particular ways. As a consequence, the interpretations of the interactions, and the understandings this evoked, also differed. The students created and made connections between alternative models of the situations, while the visual, tabular structuring of the environment, in conjunction with its propensity to instantly manage large amounts of output accurately, facilitated their observation of patterns. They frequently investigated the visual nature of these patterns, and used visual referents in their interpretations and explanations. It also allowed them to pose and test their informal conjectures and generalisations in non-threatening circumstances, to reset investigative sub-goals easily, hence fostering risk taking in their approach. At times, the learning trajectory evolved in unexpected ways, and the data illustrated various alternative ways in which

unexpected, visual output stimulated discussion and extended the boundaries of, or reorganised, their interaction and mathematical thinking. An examination of the visual perturbations, and other elements of learning as hermeneutic processes also revealed alternative understandings and explanations.

Viewing the data and the research process through hermeneutic filters enhanced the connectivity between the emergence of individual mathematical understanding, and the cultural formation of mathematics. It permitted consideration of the ways this process influences the evolution of mathematics education research. While interpretive approaches are inevitably imbued with the researcher perspective in the analysis of what gets noticed, the research gave fresh insights into the ways learning emerges through digital pedagogical media, and the potential of this engagement to change the nature of mathematics education.

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CHAPTER ONE: Introduction

He au kei uta e tae ate karo,
He au kei te moana e kore e taea.

You may dodge smoke on land,
But you cannot dodge current at sea.

Research Problem

In what ways is mathematical understanding reorganised when mathematical phenomena are engaged through digital pedagogical media; the spreadsheet, in particular?

Context

Information and Communication Technology (ICT) offers potential for transforming the nature of the learning process. The learning environment and the manner in which learners engage in tasks differ, with consequential variation in both learner activity and dialogue compared to other pedagogical media. The Internet, for instance, offers greater scope for child-centred, inquiry-based learning. It has enabled learners to connect with an extensive, eclectic array of information, opinion and expertise, albeit varying in quality. This variation itself has changed the emphasis of particular aspects of learning. Navigating these information pathways emphasises a different set of skills and ways of thinking, giving privilege to alternative approaches to learning. The need to evaluate, differentiate and synthesise becomes critical for the learner to discern the appropriateness of information.

Meanwhile, when mathematical tasks are encountered through ICT media the learner frames the interaction with the task from a distinct perspective. A digital pedagogical media might enhance or constrain the alternative, learning trajectory, and hence the learning experience, in particular ways. The affordances offered,

for instance, through the linking of symbolic, tabular and visual representations of the same phenomena, the virtually instantaneous response to the input of data, and the potential for visual reasoning (Borba & Villarreal, 2005; Smart, 1995; Tall, 2000) have all been identified and examined in various contexts, through a range of digital pedagogical media. These give rise to more generic entitlements: learning from feedback; observing patterns; seeing connections; working with dynamic images; exploring data; and 'teaching' the computer, which have also been recognised as opportunities students can expect through engaging school mathematics through ICT media (Johnson-Wilder & Pimm, 2005).

As the ICT metamorphosis is rapid, it is hard for researchers to evaluate the effects on learning, in contemporary settings. The time lag between the dissemination of research findings, coupled with the synthesis of various studies required to build a meaningful picture of the influence of ICT in the learning process for mathematics education, and the rapidly changing, commercially driven nature of software and hardware development can lead to the technology being superseded before a coherent analysis of its implications has emerged. Yet it is critical that this research is undertaken, so a pattern of implications can evolve, and be recorded. It is also true that generalisations may emerge. This study is part of an extended examination of the ways using spreadsheets as the pedagogical medium for investigating mathematics might restructure the learners' understanding of mathematical ideas. It is, however, situated in the broader frame of using digital pedagogical media in mathematics education generally.

In mathematics education, dynamic geometric software (DGS), graphic calculators, function plotters, statistical analysis software, computer algebra systems (CAS) and spreadsheets have the potential to revise the way various mathematical concepts can be presented and accessed. Yet while the formatting and capacity of these has been enhanced over time, the ways they are utilised in the learning process has evolved slowly and intermittently. There is certainly superb practice occurring, but digital technology doesn't permeate all feasible learning opportunities, and is often utilised to support traditional modes of learning. Progress is being made, momentum is gathering, and issues of equity

and access are recognised, but actual penetration into mathematics classroom practice is still generally limited.

Using the internet offers diverse opportunities for learners to engage in specific interactive applets and software, as well as collaborative approaches to data collection and problem solving (Sinclair, 2005), and this networking facility offers further possibility with alternative ways of learning in mathematics. The formation and evolution of new teacher and learner communities that can interact in a more rhizomatous network structure, beyond the relatively homogeneous environment of a classroom, school or local community, gives opportunity for richer, more diverse global perspectives in mathematics education. It provides the potential for making sense of, or generalising, in a different way. More recent, pedagogically or curriculum influenced developments e.g., interactive whiteboards and tinkerplots offer further scope, but again the transition from experimentation, to research of practice, to widespread presence into classrooms, with the associated reflective cycles, takes time to evolve.

The utilisation of these various tools, and the corresponding potential to change both the teacher's and the learner's approach, have critical ramifications for assessment. The primacy given to this aspect, whether diagnostic, formative, or for high-stakes qualifications, has tended to focus research on the effects of the technology on learning outcomes.

However, little research has been undertaken into the actual way digital technologies influence the learning *process*, as compared to the *outcomes* of learning. Critically, in light of discussion on the manifestations of socio-cultural perspectives and the hermeneutic processes in the mathematics classroom, more research would enrich the emerging picture of how individuals negotiate mathematical meanings in different learning environments, in this case using the spreadsheet as a tool for mathematical investigation. If the experience is different, the dialogue evoked is different, and the connections are conceived in different ways, in what ways might the individual understanding differ? This study is an attempt to gain further insights into how understanding might emerge

when learners engage in mathematical tasks through the pedagogical medium of the spreadsheet. Central to this question is how their learning trajectories might differ in this learning environment, and the generalities of the learning experience with other digital pedagogical media.

Therefore, a primary aim of this research was to develop an account of how doing mathematical investigations with spreadsheets influences understanding in particular ways. A key intention also was to investigate how various pedagogical media, and the mathematical discourses with which the learner filters the mathematical investigation, frame the emerging understanding. Fundamental to this was the differentiation of learners' language in a spreadsheet context. Consideration was given to how this might have influenced the negotiation of understanding. Another objective of the study was to expound a theoretical account of how learning might emerge in school mathematics. This underpinned another significant aim, which was to build a theoretical framework through which perspectives of enculturation and individual interpretation might be reconciled using the interpretation of student approaches when working with digital technology, in particular, spreadsheets. From these aims, the following research questions emerged.

Research Questions:

- How can we understand learning processes when students use spreadsheets to investigate mathematical problems?

- In what ways might the experiences differ from investigating mathematical phenomena through other pedagogical media, and how does this influence a student's learning trajectory?

- How does understanding emerge when the learning trajectories evolve in particular ways, and mathematical problems are investigated through the pedagogical medium of the spreadsheet?

- What are the commonalities in the learning experience, when mathematics investigation is engaged in through spreadsheets, as compared to other digital technologies?

- How might investigating mathematical phenomena through digital pedagogical media produce alternative conceptualisations of the mathematics involved?

Before embarking on chapters rendering the detailed inspection of existing associated research literature, and the methodology employed to examine these questions, there is benefit in introducing them so as to further contextualise the perspectives taken for this research thesis. A concise account of the transformative research process I underwent, followed by an overview of the literature was designed to assist the reader in interpreting the position taken with regard to these perspectives.

A research trajectory: a brief outline

The initial research proposal for the study, focussed on an eclectic approach to the methodology, data collection and analysis. A mixture of qualitative and quantitative approaches was originally envisaged, in pursuit of a rich tapestry of data for analysis, with a sense of validity provided by consensus, or triangulation of the findings. Participants' output and dialogue were recorded and transcribed, assessments were engaged in, interviews and surveys undertaken, observations made in situ, personal narrative and reflections were written, statistical testing undertaken, and data entered and sorted by *NVIVO* software. While several stories were beginning to emerge from the data, engagement with broader theoretical literature, and a growing disquiet about the likelihood of being able to reveal a fulsome story through these lenses, led to the adoption of a more interpretive frame.

The dichotomy evident in my perspective, its ensuing atomisation, then interplay between the evolving viewpoints associated with the two paradigms (quantitative

and qualitative) led to the examination of broader philosophical interpretations of social science research. Ratiocination needed a self-inflicted disturbance to illuminate the way forward. Engagement with these broader viewpoints enabled my own perspective to be re-envisioned. A brief traverse of the reflective personal narrative this literature evoked, illustrated this unhinging, then the subsequent refocusing. Schostak (2002) described this transformative process as a need to reframe a research project according to modified rationales. These new perspectives emerged from the exploration of the literature, the research process itself, or an ensuing combination. The researcher's selection and discussion of the literature, the methodology and analysis are manifestations of what the researcher *chose* to notice (Mason, 2002), with these *choices* framed by his/her prevailing discourses at that particular time. The space the researcher occupied at various junctures of the research process was therefore critical as the reader interprets the researcher's own evolving interpretation.

While the theoretical literature and its examination are described from the perspective that emerged through the research process, there is recognition in the first two Results chapters of the marking of these transitory positions. These two chapters (Chapters Five and Six) attend to the initial results with the intention of articulating the stories and themes that emerged, and how these features shaped the evolution of the methodology, and ongoing analysis. They also historically situate the various perspectives that I held at each particular point of interpretation, and the way those perspectives might have influenced the evolving research process. The methodology and a chapter examining this researcher evolution also detail this perceptual shift. The data were then reconsidered from these fresh theoretical perspectives, with various layers of interpretation likewise emerging. In effect, I was engaged in a hermeneutic process of research. A hermeneutic circle was evoked by the act of investigation. The data were examined through the lens of my prevailing discourses, instigating a modification of this perceptual frame. The data were then re-examined from this fresh perspective, with my interpretations, and the space from which these interpretations were drawn, evolving with iterations of the cycle. How this is manifest in the research will be evident as the thesis unfolds.

Overview of the literature

There are two substantial bodies of literature that require analysis with regards to the research questions. The first is the literature surrounding the research pertaining to ICT, in various manifestations, as a medium for mathematical exploration. The second is centred on learning theories, how they are situated within broader interpretive, social science research discourses, and how the learning process in mathematics, as interpreted in this study, is positioned within those learning theories.

Research into the way dynamic geometry software, such as *Cabri-geometry*, shapes students' understanding of geometric concepts (e.g., Laborde, 1995); the influence of CAS on learning in algebra (e.g., Kieren & Drijvers, 2006); the suitability of spreadsheets for visualisation of number patterns (e.g., Calder, 2002), and an interactive approach (e.g., Beare, 1993) has been undertaken. Yet, there is a scarcity of research on how, as a media for exploration, the spreadsheet might influence the dialogue, the investigative pathway, and hence the understanding, of students. This is a key focus of the study. Digital technologies offer new perspectives on the engagement of learners and the ways they might actually negotiate their understanding. Reconciling their use with appropriate learning theories is also central to this study. The focus is on how understanding might be shaped, when spreadsheets were used as a tool for exploring mathematical problems.

The place of discourse and the way understanding evolves through the differing media is pivotal to this. This also involves theoretical perspectives such as hermeneutics (e.g., Ricoeur, 1981), its relationship with education (e.g., Gallagher, 1992), and with mathematics education (e.g., Brown, 2001), allied with pedagogical perspectives that have evolved from interaction in the ICT environment per se. The processes involved with learning, including an examination of conceptualisation and what a 'concept' might be, and how it emerges, is also critical to this undertaking. Learning mathematics, as Brown (2001) contends, is "a perpetual state of becoming, governed through the social discourses, enacted through the individual" (p. 173). In this ascribed

interpretation of learning, ‘concepts’ are not fixed realities from which we peel the outer layer to reveal their entirety, but are more elusive, formative processes that become further enriched as learners use their temporary fixes to view events from fresh, ever evolving perspectives. The objectification of knowledge is a progressive process of noticing; an active, creative, and interpretive social process that surfaces through the interaction of a range of elements such as language, symbols, and artefacts (Radford, Bardini, & Sabena, 2007). In essence the mathematical task, the pedagogical medium, the preconceptions of the learners, and the dialogue evoked are inextricably linked. It is from their relationship with the learner that understanding emerges. This understanding is the learner’s interpretation of the situation through those various filters.

The evolution of perspectives and interpretations is not confined to the perceptual shifts associated with the emergence of an individual’s understanding. As well, the cultural formation of mathematics evolves as mathematics is envisaged in varying forms when it is engaged through alternative filters. Objectivity may be conceived through consensus, but it “represents a perceived stability of ideas, not a permanent state of being” (Confrey & Kazak, 2006, p. 319). Mathematics itself is not a collection of fixed conceptualisations and defined processes but more an historically situated, socially negotiated interpretation that transforms under the gaze of those alternative filters. If one of those filters, the pedagogical medium, provides alternative tones in the perceptions of mathematics, it is reasonable to assume that the range of mathematical experiences would also reorientate the interpretation of what mathematics might be. A key premise, central to this version of the ways understanding emerges and transforms, is that mathematics (including school mathematics) evolves from socio-cultural processes. It is an interpretation of action, interaction, and the associated reflection through which understandings emerge. Those interpretations are not fixed but are formative. Each engagement with them leads to fresh interpretation at the individual level, but also to some extent, the individual engagements extend the boundaries of, or enrich, the broader generalised mathematics discourses. The shifts in perspective at a personal level resonate in the broader understanding of mathematics, although usually their influence on the broader discourse of mathematics is miniscule.

In a similar manner, the evolution of mathematics education research is transformed to some extent through the evolution of an individual mathematics education researcher, along with the simultaneously modifying interpretations of mathematics. The notions of mathematics, mathematics education, and mathematics education research are inextricably linked in both their specific and generalised versions. They influence and are influenced by each other. The layering and sedimentation of the researcher's approach to the generation of knowledge echoes of, and is echoed by, the evolution of mathematics education research per se. Each shift in the researcher's approach resonates in the shifts in how research is subjectified with respect to alternative discourses. This underlying premise was influential in the account of this present research study as reported in this thesis.

Returning to the underpinning research problem:

What is the nature of learning when mathematical phenomena are engaged through digital pedagogical media; the spreadsheet, in particular?

There are several areas of research literature that require review in the exploration of this research problem. These are an examination of:

1. Current research and practice in the utilisation of ICT generally in mathematics; including official expectations, the ways ICT is integrated into teaching and learning programmes, and how the ensuing reorganisation of thinking might influence the understanding of mathematics.
2. Current research into the utilisation of spreadsheets in mathematics education. This will include the nature of their use, and how this shapes the learning process.
3. Perspectives on how understanding evolves in mathematics education.
4. The nature of mathematical investigation.

The next two chapters comprise a review and discussion of the literature that informs these four constituent aspects. Chapter Two considers digital technology as pedagogical media. It contemplates the scope and nature of digital technology in mathematics education, and considers how engaging in mathematics through various digital pedagogical media might fashion the learning in particular ways, including the ways that spreadsheets might influence students' learning trajectories and understanding. Chapter Three, meanwhile, is concerned with learning in mathematics education. It includes a description of the ways various contemporary philosophical perspectives in the social sciences influence the educational theory landscape, the manner in which mathematics education is situated within these broad philosophical positions, and how approaches to learning in mathematics education might traverse various manifestations of these theories. A discussion of the various versions of hermeneutics as they relate to learning in mathematics and the production of knowledge through the research process generally, is also expounded. This incorporates perspectives of the hermeneutic circle and how this applied to the learning process in mathematics as well as the research process and the evolution of the research methodology. An illustration of the hermeneutic circle as manifest in the examination of the data is also considered here.

Chapter Four describes the methodology of the study, and discusses the research methods employed and justification for their choice. It is here, as well, that the intertwined weave of connectedness between the emergence of mathematical understanding for individuals, the cultural formation of mathematics, the transformation of my theoretical frame to a moderate hermeneutic perspective, and the evolution of mathematics education research is discussed. The manner in which these elements interacted and are mutually constitutive of each other is further elaborated upon.

Chapters Five and Six, the first two chapters reporting the results and associated discussion follow. These results emerged from the evolving landscape that comprised the researcher's methodology. While the data in these chapters were not entirely perceived from the final methodological perspective that was settled on, the sifting and shaping of this data into these stories related directly to the

research questions. The stories that emerged were also central to the subsequent analysis when the moderate hermeneutic lens was evoked, and as such were both constitutive and influential in that process. Furthermore, they historically situate the evolution of perspective as I underwent a transformative hermeneutic cycle through the process of research. On both accounts they are fundamental to the examination of the research questions. Chapter Five gives an account of the observational data. It uses the data to illustrate key stories that emerged, related to the nature of the learning experience, and the consequential influence of the pedagogical medium on students' learning trajectories and understanding. Chapter Six considers the interview, problem challenge, and questionnaire data through an analogous lens to that applied to the observational data. The findings in these chapters were not examined to the extent that would be warranted if these methods were perceived as the lenses that would most productively reveal insights into the investigation of the research questions. The data were organised into the stories that emerged, with the associated discussion considered in terms of the influence of these stories on the interpretations and understandings of the students. The ways those influences shaped my interpretations and consequently the transformation of the research trajectory and subsequent analysis are also outlined. The findings here were not examined with deep incision, nor with comprehensive links drawn from in depth theoretical viewpoints. Rather they were described, analysed, and historically situated so that they portrayed themes in the data and the perspectives that I held at particular points of interpretation.

Chapter Seven examines the personal transformative process that I underwent as the researcher. As the researcher's perspective evolves, what they notice in the data evolves too. The interdependence and co-evolution of data and methodology position the researcher's perspective and explanations at various junctures, and as such are constituent parts of the data and analysis. Hence, the examination of my evolving research trajectory is central to the investigation of the research questions. How this personal transformation is formative (to some small extent) in the evolution of mathematical education research through iterations of interpretation is also considered. The three chapters that follow report the re-examination of the data through the theoretical frame and methodology that had emerged. Each describes and analyses interpretations borne of the initial

examination of the data in Chapter Five and Six, but with the fresh eyes evoked by the moderate hermeneutic lens. Further data were also collected from these alternative perspectives so as to inform and enrich the evolving interpretations. Chapter Eight is concerned with the setting and shaping of sub-goals in the investigative process when it is filtered by the pedagogical medium of the spreadsheet, and the distinctiveness of this experience. Chapter Nine analyses the visualisation element of engaging mathematical phenomena within that particular environment and gestures towards the notion of visual perturbation, the focus of Chapter Ten.

While these three chapters identify and consider particular elements of the students' engagement through the spreadsheet medium, they are nevertheless specific versions of localised hermeneutic processes. They are also influenced by, and influential of each other. A change in the learning trajectory brought about by the actual output being different to the expected output (a visual perturbation, e.g., 9.22337E+18 in Chapter Ten) is also indicative of the learner resetting an investigative sub-goal. In Chapter Eleven, the researcher draws on the previous chapters to articulate the conclusions that the research has illuminated in response to the research questions, and the central themes that have emerged. Consideration is also given to the constraints and limitations of the research and possible directions for future research that the study has revealed.

In rejoinder, in a skeletal version of the story the thesis accounts, the researcher, through the frame of the underlying discourses in the associated areas, posed initial questions, engaged with literature and in dialogue, hence interacting in a manner that fashioned a research trajectory. Reflection on these processes revealed several potential approaches to investigate the identified research questions. After the formal process of developing the proposal and gaining ethical approval was negotiated, an eclectic array of data were collected and analysed. When juxtaposed with my ongoing, evolving perspective of methodology and the way mathematical understanding emerges, some of this analysis was perceived as problematic. The perspective and theoretical frame from which the data were viewed, transformed through the ongoing interplay between the data, a broad range of social science research perspectives,

interpretation of the ways mathematical thinking and understanding might evolve, and reflection initiated through both dialogue and the writing process. A moderate hermeneutic lens was evoked. The data, including some that was subsequently collected, were then considered through fresh eyes with the analysis and interpretations re-envisioned. On returning to the data from this modified viewpoint, further insights into the stories the data revealed were perceived, with alternative conclusions drawn as a consequence. These alternative conclusions in turn influenced my perspective of research methodology, which thus continued to evolve through the iterative cycles of interpretation.

The following chapter, Chapter Two, examines literature related to the ways the learning experience, and by implication the students' understandings, might differ when mathematical phenomena are engaged through digital technologies, the spreadsheet in particular. It concerns research in mathematics education that informs the study with regards to how digital technologies might act as pedagogical media and influence and filter students' understanding.

CHAPTER TWO: Digital technologies as pedagogical media: How knowledge is reorganised through ICT activity.

Ko te pae tawhiti whaia kia tata,

Ko te pae tata whakamaua kia tina

Seek out the distant horizons,

And cherish those you attain

Introduction

The emergence of information and communication technology (ICT) media in classroom practice, has arguably transformed the way mathematical ideas are encountered in schools. Access to many key elements of school mathematics has been altered as initially calculators, and then more advanced computer software and hardware, offered new ways in which certain constructs might be created and understood. The notion of a mathematical function, for instance, will be understood differently if it emerges from applying a rule; plotting ordered pairs as Cartesian points manually; developing relationships between the dragging function and its visual effects in *cabri-geometry*; developing relationships between physical phenomena; using spreadsheets to explore numerical patterns; exploring families of geometrical transformations with the draw functions of *Microsoft Word*; or linking symbolic and graphical data in a CAS environment. Russian psychologist, Tikhomirov (1981), when discussing how computers affect cognition, argued that in the early stages of their implementation, using computers led to a reorganisation of thinking. He saw the computer playing a mediating role in learning similar to that of language in a Vygotskian perspective. These roles, with the mediators functioning as regulators of understanding through engagement and reflection, are not unrelated or independent, however; the process varies when the pedagogical medium is different. This conception of

the reorganisation of thinking, underpins Borba and Villarreal's (2005) humans-with-media model for the use of digital technologies in mathematics education.

Early exponents of ICT in mathematics education such as Tall (1985), with his graphical approach to calculus based on visualisation, recognised the pedagogical potential offered by a digital pedagogical medium. They saw opportunity for the re-envisioning of approaches to learning, and the development of environments where mathematical ideas could be explored in more interactive, flexible ways. This initial impetus was promptly followed by the emergence of both content and pedagogically driven initiatives in software design, as mathematics educators recognised the affordances offered by these learning environments. Others saw the mathematical potential in software, such as spreadsheets, which were designed for other purposes but offered rich environments for mathematical exploration and thinking. Particular characteristics of using spreadsheets such as an interactive approach, and the propensity to link multi-representations of mathematical phenomena, are examples of particular affordances or opportunities ICT avails. The progressions in software design, coupled with rapid developments in hardware and peripheral devices, have maintained the evolution of the diverse array of digital technologies that could potentially be integrated into mathematics classroom practice.

The literature indicates there are a number of ways in which ICT might be incorporated into mathematics programmes:

- Calculators, with graphics calculators having the ability to graph statistical data, functions, and their respective transformations, incorporate versions of CAS and dynamic geometry software (DGS), as well as a range of computational functions.
- Programmed mathematical environments, suitable for exploring specific mathematical areas e.g., *Logo* or *The Geometers Sketchpad* for geometry, and CAS for algebra. More recently these have incorporated several areas and enabled links between them e.g., *Autograph*.

- Microworlds, often constrained, well-defined versions of the above e.g., *LogoGrid*, but also specifically designed environments e.g., *Numbers*, that focus on localised sets of mathematics ideas.
- Internet sites, that range in form from being sources of problem-solving activities and solutions for students, to brokers for teacher planning and resources, to information conduits for sharing in mathematics education research, to distance learning environments, to succinct visual applets or exploratory environments for specific mathematics topics, to interactive games or activities that might input global data, unconstrained by localised grouping and learning contexts.
- Generic tools, which lend themselves to the investigation of mathematical problems, e.g., spreadsheets.
- Computer Aided Instruction (CAI), which typically is in the form of skill development programmes, sometimes embedded in a game context e.g., *Logical Journey of the Zoombinis*.
- Interactive whiteboards, incorporating active screens with built in programmes, e.g., *Autograph*, internet access, and interplay with input devices such as student controlled tablets.
- Other digital peripheral devices that are used for communication e.g., cell-phones, ipods, or input e.g., heat sensors.

The following section examines the literature related to the research and transformation of practice when incorporating ICT into mathematics programmes, with a particular emphasis on the position of geometry and dynamic geometry software in this evolution. This prefaces, and indicates, the affordances offered by using digital technologies, which are subsequently considered along with the ways they might shape learning trajectories in particular ways. The notion of humans-with-media, which accounts for integrated collectives that influence the learning process, will then be discussed, as will the various official stances on the inclusion of ICT in school mathematics education programmes. The chapter will conclude with an examination of a range of literature related to current practice when using spreadsheets in mathematics programmes. This section will situate the focus on the use of spreadsheets as pedagogical media within the broader framework of the previous sections.

ICT as pedagogical media: Thinking in geometry

Geometry, with its visual and construction elements lending themselves to an interactive approach, was one of the first mathematical areas to see the potential of, and embrace digital technology as a pedagogical environment. Early proponents of using computer technology in mathematics education, such as Papert (1980), whose seminal work *Mindstorms: Children, Computers and Powerful Ideas*, advocated children developing their mathematical thinking through programming in a geometrical programming package, *LOGO*, provided the catalyst for rich practical classroom experiences and the beginnings of associated mathematical research. Early material in this area e.g., Ainley and Goldstein (1988) identified both pre-constructed figures and students' construction of their own figures as pedagogical means to privilege particular features of mathematical situations with regard to *LOGO* procedures. Aspects of this early research are still pertinent to, for instance, Johnston-Wilder and Pimm's (2005) differentiation of the *exploratory* and *expressive* approaches. The *exploratory* mode is when pre-constructed documents invite the learner to explore ideas within their constrained parameters. This approach is currently manifest, for example, in the form of *applets* that are accessible on the internet or dynamic geometric figures within DGS packages, such as *cabri-geometry.* The *expressive* mode is when students create their own figures or files from scratch, enabling them to express their mathematical thinking. The distinction between these approaches and the appropriateness of their utility are still under consideration (Mackrell & Johnston-Wilder, 2005).

LOGO is described as a first-wave ICT (Sinclair & Jackiw, 2005), and as such is seen to have provided an individual learning experience rather than one associated with school geometry. One feature of this differentiation is the accentuation of non-Euclidean forms of geometry. *LOGO* produces Euclidean figures but the conceptualisation evolves through syntonic or body geometry.

Movement and time are incorporated in this process, with motion an integral part of its defining state (Stevenson, 2006). As such, it mediates forms of geometry that are not in the school curriculum. It still provides a rich dynamic learning experience and the potential to reorganise thinking in school geometry, and is used in comparative studies examining the ways geometry ideas are internalised e.g., Hoyos (2006). A. Neyland (1994) argued that not only did this mathematical learning environment develop understanding of content, it was particularly effective in the facilitation of the process strand, most notably developing logic and reasoning. He noted that students learnt to think logically through a progression of steps, and used iterative processes.

The dynamic geometry software (DGS) most commonly occurring in school geometry, *Cabri-geometry* and *Geometer's Sketchpad,* utilises Euclidean geometry. They use a more external dynamism in that the learner moves figures or their features on the screen. In both, the learner constructs Euclidean diagrams, and examines the logical dependencies between figures and associated points, and the corresponding relationships (Laborde & Laborde, 1995). The learner can interact directly in a dynamic manner with the figures they have created, or that have been created for them, through the movement of the mouse. This facility, coupled with the ability to animate figures that have long been in static 2-dimensional form (Mackrell & Johnston-Wilder, 2005) set DGS apart from pencil-and-paper technology as a pedagogical medium, and facilitate the reorganisation of thinking in geometry. A circle, for example, is understood differently according to whether it is constructed using a pencil and compass, a template, *Cabri-geometry* or *LOGO*. The notion of the circumference being equidistant from the centre, for instance, might be more obvious when *Cabri-geometry* is used compared to constructing a circle using a template.

Studies involving the dynamic geometry software, *Cabri-geometry*, (Mariotti, 2002; Mariotti & Bartolini, 1998) employ the Vygotskian (1978) notion of semiotic mediation to link technical tools to the process of internalisation. Semiotic mediation is the way in which we learn to assign meaning and to internalise that meaning. A number of studies (e.g., Mariotti, Laborde, & Falcade, 2003) have focussed on the analysis of particular attributes of *Cabri-geometry* (dragging

facility, commands available, etc.) as instruments of semiotic mediation that the teacher might utilise to introduce and conceptualise mathematical ideas. The functionality properties of the spreadsheet (*Fill Down*, use of formulae, etc.) might also be considered as potential tools for semiotic mediation of the mathematical concepts of patterning and generalisation. It follows that conceptualisation of mathematical phenomena, will be different when engaged through the particular software lens. Mariotti, Laborde and Falcade (2003) contend, for instance, that a function can be conceptualised differently using *Cabri-geometry*. Other researchers have likewise reported on the development of relational thinking when learners engage in geometry activities through DGS. Jackiw and Sinclair (2006), when discussing the learning of grade 3-to-5 children as they engaged in activities involving dragging, described how the exploration with movement enabled students to gain some understanding of continuity and abstract relations. The dynamic nature of the medium enabled understanding to emerge in unexpected ways, with the learning trajectories evolving differently than with pencil-and-paper methods. This is an aspect we will pursue with other digital technologies with, for the purposes of this study, spreadsheets considered in particular.

Other dynamic geometry environments (DGE), such as *Cabri 3D* have also been found to enhance students' ability to visualise when modelling physical constructions and motion (Mackrell, 2006). She contends that through the use of an integrated approach, including interactive demonstrations and 'pictures', visualisation helped in the emergence of 'new mathematics'. Others, for example Leung, Chan and Lopez-Real (2006), have identified how the dragging mode in a DGE system, perceived as an artifact, contributed to the conceptualisation of geometric ideas. Strasser (2006), likewise identified the drag mode, in conjunction with the macro-functionality of DGEs, as offering ways of learning geometry that are not available in pencil-and-paper environments. It is the dynamic visualisation of screen objects in these environments, operated on by dragging or the use of manipulative tools, which most significantly differentiate them from engaging in geometric thinking through other media (Mackrell & Johnston-Wilder, 2005).

Placing the emphasis on the visualisation dimension of geometry has opened opportunities for the design of software that enhances those qualities. While

developing software that makes the construction and manipulation of geometric objects in three-dimensional space possible (*3DMath*), the key elements of visualisation were privileged. Jones, Christou, Pittalis & Mousoulides (2006) reported on this process, and how covering mental images, external representations, and the means and potentialities of visualisation were given significance. This is designing software to deliberately shape the learning process in a particular way, and as such recognises the influence that the pedagogical media has on the interpretation and organisation of meaning. Conversely, pedagogical approaches can also evolve to reflect the affordances of the media. In his research with secondary-aged students, Lew (2006), suggested four stages to solve construction problems in a dynamic geometry environment, and maintained that this approach, based on a systemised analysis method, improved understanding of didactic proofs. The four stages are: recognition of the problem's conditions and goals; the analysis of what is to be solved; synthesis of that analysis in the construction of a proof; and finally reflection on the process as a whole. What was evident in Lew's research is that the exploratory nature of the interactive approach that the software afforded, along with interaction between the participants, facilitated students' understanding and reorganised their approach to deducing proofs. This reorganisation of knowledge, evidenced by the interaction between participant and medium, and the dialogue between participants, is central to this study.

Prevalent in much of the recent literature involving digital technologies in geometry (as well as in other areas, notably CAS) is the notion of instrumental genesis (e.g., Jackiw & Sinclair, 2006; Leung, Chan, & Lopez-Real, 2006; Mackrell, 2006; Strasser, 2006). In this, the DGS, behaving as a cognitive tool, is seen as an extension of the mind. Instrumental genesis arises from the instrumental approach (Rabardel, 2002) and its differentiation of an artifact and an instrument. In this version of tool use, the instrument is more than an object, but encompasses the techniques and individual mental schemes that evolve through the use of the tool and social interaction. These aspects guide both the way the tool is used and the user's thinking. Instrumental genesis is the process that describes this transition from an artifact to an instrument. Included in this notion is the symbiotic relationship between the user and the instrument: while the user's knowledge channels the way the tool is utilised, the affordances and constraints of the tool

influence the user's approach, their learning trajectory, and by implication, the nature of their understanding. It has also been applied to the relationship between the learner and the integration of CAS in mathematics programmes (e.g., Artigue, 2002; Lagrange, 2005) and with spreadsheets (Haspekian, 2005). It is of interest to this study from the perspective of how the affordances of the spreadsheet might influence the learners' approach, their dialogue and hence mathematical understanding, and how this instrumental genesis process might be part of the hermeneutic circle the learner is engaged in as their understanding evolves. While in many respects, research into the use of ICT in geometry has fore-shadowed software design, classroom practice, and the broader theoretical discussions regarding digital technologies as pedagogical media, it is nevertheless only a part of the story, albeit one that prefaces the overall discussion by introducing several of the key characters and settings. In the next section, the discussion moves to an examination of the literature where ICT is used to enhance other mathematical thinking.

ICT as pedagogical media: Other mathematical learning

Some graphics calculators have included DGS as part of their operational repertoire. Graphic calculators, due to their greater affordability than personal computers, and handheld operation, have enabled digital technology to be more easily accessed in classroom situations. They allow more flexibility and mobility in classroom organisation. They can be used for the manipulation and graphing of functions and data, while more powerful, recent versions have also included CAS capabilities. Goos, Galbraith, Renshaw, and Geiger (2000), found that the graphics calculator facilitated personal and public knowledge production. Used to engage in mathematical activity, graphic calculators also operated as conveyors of data and processes, and student partners and collaborators, to become, in conjunction with other influences such as the teacher, mediators to enhance conceptual understanding. Kieren and Drijvers (2006) identified the co-emergence of technique and theory when junior high school students engaged in algebra learning through a CAS environment. They reported on how this, in conjunction with the communication evoked, shaped the understanding.

Changes in classroom practice and affective areas such as perceived peer status have been reported in Malaysian settings where graphics calculators were utilised (Kee and Sam, 2006). However, there are areas of caution. Gardiner (2001), discussed the effects of graphic calculators and a computer algebra system (CAS) on students' manual calculation skills, and warns of possible negative effects. Heid and Edwards (2001), however, describe positive transformation in classroom activity and student behaviour through the use of CAS in the learning programme.

Recent research in New Zealand (Ministry of Education [MOE], 2006), found that the use of CAS enabled graphics calculators with junior secondary school students (13 to 15-year-olds) led to a shift in classroom pedagogy, towards a more investigative, student-centred approach. Both teachers and students reported a greater emphasis on understanding rather than applying rules and procedures, with the incorporation of more interactive, collaborative type of activities. They acknowledged that this change of emphasis was not dependent on the CAS environment, but that the availability and appropriate use of the CAS digital technology had enhanced the influence of the exploratory pedagogical approach, and the students' understanding. While these conclusions are informative, rather than directly applicable to the research undertaken, they do indicate that using ICT in mathematics programmes is consequential and does affect the learning process, while highlighting the need for ongoing research, particularly in the primary school context.

Meanwhile, spreadsheets have been found to offer an accessible medium for young children tackling numerical methods. With the potential to simultaneously link symbolic, numeric, and visual forms, they have been shown to enhance the conceptualisation of some numerical processes (Baker & Biesel, 2001; Calder, 2002). Here visualisation bridges the concrete and abstract manifestations of mathematical experiences. While some mathematicians contend that mathematics itself is evolving through its interaction with computers (Devlin, 1997; Francis, 1996), there is no consensus amongst them regarding this point. Borba and Villarreal (2005) argued that ICT emphasises the visual aspect of mathematics, and

changes the status of visualisation in mathematics education. The positive role visualisation plays in supporting conceptual understanding has frequently been advocated (Bishop, 1989; Dreyfus, 1991; Dubinsky & Tall, 1991), but visualisation has often been considered as secondary, or supportive, of a symbolic, analytical, or algebraic conceptualisation. There is growing evidence, however, that visual reasoning is itself legitimate mathematical reasoning (Borba & Villarreal, 2005). In studies (e.g., Julie, 1993; Smart, 1995; Villarreal, 2000) involving students using graphic calculators and computer software, ICT mediated the mathematical understanding, and a visual approach to reasoning was identified. The researchers also contend that this visual reasoning, initiated by interacting with the mathematics through an ICT medium, extended students' mathematical conceptualisation: "…they employed their visual knowledge to help make generalisations and solve any new problems. In doing so, they extended their mathematics beyond what was expected by the teacher and the textbook" (Smart, 1995, p. 203).

Higgins and Muijs (1999) with respect to numeracy, identified two strands of software development, one from a behaviourist approach which focused on the practice of specific numerical skills, frequently in a game context, and the other from a more investigative approach which emphasised understanding of number. They found no conclusive evidence that either had a direct impact on primary children's attainment in numeracy, and concluded that effective use of ICT would, like effective teaching, at times require use of either strategy, depending on the specific lesson objectives or the particular focus for part of a lesson.

There is much research at the secondary or tertiary level, that focuses either on computer-aided instruction packages (CAI) used to develop specific skills, or software that allows rich exploration opportunities in particular content areas, for example *The Geometers Sketchpad* and *Cabri-geometry* in geometry, the use of CAS and various function plotters in algebra and calculus, or *Tinkerplots* and *Minitab* in statistics. While some conclusions from these studies can be applied generically and are informative, it is not usually appropriate to apply their findings directly to the primary classroom. This can be illustrated by reference to Cretchley, Harman, Ellerton, and Fogarty's (1999) study of the use of *MATLAB* with students studying first year university mathematics papers, where they found

that the students had strong preferences for developing the understanding away from the computer on paper and using the software to confirm and extend their understanding. This may also apply to primary-aged children developing their mathematical understanding, but the context is significantly different, so drawing direct comparisons would be unwise. Likewise, their findings that the majority of students valued the greater clarity of understanding, and the ability to visualise and compute more easily, would seem to be applicable to enhancing mathematical understanding with the use of spreadsheets, but research needs to be undertaken in the primary setting for these conclusions to be valid.

Other researchers have found a link between the use of ICT and the development of understanding in mathematics. For example, Gentle, Clements, and Battisa (1994) have suggested that it increased the construction of higher-level conceptualisation in geometry, and Zbiek (1998) found that it enhanced students' ability to model mathematically. Chance, Garfield, and delMas (2000) reported that visualisation through the ICT medium enhanced understanding of sampling distributions, but that pre-requisite knowledge affects students' ability to learn from technology. They believed that understanding was facilitated most fully with an eclectic approach; that students needed to experience a variety of activities. Other researchers have stressed the importance of the teacher's role: to integrate computers with non-computer learning experiences, facilitate reflection, and provide the necessary scaffolding to assist the student's construction of knowledge (McRobbie, Nason, Jamieson-Proctor, Norton, & Cooper, 2000). Tall (2000), while acknowledging that the use of ICT and its effect on mathematics is at a very early stage of its evolution, found that a graphic approach to calculus, developed in the right way, led to understanding of the most subtle of formal concepts. He felt that the whole approach to some aspects of mathematics was on the verge of a revolutionary transition, which could lead to greater insights into mathematics education and change in the nature of mathematics itself. For instance, he reported that a graphic approach to calculus offered insights into far deeper ideas about differentiability.

Throughout this discussion of research into the ways digital technologies have facilitated the reorganisation of mathematical thinking, the specific situation of

the research has constrained the findings. Yet there are common themes that emerge; commonalities in the ways that tasks are engaged in, dialogue is facilitated, and learning is framed. In the next section consideration is given to some of these characteristics and their relationship with the learning process.

Affordances of digital technology: Potentialities for action

To consider how learning and learning trajectories might differ when mathematical phenomena are engaged through digital pedagogical media, characteristics of that engagement need to be examined. In what ways might the learning experience be different from engaging mathematical phenomena with other media such as paper-and-pencil? This section focuses on some common affordances that ICT offer across a range of platforms and software. It considers research undertaken across a broad range of settings with varying ages and mathematical areas, where the participants used a diversity of digital media including CAS, dynamic geometry software (DGS), the internet, spreadsheets, and games. While the specificity of the particular context is significant to the findings of each, there were common themes that emerged across contexts, with a variety of digital technologies.

Affordances in a digital environment are the opportunities that the environment offers the learning process. They may facilitate or impede learning. Brown (2005) identified the facilitation of an exploratory approach, providing multiple strategies, and the promotion of dialogue when three affordances offered to year 11 students as they engaged mathematics tasks in a technology-rich teaching and learning environment. Affordances are a potential for action, the capacity of an environment or object to enable the intentions of the student within a particular problem situation (Tanner & Jones, 2000). We might consider them as perceived opportunities offered through the pedagogical medium in relationship with the propensities and intentions of the user.

Affordance implies the complementarity of the learner and the environment. They are not just abstract physical properties (Gibson, 1977), but the potential

relationships between the user and the 'artefact' (Brown, 2007). Important in this discussion is the symbiotic relationship between the digital media and the user. While the digital medium exerts influences on the student's approach, and hence the understanding that evolves, it is his/her existing knowledge that guides the way the technology is used, and in a sense shapes the technology. The student's engagement is influenced by the medium, but also influences the medium (Hoyles & Noss, 2003).

One aspect that has often been associated with digital environments is the notion of multiple representations. The ability to link and explore visual, symbolic, and numerical representations simultaneously in a dynamic way has been recognised extensively in research. Borba and Confrey (1996), for example, contend that this aspect, particularly for topics including functions, facilitates the co-ordination of established representations, enriching the conceptualisation, and the way functions were understood. Ainsworth, Bibby, and Wood (1998) suggested that multiple representations promote learning for the following reasons: (a) they highlight different aspects; hence, the information gained from combining representations will exceed that gained from a single representation; (b) they constrain each other, so that the space of permissible operators diminishes; and (c) when required to relate multiple representations to each other, the learner has to engage in activities that promote understanding. Meanwhile, Sacristán & Noss (2008) illustrated how the engagement of computational tasks in a carefully designed microworld might lead to different representational forms (such as visual, symbolic and numeric); a process that they called *representational moderation*. In a large-scale study involving students doing problem solving in classrooms where a range of digital technologies were available, Santos-Trigo and Moreno-Armella (2006) found that students' construction of mathematical relationships was enhanced. They also identified how using dynamic software generated particular questions that facilitated the development of conjectures. Others, such as Tall (2000), also considered multiple representations when discussing attributes of using digital media that influenced understanding. Multiple representations, through interactive digital environments such as applets, and the designing of games have also enhanced the learning process (Boon, 2006; Confrey, Malone, Ford, & Nguyen, 2006). Boon (2006) reported on

the development and use of java applets in the Netherlands, while Confrey et al. used multi-representational software with children in under-resourced schools, to develop mathematical ideas and thinking, as they constructed their own animated games. The aim was to develop their proficiency in the underpinning mathematics so as to enable them to eventually pursue study of advanced mathematics.

Associated with this affordance is the idea of visualisation. While the debate is inconclusive as to the positioning of visualisation in mathematics (e.g., Jorgenson, 1996; Thurston, 1995), there is greater consensus regarding the positive role of visualisation or graphic approaches in the facilitation of understanding in mathematics education (Baker & Biesel, 2001; Calder, 2004b; Dreyfus, 1991; Olive & Leatham, 2001; Villarreal, 2000). Similarly, in various studies involving DGS, the dynamic, visual representations enhanced the understanding of functions (e.g., Mariotti, Laborde, & Façade, 2003). In a study of students' understanding of key aspects of geometric transformations when engaged with *The Geometer's Sketchpad,* Hollebrands (2003) reported the development of deeper understandings of transformations as functions.

Digital technologies can also manage large amounts of realistic data more easily than pencil-and-paper technology, allowing students to more easily explore social and political debates through a mathematical lens (Ridgeway, Nicholson & McCusker, 2006). They can remove elements of simple, repetitive computation so that more in-depth thinking and consideration of over-arching issues can be undertaken (Deaney, Ruthven & Hennessy, 2003; Ploger, Klinger & Rooney, 1997). They often allow the learner flexibility to quickly rearrange information and re-engage with activities from fresh perspectives (Clements, 2000). In an ongoing study of how primary school-aged students solve problems using spreadsheets, Calder (2005) has described how the particular nature of the spreadsheet environment framed the emergence of subgoals in the investigative path.

The learners' pre-conceptions, both mathematical, and of the medium, appeared to influence the approach they have taken to using the digital technology. Chance

et al. (2000) found that visualisation through a digital medium enhanced understanding of sampling distributions, but that pre-requisite knowledge affected students' ability to learn from technology. They also concluded that the facility of digital media to immediately test and reflect on existing knowledge was an influence on the learning process. This is consistent with other findings (e.g., Beare, 1993; Deaney et al., 2003). The almost instantaneous nature of the response in a digital environment, coupled with the interactive nature of the engagement, allows for the ease of exploration of ideas. Discussion is stimulated, as the results of prediction or conjecture are viewed rapidly and are more easily compared. This enhances the emergence of logic and reasoning as students investigate deviations from expected output, or the application of procedures. Students also required greater accuracy when applying procedural structures, to be more explicit with entering mathematical manipulations (Battista & Van Auken Borrow, 1998).

Others have indicated that these affordances, when facilitated appropriately by the teacher, may lead to students exploring powerful ideas in mathematics, learning to pose problems, and create explanations of their own (e.g., Baker, Geerheart & Herman, 1993; Sandholtz, Ringstaff & Dwyer, 1997). They reported improved high-level reasoning and problem solving linked to learners' investigations in digital environments. In a study of grade three children using spreadsheets to explore fractional number problems, Drier (2000) reported that the students reinforced and extended their rational number knowledge, while exploring many mathematical concepts in an integrated manner. Ploger et al. (1997) concluded from their study that students learnt to pose their own problems and create personal exploration through investigating in a digital environment. Gentle et al. (1994) suggested they increased the construction of higher level conceptualisation in geometry.

Tension, evoked when expectation of output conflicted with pre-conceptions, also promoted a productive form of learning. Drijvers (2002) contends that cognitive conflicts that arose when high-school students used CAS when learning algebra, became an opportunity to enhance learning, rather than impede understanding. In a study involving tenth grade students learning algebra in a

task-based CAS environment, Kieren and Drijvers (2006) reported a relatively seamless integration of technique and theory, and significantly that some of the most productive learning occurred when the CAS techniques produced data that conflicted with the students' expectations. They did qualify this with the requirement that teachers needed to manage the process appropriately, for this tension to enhance learning. Discussing mathematical thinking, when using digital images, Mason (2005) maintained that the selection and undertaking of a particular action is monitored in relation to the response meeting the expectation. When the expectation is not met, the tension might provoke further reflective engagement. He warned though of the need for space for the dissonance to emerge, cautioning that this may not occur when the transition between images is too frequent. In this regard, an advantage of working in an exploratory digital environment is that the cognitive conflict is predominantly non-judgemental (Calder, 2007). Calder also reported on the initiation of learners' informal conjectures in a spreadsheet environment, when the visual output produced unexpectedly differed from the ouput that was anticipated.

Attributes, such as the interactive nature of the engagement and the multi-representation of data, coupled with appropriate teacher intervention, enable the learner to not only explore problems but to make links between different content areas that might otherwise have developed discretely. They allow students to model in a dynamic, reflective way, and enhance students' ability to model mathematically (Borba & Villarreal, 2005; Zbiek, 1998). They also foster risk taking and experimentation (Calder, 2002), allowing space for students to explore. This exploration requires some scaffolding, however as it may not occur spontaneously. The visual image may provide the stimulus, but it is the subsequent thinking that is key to the learning process. Imagining consequential possibilities are part of that response. Mason (2005), further contends that: "When surprise is encountered imagination mobilises further powers to explain or make sense of what has happened" (p. 225). Others view the integration of reflective, analytical thinking with a more intuitive, creative approach as being necessary for the enhancement of powerful mental conceptualisation (Meissner, 2006).

Using the internet offers diverse opportunities for learners to engage in specific interactive applets and software, as well as collaborative approaches to data collection and problem solving (Sinclair, 2005), and it is the affordances made available through this networking facility that offer further potential with ways of learning in mathematics. The formation and evolution of new teacher and learner communities that can interact in a more rhizomatous network structure, beyond the relatively homogeneous environment of a classroom, school or local community, gives opportunity for the development of richer, more diverse global perspectives in mathematics education, and the potential for making sense of, or generalising, in different ways. The internet provides the core conduit for interaction that might evolve centrally, or alternatively from communication from or between nodes outside the central initiatives, allowing the sporadic emergence of new central clusters of co-learners. Using the internet to explore ideas and communicate has also been noted in early childhood settings, with children working on an integrated unit on energy indicating the enjoyment of working with digital tools, and the opportunities for mathematics exploration the unit allowed (Yelland, 2005).

The effect on student engagement and motivation when using ICT in school mathematics programmes has also been noted. Higgens and Muijs (1999) found much work pertaining to the positive effects on motivation and attitude, and while this enthusiasm might relate to the novelty factor initially, it can't be ignored, given the correlation between students' attitudes to learning in mathematics, and their understanding. Other researchers have likewise found positive motivational effects through using digital technologies in mathematics programmes (e.g., Hoyles, 2001; Kulik, 1994, in his meta-analysis of computer based learning; Lancaster, 2001; Sandholtz, et al., 1997; Schacter & Fagnano, 1999). Calder (2001), in a research report to the MOE, likewise noted the positive motivational effects on students of integrating ICT into a mathematics programme. Anthony Neyland (1994) in his discussion of *LOGO,* observed that it promoted high levels of concentration and self-motivation.

The almost instantaneous nature of the response with ICT, once something has been thought through and the data entered, has the potential to facilitate learning

in mathematics. It allows for relative ease when exploring ideas in problem solving (either numerically or visually), and stimulation of discussion as the results of prediction or conjecture are viewed so rapidly, allowing them to be more easily compared. This aspect facilitates the development of logic and reasoning through the above, with students promptly seeing the effects of gaps or errors in their logic or application of procedures. Chance et al. (2000) found that "the establishment of cognitive dissonance appears to be a crucial component to effective interaction with technology, providing students with the opportunity to immediately test and reflect on their knowledge in an interactive environment" (p. 30). Shifting the computational responsibility to the computer also allows the learner to explore and focus more on conceptual understanding.

The notion of entitlement describes the opportunities students can expect through engaging school mathematics through ICT media. Six major opportunities were identified by Johnston-Wilder and Pimm (2005): learning from feedback; observing patterns; seeing connections; working with dynamic images; exploring data; and 'teaching' the computer. They illustrated each of these with specific examples across a range of contexts. As mentioned previously, they also discussed two approaches; the *exploratory* mode with pre-planned documents, and the *expressive* mode where students create their own documents to express themselves mathematically. These two approaches to task design and learning, need to be considered alongside the appropriateness of particular software to the learning experience. Each approach may need to be utilised at particular times, as the situation may require some structured direction to develop particular content or the medium's operative functions, while on other occasions, an open exploratory space with students creating their own versions of models within the environments would be best suited to optimise the thinking.

In their discussion on understanding and projecting ICT trends in mathematics education, Sinclair and Jackiw (2005) considered the impact of three waves of development with the future emphasis on relationships amongst learners, their immediate environment, and the world beyond the classroom. The writers contend that by attending to the roles future ICT might play in the relationships of those involved in individual and group learning situations, these future ICT

will be more meaningfully integrated into classroom culture.

Digital technologies, if used appropriately, enable mathematical phenomena to be presented and explored in ways which afford opportunities to initiate and enhance mathematical thinking, and make sense of what is happening. They allow the learner potential to look through the particular to the general (Mason, 2005). When the learning experience differs with digital technology, we can assume that learning trajectories and understanding will also differ. The examination of this notion is central to this thesis. The digital technology doesn't operate in isolation, however. Its influence is inextricably linked to the pre-conceptions of the user, other societal and cultural discourses, and the nature of the learning process. In the next section we examine a version of how these contributing aspects might facilitate a reorganisation of mathematical thinking.

Humans-with-media

Borba and Villarreal (2005) discussed the notion of humans-with-media, which they see as collectives of learners, media (in various often collaborating forms) and other environmental aspects e.g., mathematical phenomena, other humans, other technologies. This notion will be briefly examined and then situated with digital technologies as pedagogical media, and the perspective taken on learning in mathematics. They utilised a Tikhomirov (1981) perspective that claims the computer plays a mediating role, in the reorganisation of thinking, and thus understanding. This mediating role is comparable, but not the same as Vygotsky's idea (1986) that language mediates thinking. Borba and Villarreal (2005) saw understanding emerging from the reconciliation of re-engagements of the collectives of learners, media and environmental aspects with the mathematical phenomenon. They viewed these collectives in a dynamic way where the collective not only influences the approach to the mathematical phenomena, but is itself transformed by that engagement. "In our perspective, the experiences with computer technology, and the co-ordination of these experiences with other media, reorganises thinking and transforms, in a recursive way, different human-with-media collectives" (Borba & Villarreal, 2005, p. 167).

As each engagement re-organises the mathematical thinking, and initiates a fresh perspective, this in turn transforms the nature of each subsequent engagement with the task. This also suggests that the process is ongoing, and echoes the hermeneutic circle. The humans-with-media notion seems to be a manifestation of the predominant mathematical discourse. It is the collective of objects/ideas from which the mathematical discourse in a particular domain emerges. This provides the lens through which the mathematical task is engaged. The engagement with the task, and the tension or opportunities this evokes, reorganise the thinking through the ensuing dialogue and action, what they say and what they do (Ricoeur, 1981). This transforms the discourse and hence the humans-with-media collective.

Borba and Villarreal's (2005) observation, that this process is recursive, is also indicative of the cyclical process of the hermeneutic circle, as the learner oscillates between the part (mathematical phenomenon/activity) and the whole (humans-with-media collective). Although they described the transformations of the humans-with-media collective as being recursive, they contend that each of these transformations also results partially from the experiences (engagement) with computer technology. The subsequent co-ordination with other media (including oral dialogue), that reorganises the thinking (changes the perspective) leads to this transformation. Implicit then is the contention that this engagement and reorganisation of thinking is also ongoing and self-repetitive, at least until some reconciliation is reached.

Borba and Villarreal (2005) are, therefore, alluding to the medium as being significant in the reorganisation of thinking and, as a consequence, understanding. They contend that because of the sometimes unpredictable nature of the learner's interpretive perspective, "media, therefore, condition the way one may think, but do not determine the way one thinks" (p. 16). The computer technology influences the engagement and ensuing dialogue in particular ways, that lead to a reorganisation of the learner's prevailing discourse in that domain. The learner through self-reflection, through dialogue with others, or a combination of both, then resets their sub-goal and re-engages with the task from the newly situated perspective. This iterative process continues until there is

resolution of some form. The iterative process of the hermeneutic circle and the ensuing evolution of understanding resonate of the humans-with-media notion and the corresponding reorganisation of mathematical thinking.

While research, exemplary classroom practice by enthusiasts, commercial development, or a combination of these can provide the impetus for change in overall classroom practice, policy and funding are critical in the change process at a national level. Some official perspectives on the use of ICT in mathematics are now briefly considered, before the spotlight is directed to the use of spreadsheets as pedagogical media, the particular focus of this study.

Official perspectives

Various educational and political institutions advocate the inclusion of ICT in mathematics classroom practice. The current New Zealand mathematics curriculum document (MiNZC) assumed ICT will be available and used at all levels in the teaching and learning of mathematics (MOE, 1992). It also maintained that computer software, such as graphing packages and spreadsheets, enables students to focus on the mathematical ideas rather than on routine computation, and presented effective environments for mathematical experimentation and open-ended problem solving (MOE, 1992). The revised New Zealand Curriculum (MOE, 2007), which is mandatory from 2009, also accentuates the potential of ICT in general to "assist with the making of connections by enabling students to enter and explore new learning environments" (p. 36). It indicates the possibilities of initiating or joining learning networks beyond the confines of the classroom, the enhancement of learning through the affordances ICT offers by, for example, saving time, and how it might open up novel, alternative approaches to learning.

In the United Kingdom (UK), the Smith report (DfES, 2004) *Making Mathematics Count,* emphasised the imperative to incorporate the use of ICT into the teaching and learning of mathematics. It recommended that teachers be "fully informed about the role and potential of ICT to enhance the teaching and learning

of mathematics, and have access to state-of-the-art software" (p. 122). In Singapore, the revised junior college mathematics curriculum implemented in 2006, specifically identified graphics calculators, amongst other digital technologies, as important in the teaching and learning of mathematics, particularly in advanced level topics. State educational administrative bodies in the US and Canada (e.g., Ontario Ministry of Education, 2005) emphasised the desirability of incorporating ICT into school mathematics programmes. Teacher associations, such as the National Council of Teachers of Mathematics (NCTM) in the USA, and their Australian and New Zealand equivalents (AAMT and NZAMT, respectively), advocate the integration of ICT into classroom practice also. Researchers in other European e.g., France, Italy, Norway; American e.g., Mexican, Brazil; and Asian countries e.g., China, Korea, have likewise identified either legislated increased emphasis on utilising ICT in mathematics education through gazetted curricula, or in-depth, large-scale government-funded research into aspects of its use. At a recent International Commission into Mathematics Instruction (ICMI) study conference, Technology Re-visited (2006), partially funded by UNESCO, researchers from every continent, including delegates from thirty countries, reported on investigations into the use of ICT in mathematics education. The official approach, both political and institutional, is global.

While MiNZC places an expectation that technology will be utilised in the learning of mathematics in New Zealand classrooms, and specifies the use of both calculators and computers, observation in schools suggest that ICT is still only used intermittently in classroom mathematics and only usually when the teacher has the knowledge, confidence, accessibility and inclination to actually incorporate it into their programme. The lack of such knowledge, confidence and so forth constitutes a considerable impediment to overcome, and acts as a disincentive to change teacher practice. Burns-Wilson and Thomas (1997), for example, identified inadequate teacher professional knowledge in this area and teachers' lack of confidence in using technology with the appropriate mathematics content as the most significant barriers. Thomas, Tyrrell and Bullock, (1996) found in their research into the implementation of computers into classroom practice, that the teachers' overriding concern was regarding their use in mathematics education, rather than the actual use of the computer. This also

highlighted the need to develop teachers' experience and a range of resources in this area. More recent professional development for teachers in ICT has included subject-specific, as well as software-specific workshops (e.g., Calder, 2000), and classroom practitioners have predominantly signalled the need for this change. As recently as 2002, only 25 percent of British schools were reported as utilising ICT effectively in teaching mathematics (Ofsted, 2002). The immersion into classroom practice then has been erratic and governed to some extent by teacher knowledge and intention. There is substantial, highly successful practice occurring though, momentum is growing, and shifts in the nature of media, including pedagogical media, pervade the way we live. The next section examines literature associated with using spreadsheets in mathematics education.

Spreadsheets in mathematics education: current research and practice

Spreadsheets have given mathematicians and mathematics students a tool to extend the capacity and speed of computation. This has enabled students to better focus on the underlying mathematical ideas rather than on routine mathematical manipulation (MOE, 1992). They allow for the exploration of mathematical concepts and problems in different ways. Students can, for instance, explore optimisation problems by quickly calculating and scanning a range of inputs in a logical, sequential manner, compared to a more time-consuming guess-and-improve approach or the use of calculus. This exploration leads to a more intuitive conceptualisation in a numerical context, that the student is already familiar with, and later helps develop understanding of the more procedural, algorithmic, calculus approach.

While there is a reasonable amount of research into using spreadsheets with secondary mathematics students (e.g., Masalski, 1990; Russell, 1992), there is a scarcity of research involving primary-age pupils. Ploger, Klinger and Rooney (1997) investigated the use of spreadsheets in developing algebra thinking in a fifth grade class. They found that children learnt to pose problems and to create their own explanations while using spreadsheets to explore powerful

mathematical ideas. Unencumbered by numerical computation involving large or decimal numbers, and using formulae in meaningful ways, the young children gained access to the predictive quality of algebraic thinking. This allowed them to pose rich 'What if...?' questions.

Other aspects of the mathematics education potential of spreadsheets are succinctly summarised by Beare (1993) who concluded that:

> *Spreadsheets…have a number of very significant benefits many of which are now apparent. Firstly they facilitate a variety of learning styles which can be characterised by the terms: open-ended, problem orientated, constructivist, investigative, discovery orientated, active and student centred. In addition they offer the following additional benefits: they are interactive; they give immediate feedback to changing data or formula; they enable data, formula and graphical output to be available on the screen at once; they give students a large measure of control and ownership over their learning; and they can solve complex problems and handle large amounts of data without any need for programming … (p. 123).*

These attributes, coupled with appropriate teacher intervention, enable the learner not only to explore problems, but to make links between different content areas that might otherwise be developed discretely. They allow students to model in a dynamic, reflective way. Funnell, Marsh and Thomas (1995, p. 231) contend that: " by interacting with a computer programme which, as well as showing some of these different algebraic, linguistic and graphical representations, actively encourages students to relate one to the other through investigation, may assist them to construct linked mathematical cognitive structures".

While acknowledging that spreadsheets were designed for accountancy or financial purposes rather than mathematics education, S. Johnston-Wilder and Pimm (2005) nevertheless argued that spreadsheets offer important facilities to enhance mathematical teaching. The visual and interactive elements of working in a spreadsheet environment as well as the ability to explore number patterns,

solve equations both numerically and graphically, operate on and transform vast amounts of data, and then represent them graphically for analysis are, they contend, particular affordances of the spreadsheet environment. Monaghan (2005) identified the use of *iterative refinement* as an element of thinking algebraically that the spreadsheet is particularly suited to. Meanwhile, P. Johnston-Wilder (2005) while discussing spreadsheets use in statistics acknowledged its usefulness, but warned of its propensity to mislead with novice learners in this area due to structural aspects of the graphing process and the requirement that the student aggregate the data within frequency tables before graphing.

Fuglestad (1997) studied the use of spreadsheets in 10- to 14-year-old, Norwegian students' understanding of and performance with decimal numbers. She found that once a few basic skills were developed in the functioning of spreadsheets, the major part of the students' work and their discussion was about their understanding of decimals. The children made some exciting discoveries, particularly in the areas of multiplying and dividing by decimals.

Similar advantages have been found in the development of algebraic thinking (Ploger, et al., 1997). The use of a spreadsheet allowed children to explore number patterns algebraically. Their earlier work (1996) with children generating number patterns and times tables, demonstrated how children could see the consequences of algebraic transformations on familiar numbers. Healy and Sutherland (1991), and Battista and Van Auken Borrow (1998) also found children working on spreadsheets in a familiar numerical context, while operating with algebraic reasoning, facilitated the development of algebraic thinking. Both studies advocated teacher intervention to encourage reflection on the meaning and effects of syntax, to ensure that children develop their thinking beyond the simple procedural stage. Other researchers have identified how the use of spreadsheets in the preliminary stages of algebra courses enhanced conceptual understanding of equations and their solutions (e.g., Tabach & Friedlander, 2006). They advocated that spreadsheets be utilised in mathematics programmes beyond the investigation of variation and patterns, but also in the areas of relations and transformations.

Battista and Van Auken Borrow (1998) described three levels of sophistication in thinking about number procedure: performing it, abstract application to numerous cases and thirdly, a multifaceted understanding that allowed students to reflect on, decompose and analyse the numerical properties. This decomposition and reconstruction of numerical quantities is the beginnings of algebraic thinking. The part-whole strategies described in various numerical frameworks (e.g., Fuson, 1992; MOE, 2001; Steffe, 1992; Wright, 1998) likewise utilise this decomposition and reconstruction of numerical quantities and certainly seem to also describe the beginnings of algebraic thinking.

The use of the spreadsheet as a tool for problem solving to explore situations that contain number patterns, facilitates the development and writing of formula to develop those patterns. This direct application of procedures to a prescribed spreadsheet methodology, coupled with the immediate feedback given, also develops children's algebraic thinking. Healy and Sutherland (1991), after working over four years with pupils in classrooms, strongly advocated the use of spreadsheets in the development of algebraic thinking. They believed that much of the seemingly difficult algebraic concepts could be engaged within a spreadsheet environment, particularly the idea of negotiating and expressing a generalisation.

While those who support the use of spreadsheets to develop algebraic thinking describe the generalisation of numerical patterns as a key aspect of that development, there are aspects of numeracy and number investigations that are also suitable for exploration using spreadsheets. Several mathematics education researchers (e.g., Baker & Biesel, 2001; Drier, 2000; Hyde, 1998; Manouchehri, 1997; Sgroi, 1992), have utilised spreadsheets to help children develop a better understanding of various numerical concepts such as equivalent fractions and exponential numbers, and in doing so have gained some insights into the way children's understanding develops.

The speed of a computer's response to the input of data facilitates their suitability for developing mathematical reasoning. When students can observe a pattern or

graph so rapidly after they input some values, they develop freedom to explore variations and, particularly with teacher facilitation, learn to make conjectures, then pose questions themselves. This facility to immediately test predictions, reflect on outcomes, then make further conjectures, not only enhances the students' ability to solve problems and communicate mathematically, but it develops logic and reasoning as students investigate deviations, or the application of procedures. Chance et al. (2000) found that this, coupled with the speed of computation, allowed the learner to concentrate more on conceptual understanding. Baker et al. (1993) and Sandholtz et al. (1997) also reported improved high-level reasoning and problem solving linked with this capability.

Motivation was an aspect of learning experienced by students when computers were integrated into their mathematics programmes. For some, the pure novelty of the learning experience in a fresh context seemed to allow them to break from the constraints of their previous accumulation of mathematics learning, some or all of which may have been negative. For others, there is the intrinsic motivation that is fostered by the capabilities the spreadsheet allows the learner; that is, the potential to investigate complex problems in a reflective manner, to see visual representations of data simultaneously with symbolic forms, and the interactive nature of computer usage per se. Several research studies into the use of spreadsheets in classroom programmes have identified this motivational aspect for students (e.g., Drier, 2000; Funnell et al., 1995; Healy & Sutherland, 1991; Manouchehri, 1997; Orzech & Stetton, 1986). Where motivation is based superficially on novelty, its sustainability would be limited if the spreadsheet (as advocated) was always available as a tool for problem solving.

Giving the learner the scope to visualise both in tabular and graphical form clearly gives the spreadsheet a major advantage as a learning tool. Baker and Biesel (2001) found some advantage to a visual instructional style, modelled by spreadsheet usage, in their investigation of how children best understand averages. NCTM has also advocated the use of spreadsheets for their support of a visual instructional style (NCTM, 2000). Olive and Leatham (2000), in their work with pre-service teachers, found that most thought visualisation was the most beneficial aspect of students using computers. Lemke (1996) maintained

that visual-graphical representations available in software such as spreadsheets have the potential to allow students to develop mathematical concepts and relationships. McRobbie et al. (2001) contend that the representation of information in both textual and visual forms offered when using spreadsheets had the potential to provide a multi-media environment, which allowed more effective learning. Seeing an immediate change to a graph, when a table value is altered, is certainly a powerful method of imaging the relationship between the two.

Researchers have identified other benefits that spreadsheets offer within investigative approaches. These include its interactive nature (Beare, 1993), its suitability for linking concepts (Funnell et al., 1995), and its capacity to give immediate feedback (Calder, 2004b). Others (e.g., Ploger et al., 1997) allude to this propensity to foster an investigative approach in developing algebraic thinking. They have found, significantly, that young students learn to pose problems and to create explanations of their own. Manouchehri (1997) reported similar findings, while Wilson, Ainley, and Bills (2004) contend that spreadsheets give opportunities for the conceptualisation of algebraic variables.

These aspects, coupled with the speed of response to inputted data, appear to give learners opportunities to develop as risk takers. Students made conjectures and immediately tested them in an informal, non-threatening, environment. This permitted the learners the opportunity to reshape their conceptual understanding in a fresh manner, to reorganise their mathematical thinking. Improved high-level reasoning and problem solving linked with this capability have been reported in more general research into using ICT in mathematics (Baker et al., 1993; Drier, 2000; Sandholtz et al., 1997). The capacity to provide instantaneous feedback also allows for conjectures to be immediately tested and perhaps refuted. The spreadsheet medium supported the investigation in a particular way as this attribute enabled the participants to set, and then reset sub-goals, as they worked their way through the investigation (Calder, 2005). The spreadsheet enabled different kinds of examples to be tested, compared and contrasted, within a particular frame.

Battista and Van Auken Borrow (1998) noted another positive attribute of students developing spreadsheets to solve problems, namely greater accuracy in computation and the application of procedural structures. They claimed that having students create then enter mathematical procedures into a spreadsheet environment required them to be more explicit than they might usually be. There is a need to balance the development of spreadsheet skills to enable entry into the spreadsheet environment, with the development of mathematical thinking. Burns-Wilson and Thomas (1997), Healy and Sutherland (1991), and M. Neyland (1994) all acknowledged that for the students to eventually work independently with the spreadsheet as a tool, they initially required an orchestrated sequence of skill development embedded in mathematical contexts. The aim should be for this approach to quickly be replaced by appropriate mathematical problems that facilitate the use of spreadsheets, and for the skill development to then be only driven by need, that is, for the approach to undergo a transition from the *exploratory* to the *expressive* mode. As Funnell et al. (1995) found, "Initially teachers should not expect the students to invent and develop their own spreadsheets but, as they gain experience and gradually build up skills, this could become possible" (p. 233). For the spreadsheet to be an influential pedagogical medium with investigative approaches to learning mathematics, this would certainly be desirable.

Summary and Implications

The literature has suggested several key aspects with implications for this study in terms of the nature of ICT and spreadsheet usage in contemporary classroom settings, and influenced how other researchers have hypothesised and reached conclusions on the ways learning is conditioned in these differing contexts. While each of the studies was informed by historically and socially situated contexts there were, nevertheless, common features to the affordances digital media offered the learner. While these have been discussed in more detail in previous sections, they are worthy of synthesising into a brief, succinct summation.

Both the visual (Borba & Villarreal, 2005; Calder, 2002; Laborde, 1995), and interactive (Funnell et al., 1995; Mackrell, 2006) nature of the media have contributed to the shaping of mathematical understanding in a distinct manner which is different from pencil-and-paper approaches. The research also indicated that the propensity to see and engage with multi-representations of data (numerical, symbolic and visual), to manipulate and transform large amounts of realistic data, and to foster the links between content areas promoted the learner's use of prediction, conjecture making and problem posing. The speed of response to inputted data, allowing the results of prediction or conjecture to be considered more rapidly, stimulated discussion and encouraged risk taking and experimentation. The dynamic nature of the environments and the enhanced ability for students to model, allowed the learner flexibility to rearrange information and re-engage with investigation from fresh perspectives. The literature reported that these enabled the facilitation of higher-level conceptualisation, developed logic and reasoning, and extended mathematical thinking, across a broad range of levels and contexts. Enhanced levels of motivation were likewise reported in a diverse range of situations, with several researchers noticing the digital media gave the students a large measure of control over the learning process. Significant too, was the contention that the teacher played a critical role in the emergence of understanding, and the frequently noted symbiotic relationship between the medium and the user. Importantly, the literature gave an account of the way digital technologies, acting as pedagogical media, allowed the learner to envisage the mathematics in a different way. They facilitated the reorganisation of mathematical thinking and pedagogical knowledge.

The *affordances* and *entitlements* facilitated by the use of ICT, and spreadsheets in particular, in mathematics programmes differentiate the learning experience from those engaged through other pedagogical media. How the learning experience and the nature of understanding are different is central to this thesis. If the same stimulus evokes a range of social interactions and dialogue when approached through varying pedagogical lenses, and if understanding is negotiated through the sense making of that dialogue, it is reasonable to conclude that the mathematical thinking and understanding will differ also. The analysis of

the dialogue, in conjunction with concurrent mathematical understanding, and the activity the learner engages in, may reveal if this epistemological thesis has validity. It will at the least enhance the limited body of research and understanding in this particular sphere.

CHAPTER THREE: Learning Theories in Mathematics and their Broader Constituent Influences

E kore te totara e tu noa I te parae engari

me tu I roto I te wao-nui-a-Tane

The totara tree does not stand alone in the field,

but stands within the great forest of Tane

Preamble

Human behaviour has long been the object of study and speculation, yet the formalisation of the human sciences into a coherent, recognised body of disciplines has been more recent, and its gestation fraught with political and philosophical contradiction. The great social philosophers, their debate, and related ongoing reflective commentary, contribute implicitly to the emerging theories of learning. This chapter begins with a brief description of a hermeneutic perspective on the learning process. The purpose of this section is to indicate the researcher's viewpoint, which can then be situated within the discussion of the broader perspectives from which it is constituted. These broad epistemological notions are formative and influential in the emergence of the hermeneutic frame employed in the examination of the research questions. An intention was to accentuate the connectedness of these influences.

The chapter threads a theoretical trail; from broad social science philosophical beginnings, through the influence of various educative referents, leading to a brief discussion of how research framed by constructivist and socio-cultural discourses has influenced mathematics education research. In particular, the acquisitional theoretical frame of Piaget, and Vygotsky's socially orientated one,

are situated within literature associated with the ongoing evolution of learning theories in mathematics (Sfard, 1991). A section investigating how those positions might be reconciled or enriched through a hermeneutic lens follows, with an analysis portraying the researcher's contention that a hermeneutic interpretive lens provides a productive filter for analysing the material generated. Central to how this might manifest through the activity of the participants in this study, is an understanding of the hermeneutic circle, and the ways the data illustrated this process. Hence, a discussion of literature concerned with the educational implications of this notion is incorporated. The literature surrounding the nature of mathematical investigation, in which the participants engaged, is also considered.

The chapter provides an examination of the literature surrounding the hermeneutic perspective that frames the thesis, as well as a discussion of literature around the more wide-ranging discourses that inform the consideration of learning theories. These social and historical discourses are pervasive in the cultural evolution of learning theories in mathematics education, and are central to the examination of hermeneutics when it is envisioned in an educative sense. We never escape those socio-cultural discourses of tradition and authority that police the boundaries of more localised perspectives. They are interwoven and influential in the version of hermeneutics employed in this research to gain insights, and to better understand, the ways new knowledge emerges. The chapter begins with a discussion of the literature associated with those broader influences, with each subsequent section informed by the previous one as the discussion threads a pathway from more expansive positions, through other formative influences, and increasingly refined interpretations of the hermeneutic perspective, to the illustration of the hermeneutic circle. Firstly, the brief overview of hermeneutics is outlined.

Hermeneutics is understood as the theory of interpretation of meaning, and in a classic sense is drawn from the context of the written medium. More recently though it has been invoked in the sense making used in the interpretation of language per se. While it was traditionally perceived in relation to the interpretation of text, Ricoeur (1981) rationalised spoken and written language

through the definition of dialogue or discourse; “It is as discourse that language is either spoken or written” (p. 197). It is not that they are the same, but that they have commonalities. There are a range of historical and philosophical positions that help situate hermeneutics, but a common strand is that in the process of interpretation no one facet exists in isolation. Each, whether author, text, listener, meaning, etc. has its own cultural, sociological, historical elements that fashion the interpretive process.

Conservative hermeneutics contends that the aim of interpretation is to transcend historical bias and replicate the author‘s intended meaning; an objective interpretation. Proponents would argue that through rigorous application of techniques, the author’s intended meaning can be extracted. A moderate perspective of hermeneutics, however, not only acknowledges the influences of time and space, but also those of the conditioned prejudices that are embedded in language. “They are the changing biases of various traditions which are not past and bygone but are operative and living in every reader and every text” (Gallagher, 1992, p. 9). As the interpreter, we are constrained by our own language, but also by the language of the author, and the discourses that pervade both of these influences. “Understanding is always under the influence of history” (Gallagher, 1992, p. 90).

Central to this interpretive process is the hermeneutic circle. This describes the process of the interpreter moving cyclically from the part to the whole, then back to the part and so forth, until some manner of resolution or consensus emerges. It is the circularity between present understanding and explanation, where the explanation gives rise to a change in perspective, which in turn evokes a new understanding (Brown, 2001). Within the learning context, the whole can be aligned with the various discourses or schema the learner brings to the situation, and the part with the specificity of the situation they confront (perhaps in the form of a particular learning activity). The learner’s engagement oscillates between their prevailing discourse and the activity. With each of these iterations their perspective alters, and as they re-engage with the activity from these fresh perspectives, their understanding evolves.

Moderate hermeneutics and the hermeneutic circle will be examined more closely at a later stage in the chapter, but first the work of seminal social science philosophers inherent to these notions is considered.

Broader views of reality

By viewing mathematics education through an interpretive lens, there is acknowledgement of two fundamental aspects of interpretation: firstly, that there is an historically situated, socio-cultural space the interpreter occupies from which they make their interpretation, and consequently, having interpreted phenomena, that space or position is transformed to some extent. From a poststructuralist viewpoint the interest is in investigating the historically situated nature of knowledge creation and its validation, and the strategic purpose for that transformative practice being founded on the maintenance of power (Foucault, 1984; Philp, 1985). The first aspect, considering knowledge as being framed by historically situated discourses, legitimises a view of mathematics beyond that of being irrefutable fixed truth, to one of being an evolving process negotiated through a social consensus of language. This perspective "begins with the problem of unmediated access to a *transparent* mathematical reality, shifting the emphasis from the critical learner as the site of original presence, to a decentred relational complex process" (Walshaw, 2001, p. 28). It is this underpinning principle that guides this thesis, that varying the pedagogical medium will lead to the evoking of alternative frames and underlying discourses, hence rendering the learning experiences and ensuing dialogue in a different manner, and allowing space for the restructuring of mathematical understanding; for alternative ways of knowing. The following sections will address the validation of this contention, and the evolution of this perspective of learning that has emerged through the research process. How the data were examined through this theoretical lens will be attended to in the methodology chapter.

The poststructuralist notion that the construction of knowledge has strategic motivation gestures towards the particular role of power in what informs, guides and restricts mathematics education, while attending to potential ways in which it

might be re-envisaged; how alternative understandings might be investigated, realised, and articulated (Klein, 2002). By recognizing its constitutional influence in what is defined as mathematics and mathematics education, the examination of that influence paves the way for the restructuring of the mathematics and mathematics education landscapes. Foucault's (e.g., 1984) poststructuralist perspectives don't proffer alternative theories, but are perceived as interrogative practices to problematise and challenge existing assumptions (Walshaw, 2001). Foucault provided a critique of the way modern societies control and discipline the population by sanctioning the knowledge claims and practices of human sciences. He argued that the social sciences have subverted the classical order of political rule based on sovereignty rights. This new regime of power is based on 'norms' of human behaviour. This establishment of normality provides a framework for the vast area of deviation from it. It has made us subjects, and subjected us to laws: of economy, social behaviour, speech etc. (Foucault, 1984; Philp, 1985). Our position and perspectives are maintained by these underlying influences that pervade the actions and responses of the individual. The assumption that learning is based on discursive practice seems a logical extension of his argument as those power cliques look to maintain their position. He saw language as central to this, although he has a fundamental mistrust of its influences, and saw the relationship between words and objects as inherently partial (Foucault, 1984).

Language is seen as a common factor in the analysis of social and individual meanings (Weedon, 1987) with its crucial role in the constitution of social reality making language critical in the contestation of meaning (Letts, 2006). One interpretation of the poststructuralist discourse, very simplistically situates it as one in which all phenomena are linguistic constructs. Knowledge and understanding only exist to the extent that they can be described. Mathematical knowledge emerges from linguistic, discursive activity; it becomes " a set of fundamental rules which define the discursive space in which the pedagogical relation exists" (Walshaw, 2001, p. 30). Those portrayals are functions of the space the participant occupies, as much as the nature of the interactions. Those spaces evolve as participants position themselves according to the diverse influences that pervade their previous experience (MacLure, 2003). As they move

in time through continuous space, there is a constant process of internalisation occurring as they connect with their environment and interact with it through language. A participant's conceptualisation is not only shaped by this process but also shapes it. The participant and the process are symbiotically meshed. That mathematics and other traditional pure sciences are social constructs rather than descriptions of reality proffers a model of learning based on the negotiation of meaning (Brown, 2001), or perhaps enculturation into a social practice. This unhinges rigid notions of mathematical truth and permits a perspective of mathematics as a flexible, contestable position (Klein, 2002). A corollary to this is that mathematics understanding can be envisioned by the way it is engaged, with the pedagogical medium crucial to the nature of that engagement. This position sanctions the contention of this thesis that digital technologies acting as pedagogical media reorganise mathematical understanding. In particular, that investigating mathematical problems within a spreadsheet environment leads to alternative learning trajectories and understanding of the mathematics involved.

Discourse is a theme that threads through Foucault's work. Foucault perceived discourse as a system of possibility for knowledge. He rejects the conventional elements of analysis and interpretation. As Philp (1985) writes in a commentary of Foucault's work: "The relationship between word and things is always partial and rooted in discursive rules and commitments which cannot themselves be rationally justified" (p. 70). Everything is framed and contextualised by the pre-conceptions and intentions of the user, yet interpreted through the lens of the receiver's prevailing discourse. This appears consistent with the contention that understanding evolves from the negotiation of meaning, and that the situating of learning is within the context of the experience. The notion of discourse arises frequently in this discussion. It is opportune then to briefly intermit the key thrust of this underlying theory, to clarify possible perceptions of discourse.

Discourse

Research in mathematics education often refers to discourse; yet a closer examination indicates a plethora of interpretations associated with this term. In

this section, I would like to examine several aspects of discourse and begin a tentative clarification of the notion of discourse for the purpose of this thesis.

An initial clarification is centred on the differentiation of discourse as dialogue, or linguistic interaction, and discourse in the poststructuralist sense. The post-structural account of discourse portrays a version that incorporates a way of being in the world, including elements of social and cultural values, beliefs and attitudes, with language, knowledge and understanding integrated, often regarding the constitution or marginalisation of subjects (e.g., Gee, 1999; Luke, 1995). Discourses are perceived as constitutive for meaning and subjects, or for the regulation of institutional or societal conduct (MacLure, 2003). Foucault (1972) portrayed discourses as formative of the entities to which they attend. While dialogue is inherent to such a notion and permeates those various aspects, or is intrinsically influenced by them, this interpretation is broader and is bound to notions of power within societies. The linguistic discourse, meanwhile, concentrates more on the structure and meaning of texts, written or spoken, and is concerned primarily with what people actually say or do (MacLure, 2003). Analysis of this discourse reveals complex sets of rules or conventions for particular situations, both formal and informal, which are born of the broader influences. Classroom dialogue, for instance, has different conventions to playground dialogue despite both being in a school context. This difference is due to the broader societal and cultural influences, but also because of where such aspects as power are to be found in varying situations.

Put in a simplified version, the discourse in the linguistic sense is fashioned into its particular form through the frame of the participants' fore-conceptions; their prevailing discourses in the broader sense. Conversely, the broader view is influenced by dialogue in a formative, organic way. The interaction between dialogue (amongst other aspects), and the prevailing discourse in that particular area, repositions the participants' understanding/interpretation hence their prevailing discourse is also adjusted. So the two interpretations of discourse are inextricably linked, but different. Some theorists (e.g., Lee & Poynton, 2000; MacLure, 2003) also make a distinction in terms of ancestry; the poststructuralist version coming from European philosophy and the linguistic from Anglo-

American linguistic theory, and that the people from the two cultures think differently and have different belief systems and allegiances.

Others have distinguished these accounts as being focussed at a micro level, the linguistic version, and macro level, the poststructuralist version (Luke, 1995). Gee (1999) described this differentiation as discourse with a capital 'D' for the broader socio-cultural interpretation, and discourse with a small 'd' for the more localised linguistic interpretation. The place of language in the shaping of a version of reality, allied with the notion of difference that enables the signification of a term to be made, is central to the linguistic interpretation, but is important nonetheless in poststructuralist theory. The distinguishing feature is that rather than being perceived as a coherent, structured system, poststructuralist theory sees language itself as being partial, and understanding as an interpretation which is situated by the space (cultural, societal, political etc.) that the interpreter occupies at the given time. This implicates the historical differentiation to the perceived reality too. What we 'are', and how we perceive, are reflections and refractions of these various discursive lenses.

> *Poststructural theorists argue that subjects are constituted within discourses that establish what is possible (and impossible) to 'be'- a woman, mother, teacher, child, etc.- as well as what will count as truth, knowledge, moral values, normal behaviour, and intelligible speech for those who are 'summoned' to speak by the discourse in question (MacLure, 2003, p. 175).*

It is important to note that any interpretation of discourse, especially one through a poststructuralist lens, will be perceptual only, and notions of definition or differentiation are inherently ambiguous when viewed through such a lens. If one account considered that language is partial then how could anything but a tentative discussion about it occur? It is, nevertheless, a notion that takes many forms. These range from being fore-conceptions; the interwoven parcel of historically situated social, cultural and political traditions that permeate our engagement with phenomena (including the reconciliation of our interpretations, as per Gallagher's (1992) meshing of moderate hermeneutics and education),

through to Foucault's (1972) description of discourse as "practices that systematically form the objects of which they speak" (p. 49). This is more than influencing or framing understanding or interpretation; it is the practices that constitute subjects, and create meaning, while regulating conduct within institutes and disciplines. Foucault (1979) proffered that the individual is fabricated into the social order, while Luke's (1995) illustration of a child's 'identity papers' being watermarked suggests the enabling as well as constraining nature of this account. Brown (2001) while offering a poststructuralist version of learning within his hermeneutic rendition nevertheless implicates the influencing rather than constitutional role of discourse "Indeed there are many forms of mathematical discourses each flavoured by their particular social usage" (p. 26). In his discussion of the enculturation of children by their parents through access to 'mainstream' discourses, Gee (1999) likewise tends towards the influencing flank of this imaginary continuum. Both acknowledge the constitutional element to discourse and the learner's/child's influence in the re-constitution of a discourse after their engagement through it. This also echoes of various renditions of the hermeneutic circle (Brown, 1996; Gallagher, 1992; Ricoeur, 1981).

In seeing learning as a process of interpretation, with understanding and 'concepts' being states that are in ongoing formation, rather than fixed realities that need to be reached, the version of discourse that tends towards its influencing nature, seems a more useful instrument for reconciling the enculturation aspects of learning and the formation of individual interpretation. This recognises that our understandings, and who we are, evolve by cyclical engagements with phenomena through the constant drawing forward of prior experiences and understandings that are consequently influenced by that engagement.

In the educational context, Gallagher (1992) contends that this version of the hermeneutic circle uses the notion of discourse in this manner:

> *Learning, as much as teaching, is possible only on the basis of traditions. In learning, the student is brought into certain preconceptions which serve to orient her toward the subject matter.*

> *The fore-structure of the student's understanding is conditioned by the traditional preconceptions which are offered under the sign of authority (p. 94).*

Understanding in mathematics is not the conclusion of a natural maturation, or a sequential development that is universally human, but a specific and persuasively created discourse in which power and control are etched (Walkerdine, 1988). While there are mathematical, social, political, and cultural discourses associated with investigating number problems in a spreadsheet environment, there are also those associated with the particular pedagogical medium, and with approaching traditional situations through new sets of eyes. These discourses tend to be constitutional to the engagement and subsequent re-positioning of perspectives. In this study, an analysis of whether the participants' pre-conceptions in numerical, algebraic and proportional thinking were re-organised in particular ways by the engagement through the spreadsheet medium was undertaken. A determination of the manner in which their subsequent re-engagements were then framed by new perspectives, and their learning trajectories influenced by the pedagogical medium ensued. Likewise, the dialogue evoked by the engagement was examined to ascertain ways it may have led to alternative conceptualisation and understanding.

Although one interpretation of discourse can't be discounted in the clarification of an interpretation of the other, for the purposes of this thesis I am referring to discourse in the broader, macro sense, and using terms such as dialogue for the linguistic, micro version. Discourse is a way of being in the world that integrates words, acts, values, beliefs, attitudes, and social identities as well as knowledge and understanding. While recognising the role of discourse to constitute subjects, and the inherent nature of power within these aspects, the account of it used in the thesis will lean more towards one of discourse being traditional pre-conceptions that condition the learners' interpretations and the spaces they occupy. An inspection of literature associated with other theoretical viewpoints that have influenced my position on how understanding evolves in mathematics education will now preface the discussion of two key perspectives in

mathematics education. This discussion will bridge various social science discourses to the emerging view of mathematics education.

Other formative theoretical influences

Habermas (1976) tried to reconcile hermeneutics the theory of interpretation, and a worldview where outside influences e.g., political forces, distort the perspectives employed. In his approach to critical social theory, he contends that questioning the existing ways of doing things enables an evolution to the understanding of those outside influences that may have previously been explained as spiritual phenomena only. He saw this evolution as linked to the way things are described (Brown, 2001). Hermeneutics stresses that to understand human behaviour we have to interpret it's meaning (Gadamer, 1976). We have to grasp the intentions and reasons people have for their activity. In the classroom setting we need to recognise and attribute the elemental causes of activity and dialogue, as well as describing and analyzing them. "Truth is the promise of a rational consensus" (Giddens, 1985, p. 130), but how can we differentiate this from one based on power, or customs and traditions?

Habermas advocates that power is a critical measure of existing interaction: it can highlight where consensus is based on tradition, power or coercion (Giddens, 1985). Other philosophers likewise flag the juxtaposition of perceived freedom of choice and the power hierarchies or traditions that actually shape those 'freely' made decisions. It is the influences of these discourses; of culture, society, and tradition, with all the historical, political and power and submission voices they resonate, that frame the mathematical and media pre-conceptions that each learner brings to mathematical phenomena and activity.

Levi-Strauss (1973) saw interpretation of events as a matter of making sense through communication codes. More radically, through his perspective of structuralism, he contends that life, or knowledge of it, equates with language (Boon, 1985). He transposed the linguistic model to other disciplines, on the premise that those domains are themselves social constructs, constrained by

systems of communication. This situates mathematics education per se as a linguistic derived construct while the interpretation of mathematics activity, and some understanding of the participants in that activity would be accessible through their dialogue. However, it seems that these mathematical structures are imposed from definitions or concepts derived by the power structure itself. They don't evolve in a vacuum; they themselves evolve as a result of cultural norms and social interaction, as well as fate, which might put a certain intellect in conditions that allow these ideas to manifest. It seems reality, while perhaps a negotiated shared vision, is dependent on a consensus which may or may not be the same for everyone, and may be arrived at through dominance or power derived from knowledge or status (Boon, 1985).

Gadamer was a hermeneutic philosopher who argued that understanding has two perspectives that frame its definition: Firstly, as an holistic process reconciled by a multifarious framework, and secondly, as an dynamic process of encounter and response. He stressed that understanding is a matter of commitment. Gadamer (1975) argued that it is preconceptions and prejudices that make the understanding possible in the first place. He talked of projection based on a common sphere of experience and viewed hermeneutics more broadly; as a fundamental dimension of all human consciousness, grounded in the concept of lived experience (Outhwaite, 1985).

It seems rational to argue that we interpret our approach to everything through that lens that is our present state, prejudices and all. Even when we experience quite cataclysmic events or have life-changing experiences, the catalyst or readiness for changes in understanding or perceptions, are embedded in our initial viewpoint. Even though it might be an individual construction, or social enculturation that brought us to that point. In mathematics education the learner brings a set of preconceptions and understandings to the new situation. These fashion the interpretations and hence the nature of the engagement in specific ways. In this particular study, we are concerned with the learner's preconceptions of the pedagogical medium, and how these in conjunction with the affordances offered by the medium itself, promote distinct pathways in the learning process.

The phenomena studied by the social scientist are crucially bound up with (though not identical to) the interpretations of them given by the members of the society being studied. Alfred Schutz insisted that the social scientist's data "are the already constituted meanings of active participants in a social world" (Schutz, 1972, p. 10). Wittgenstein (1963) argued that the meaning of any utterance is a matter of its use and therefore, the understanding of any action or dialogue is dependent on the context in which it occurs. It would seem to follow from this argument that differences in context will affect understanding, even if the stimulus is constant. Allied to that is the contention that the pedagogical medium through which the action or dialogue is evoked, will also influence the nature of the understanding.

The difficulty is that the language used in that dialogue is not exclusively drawn from the learner's perspective, but implicitly is coloured, or even shaped, by the viewpoints of previous users of the language and, in fact, society's norms for the connotations of that language. An individual's viewpoint can't be seen as discrete from the communal perspective in which it was derived. As Brown (1996) expanded: "As inhabitants speaking of our world, we may describe our experience, yet these descriptions are imbued with societies' preferred ways of saying things and conditioned by our tradition of seeing our world through positivist frames" (p. 116). Although these positivist influences may gradually be diluted as interpretive perspectives become more prominent, they nevertheless always remain to some extent in an enculturation process, even if an influence of contrast. With the realisation that a positivist approach to investigating the human sciences does not reveal full testimony, it is claimed that analysis of human behaviour should include an attempt to recover and interpret the meanings of social actions from the point of view of the agents performing them (Skinner, 1985).

Walkerdine (1988), in her investigation of the way young children learn mathematics, introduced a broad spectrum of ideas based on linguistic and psychological perspectives. She initially discussed Saussure's representation of a sign system showing the relational nature of the signified and the signifier as being arbitrary, and questioned what this relationship evoked. Lerman (2001)

described meanings as opaque rather than transparent; what they signify for the interpreter can't be taken for granted. One aspect Walkerdine examined was whether the visual stimuli that manifests a particular connotation for one beholder, would produce the same nuances for others, given the context is consistent. Meaning then is associated with more than the mathematical activity, but is framed by the individual mathematical and pedagogical discourses that police the associated interaction. While in this version of mathematics education the context clearly influenced this discourse, the discourse and sense making associated with it likewise shaped the context, and both are shaped through a larger more pervading lens of social practice. " If reference is variable, then comprehension itself varies, and is not an all or nothing phenomenon" (Walkerdine, 1988, p. 12).

Walkerdine also maintained that the shift to the belief in the power of reason with, in mathematics education, its roots in the child-centred learning approach, is a shift in the perceived regulation of citizenship: a shift from the more overt, demonstrably authoritative power model and its inherent expectations, to one where the perception was outwardly of choice and freedom. She contends that the constrained and manipulated freedom that underpins the layers of choice was nonetheless regulatory. There is a sense of inevitability surrounding the way institutional discourses pervade individual preconceptions. To some extent this is an element of enculturation. It seems a logical extension, that an educational institution, or any institution, however loosely bound, will reflect in some form the political context in which it is set, whether in a submissive or reactionary form (MacLure, 2003). Underpinned by a hierarchy of conceptual development, which depends on perception for cognition, Walkerdine contends that the references used in social discourse were not universal, " … but rather an aspect of the regulation of social practice which form the daily life of young children" (Walkerdine, 1988, p. 11). New Zealand schools are no different, with political, regulatory and societal discourses holding sway in conjunction with mathematical and epistemological influences. While recognition of these determining features and the way they frame the learner's perspectives is significant to this thesis, the influence of the learning medium on the learner's

engagement, and the manner in which this relationship interacts with language to shape the learning trajectory, and hence the evolving understanding, is the principal focus of this study.

This draws the discussion to phenomenology as it relates to mathematics learning. By envisaging mathematics as a social construct, as something arising in social activity, Brown (1994) reasoned that meanings of phenomena were located in particular contexts. He maintained that meanings are attributed to phenomena during the gaze of the individual through the lens of their personal perspectives. Understanding, in mathematics for the purpose of this study, emerges through the interpretations of phenomena, and while consensus of meaning evolves through language, no interpretation is ever final. The way that an object is encapsulated in the language of the subject, determines the interpretations that are evoked, but it requires a temporary fixation of time to allow interpretation to occur (Ricoeur, 1981). Hence, understanding in mathematics can be seen as the evolution of historically positioned meanings dependent on the spaces from which they are observed and the media through which they are encountered (Brown, 2001).

These broader theoretical positions gesture towards the interpretive perspective privileged by this thesis in the production of knowledge, and applied to the analysis of the data. They give validation to the fundamental premise of interpretation that there is an individual, historically situated, socio-cultural space the interpreter occupies from which they make their interpretation. The discussion also considered the assumptions that underpin the consequential notion, that having interpreted phenomena, that space or viewpoint is transformed to some extent. Two of the principal constituent influences to learning theories in mathematics education will now be considered, leading to an examination of how the juxtaposition of their perspectives might enable them to be reconciled through an interpretive lens.

Perspectives on learning in mathematics education

There are many theories of learning that are prevalent in the literature of learning in mathematics education. The following is a brief discussion of two key perspectives, those of Piaget, which is fundamentally an acquisitional perspective of learning, and Vygotsky, a participatory perspective, and how they might be reconciled within a hermeneutic view of learning.

Piaget discussed the notions of assimilation and accommodation as learners interact with their immediate environment or situation. He described an ongoing process of "...assimilation of objects to schemes of action and accommodation of schemes of actions to objects" (Piaget, 1985, p. 7). He portrayed these as occurring within a developmental framework of age related stages that set parameters for intellectual growth. When a disturbance or tension emerges between their present understanding and a situation they encounter, the learner's thinking evolves to balance that disequilibrium. He proposed a notion he called equilibration, a way to cognitive change via multiple disequilibria and re-equilibrations (Piaget, 1985). He portrayed equilibration at several levels: their interaction with their world (as described above), interactions between sub-systems (schema) related to objects or actions, and equilibration between the sub-systems and their overall system of conceptual understanding. Learning can involve change in any of the three levels of equilibration (Piaget, 1985) and, he postulated, occurs when schema are re-organised through alteration (accommodation) or addition (assimilation). Inherent to this version, is the perception that any reorganisation will mean the subsequent re-equilibrations will be from fresh perspectives. The space the learner occupies will be different from that prior to reorganisation, and the multiple, ongoing engagements sustain the cognitive change.

Constructivism is a perspective where the learner actively constructs the knowledge, and the learning is a process of adapting one's view of the world as a result of this construction (Confrey & Kazak, 2006; Simon & Schifter, 1991; von Glaserfeld, 1989). Learning, as per the constructivist version, can be construed as individual cognitive reorganisation (Lerman, 2001), although he argued the limitations of constructivist theory due to its marginalisation of the socio-cultural dimension. Other researchers have given primacy to Piaget's conception of

reflective abstraction as an apparatus for cognitive development (e.g., Battista, 1999; Vergnaud, 1990). Confrey and Kazak (2006) in an overview of the emergence and evolution of constructivism perceived it as a 'grand theory' with ten key principles. Through these, they maintained that constructivism could effectively account for various classroom practices through a series of bridging theories such as Realistic Mathematics Education (Gravemeijer, 2002) or theories on mathematical thought that link learning processes inextricably with conceptualisation (Sfard, 1991). Sfard (1991) termed the state of envisaging the process as a mathematical object as *reification*. Meanwhile, Gray and Tall (1994) used the notion of *procept* to link process-based understanding with object-based understanding. Social constructivism, with its negotiated meaning, depends on the commonalities the group traverses. While there is still a connection between activity and learning, it is the dialogue that arises as a result of the activity that leads to understanding (Bishop, 1988; Resnick, 1989; Schoenfeld, 1992). Others, (e.g., Steffe & D'Ambrosio, 1995) argued that it is the use of situations that involve assimilating generalisations that lead to understanding.

While Piaget's viewpoint is consistent with the notion of a personal constructed perspective of learning, Vygotsky saw learning as socially situated. He saw social participation evolving through transformative processes to become understanding. Vygotsky advocated that individual knowing stems from relations between individuals, from human interaction (Vygotsky, 1978). As these relations are situated in particular times and places, the learning becomes socially and historically rooted. With varying, ongoing interpersonal experiences and consequential reflection, interpersonal events can over time become intrapersonal knowing, appearing to be increasingly abstract, but still tied to the series of events from which they are manifest. Vygotsky depicted the transformation of an interpersonal process into an intrapersonal one as "the result of a long series of developmental events" (Vygotsky, 1978, p. 57).

As this view of learning has cognition tied to particular situations of practice, it challenges the notion of constructed, abstract concepts that might be transposed into varying contexts. Various commentators construe Vygotsky's tenets in this manner e.g., "Learning is located in co-participation in cultural practices" (Cobb,

1994, p. 4). Vygotsky's articulation of the perception of tools as mediators and the semiotic mediation of language provide an historically situated, socio-cultural version of the process of understanding (Lerman, 2006). Research involving the utilisation of ICT in mathematics education often utilise this frame in accounting for alternative cognitive internalisation through the mediation of cultural tools (e.g., Arzarello, Paola & Robutti, 2006; Marriotti, 2002, 2006). Meanwhile, Radford, Bardini, Sabena, Diallo, and Simbagoye (2005) described the active re-interpretation of signs by students interpreting graphs of movement, as they reconciled their informal interpretations with historical socio-cultural meanings through classroom interactions. Participation in social interaction leading to knowing might be direct, or from observational viewpoints, or from internalised conversation (individual thinking), but Vygotsky perceived learning as the internalisation of social processes. In an educative sense, these social processes may be evoked by phenomena or perturbation.

The re-conceptualisation of mathematics learning theory from being one of an individual's construction of understanding to that of their enculturation, with mathematics perceived as a social construct, has evoked a pedagogical tension (Brown, 2001). The notion of enculturation, with the teacher as facilitator, places greater emphasis on the dialogue and therefore, the language in which the understanding is negotiated. An individual's understanding is more deeply embedded in the collective sense made of the various mathematical stimuli and the relationships developed between students, and students and the teacher, than merely the construction of meaning. The location of the learning also has greater significance (e.g., Arzarello et al. 2006; Confrey & Kazak, 2006).

Cobb (1994) argued that the two viewpoints are complementary rather than mutually exclusive. He advocated that the socio-cultural perspective informed theories of the conditions for the potentialities of learning, while theories developed from the constructivist viewpoint focused on what students learnt and the associated processes. It appears they are perhaps even more intimately entwined if one considers that an individual's construction can only occur within a social framework. Confrey and Kazak (2006) likewise argued that learning in mathematics involves both activity and socio-cultural communication interacting

in significant ways. They contend that neither influence is privileged, nor in fact can be separated, as we are simultaneously participants and observers in all enterprise, at all times. In a similar manner, the objectification of understanding can be perceived as being underpinned by the interplay of *typological* meaning (language) and *topological* meaning (visual figures and motor gestures) (Radford, Bardini, & Sabena, 2007)

Brown (1994) seeks further clarity with examination of these issues from a contemporary hermeneutic perspective. From there he sees the formations of understanding evolving from both individual and collective interpretations of mathematical stimuli. These understandings develop through social activity and discourse, with all the historical, political, and cultural influences that such an interpretation implies; "…the individual human subject perceives the world phenomenologically, that is, he or she sees the world comprising phenomena having particular meanings to him or her in particular contexts" (Brown, 1994, p. 145). It follows that identical stimulus enacted upon in various pedagogical media will lead to different understandings no matter how subtly differentiated that might be. The differentiation is evident in the types of dialogue, both formative and explanatory; and the links made to other concepts i.e. how the learner embeds the understanding in their existing schema, and how they might utilise these concepts or approaches in later mathematical investigation.

Research is also beginning to identify alternative areas of social-cultural theory that are emerging as current themes. The notion of identity, and how it behaves when social structures associated with a transformative process are in a state of flux, was considered to be crucial to the learning process (e.g., Lerman, 2006; Walshaw, in press). Drawing on Boaler's (2003) reference to the 'dance of agency' at the intersection of knowledge and thought, Lerman contends that the teacher's task is to lay a mathematical identity among the sedimentation of personal identities. The work of earlier researchers underpin this approach (e.g., Lave & Wenger, 1991) who discussed learning in terms of a construction of identities, with the learner a participant in a socio-cultural world, and learning a process emerging from activity by specific people, in particular circumstances. One of these circumstances will be the learning environment with particular

attention given to the pedagogical medium, as different modalities of learning affect the emergence of these mathematical identities (Boaler & Greeno, 2000).

Other researchers (e.g., Lesh & Doerr, 2003) foresee the emergence of modelling, as a key evolution of constructivism beyond individual cognitive composition. They reasoned that seeking generalities and consistencies within data and multiple-representations emphasises conjecture, promotes dialogue, and allows for the development of internal relations and meanings. They contend this is of particular relevance to environments where digital media-based investigation is conducted, while acknowledging that the examination of model-based reasoning is still in its early stages (Confrey & Kazak, 2006).

Despite the fundamental cleavage between Piaget's developmental theory and Vygotsky's socially situated viewpoint of learning, when we attempt to reconcile these seemingly polarised perspectives of the learning process through an interpretive lens, several commonalities emerge. Hermeneutics and a hermeneutic perspective to learning will now be considered with regards to the further insights into learning they might avail.

Hermeneutics

Hermeneutics is understood as the theory of interpretation of meaning (Gallagher, 1992). It originally emerged from the examination of the meaning of texts, as the question of whether the text possesses the meaning, or the meaning resides with the reader, was explored. The text presents itself to the recipient, not with an absolute, context-free veracity, but as something that evokes a response. The response is an interpretation inherently filtered by the fore-structures the recipient views the text through. What the reader understands is dependent on the various historically situated, socio-cultural discourses that frame their perspective. As Brown (2001) asserts: "The hermeneutic task can be seen as an uncovering of meaning, but an historically situated meaning dependent on the media and experiences through which it is observed" (p. 24). Each perspective brings its own whakapapa or lineage.

While hermeneutics was traditionally perceived in relation to the interpretation of written text, it has more recently been envisioned with wider connotations. Ricoeur (1981), reconciled spoken and written language through the notion of discourse in the linguistic sense; "It is as discourse that language is either spoken or written" (p. 197). It is not that spoken and written language are the same, but that they have commonalities, and behave in a similar manner in the hermeneutic process. Mason (2002), likewise viewed text in the broadest sense when he discussed the hermeneutic circle in relation to utterances, while Brown (1996) and Gallagher (1992) similarly utilise a notion of a broader dialogical interaction in their discussions of hermeneutics and education. Others (e.g., Gadamer, 1976) use language in the more expansive sense, or see all interpretation as being linguistic (Brown, 2001).

Hermeneutics is the theory of the operations of understanding (Ricouer, 1981). It can be understood as the manifestation and restoration of meaning that a person makes sense of in a personal way, or as a demystification or reduction of illusion. The first aspect resonates with a perspective of personal construction, while the second, reduction of illusion, echoes of aspects of enculturation.

Conservative hermeneutics aims at eliciting the precise intended meaning of the author. It proffers a view of interpretation that seeks to transcend historical influences so the recipient replicates the author's interpretation; an objective ascription. Advocates contend that through rigorous application of techniques, the author's intended meaning can be obtained. Proponents of moderate hermeneutics meanwhile see interpretation inextricably embedded in the discourses from which the interpreter frames their perspectives. They acknowledge the societal and cultural influences of an historically situated version of the 'text'. They also recognise the conditioned prejudices that are embedded in language; the language of both the author and the recipient. "They are the changing biases of various traditions which are not past and bygone but are operative and living in every reader and every text" (Gallagher, 1992, p. 9). The interpreter is constrained by their own language, as well as the language of the text. Understanding can't evade the influences of history and tradition, nor the

medium through which it is evoked. As the complete reproduction of the author's thought can't be accomplished, unqualified understanding is therefore theoretically impossible. Rather like a limit in calculus we might get very near to an absolute position without one ever actually emerging. As soon as we fix our perspective to engage in the mathematical phenomena, a fresh perspective is evoked. In this way, understanding is a process rather than a position and a 'concept' is a shared consensus rather than an irrevocable truth. We will discuss this notion further when we more fully address the hermeneutic circle.

Radical hermeneutics as practised by de-constructionists, and poststructuralists such as Foucault, is doubtful of any link being made to the original meaning. The aim for proponents of radical hermeneutics is not to reconstruct another version of the meaning, but to show that all versions are relative and conditional. Critical hermeneutics aims at political and economic emancipation using hermeneutics to breach false tenets in these areas, and therefore liberating a prejudicial free consensus. In contrast to radical hermeneutics the contention is that given the right conditions, the hermeneutic constraints of our limited historical situation can be transcended, for example with Marx's notion of communism, or an ideal consensus. Deconstructionists, contrastingly, perceive that no interpretation can be trusted, that all are underpinned by false precepts promulgated by prevalent power structures.

Gallagher (1992) sees the relationship between interpreter and tradition as being an anterior relation; tradition not only operates behind the interpretation influencing its particular manifestation but also ahead of the interpreter; it is part of what the interpreter brings to the process. He advocates that "language plays the role of medium or vehicle by which traditions enter interpretation" (p. 100), and suggests that "language conditions all learning" (p. 173). For Dewey, there is an intrinsic connection between language and meaning, "Meanings do not come into being without language" (Dewey cited p. 119). This is not advocating a causal relationship, but that the two are inextricably linked.

In conservative, moderate, and critical hermeneutics there is a degree of trust in language to enable a consensus. Even critical hermeneutics, which has an

underlying suspicion of the purpose of interpretation, entrusts language as the vehicle through which the constraints and authority can be emancipated from. In contrast, radical hermeneutics, de-constructivists, are suspicious of language. They "would argue that the only truth is untruth, that all interpretations are false, that there is no ultimate escape from false consciousness" (Gallagher, 1992, p. 22). They do not attempt to provide a solution or framework on which to develop a legitimate interpretation, but look to dislocate or shatter all interpretation. In this way Derrida (1978), for instance, opposes the transformative process of interpretation, as he is suspicious of its fundamental tool. This version of interpretation would see it as more an exploration or play of possible meanings. Interpretation to attain the author's original meaning is not possible; the most the reader can hope for is to "stretch the limits of language to break upon fresh insight" (Gallagher, 1992, p. 10). There is not an original truth or reality beyond language. Language is constitutive of any perceived reality. It is more than a communicative medium, being primordial in the emergence of identity (Walshaw, in press). The interpreter is also suspended within language and traditions that as such offer more fluid versions of consensus of meaning, rather than fixed interpretations. Consensus will be fraught with the pervading power discourses of its constituents, something that is possibly unknowingly imbued in their perspective. In the educational context we need to consider if the transformative process is one evoked through a trust in the process of negotiation of consensus of meaning, or conversely that any transformation must be treated with suspicion that the underlying cultural/political discourses are so pervasive as to render any consensus meaningless in terms of individual interpretation or sense making. Either way we cannot disregard the extent language permeates the evolution of understanding. "Whatever 'the real' is, it is discursive" (Lather, 1991, p. 25).

Various philosophical perspectives help situate the range of hermeneutic positions, but a common feature is that in the process of interpretation no one aspect exists in isolation. Each, whether author, text, listener, medium etc. has its own cultural, sociological, and historical influences which shape the overall interpretive process. It is the acknowledgement of these influences in both the production and interpretation of 'text', allied with the emancipative propensity

entailed, that gesture towards the moderate account of hermeneutics. The following section scrutinises moderate hermeneutics and its relationship with education.

Moderate Hermeneutics and Education

While each of the various versions of hermeneutic theory can be rationalised with corresponding social science theoretical discourses, the moderate hermeneutic perspective seems to resonate most eloquently with both my personal philosophical perspective and that of various mathematical research that informs this thesis. Gallagher (1992) contends that each of these versions, in different but complementary ways, might present profound understandings of educational theory:

If education involves understanding and interpretation; if formal educational practice is guided by the use of texts and commentary, reading and writing; if linguistic understanding and communication are essential to educational institutions; if educational experience is a temporal process involving fixed expressions of life and the transmission or critique of traditions; if, in effect, education is a human enterprise, then hermeneutics, which claims all of these as its subject matter, holds out the promise of providing a deeper understanding of the educational process (p. 24).

In the educational context we need to consider whether the objective is for the learner to reproduce the meaning of the teacher/text (and if this is possible) or whether the objective of the teacher/text intervention is to facilitate the learner's unique interpretation? Interpretation is not just determined by the lens through which the interpreter filters the phenomenon, but also by where they are situated. The spaces they occupy at various junctures have cultural, social, political and economical contexts that permeate their interpretation. This might be orchestrated e.g., with propaganda, or manifest more organically e.g., the evolution of customs. The question concerning this, when engaged in the hermeneutic process, is to what extent are these influences reproduced in understanding, a

process that evokes domination, and to what extent are they transformed, a process that evokes or has potential for emancipation? Similarly, in the educational context, can the reflective process transcend these political and authoritative influences? Our response to this will be guided to some extent by our approach to educational theory. A critical approach to education will maintain that the power of reflection has the propensity to fragment structures of power and authority, in educational processes and institutions. Conversely, approaches consistent with moderate hermeneutics hold that structures of power and authority inevitably underpin educational experience (Gallagher, 1992).

Returning to the interpretation of learning used in the introduction, we ascribed to the following view of understanding: that 'concepts' are not fixed realities we peel the outer layer from revealing their entirety, but more elusive, formative processes that become further enriched as the learner uses their temporary fixes to view events from fresh, ever evolving perspectives. In essence the mathematical task, the pedagogical medium, the pre-conceptions of the learners, and the dialogue evoked are inextricably linked. It is from their relationship with the learner that understanding emerges. This understanding is their interpretation of the situation through those various filters. Understanding emerges from cycles of interpretation, but this is forever in transition: there may always be another interpretation made from the modified stance. A moderate hermeneutic discourse provided a productive filter for analysing this version of learning.

An individual's pre-conceptions or underlying discourse in a particular domain, influence their interpretation. This is similar to the idea of existing schema in the Piagetian viewpoint, and echoes of Vygotsky's recognition of the crucial role of social regulation and the social constitution of a body of mathematical knowledge (Berger, 2005). The learners bring a series of socially situated ideas that are embedded in the associated signs. Further, Vygotsky argued that the child does not spontaneously develop ideas discrete from their social context: "He does not choose the meaning of his words…The meaning of the words is given to him through his conversations with adults" (Vygotsky, 1986, p. 122). A moderate hermeneutic perspective enables both viewpoints to be reconciled through the

notion of discourse. The learner brings historically situated social, cultural, and political perspectives to the learning process.

Within the vast array and diversity of classroom experience there is nevertheless an interchange of learning, the scope and nature of which will differ with the model. Although complex in its various manifestations, put simply the interchange of learning in the classroom situation is an interchange of interpretation. There is the interchange of interpretation, and thus learning, between teacher and pupil, pupil and pupil, and teacher and teacher that seems apparent, but also between the teacher and pupil with the pedagogical presentation. This model over-simplifies the full complexity of the classroom situation even given the social, cultural and historical discourses each bring to it. "The classroom is a curious and amorphous discursive space therefore-expanding and contracting under the pressures of different discourses that police its boundaries and construct its interiority in disparate ways" (MacLure, 2003, p. 11). Complexity exists within perceived and demonstrable interpretation as well. "The teacher's understanding and her pedagogical presentation may, and usually do, differ" (Gallagher, 1992, p. 39). Although the interpretation of the presentation and the reconciliation of a consensus are indicative of the learning process, an echo of Piaget's assimilation and accommodation, this does not necessarily happen. The pupil may be stimulated to move to a different, unintended direction or be misled by, or examine or interrogate the presentation, but interpreting is implicit to the process that evolves. Similarly the pedagogical medium evokes an interpretive response from the pupil. That the pedagogical medium might influence the interpretation and thus the understanding is central to this thesis.

Piaget suggested that disturbance results in cognitive change as the learner looks to re-establish a state of equilibrium, while central to Vygotsky's theoretical position on learning is the learner's participation in social processes. This contrasts with Piaget's notion of the disturbance of existing structures, but there is place for dialogue and negotiation of consensus to emerge from interaction with new phenomena. The pedagogical medium might likewise evoke particular social responses. The hermeneutic circle combines notions of language and

structure in emphasizing interpretation through the development of individual explanations (Gadamer, 1989). The learner develops explanations based on their interpretations of the phenomena. Their explanation then meets resistance from broader discourses, understanding evolves and the explanation alters. There is always a gap between the interpretation and the explanation, and this provides the space for understanding and learning to occur. The gap between 'the real' and 'the imaginative' allows objects or concepts to be signified. Without what Derrida (1978) termed *differance*, there would be no gap across which desire might spark. "Difference, distance, and paradox lie at the heart of meaning, being and reality. The abyss is not an avoidable error of relativist thinking, or an accident of careless philosophising, but a structural necessity" (MacLure, 2003, p. 4). Without that gap there would be no meaning; no intervening difference that would allow one word to signify another. We need space to play between the familiar and the unfamiliar so that interpretation can manifest (principle of distanciation). To have dialogue, things have to pass back and forth between perspectives. The gap opens up spaces for knowledge to exist that would not otherwise be able to evolve. Brown (2001) discusses the spaces that emerge in conversation or activity, and considers them as gaps in which individual interpretations might be made. Learning becomes an exchange of narratives between interpretations of the world and existing explanations.

The hermeneutic circle

A central principle to the hermeneutic process is the hermeneutic circle. This was originally perceived as the circularity of the interpretative process as the focus shifted from the parts to the whole to the parts until a unity or consensus of meaning emerges. It has been conceived as a constant modification of the fore-structures of experience (Gadamer, 1976), which might be either fulfilled or disappointed. With fulfilment, the evolving fore-conception would be reinforced and be maintained as an interpretive influence; if disappointed the fore-conception is re-envisioned, with each revision conditioning the understanding

(Gallagher, 1992). Hirsch (1987), made use of psychology terminology in the application of his model of 'corrigible schemata' to the hermeneutic circle; schemata which he contends are radically modifiable and responsive to context. The notions of existing schema, and historically and socially situated discourses, both echo of the hermeneutic circle as the pupil oscillates between the various discourses or schemata they bring to the situation, and the specificity of the situation they confront; that is, they move from the whole to the part with understanding shifting with each iteration. The constant modification of the schema is what the process of interpretation involves. In terms of the hermeneutic circle the meaning of a part is understood only within the context of the whole; the whole is never given without an understanding of the parts. "Every revision of the schema involves a recasting of meaning" (Gallagher, 1992, p. 64). The pre-conceptions or schemata guide the learner's attempts to understand, but within that is a notion of constraint. The unshackling or broadening of these schemata is a key aspect of the learning process and it's the task of the teacher to create conditions that allow these pre-conceptions to be reshaped. "If the context of the learning is not set up on the basis of the child's pre-conceptions… the communication fails" (Gallagher, 1992, p. 79). Piaget and Vygotsky likewise see the role of the teacher, in the broad sense, as being central to the learning process.

With the data examined in this research, the participants oscillated between the discourses summoned by school mathematics, language, and other broader social influences, and the activity with which they were engaged. Not only was their understanding negotiated through these filters and that of conversation within their group, but with 'conversation' with the pedagogical medium of the spreadsheet. They moved between fore-structure and their immediate reality. "The circular, dialogical structure of the teacher-student communication is maintained by the difference between the fore-structure (schema) operating in the students comprehension and the fore-structure that conditions the pedagogical presentation" (Gallagher, 1992, p. 75). Each iteration of the hermeneutic circle transformed their interpretation of the situation, while the pedagogical medium also influenced their approach, and inevitably their interpretation and negotiation of consensus of meaning. This perspective enables Piaget's position of "multiple disequilibria and equilibration" to be viewed as a cyclical process alternating

between disturbance and reconciliation with existing schema as understanding emerges. It also allows for Vygotsky's view of learning as: "the transformation of an interpersonal process into an intrapersonal one is the result of a long series of developmental events" (p. 57). This ongoing series of events moves between the phenomenon and the dialogue it evokes, with each iteration of social interaction a transformative process that shifts the interpretative frame to a new space from which the phenomena is viewed.

Ricoeur's (1981) notion of the hermeneutic circle emphasises the interplay between understanding and the narrative framework within which this understanding is expressed discursively, and which helps to fix it. While these 'fixes' are temporary, they orientate the understanding that follows and the way this comes to be expressed. In seeing understanding as linguistically based, it is appropriate that student dialogue and comment will provide the source for the interpretations of their mathematical understanding, in the domains considered in the research. Ricoeur (1981) parallels the relationship between spoken and written discourse, with action and the sedimentation of history. "History is this quasi-'thing' on which human action leaves a 'trace', puts its mark" (Ricoeur, 1981, p. 209). In this case, the evolving history of the learner is a collaboration of their dialogue and the corresponding action. A hermeneutic viewpoint allows the incorporation of dialogue and actions, as the links between what was being said or written, and the participants' investigative approach, were examined in terms of their interpretation of the mathematical phenomena. The data are hinged to the discourse that constituted its production and analysis. An illustrative excerpt will give insights into the ways understanding might emerge when the learner interacts and interprets through these various filters.

Illustration of the hermeneutic circle

The following excerpt illustrates how a hermeneutic circle models the process by which learners come to their understandings. It applied to a localised learning situation drawn from the study, which involved a pair of pre-service teachers investigating the 101 X activity (see Figure 1 below). It demonstrates how their

generalisations of the patterns, and their understanding, evolved through interpreting the situation from the perspective of the preconceptions that were brought forth by their underlying discourses in the associated domains. These interpretations were from the perspectives summoned by personal discourses related to school mathematics, language, the pedagogical medium, and other socio-cultural influences. They influenced the manner in which the participants engaged with and then investigated the task, while the interaction with the task and subsequent reflection shifted their existing viewpoint, it repositioned their perspective. The participants then re-engaged with the task from that modified perspective. It was from this cyclical oscillating between the part (the activity) and the whole (their prevailing mathematical discourse), with the associated ongoing interpretations, that their understanding emerged. The excerpt also indicated elements that emerged through the moderate hermeneutic gaze that will be more fully addressed in Chapters Eight, Nine, and Ten, that is, the stimulation of sub-goals in the investigative process, the use of visual referents to generalise the noticing of mathematical patterns, and the visual perturbations evoked by the actual visual output conflicting with the expected output. These were specific instances of localised hermeneutic processes, but while individually identified, they were interwoven with each other. The participants dialogue and output were the data used to illustrate these emerging themes.

101 times table

Investigate the pattern formed by the 101 times table by:

- **Predicting what the answer will be when you multiply numbers by 101**
- **What if you try some 2 and 3 digit numbers? Are you still able to predict?**
- **Make some rules that help you predict when you have a 1, 2, or 3-digit number. Do they work?**
- **What if we used decimals?**

Figure 1: 101 times table task.

They begin the task:

Clare *Investigate the pattern formed by the 101 times table. When you multiply numbers by 101.*

Diane *Times tables - so we just go like 2 x that and 3 x that.*

Their initial engagement and interpretations are filtered by their preconceptions associated with school mathematics. "Times table" is imbued with connotations for each of them drawn from their previous experiences. The linking of the term to "multiply numbers" and " 2 X that and 3 X that …" brings to the fore interpretations of what the task might involve. These position their initial perspectives. Their preconceptions regarding the pedagogical medium were also influential. It was from the viewpoint evoked by these preconceptions that they engaged with the task.

Clare *Just try 2 first, so one then two in that cell. Now go down.*

The monitor displayed:

A
1
2
3
…

Diane *It'll be 2 times, no 101 then, 202.*

They entered the following:

101
202

Clare *Yeah but couldn't we just go times 2 or 101 times.*

Diane *Yeah just do that.*

Clare *You go equals, 101 times 2. Then you click in there. Oh man we did it. Now what are we going to go up to?*

Their engagement with the task, and the dialogue this evoked, was influenced by their understanding of the situation, the mathematical processes involved (e.g., the patterns), and the pedagogical medium. This interaction has shaped their underlying perspectives in these areas and they re-engaged with the task from these fresh perspectives.

They re-entered the data with a change to the format to give the following:

A	B	C
101	1	101
101	2	202
101	3	303
101	4	404
101	5	505
…	…	…

Diane *What we did was, we got 101. We went into A1 then we typed in 101. Then we typed in B1, and then we typed in equals A1 then the times sign then two. Then we put enter and we dragged that little box down the side to the bottom to get all the answers. That gives you the answers when you multiply numbers by 101. We multiplied two by 101. You get 202.*

Clare *So you get the number, zero, then the number again. The next thing is to try other numbers. Like two zero, twenty.*

They articulated an informal conjecture for a generalised form of the pattern, based on the visual pattern revealed by the spreadsheet structure, in conjunction with other affordances of the medium (e.g., instant feedback), and their

mathematical preconceptions. They investigated the situation further from this fresh perspective.

Diane	*So if we do two-digit numbers can we still predict?*
Clare	*So we'll do like ten times 101. That's a thousand and ten.*
Diane	*Shall we try like 306.*
Clare	*No, we'll try thirteen, an unlucky number. That'll be 13, zero, 13.*

They enter 13 then drag down:

101	13	1313
101	14	1414
101	15	1515
101	16	1616 etc

.

Diane	*Wow!!*
Clare	*Cool*
Diane	*Now putting our thinking caps on.*

They had anticipated an outcome of 13, zero, 13 (13013) when 13 was entered, consistent with their emerging informal conjecture, yet the output was unexpected (1313). There was a difference between the *expected* and the *actual* output, initiating reflection and a reorientation of their thinking.

Clare	*Making some rules that help you predict. That would be like the answer you get.*
Diane	*Like the 101 times table. Like we've got pretty much the 101 times table up on our screen because we just did that.*
Clare	*We had the number by itself then we saw that it was the double. So with two-digits you get a double number. What if we had three-digit numbers?*

Diane *Lets try 100. That should add two zeros. Yeah see. OK now. Now copy down a bit.*

101	100	10100
101	101	10201
101	102	10302
101	103	10403
101	104	10504
101	105	10605
101	106	10706
101	107	10807

Clare *Wow, there's a pattern. You see you add one to the number like 102 becomes 103 then you add on the last two numbers [02, which makes the 103, 10302. So 102 was transformed to 10302].*

Their engagement with the task has evoked a shift in their interpretation of the situation. The alternating of their attention from the whole (their underlying perceptions) and the part (the task), as filtered by the pedagogical medium and their interaction, was modifying the viewpoint from which they engaged and the approach with which they engaged the task. It was from their interpretations of this interplay of influences that their understanding was emerging. This cyclical oscillation from the part to the whole continued with their viewpoint refining with each iteration.

Diane *Yeah, it's like you add one to the hundred and sort of split the number. Try going further.*

They dragged the columns down to 119 giving:

101	108	10908
101	109	11009
101	110	11110
101	111	11211
101	112	11312
…	…	…
101	118	11918
101	119	12019

Clare *You see the pattern carries on. It works.*

Diane *Look, there's another pattern as you go down. The second and third digit go 1,2, 3, up to 18, 19, 20 and the last two go 0, 1, 2, 3, up to 19. Its like you're counting on. Try a few more.*

101	120	12120
101	121	12221
101	122	12322
101	123	12423

Clare *Right our rule is add one to the number then add on the last two digits. Like 123 goes 124 then 23 gets added on the end 12423-see.*

Diane *OK lets try 200. That should be 20100*

They enter 200, getting:

101	200	20200

Oh…it's added on a 2 not a one.

This unexpected outcome evoked a tension with their emerging generalisation, instigating reflection and renegotiation of their perspective. The direction of their investigative process shifts slightly; they propose a new sub-goal or direction to their approach and investigate further.

Clare *Maybe its doubled it to get 202 then got the two zeros from multiplying by 100. Try another 200 one.*

They enter 250 then 251 with the following output:

101	250	25250
101	251	25351

Diane *No it is adding two now-see 250 plus 2 is 252 then the 50 at the end [25250]. Where's that 2 coming*

from? Is it cause it starts with 2 and the others started with 1 [the first digit is a two as compared to the earlier examples where the first digit was a one]. See if it adds three when we use 300s.

They enter in the following:

101	300	30300
101	350	35350

Diane *Yes! Now 351 should be 354 and 51, so 35451. Lets see.*

The enter 351

101	351	35451

Clare *OK then will you add 4 for the 400s? Lets see.*

They enter some numbers in the four hundreds getting the following output:

101	400	40400
101	456	46056
101	499	50399

Clare *That last ones a bit weird, going up to a 5*

Diane *Its adding 4 though. See, 499 plus 4 is 503 and then the 99 at the end. Now how do we put this. It adds the first number to the number then puts the last two digits at the end. We'll put some more 400s in to see. 490 should be 49490 and 491, 49591. Try.*

They entered those two numbers and then dragged down to get the following:

101	490	49490
101	491	49591
101	492	49692
101	493	49793
101	494	49894
101	495	49995

101	496	50096

Diane *Yeah it's working all right.*

Clare *Seems to be. What's next?*

Diane *What if we use decimals.*

The participants have negotiated a lingering consensus of the situation: one borne of their evolving interpretations as they engaged the task from their preconceptions in the associated domains. The ensuing interaction and reflection evoked subsequent shifts in their perspective. They subsequently re-engaged with the task from these modifying perspectives. Each iteration of the hermeneutic circle transformed their interpretation of the situation, with the spreadsheet medium influential to their approach, interpretations, and inevitably their consensus of meaning. The mathematical understanding that emerged was inevitably a function of the pedagogical medium employed, in this case the spreadsheet, and the interplay of their interactions as framed by their underlying discourses.

The various discourses frame the learner's attempts to understand but implicit to this is the notion of constraint. The challenging of preconceptions in a critical or at least reflective manner is an aspect of the learning process with the teacher's task to create conditions that allow these preconceptions to be reshaped. These schemata function in the same way as Husserl's notion of horizon, "supplementing the missing profiles with a pattern of meaning, that is constructing a perceptual interpretation" (Gallagher, 1992, p. 63). If a precept of education is transformation, then implicit to this is the notion of moving from the known to the unknown, the familiar to the unfamiliar. Gallagher reasoned that it was the context of the familiar, from which we negotiate the understanding of the unfamiliar, with this context provided by the operations of tradition through language. Learning about something unknown always involves a preconception of what the unknown could be, given our prior experience and our prevailing discourse. Our interpretation of phenomena is either challenged or is reconciled by what we already know. Our particular lens tints our interpretation.

The situation we hold, our positional viewpoint, influences the sense we make of unfamiliar phenomena. Likewise the interpretations made by the participants, the researcher, and the readers were influenced by the space they occupied at that particular juncture and might have varied in different times. Hence, the authorship of the data may be denied and "the entire process of data gathering, together with the data, seen to be a composite artefact regulated by arbitrary historical currents" (Sanger, 1994, p. 178). The data is not only inextricably linked to that history; the outsider's view is limited without it. Gallagher (1992) saw the hermeneutic situation as being a localised interpretation, with the interpretive practices evolving within the local context. A hermeneutic frame might be prescribed, but only to a local context with which it subsequently becomes tied. The layering of these local hermeneutic situations informs the macro position, but each retains specificity to its evolution.

The mathematical phenomena with which students engage, the classroom culture, and the pedagogical medium through which they interact will all influence the nature of any transformation. Mathematical tasks that evoke an investigative approach are engaged in school mathematics, and in this study, to best facilitate understanding. The characteristics of this approach and the rationale that underpin its use are briefly considered.

The nature of mathematics investigation

Problem solving and using investigative approaches to teaching mathematics is seen as a way to engage students in the process of mathematising. Mathematical tasks that promote mathematical investigation introduce key mathematical ideas and allow opportunity for the learner to become engaged in mathematical thinking (Marton, Runesson & Tsui, 2004). Opportunities that have been identified in the rationale for an investigative approach are the cultivation of the skills of mathematical enquiry and argumentation, the consolidation of conceptual understanding through engagement of ideas in unfamiliar circumstances, and the encountering of novel mathematical situations that evoke

the emergence of new ideas or perspectives (Ruthven, 2001). The process of mathematical investigation, which is fundamental to what mathematicians do, stresses activity such as forming conjectures, justification, reflection and generalisation (Ponte, 2001). Neyland (1995) contends that being engaged in mathematical activity is to partake in the process of mathematising. If this mathematical activity occurs in a learning environment where discourse on mathematical concepts and conjecture are valued, then understanding is even more likely to be enhanced. Meanwhile, Anthony and Walshaw (2007a) contend that mathematics teaching should permit students opportunities to think in creative, critical, and logical ways while also developing the skills required to investigate problems and better understand the world.

The influential Cockcroft report, *Mathematics Counts* (Cockcroft, 1982) and Mathematics in the New Zealand Curriculum (MiNZC) (MOE, 1992) both advocate the use of problem solving and investigation, and for them to be intrinsically woven with mathematical content in school mathematics programmes. This approach emphasises the processes involved in mathematics, not the content exclusively. It allows for contextualisation of the learning and purpose for its actualisation. It allows the student to behave in ways more aligned to what mathematicians actually do (Holton, 1994; Neyland, 1995). It will promote mathematical conjecture and evoke dialogue that analyses these suppositions, and facilitates consensus in interpretation. Likewise, in its revised form *The New Zealand Curriculum* (Ministry of Education, 2007) maintains the learning of mathematics involves the creation of models, the posing and justification of conjectures, and the forming of generalisations. It also advocates problem solving generally as a means to promote thinking.

An investigation is similar in its characteristics to problem solving, for instance, in the requirement to interpret the problem in a mathematical sense, and to choose strategies (Schoenfeld, 1992). However, an investigation is more an extension of a problem, and it is more likely to contain exploration and generalisation. They are frequently open-ended and as such offer a range of opportunities for students to process and formulate alternative responses. Learners are compelled to "engage in additional problem definition and

formulation in order to proceed" (Anthony & Walshaw, 2007a, p. 107). In an investigation, students are essentially theorising mathematics or creating and examining mathematical conjecture (Holton, 1994).

Polya's (1945) four step approach to solving problems; understand the problem, choose a strategy, solve the problem, and look back, has become synonymous with the investigative problem-solving approach, as has his advocacy of developing and utilising a range of strategies. Holton (1998) and Lovitt (1991) likewise recommend a comparable methodology. While engaged in mathematical investigation, students must interpret the task and then structure their thinking accordingly (Holton, Spicer, Thomas, & Young, 1996), pose conjectures, then communicate and justify their approach and understanding (Carpenter, Franke, & Levi, 2003).

Mathematical conjectures often have speculative beginnings and as Dreyfus (1999) implies, have elements of logical guesswork. Researchers often consider them as generalised statements, containing essences distilled from a number of specific examples (e.g., Bergqvist, 2005). They are often contextualised and constrained by defining statements, for which they hold true, unless identified as false conjectures. They can be tested for accuracy by various approaches including abstraction (e.g., algebraic or geometric proof), inference, or counter example. In their embryonic form they emerge as opinions, mathematical statements, generalisations, or positions. These can then be challenged or confirmed with explanation, leading to mathematical thinking. The development of mathematical conjecture and reasoning can be derived from intuitive beginnings (Jones, 1998, 2000). Jones and others (e.g., Fischbein, 1994; Schoenfeld, 1985), contend that deductive and intuitive approaches are not exclusive, but can be mutually reinforcing. While discussing mathematising in a geometrical context, Hershkowitz (1998,) likewise, suggests that visual reasoning is more than just a support, or catalyst for developing a proof. It can underpin the approach taken to generalisation, and be its proof and verification in one process.

Despite summaries of the literature showing that, in general, students do not provide a sound basis for proof, Dreyfus (1999) believes that even primary aged

children show the seeds of mathematical reasoning. There are varying degrees of sophistication in the formation of conjectures, as they manifest in dialogue. Building on Chinn and Anderson's classroom discourse model (1998), Manouchehri (2004), described the nature of arguments offered in mathematical discourse; the simplest being an individual stating a position and a supporting explanation without any reflection, either confirmation or challenge, by other group members. More sophisticated forms of conjecture emerged through exchanges relating to the mathematical explanations. Students participating in the research for this thesis demonstrated collective argumentation, as they negotiated the meaning of the output produced. Collective argumentation occurs when two or more individuals justify their conjecture through interactive dialogue (Krummheuer, 1995; Yackel, 2002). This present research study also illustrated how actions, diagrams, and notation function alongside verbal statements in an argumentation (Yackel, 2002). The students participating in this ongoing study used the computer output, and their subsequent actions, to help substantiate their claims.

A more advanced form of conjecture occurs as students offer counter-examples, or when they identify similarities between two mathematical explanations (Manouchechi, 2004). Chi (1997) asserts that such exchanges need not be harmonious, and that arguments refuting others' explanations are effective learning mechanisms. The learner's perturbation, as a result of gaining immediate access to counter-intuitive outcomes to inputted data, can create a tension that might subsequently influence the investigative process. In this present research study, this was illustrated by the data when students reflected on this tension, and through the discussion it evoked, reset their sub-goals (Nunokawa, 2001). The data was examined for signs that the distinct features of the spreadsheet environment were influential in the setting of sub-goals, and how the investigative trajectory may have been shaped in a particular way.

The acceptance of investigating mathematical problems as a critical part of a meaningful mathematics programme, is not only evidenced by various theorists and curriculum statements such as those discussed previously, but a perusal of mathematics education literature, with links to this approach, emphasises the way it is inextricably linked to school mathematics (e.g., Bennett & Nelson 1994;

Ducolon, 2000; English 1992; Gill, 1993; Jensen, Whitehouse & Coulehan, 2000; Perso, 1997; Rowan & Robles 1998).

Watson and Mason (2005) advocate the use of tasks that facilitate the generalisation process through requiring students to investigate invariants and variation. The open-ended nature of investigative activities promotes mathematising and mathematical thinking (Sullivan, Warren, & White, 1999), while evoking experimentation and creative approaches in the generation of solutions (Holton, Ahmed, Williams, & Hill, 2001). Tasks that involve the use of complex, non-procedural thinking, or those that promote generalisation through offering opportunity for students to make comparisons and analyse variation, enhance opportunities for mathematical thinking (Anthony & Walshaw, 2007b). The tasks used in this research, while containing an instructional element initially as the students became familiar with the spreadsheet environment, were investigative in nature so as to promote mathematical thinking and dialogue.

Some concluding comments

Hermeneutics can be understood as the manifestation and restoration of meaning that an individual makes sense of in a personal way. Conversely, it can be understood as demystification, as a reduction of illusion. Are these two perspectives mutually exclusive? It seems that the first is indicative of an encoding process; something only exists if it is socially constructed, whereas the second is a decoding process, something already exists, through varying reasons (e.g., they may have been socially constructed) but the understanding is the unraveling of the layers. Discourse allows the learner to decode other (including expert) viewpoints that reveal their understanding (i.e., enculturation). It seems there is some common ground between these two perspectives. If we argue that everything is individually constructed, what is it that brought the individual to the point of readiness? It is the vast prelude of experiences and commonalities of understanding that have been previously negotiated; that is, the enculturation of the individual into those aspects that influenced the individual's perceptions.

Meanwhile, Brown has sought to soften the individual/social divide with a phenomenological formulation that has an emphasis "on the individual's experience of grappling with social notation within his or her physical or social situation" (Brown, 1996, p. 118). This is consistent with Vygotsky's view of learning as the internalisation of social processes, but also sanctions Piaget's emphasis on the individual and his notion of equilibration, if we consider the "grappling with social notation" as part of attending to an action, problem or interaction. Piaget uses assimilation and accommodation as vehicles for how existing schema or pre-conceptions evolve. A moderate hermeneutic perspective would see social processes and interpretation as inextricably immersed in those practices.

Hermeneutics, like education, is complex and one might argue that attempts to locate them both in specific philosophical positions only detracts from essential ambiguity. "In every case interpretation involves something that is less than absolute; it is always something imperfect and incomplete" (Gallagher, 1992, p. 348). He contends that interpretation is always a balance of constraint (with tradition) and transformation (of tradition). This balance is more the process of balancing rather than reaching a point of absolute balance. There is a play between familiar and unfamiliar horizons. The notion of the hermeneutic circle allows for Piaget's ongoing process of multiple disequilibria and equilibration (Piaget, 1985) while remaining consistent with Vygotsky's view of a long series of developmental events transforming interpersonal into intrapersonal processes, on the way to becoming individual knowing. There is a suggestion of an ongoing cyclical process oscillating between existing perspectives and new events.

Individual interpretation is also implicit in these perspectives; interpretation that is filtered by existing frames or discourses. As Brown so succinctly contends: "The social world is accommodated by focusing on the perspective the individual has of this and the possibilities open to them within the world they see" (Brown, 2001, p. 251). The place of dialogue has likewise been emphasised throughout this discussion, and the moderate hermeneutic perspective enables us to situate this within both theoretical positions. The role of phenomena, possibly evoking tension or perturbation, is also realised when we reconcile the approaches

through a moderate hermeneutic perspective, either directly as per Piaget, or through the stimulation of dialogue. The relevance of this, and the emphasis on environmental factors, likewise highlights the position of the teacher, and pedagogical media in the facilitation of learning.

Understanding emerges from both individual and collective interpretations of mathematical phenomena. It develops through social activity and dialogue, with all the historical, political, and cultural influences that implies. The mathematical activity is inseparable from the pedagogical device as it were, derived as it is from a particular understanding of social organisation, and hence the mathematical ideas developed will inevitably be a function of this device. Such pedagogical devices should be regarded as worthy objects of mathematical learning insofar as school mathematical learning is largely carried out in support of the student's later engagement in mathematically-oriented social activity (Brown, 2001). Attending to these fundamentally different perspectives of the learning process through the lens of moderate hermeneutics allows some reconciliation of their basic tenets, while also enriching the moderate hermeneutic position as a way to enhancing understanding of the learning process. There is no absolute truth waiting to be discovered, but an evolution of socially and historically situated individual ways of knowing.

In concluding, the rationale that supports the data being viewed from a moderate hermeneutic perspective is briefly outlined. Firstly, hermeneutics is the theory of interpretation of meaning. In the educative sense, this is implicit to understanding. Interpretation, of text in the broad sense, of associated reflective dialogue and action, of any of the diverse range of communications and phenomena that permeate the rich milieu of the classroom, is how a shared understanding is manifest. This shared understanding may have conceptual, processing, emotional or physical elements. Understanding, and by inference learning, is central to what education is.

For those who embrace a socio-cultural viewpoint (e.g., Lerman, 2006), the dilemma of enculturation, as opposed to personal construction of understanding, is reconciled by the notion that personal understanding emerges in a social

context. A moderate hermeneutic perspective acknowledges the historically situated, socio-cultural discourses that the learner brings to the learning 'situation', while also accepting the political and institutional influences that pervade these discourses. It allows for a personal interpretation of a social or linguistic interaction, containing elements of socio-cultural learning theory while recognizing the personal individual assembling or structuring associated with this. Fundamental to this perspective is the notion of the hermeneutic circle, which describes the learning process, and sits particularly comfortably with the learning trajectories that evolve in the investigation of mathematics phenomena.

Complicit to this process is the notion that understanding is filtered through prevailing discourses, the pedagogical medium and active participation, including language. This has particular resonance with the purpose and questions of this thesis. The notion of a concept as an evolving process, and acknowledgement of the trust imbued in language to broker consensus, are also aspects that enhance the interpretation of the learning process that this research is situated within. The moderate hermeneutic perspective also gives recognition to the idea that socio-cultural influences are reproduced through the educative process in a transformative manner, rather than the educative process being used to fragment power and authority as per radical interpretive theories.

The nature of hermeneutics, and its broadening from the classical viewpoint of understanding derived from written language, to one cognisant of the notion of discourse (Gadamer, 1975), and its mediation with a phenomenological viewpoint (Ricoeur, 1981), was considered. A moderate hermeneutic perspective frames the interpretive approach taken in this study. The study contends that this approach also reconciles several key aspects of acquisitional and participatory theories, used by mathematics researchers and practitioners to examine how mathematical understanding evolves.

Philosophers as diverse as Foucault and Habermas discuss how power hierarchies, or tradition, might shape understanding through limiting the nature of the dialogue (Giddens, 1985; Philp, 1985). This is also consistent with the notion of understanding being situated within the social context that initiates the

learning process. The discourses that evolve in various pedagogical contexts are determined by language that is implicitly shaped by previous users and the community from which it derives. This dialogue is constrained by societal norms for the structure of language in that particular context (Brown, 1996). It seems a logical implication then, that varying the pedagogical lens will evoke different linguistic phenomena, and thus the negotiation of meaning will likewise vary. By examining the participants' dialogue as they engaged in the tasks through the pedagogical medium of the spreadsheet; by observing their actions; and by analysing their reflections, it was intended that insights be gained into the ways investigating mathematical problems with a spreadsheet might influence their mathematical understanding.

This prefaces the next chapter in which the methodology and the approaches employed to obtain the data are described and examined.

CHAPTER FOUR: Methodology

Whaia e koe te iti kahurangi

Ki te tuohu koe me maunga teitei

Seek that which is precious,

If you are to bow down

Let it be to a lofty mountain

Introduction

The previous chapter examined the theoretical perspectives, and associated literature, that underpins the methodology of the thesis. It provided a framework in which to situate the key elements of the research. The interpretive stance that emerged evolved through the research process itself and as such became constitutive in the data production, while also providing a lens through which the data was considered. This chapter contemplates these two aspects of the methodology. Firstly, the transformative process of the research and how the revisioning of the researcher's approach to the analysis and associated ways of knowing, led to varying perspectives. The historical marking of the results and discussion evidenced these evolving perspectives as they emerged through a variety of analytical lenses. This illustrated a hermeneutic process, with cyclical engagements involving both the theoretical literature and interpretation of the data modifying the dominant research discourses, with iterations of the hermeneutic circle. The evolving theoretical framework emerged as these modified discourses were subsequently used to re-engage with literature and data. It is important that these perspectives were historically indexed as they evolved, as they articulated the cultural, philosophical and ideological basis for the perspective the researcher held at each particular juncture (Zevenbergen & Begg, 1999). The theoretical framework is a dynamic, formative notion that shapes the research and is shaped by the research. Guided by the literature and theoretical

viewpoints, the research framework was refined within the unique practical context.

As rehearsed in the previous chapter, mathematics is not a fixed reality beyond the scope of human influence. It is better envisaged as a socio-cultural way of thinking. It is a shared ongoing interpretation of situations and perspectives, some of which have been embedded in our traditional beliefs (e.g., five plus one is six) that we treat them as reality. They have become 'truths' by the repeated communal consensus of interpretation. Mathematics is an evolving set of perceptions, seeming to become more complex on its peripheries, yet more refined in its core identities, with each iteration of interaction, reflection and interpretation by its users. The elements of mathematics that are engaged transform the perceptions of the person interacting with the mathematics, but likewise those elements are transformed by their engagement with mathematicians, learners, or researchers, even if only by a minuscule amount. The boundaries of mathematics are expanding or becoming more refined through that interaction. The socio-cultural formation of mathematics can also be envisaged as a hermeneutic process, one where iterations of engagement, reflection, interpretation, then re-engagement from modified perspectives fashion those emerging theories.

In this study, for example, the affordance of the spreadsheet environment to more easily manage large amounts of data, opened up opportunities for the students to explore the activities in alternative ways to the approaches they might have employed in a typical classroom setting (i.e. one with students working at tables or groups of desks, using pen-and-paper technology, with equipment available). The Year 6 pupils, for instance, were able to generate and manipulate large amounts of numerical output within their spreadsheet models of the situations that would not be practical in the classroom setting. They could investigate and interpret the mathematical phenomena in an alternative manner hence the boundaries of their mathematical investigating, and by implication their understanding, were extended. As a consequence, the verge of what constitutes school mathematics, and mathematics itself were also extended. Each iteration of interpretation evoked by this alternative filter was simultaneously iteration in the

cultural formation of mathematics. Mathematics had become a slightly modified version of its previous self. The perceptions of mathematics associated with the way we express ourselves and make sense of our world (Radford et al., 2007) had changed. The spreadsheet environment's facility to manage large amounts of data quickly and accurately, also allowed the students access to different types of situations and problems, and to investigate mathematics in more realistic contexts (e.g., Ridgway et al., 2006). In a similar way, this had also transformed the nature of school mathematics. The participants' perception of what mathematics is and therefore general perceptions of mathematics have been altered. The engagement, reflection, and transformation of perspectives at an individual level resonate (no matter how slightly) in the general perspective.

This hermeneutic process echoes the viewpoint of learning in mathematics education articulated in Chapter Three. Implicit to this perspective is the inextricable linking of mathematics, learning in mathematics, and the research of mathematics learning. They are mutually formative practices, and evolve in an interactive manner. Viewed from this perspective, the reshaping of mathematics through alternative filters, the reorganisation of mathematical understanding through engaging mathematics phenomena with digital pedagogical media, and the transformative research process the researcher undergoes, also have a symbiotic relationship. As such, they were each constitutive of the methodology that could be productively employed in the investigation of the research questions. The following excerpt gives insights into that relationship.

A group of pre-service teachers was exploring the 101 X task with the spreadsheet available. They read the explanation of the task before beginning the investigation process:

Kyle *I haven't predicted. I was just going to put in A1 times 101 and drag down (does it).*

The following output was produced:

A	B
1	101
2	202
3	303

4	404
…	…
14	1414
15	1515
16	1616

The pedagogical filter through which it was engaged shaped the group's initial interaction with the task. They have immediately used the functionality and affordances of the spreadsheet environment to explore the situation. Their approach, as demonstrated by the dialogue, was not to try an individual example as might be expected with a paper-and-pencil medium, but to form a symbolic model of the situation designed, with the spreadsheet's functionality in mind, to create a visual model - a table of related, consecutive, numerical values. The spreadsheet medium has led them to investigate in an alternative manner, expanding the potential strategies for mathematical investigation and the scope of mathematics. Their interpretations and understandings were different, and articulated in visual terms, e.g., the type and position of the digits:

Kyle *[referring to 44440, the output from 44]. Its like double the number, but with zero added on.*

The 'double the number' comment refers to a repeating of the digits rather than doubling as a process, again accentuating the visual element to their interpretation. As well as the spreadsheet environment expanding the potential to mathematise and the types of understandings that might emerge, the pre-service teachers' investigative processes e.g., 'drag down' indicated the need for alternative research approaches. For this study, the approach to collecting data required the collection of synchronous data relating what they said (the taped dialogue) with what they did (the printouts of their output). Methods for data collection will also need to evolve as the nature of mathematics, and the ways it is understood, evolve. The desire by researchers to develop ways to more accurately collect synchronous data generated in digital environments, so as to gain more insightful interpretations of the learning processes that emerge, is symptomatic of the connectivity between the evolution of mathematics, learning in mathematics, and research in mathematics education.

The second part of the methodology chapter addresses the methods by which data were collected and analysed at various points in time. It includes the initial approach taken to obtain the *desired* data, then the refining of the data analysis according to the modified rationales (Schostak, 2002). It is the story of how it was intended to produce the understandings and knowledge through the approaches taken. These approaches by necessity are inextricably linked to the emerging theoretical frame, as both are constituents of the hermeneutic circle that is the research process, in the version of research privileged in this study. Meanwhile, a local hermeneutic circle also occurred, with the interpretation of the data as the students engaged in the process of evolving their mathematical understanding. This chapter therefore, is the crafting of the rationale for the approach taken, and an introduction of the resources used to elicit the understandings that emerged.

An interpretive perspective

The research undertaken was located in classroom settings. The complex milieu that is the classroom requires an approach to research that acknowledges that complexity; one that recognises that attempts to reduce this intricacy of relationships, and the multitude of underlying socio-cultural influences and discourses, to single constituent elements disregards the interdependence of these relationships and the multifaceted nature of human interactions. These relationships are as eclectic as the situations and environments from which they arise. Meanings that emerge from the inter-relationships between students, teacher, classroom phenomena, pedagogical media, and all the associated influences and underlying discourses, can be lost if situations are fragmented and reduced for the perceived purposes of some unattainable objectivity. The understanding of the inter-connected features, including the methodology of inquiry is impoverished by a reductionist approach (Kinchloe & Berry, 2004). The situating of learning within a social context is not the only influence that gestured towards the utilisation of an interpretive paradigm. Beck (1979) discussed the purpose of social science as being immersed in interpretive

perspectives. He contends that the purpose of social science is the comprehension of social reality through various perspectives, arguing that social sciences do not reveal ultimate truth, but negotiated human definitions of reality, and the examination of these evolving virtualities. They allow some sense making and enable clearer understanding of situations. They are concerned with explanation and clarification of the world that humanity has created around itself; a world that is multi-layered and socially constructed, with events and the people and circumstances that constitute them, uniquely situated in a particular time and context. To comprehend the reasons for particular individual interpretation, and the ensuing action it instigates, requires an insider's viewpoint, one where the researcher is able to share, or at least understand, the individual's experiences.

Questions concerned with understanding the process of learning, situated within classroom learning locations, and associated with mathematical understanding, evoke a qualitative methodology. As researchers have identified and investigated further aspects of the learning process as functions of the dynamic relationships and specific contexts in which the learning takes place, so their methods of research have changed. They needed to collect data that enabled them to more fully understand features while set in the appropriate context, rather than trying to be context free; that is by observations (e.g., Cohen, Manion & Morrison, 2000), case studies (e.g., Bassey, 1999), or interviews (e.g., Kvale, 1996). These approaches were hence considered significant for this study as I sought, within a classroom environment, to better understand the learning process as students engaged in mathematics investigations through the pedagogical medium of the spreadsheet.

As the dynamic relationships and situational contexts were acknowledged as significant aspects of the learning process, the requirement to collect data set within the learning environment was recognised. Naturalist approaches (e.g., ethnomethodology) are concerned with interpreting everyday phenomena. While manifest as both linguistic and situational interpretations, they are cognisant of the uniqueness pertaining to situations of occurrence, and a commitment to methodology. Burrell and Morgan (1979) have discussed the notion of typification of everyday experiences, as a way of making sense of social orders,

and synthesised phenomenology with this, through interpreting multiple realities, each constrained by its situation. Observation and description were significant in this process, and inherent in both is interpretation. As discussed in Chapter Three, hermeneutics, the study of interpretation, not only proffers a way to better understand a localised learning situation; it also allows the researcher to better understand the methodology of research. Central to this are the worlds of both the participants and the researcher. Just as the participants bring their historically situated pre-conceptions and discourses to each situation, so to does the researcher. There are multiple versions and interpretations of situations, and multiple perspectives from which these interpretations are evoked. As such, the space the researcher occupies in each version of their interpretation is as much a part of the data as the observations themselves (Brown, 2001; Mason, 2002). Much recent research in mathematics education has drawn on contemporary social science research and given greater emphasis to "the positionings, motivations, discursive formations and emotions of the researchers involved" (Brown, 2008).

As an illustration, consider a set of data collected when a group of students were using spreadsheets to investigate the patterns formed by the one hundred and one times table (see Figure 1). This was in the form of transcripts of their dialogue and the output they produced on the monitor: what they said and what they did. When it was initially analysed, it told a story of the students applying a visual lens to the number patterns that emerged for them. Underlying discourses led me to notice that aspect, to bring it to the foreground. When the same data was examined at a later stage of the study the hermeneutic circle was employed as alternative discourses were now privileged. The story that emerged most recently reflected the students' understanding as it evolved through iterations of engaging with the tasks and the consequential repositioning of their perspectives. These modified perspectives in turn framed the subsequent re-engagement with the tasks. The data was still in the same original form, but the researcher's viewpoint had been transformed, thus the interpretation and discussion were different. Another researcher might have privileged other perspectives.

Narrative frameworks, from which accounts are fabricated, temporarily fix these historically positioned interpretations. Interim 'fixes' of the phenomena that allowed the provisional interpretations at the local level are likewise providing temporary 'fixes' of the emerging methodology. Within this recognition of the evolving perspectives is the notion that there is no absolute truth waiting to be revealed by the appropriate methods, but rather an unveiling of fragmented perspectives that elucidate the researcher's understanding. This echoes a post-structuralist position that perceives any meanings as partial, and the occupation of interpretation to " keep the trembling and endless mirror play of signs and texts in play" (Caputo, cited in Gallagher, 1992, p. 278). Ways of knowing are discursive, and selves are multiple, fragmented, and constrained by their dominant discourses (MacLure, 2003). In a version of mathematical learning flavoured by this perspective, mathematics is a social construct premised on previous interpretative stances (Brown, 2001). From this perspective, an examination of the learners' preconceptions, their interpretations (as manifest in their dialogue and actions) and how they subsequently re-engaged with the activities gave insights into the layering of meaning as their understanding evolved.

Situating meaning making as a process of consensus dependent on language, cultural conventions and metaphors, emphasises the social aspect of interpretation (Mason, 2002). Kinchloe and Berry (2004) have likewise maintained that meaning making "cannot be quarantined from where one stands or is placed in the web of social reality" (p. 82). They also advocated that interpretive research involved the connection of the subjects to their prevailing discourses, the acknowledgement of the researcher's perspective and his/her relationship with the participants, and embedding the sense making in human experiences and interactions. Meanwhile, Brown (2001) saw the hermeneutic task as a revealing of meaning, "but an historically situated meaning dependent on the media and experiences through which it is observed" (p. 4). This indicates that the interpretation of dialogue and the associated negotiation of generalisations, are rich ingredients in the research process, particularly when accompanied by the articulation of corresponding researcher perspectives.

The emphasis on the researcher perspective, however, is precursory to the foremost critiques of the interpretive methodologies. If the notions of behaviour and interpretation are constrained only by the participants' viewpoints, there is the danger of partiality or incompleteness, through the limitations of the construction of those viewpoints (Bernstein, 1974; Giddens, 1976, Layder, 1994). However, while the recording of phenomena is significant in the developing portrait of interpretation, the insights of participants are crucial. Other concerns with applying an interpretive lens to research involve the subjectivity of the researcher. Key elements of the research process, observation, description and analysis require selection. The researcher's underlying preconceptions and intentions may influence those selections and if the researcher's perspective is part of the interpretation, part of the data and analysis, objectivity is hard to reconcile (Mason, 2002). Mason also wondered, given that data is a construction by the researcher, whether they would compose the identical record in the same situation again. Likewise, he identified potential for the mingling of the data which emerges from the analysis, and the original phenomenon; "the complex interplay between story and experience" (Mason, 2002, p. 228). Language is the vehicle of the data, description, and analysis, and as such these are inherently subjective. Language is connotative by design and the interpretations it permits become constitutive of ongoing meanings. Understanding arises from consensus borne of the engagement and interpretation of phenomena, with each interpretation influenced by, and influential, in the ongoing process. Interpretation provokes possible explanations, but there is always a productive gap between interpretation and explanation that provides the space for understanding to emerge. This space allows the play between the familiar and the unfamiliar from which interpretation evokes new thinking. The principle of distanciation emphasises that all interpretation is transformative to some degree, but never in an absolute way (Gallagher, 1992). As Brown (2001) has discussed in a separate, but associated, context, "…understanding evolves continuously but is represented through tangible product, capturing the moment, such as pieces of writing, calculations, diagrams and so forth" (p. 98). It is the reconciliation of these snapshots of tangible phenomena through discourse that enhance that emerging, yet dynamic understanding. The educational researcher must likewise be concerned with the tangible and the interpretive.

Ethnographic research is concerned less with predictive generalisations, than with the formation of generalised descriptions, the interpretation of events. The researcher's perspective is not the sole contributor. As LeCompte and Preissle (1993) contend, "...meanings are accorded to phenomena by both the researcher and the participants; the process of research, therefore is hermeneutic, uncovering meanings" (pp. 31-2). This does not mean a purely subjective, record of events fashioned through the personal filters of the participant or researcher. Methodologies have emerged that help alleviate validity and consistency concerns: models with commonalities of design (LeCompte & Preissle, 1993; Lincoln & Guba, 1985). Discourse analysis, recordings, notes made *in situ,* observations and interviews all have interpretive elements that give crucial insights, and if consistent and collaborative, layer the sedimentation of a valid history of events. Some researchers (e.g., Kinchloe & Berry, 2004) advocate the use of *bricolage* as an educational research methodology. This approach, premised on a critical hermeneutics perspective, contends that research methods should by necessity be eclectic if they are to examine the complexity of educational processes in meaningful ways. Bricoleurs actively assemble their research methods from the available, deemed-appropriate strategies that are afforded by broad social science research paradigms, including practical, theoretical and interpretive approaches. A range of methods was engaged in this study to elicit better understanding of the complexity of learning and the ways students came to their understandings, and a hermeneutic circle was enacted through the practice of research. However, for me as researcher, a moderate hermeneutic perspective emerged as being most productive, one that privileged the transformative view of education rather than the emancipatory one ascribed by critical hermeneutics and the *bricolage.*

There was also the need to gain understandings of the learning occurring at an individual level and the possible reasons for this, that is, the understanding of actions or implications rather than causes. This too indicated the need for elements of an interpretative paradigm. To gain insights into, and an understanding of, the learning that might occur for individuals, observations in the learning environment and interviews with participants were used to provide

important information. Clandinin and Connelly (2000) outlined an historical perspective of narrative inquiry, and demonstrated the process, by recounting what narrative inquirers do. They contend that narrative enabled the researcher to investigate experience in a way that situates change or the learning within the context it occurs, or the narrative it is derived from. Like Geertz (1995), they appeared to see understanding evolving concurrently, but not necessarily in parallel, over a range of perspectives both phenomenal and attitudinal, as change inevitably occurs and has effects. Their thesis is that education is a form of experience, and narrative is the way of representing and comprehending experience (Clandinin & Connelly, 2000). In educational research settings, narrative inquiry is usually associated with the stories of reflective practitioners (Schon, 1983), or action research where practice, closely meshed with theory, induces deliberate aspects of actuating teacher change (e.g., Somekh, 2001). The research undertaken in this study contained elements of a narrative inquiry approach in the stories told, both individually and in groups, of the participants' perceptions of the learning process.

The researcher's viewpoint is implicitly situated with the interpretations of the data. However, the lens that is our present state is not constant. While imbued with the social, political, and cultural influences that shape its perspectives, it also shifts in its construction over time, and with varying audiences. Geertz (1995), within an anthropology context, maintained that it is not only the phenomenon that changes over time; the onlooker's viewpoint changes too. He identified the setting in which the phenomena occurred, its intellectual and moral justification, and the nature of the discipline the onlooker is viewing from, as also shifting. Sanger (1994) added a further view:

> *For the post-modernist language philosopher, data are arbitrary and are therefore vulnerable to a wide variety of analytical operations. The authorship of the data, in the form of the actor's statements, may be denied and the entire process of data gathering, together with the data, seen to be a composite artefact regulated by arbitrary historical currents (p. 178).*

While varying influences might pervade the data, methodology, and interpretative analysis, the researcher, while recognising this, can nevertheless only investigate that which is offered through his/her current lens. The sorts of spaces the researcher occupies, and the extent to which these may be delusory or illuminating depending on the story they are telling and to whom, are precursory to the data itself. The data is seen through varying sets of eyes. It is important to understand how those eyes see, and how they produced the objects they described. The researcher's personal narrative was a vehicle for revealing those fragmented perspectives and for giving insights into how the analysis was a function of those personal viewpoints, on any particular occasion. The personal narrative and the transformative process the researcher inevitably experiences are more than illuminating; they are part of the data itself and fundamental to the methodology. An ongoing diary of reflections, the documentation of interactions with supervisors, along with the writing and presentation of papers all constructed and promulgated this personal narrative.

Mathematics education research is an ongoing, evolving process with each individual engagement in research extending its boundaries. At the individual level the researcher undergoes a transformative process (e.g., Mason, 2002; Schostak, 2002) as they initially envisage their study from preconceptions drawn from their prevailing discourses. Their interaction with the literature, data, and colleagues, with its associated reflection, modify the researcher's perspective. The space from which they perceived the research shifts, and they re-engage from a modified position. This process is hinged to the evolution of mathematics at both the individual level (for researcher and participants) and the broader, more general understandings. The individual research process is informed by preconceptions borne of those mathematical discourses as well as the discourses in associated areas e.g., social science research. Likewise, the ongoing formation and transformation of mathematics education research is influenced by the cultural formation of mathematics as it adjusts through interpretation at the individual level, and the transformative process of the individual research trajectory. They are mutually influential of each other, and in both the individual and broader forms research methodology evolves through cycles of interpretation, as attention oscillates between engagements through the gaze of

underlying perspectives, to modification of those perspectives through that engagement.

For instance, the students in this study engaged with the tasks from the perspective of, and through their preconceptions in, the associated areas. Seeing the output of their mathematising in the visual, tabular form of the spreadsheet modified those preconceptions as they made interpretations of their interaction. In the following brief excerpt, two pupils were investigating the 101 X activity (see Figure 1).
They had produced the following output:

1	101
2	202
3	303
4	404

Tim *So it's the number, then a zero, and then the number again*

Carl *Yeah, yeah. 5 will be 505, 55 would be 55055. Drag down.*

...	…
13	1313
14	1414
15	1515
16	1616
17	1717
…	…

Carl *What? It's just repeating.*

Tim *Like doubles, so 18 would be eighteen, eighteen and 55 would be fifty-five, fifty-five.*

They continue refining their generalisation through the modification of their perceptions as they interpret the outcome of their engagement and adjust their perspective. Their generalisations are based on the number and positioning of the digits. They have used a form of visual reasoning to generalise the pattern (Presmeg, 1986). They then re-engaged with the activity from a fresh perspective

with the interpretation and understanding evolving in this ongoing manner. The broader discourse of mathematics (in this case visual reasoning) was likewise transformed (albeit slightly) by this engagement. The boundaries of mathematics per se were extended, or existing positions enriched, by that engagement. Other pupils commented in the interviews on the way the spreadsheet environment assisted their interpretation, e.g.,

> *Chris* *Columns make it easier – they separated the numbers and stopped you getting muddled. It keeps it in order, helps with ordering and patterns.*

This cultural formation of mathematics evolved as the mathematics phenomenon was engaged with the subsequent interpretations influencing the way mathematics was perceived.

The individual engagements of the students were also influential on my researcher perspectives and interpretations of the data, and the research methods that were employed. The analysis of the initial data revealed this emerging story around the affordance of the spreadsheet environment to structure the output visually. This analysis of the data, in conjunction with other constitutive influences e.g., the research literature, modified my approach to a more interpretive perspective. I looked to research methods that would give alternative insights into these visual interpretations as the pupils' attention shifted alternately from preconception to interaction. Viewing the data through this lens gave further insights into the investigation of the research questions, in particular, the ways understanding emerged for the pupils, and the ways the pedagogical medium of the spreadsheet influenced their understanding. Mathematics education research was modified simultaneously as I engaged in research practice drawn from my existing prevailing discourses in mathematics education research, engaged in the research process, and then modified my perceptions of mathematics education research. The individual transformational research trajectory resonates and modifies mathematics education research per se. In this case, the collegial dialogue, writing papers and presenting at conferences, and writing articles for journals, that indicated this visual, tabular structure and its influence on the

research process I employed to productively interpret the situations, has extended to some small extent the boundaries of mathematics education research.

In rejoinder, the mathematising at an individual level, the cultural formation of mathematics, the individual research process, and the evolution of mathematics education research, are all inextricably linked, they are mutually influential of each other. Their relationship is symbiotic. This relationship is also evident in the manner in which each emerges through iterations of interpretations drawn from preconceptions and associated discourses, with the subsequent modification of these perceptions and discourses through that interpretation and interaction. They all evolve through cycles of interpretation.

Within the initial standpoint of the research questions, there was a desire to explore the links between using spreadsheets and understanding, allied with the requisite of better understanding of how the use of spreadsheets as pedagogical media influenced the engagement with the tasks and the understanding that evolved. This included how investigating in a spreadsheet environment might have reorganised the ways the participants understood the ideas involved. The validation of results when different approaches to data collection support each other's stories authenticated the collection of some data suitable for quantitative analysis. The availability of appropriate assessment and analysis instruments, the scope for controlled intervention, and access to a population sample allowed some collection and analysis of this data. The combination of methods offered a richer texture to the emerging picture, as well as some further tentative validation of the findings. Therefore, while a qualitative paradigm underpins the methodology, some quantitative methods were also utilised when appropriate. Statistical analysis of the Otago problem challenge results and quantification of the participant surveys became part of the developing story. These quantified measures required consideration through a reflective lens so as to assimilate them into the pervading methodology. Concurrently, the researcher examined the perspectives assumed, and the spaces occupied, by the participants and himself, as he analysed this quantitative data and reconciled it with the emerging interpretive and reflective viewpoints.

The original intended methodology then was an ethnographic one that also included elements of interpretations of quantitative data. Data were gathered through the methods that are outlined in the following sections. Through the research process, including cyclical engagement, interpretation and reflection on literature, the data, and the exploration of alternative methodological, philosophical and theoretical positions, the thesis was reframed from an interpretive perspective; specifically a moderate hermeneutical one, as rationalised in the previous chapter. This hermeneutic process in the form of the hermeneutic circle emerged in two central ways; the methodology and the approach used to analyse the data. The emergence of the underlying research methodology through iterations of a hermeneutic circle, and its associated rationale have already been well rehearsed. This ongoing transformation of the research process became part of the data and evolving discussion. Hence a chapter, Chapter Seven, was used to portray and rationalise that process and the influences of each historically situated researcher perspective. Critical too is the description and rationale for the original approach, and the signposting of the discussion and subsequent analysis at those particular junctures. These, and associated informal reflections and writing for publications, were influential in the hermeneutic process. They are constitutive of the methodology, but in themselves are significant to the findings of the project and elucidation of the understandings gained. Chapters Five and Six will discuss some of these initial findings and draw some tentative conclusions about the stories that emerged. The data was then re-examined through a moderate hermeneutic lens that illuminated alternative perspectives, and created alternative understandings through the application of this interpretive approach. This placed an emphasis on the original methods and approaches, as it is from their relationship with the original phenomena that the original data arose. The following sections give an account of those approaches.

Participants

The research for this thesis is part of an ongoing research programme exploring how spreadsheets might function as pedagogical media. This has included a range

of research situations, two of which are associated specifically with the thesis. These are both described below. Primacy has been given to the first situation that was instigated for the specific purpose of investigating the particular research questions of this study. It provided an opportunity to examine the ways learners engaged in mathematical activity through the pedagogical medium of the spreadsheet, and how this might influence the learning trajectories and understandings that emerged. The second situation described was influential in the formation of the research questions, but while envisioned as a pilot study for the project, it became an elemental constituent of the varied texture of the data and analysis. In both situations the participants were familiar with the researcher, and had previously investigated, as part of their respective mathematics education programmes, mathematical phenomena with him present.

The participants in the principal situation were drawn from year six students, attending five partnership schools associated with the University of Waikato at Tauranga campus. They were at the time involved in a collaborative project offering programmes to develop gifted and talented students in their schools (Beach Brilliance). There were four students from each school (five from one school), who had been identified through a combination of problem-solving assessments and teacher reference. There were twelve boys and nine girls. The schools' socio-economic ranking ranged from decile one to decile nine, where decile one is the lowest socio-economic rank, and decile ten the highest. The decile ranking is an indication of the relative affluence of a school's contributing community. The pupils came from a range of socio-economic backgrounds. Two of the schools were full primary (Years one to eight) schools, while the other three included students from Years one to six. The participants were located in a classroom situation that included seven computers with spreadsheets as available software. This was the typical working environment for two of the schools, while the other three schools had three or four computers in each class at this level. For the students from those three schools, the computer access was therefore marginally less constrained than their usual class situation. However, the group was familiar with this particular classroom, having worked there on previous occasions that year, engaging in rich mathematical tasks and investigations as part of the Beach Brilliance programme. For the research project, the students

worked on a programme of activities using spreadsheets to investigate mathematical problems, predominantly suitable for developing algebraic thinking. They were observed, their conversations were recorded and transcribed, and their investigations were printed out or recorded. There were school group interviews, and interviews with working pairs. As well, their results for the Otago Problem-solving challenge were monitored and analysed in terms of the development of understanding or use of strategies over the year, before and after the spreadsheet sessions. They undertook a survey based on opinion and motivational considerations. Some on-going data was also gathered over a longer-term period (eighteen months) with three of the groupings, allowing for some case-study styled data to emerge. Observations of, and the recording and transcribing of participants' conversations in ensuing Beach Brilliance groups also further enriched the data set and understandings gained. In the results chapters and associated discussion specific to them, this group was referred to as the 'pupils'.

The second situation involved pre-service teachers who used spreadsheets as part of their mathematics education programme. This research was part of a project involving members of my mathematics education department, which sought, amongst other objectives, to explore the mathematical discussion evoked by different pedagogical situations and media: How the situating of mathematical experiences might shape the dialogue, and filter the understanding of mathematical phenomena and their approach to teaching mathematics. Each researcher was able to address this question in a context and manner that suited their particular interests and intentions. My research emphasis here was to address this fundamental theme with spreadsheets as the pedagogical medium, as part of an ongoing investigation into how learning in this environment might influence understanding, and as a means to examine potentialities for this thesis. It was considered as an informal pilot study, with direct implications for the thesis data and interpretations. The participants for this phase of the study were volunteers from a class of forty, mainly mature-aged students in the primary pre-service teacher programme. Three groups of three first-year students worked in a typical, classroom, group setting with counters, calculators, and pen and paper available, and three groups, from the same class, simultaneously worked in an

ICT laboratory, doing identical investigations using spreadsheets. All participants had previously worked in both settings, although the classroom setting was the typical location for their mathematics education classes. This situation involved a similar approach to data collection as the first; that is, observation, recorded and transcribed dialogue, written or printed output to problems, interviews, and questionnaires. A reflective journal was maintained, along with observational notes recorded *in situ*. This group was referred to as the 'pre-service teachers'. When the discussion involves the pupils and pre-service teachers, the collective group is referred to as the 'students'.

While my ongoing relationship with both participant groups might be considered problematic in some methodologies regarding objectivity, there was no intention with this research to obtain an objective stance from which to draw generalisations for predicting behaviour in other contexts. The aim of the research was to gain insights and better understand the learning process with spreadsheets as it occurred in the appropriate context, not to gain generalisations that might be context free. As discussed in earlier sections, an interpretive perspective emerged as the most useful way to gain insights, meanings, and understandings to address the research questions. In an interpretive methodology the context and the researcher's perspective are functions of the data as much as the participants themselves. While it could be argued that these are 'typical' mathematically able ten-year-olds from a representative sample of schools located in provincial New Zealand cities, this is not an imperative for the purposes of the research. The participants might be described as a convenience sample as they were accessible in relatively authentic settings in which my intervention was minimised. They were also selected because they gave a range of opportunities to explore the questions in settings that offered meaningful insights. The next section will consider the approaches that were initially engaged to generate data that I envisaged would facilitate the creation of knowledge with which to address the research questions.

Research methods

This section describes the methods by which data were gathered to address the questions posed. It is an introduction to the resources utilised to educe the understandings that emerged. The research questions for this study centred upon the participants' learning experiences, when mathematics phenomena were encountered through the pedagogical medium of the spreadsheet. Allied to this were the understandings that emerged for the students in that learning environment. Hence the study was situated in classroom settings and initially approaches were used to gathering the data that involved observation, description and reporting. The inquiry attended to understandings and meanings, and with context profoundly implicated in meaning, a natural setting was considered most illuminating. However, the intrusion and associated influence of the researcher was inevitable. In their description of situations and occurrences, the researcher is influential in any experience by their presence (Mason, 2002). As such, they become a constituent of the data, but an aim was to minimise my intrusion, and while this presence would exert some influence, any ensuing effect was not the focus of the observations. A multimode approach to data collection was engaged in attempting to gain a rich, yet broader story of the learning process that was evoked. A thick tapestry of data was envisaged, with any ensuing consensus permitting a sense of validity. Commonalities that emerged through alternative methods might also enrich the understandings and patterns that were noticed within the data. Following is a description of the approach that was undertaken.

Procedures in which the research participants were involved

Participants were involved in the following procedures:

- Observations
- Activities using spreadsheets, as part of their programme
- Individual assessment tasks
- Interviews
- Questionnaires

Observations

Observation gave the opportunity to collect data within more natural settings; the usual learning environments where the participants engaged in mathematical investigation. It allowed for information and understanding to be gained regarding organisational aspects of these settings, including the physical environment, organisation and characteristics of the people involved, resources, and pedagogical styles (LeCompte & Preissle, 1993). Significantly for this study, it enabled me to gather data on the interactions that were taking place. Audio-recordings of verbal interactions were made as groups engaged in the tasks, informal ongoing notes made, and logs and reflections of events or around certain themes were written. As mentioned, the influence of the researcher's presence needs to be acknowledged, along with the emphasis given to the various aspects of noticing; the features that are brought to the foreground or privileged to some extent. We are selective about what we notice, what we distinguish from its surroundings, and researchers might inherently perceive phenomena through the selective lens by their theoretical predispositions (Mason, 2002). All the audiotapes were transcribed verbatim (with checking) and positioned with the corresponding output and printouts. With the other observations, which incidents were chosen to be recorded, the emphasis given within that recounting, and the communication of them are subjective and inevitably a function of the researcher's perspective. It may be sub-conscious influences that determine salient or typical features (Mason, 2002). We cannot eliminate these influences and, as per the discussion of an interpretive methodology, it is not a necessity in our desire to gain insights and understandings of a process, rather than identifying objective, predictive rules. It is a matter of awareness of their potential influence in the sedimentation of understanding, and recognition that these more informal observations are illustrative rather than formative of generalities.

Spreadsheet Activities

The students participated in four two-hour sessions, once a week, over four weeks, using spreadsheets to investigate mathematical problems. They had some initial instruction on using spreadsheets. Although aspects of the early sessions involved mathematics investigation as a context for familiarisation of the spreadsheet operational mode, the learning emphasis was on using them as a tool to explore the mathematical problems. These sessions took place at one of the partnership schools, in the same classroom where the group had gathered previously to participate in mathematics learning activities. The activity sessions were recorded and transcribed, and printouts and written material was collected and linked to the transcriptions (See Appendix A for example of activity). There were three follow-up sessions involving three pairs of the year six students from three of the schools, chosen for convenience reasons, and their initial willingness to articulate their approaches and conceptual understanding. They weren't the only children to meet these criteria, and were more typical of the participants than atypical, but some consideration of their availability due to the ongoing school programmes was taken into account.

All of the participants were involved in an on-campus day, based at the University of Waikato's Tauranga campus, where they participated in a range of mathematical experiences, predominantly investigative in nature, and including some with the spreadsheet as the pedagogical medium. A similar approach to data collection was used in subsequent years, with other groups of ten-year-old students involved in the Beach Brilliance programme.

Individual assessment tasks

The students did a number of problem-solving activities as part of the Otago mathematics problem-solving challenge. These activities were part of the ongoing programme in which they were participating. Their results can be analysed comparatively as part of a large national population of able Year six students. They did these assessment activities on three further occasions: 15 June, 27 July and 24 August as part of the year's challenge, and on two other occasions

with similar moderated tasks in November after completion of the spreadsheet sessions (See Appendix B for example).

Interviews

Interviews involve an exchange of perspectives and interpretation. They are the interchange of views, and position human interaction as central to knowledge production (Kvale, 1996). They allow the opportunity for better mutual understanding of meanings and intentions through this exchange, and might also allow the researcher access to further reasoning and motivation of the participants. While some forms of interview endeavour to maintain consistency to gather data with a degree of comparability across contexts, the aims of this research supported a more open-ended, semi-structured interview as the researcher sought to clarify descriptions and explanations, and share interpretations (Kvale, 1996). This more informal, open approach allows a researcher to better understand the participants' perspective on their own terms. Mason (2002) identified a range of styles of interview across the spectrum of researcher intention. He described the range of such a spectrum from those approached as 'fishing expeditions' with the hope that something striking might emerge, to those carried out to justify preconceived theoretical positions. He advocated an intermediary stance of using interviews to investigate theories-in-action. Open-ended questions give opportunity for flexibility with the responses; they sanction probing for clarification, and allow for unanticipated responses. In these three regards they were considered more suitable for my intentions than either fixed-alternative or scaled-type questions. Once more, there is an inherent subjectivity associated with the researcher's role in question selection and the trajectory of the interview through probing questions, but the rationalisation for the acceptance of this aspect has been discussed previously. The students were interviewed on two occasions to ascertain the approach they used to solve the problems in more detail: Once, before the spreadsheet activities (in August), and once after (in November). The interviews were in groups of four for approximately twenty minutes and were recorded and transcribed. They took place at the pupils' respective schools (See Appendix C for interview questions).

Questionnaire

The year six pupils completed questionnaires at the end of the spreadsheet activity sessions (in October), to gain insights into how using the spreadsheets affected their attitude and approach to doing the mathematical problem solving (See Appendix D). Questionnaires were used to complement the range of apparatus used to produce data to investigate the research questions. They provided structured data, and allowed a degree of anonymity for the respondents; an aspect considered beneficial regarding participants sharing their attitudes and perceptions. They gave opportunity to obtain data that could be quantified and compared (Cohen et al., 2000). As such, they afford information that is alternative in nature to some of the other methods, but which could augment the overall emergent insights and patterns in the results. A mixture of closed-response, open-ended, and rating-scale type questions were utilised. The rating-scale questions incorporated some level of sensitivity and differentiation with the participants' responses, although a defined-terms comparative scale was used, rather than a numerical one, as per Likert-style scales. Limitations with rating scale questions include the tendency of respondents to avoid extremist responses, the questions being researcher derived may not give full scope to the participants' views, and the meanings given to the terms used and the intervals between them by the participants. Some of these aspects were alleviated by the inclusion of associated open-ended questions. The age of the respondents also dictated the language and type of question used to some extent. They needed to be able to understand the questions and be able to form responses that would reflect their perceptions. I had used a similarly styled questionnaire with research involving eight–year-old children, and drawing on that experience thought the level was appropriate for this particular research.

Approach to Data Analysis

The conversations of the participants, while they negotiated both the context and the investigation of each intervention, were audio taped and transcribed. These then became the dialogue to be analysed. Checking by myself, and the

transcriber, was undertaken to ensure accuracy of the transcription, but an interpretative constituent is implicit to analysis of dialogue, and the researcher needs to be mindful of misunderstanding. It is, nevertheless, an effective way to gain critical insights into the participants' thinking. Attitudinal considerations, often associated with motivation in learning, were also considered, and these likewise required an ethnographic, interpretative approach to gain some clarity to the insights gained. Some attitudes became manifest through the recorded dialogue and inter-relational conversations, while the questionnaires and interviews revealed other insights.

The dialogue and interviews were then analysed by two independent but comparative mechanisms. The derivation of the filters used to sift the data produced by these research methods were related, and evolved in an ongoing formative process, but the sifting mechanisms were applied independently. The initial examination and screening considered emerging patterns of responses to the phenomena, and the familiarisation and engagement with the mathematical tasks. Consideration was given to aspects illuminated through previous research studies. Features, such as, the nature of any technical language, the quality of articulated reasoning, the amount of conversation, and differences in the discussions pertaining to the specific pedagogical approaches employed were starting points for this process. For example, was there evidence of understanding emerging through the visual aspects predicated by the spreadsheet's latent facility to display symbolic, numeric and visual representations simultaneously?

Characteristics and patterns were identified and a tentative list of features derived. Assorted snippets of inter-related data with some preliminary, informal examination were compiled. For example the following dialogue from a group of Year six pupils:

B: *So we need to think of a rule.*

A: *It's like double the number. It's nineteen, nineteen.*

B: *What about twenty? Oh you'll get twenty, twenty.*

And an interview response to the question: "What were the maths ideas the spreadsheet helped you with most?"

"It helps when you look at patterns. You just type it in and see the whole pattern", were both indicative of the participant using a visual lens to pose, then test a conjecture. An assortment of scenarios was developed from these snippets of dialogue that illuminated various aspects. This initial analysis also revealed some other potential aspects for consideration.

A list of these aspects was formulated to form the basis of the second tier of scrutiny. These were used as the primary trees and nodes for the *NVIVO* data coding. *NVIVO* is a derivative of *NUD*IST*, a software package for analysing qualitative data. The encoding process revealed the need for some refining of these nodes with the concluding form as below:

Table 1: Tree and node structure for final NVIVO data coding.

1. Initial approach:	A	visual i.e. table, sheet
	B	formulaic
	C	response due to medium
	D	Fill Down
2. Generalisation:	A	visual
	B	other
	C	implied
3. Conjectures:	A	pose and test
	B	develop
	C	reset (change the course of investigation)
	D	different response due to medium (including technical language)
4. Negotiation of meaning:		
	A	requirements of the task
	B	approach to task

C investigation of task

D synthesis of concepts

5. Spreadsheet structure shaping approach

6. Risk taking encouraged

7. Re-conceptualisation: A of problem

   B of data

8. Motivational/ enhance student interest.

9. Use of technical language:

   A to negotiate sense of the task

   B in initial approach

   C in developing ideas

   D in drawing conclusions

Other attributes such as document type (in-class dialogue or interview), gender (where identifiable), school, and group size overlaid these nodes to offer further potential points for differentiation.

The *NVIVO* coded data were reviewed and scrutinised for potential errors or misrepresentation through the placement of data. There was data that intersected with more than one node and judgements made where an item might not categorically fit with a particular node, but each section of data was apportioned to the most appropriate category. The data were then viewed within the various nodes, with consideration given to its contexts and form. Scenarios evolved from emerging patterns and these formed the basis for further analysis and reflection.

Pre-service teacher data

For the pre-service teacher data, three groups of three first year pre-service teachers worked in a typical classroom setting with counters, calculators and pen and paper available, and three groups, from the same class, worked independently, in an ICT laboratory, doing the same investigation using spreadsheets. Their discussions were audio-recorded and transcribed; each group was interviewed after they had completed their investigation; and their written recordings were collected. This data, together with informal observation and discussions, formed the initial basis for the research. Five weeks after the first data was gathered, a similar approach for data collection was used, with the students using the same medium, but a different investigation. Analysing both tasks provided greater depth to the data as the participants had undertaken more investigative work in the interim, and the data collection was hopefully less intrusive the second time. The participants were interviewed in groups following the investigative work to ascertain their perceptions of the learning process, their understanding, and some affective or motivational elements. The approach to the analysis was similar to that engaged with the other group of participants as described in the preceding sections. This data was part of the discussion.

Otago Problem Challenge

Only limited analysis of the children's approach to these mathematical problem-solving tasks could be undertaken, comparing their investigative methodology before and after the spreadsheet sessions. Firstly, of the five sets of questions the participants completed only the fifth set, SET5, occurred after the spreadsheet sessions were facilitated. As well, there was the eclectic nature of the tasks in terms of content knowledge and the aptness of strategies to solve them, and some of them were not suitable for spreadsheet investigation. The need for consistency in the administration of the tasks (e.g., they were done individually in a classroom setting without spreadsheets available, whereas the spreadsheet work was done collaboratively) also meant that attempting to establish causal links between the use of spreadsheets to investigate mathematical problems and an actual change in investigative strategies is tenuous. Some observations are

nevertheless pertinent and of interest. Assessment tools when completed were marked, recorded, and then filed. Data were analysed as a group, and comparative to the population of Otago problem challenge participants in 2004. Descriptive statistics were calculated and tabulated. Because the sample size was less than thirty, the students' t-test was used for the difference of means (Mendenhall & Ott, 1980).

This whole data set also required elements of reflective interpretation as it was reconciled with my evolving perspective on the research process. The assumptions made about the nature of the data, and any generalisations perceived, needed to be clearly articulated when the results from this analysis were examined and discussed. An anthropologist's lens was required to unpack, and historically situate these assumptions, as well as delving into the researcher and participant perspectives that this type of data and analysis presumes.

Questionnaires

The data from the questionnaire was collated, and recorded in tabular form for analysis. Although the study did not utilise a grounded theory methodology, to some extent the understandings of the learning situation and any patterns or generalisations that emerged, arose from the data. As such, the use of some grounded theory methods such as open coding were appropriate due to the longitudinal nature of the data collection (LeCompte & Preisle, 1993). This allowed opportunity for modification of the data groupings if required, with potential for redefinition based on evolving participant or researcher perspectives. The data was collated and examined to identify any trends in the responses, and to form some general descriptions of the data. However, with regards to the evolving perspective on the research process, the assumptions made about the nature of this data and any perceived trends; how this approach produces knowledge in a particular way, needed to be clearly articulated as part of the examination and discussion of results.

Personal Narrative

As discussed earlier in this chapter, educational research has frequently been grounded in approaches embedded in hermeneutic cycles. Action research and reflective practitioner processes are functions of the researcher interacting with the data or phenomenon, reflecting on the process and reshaping their perspective, then viewing the data through 'a fresh set of eyes'. They not only view the existing data differently, but by the nature of the process create further supplementary data. The methodology is part of the data and the researcher is part of the methodology. The space the researcher occupies shapes how the research is conceived, and how data is analysed, then reported. The evolving set of reflective spaces creates a fragmented set of researcher perspectives. My personal narrative was a vehicle for illuminating the fragmented perspectives employed and to situate those personal viewpoints, at any particular occasion. The personal narrative, and the transformative process I experienced were more than enlightening; they were part of the data itself, and fundamental to the methodology. Brown (in press) also discussed the gaps, that which is not present, in these viewpoints as being revealing. He suggests that the reflective stories the researcher tells provide data for analysis. Yet these reflections are influential and composite to the developing research, not mere neutral reflections. Analysis can be directed at examining the nature of the truth told and how this truth might be partial and seen as cloaking or activating alternative stories.

Proponents of radical hermeneutics and poststructuralists might contend that these gaps are deliberate, the consequence of 'other' predominant political and power discourses. They envisage knowledge and social realities adopting multiple portrayals or characterisations, determined by societal positions and the associated discourses (Burton, cited in Walshaw, 2001). Kinchloe and Berry (2004) refer to the notion of axiology in their discussion concerning perspectives of values and moralities and how they are allotted primacy. They contend it allowed the repositioning of the central dominant perspectives in the continuous scrutiny of meanings and interpretation. Researchers need to understand their location in a complex web of influences and use these varying positions to

discern their personal role in the interpretive process (Kinchloe & Berry, 2004). It's similar to a movie of an event. Cameras position the viewer in a range of perspectives, with each seemingly fluid rendition actually an historically sequenced series of snapshots. The researcher, as director, not only positions the perspectives, while receptive to unexpected ones, but edits what is noticed down to the detail of the single snapshot, to create the account of the event. These fragmented perspectives, influenced by the director's particular prevailing discourses, are reconciled through their overall interpretation; the story they create. Likewise, the researcher creates a story envisioned through fragmented perspectives of an historically situated series of snapshots. To continue the metaphor, the actors, director and audience are all aware of this version of truth, even if they temporarily suspend their sense of reality. They are also mindful of the director and actors' role in the rendition of the event, and that both are constitutive of its unfolding, while the audience makes a personal interpretation of the movie. Likewise, the researcher and the participants are both constitutive of the data and its examination, while the 'audience' will interpret the research through their own personal lens. The filmed event, meanwhile, might be integrated into a multitude of ongoing stories within varying contexts and histories, while influential in the ongoing stories of the audience. The research likewise is influenced by and influences other stories (as do the historically situated 'snapshots' from which it is composed). It becomes part of the fabric of stories concerned with the ongoing emergence of knowledge across the breadth of themes or perspectives through which it traverses e.g., mathematics education, using digital technologies, children's learning, methodology and methods, the researcher's perspective, the supervisor's perspective, the participant's and reader's perspectives, etcetera.

As all experience is mediated, an examination of the layering of the discursive formation of subjectivities uncovers socially constructed accounts of understanding. Since subjectivities are fluid and always evolving, understanding, meaning making, and knowledge are not set but are always in a formative condition. Theories are ever-changing forms of insight that might gesture towards a view of reality that is not describable in its totality (Mason, 2002). A personal narrative can depict this evolution of perspectives: it articulates the path the

researcher traverses. Hence it informs both the researcher and the 'audience' of the researcher perspective at various junctures of the research process.

The personal narrative emerged and transformed through various practices. The documentation of interactions with supervisors was potent in that my personal grounding was challenged in light of readings, discussion, justification, and negotiation of shared perspectives. Having three supervisors of varying philosophies and the use of video conferencing enhanced the utility of this procedure. An ongoing diary of reflections, particularly related to the vast array of philosophical and generic social science perspectives was also a powerful transformative tool. It not only shuffled my existing viewpoint, it enlightened and drew on fresh paradigms. The relatively 'free-flow of consciousness' writing associated with these diarised accounts loosened historically settled constructs and enhanced the reflective trajectory. Meanwhile, the writing and presentation of papers more formally signposted these constructions and promulgated this personal narrative. Papers were presented at the conferences of the Mathematics Education Research Group of Australasia, International Group for the Psychology of Mathematics Education, International Centre for Applied Research in Education, International Commission of Mathematics Instruction (ICMI), British Society of Research into the Learning of Mathematics, New Zealand National Numeracy hui, University of Joseph Fourier, University of Warwick, and Manchester Metropolitan University conferences or presentation days. Some of these were double blind peer-reviewed and published in conference proceedings. Articles have been published in Teachers and Curriculum (2004) and Mathematics Education Research Journal (2006), with a collaborative chapter for the ICMI: Technology Revisited book currently in process. Ongoing interpretations have therefore been reviewed and subjected to peer scrutiny. This evolution has influenced and been influenced by the research process.

Summary

While commencing the research from a predetermined disposition to methodology and the approach that would be employed, it soon became evident that a more formative, interpretive methodology would best suit the investigation of the questions and permit greater elucidation of the understandings and situations to be investigated. The writing and reflection processes were influential in this. The reflective and written processes can be described in terms of their function as data generation, as well as their explicit contemplative and synthesising attributes. The corollary being that the approach to analysis has to be more formative and evolutionary in nature. Therefore, the methodology must be structured to allow for this ongoing fermentation of approach and accommodate non-predisposed trajectories. That is not to trivialise a structured, clearly delineated path to data collection as outlined in the parts of the methodology; this is critical, and the interpretive, qualitative methods underpin the whole thesis. There is imperativeness however, to formally incorporate this data-producing feature of the reflective process into the methodology, as it illuminates key features of the theoretical perspectives taken, and situates the understandings gained within these. It is also instrumental in the reconciliation of the eclectic, and chronologically evolving, disjointed points of reference the researcher inhabits at various junctures.

The initial rationale for combined usage of quantitative and interpretative methods is well rehearsed, and allows some confidence in the analysis regarding the mathematical understanding. It was envisaged that more fulsome insights into the understandings would be facilitated, and how they transpire individually, from group and class situations. The various approaches were all formative in the evolution and emergence of the underpinning methodology, a moderate hermeneutic perspective. The personal narrative was an instrument through which this perspective emerged; some of the conclusions were a function of the reflective stance to this data. This chapter examined the methodology that evolved from the theoretic frameworks to which the researcher subscribed, that best suited the research situation and research questions. It described the rationale for the approaches and methods employed in the examination the research questions and elucidation of understandings and insights. The next two chapters include the results and analysis of some of the earlier findings and discussion

before the moderate hermeneutic lens was evoked. They are constitutive in both the methodology and the subsequent analysis, and hence are important in the articulation of the interpretation and understanding of the issues the research questions raise. Chapter Seven describes the transformative process that I underwent and the emergence of my current perspective; the lens through which the data was subsequently viewed. This prefaces the further results, discussion and analysis chapters that follow.

CHAPTER FIVE: Results and Discussion

Initial findings and analysis

Ka mimiti nga puna o Hokianga
Ka toto nga puna o Taumarere
Ka mimiti nga puna o Taumarere
Ka toto nga puna o Hokianga

Should the springs of Hokianga run dry,
The springs of Taumarere will flow
Should the springs of Taumarere run dry,
The springs of Hokianga will flow

Introduction

The research questions for this study were centred on the nature of the students' learning experiences, when mathematical phenomena were engaged through the pedagogical medium of the spreadsheet. How those experiences might have been influenced in particular ways, and the manner in which the students interpreted the situations through the various filters associated with that engagement, are implicit in the examination of those questions. The understanding that emerged for the students from their interpretations within that learning environment permit insights into the reorganisation of mathematical thinking evoked by the spreadsheet medium. The enquiry attended to understandings and meanings, and with context profoundly implicated in meaning, a natural setting was considered most appropriate, given the intrusion of the researcher. Hence the study was situated in classroom settings and approaches were initially used to gather the data that involved observation, description, and interpretation. Some quantitative methods were also employed. The next two chapters examine the results from these initial approaches to data collection in response to the research questions. It includes the results, discussion, and analysis of some of the earlier findings

before the moderate hermeneutic lens was employed. While to some extent these findings were superseded by those drawn through the more interpretative frame, they are nevertheless constitutive to both the methodology and this subsequent analysis, and hence are significant in the interpretation and understanding of the issues raised by the research questions.

A multi-modal approach to data collection was employed to gain rich, yet eclectic data regarding the learning process. Each of these approaches is considered in the following sections of the chapter. The description and preliminary analysis of the data is reported, together with my historically situated perspectives, and any perceived constraints with the type of knowledge the approaches allowed. Although the examination of the methods in Chapter Four rationalises the need for formative elements to the analysis of the data, the semi-structured, preconceived methods outlined were essential in the creation of knowledge to explore the research questions. Not only did they offer preliminary indications of the stories and versions of the ensuing data, but the observations, recorded data, printed output, and interview responses were the data re-examined through my evolving perspectives. They are directly constitutive of the analysis in this manner, but also indirectly through their influence on the interpretive space that was occupied at various points in time. Commonalities in the themes that emerged during the research process through the alternative methods were also considered, as they enriched the understandings and patterns that were noticed within the data. In the following sections, the results and some preliminary discussion of the observations are considered. This is followed in Chapter Six by a description of the interview data. The participants are identified as 'pupils' when the data is from ten-year-old primary school students, and 'pre-service teachers' when the data relates to the pre-service teaching students. The analysis of the Otago problem challenge results, with an associated consideration of the type of knowledge this produced, continue the chapter, along with a discussion of the questionnaires completed by the students. Chapter Six concludes with a section that examines the commonalities that emerged through the alternative methods, and attempts to situate the various perspectives historically, at each of these points of discussion and reflective analysis, within my evolving interpretation and the thesis overall.

Observations

The observations that took place in the study were varied in their intent and subjectivity. The most prevalent approach to data creation using observation was the audio-recording of the dialogue as the students engaged in the investigative activity. While this tended towards the non-intrusive rather than participatory end of the continuum of observation, the students were aware of the recording devices and my presence in the classroom, both of which must have influenced their behaviour and interactions. These episodes, and the associated output, were unstructured in approach, permitting the themes and stories that emerged to be more derived from the data than any preconceived notions. As well, there were observations made in situ describing the classroom situation, events, and behaviour, logs containing recollections of the episodes, and logs focussed around particular themes that had begun to emerge. The data from these various observational procedures were shaped around themes related to the research questions, specifically regarding the differentiation of the learning experience when mathematics investigation was encountered through the pedagogical medium of the spreadsheet. How might the experience differ from that with other media? Are the students focussing on different elements or at different levels of understanding as a result of their access being filtered by the spreadsheet medium?

At this stage, the data were viewed through a socio-cultural frame, as it was examined for evidence of occurrence or episodes of social interaction that illustrated that differentiation of learning experience, rather than envisioning whole ongoing episodes as the layering of interpretation and negotiated shared understandings. These episodes weren't situated within other connected evolving stories either. The data did nevertheless speak of key contributing aspects of those more holistic interpretations of the ongoing situations. The data related to these themes are discussed in the following sections.

Initial engagement

One aspect that surfaced very consistently through the sifting of episodes was the approach taken by the students in their initial engagement with mathematical tasks. Across a range of tasks, the students, sometimes after a brief familiarisation with the problem, immediately attempted to either generate tables or columns of data, frequently through the development of formulae and the fill-down function. These tables were subsequently analysed for patterns. The following excerpts illustrate how the pedagogical medium of the spreadsheet influenced the preliminary encounters with the tasks, and hence conditioned the emerging learning trajectory.

The following scenarios relate to the activity the 101 times table (see Figure 1). The first sets of data considered the distinctions between groups of pre-service teachers working with the same activity in similar time frames, but through contrasting pedagogical media: some with spreadsheets available, and others in 'typical classroom' settings. The groups working in the spreadsheet environment tended initially to perceive that the bigger picture was most easily accessed through entering a sequential, formulaic structure into the spreadsheet, before visually analysing the data for patterns. For example:

Kyle: *I haven't predicted. I was just going to put in A1 times 101 and drag it down (does it).*

A	B
1	101
2	202
3	303
…	…
…	…
14	1414
15	1515
16	1616

Josie: *So we're investigating the pattern of 1 to 16 times 101.*

This group moved straight into the spreadsheet environment to begin the investigative process. They drew on their understandings of the spreadsheet medium to create a table that they perceived as being central to the task. This was used to both familiarise themselves with the intent of the task, and for working through the actual investigation.

Another spreadsheet group illustrated this approach of producing a table of values in the spreadsheet to scrutinise for general qualities.

Jan:	*[Reads the task] Do you understand what that means?*
Rita:	*101, 202, 303, 404, and 505 onwards, because it is one times the number. It's straightforward in terms of doing the spreadsheet. It should continue to show that pattern throughout.*
Jan:	*We've got 101 times what?*
Rita:	*Times one, times two, times onward. Drag it down to the box in the bottom right corner and see what happens. I think it will probably pick it up [the pattern]. I imagine it'll be 1010 for ten. It'll be interesting when it gets to eleven, twelve, thirteen etc. Yes, 1313, 1414. It continues to show that pattern all the way through.*
Jan:	*We're up to 17.*

The pre-service teachers made sense of the situation and proceeded to investigate it through the scrutiny of the tabulated output. They indicated their expectation of a pattern, which might possibly lead to them forming a generalisation and expected that the spreadsheet would allow them to access that pattern quickly.

In contrast, the conversations in the classroom situation began with a group member initiating the negotiation of the meaning and requirements of the activity. This initial negotiated sense making started with a single discrete numerical example. These pre-service teachers used this not only to begin the

process of solving, but also to help determine the nature of the task; what it was asking them. For example:

Sarah: *So if we had twenty-three times a hundred you would have twenty-three hundred…Lets say we do twenty-three times a hundred-and-one, we would get twenty-three hundreds plus twenty-three ones*

Hemi: *Does it look right?*

Sarah: *Yes that is what I would guess it to be. Like if it was eleven times a hundred and one it would be eleven hundreds and eleven ones.*

While this is clearly the precursor to the process of generalising, the students needed to then verify these and other examples before using more recognisable language of generalisation. A second classroom group likewise went initially to a single example although they took a more sequential approach.

Justin: *What if we went one, two, three, four, five, six and multiply it by one hundred and one?*

Likewise in their recorded thoughts after the investigation, this approach was highlighted.

Carl: *We went through one at a time and solved them. We solved them on paper and we solved them with a calculator.*

This approach was evident with another group too. For example:

Eru: *Shouldn't we work through each one?*

These groups looked to evaluate individual numerical examples; to build up a numerical picture, usually in a written tangible form, before trying at a later stage to order it, analyse, and look for generalisations.

Meanwhile a group of pupils, working on the same investigative activity in the spreadsheet environment, moved almost immediately to the spreadsheet format, couching their ongoing investigation in a tabular frame.

Adam: *101 and then…Now 2 digit numbers. So we've got in the A column we have 101, in the B we have 1 to 15, and in the third column we have a formula.*

This produced the following output:

A	B	C
101	1	101
101	2	202
101	3	303
101	4	404
…	…	…
…	…	…
101	12	1212
101	13	1313
101	14	1414
101	15	1515

Beth: *Oh that's interesting – look at that. The numbers just repeat themselves.*

The students utilised the constructs of the spreadsheet environment to interpret the task as part of the familiarisation process, and then consequently to frame their emerging investigative trajectory. The structured tabular form of the spreadsheet output unfastened possibilities by shaping the output in a particular manner in their opening interactions with the task [*"Oh that's interesting – look at that. The numbers just repeat themselves"*]. Their interpretation and explanations thereafter were influenced by the nature of that engagement.

It was noticeable that the students were willing to enter something into the spreadsheet immediately. There was little attempt, in general, to negotiate the task situation through discussion or pencil and paper methods, although some individual processing of the task requirements must have occurred. For example:

Ali:	*So, we've got to type in 101 times.*
May:	*How do you do times?*
Ali:	*There is no times button. Oh no, wait, wait, wait.*
May:	*There is no times thing. Isn't it the star?*
Ali:	*=A1*101. Enter.*

When students engaged in other tasks, there was a similar trend to the appointment of the spreadsheet's functional propensities in the initial framing of the investigation. The following excerpts are related to an activity set in a scenario that allowed the students to explore different ways that they could get a pocket money allowance.

All that cash!

Congratulations. You have just won a competition that gives you pocket money for 20 weeks. You have to choose out of three options:

1. **$200 each week for the 20 weeks.**
2. **One cent the first week, doubling each week. Two cents the second week, and so on.**
3. **$40 the first week, and then $20 extra each week after that. $60 the second week, $80 the third, and so on.**

Investigate the various options and choose which one you would prefer. Explain your reasoning.

Figure 2: All that cash task.

In this excerpt, the pupils' opening interactions as they familiarised themselves with the task, were premised more on their mathematical preconceptions, yet the spreadsheet's influence was promptly exerted, as demonstrated by the insertion of, for instance, "That's sum" and "You can go equals A2 times two" into the dialogue.

Fynn:	*It's two hundred times twenty.*
Jane:	*That's sum. That one is one cent for the first week and then it doubles each week. Point zero one times two.*
Fynn:	*Point zero one is your first week and then point zero one times two would be your second week.*
Jane:	*Yes that is correct. So we have that in brackets again. That was the first one.*
Fynn:	*That is your second week.*
Jane:	*We only have one cent the first week. We need to figure how to get the cents to increase over twenty weeks.*
Fynn:	*You can go equals A2 times two.*
Jane:	*A is one column and B is one column right?*

They have incorporated elements of the spreadsheet environment into their initial familiarisation, and subsequent framing of the interactions with the task. The data produced from a group of pre-service teachers illustrated this formative, fusion of influences in their initial approach too.

Tim:	*I don't understand.*
Rewa:	*See two hundred, two hundred and two hundred. Auto sum it and its four thousand dollars.*

The pupils had incorporated the notion of adding groups of two hundred, with the spreadsheets capacity to undertake that task and calculate it through the auto sum function.

The next excerpt of dialogue and output related to investigating the second option. The pupils began by interacting within the spreadsheet environment but were motivated by the tension between the output and what they had expected. They appeared to rethink their interpretation of the task, then reorganised their

approach to its investigation. The pupils initially began to enter the counting number sequence into the spreadsheet.

1
2
3

One of the pupils, Mike, using his current understandings immediately had a conflict between the output, and his more global perspective.

Mike: *Hey there's a bit of a twist, look, third week he gets 4 cents. We'll have to change it.*

His partner Jay started to enter data into cell A3.

Mike: *No, no, no we'll have to be in C (column C of the spreadsheet).*
Jay: *Equals A1 plus one*
Mike: *No, no*

The properties and functionality of the spreadsheet environment structured the way they engaged with the task initially, and then conditioned their subsequent interactions with the task and negotiation of meaning. It also contained elements that led them to re-negotiate their sense of what the task was about - their interpretation of the task rather than just engagement in its investigation.

This next extract from an episode is related to a traditional investigation based around the story of the Grand Vizier Ben Dahir choosing his reward for inventing the game of chess. He wanted a grain of rice for the first square on the chess board, two for the second, four for the third and so on, doubling every square up to the sixty-fourth. The investigation introduced the story, then posed questions as outlined below:

Rice mate

1 **Work out how much grain Sissa is owed by the king**

2 **Estimate how many grains a metric cup holds. Use the rice and scales for this.**

3 **The world population is 6,221,409,060**

4 **Given a cup of rice will feed one person for a day, approximately how long will the rice on the chess board feed the world for.**

Figure 3: Rice mate task.

This investigation was introduced to a group after the pupils had already had some experience of using the spreadsheet. They were less tentative regarding the operational aspects of using it; for example, they were more comfortable generating formulas, and had an expectation of what output they might get based on some accumulated experience.

Ana:	*It goes 1, 2, 4, 8, 16 …., so its doubling.*
Lucy:	*=A1 times 2.*
Ana:	*Is that fill-down.*
Lucy:	*Go down to 64.*
Ana:	*Right go to fill, then down.*

They made an initial interpretation of the problem, and immediately saw a way the spreadsheet would help them explore the problem. Another group likewise moved immediately to investigating with the spreadsheet, but required some exploration and negotiation to produce the envisaged format.

Paul:	*OK, just type in one.*
Sue:	*Oh, what about equals?*
Paul:	*Right.*
Sue:	*A1 becomes two.*
Paul:	*You have to write in the number first [Sue enters 1 into the spreadsheet]. Now A1 times two where is the times button?*

Sue: *Times is the star button.*

Paul: *A1 star 2 [he enters A1 * 2].*

Despite some formatting considerations, the pupils were endeavouring to produce a formula. While their dialogue and engagement continued until the data was in the desired format, these initial interactions illustrated their intention to develop a formula for the purpose of exploring more than one individual case. In fact, after exploration and reflection, and with some assistance, they produced an extensive table to analyse. The pedagogical medium was shaping their initial engagement and the consequential investigative trajectory, while the activity was simultaneously influencing the way they used the spreadsheet. It illustrated the symbiotic relationship between the pedagogical medium and the engagements with the tasks.

The final excerpt utilised in the illustration of the ways the spreadsheet framed the initial engagements with the task is related to the following task:

Dividing 1 by the Counting Numbers

When we divide 1 by 2, we get 0.5, a terminating decimal.
When we divide 1 by 3, we get 0.33333…, a recurring decimal.

Investigate which numbers, when we divide the number 1 by them, give terminating, and which give recurring decimals.

Figure 4: Dividing one by the counting numbers task.

In this illustration, the pupils reflected on the task to gain some sense of the situation. After this brief, initial familiarisation, they entered data and began to explore within the spreadsheet environment.

Sara: *One divided by one is one - it should be lower than one.*

Jay: *Try putting one divided by two, and that should be 0.5.*

They then entered 1 to 5 in column A and =A1/1 in column B to get:

1	1
2	2
3	3
4	4
5	5

Sara: *Is it other numbers divided by one or one divided by other numbers?*

Jay: *Lets recheck.*

Sara entered =A1/4 and got the following output:

1	0.25
2	
3	

Jay: *Umm, we're not going to get change…we'll have to change each one.*

The pupils' initial engagements, while enacted within the spreadsheet medium, were problematic in that their preconceptions of using formulae with the spreadsheet conflicted with their mathematical predispositions and constrained the way the output was being produced. They appeared to sense that there should be a way to produce a table of values easily to explore, so they shifted between their mathematical understandings and their knowledge of using formulae in spreadsheets, to construct the table they imagined would allow them to explore the problem effectively. They explored other formula such as =B1/(4+1), before settling on the one they thought would be appropriate.

Jay: *Oh now I see =1/A1*

The spreadsheet environment shaped the pupils' sense making of the task while providing the environment to test the initial versions of their formula promptly.

Discussion of initial engagement

Having the spreadsheet as the environment for exploration appeared to afford quite distinctive initial approaches to the familiarisation of the tasks and the subsequent framing of the investigating process. In the informal evaluations of the data that were produced in the two different situations (the spreadsheet environment and the classroom environment), some differences became evident. The groups using the spreadsheet, after an initial perusal of the problem, appeared to look immediately for formulae to generate tables of values. They predicted, then verified within this tabular structure, to ensure that it was appropriate for exploring the problem, and then moved more directly to the generalisation phase. The use of a spreadsheet-generated table of values to predict, explore, then reflect is discussed in a separate section of the chapter. It appeared the initial framing of the task through this particular pedagogical lens gestured clearly towards the use of a tabular structure for the ensuing investigation of the tasks. Those working in the classroom setting discussed the problem, while trying one or two explorations with the numbers. As they made further sense of what the problem was about, they began to predict, verify and reflect with a discrete numerical example, before invoking a recording approach that enabled them to make generalisations more easily.

As evident in the preceding section, dialogue in each situation demonstrated a contrast in the initial approach to engaging in the mathematics. In the spreadsheet setting, the data told a story of using the spreadsheet to get a broad picture, with the formulae and copy-down functions used to create a numerical table that could then be analysed for patterns. The participants in these groups looked straight away to generalise a formula that they could enter and *Fill Down*. Their language reflected this, but the interactions also contained more language of generalisation, and it took them generally fewer interactions to start a more formal generalisation process. This may have been because the spreadsheet promptly produced a relatively large amount of data with which to explore the task, compared with the

classroom setting groups. Those working in the classroom setting used a discrete numerical example to engage in the problem, to make sense of the requirements of the problem as well as initiating the process of solving. They tended to try, confirming with discussion as well as another method (e.g., the calculator), prior to moving more gradually into the generalisation stage as their set of discrete examples increased enough to allow comparisons to be made. Their initial dialogue seemed more cautious, and contained comments requiring a degree of affirmation amongst group members before moving into a more formal approach to generalisation. While superficially these approaches had equivalent features, there was a contrasting approach to the initial exploration and making sense of the problem, that not only affected the subsequent learning trajectory, but the type of conversation that occurred and consequently, possibly the understanding of the mathematics.

Some caution needs to be exercised with this differentiation and the motivation for the students to embark on this distinctive approach to their initial engagements with the tasks through the spreadsheet lens. Firstly, they had worked and explored mathematics tasks in a spreadsheet environment previously with me, and although we had also worked frequently in other environments, the availability of the computers with spreadsheets, and my presence may have suggested the spreadsheet as a suitable avenue of their initial engagement. The types of tasks that were selected required them to be appropriate for investigation in the spreadsheet environment (e.g., an investigation involving geometrical transformations would probably be problematic at this age level in the spreadsheet environment), so there may have been recognition of this suitability by the students, which might have evoked that initial response. Their previous experiences with spreadsheets may have permitted them to recognise the advantages of using the spreadsheet in those types of situations. These influences would similarly be applicable to the discussion of all the identified aspects considered in this chapter, and are unavoidable with observation, and the employment of an interpretive frame. However, this research was not intended to establish objectivity through reductionist methods, but acknowledges the complexity and situated influences of the learning environment. It investigates questions to do with meanings and explanations that might emerge from

interpretation of these interconnected aspects and relationships. There is an imperative nevertheless, for this perspective and the acknowledgement of the researcher's intentions be articulated in conjunction with this discussion.

The excerpts cover a range of tasks and within this discussion of the data several versions of initial engagement with the tasks are illustrated. These include the student groups who, in the first instance employed the approach of developing a formula or table directly. The data also depict approaches where the students drew from mathematical understandings then, following a single preliminary interaction, engaged with the spreadsheet. Likewise, that initial familiarisation was sometimes to clarify the intent of the task, before engaging with the spreadsheet. Sometimes the groups moved straight to the spreadsheet and used that as the medium for familiarisation, while often they would reposition their goals or intentions with the investigation after those preliminary spreadsheet contacts. Some data indicate that the initial familiarisation process was undertaken within the spreadsheet environment. With another of the illustrative groups, the spreadsheet's affordance of providing almost instant feedback to input, facilitated the exploration of various formulae to produce possible tables of data that might be reconciled with their underlying mathematical discourse.

Encompassing these variations of initial engagement was the students' enacted intention of using the spreadsheet as the medium for exploration and the rapidity (frequently immediate) of that utility. The distinctive, initial engagement through this pedagogical lens fashioned the learning trajectory; it framed the subsequent interactions and interpretation as the students envisioned the investigative process through that lens. On the other hand, the data were also indicative of the activity influencing the understanding of the medium; their relationship was mutually influential. This echoes of the notion of instrumental genesis (Hoyles & Noss, 2003); that is, the student's engagement is shaped by the medium, but also shapes the medium.

The influence of the initial engagement through the spreadsheet permeated the consequent ongoing interactions. One particular approach it engendered was the

use of tables or columns of data to structure the investigation process. This aspect is examined in the next section.

Investigating within the spreadsheet's structured, tabular format

Several of the excerpts used as data in the previous section illustrate the manner in which the students fashioned their interpretation of the situation they encountered by the generation and subsequent reflection on a table of numerical output (e.g., with Adam and Beth). These tables were typically generated by a formula; they were a function of the formula generated to model the situation as interpreted by the students. As such they were relational, and while the formula was representative of that relation, the students typically viewed the output as a means to interpret, explore, and explain relationships linking outputs within the table, as well as the output as a function of the input. The passages cited in this section illustrate the manner in which the students investigated the tasks once the tables had been generated.

The first two excerpts relate to the task "All that cash!" (see Figure 2). In the first one, the group of pupils used *Copy* and *Paste* rather than *Fill Down* to generate the table for the second option for receiving the prize money (one cent the first week, doubling each week). They had already had some dialogue and interaction regarding formulas that might have generated the desired output, and had used the spreadsheet to calculate the first ($200 each week for the twenty weeks) and third ($40 the first week, and then $20 extra each week after that) options. It illustrates how the pupils used the relationship between outputs to form their generalisation. They used an iterative approach with each term generated from the previous term. In my view, the *'Copy and Paste'* approach to generating the table, indicated they had applied mathematical and spreadsheet preconceptions to their interpretations. From a mathematical perspective, it indicates recognition of a pattern, a sequence of numbers, while also indicating an awareness of the functionality of the spreadsheet (albeit an elementary one) to generalise and to model the situation.

Jane: *I'll do that copy and paste again like we did last time. That is forty plus twenty. It is sixty dollars the second week and three weeks and so on. It's the same principle though isn't it? Forty, sixty, eighty.*

The onscreen output was:

A	B
0.01	40
0.02	60
0.04	80
0.08	
0.16 etc.	

Fynn: *It'll be a hundred dollars.*

Jane: *Twenty, forty, sixty, eighty, a hundred, a hundred twenty.*

Both of these comments seem to indicate the visual tabular structure, stimulating a response drawn from the students' preconceptions associated with multiples of twenty. Fynn used his preconceptions to predict the next term in the pattern (100), while Jane does a similar rehearsing of known number facts. Interestingly, she began at twenty, perhaps indicating the recognition and situating of the output within that known sequence. It was the visual table of numerical values that appeared to have evoked that response. They continued until week 20, which gave $420, which along with the completed option three gave the subsequent output:

A	B
0.01	40
0.02	60
0.04	80
0.08	100
0.16	120
0.32	140
…	…
…	…

1310.72	
2621.44	
5242.88	

Fynn: *Number two is going to be the best option.*

Jane: *I think you would be correct. Yes one cent doubled is excellent.*

Fynn: *How did you solve the problem? In number one it's just two hundred dollars a week for twenty weeks. Two hundred times twenty is four thousand. Number two, how did you work that out?*

Jane: *We started with one cent. Our first equation was sum equals one cent. The second equation we worked out was the sum equal one cent times two.*

Fynn: *Then we just dragged the calculator down for twenty weeks, which gave us the doubling effect across twenty weeks and then used auto sum to add the whole lot together to give us the total.*

Jane: *I thought option three would have been better but evidently not.*

Fynn: *I looked initially and thought option three but obviously that doubling effect of option two is powerful. It would be nice to get five thousand two hundred and forty two dollars for us.*

The spreadsheet constrained the output within its tabular structure. The data were in adjacent columns that enabled them to be compared more directly. In this regard, the affordances of the spreadsheet influenced the students' interpretation. A group of pre-service teachers was working with the same task. The excerpt involving them likewise demonstrated how the output in columns seemingly allowed the data to be more easily interpreted.

Dean: *Yes. See what happens if you drag it. Take that one there and drag it down and see what happens, if it*

doubles. Grab the corner and drag it down one. If you drag it all the way down it should do it all on its own.

B
0.01
0.02
0.04
0.08
…
…
2621.44
5242.88

James: *Option two looks quite good. Its twenty weeks. Excellent. We got one cent for our first week, two cents for the second week, four cents by the third week and doubles that each week after that. Our total sum is…*

They totalled up the column using the *Sum* function giving $10, 485.75c. James continues:

From option one we started with four thousand dollars in total, option two we've got an option of five thousand two hundred and fifty dollars, which is even better.

In both of the previous passages, the way the data was structured within the spreadsheet format fashioned it in a particular manner. This seemed to influence the nature of their interpretation as the table structure enabled the output to be more directly compared. This aspect, allied with the propensity to rapidly calculate the sum of values within the table, motivated them towards this approach. They appeared quickly and easily to notice the relationship between numerical data within the table, observe patterns, and predict, enabling them to interpret and make decisions from a particular frame. While they might have used a number of computational approaches and produced a similar, hand-written table

of values without using the spreadsheet, the spreadsheet environment appeared to facilitate this approach more directly. It is an affordance of the spreadsheet environment. Once the tables were generated, they seemed to condition the students' interpretation and subsequent decision-making. Their thinking, as evidenced by their dialogue and actions, was organised in ways that were different from how they may have been with other media. This suggests an extending of the margins of what is usually perceived as school mathematics for that ten-year-old age level. The experience unhinged opportunities not available through other pedagogical media.

A group of pre-service teachers, working on the 101 times table task (see Figure 1), generated a table of numerical output by entering a sequential, formulaic structure into the spreadsheet before visually analysing for patterns. They produced the following output:

A	B
1	101
2	202
3	303
…	…
…	…
14	1414
15	1515
16	1616

Josie: *So we're investigating the pattern of 1 to 16 times 101.*

This appeared to be a relatively direct path to the patterning approach while the table structure also assisted with the sense making of the purpose of the task. They viewed the output from this tabular perspective, and their dialogue illustrates how, several comments later, this group had recognised a pattern, and explored it further based on the rule for their pattern.

Kyle: *It goes up in hundreds plus one.*

Josie: *It's because one doesn't change the multiplying.*

Kyle: *Its always one so you are adding on.*

Josie: *So 2 times 101 is 202.*

Kyle: *101 times 2 is 202? When you multiply numbers by 101, you also notice 3 times is 303 and 4 times is 404. So if you went 20 times 101 it would be 2020.*

The visual pattern formed within the table formation seems to have prompted the beginnings of an informal conjecture premised on the visual attributes of the output. Josie applied that to a number outside the scope of their table.

Josie: *If you did a huge number like five hundred times 101 it would be 500500 wouldn't it?*

Kyle: *Lets have a look. It's 50500 and it's just shown that it doesn't do that.*

The output was different from what they had expected, and it seemed to lead their investigation in a different direction, as they investigated the patterns formed by multiplying three-digit numbers by 101. Once an initial formative conjecture had been fashioned, the application of that to a fresh situation (the three-digit numbers) created a tension that initiated a change of perspective, and hence led the ongoing investigation down a particular pathway. The spreadsheet medium appeared to provide opportunities to interpret in a specific manner, while in other instances, it created tension that facilitated further investigation, reflection, and associated mathematical thinking.

The next group, also pre-service teachers, explained their generalisation in slightly different terms, but still seemed to illustrate the spreadsheet's affordance, when the output was structured in tables, to induce interpretation framed in visual terms. Again the particular nature of the engagement with the task through this pedagogical medium led the investigation and explanations being shaped in a particular manner. Their table was similar to the previous ones associated with this investigation.

A	B
1	101

2	202
3	303
4	404 etc.

Jan: *Make some rules that help you predict when you have a 1-, 2-, 3-digit number. Do they work? Well we did the single digits. The first one we did was 101 times one equals exactly what it is. 202 times two is exactly what is it, 202 [this was a slight misreading of the output, rather than an interpretational aspect].*

Rita: *So the first and last digits match the number that you are multiplying by for single-digit numbers. Once you get on to two-digit numbers, the first, second, third and fourth digits match the number that you are multiplying by - like 1818 for 18.*

Jan: *For our three-digit numbers we've got 101 times 120. The last digits add up again?*

Rita: *Yes, the first digit and the last two match the number that you are multiplying by.*

It seemed to me that the generalisations expressed in this manner were based on their interpretation of the table of values. This interpretation and the associated explanations were articulated in visual terms; that is, to the position and matching of the digits when the output was compared to the input. The continuing investigative process was then framed by this interpretation. Therefore, it is appropriate to conclude that the tabular nature of the spreadsheet environment had shaped the learning trajectory and fashioned their thinking and understanding in distinctive ways.

In the next excerpt, where the "Rice Mate" investigation (see Figure 3) was being explored, the tabular structure constrained the nature of the output in a way that affected the ongoing investigation. However, it also, due to those constraints, produced data in a form that opened opportunity to explore new content

knowledge. The pupils used a doubling formula and *Fill Down* to produce a table as shown below:

1
2
4
8
16
…
…
2147483648
4294967296
8589934592
1.718E+10
3.436E+10
6.8719E+10
1.3744E+10
etc

Jim: *The computers going off what we want.*

They sought my input and negotiated a sense of the numbers based on the E+10 indicating where the decimal point would be, and the use of place holding zeros. I also explained to them that the scientific form was used to allow the spreadsheet to display very big and very small numbers. They wrote the number down on a piece of paper and inserted commas: 13,744,000,000

Joc: *Thirteen billion, 744 million.*

They continued with the investigation but now seemed to have comfortably incorporated the scientific form into their content knowledge, albeit tentatively at that stage.

Jim: *9.22E+18. So that decimal point goes up eighteen places.*

Joc: *So that would be 922 and sixteen zeros - a really big number.*

This excerpt illustrates, amongst other aspects, the facility of the spreadsheet structure to present the output in a manner that allowed opportunity for the pupils to engage with new content knowledge and then utilise it in a meaningful way. The tabular structure had influenced the ongoing form of the investigation by facilitating the engagement of the new ideas and the subsequently altered perspective. It revealed prospective student learning trajectories particular to this pedagogical medium and gave potential extension of the conceptual envelope associated with this number domain.

In the final illustrative excerpt for this section, the group of pupils engaged with a task "Save Some, Spend Some", where students in the activity were considering different scenarios for saving to buy a DVD player. In this episode, the pupils used the table of values in an interactive manner to explore the output and compare it against a fixed mark. They were interested in the input of the various scenarios that produced output around a given value. There were also some personal value decisions to be considered as well.

Tama:	*No. We don't add them together - it says each one. Like when will each person have enough money to buy a DVD. How much is a DVD? $240, so we'll have to highlight many squares, lets say go down to 25.*
Rachel:	*Oh I see it.*
Tama:	*Formulate one at a time. No, we should do it all at once. Oh that's more than enough, cool, way more, Yes cool, that's way more that amount. So after 25 weeks she has $262.*
Rachel:	*I say 24 weeks we don't count the first week, we've got week then ..*
Tama:	*After 24 she's got…Daniel had it at week 24 then.*

They then explored another scenario:

Rachel: *We go way back up to week 7. So what's the formula now? =0, =, =B7 +*

Tama: *334, she gets there by the fifteenth week.*

And with further exploration and negotiation, the third scenario was considered:

Rachel: *From week 7 onwards – won't that be C9+10.*

Tama: *Right, do we get $155…147, 146…174.*

Rachel: *=C5+11. Highlight down. Fill Down. There, he has enough.*

This episode illustrates how the pupils' use of the tabular structure within the spreadsheet, to investigate various scenarios, appears to have influenced the interpretation of the situation and the choices the students made, thus guiding the nature of the ongoing investigative process. In this situation they were concerned with the comparison of input that produced the range of values in the table, as compared to output only.

Discussion related to the spreadsheet format

This section of the chapter considers the ways in which students negotiated a pathway through an investigation, having created a table of numerical values as part of their initial engagement. The data excerpts used illustrate the manner in which the students fashioned their interpretation of the situation they encountered by the generation and subsequent reflection on these tables of numerical output. These tables were typically generated by a formula, and were a function of the formula generated to model the situation as interpreted by the students. The students engaged with the tables in various ways, although typically they viewed the output as a means to interpret, explore, and explain relationships linking outputs within the table, as well as the output as a function of the input. The data passages cited in this section illustrate the manner in which the students approached the tasks once the tables had been generated.

While the tabular structure constrained the output, and as such may have limited some investigative aspects, nevertheless it seemed to offer opportunities as well. It allowed students more directly to compare adjacent values or columns, highlighting aspects of the patterns they were attempting to analyse. Sometimes this was between outputs and at other times the focus was on inputs. This facility to compare so readily left space for other influences: For example, personal value judgements might have been more accessible and influential in the investigative process. The students could operate on a table of values that, coupled with other affordances such as the almost immediate response to the input of data, allowed them to interpret and make decisions far more readily. They might more quickly and easily have perceived relationships between numerical outputs within the tables, to have more readily seen patterns on which to base their predictions and generalisations.

In several of the excerpts the tabular structure caused tension or perturbation with the students as the output was vividly shown to be at odds with their present mathematical understandings. This sometimes enabled them to make better sense of the intentions of the task or at times, in my view, led them to the engagement and reconciliation of new content knowledge. The tabular structure appeared to facilitate the interpretation of the patterns in visual terms, with the position and visual pattern of the digits given primacy. This visual aspect of their interpretations and explanations led to predictions framed by visual patterns. However, when the visual pattern was transformed with a change or extension of the input, the tension evoked prompted them to consider their position and reinterpret from a different perspective. This is consistent with the previously reported findings of other studies (e.g., Baker & Biesel, 2001; Borba & Confrey, 1996; Sacristan & Noss, 2008) that identified the multi-representational nature of data display afforded by digital media and the propensity for interplay between those representations. In those studies, two or more of the symbolic, numeric, and visual (e.g., graphical or diagrammatic) forms of the mathematical phenomena were linked. In the episodes reported in the present study, it was a symbolic form (the formula), numerical, and visual in the form of the structured table that were connected. In several instances, the students also graphed the numerical data, but

frequently the visual tabular structure helped to unlock the patterns they were seeking, and allowed them to pose informal conjectures to explain those patterns.

The table format of the spreadsheet, once invoked, appeared to have influenced the nature of the students' engagement with the tasks in the particular ways described above. This seemed to lead to the investigative trajectory being framed then fashioned through those particular visual influences. The affordance offered by this pedagogical medium appeared influential in the students' interpretations and explanations of the situation. There were other affordances of this particular pedagogical medium the data illustrated that were noticed. The following sections address each and illustrate them with a brief, single episode.

The facility to manipulate large amounts of data

At various junctures in the research process, the students recognised the spreadsheet's facility to undertake a large number of computations almost simultaneously, frequently through the application of a formula to produce a table of output. This appeared to promote that particular approach to the initial engagement and ongoing investigative process. From my perspective this allowed the students to focus more on the broader issues of the investigations based around the mathematical thinking, such as generalising, predicting, forming and testing conjectures, rather than having to spend considerable time on computational tasks to produce sufficient comparative data for those purposes. This particular characteristic of the environment seems to have facilitated the use of tables of output, and enhanced the subsequent interaction evoked by the engagement of those tables. In the following episode this propensity seemed to be accentuated. The pupils were investigating the task "Dividing one by the counting numbers" (see Figure 4). They had already generated a table of values and formulated an emerging theory. After several further interactions and refinements, Sara notices something in the table of values:

Sara: *If you take these numbers out they double and the answer halves.*

Jay: *That makes sense though, if you're doubling one, the other must be half.*
Like 125 0.008; 250 0.004.

Sarah: *What's next? Let's check 500.*

Jay: *Let's just go on forever.*

They generated a huge list of output, down to over 4260. The nature and structure of the spreadsheet enabled them seamlessly, yet intentionally, to generate large amounts of relevant data, thus fashioning the emerging theory.

Jay: *500 0.002; 1000 0.001.*

While it would have been possible for the pupils to work out each of these computations individually and record them manually, this would have had limitations both in terms of motivation and interest for ten-year-olds. It probably would have disrupted the flow of their interpretive thinking and might also have incurred some computational errors. The facility to manipulate large amounts of data permitted the generation of tables, with the influence of these on the investigative process well rehearsed previously in this chapter. When the students wanted to explore within a table structure or their investigative trajectory gestured towards it, it enabled them to explore a range of data that wouldn't have been feasible in a typical classroom, pen-and-paper environment. It appeared this characteristic of the spreadsheet medium influenced the manner of the students' engagement and learning in a particular way. An observed comment from one of the pupils also recognised this characteristic:

Sara: *You have unlimited room. You can go forever [seemingly]. You can fill out a whole lot quickly that you can't do with a calculator.*

The 'quickly' aspect to the comment shows the way towards the next section where the spreadsheet's attribute of giving almost instantaneous response to the input of data is considered.

Giving immediate feedback to the input of data

The almost instantaneous nature of the response in the spreadsheet environment, coupled with the interactive nature of the engagement, allowed for the ease of exploration of ideas. The facility of the spreadsheet medium to immediately test and reflect on existing knowledge was an influence on the learning process. Again, this attribute had been evident with the earlier sections of this chapter and was identified in the literature review as an affordance of digital technologies in general. The episode below is specifically illustrative of this characteristic and its influence on the learning pathway. The pupils were investigating the 101 times table task, and having explored it with a table, have been through several iterations of their conjecturing approach. They considered what the pattern might be if decimals were used:

Beth: *Okay do a few with decimals 4.35.*

They entered 4.35 into their workbook producing the following output:

101	4.35	439.35

Adam: *Try a higher one 43.5.*

101	43.5	4393.5

Adam: *4393.50, a whole new can of worms here.*

Beth: *Although the numbers look the same.*

They considered the output as it appeared on the screen:

101	435	43935
101	4.35	439.35
101	43.5	4393.5

They inputted another:

101	0.435	43.935

Beth: *They are the same numbers but just with the decimal.*

The pupils were able to test values and obtain an immediate response. This allowed their predictions and evolving conjecture to take shape, as output was able to be considered quickly and either discarded or folded within their interpretation. Discussion was stimulated, as the results of prediction or conjecture were viewed rapidly and were more easily compared and reflected upon. This enhanced their use of logic and reasoning as the pupils investigated, then endeavoured to explain deviations from the expected output, or opportunities that the output evoked. An observed comment from another situation further emphasised this attribute:

Tama: *Highlight the row…Bingo. Just highlight and do it. Its done.*

The spreadsheet's facility of giving an almost instantaneous response when data was inputted into a formula enabled the students to be more interactive and responsive to the output. They were able to test their emerging formative theories quickly and model situations relatively easily. In my view, this meant they took a more exploratory approach and seemed to be more willing to try and then engage or discard as appropriate, compared to situations when there might have been a greater investment of time in computational aspects.

The nature and effect of technical language

The language used by the students included technical questions and statements, primarily regarding spreadsheet operation. The students needed to negotiate a shared understanding of these alternative versions and appeared to be able to do so through their engagement with the tasks, including the dialogue with me and other students. The spreadsheet approach, perhaps due to the actual technical structure of the medium, seemed to lead more directly to an algebraic process, with the language interactions containing both algebraic and technical terminology. This also introduced a difference in terms of the technical language utilised. Did this alter the

way the students negotiated their informal conjecture or proceeded to analyse it? "Drag it down" is functioning language rather than mathematical, but the inference is clearly that there is a pattern, which might possibly lead to a generalisation. It seemed to me that the students assumed that the spreadsheet by nature would have enabled them to quickly access that pattern. The following episode illustrates some of the ways the students used technical language and possible implications for their research strategy. The pupils were investigating the "All that Cash" activity (see Figure 2). They had begun some exploratory activity and entered several formulae that produced output related to the doubling of numbers.

Jane:	*Copy and paste.*
Fynn:	*It's better to double the cell than double the amount. Instead of going point one times four, just go the sum of B2 times two.*

They were using words like *Copy* and *Paste*, *Cell* and cell references such as "B2" in ways they both had an understanding of. Not that this understanding was the same for both the pupils, but there are some similarities in the meanings and intentions of the words between the two of them. Their dialogue continues:

Jane:	*The sum of B2 times that.*
Fynn:	*Take out the zero point zero one because all you are doing is doubling the cell.*
Jane:	*It needs to be bigger. Copy. B3 times two, correct?*
Fynn:	*Yes. See what happens if you drag it. Take that one there and drag it down and see what happens, if it doubles.*

The technical language seemed to be a feature of the dialogue between students in the spreadsheet environment, and the use of this language opened opportunities for possible directions to be explored. 'Copy and paste' and 'drag it down' carry meanings for the students beyond the confines of this study but they indicated how the students might have investigated, or had opportunities to explore, in ways specific to the spreadsheet medium. Considered through this viewpoint, the words

are both imbued with connotations of iterative patterning and possible associated generalisation. These connotations were particular to the students involved and the situation in which they were engaged. They were historically and contextually situated. Even so, it is reasonable to assume that for the participants, their use evoked a unique understanding, but one related to patterning that hence influenced the way they interpreted the task and their subsequent engagement with it. This would likewise have influenced their investigative pathway and the understandings that might have emerged.

Conclusions

As the research questions were concerned with the students' learning experiences and the environment in which the mathematical tasks were engaged, the research was situated in classroom settings. Observation allowed the data to be obtained in these more naturalistic settings; settings similar to the learning environments the students would have typically engaged with school mathematical tasks. A conclusion was that data from these settings would inform the examination of the research questions. However, no matter how unobtrusive the observer's position, unless the participants were completely unaware that observation was occurring (and this might have compromised ethical considerations) their behaviour would be affected by an awareness of the observer to some extent. The participants' interpretations and choices would inevitably have been influenced by the presence of the researcher. Likewise, no matter how unstructured the observational approach, the observer's perspective would still have been influenced by the selectivity of the noticing process (Mason, 2002). While there must, by the nature of the act, be a degree of subjectivity in any interpretation or explanation of observation, what we notice when we observe is value- or theory-laden from our fore-structures or underlying discourses. The gaze of the researcher is implicit in the data.

Ethnographic research is concerned less with predictive generalisations, than with the formation of generalised descriptions and the interpretation of events. The researcher's perspective is not the sole contributor: there is also the need to gain understandings of the learning occurring at an individual level, and the possible

reasons for this. That is, the understanding of actions or implications rather than causes. This also indicates the need for elements of an interpretative paradigm. To gain insights into, and an understanding of, the learning that might occur for individuals, observations in the learning environment were used to provide information central to the research questions. Understanding can only ever be historically and socially situated, so insights and interpretation associated with understanding will be best contextualised within the related setting. Objectivity in observation and interpretation is not measured by the degree of separation between the observer and subject, but might occur when there is negotiable agreement of interpretation, meaning, and significance (Mason, 2002).

Given those constraints, the observational data nevertheless illuminated what could be regarded as fundamental features of the spreadsheet as a pedagogical medium, when students investigate mathematical tasks. The students were inclined towards a reflective, cyclical approach involving making sense of the problem, prediction, verification, reflection, and generalisation. Discussion, and interaction with the medium and the task were intrinsic to that approach, and for each group these were invoked throughout the episodes at different stages, for different purposes. Further iterations of the cycle occurred in varying degrees, followed by communication of a perceived solution in terms of the problem's context.

The spreadsheet environment drew a distinctive response to the initial engagement with the tasks. While each episode was unique, a tendency almost immediately to engage the spreadsheet by the creation of formulae or tables of numerical output at the initial stages, was evident throughout the data. This was part of the familiarisation or making sense of the problem stage, while framing the trajectory the students navigated through the tasks. As described in the discussion of this aspect above, there were various permutations of the nature and chronology of the engagement, reflection and articulation, but a common thread was the participants' intention of quickly generating output to explore for patterns, and this initial exploration influencing their ongoing interpretation of the purpose and the meanings implicit to the tasks. The initial engagement was borne of their preconceptions and current understanding in the domains associated with each task and the environment, while these in turn were influenced by that engagement. The excerpts

illustrated occasions when the student groups employed an approach of developing a formula or table directly in the first instance; drew from their mathematical understandings, then following a single preliminary interaction engaged with the spreadsheet; used their initial engagement with the spreadsheet as the medium for familiarisation, often with an associated repositioning of their goals or intentions; and explored various formulae to produce possible tables of data that might be reconciled with their underlying mathematical discourse. Within these variations sits the distinctive response of an initial engagement with the spreadsheet and its associated affordances.

The distinctive, initial engagement through this pedagogical medium influenced the learning trajectory, framing the subsequent interactions and interpretation as the students envisioned the investigative process through that lens. It permeated the subsequent ongoing interactions. One particular approach it engendered was the use of tables or columns of data to structure the investigation process. This affordance of the spreadsheet influenced the way the students interpreted and explained their emerging generalisations and informal conjectures. While the spreadsheet environment, in conjunction with other socio-cultural influences, filtered their thinking it appeared that the visual structure of the tables was influential in the investigative process. The students were able to compare more directly adjacent values or columns, illuminating characteristics of the patterns they were attempting to analyse. These patterns were manifest as relations between output, between input and output, and at times between inputs. This facility to compare so readily left space for other influences such as the personal value judgements that might hence have been more accessible and influential in their reflection and decision-making. In some of the illustrative excerpts, the students operated on the table of values and this, coupled with other affordances such as the almost immediate response to the input of data, allowed them to interpret and make decisions far more readily. They might have perceived the relationships between the numerical outputs within the tables differently. From my perspective, the students conceived the patterns on which they based their predictions and generalisations, through a visual lens. They articulated their explanations in visual terms as well.

In several instances the output generated in the table structure caused tension or perturbation with the students as it was clearly at variance with the adjacent output or their prevailing mathematical understandings. This sometimes enabled them to make better sense of the intentions of the task or at times facilitated the engagement and reconciliation of new content knowledge. The tabular structure appeared to condition the interpretation of the patterns in visual terms, with the position and visual pattern of the digits given primacy. It appeared that this visual aspect of their interpretations and explanations led to their predictions being shaped by visual patterns. However, when the visual pattern changed due to a variation of input, the tension evoked prompted them to reconsider their emerging theory and reinterpret it from a different perspective. The table format of the spreadsheet, once invoked, appeared to influence the nature of the students' engagement with the tasks, and from my viewpoint, this seemed to lead to the investigative trajectory being framed then fashioned through those particular influences. This affordance offered by the pedagogical medium appeared influential in the students' interpretations and explanations of the evolving situation.

The tendency to develop tables of data to model the situations was supported by the spreadsheet's facility to manage and operate on large amounts of data simultaneously. This coupled with the propensity of the spreadsheet medium to allow the students to test their predictions and conjectures immediately, then reflect on existing knowledge, was an influence on the learning process. This is consistent with other findings (e.g., Deaney et al., 2003). These affordances also permitted the students to focus more on the broader issues of the investigations based around mathematical thinking, such as generalising, predicting, forming and testing conjectures, rather than having to spend considerable time on computational tasks to produce sufficient comparative data for those purposes.

The spreadsheet groups also used more algebraic and technical language, for example, formula, while the pencil and paper groups had more numerical references. While the two were linked, the differences in language were probably a reflection of the distinctive approach engendered by the two settings, rather than the differences in language evoking distinctive approaches. Whether this negotiation of procedures, and the different style of social interactions initiated, changed the

approach to the mathematical dialogue is difficult to ascertain, but considered in conjunction with other aspects, it certainly seemed to lead to a different contextualisation of the mathematical ideas. The students investigating within the spreadsheet medium also moved more quickly to the generalisation process - they fashioned a faster moving account of their interpretations.

The observational data in this study illustrate that different pedagogical media provide a distinct lens to contextualise the mathematical ideas, frame the mathematical exploration, and condition the negotiation of the mathematical understanding. The affordances of the spreadsheet environment provided a particular flavour to the students' approaches, interpretations and explanations. While this appeared to have unfastened opportunities, it also constrained the nature of the engagement. Nevertheless, the learning trajectory, and by implication the understanding, had distinctive features when the tasks were encountered through the spreadsheet environment. The manner in which they approached the investigative process differed, and fresh ways to engage with the mathematical phenomena were evoked. As well, the students were accessing mathematics ideas that would not have arisen through alternative media. The students' actions and associated dialogue indicated that the pedagogical medium of the spreadsheet allowed them to organise and reorganise their thinking in a distinctive, alternative way. The map of their mathematical perception was re-orientated, anchored by different features. Engagement of the tasks through this medium unhinged opportunities not available in other media, and extended the boundaries of school mathematics for those students. This, by inference, also extended the boundaries of mathematics itself. The data provided examples of how mathematics is re-configured through specific educational experiences. This is an aspect that has links to the socio-cultural formation of mathematics, and is addressed further in the conclusions.

In Chapter Six the results of the interview process are described, followed by an analysis of the Otago problem challenge and questionnaire data in terms of how they relate to the examination of the research questions.

CHAPTER SIX: Initial Findings and Analysis

Ruia, taitea, kia tu
Ko taikaka anake
Strip away the bark
And expose the heartwood

Interviews

The students were interviewed on several occasions in groups, over approximately twenty- to thirty-minute time periods. The interviews occurred both before and after the spreadsheet sessions, in classroom settings with which the students were familiar. They were semi-structured in nature, containing a range of open-ended questions (see Appendix C). Some forms of interview endeavour to maintain consistency, and give primacy to data that the researchers might wish to compare across contexts. However, the aims of this research supported a more open-ended, semi-structured interview style, as the researcher sought to clarify descriptions and explanations, and to share interpretations. This more informal, open approach was intended to give the researcher better insights, to better understand the participants' viewpoint, in their own terms. Interviews involve an exchange of perspectives and interpretation. They are the interchange of views, and position human interaction as central to knowledge production (Kvale, 1996). This exchange allowed the opportunity to negotiate a shared understanding of meanings and intentions, and offered me the opportunity to access the motivations of the participants.

The students' responses and comments in the interviews were clustered around themes that the students had recognised as being distinct in their approach, as they used spreadsheets for mathematical investigation. They highlighted aspects that the students identified as being influential in their engagement with the tasks through using the spreadsheet medium. The

majority of the interview data is from the pupil groups. When it is from the pre-service teacher groups, this is indicated. Some of the data were direct responses to questions in the interview schedule (see Appendix C), or subsequent probing questions. As such, the researcher's perspective and intentions colour the nature of their responses to some degree. They were the students' thoughts nevertheless, and provided insights into their thinking regarding various aspects of their engagement.

Students' previous experiences with spreadsheets

The first question in the interview was an introductory one regarding the students' prior experience of using spreadsheets in mathematics. The majority of students had neither used spreadsheets with mathematics nor had any previous experience with them in any other context. Of the ones that had, most had used them to list data and draw graphs, while some had also used them in computational and modelling situations. The following comments were typical of those:

Adam: Graphs, I've used them for graphs.

Diane: Adding up, drawing graphs

A number mentioned exposure to their use in mathematics-related activities at home. For example:

Jay: Yes, at home I have.

Chris: My brother made a timetable for school once; you can make calendars.

Jo: My Aunty uses it at work… sometimes people use it at home for things like their budgets and things like that.

Jeff: When we were building a house, that's what we did; I put all the costs down of what I was going to spend and then what I actually spent (pre-service teacher).

While these experiences would have influenced their engagement in the study, it is my belief that the ongoing interactions, and other associated discourses, would have folded into the emerging understanding to a greater extent than the influences of the earlier contact. Each of the following sections is centred on a particular aspect of the learning experiences as illustrated by the interview data.

Students' initial engagement

One aspect evident from the data regarding the initial engagement with the task was the regularity with which the students immediately employed the spreadsheet medium. This was consistent with the observational data. In response to the question: "When you saw the problem, how did you think you would start?" the students' data from the interviews illustrated this trend. In many instances from the students' viewpoint, this initial engagement involved attempts to model the situation with a formula. For example:

Fran: Thought of a formula.

Cam: You had to think of a formula and sometimes it was hard to think of one and you would have to get it right otherwise it wouldn't work.

Ben: Because of spreadsheet, we went straight to formulas, looked for pattern, for a way to make the spreadsheet work.

Some of the students who just responded "formulas" were asked the following probing question: "And how did you start thinking about what formula you would put in?"

Alan: The letters and the numbers along the top and down the side. You put the = and then you put like the question in; like if it was 22 – 7 + 6.

Fran: I pretty much just looked at the question and tried to work out what it was asking me to do, either times it or divide it.

For other groups there was a brief, preliminary phase of making sense of the intentions of the task. For instance:

Sara: Re-read to get into the maths thinking, then straight to a spreadsheet formula.

Beth: I looked at how it was written down and looked at all the patterns; then I sorted it out in my head then put it down [the formula] and if it wasn't right then try another one. Experiment.

It was also clear from their dialogue and responses in the interviews, that the spreadsheets had provided not only a unique lens to view the investigation, but had possibly drawn a distinctive response in terms of investigative practice. Students experimented with various formulae within the spreadsheet environment. For example:

Cam: We put something and had a look and if it wasn't right, I'd just do another one and keep going.

Greg: I type what I think and try it.

The following excerpt with a group of pre-service teachers included their responses to some probing questions:

Adam: Um … I looked at it for a bit, tried a few formulas and found out what one is correct.

NC (interviewer): And how did you make a decision on which one was correct?

Beth: Just testing them out and then we thought that would be … that we would point out the answer and type it down.

NC: Can one of you say that again; just explain what you did?

Adam: Like what we did is we tried a few formulas. To start off with we like typed in a few formulas that we thought it might be, and then went through and got the correct one, which got us the answers.

This pre-service teacher's perspective of "the correct one" is interesting. There was some interplay and negotiation between their mathematical understandings in the associated domains, the student group, and their engagement with the task through the medium. This facilitated this later reflection, a seemingly simplified perception of getting "the correct one". Likewise:

Ana: Looked at how is was written down and look at all the patterns; then I sorted it out in my head then put it down, and if it wasn't right then try another one. Experiment.

This following excerpt similarly illustrated the opening investigative approach unfolding within the spreadsheet environment:

Awhi: I preferred thinking something about what I needed to do, then take it and highlight it down and then the whole table is there, which would help me.

NC: How did you know it was right?

Awhi: I used trial and error.

A small number of groups in the first instance entered number values to make a table that they could then analyse within that structure. These instances seemed to indicate the students' recognition of the value to be gained from using the spreadsheet, perhaps because of their interpretation of the particular

circumstances of the situation. For instance, the spreadsheets had been used previously with investigative activities I had undertaken with them, so the students might have assumed an expectation to use them again.

> Tom: I would find the numbers I need and type them down on the spreadsheet.

It might also have been their preferred medium for interpreting the intention of the activity, or some inherent or explicit feature of the activity itself. This appeared to be the reason for the choice of approach articulated by another group:

> Liam: When I first read the problem it seemed like it would be good for the spreadsheet.

Within this range of what the students described as their initial approaches to engaging with the task, there was consistency with the data to support the contention that the students moved promptly to an initial engagement with the spreadsheet. As indicated in the discussion of the observational data, there are a number of possible contributing reasons for this: the spreadsheet environment being accessible, and projected as having primacy among the available pedagogical media; influences associated with the power and expectations of the researcher or the particular group of students that were present; the nature of the tasks selected; as well as the students deciding that the spreadsheet was the most suitable approach to investigating the task. Anyone of these might contribute in varying degrees to the individual and group decision making in this regard. The fact the interview data supported the observational data in the illumination of this aspect is noteworthy also.

Structure of the output within a spreadsheet workbook

Once the spreadsheet was engaged, there were elements of the spreadsheet structure that, in my opinion, fostered a distinctive approach to the research process. The retrospective reflections of the students also indicated that the

tabular or column structure within the spreadsheet influenced their subsequent engagement. The interview approach to data collection provided insights into this particular aspect that assigned greater emphasis to the students' perspective and interpretation, than the observational data permitted. Several commented that it was aligning the numbers into columns or rows that enabled them to make better sense of the output. For instance:

Chris:	Columns make it easier - they separated the numbers and stopped you getting muddled. It keeps it in order, helps with ordering and patterns.
May:	I found it useful.
NC:	In what way?
May:	It was faster. Because it was… it put the numbers into rows.
NC:	Was there anything in the spreadsheet that made it good to work with - apart from the speed. The way you work?
Jack:	Well there's the columns
James:	Yeah, that made it easier to see.
Jack:	Easier to understand, yeah, heaps easier to understand; helps you work out the answer. You could put in like =A1 + 11 and it would Fill Down. It helped quite a bit; the answer is there.

One group found the aligning of the numbers into columns helpful, but also problematic in one aspect, although this might be partially attributed to other formatting factors of the spreadsheet structure than just the columns:

Kyle:	Being in a table and pattern was really helpful until it got into the funny form (scientific notation).

While another found an organisational satisfaction in the format that may have had some influence on their approach:

Cass: You couldn't have messy work. It was all tidy and in lines.

Several groups referred to the tabular structure enabling them to operate on large amounts of data simultaneously:

Sam: Putting numbers in the columns and it calculates them by itself.

Jo: When I changed a number, it changed all the numbers itself.

Helen: Helps you to do quicker columns and work things out.

Others mentioned the way the visual table structure gave a more fulsome, holistic overview of the data, which in their opinion, was advantageous to the investigative process:

Kate: Yes, I used it (columns) for keeping track of the figures; seeing where you were in the whole thing.

Cale: Just the way it's displayed, everything is done; you don't have to look back or anything, it's all there.

The data suggested that the students found the facility of the spreadsheet to organise the data into columns or tables as enhancing their learning pathway. Some of their comments referred to the speed or ease of computation, which will be elaborated on in a further section. It appeared that the opportunities the tabular structure afforded the learner through the visual arrangement of relatively large amounts of consecutive output facilitated the recognition of patterns. This aspect is considered in the next section.

The recognition of patterns

The data indicated that the students found the spreadsheet environment enhanced their noticing of the patterns within the output. It appeared the structured format of the tables and columns increased the clarity of that noticing. The comments from the students in response to the question: "What were the maths ideas the spreadsheet helped you with most?" revealed this facet. For example:

Stu:	Definitely emerging patterns (pre-service teacher).
Greg:	That kind of maths, the breaking it down thing. I guess problem solving, trying to find a pattern and figure out what it was.
Deanna:	You could see the patterns easily so it helped you with the maths, like adding things up and getting the formula.
Ben:	Finding the rules and patterns.
Hine:	Fill Down is really good. When you fill down you could easily see the pattern when you looked at the answer.
Mark:	The spreadsheet helped because all you had to do was to put in 200 or the formula and Fill Down. You could see the pattern (pre-service teacher).
Nell:	You could see the answer and the pattern straight off. We couldn't properly see the answer until we had the whole pattern.

While several of these comments referred to aspects the students considered to be interrelated such as "finding the rules" or "getting the formula", central to the comments was the notion of the pattern and the students' noticing of it

within the structured output. The last comment by Nell made reference to the whole pattern. This was also a feature of other responses:

Ellie: It helps when you look at patterns; it saves you writing it all down, you just type it in and see the whole pattern.

Ben: The fact that it was all there at once. You could see it all there. So you could see it all written down.

Jay: It helps when you look at patterns. You just type it in and see the whole pattern.

The following excerpts included some probing questions and illustrated the students' perceptions that having "everything there" made the noticing of the pattern easier and quicker, while the comparison of neighbouring values also appeared to be implicit in their consideration of this idea and the manner in which the pattern was noticed.

Ata: Algebra, to work out numbers and patterns.
NC: How did it help you with patterns?
Ellie: Filling down. You could look down and see it was going up in threes or whatever and in the pocket money you can see it change to going down in eleven; all the way down. It was easier to find the answer and quicker.
NC: What was it about the spreadsheet that helped you get to the answer quicker?
Ata: Everything there. It was all there
NC: Did that help you to see the pattern?
Ben: Yes.

In the following excerpt, the student also mentioned how the noticing of output that contrasted with the emerging pattern, also assisted their interpretation. In their view, this noticing of the contrast was also facilitated by the tabular format enabling them to see the output in a structured manner.

Ata: Just filling down stuff so you don't have to type in each square. It meant you could look at a whole page of things.

NC: What could you do once you had the whole table down, what does that help you do.

Ben: Um, you could look at it to find where the problem was. How to find out you look back on it and see.

Likewise:

Ellie: Yes, because you could see the patterns, and you could see if you had done something wrong because it was all out on that page.

From my perspective, it looked as if the tabular structure within the spreadsheet environment facilitated the noticing and visual interpretation of patterns in the output. The data from the students' responses indicated this and that having a range of output in an ordered visual array was conducive to that noticing. As Zane commented:

Zane: [The spreadsheet] displays it really good so you can understand it, see what's going on, and it was quick.

Another feature that has been signalled by the data was that the spreadsheet did much of the computation for the students. The next section considers this and associated facets.

Computation

The spreadsheet allowed for the simultaneous computation of columns of data. This changed the nature of tasks that included large numbers of computations or number operations with large or rational numbers. The spreadsheet's facility to compute quickly and accurately influenced the nature of the engagement with the tasks. There are three student positions to be considered in this regard. The first is that this rapid, accurate computation was an advantage as illustrated by the comments below:

Ben: Easier because the computer basically did most of the 'times' and the dividing for you.

James: Um. It helped us with like using the formulas and like how you get a number and multiple it down the bottom.

Sophie: It helped with adding. Fill Down too!

Tia: The 'summary' tool – sum of all numbers was useful.

Jack: I know what you mean because the computer did it. It meant it did something you didn't have to do.

One student referred to the use of cell references in operational formulas:

Greg: You didn't have to write the number again – you can just put down the cell (pre-service teacher).

Others recognised this attribute, but expressed concerns over the long run effects of this on students' mental computation. This second student position appeared very reflective, but may have been linked to the emphasis given to mental calculation given by the New Zealand Numeracy Development Projects, that were the basis of the classroom programmes for most of the pupils.

Ben: Didn't really help you with that [computation] because the computer was doing it for you so you didn't have to work it out, you could just Fill Down.

Another group discussed this aspect with similar concerns:

Fran: In the long run, it would be harder to do your additions, because you are not really learning it.

Ellie: No, because the computer can do it for you.

Finn: If you're doing adding, you don't have to think about it, you just write it on the spreadsheet and then it's there.

While there were concerns expressed about the eroding of mental computational skills, other students recognised opportunities associated with the computational facility. The following excerpts illustrate this third student position:

Ata: It gives you time to work out the other questions, so you could concentrate on the thinking part of it.

Ben: Like patterning or …..

Ata: Word maths problems. It helps you solve it.

Luke: It helped me during my problem solving, it does the adding up for you so all you have to do beforehand is find out what to put in, insert into it what you want to find out. Once you know what you need to do, it gives you the answer automatically.

Whitu: I found it helpful that it could calculate itself and I had more time to work on the problem.

Other students referred to how in their view, the spreadsheet's capacity to do simultaneous computations on large amounts of data made the task easier for them:

Kerry: Yea, it's easier than actually going down and figuring each one out.

Greg: I like the way it is easier. It'll calculate by itself. The spreadsheet does most of the work for you. They helped with operations (pre-service teacher).

These last two comments indicate the focus of the next section, which considers the ways in which the students indicated the spreadsheet pedagogical medium made the engagement with the tasks faster and easier.

Faster and easier modelling

The data in this section was drawn from the responses to a range of prepared and probing questions as well as comment that arose out of general discussion at the interviews. The question: "What type of activity did you find the spreadsheet most useful for?" led to some of the data, while other data surprisingly emerged as rebuttal to the questions: "Did it make any work harder? If so, what did using the spreadsheet make harder?" Students' contrary contention to this question appeared to accentuate this aspect of the spreadsheet environment making the process faster and easier for those students. There were responses that indicated the students felt working with the spreadsheets was faster or quicker. It is assumed that this is in comparison to the typical approach taken in their mathematics lessons. This predominantly involved using pencil-and-paper methods, with equipment (including calculators), and games also components of their mathematics programme. The pupils would also have computers available in their classrooms. Below are some of their responses:

Bree: You don't have to go through that whole process to find the answers first. It is a lot quicker (pre-service teacher).

Awhi: Putting in a formula then filling down, saved time making tables. It sped things up a bit. You got into stuff a bit quicker.

Ata: It allowed me to get things done quicker.

Kyle: So much quicker. You only have to do one formula. Takes away all the hard work. Much quicker than a calculator too because you still have to put it all in and add it up (pre-service teacher).

Ellie: It was faster. Because it was… it put the numbers into rows.

Ben: It's doing most of the work for you.

The students alluded to the table format, the use of formulae, and the attributes related to them in their reference to the spreadsheet speeding up the investigative process. This and the facility to manage several computations simultaneously were also evident in the comments regarding the spreadsheet making the process easier.

Ben: It was easier cause instead of writing everything down you could just type the formulas and you press Fill Down and it's quicker and you don't have to write down every number.

James: It didn't sort of burn your brain, it was just type in a couple of things and it gives the answer straight off.

Ata: Then that was easier [using the spreadsheet]. I found it useful to multiply and divide. I found it useful.

Kerry: Easier getting my head around it because on the spreadsheet, you just type it in once and drag it down (pre-service teacher).

Whitu: The formula does it so easily.

Mike: On the spreadsheet you just type it once and drag it down.

While the data illustrated the students' opinion that engaging in the investigative tasks through the medium of the spreadsheet made the interaction with the tasks quicker and easier, several commented on how this allowed them

more time for reflection or to approach the tasks in an alternative manner. For instance:

> Deanna: I still like working with paper a bit but with spreadsheets, it was easier to find out how much money you have left: To be able to look at different things over time.
>
> Ben: It was easier 'cause instead of writing everything down you could just type the formulas and you press *Fill Down* and it's quicker and you don't have to write down every number.
>
> NC: Did that extra free time allow you to do more?
>
> Ben: It made it easier for me because I'm not the fastest at things like that but I am a fast typist, so did it give me more time to think? Yes. I find it was easier using computers rather than writing things down because it's quicker. Spend more time to think and concentrate on it.

One student referred to the computation being easier, but the corresponding setting up of the spreadsheet as being challenging. It also appeared to make it easier for them in a physical sense.

> Ellie: Mainly you didn't have to do as much work, a different type of work, easy maths but hard spreadsheeting. You can't get a sore hand from writing.

Another commented on the ease, but was not fully utilising the functionality of the spreadsheet. Instead of using a formula and *Fill Down*, they were just entering the numbers, and possibly still found the format useful to view the data with.

> James: Yeah, it makes it a little bit easier to type in. You just type that number, return; that number, return.

That example from James highlights the requirement for an understanding of the functionality of the spreadsheet in order for the students to use it to its full potential. This was something that evolved through purposeful activity and discussion, but sometimes had an accompanying affective dimension. For example:

> Leah: Today, I didn't feel like that. As soon as we'd done that first column, I was thinking 'this is good, much quicker and easier. I could do this all the time.' I thought differently once we'd got that nailed, which we did this time (pre-service teacher).

There were several comments that attended to the difficulties some students at times encountered. At times this caused frustration that led to an entrenchment of their current approach, while at other times it provided motivation to work through those elements. They were usually in response to the questions that referred to any perceived difficulty the spreadsheet might have engendered.

Difficulties with the medium

The difficulties some of the students encountered and identified in the interview data were related to the techniques involved in the operation and functioning of the spreadsheet. While it appeared that all of the students encountered difficulties or aspects (both with technique and mathematics understanding) that were challenging at times, this would be considered a normal occurrence for students using the investigative process. The articulation by the particular students in the data regarding having difficulties with the application of techniques implies that in their perception it was significantly problematic; more than the perturbation they would expect in the investigative process. They appeared to centre on techniques and the use of formulae. For instance:

> NC: Was there any maths or work that the spreadsheets made it harder for?

Ata: Not really, but sometimes I didn't really understand it to start off with.

NC: How to use the spreadsheet or the maths?

Ata: Both really. It was hard to find out how to do the percentages and values. We tried to do it on the calculator too [the computer's calculator], but the calculator wouldn't take percentages.

Sophie: [It was] tricky getting used to the spreadsheet.

Ben: Making graphs was hard for me, I always forget how to make them.

The following pre-service teacher also found the transition to a new pedagogical medium problematic and exerted a degree of resistance to engaging with the new medium that might have influenced his perceptions.

Stu: I kept saying to Rewa could you explain that because she was very focussed on the end result. I got lost. Technology did take over. I wouldn't have allowed that myself. I would've done it on paper. I wouldn't have worried about the spreadsheet. I reflect back to that first time we did it, I still find using another medium like that confusing. Why bring in something more complex when trying to solve basic problems? (pre-service teacher).

In my view, this indicated that his thinking had been swamped by the functionality techniques of the medium. He was more concerned with how to operate within the environment than the mathematical ideas. It seemed as a result of this that his focus was more on spreadsheet techniques at the expense of conceptual understanding. The following comment refers to the same aspect, but demonstrates an alleviation of the concern through the engagement of the medium.

Leah: The technology worked smoother for us today. Last time we hadn't done spreadsheets and that overtook everything. I think I was focussed on that and making sure it did go smoothly today (pre-service teacher).

A particular aspect related to the students' perceived difficulties involved the generation and use of formulae.

Ellie: I found it easier to use paper.
Ben: Yes, it was quite challenging trying to use the formulas.

The following discussion focussed on the formula but was related to it not appearing on screen as the group worked with it. As such, it appears the students were concerned with a functionality aspect rather than a mathematical one.

Ben: When you are working, it gets a bit confusing because when you work at it on the spreadsheet all the workings are hidden behind the number. You do the working [formula] but then it hides from view. The formula was the main thing to find.
Ata: I think they [spreadsheets] help too much, because the numbers were just there, but it was the formula that was what you had to work out.
Ellie: You look at the outcome. The numbers are there, but then you have to look at the working: the formula that got you the numbers.

With the following comment it was both the mathematics and the medium that caused consternation. They seemed to have difficulty finding a footprint in their preconceived understandings of either which would allow them to step further into unfamiliar aspects of the investigation.

Greg: That whole problem did my head in because we used the spreadsheet. On top of that we were trying to work out

the formulas. It was too confusing for me (pre-service teacher).

While the questions: "Did it make any work harder? If so, what did using the spreadsheet make harder?" predominantly elicited contrary assertions, that the spreadsheet environment made the tasks easier, there were nevertheless concerns raised around the operating techniques of the spreadsheet. These were linked to the perception of the medium inhibiting the engagement with the broader mathematical ideas. This might have arisen through individual preconceptions about the medium or the associated mathematics or contexts, the nature of the interactions, or it may have been that tensions arising out of the investigative process hadn't been reconciled at the time the questions were asked. Alternatively, perhaps the techniques required were experientially or conceptually beyond the scope of those particular students, or it was a combination of any of those three facets. Instrumental genesis, the transition of an artifact to an instrument, with the development of techniques and schemas that evolve while using it, appears to be necessary if the focus is to shift from functionality techniques to mathematical understanding. This data shows that it is not always a straightforward, unproblematic process.

Making the learning fun or interesting

Another theme that emerged from the interview data involved points of view regarding how working in a spreadsheet environment might have enhanced the fun or interest dimension of the learning experience. Again there was a degree of polarity in the perspectives. Those that indicated an unenthusiastic opinion are considered firstly:

Deanna: I didn't find it more enjoyable because I still find writing it better; and doing it in your head.

Ellie: I think working on paper is a bit more fun because if you know lots about spreadsheet then you might be enjoying

doing, but I'm not really sure how to use spreadsheets very well.

The last comment appeared related to the student's confidence with the functionality of the spreadsheet, while for others it was the challenge of engaging in the tasks through a fresh approach that evoked interest.

Ata: It's fun learning to use the controls. It took me a while to learn to the apple control. You have to learn how to use the controls. It was fun learning about how to use the formula.

Fran: Yes, I liked doing it that way [with the spreadsheet].

Rachel: I think I liked using spreadsheet to work out some problems, I think it's good because it makes me think more harder.

The following excerpt includes some prompting that drew out features of the students' reasoning. The first links back to an aspect considered during discussion of the medium enhancing the speed and ease of the engagement.

NC: Did you find it enjoyable working on the spreadsheets?

Ben: Yeah, it was; it was easier and wouldn't take as long.

Ana: It was easy to do the timetables because all you had to do was click and drag.

NC: So that made it…

Ana: Made it faster and easier.

Ben: Yes. Finding the answers quicker.

NC: What sort of answers? What were you doing, what sort of maths were you doing to find the answers?

Also:

Ben: We were using *Fill Down*, and formulas and numbers, and I found it quite a bit easier than writing down problems.

Jay: It was fun because using the computer; it does a lot of things automatically.

In the following excerpts, the students connected their interest to the challenging elements.

Tony: The formulas were interesting, that one when you had to work out formula like A1 X 2, last week. Or you could put in A1 X 2 + 3.

NC: What was interesting about that?

Tony: The working out of it, you had to find the equations; to investigate it a bit more.

Jack: You put the output than you missed a column.

James: I learnt about the Fibonacci formula; it was hard, but I liked the Fibonacci formula.

Also:

Tony: It was more enjoyable because it saved me, like when you got the numbers down you could just delete them easy, it did all your times and multiplying for you.

Ana: Everything you did was sort of a…- you saw a game, a bit of a challenge.

Ellie: It was like easy. In the class everyone's at different standards, so we do easier work, but when we're here it's more challenging.

The final observation indicated the part that purpose provided in the fostering of interest.

Ana: I found it enjoyable because I got to practice the computer and maths at the same time. I used the computer for something instead of just learning about

them, which is sometimes boring. So it was better. We actually used the computer skills for something.

The data in this section is concerned with how working in the spreadsheet environment might have influenced the students' attitudes towards the mathematical activity. They reference the technical aspects with both negative and positive connotations, while making connections with the ease and speed of computation afforded by the spreadsheet. Also linked to this, was the interest and enjoyment gained by the challenging aspects evoked by the pedagogical medium. This was at times linked to the students having more space for the reflective process, but at others to the students viewing the data from alternative perspectives. This second point resonates with the discussions regarding the initial engagement and the tabular structure. Student confidence appeared to be enhanced by their enjoyment and an interested disposition, while confidence enhanced the learner's propensity to take risks. In the next section that aspect is considered.

Risk taking

The data appear to indicate a greater propensity for exploration and risk taking engendered by the spreadsheet environment. This is consistent with other findings (Beare, 1993; Sandholtz et al., 1997; Calder, 2001, 2006). The responses seem to be primarily related to the functional or formatting affordances of the spreadsheet. For example:

Fran: Using a spreadsheet made it more likely to have a go at something new because it does many things for you. You have unlimited room. You can delete, wipe stuff out.

Tony: It was easy to try things – saved you rubbing it out, you press delete and it's gone. What else was good about it? – trying things out.

Ben: We tried a couple of formulas and none of them were right but we could see what the formula might be, so we could change it around a bit.

Ant: Yeah, like when we had to on the first activities when Dan had 8 then he had 11 we had to find what was different – we could try things out and see if that worked and change it.

Sophie: I always find it good for me. I can put something in and if it's not quite right, I can change a couple of things and bang, it changes it automatically and I don't have to start from the beginning again.

These student comments reflect a certain comfort with trying things, knowing they can be easily modified, and with an awareness of the rapidity of that modification process. It seemed there was an implicit reference to the encouragement of experimentation as well, through the facility to model situations in various ways, for example formulae or tables, coupled with that ease and speed of modification of those models. They also appeared to be more able to easily experiment with new ideas that arose during the process, such as Jam's exploration of the Fibonacci sequence or other new ideas/approaches as illustrated by the comments below:

Whitu: Through doing the work I found about the power of ten and tested it out; used it.

Deanna: Good for ones like tracking the money, but if you forget something in the formulas that's wrong, you can just change it; and you can enter future ones to see what will happen.

Some used the spreadsheet for more usual investigative approaches but nevertheless found the spreadsheet conducive to that practice. For instance:

Bree: It took awhile to figure out what we were doing. How did we solve the problem? It was trial and error. We did one digit, two digits, three digits. It was in the three digits that we started to figure out what the pattern was. There was a lot of trial and error. What if we try this number? What if we try that number? (pre-service teacher).

Tony: Me, and a guy Thomas, we were playing around with the graphs and you could find out what different graphs are used for.

Aspects related to investigating in the spreadsheet environment such as the tabular format for output, the immediacy of the response to input, the facility to compute large amounts of data simultaneously, and to modify various elements quickly and easily, all engendered confidence in students to try things and take risks. Confidence is a very personal condition however, and is borne of a layering of interactions and interpretations, some seemingly unrelated to the situation in which the researcher might have noticed the confidence or lack of confidence. Two people given the identical spreadsheet experience would have distinctive responses invoked by the experience. One student might feel very confident to try new approaches, and another not at all confident. Nevertheless, the environment had the potential to enhance the students' willingness to take risks. It was also a relatively non-threatening, easily managed environment. This would also seem to make it suitable for encouraging risk-taking.

Conclusions:

Several themes emerged from the interview data that, in the students' opinion, appeared to have made the learning experience distinctive. The students' initial interaction with the tasks was invariably through the gaze of the spreadsheet. While there was an element of familiarisation with the intentions of the task, this was frequently undertaken within the spreadsheet environment. This initial engagement shaped the subsequent interactions and framed the learning

trajectory in a particular manner. The students in general articulated their perception of the structure and operational aspects of the spreadsheet, manifest in the tabular format and the use of formulae and the *Fill Down* function, as facilitating their noticing of the patterns in the output. The generation of related values in an interconnected array was a characteristic of this pedagogical medium that appeared, for the students, to lend itself to the notion of patterns and the interpretation of them. The actual tabular structure, the layout, and the sequencing of the data within that, was important from the students' perspective in enabling them to visualise the patterns, to more easily generate and analyse patterns, and to test their conjectures within that structure.

The students also felt that the table format and application of formulae made their engagement with the task both easier and quicker. The speed of calculation, especially multiple calculations with 'untidy' numbers, freed the student from the computational fetters that were part of investigations that required many computations, for example with the 'Terminating or recurring decimal' task. This allowed them to work or focus more on the mathematical thinking and broader issues, without fear of a computational error, or being unduly restricted by the time taken to compute the necessary amount of input, for the investigation to be meaningful. Several students also linked this property to being able to reflect more on the process or the problem itself. Several indicated that it made them think harder or in different ways. These aspects, coupled with the facility to give immediate feedback to inputted data and the non-threatening nature of the learning environment, illuminated the medium's suitability for encouraging risk taking and a more exploratory approach.

While the interview data was the students' perceptions, told in their own terms, there were constraints nonetheless regarding the nature of those comments. Firstly, many were responses initiated by the semi-structured interview that had been constructed. The selection and wording of these questions, and any probing questions, would have been influenced by my preconceptions and underlying discourses in the related domains. Despite efforts to structure the questions to avoid leading the students' responses, the researcher's view would have coloured the intent of the questions, and by implication, the responses. As

well, the students' comments would have been their perceptions of the situation and what they did. Their interpretation of events might differ from that of an outside observer or another participant. What they noticed would have been framed by their own underlying discourses. It was also an historically and contextually situated opinion. Their comments might have differed at a different time, in different circumstances. Finally, the interviews were undertaken in groups with myself, as researcher, present. There are power discourses associated with the dynamics of any group. The nature and intent of their responses might have been influenced by the relationships within the group they were interviewed with, and by having the researcher present. These are aspects that are complicit to the collection of qualitative data. As discussed previously we can't eliminate these influences and nor would we want to. We do need to acknowledge them, however.

The students' responses and comments illustrated the ways in which they perceived the learning experience as being distinctive when encountered through the pedagogical medium of the spreadsheet, and how it shaped their learning trajectory and as a consequence affected their understanding. While this influence varied, and would have been unique for each individual student, it still appeared to be an influence to some extent. As one student articulated it:

> Bree: I was thinking I needed a pen because it would have been easier if I'd written things down. But I think if I had used a pen it would've been a different approach (pre-service teacher).

A different approach: by implication a different approach would have organised the thinking in a different manner and might have facilitated different understandings.

Problem Challenge

Introduction

The twenty-one pupils involved were entered into the 2004 Otago problem solving competition: Problem Challenge 2004. This section outlines some of the data and analysis that it produced, and situates that within the context of the whole study. While the rationale for utilizing this approach and the intentions of the research regarding it were addressed in the methodology chapter, it is important to articulate the space the researcher occupied at that particular juncture so that is accessible to the reader, and to position that perspective within the broader methodological lens that is being applied to the thesis.

The research questions centred upon the ways mathematical understanding might be reorganised when mathematical phenomena are engaged with through the pedagogical medium of the spreadsheet. It was originally envisaged that an eclectic approach to data collection would best inform consideration of these questions, including the statistical analysis and discussion of comparative data. By comparing the pupils' results before they engaged in the spreadsheet sessions with their results after, it was perceived that further insights and perspectives on the influence of the spreadsheet environment might be produced. It was felt that this would enrich and expand the production of knowledge regarding the research questions via the engagement of divergent mechanisms. By viewing the data from alternative perspectives I reasoned that the scope of data would be enriched, and by implication, the understandings also. Another consideration in the selection of this particular tool was that the Beach Brilliance mathematics group had been involved with the Otago problem challenge in the previous three years, and there was access to records of the national data from its inception in 1991. The participants in this research were the ten-year-old pupils. This section begins with a capturing of the problem challenge experience through a particular set of eyes, then discusses that perspective, and how it is situated within the overall methodology.

Problem Challenge

This mathematics problem solving competition is aimed primarily at mathematically able intermediate-aged school children in years 7 and 8 (ages 11-13 years), but is also of interest to mathematically gifted children in year 6 (ages 10-11). Children participating in the competition attempt to answer five questions in 30 minutes on each of five problem sheets, which are done about a month apart. They do the problems individually, but they can share their answers and strategies in small groups afterwards. Note that all three levels (years 6, 7 and 8) attempt the same problem set, although there are separate awards for each of those levels.

For each SET, a summary of the overall results is collated, so that schools and participants can evaluate pupils' relative progress. However, individual school results are not collated or publicised. All children taking part receive a certificate of participation. As in previous years, there was a Problem Challenge SET of five questions given each month from April to August. For 2004 the dates were: SET1- 6 April; SET2 – 11 May; SET3 – 15 June; SET4 – 27 July; SET5 – 24 August.

Beach Brilliance Participation

The Beach Brilliance group had four, one and a half hour sessions at one of the schools to develop their approach to problem solving. Most of this work was done in groups, but with a mixture of group and individual recording and reporting of findings and results. The emphasis was on developing and celebrating creative, diverse approaches, as well as recognising and practicing more commonly used approaches at this age; for example, guess and improve, or forming a table. There were two further sessions, that followed the classes involving the utilisation of spreadsheets as an investigative medium. This analysis and discussion was written up in 2005. The 2004 overall results for the children participating in the research were as follows: N.b. only 20 of the pupils completed all five sets. For this group, that meant there were four certificates of

excellence (top 11% of participants nationally), ten certificates of merit (the next 25%) and six certificates of participation.

Table 2: Otago problem challenge results.

Score (/25)	No. of Children
7	1
8	2
9	1
10	2
11	3
12	3
13	0
14	2
15	2
16	1
17	1
18	1
19	1

Analysis of problem sets

The following table shows the results for the population of participants as percentages:

Table 3: 2004 National Otago problem-solving competition results.

Qn = percentage of correct answers for question *n*.

Tn = percentage of students getting a total of *n* questions correct.

Cn = percentage of students getting a total of *n* or more questions correct.

	Q 1	Q 2	Q 3	Q 4	Q 5	T 0	T 1	T 2	T 3	T 4	T 5	C 2	C 3
SET 1	82	73	62	33	8	6	14	25	31	19	5	80	55
SET 2	86	77	44	29	22	6	15	29	25	16	9	79	50
SET 3	81	74	46	47	19	7	14	24	25	21	9	79	55
SET 4	93	61	56	46	9	4	19	22	22	27	6	77	55
SET 5	85	33	**60**	10	6	10	26	35	**21**	6	2	64	**29**

(e.g., for **SET5**, for example, **60**% got question 3 correct (**Q3**), **21**% got three questions correct (**T3**), and **29**% got three or more correct (**C3**))

Application of Problem Challenge to the Present Study

Only limited analysis of the pupils' approach to the problem-solving tasks, comparing their investigative methodology before and after the spreadsheet sessions, could be undertaken. Firstly, only SET5 occurred after the sessions were facilitated. As well, there is the eclectic nature of the tasks, in terms of content knowledge and the aptness of strategies to solve them; some of them were not suitable for spreadsheet investigation. The differences in the administration of the tasks (e.g., they were done in silence and individually in the classroom setting, and collaboratively with the spreadsheets), also meant that attempting to establish causal links between the use of spreadsheets to investigate mathematical problems, and an actual change in investigative strategies, was difficult. Some observations are nevertheless pertinent and of interest.

The tasks taken immediately prior to the spreadsheet work (SET4) and immediately after (SET5) were analysed. They were scrutinised to see whether they included aspects that were suitable for investigation with a spreadsheet. This suitability for investigation with spreadsheets might be due to them having the potential to be investigated by using a table of values or a similar visual structure. It also included problems that required manipulation of large amounts of

numerical data, or large numbers or those with decimal values; or those where a connection between visual, numerical and a generalised form was advantageous; or those with an aspect that might be enhanced by some initial generalisation, or form of algebraic thinking. They might have contained one or several of these features. Others were identified as not being conducive to investigation with a spreadsheet; that is, they offered no explicit advantage or contextual lead in using the spreadsheet, or in the type of thinking or investigative approach use of a spreadsheet might engender; for example, those involving interpretation of geometric shapes.

In SET4, undertaken before the spreadsheet sessions, questions 3, 4 and 5 included aspects that might be suitable for investigation with a spreadsheet. For Q3, 65% got it correct (c.f. 56% of NZ overall); for Q4, 40% (c.f. 46%); and for Q5, 0% (c.f. 9%). This compared with those not conducive to spreadsheet exploration: 95% for Q1 (c.f. 93%) and 45% for Q2 (c.f. 61%).

Table 4: Percentage of correct answers in SET4, 2004.

% Correct SET4		
Question	Participants n = 21	National Results
1	95	93
2	45	61
3	65	56
4	40	46
5	0	9

Q 3,4, and 5 had aspects that may be suitable for spreadsheet investigation.

Bar Graph of 2004, SET4 results.

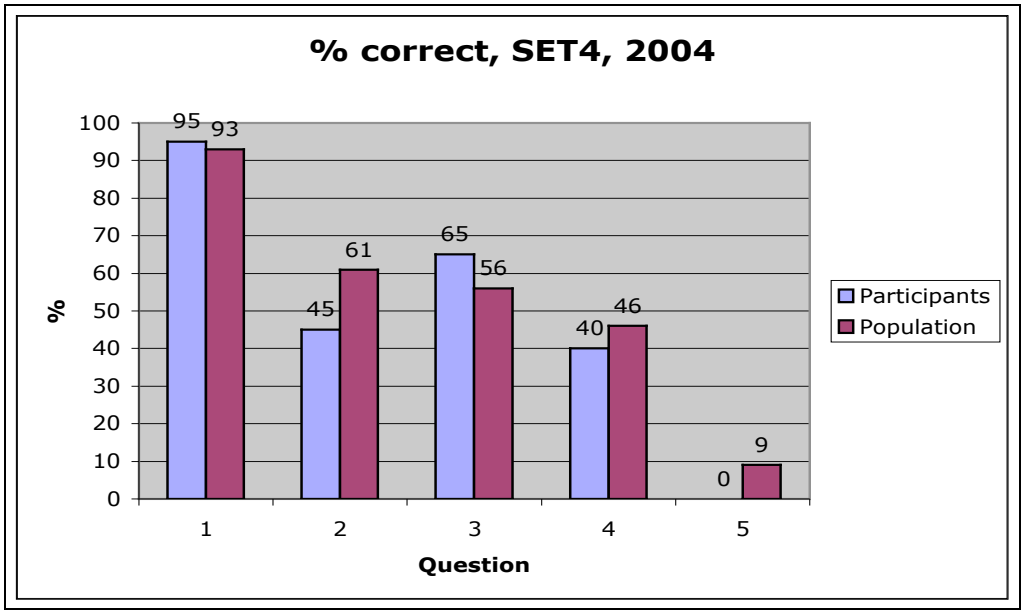

Figure 5: Percentage of correct answers in SET4, 2004.

There was a small, expected difference between the percentages correct between the research study's participant group and the NZ overall population cohort from 2004: some, where the population percentage was higher and some, where the study's participant group was. Certainly, with regards to those questions suitable for investigation with a spreadsheet there was no clear pattern evident.

None of the participant group got question 5 correct. This was typical of all sets, including SET5, 2004, which is analysed below. Question 5 was usually the most difficult question, even for the year 8 students. It frequently contained conceptual aspects that the year 6 children would not be familiar with in their usual classroom mathematics programme, or that required higher level mathematical thinking to distinguish between the more able year 8 students.

For SET5, of the five questions, Q1 and Q3 contained aspects that may have been suitable for investigation using the spreadsheet. The study's entire participant group got Q1 correct and 80% of them got Q3. This compared with 85% and 60% respectively for the whole population on these two questions. Both

percentages were substantially higher than the population percentages for those questions. The other questions, one requiring an understanding of the mean and the other two proportional thinking, were not conducive to spreadsheet exploration. For those questions, 25% got Q2 correct (c.f. 33% of population); 10% Q4 (c.f. 10%); and 0% Q5 (c.f. 6%).

Table 5: Percentage of correct answers in SET5, 2004.

% Correct SET5		
Question	Participants n = 21	National Results
1	100	85
2	25	33
3	80	60
4	10	10
5	0	6

Q1 & 3 had aspects that may have been suitable for spreadsheet investigation.

Bar Graph of 2004, SET5 results.

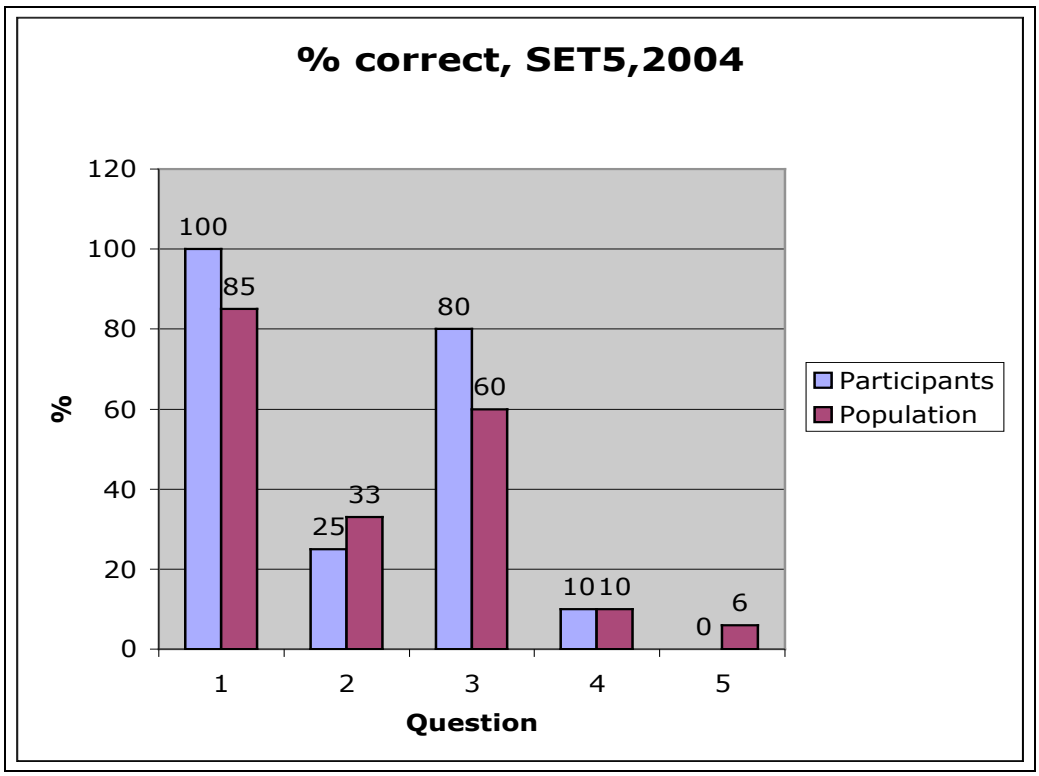

Figure 6: Percentage of correct answers in SET5, 2004.

The only two questions from either SET4 or SET5 where the research study's participant group scored 10% or greater accuracy than the NZ cohort population, were the two which had elements suitable for investigation by spreadsheet, that were taken after the spreadsheet sessions (SET5, Q1 & 3). This suggested that the sessions involving the spreadsheets for mathematical investigation assisted in enhancing their approach to solving these types of problems. This might perhaps have been due to the spreadsheet sessions enhancing their capacity for the organisation of the data, or facilitating the comprehension or processing of the problem, or perhaps affording the potential for the participants to explore some content knowledge or process in a unique way.

There are far too many other potentially compounding variables to draw any causal inferences, however. For instance, the questions may have been more accessible for that age group's content knowledge. One school or class, from which the participants were drawn, may have been involved in some unrelated content knowledge, or strategy approach, that gave them particular advantages. This may have skewed the data for that assessment item, or the participants may have encountered a similar style of problem in previous sessions. However, it is interesting to note the tentative relationship between the spreadsheets sessions, and the higher percentage of correct answers for the spreadsheet-related questions after these sessions. This perhaps enriches the research landscape for that particular aspect.

It did consolidate, if only to a small degree, an emerging picture showing that the use of the spreadsheet enhanced certain facets of the sense making and investigation of mathematical activities. This concurred with scrutiny of other data in this particular study (the in-class dialogue and interviews) and is consistent with other researchers' findings (e.g., Ploger, Klinger & Rooney, 1997; Tabach & Friedlander, 2006).

Comparative Analysis of Results

There was further data gathered when the pupils visited the university campus for an on-campus day. They were divided into two equal groups to do SET4 and SET5 from 2003. In the morning of the on-campus day, both groups did Set 4. One group (group A) did it in a classroom type of environment with access to typical classroom equipment: a calculator, blocks, pencil and paper, compass, ruler etc. The other group (group B) did SET4 in the computer suite. In the afternoon they did SET5 from 2003, but the groups swopped environments, that is, group A were in the computer suite, and group B in the classroom setting. This allowed for analysis of the data, both between the groups and comparative with the 2003 national data set.

Results

SET 4; 16 November, 2004; Morning of the Beach Brilliance on-campus day.

Questions 1 and 3 were identified as having some aspect that would be suitable for investigating with a spreadsheet. In both these particular questions generating a table of values was one possible approach. Q2 used multiplicative thinking, Q4 involved number sense and Q5, network theory.

Table 6: Results from the BB day, a.m. classroom group (Group A):

Student	**Q1**	**Q2**	**Q3**	**Q4**	**Q5**	**Total**
JP	X	√	X	√	X	2
BP	X	√	√	√	X	3
TJ	√	√	√	√	X	4
RA	√	√	√	√	X	4
CK	√	√	X	X	X	2
SG	√	√	√	√	X	4
BF	√	√	√	X	X	3
LD	√	√	√	X	X	3
%correct	75	100	75	62.5	0	

Two of the participants (BF & CK) used a table structure for Q1, and they and two others (JP & TJ) said they would have used a spreadsheet for number one, if it had been available. Two (JP & BP) said they used a calculator for a question (Both with Q1 and another Q4 and to check Q3). None of the participants used the other equipment available. They used pencil and paper with a mixture of diagrams, calculations, tables and guess and improve strategies and for recording.

Table 7: Results from the BB day, a.m. spreadsheet group (Group B):

Student	Q1	Q2	Q3	Q4	Q5	Total
JE	√	√	X	X	X	2
EA	√	√	√	X	X	3
JH	X	√	√	X	X	2
JM	√	√	X	X	X	2
EB	√	√	√	√	X	4
MT	√	√	√	√	X	4
MM	√	√	√	X	X	3
ES	√	√	√	X	X	3
EV	√	√	√	√	X	4
%correct	88.9	100	77.8	33.3	0	

All of the pupils used the spreadsheet to investigate Q1 and three (MT, MM, ES) used it for Q3. One (ES) used the calculator function for Q4. They found it helpful for Q1 because it made a table; it filled it in for you and saved time. Seven of them (JH, JM, EB, MT, MM, ES, EV) said they found using the spreadsheet to solve the problems was enjoyable and two (JE, EA) said it was OK.

SET 5; 16 November, 2004; Afternoon of Beach Brilliance on-campus day.

Questions 1, 4 and 5 were identified as having some aspect that was suitable for investigation with a spreadsheet. Questions 2 and 3 required logic, and guess and

improve as the most suitable strategies.

Table 8: Results from the BB day, p.m. classroom group (Group B):

Student	Q1	Q2	Q3	Q4	Q5	Total
JE	√	√	√	X	X	3
EA	√	√	√	X	X	3
JH	√	√	√	√	√	5
JM	√	X	√	X	X	2
EB	√	X	√	X	X	2
MT	√	√	√	√	X	4
MM	√	√	√	X	X	3
ES	√	√	√	X	X	3
EV	√	√	√	√	X	4
%correct	100	77.8	100	33.3	11.1	

Six of the pupils (JH, JM, JE, EB, EV, MT) used a calculator to solve Q4; none of them used any of the equipment; and four (JH, MM, ES, JM) said they would have used a spreadsheet for Q4 if one had been available; one (MM) with Q3 and two (JH, JM) with Q5.

Table 9: Results from the BB day, p.m. spreadsheet group (Group A):

Student	Q1	Q2	Q3	Q4	Q5	Total
JP	X	X	X	X	X	0
BP	√	√	X	X	X	2
TJ	√	√	X	X	√	3
RA	√	√	√	X	X	3
CK	√	X	√	√	√	4
SG	√	√	√	X	X	3
BF	√	X	√	X	X	2
%correct	85.7	57.1	57.1	14.3	28.6	

N.B. pupil LD was unavailable for the afternoon.

Six of the pupils (BP, CK, RA, SG, JP, TJ) generated a table within the spreadsheet to investigate either Q4 or Q5. These involved either combinations or the pattern formed by the factorials. The pupil JP spent a lot of time setting up the number square problem without actually using the spreadsheet to solve the mathematics. Two others did likewise. They used the spreadsheet for the setting up and presentation of the number square, which took time, without actually using their set up to solve the mathematical aspects of the problem.

Student t-tests

Student t-tests for comparing two population means for small samples were undertaken to see if there were any significant differences in achievement in the Otago Problem Challenge results between the two groups.

The first (Test 1) was for the participants' overall results throughout the whole challenge to ascertain whether one group had better performance at this particular form of problem solving, which might then have been reflected in comparisons of specific sets of questions.

Test 1

Mean (Gp A) = 11.5, mean (Gp B) = 13.52; t-test, t = 0.24, (df = 15). There is insufficient evidence to indicate a difference in the overall scores of the two groups.

The next two tests compared results from SET4; taken in the morning. The first, (Test 2), compares the marks out of 5 for the whole set, and the second, (Test 3), just the scores in the questions with some aspect suitable for spreadsheet investigation, that is questions 1 and 3.

Test 2

Mean (Gp A) = 3.13, mean (Gp B) = 3; t-test, t = 0.77, (df = 15). There is insufficient evidence to indicate a difference in the overall scores of the two groups for SET4.

Test 3

Mean (Gp A) = 1.5, mean (Gp B) = 1.67; t-test, t = 0.61, (df = 15). There is insufficient evidence to indicate a difference in the scores of the two groups for the SET4 questions with some aspect suitable for spreadsheet investigation.

Although there was insufficient evidence to indicate a difference between the two approaches with both tests, it was interesting to note that the group with the highest mean changes when we shifted the analysis from all questions, to focusing to those that are conducive to spreadsheet investigation. For the full set of questions, the group A (classroom approach) mean was 0.125 (or 4.17%) higher than the group B (spreadsheet approach) mean. Yet for the questions with some aspect suitable for spreadsheet investigation, the group B mean (spreadsheet approach) was 0.167 (11.13%) higher than group A, (classroom approach). This suggested that investigating in a spreadsheet environment enhanced achievement in the problem challenge, with the questions with some suitability for spreadsheet exploration, as if there were no advantage there would be an expectation for group A to remain about 4.17% higher. However, this difference was not statistically significant.

The final two tests compared results from SET5, taken in the afternoon. The first, (Test 4), compares the marks out of 5 for the whole set, and the second, (Test 5), just the scores in the questions with some aspect suitable for spreadsheet investigation, that is questions 1, 4 and 5.

Test 4

Mean (Gp A) = 2.43, mean (Gp B) = 3.22; t-test, t = 0.20, (df = 14). There is insufficient evidence to indicate a difference in the overall scores of the two groups for the SET5.

Test 5

Mean (Gp A) = 1.5, mean (Gp B) = 1.31; t-test, t = 0.74, (df = 14). There is insufficient evidence to indicate a difference in the scores of the two groups for the SET5 questions with some aspect suitable for spreadsheet investigation.

Again, while there was insufficient evidence to indicate a difference between the two approaches with both tests, it was of interest to note that the group with the highest mean changes when we shift the analysis from all questions, to focusing to those that are conducive to spreadsheet investigation. For the full set of questions, the group B (classroom approach) mean was 0.79 (or 32.7%) higher than the group A (spreadsheet approach) mean. Yet when only considering the questions that included some aspect suitable for spreadsheet investigation, the group A mean (spreadsheet approach) was 0.19 (or 14.3%) higher than group B, (classroom approach). Similar to comparisons of the SET4 data, this suggested that investigating in a spreadsheet environment enhanced achievement in the problem challenge, with questions containing some suitability for spreadsheet exploration, for if there was no advantage there would have been an expectation for group B to remain about 32.7% higher, rather than group A being higher. This difference was not statistically significant.

The two tables below compared the percentages correct for each question in the sets, for the classroom group, spreadsheet group, and the national group all pupils that took that set in 2003.

SET4

Table 10: Percentage of correct answers, SET4, 2003

	Q1	Q2	Q3	Q4	Q5
Classroom	75.0	100.0	75.0	62.5	0.0
Spreadsheet	88.9	100.0	77.8	33.3	0.0
National	80.0	74.0	58.0	46.0	8.0

SET 5

Table 11: Percentage of correct answers, SET5, 2003.

	Q1	Q2	Q3	Q4	Q5
Classroom	100.0	77.8	100.0	33.3	11.1
Spreadsheet	85.7	57.1	57.1	14.3	28.6
National	81.0	72.0	59.0	30.0	22.0

For SET4, the spreadsheet achieved a higher percentage in questions 1 and 3 only, the two questions identified as having some aspect suitable for spreadsheet exploration. For SET5, the spreadsheet group had a higher percentage in question 5 only, a spreadsheet applicable question. Noteworthy though is that one pupil (JP) got completely immersed in using the format function of the spreadsheet to set up the cross number puzzle in question 1. This resulted in him not finishing any questions in the time allowed, influencing the overall results.

The spreadsheet environment again appeared to enhance the investigation of questions with some aspect conducive to spreadsheet investigation, except in that particular case for that participant.

Concluding remarks:

The comparison of the pupils' results on the Otago problem challenge sets taken before and after the sessions on investigating in the spreadsheet environment, appears to indicate that the spreadsheet work enhanced their results with the questions that contained elements suitable for exploring with the spreadsheet medium. Out of the two sets, the only questions where the percentage of pupils getting the question correct was noticeably higher than the national percentage were the ones with an aspect suitable for exploring with a spreadsheet taken after the spreadsheet classes, that is questions one and three from SET5. With the data produced from the 2003 problem challenge questions undertaken retrospectively in the contrasting settings, it appeared that working in the spreadsheet environment likewise influenced the results positively. When the mean number correct for pupils working in the spreadsheet environment was compared to the means of those working in the classroom setting, there was a noticeable shift when considering only the questions that were suitable for spreadsheet exploration.

Even given the limitations of this type of analysis for considering learning in classroom settings, there were constraints on the analysis that need to be recognised. Firstly, it was me who determined which of the questions had an element suitable for investigating in a spreadsheet environment. While I have

had considerable experience in this area it was nevertheless a subjective decision, perhaps influenced by the type of activities I was used to getting pupils to work on with spreadsheets. Also, because the questions conducive to spreadsheet exploration were considered in the comparative data, it might be argued that the spreadsheet work by definition would enhance the thinking in that area. The time constraints for the assessment might have inhibited the students' responses. Finally, given that it was quantitative analysis and as such attempts were made to isolate variables, with human subjects and the complex history of experiences they bring to each situation, the analysis pertained to only a limited perspective of the learning situation. How we might reconcile this issue needs to be considered at a later stage. While in this case these differences were not great enough to be confident they were not due to chance, they seemed nevertheless to be indicative of, and supportive of a trend in the data: that working with spreadsheets improved the learning experience.

Of considerable interest, however, was the nature of that data and how it might have informed the research questions. While the analysis interpreted through one lens supported the tentative picture that was emerging from the various forms of data, just how meaningful this was, what assumptions were made to ascribe those particular meanings, and how it might be reconciled with the qualitative data were important aspects to consider. This type of statistical analysis is a genre borne of a scientific, positivist paradigm. As such there was a premeditated desire to strip away the complexity of the learning situation and make comparisons regarding one variable. Therefore, there was a tendency to isolate variables, by manipulating other influences and situating the research in controlled settings. With the problem challenge data, the setting was an individual assessment situation done in silence rather than a collaborative one with considerable verbal interaction that had typified the learning environment. The activities were constrained by the type of question selected by the problem challenge administrators that suited their particular perspectives and the designated organisational parameters; for example, a fixed time allocation. As well, comparisons were made between the particular types of question deemed suitable for exploring with spreadsheets, so as to eliminate the variables that might have complicated the results. The atomisation of the variables meant that

there was the potential loss of valuable data and insights associated with the interrelationships between variables. By constraining the environment to remove external influences, the notion of the context being implicit in any interpretation or understanding is compromised, and any conclusions drawn would be limited in scope and meaning.

With the research being undertaken within a qualitative paradigm, and the desire to obtain data in naturalistic settings, it might be less problematic to disregard this data as relatively meaningless to the classroom situation. However, the research questions are to do with gaining an understanding of the influence of the spreadsheet environment on the learning situation and the associated meanings for the students, and given the constraints above, the data does inform that discussion. It also informs the transitions I underwent in my understanding of the research process. The intent of this research was to make sense of, and better understand, the ways students traverse learning pathways and understand mathematical ideas when encountered through the spreadsheet medium, not to offer a causal relationship between single isolated attributes. From the perspective of this research, data are always historical situated in the context from which it emanates. If we view this data, given its constraints, as informing the research question and researcher's perspective at a particular juncture of the study then it would appear to be worthy of consideration.

Questionnaires

Questionnaires were used to produce data to inform the investigation of the research questions in alternative ways. They offered respondents a certain degree of anonymity; a beneficial aspect when considering that participants shared their attitudes and perceptions in group interviews. They gave opportunity to obtain comparative data that, while alternative in nature to some of the other methods, might augment the overall emergent insights and patterns in the results. A mixture of closed response, open-ended, and rating scale type questions were utilised. The rating scale questions incorporated a defined-terms, comparative scale rather than a numerical one as per Likert-style scales. Because they were

researcher constructed questions, there may not have been full scope to elicit the participants' perspectives, and the meanings given to the terms used may have differed from the meanings given to them by the participants. Some of these aspects were alleviated by the inclusion of associated open-ended questions. The questionnaire (see Appendix D) also included questions that allowed the students the opportunity to evaluate the ways the spreadsheet work might have assisted their understanding of specific aspects of their class programme, and the extent to which they enjoyed the work. The questionnaires were given to the Year six students only, with twenty-one pupils completing them, twelve boys and nine girls.

Collation of the questionnaire responses indicated some fairly clear attitudes to the use of spreadsheets in the mathematics programme. All but one of the pupils felt that the activities with the spreadsheets had helped them to understand some of the maths, with the one who hadn't responded positively, answering with a "yes and no". This was interpreted as meaning that it helped them in some instances, but not in others, probably signifying that they did, in fact, consider that the spreadsheet had facilitated their mathematical understanding to some extent. From their own perspectives, all of the pupils had enjoyed doing the work on the spreadsheets. Most (61.9%) needed a little help to do the activities while eight pupils (38.1%) indicated they required no assistance. The most frequent reason for requiring "a little help" that they articulated, was to assist with "some parts of the formulas" (36.4%). Several (18.2%), needed assistance at times with aspects related to graphs, the command key, or their initial interpretation of the task, while one pupil indicated they required assistance with one of the activities.

There was a large degree of diversity in the mathematics areas that the pupils felt the spreadsheet work offered greatest benefit, although 20% indicated that it helped with the computational operations, and individual pupils mentioned other facets of number work; negative numbers or decimals. 20% of them also specified that it enabled them to better understand patterns, with related ideas such as algebra, formulas, and rules also designated. Two pupils (MT, HH) pointed towards the spreadsheet allowing them to learn different ways of doing mathematics, and one replied "everything". The responses to the question

regarding how they used spreadsheets to solve problems were also eclectic, but there seemed to be an obvious perception related to the use of formulae and patterns (57.1% responded that the use of formulae and patterns helped them to solve the problems in the spreadsheet environment). Other aspects, related to processes such as setting up tables, guessing and checking, and promoting discussion, were also reported. Four pupils (SG, BF, MT, TJ) regarded number operations as their preferred manner for the spreadsheet's use. There was also a range of responses to the question asking them to consider what they found most useful in the utilisation of spreadsheets, with 17.9% responding that each of the following was most useful: the *Fill Down* function, the use of formulae, seeing the patterns, and the spreadsheets propensity to calculate by itself. Other aspects considered useful by the pupils were the graphing function, the ability to calculate, and that they were quicker; while three pupils (EV, MT, SS) found the characteristic of the medium to permit users to see the whole picture with everything linked as most useful. With the continuum regarding the utility of the spreadsheet compared to alternative problem-solving lessons (see Appendix D), 95.2% situated their response to the right of the midway mark (the same amount of usefulness), that is, in the region designated as being useful to a lot more useful, with 23.8% marking the extreme end of the continuum (a lot more useful).

The breadth of response in these two categories was not surprising. As the pupils worked relatively independently of the teacher, and as their learning needs associated with understanding of the mathematics and the investigative processes would be individual, the extent of benefit or difficulty they experienced would be as diverse as their own individual learning requirements. Their understanding and interpretations were unique. Even though they worked in groups and discussion was encouraged, they would have met individual barriers and had individual breakthroughs in understanding as they made sense of the ideas they encountered. In a similar way, they would have brought their own preconceptions and underlying discourses to the reading and interpreting of instructions. Each would also have had an individual aptitude and experience in using ICT and spreadsheets, which would have impacted on these aspects to some extent. For instance, one pupil in the interviews said they had a computer at home, that her family members employed the spreadsheet for private and work-related uses, and

had been showing her ways to operate them at home, while another's only experience was the in-school sessions that had been facilitated.

All the students took pleasure in using the spreadsheet, with the most enjoyable aspects being the engagement with the activities (33.3%), and the writing of formulae (27.8%). Finding rules in the number patterns or adding (11.1% each) were reported, with the use of graphs, the different formats available, or simply working on a computer also given as singular aspects of the spreadsheet and number investigative work, which the pupils reported they had enjoyed. On the continuum, 95.2% of the pupils reported the spreadsheet sessions as enjoyable to a lot more enjoyable than their other lessons involving problem-solving approaches, with 33.3% of their responses being situated to the most right hand point of the continuum indicating they considered the spreadsheet work considerably more enjoyable than other maths problem-solving lessons.

My observation was that the children needed progressively less assistance with both the interpretation of the activities, and the actual spreadsheet skills as the sessions evolved. Their underlying personal experiences and emerging expertise with the functionality of the medium possibly helped in this regard. As well, from my perspective, their ongoing trust in the medium and the evolving dynamic of the groups enabled them to work in a more confident manner with greater willingness to take risks and explore potential solutions. The positive attitude engendered by using spreadsheets, and the student motivation associated with this, are consistent with other researchers' findings (Calder, 2002; Hoyles, 2001; Lancaster, 2001; Sandholtz et al., 1997) who all reported positive student motivation. Higgins and Muijs (1999) likewise noted various references to the positive effects of motivation in their discussion of the use of ICT in mathematics. They also cautioned that some of this motivational effect could be the result of the novelty of the learning situation initially, but even so that its effect was sustainable and of consequence.

The correlation between motivation and learning appears self evident, but to be motivated is a complex condition and as individual as the learning process itself. As Lefrancois (1997) discussed, it has origins in instincts and arousal, and is

inextricably linked to self-efficacy. Keith and Cool (1992) in their study of causal effects in the achievement scores of 25,000 students, found motivation was a factor that had strong indirect effects on achievement. Marsh, Parker and Barnes (1985) reported on the consistency and depth of research that linked academic self-concept to academic performance. The motivational aspects of the spreadsheet work, particularly when self-identified, can be considered as a factor that enhanced the learning process. Several pupils noted that the practical nature of working on the computer enhanced their enjoyment or ability to engage in particular facets of the work. Pupils also commented that the use of formulae and the *Fill Down* function, which allowed them to generate patterns, were most useful in the noticing and interpretation of the patterns. The work on the spreadsheets appeared to accentuate the links between visual, symbolic, and numerical models. It enhanced some aspects, and allowed them to process their understanding in various ways.

The capacity to edit easily was another practical aspect noted by pupils. This also facilitated their willingness to explore and take risks, particularly when coupled with the speed of response and the intimacy of working with a partner on a task, rather than in a whole-class situation. Risk-taking and relatively unrestrained exploration of mathematical ideas are key features of effective problem solving. This investigative approach, fostered through the points above, might have encouraged the pupils to experiment with different strategies. However, it is not clear whether this transition was directly related to the actual medium itself, or the change of approach, which gave them an opportunity to reconstruct their interpretation and understandings. Analysis of the questionnaires also confirmed the appropriateness and apparent effectiveness of the *Fill Down* function, coupled with the generation of formulae, as ways the pupils regarded as most useful in the exploration of patterns that were associated with the activities. These were the most commonly offered responses to the questions that considered what they thought offered the most utility in the spreadsheet environment.

Percentage values have been attached to the questionnaire data at various stages of the discussion. While this might give some indication of commonality of response, there are assumptions associated with this such as there being a shared

understanding of the terminology and the intent of the questions. Likewise the responses have been interpreted through a particular researcher lens, perhaps coloured by personal preconceptions of what the data might reveal. This version of the discussion is coloured by personal interpretation, which in turn was framed by prevailing discourses at that particular juncture. The quantitative data was still viewed as contributing to an emerging series of readings of the data, but I was starting to have doubts about the validity of some of the contentions when the data was removed from the context in which it was historically and culturally situated. Nevertheless the data were the participants' views on the researcher's questions, and they informed the research questions within this range of constraints. The open-ended nature of much of the questionnaire also meant there was opportunity for the pupils to communicate their own perspectives in their own terms. It appears that the questionnaire data did indicate that in the participants' view the spreadsheet environment did open up alternative ways of viewing and engaging with the tasks, giving opportunity for the students to gain different understandings as the affordances of the spreadsheet environment permitted alternative learning trajectories. Analysis of this data also created opportunity for reflection on my own approach to the methodology, and to re-envisage the approach to the creation of knowledge that informed my engagement with the research questions.

PMI (Plus/Minus/Interesting): An informal organisational structure

Through incidental conversation, the school students indicated that they were very familiar with the informal organisational structure known as a PMI. This was used in all their classes as a means to focus their opinions regarding a topic. The approach involved the listing of what they perceive as being positive aspects (plus), negative aspects (minus), and interesting aspects of a particular topic. These lists were then typically used as a basis for group or class discussion regarding an evaluation of that topic. Once aware of this relatively consistent approach to an evaluative process that the school participants were all familiar with, I made an impromptu decision to get them to do a PMI on the back of the

questionnaire sheet. The breadth of responses was typical of this organisational structure, partially due to the simplicity of the structure in that just three broad aspects are considered (i.e. plus, minus, and interesting) and also to the open horizon approach implicit in this organisational matrix. The pupils could list positive, negative or interesting comments associated with any aspect of the spreadsheet work. Overall from the twenty-one participants, there were forty-two "plus" responses recorded, fifteen "minus" responses, and thirteen "interesting". There were three broad categories identified that further differentiated the nature of the responses: those concerned with the mathematics engaged with, those related to approaches to learning or to pedagogy, and those concerned with the practical aspects of the digital technology interface. Below is a description of the responses in the three categories.

"Plus" responses

The most prevalent aspect the pupils listed which related to the mathematics involved, was that the spreadsheet work made the maths easier, with two responses also indicating it made the mathematics work faster. Two pupils suggested that the environment enabled them to better recognise patterns in the output, while three found using formulae a positive facet of the medium. Other positive comments under the plus heading concerning the mathematics involved were that the spreadsheet was helpful and useful for number work, good for solving problems, and helped with operations involving decimals and fractions. In terms of the learning process, five identified the games and activities as being positive aspects with the challenge involved with these mentioned twice, while two of them indicated that working in the spreadsheet medium made them think harder. Working with a partner was another positive aspect identified and while that feature is not specific to the spreadsheet environment, the pupils identified it in the context of their work in this medium. Other positive aspects related to the learning or pedagogical elements of the activities identified by the pupils were the complexity of the tasks, the fact you didn't have to write, that it was fun, and that they were learning new ways to do maths. The aspects they identified that they learnt about with regards to the digital technological interface were

spreadsheets (three pupils), imacs (two pupils), different formats, and different computers.

"Minus" responses

In terms of the mathematical elements of the investigative work they did, three pupils noted that some bits were too easy, while one indicated that it was hard to work out the investigation while working with the spreadsheet. The other responses that pupils attributed to a negative experience in this element were operations and percentages. Interestingly these were both identified as positive attributes as well, indicating the personal nature of the engagement and influence in this domain. Two other pupils recorded that a minus was that they knew some of the 'stuff' already, while other responses associated with the learning process were that it was not really learning, was not as fun as doing it in your head, that there were too many people working at once, and that sometimes the graphs took longer. Regarding the operational aspects of the medium, one pupil noted a difficulty with getting the 'hang' of the environment, while another found making the formulas operate was a bit confusing sometimes.

"Interesting" responses

Four pupils responded that using the formulas was an interesting component of the investigative work, while two others found everything interesting. Other comments were that the graphs were interesting, the way it works it out for you, the games, *Fill Down*, and learning about spreadsheets. Two noted that the interesting aspect of the spreadsheet environment for them was having another way to do the maths.

Discussion

Although the PMI approach was different from the questionnaire, and the nature of the comments more diverse, perhaps due to the more open response structure, some commonalities with the questionnaire data nevertheless emerged. Both approaches allowed the participants to articulate personal perceptions and points

of view in their own terms. They facilitated the communication of attitudes and perspectives, allowing the possibility of fresh insights to be gained or the enhancement of interpretations that emerged from other data.

In both the PMI and questionnaire response, participants indicated working in the spreadsheet environment seemed easier and faster, with the complexity and accuracy of number operations mentioned in this regard. The *Fill Down* function and using formulas were both noted as being useful and interesting features, although in both approaches there was a pupil who found some aspects of working with the formulas difficult. Other participants indicated that the challenge and the need to think harder were positive aspects of the environment, while pupils in each approach commented on the way it allowed them to engage with the mathematics in alternative ways. The comments regarding the work introducing then facilitating the participants' understanding of different features of digital technology (e.g., using spreadsheets), suggested that their repertoire of investigative approaches had been extended. For those participants, this might have indicated that they have adapted their approach to some extent and perhaps implied a reorganisation of their approach to investigative work. These comments were also evident in the questionnaire and interview data. Likewise in both, the activities and games were seen as positive features, accentuating the interactive nature of the spreadsheet as a positive affordance; something given primacy in reports of other research (e.g., S. Johnston-Wilder & Pimm, 2005).

The PMI included comments that the spreadsheet work was too easy in places, something that didn't emerge from the questionnaire. An extensive range of responses related to the ease or difficulty with both the mathematics and the medium wasn't unexpected, however, as each of the pupils brought their individual preconceptions and underlying discourses to the situation. Each pupil had an individual experience that was framed by those discourses and the contexts associated with the activities. The research was not attempting to extract generalisations through the analysis of these data but was endeavouring to better describe the situation, and to inform the discussion regarding the ways that engaging in mathematical tasks through a spreadsheet medium might elicit

alternative responses, with participants traversing varying learning trajectories and understanding possible.

Taken in isolation, removed from the particular situation, it is hard to ascribe an interpretation to these pupil comments. As well, the meanings given to their comments might have been at variance to the meanings that I attributed to the same words. Their comments did nevertheless represent the attitudes of individual participants towards the spreadsheet environment and further enlighten our understanding of their perspective when they engaged the tasks through the spreadsheet medium. Therefore they did inform the study in meaningful ways, but simultaneously indicated concerns about the removal of the data from its historically situated context, and an emerging preference for a more holistic interpretive examination of the data.

Overall conclusions from the initial analysis

As discussed in each of the previous sections all of the methods undertaken (observation, interview, problem challenge analysis and questionnaire) produced data that informed the research questions, albeit in different ways. There were constraints and layers of assumption associated with each of them, but each also provided opportunities to view the research situation through an alternative lens. Different methods presented alternative filters and the potential to enrich or expand the research process by the utilisation of divergent mechanisms. They provided avenues for further insights and perspectives in relation to addressing the research questions. While each approach captured the situation from a particular perspective, through a particular set of eyes, it was important for those viewpoints to be articulated and historically situated in the overall evolution of influences and interpretations related to the research questions and the associated ways of understanding that emerged. These varying perspectives and analyses also informed the research process - they were constitutive in the evolution of the emerging methodology as my attention oscillated between examining the data

through these alternative filters and engaging in broader perspectives on the research process within the social science domain.

Despite the interpretations of the data emerging from differing perspectives, patterns and commonalities in the explanations evolved from the analyses these alternative filters evoked. Through both my perception of the events and responses, and the participants' versions of the various situations, several common positions were evident. The initial engagement with the tasks was most frequently through the spreadsheet medium. Even when there were some preliminary encounters through other means; for example, dialogue showed the participants promptly began interactions within the spreadsheet environment. Both these initial encounters and the ongoing interactions were framed, amongst other influences, by the visual, tabular structure that is particular to output generated in the spreadsheet setting. The tables of numerical output appeared to allow the students to interpret, investigate and explain patterns and relationships in the data more readily. Another characteristic that frequently arose was the speed and ease the spreadsheet afforded in the manipulation and computation of numerical data, often linked to what the students perceived as more difficult forms of numbers (e.g., decimal values), or the management of large amounts of data simultaneously. This often seemed to alleviate restrictions posed by computational aspects of the investigations and permit the students to attend to more challenging aspects or broader interpretive elements of the investigations.

The students and researcher also all recognised that the interactive nature of the environment, coupled with the speed of response to inputted data, seemed to provide an alternative way for the investigative process to evolve. It appeared that the spreadsheet environment gave opportunity for the learning trajectories to evolve differently than with other media, with the consequential interpretations and understanding possibly differing as well. These elements of engaging with the tasks through the spreadsheet medium and the confidence seemingly promoted by the particular nature of the experience also, from my perspective, engendered an attitude of risk-taking that was both overtly demonstrated and commented on by the participants. The visual aspect of the exploration and interpretation of the various situations appeared, in conjunction with other

elements of the engagement, to facilitate the envisaging and enactment of the investigative process in particular ways. This, according to both participant and researcher versions of the interpretation of the situations, subsequently facilitated a reorganisation of the participants' thinking and approaches in this particular type of mathematical activity. Thus the eclectic nature of the data, given the varying constraints and assumptions inherent with each, assisted with informing the examination of the research questions.

Each of the methods provided a temporary fix on the situation; they were historically situated accounts in terms of the analysis of the data as well as being provisional stances in terms of the research process. These interpretive perspectives of the situation at various junctures are not necessarily reconcilable however, nor must they evolve in a sequential manner. The researcher might have a fragmented understanding of self, one that is different for each situation. There is a layering of perspectives that accumulate and interweave to become the version of reality at any particular time, but there is no one correct version of truth to be realised eventually. The researcher's engagement with the research process is transformative and offers a reorganisation of ideas regarding research as well as different versions of explanation. The process sanctions an ongoing regeneration encapsulated by the co-evolution of perception and phenomena. The way this transformative process might have evolved, how various discourses might have shaped the process, and the manner in which it was shaped by language were all aspects that required deliberation. The next chapter examines the transformative process that I, as the researcher, underwent in trying to reconcile those various perspectives, those fragmented views of self.

CHAPTER SEVEN: A fragmented view of mathematical research

He maramatanga to tenei whitu

He maramatanga ano to tenei whitu

Each star has its own luminance or

presence in the sky

Introduction

The research process, and the examination of the phenomena are co-dependent. They are each attributable to the other's constitution. The various lenses applied were renditions borne of prior research experiences, allied with the underpinning discourses associated with each. These shaped the data to some extent, but in turn were shaped by that engagement with the data. There was a co-evolution of methodology and data as my gaze alternated between the examination of the data, and the interpretations and explanations associated with those examinations. The articulation of that evolutionary experience positioned the explanations in the historical and cultural contexts from which they were derived, and which thus gave them their meanings. This ongoing development of the methodology also, through its constitutive dimension, became part of the data in itself suggesting an examination and interpretation of that evolution would likewise inform the consideration of the research questions. This chapter focuses on the transformative process that came about as the research methods were engaged. It investigates a personal researcher narrative within the study, and how various research discourses fashioned the production of knowledge during this undertaking.

In its embryonic form, the research investigated how using ICT, in particular spreadsheets, might influence the ways students engaged with mathematical

phenomena. The initial research proposal envisaged an eclectic approach to the methodology, with a mixture of qualitative and quantitative approaches. A rich interlacing of data was sought, perhaps engendering some sense of validity encompassed within possible consensus between the findings that might emerge. As the data was explored, a tension emerged between my assumptions about learning in mathematics, and the truths assumed by the research methods used. The research emphasis then shifted to how participants' language might have framed their interpretation and approach, when using spreadsheets in mathematical investigation. The research asked: How was the language of this perspective fashioned by the environment? How did this language shape the interpretation?

The research then began to assume a more reflective perspective in exploring how the fragmented viewpoints of self might have been reconciled within the research process. The chapter continues with a discussion of a localised hermeneutic circle and how the process of interpreting the mathematical phenomena depicted, depended on the pedagogical media through which it was engaged and on the research media applied to this. How this personal traverse of methodology and approaches resonated with broader transformational influences on knowledge production was also central to its purpose. For this transformative process to evolve, there required the sifting of personally held perceptions through the predominant discourses that reshaped those perceptions. The emerging research perspective was a function of each previous philosophical space the researcher had inhabited, and they in turn were participatory contributors to each new perspective. Various commentators (e.g., Kincheloe & Berry, 2004; Ranciere, 2004) have discussed the complexity of this process, and how new cultural perspectives are envisioned from earlier personal influences. "Countless acts of meaning making have already shaped the terrain that researchers explore" (Kincheloe & Berry, 2004, p. 31). You never escape the influences that give you your own space. By examining the personal, transformative process that evolved through this research project, insights might be gained into how it contributed to what constitutes knowledge production about mathematics educational phenomena in a digital world.

Research is a function of its specific cultural-historical traditions displaying both reflective and transformative dimensions. While the transformative aspect is often explicit as part of a broader assessment procedure, it also evolves informally, and is clearly connected to the unravelling of the subject of the research. The open horizon implicit to this transformative process is characteristically educative. "…education must strive to open new dimensions for the negotiation of the self. It places students on an outward trajectory toward a broad field of possible identities" (Wenger, 1998, p. 263). Whether it is the phenomenon or process that is the motivating purpose, they are symbiotically linked, and inherent in the transformation. They variously open insights into the research and offer potential for more fulsome understanding. It is the researcher perspective then that enables the recognition and extent of transformation, and this perspective is embedded in their personal socio-cultural tradition. How the research might lead to changing perspectives and how the researcher navigates, and assimilates these emerging perspectives into their understanding, is complex and individual.

This chapter views a personal research transformative process yet as such informs a wider perspective. The first section outlines my initial positioning in previous research, derived from quantitative roots, and the change in perspective evoked by the tension between the influences within which the methodology and pedagogy were embedded. The next section situates the emerging perspective within a phenomenological discourse. A collaborative approach to research evoked the notion of the production and internalisation of meaning as social constructs when cultural artefacts, such as language, were engaged with through pedagogical interfaces. There remained an element of perturbation however, by the dissonance between the more illuminating interpretative methods and their situating within prevailing social, cultural and political discourses. While several stories were beginning to emerge from the data, engagement with broader theoretical literature, and a growing disquiet with my confidence to be able to reveal a fulsome story through these lenses, led to a more interpretive frame being adopted. The third section outlines attempts to resolve this perturbation through engaging in broader social science philosophy, and embracing a more

reflective trajectory. As personal narrative evolved, a tension again was evident as the eclectic nature of the varying perspectives led to the examination of what was illuminating and what was delusory. The space occupied by the researcher at various junctures was precursory to the data itself. We have a multiple, fragmented set of distinct selves arising from the linguistic and social arrangements in which they are situated (Ernest, 2004). How my fragmented views evolved and were reconciled is discussed within the hermeneutic perspective that emerged, as the data and broader theoretical perspectives were alternatively engaged. It is from this refined yet even now evolving frame that the subsequent analysis is suspended.

The disturbance of a prevailing personal discourse

The research embarked on this project stems from a previous research study that investigated how children might learn number concepts and processes when they were encountered through spreadsheets. The research traditions in which the methodology employed during this previous research was predominantly embedded, were complicit in a quantitative paradigm. Although not articulated as such, the seeking of causal relationships between intervention and effect, the constraint of examining variables to pinpoint this cause, and the justification of sampling methodology in the pursuit of generalisation, were founded in personal experiences of quasi-scientific research and formalised statistical testing. While qualitative methods were also engaged, the discussion appeared to indicate that the interpretative paradigm while offering a more viable approach to producing data in this field was given the role of enhancing the quantitative outcomes, or giving them some validation through the concurrence of explanation, rather than being the actual research picture, per se.

The following account from the earlier work gives a flavour of this perspective:

> *The growing realisation, through observation and reflection, that qualitative methods, such as ethnography, enrich the understanding of what is taking place in an environment as relationally complex as*

> *a classroom, has altered the nature and methodology of research in mathematics education. As ensuing action, based on new methodology, has likewise been observed and reflected on, the evolutionary development cycle of a more appropriate research methodology has continued. This has led to research that has closely meshed theory with practice, and contained deliberate aspects of actuating teacher change (Calder, 2002, p. 30).*

Also perceptible in the interpretation and explanations offered in this earlier research was the intrusion of language that reflected a more behaviourist foundation. Words such as 'growing', 'altered', and 'contained' indicated an expectation of action and reaction. Looking for a causal link was an implicit expectation. While the use of observation and interviews endeavoured to contextualise the data, the underpinning thrust, as manifest through language and action, was quantitative. These earlier conceptions of the research process shaped perceptions and actions at the time and set the parameters for subsequent engagement. Recognition of the value of an interpretive paradigm was clearly evident, however. Reference in the literature to Zuber-Skerritt's (1996) application of action research to organisational change theory, the articulation of the need to understand actions or implications rather than causes, and the underpinning of the research by situated constructivist – interactionist philosophies (e.g., Cobb & Bauersfeld, 1995) all indicated this acknowledgment. Yet there was still the notion of the interpretive perspective enriching the evolving picture, rather than being the illustrator, being influential rather than a constitutive element of the methodology.

Critical to this acknowledgment of the significance of interpretive methods was a personal philosophy, substantiated by the research, of the way children learn mathematics. These perspectives focus on how mathematical understanding and knowledge evolve. Yet as the data was analysed, and as interpretations began to emerge, this personal philosophy evoked tensions between what was supposedly valuable from my research perspective, and what was more illuminating to the research question in terms of the children's understanding and approach. The most worthy insights into the way the children enhanced their understanding,

through the use of spreadsheets, came from observing them and recording their discussion and comments, not from the hypothesis testing of the difference in their framework level pre-intervention and post-intervention. Typical student comments gave more perceptive insights into the children's understanding and how they made sense of the artefacts. For example,

> *Counting Back* [worksheet activity] *helped me to learn to do it in my head better. Usually I count it on my fingers instead of in my head, but it helped me to see it in my head better.*

These student comments were more illuminating of the ways they made sense of the phenomena than:

> *A Wilcoxon signed ranks test indicated a significant improvement, between testing times, for the enhanced class in their content score ($z=-2.996$, $p<0.01$), and MiNZC level ($z=-2.828$, $p<0.01$)(Calder, 2002, p. 45)*

A formal test result of this nature provided a relative measurement of a pre-defined variable, but no elucidation of how the children were thinking. Measurement is a worthy process, but only useful if its particular focus is appropriate. The focus of this research was on *how* the children's understanding might be different. Examination of the varying discursive domains that emerged from the data was undoubtedly more informative in addressing that aim. It was evident that the emphasis on quantitative methodology had constrained the data and the subsequent analysis. The conclusions arrived at in the research indicated that surfacing of a personal ideological transformation.

Also recognised was the tension between a research methodology suited to the atomisation of knowledge and its sequential transmission, and one suited to a more holistic, investigative approach where the learner negotiates their understanding through their interactions with others. "Research can only identify and describe knowledge construction if its methods fit what we know about the process of knowledge construction and the learning environments in which it

occurs" (Somekh, 2001, p. 168). My researcher perspective had shifted; through an understanding of the pedagogical process, and the tension with this that the initial research approach evoked.

Towards a more interpretive approach

A way was sought to reconcile earlier research landscapes that were frequently influenced by what was perceived as a socio-constructivist approach, with the disturbed research methodology. While these thoughts percolated through various philosophical and methodological filters, an intervening research opportunity arose. A collaborative project evolved. It concerned my mathematics education department, their current practice in both teaching and research, and the approaches to mathematics education of their pre-service students. Specifically, its focus was on how these students might cultivate their discussion of mathematical teaching and learning. It aimed to facilitate the development and evaluation of a 'social model' of pre-service teacher education, one that emphasised the enhancement of mathematical dialogue. A simultaneous objective was fostering the department's capacity to research and reflect upon their own tertiary teaching practice, towards fuller participation in the broader mathematics education research community. Interestingly, there was a bilateral nature to these objectives. There was both the phenomenon and process aspect to the purpose, and explicit acknowledgment of their relationship.

The focus on the mathematical discourse of pre-service students allowed for an eclectic approach from the research team. Bicultural mathematics education, on-line learning, the affective domain, and using ICT for mathematical investigations were individual contexts in which the broader objectives were embedded. Methodologically, the project adopted a phenomenological perspective on teacher participation, reporting it using an interpretative and generative hermeneutic process through which the pre-service teachers gradually organised their experiential world (Brown, McNamara, Jones, & Hanley, 1999). An emphasis on how the use of spreadsheets might have enhanced understanding in mathematics continued, but being part of a collaborative research community

in which the emphasis was on the participants' discourse, offered potential to reconcile tensions with respect to my research methodology, and the perturbation created by the alternative paradigm. The approach was compatible with my pedagogical stance in mathematics education, one that privileged mathematical investigation as being conducive to enhancing understanding.

The research undertaken considered the spreadsheet as a pedagogical artefact to facilitate discussion, as part of the process of understanding, alongside an alternative approach based around pen and pencil methods. The ways in which the conversations in the different environments filtered the understanding of and the approach to the investigation was central to the research. The inquiry methods being advocated emphasised more explicit facilitation of social interaction between learners. With an emphasis on working in groups and verbalising the interpretations of mathematical situations, negotiation of understanding was encouraged. Given the socio-political context in which this dialogue was framed, it was nevertheless the dialogue that elicits the negotiation of meaning.

Discussion that occurred within the context of mathematical activity facilitated learning, with the teacher, as the agent of enculturation, playing a key role in support of this. Similarly, the influence of the researcher perspective with its inherent political, historical, cultural and social hues coloured the landscape in which the discourse was constrained. This sort of perspective activated interplay between the task of the individual learner and the way in which that is understood as an engagement with a more social frame. Cobb (1994) has highlighted the pedagogical tension between the perspectives of mathematics education being perceived as a notion of enculturation, as compared with one of individual construction and the theories that have been invoked in support of these. Meanwhile, Brown (1996) has offered a phenomenological formulation with an emphasis on the individual's experiences within the pedagogical environment in which they are immersed, as their engagement is framed by prevailing discourses.

The place of discourse, and examination of the dialogue initiated through the differing pedagogical media were seen as pivotal to this research project. The following excerpt gives some illumination regarding this aspect:

> *Most significantly, the social interactions appear to shape the analysis of the patterns in distinct ways. Given that the path to, and manifestation of, the patterns differ, the conversations indicate a different approach once the patterns are viewed (Calder, 2004a).*

From the emerging personal perspective, if the mathematical conversation and the negotiation of learning were different, then the learning experience would be different, and alternative understandings would emerge. With the participants' dialogue and corresponding action firmly positioned as critical data, the research methodology was modified to acknowledge this. Interpretive methods set in more naturalistic settings were used. In this aspect, the different settings filtered the conversation and approach, and by inference, the understanding. The mathematical understanding is a function of the social frame within which it is immersed, and the social frame evolves uniquely in each environment (Brown, 1996). The study demonstrated that the different pedagogical media provide a distinct lens to contextualise the mathematical ideas, frame the mathematical exploration, and condition the negotiation of mathematical understanding. Inevitably, a social frame did emerge in both the classroom and the spreadsheet environments. The dialogue in the two settings revealed varying approaches and understandings. Significantly, the social interactions influenced the interpretation of the patterns by distinctive means. Given that the process by which the patterns emerged differs, the dialogue indicated the participants utilised different approaches to the analysis and explanation of those patterns. Those using the spreadsheet took a more visual approach. They observed and discussed visual aspects, for example the situation of the digits:

> Kimi: You take the zero out. What about when you get to the three digits? Was that 223? So is the middle number still a double? Okay, so when you've got three digits you get two, two, five, two, three.

Those using pencil and paper were more concerned with the operational aspects that generated the patterns. For example:

> Tom: Basically if you times your number by a hundred and then by one you would add them together and get your answer.

The approach taken by participants in the two settings was enlightened through discourse analysis and observation, with the data situated in a relatively natural context. The differences were revealed through the appropriateness of the methods. Earlier approaches that had been utilised would have left layers of insight undisturbed. Analysis of the dialogue revealed participant intentions that may have been overlooked in the earlier study. Taking an interpretative approach led to nuances in the differentiation of analysis that weren't evident in the earlier research. Recording the discourse and observing the participants allowed a more fermentative analysis and understanding to evolve. Generalisation was distilled from the existing ingredients, analysing what was there, rather than the prescriptive manipulation of controls to produce the data as attempted in the previous study.

Yet there was still intervention; there was still an element of control with employment of the two situations, and there was still comparative analysis. In this research, however, the data was collected with more naturalistic methodology and the data shaped the research and analytical tools, as well as the selected tools shaping the data. A quantitative overhang permeated the underlying methodology, however: the use of a control variable between the two settings. The perceived need for control, and the impediments encountered in the attempts to document it might have obscured the complexities that I was endeavouring to reveal. As a more phenomenological perspective was explored, it became evident that consideration of various hermeneutic perspectives within the broader social science context might help to reconcile these conflicting interpretations.

A reflective trajectory

As the investigation of the principal situation from which the research questions were being examined was considered, a fusion of qualitative and quantitative methods emerged as an appropriate means of gathering data. The dichotomy evident in the research perspective, its ensuing atomisation of the methods, then interplay between the evolving viewpoints associated with the two paradigms (qualitative and quantitative), led to the examination of broader philosophical interpretations of social science research. Ratiocination needed a self-inflicted disturbance to illuminate the way forward. Engagement with these broader perspectives enabled the research methodology to be re-envisioned. The personal reflective narrative evoked by this literature became data but also illustrated this unhinging, and the subsequent focusing that emerged. The notion that mathematics, and other traditional pure sciences, are social constructs rather than descriptions of reality proffered a model of learning based on the negotiation of meaning, or perhaps, enculturation into a social practice. Discourse was a theme that threaded this interpretation. This interpretation of the learning process influenced the re-emergence of a research methodology that might be perceived as a system of possibility for the production of knowledge.

Hermeneutics, the theory of interpretation, stresses that to understand human behaviour we have to interpret its meaning (Gadamer, 1976). We have to grasp the intentions and reasons people have for their activity and as Giddens points out "Truth is the promise of a rational consensus" (1985, p. 130), but how can we differentiate this from one based on power or tradition/custom? Power is a critical measure of existing interaction: it can highlight where consensus is based on tradition, power or coercion. This flags the juxtaposition of perceived freedom of choice and the power hierarchies or traditions that actually shape those 'freely' made decisions. The space the researcher occupies at particular junctures not only shapes the interpretation of the data through the prevailing discourses with which the researcher engaged, this engagement simultaneously shapes the discourse. It appears that reality, while perhaps a negotiated shared vision, is dependent on a consensus which may or may not be the same for everyone, and may be arrived at

through dominance or power derived from knowledge or status (Boon, 1985). It seems rational to contend that we interpret our approach to everything through that lens that is our present state, prejudices and all. Even when we experience quite cataclysmic events or have life-changing experiences, the catalyst or readiness for changes in understanding or perceptions, are embedded in our initial viewpoint. The phenomena studied by the social scientist are crucially bound up with (though not identical to) the interpretations of them given by the members of the society being studied. The social scientist's data "are the already constituted meanings of active participants in a social world" (Schutz, 1967, p. 10).

Hermeneutics can also be conceived of as the theory of the operations of understanding. While this has historically been perceived with regards to the interpretation of text, Ricoeur (1981) rationalised spoken and written language through a definition of discourse as interactive dialogue. He contends that through the notion of discourse, language can be either spoken or written. It is not that these forms are the same, but that they have similarities. Gallagher (1992) maintains that hermeneutics examines human understanding in general, including social processes. There is interplay between action and the sedimentation of history. History (in both specific and general terms) evolves as ongoing human action leaves a residual or mark (Ricoeur, 1981). Through interpretation, action is objectified and transformed into a temporary fixation of meaning. Within the interpretive process, the emergence of fresh meanings and possibilities for the interpreter, is permitted by the distance maintained between the interpreter and the object of interpretation (Gallagher, 1992). In this case, the evolving history of my research perspective was a collaboration of the underpinning discourse in this domain, and the corresponding action it evoked. A hermeneutic viewpoint allowed the incorporation of discourse and actions, as the links between the research approach and what was being said or written, were examined in terms of the interpretation of the mathematical phenomena and the research methodology. The data was hinged to the discourse that constituted its production and analysis. Aligned to this version of methodology was the notion that data are arbitrary and are therefore susceptible to a wide range of analytical operations. The authorship of the data may be refuted and the entire approach to data gathering, together with the data, seen as a composite artefact regulated by arbitrary historical frames

(Sanger, 1994). The data was not only inextricably linked to that evolving methodological perspective, the outsider's view was limited without it.

These, and other philosophical perspectives, illuminated the influences that might pervade the data, methodology, and interpretative analysis. The researcher, while cognisant of these influences, can nevertheless only examine that which is presented. The differing views of data and the authenticity of the research for its audience, depend on the extent to which data are acknowledged as authentic versions of events (Sanger, 1994). As research is a social construct, it is the linguistic systems that define the perspectives the researcher might take. The environment inhabited by the participants was seen through these alternative filters, and the data is as much a function of how the researcher sees it, as how the teachers and children see it. The research perspective shaped the data, and the data shaped the research perspective. Within an anthropology context, Geertz (1995) maintains that it is not only the phenomena that changes over time; the onlooker's viewpoint changes too. He identified the setting from which the phenomenon occurred, its intellectual and moral justification, and the nature of the discipline from which the onlooker is viewing, as also shifting. The researcher's personal narrative, therefore, is a vital aspect of any understanding that may emerge.

Up to this point in the research process, amongst the contemplative aspects of researching, both data and methodology had been reflected on, notes made *in situ* for later review, supervisor and research diaries maintained, data shaped and restructured in varying forms, and papers written and presented. A more formalised approach to reflection, a personal narrative, was now viewed not only as a reflective process, but as part of the data itself. Personal narrative enabled me to investigate experience in a way that situated change within the context in which it occurred, or the narrative from which it was derived (Clandinin & Connelly, 2000). Through the transformative process, a more formative, reflective perspective on methodology and knowledge production emerged. It seemed evident that an approach to methodology, which was contextually embedded, interpretive in nature, and included a clear articulation of the researcher's perspective, was necessary to embrace a more inclusive story of the

concerns of the research project. The stories of all participants should be considered and valued, with the situating of those accounts implicit in the findings. Yet the individual's accounts are fragmented by time and perspective, and one sense of reality is superseded by another. This fragmented discourse needed to be fused at a personal level, but with the reconciliation process mirrored generally, as sense making in communities of practice emerged.

Reconciling the researcher's fragmented perspective

So how might these varied insights be rationalised? An eclectic array of data had been gathered; some seemingly as evidence of a story that was possibly preconceived, some more organic and metamorphic in nature, some quantitative, some interpretive, and some reflective in form. How was this informative? How was the emergent picture illustrative of the experience? How was it representative, or did the varying perspectives diverge to the point that they concealed that which they were trying to clarify? The sorts of spaces the researcher occupied, and the extent that these may have been delusory or illuminating depending on the story they were telling, and to whom, are part of the data itself. We see data through varying sets of eyes. It is important to understand how those eyes see, and how they produce the objects they describe.

The initial production of data was framed by a version of a socio-cultural discourse that gave primacy to those underlying influences in the production of knowledge. A compilation of qualitative and quantitative approaches was deemed most appropriately to produce the data that were informative of the research questions. As Chapters Five and Six show, they did suit this purpose to varying degrees, but each had particular constraints and assumptions within which they were positioned. Through the ongoing reflection and analysis of the data, coupled with the broader perspectives engaged with through the blend of social science and interpretive methodologies, a more interpretive frame emerged. Reflective writing and personal narrative became constituent voices in the interpretation and explanation of the data, while simultaneously shaping the research methodology.

There was also recognition that while much of the initial data was set in relatively naturalistic settings and gave opportunity and emphasis to the participants' voices, the gathering of elements of the data around emerging stories, themes or metaphors, removed the data from the situations in which they were embedded to some extent. This appeared to inhibit the layering of multifarious influences that were endemic to the evolution of the data when the participants shifted their noticing from the investigation of the task through their prevailing discourse or the media, to reflection and explanation of these influences due to that particular engagement. The participants' focus oscillated between the task and their perceptions borne of underlying discourses, with each iteration flavouring the maturation of understanding. A version of a moderate hermeneutic perspective emerged that sanctioned this layering of interpretation with the socio-cultural influences, and acknowledged that these were embedded in language. Both the speaker/writer and the listener/reader were conditioned by their personal historical circumstances and language, and so was their active participation in the interpretive process. This version of the hermeneutic perspective also took the optimistic view of interpretation that positioned the audience as a creative participant in this process.

The research process allowed some leavening; it was a transformative process that was imbued with the researcher's range of perspectives. Examining the participants' and researcher's viewpoints appeared to have mediated the learning, the understanding of the mathematics, and the research process. Yet Atkinson, Brown and England (in press), discussing Lacan suggested that the way in which we see ourselves fitting in is always delusory, and the selves we see in different situations do not get reconciled with each other. We have a fragmented view of self; a different one for each sort of situation we find ourselves in. For a meaningful picture to evolve from these fragmented views, the researcher needs to clarify explicitly the lens through which he/she was viewing the data and how the linguistic conventions, and structures within which it exists, influence the other representations. If the intentions were sincere and clearly articulated, and there was honesty in the researcher's interpretation that other representations validate, then the audience should be able to decipher any misrepresentation of the data from their own viewpoint. Both the researcher and the audience are seen

as being in the midst of an ongoing metamorphosis, so an analysis can only represent a particular truth at a given time.

Alternatively, if philosophically we determine that any findings, no matter what, are delusory, then we might conclude that research is pointless anyway, as its conclusions are only determined by the linguistic conventions it embraces, with all the power and cultural implications that involves. Unfortunately, the pursuit of rationalising, then alleviating, this perspective might lead us back to the consideration of control, with its characteristic associated problems for education research outlined earlier. It might also lessen the authenticity we associate with significant interpretative research into the human condition that could never otherwise be captured. Better to have the diversity of perspectives, and the rich, eclectic array of interpretation that the researcher reconciles through the articulated research media, and the audience views through constrained but open eyes.

The research media applied did appear to influence the data and likewise the data influenced the research media. They are inextricably linked, but if the audience is informed of the orchestration, if they have awareness of what may be illusion and what is perceived reality, then they can still value the more fulsome understanding elucidated by the account. If the research approach and analysis can't be rationalised by the audience's view of reality, then it won't be recognised anyway. The challenge for the researcher is to mediate their perspective, so that it is valid in the varying audiences' perspective, while not compromising their own personal view of reality.

It was apparent that personal perturbations had initiated or provided a catalyst for the evolution of the research methodology at an individual level. This characteristic seemed complicit with a more global perspective of knowledge production as research communities look to understand more relationally complex situations. The macro level resonates in the personal account. Tension, arising from varying perceptions of reality, and opportunities, arising from possibilities surfacing through these distinctive transformative processes, can lead to the emergence of a more illuminating sense of knowing and approach to

knowledge production. A hermeneutic circle was evoked by the act of investigation. The data was examined through the lens of prevailing research discourses, initiating the amending of this perceptual frame. The data was then re-examined from this fresh perspective, with the interpretation and the space occupied evolving with iterations of the cycle, as my gaze alternated between the examination of specific data and a broader perceptive horizon.

The hermeneutic situation was authored by my circumstances, as much as by the localised situating of the object to which the interpretation was tied. While these frames were both limiting and enabling when viewed from various perspectives, the interpretations also fed forward into the next iteration of interpretation through the transformation of conceptual frames and traditions. The following three chapters discuss this version of a localised hermeneutic circle as applied to the data. Each chapter interprets situations within the investigation derived from the initial analysis, and the associated stories and traditions that emerged from that process.

CHAPTER EIGHT: Interpretation and the setting and examination of sub-goals.

E kore te patiki e hoki ki tona puehu

The flounder does not go back to the mud it has stirred

Introduction

One aspect of the data that gestured towards it being viewed through a hermeneutic lens was the manner in which participants, interpreting from their fore-structures, negotiated an understanding of the investigative situation, and then navigated pathways through temporary reconciliations between their interpretations and explanations. The learners' fore-structures are the preconceptions, drawn from their underlying discourses that shape their interpretations. In the initial analysis, the speed of response by the computer to inputted data and the filtering of emerging patterns through a visual lens, were identified as elements of the learning experience in this particular medium. The data certainly supported those interpretations, but they were elements of the data that were viewed in relative isolation from the contiguous information; brief illustrative snippets as opposed to continuing episodes. Viewing the data from a hermeneutic frame, as a local hermeneutic circle, led to a more holistic, ongoing interpretation. The episodes, and the manner in which the activities were engaged, are historically positioned: functions of their past, constituent of their present, and conditioning of their future. As such they were considered formative influences, indicative of an evolving understanding, and were examined more inclusively, with historical contextualisation, than in the previous analysis. The ways students made initial sense of an investigative situation when approaching it through the pedagogical medium of the spreadsheet were considered in this chapter, and how subsequent learning trajectories were conditioned by those initial exchanges. It examines the approaches in which participants engaged, and how their preliminary responses were shaped, and their sub-goals framed, by the

features of the spreadsheet setting. It also explores the manner in which this might have filtered their understanding and conjectures.

Investigation of a mathematical situation, whether one contrived as a 'school maths' model or one necessitated by real life circumstances, requires an aspect of familiarisation. Polya (1945) was the first to formally articulate this 'understand the problem' stage in his four-step approach to problem solving, but contemporary mathematics educators maintain the validity of this initial step (Holton, 1998). What am I trying to find out? What information do I have? How do I gather more pertinent information? What picture is beginning to emerge? These questions may be part of that familiarisation process, and the individual's response to the mathematical phenomena that will condition the shape of the investigative process.

This familiarisation process isn't distinct from the solving process however, nor is it necessarily chronologically placed prior to the commencement of that process. Nunokawa (2002), discussing Resnick's concept of sub-goals in solving more complicated problems, observed that these aspects were intertwined. He noted that the settlement of sub-goals was conditioned by the learner's understanding of the situation, but also that the sub-goals settled on by the learner influence her interpretation of the problem situation. Sub-goals are generated as part of the familiarisation and re-familiarisation of the problem, and where the learning is situated will influence the specificity of their production. This is indicative of a localised hermeneutic circle. The learner's initial engagement with the problem is conditioned by their existing mathematical understandings, the medium through which it is engaged, and their fore-structures in those particular domains. They interpret the task from the perspective of the whole, their prevailing discourses. Having engaged with the task in their initial 'skirmishes', they then re-envision their broader perspectives, and re-engage with the task, the part, from a new modified viewpoint. This allows them to set new sub-goals, according to Nunokawa and Resnick's version of the process, which after task-focused activity, modifies their perspective once more. Their understanding evolves from cycles of this iterative, interpretative process. The data in this chapter illustrates this cyclical process as the students interpreted the part (the

task) from the whole (their preconceptions, borne of their prevailing discourses) with subsequent reorganisation of their thinking, and interpretation through fresh perspectives.

It is important to be aware of how using a spreadsheet might have constrained the investigative process, by influencing the generation of sub-goals, as well as their previously identified potential to open up investigative opportunities (Beare, 1993; Calder, 2004a; Drier, 2000; Ploger et al., 1997). Zbiek (1998) meanwhile, established that ICT has enhanced students' ability to model mathematically, while Chance et al. (2000) found that its use enriched students' ability to problem solve and communicate mathematically. Providing an environment to test ideas, link the symbolic to the visual, link the general to the specific, give almost instantaneous feedback to changing data, be interactive, and give students a measure of autonomy in their investigation, are other opportunities afforded that facilitate an investigative approach. The current study was designed to explore how the pedagogical medium of a spreadsheet, used as a tool for investigation, might have influenced the learning experience and how processing mathematics in this way might have reorganised children's mathematical perceptions and understandings. One purpose of this chapter was to identify the ways participants approached the mathematical investigations as they negotiated the requirements of the tasks, and how this might have filtered their conjectures and generalisations.

Central to this is the participants' dialogue as they negotiated the meaning of the tasks. By examining the participants' verbal interactions as they engaged in the tasks, by observing their actions, and by analysing their reflections, insights were gained into the ways investigating mathematical problems with a spreadsheet might have influenced their understanding of the problem. As they negotiated the requirements of the tasks and explored possible solutions, a more fulsome picture of the ways participants framed their conjectures and generalisations evolved. There were three areas considered in response to the research questions:

1. How did the students negotiate their understanding of the task and whether their initial responses, were shaped by the spreadsheet environment?
2. In what manner did this initial familiarisation and the subsequent exploration process lead to generalisation, posing of conjectures, and the resetting of sub-goals?
3. In what ways did investigating in a spreadsheet environment fashion the children's approach to investigation in general?

If engaging the mathematical tasks through the spreadsheet medium permitted the learner alternative ways of envisioning the intentions of the task and then navigating the investigative process in particular ways, it is reasonable to assume that their thinking was conditioned by these alternative engagements. The actions of the students and the accompanying dialogue were examined in more extended excerpts to help determine how the learning trajectory might have evolved as the students' gaze moved between their underlying perceptions and interaction with the task. Consideration was given to whether the sub-goals they articulated through their interactions were shaped in particular ways by the pedagogical medium.

The first sets of data refer to an activity based around exploring the products when multiplying numbers by 101, the **101 X table** activity (see Figure 1).

Evolving learning trajectories through the generation of sub-goals

In this first episode, attention was drawn to the manner in which the immediate engagement of the spreadsheet to produce tables of numerical output, framed the investigation through a visual, structured lens. This structure subsequently suggested a pattern or relationship and led the pupils, through predictions, to pose informal conjectures with the employment of visual referents. The episode will be used to demonstrate that the spreadsheet environment influenced the negotiation and

settlement of their investigative sub-goals. It also illustrated how the pupils drew from their underlying discourses as they engaged with the tasks, interpreted the situation, and then explained their activity. The manner in which these discourses were subsequently transformed through that engagement was also demonstrated. As part of the investigative process, the pupils settled on fresh sub-goals from their modified perspective, invoking a local hermeneutic circle as their gaze oscillated between the underlying evolving discourse and the mathematical phenomena. It was noticeable that the pupils were willing to immediately enter something into the spreadsheet. There was little attempt, in general, to negotiate the task situation through discussion or pencil and paper methods, although some individual processing of the task requirements must have occurred. For example:

Awhi:	*So we've got to type in 101 times.*
Ben:	*How do you do times?*
Awhi:	*There is no times button. Oh no, wait, wait, wait.*
Ben:	*There is no times thing. Isn't star?*
Awhi:	*=A1*101. Enter.*

This approach was confirmed with responses in the interview:

Awhi:	I preferred thinking something about what I needed to do, then take it and highlight it down and then the whole table is there, which would help me.
Adam:	What we did is we tried a few formulas. To start off with we like typed in a few formulas that we thought it might be, and then went through and got the correct one, which got us the right answers.

It appeared the actual spreadsheet environment provided the impetus to take this initial approach. Another pupil commented:

Dee: Because of the spreadsheet, we went straight to formulas, looked for a pattern; for a way to make the spreadsheet work.

Not only had the use of spreadsheets led them to explore in a seemingly stylised procedure, it also led to an immediate form of generalisation. To generate a formula that models a situation is to generalise in its own right, but to consciously look to *Fill Down* ("highlight it down"), or create a table of values was also indicative of an implicit cognisance of a pattern; of an iterative structure that was a way into exploring the problem.

Awhi and Ben continue:

Ben: *202.*

Awhi: *Now let's try this again with three. OK, what number do you think that will equal? 302?*

Ben: *No, 3003.*

They drew on their prevailing discourses in several inter-related areas: number structure and patterns, expectations in school mathematical situations, number operations, and the spreadsheet environment. As they attended to their activity associated with the task, their understandings and persuasions from these individual broader frames had influenced which aspects were brought to the fore, which aspects were given primacy in the process of predicting. They copied the formula down using the *Fill Down* function to produce the output below:

1	101
2	202
3	303
4	404
5	505 etc.

Ben: *Oh no, 303.*

The output was different from the predictions that their prevailing discourse had framed. The pupils appeared to use the table structure as a means to interpret the situation. It allowed them to more easily notice the relationship between the input and the output, and the ensuing pattern of the output values. Their perspective evolved and they re-engaged with the task from a fresh, modified stance.

Awhi: *If you go by 3, it goes 3 times 100, and zero, and 3 times 1; 303.*

The pattern that Awhi articulated was consistent with the output that the spreadsheet produced. Their informal proposal was confirmed and they reset the direction of the investigative trajectory accordingly. They were immediately into the business of predicting and confirming in a confident, relatively uninhibited manner. They explored a range of two-digit numbers. They began to pose conjectures, and test them in an informal approach:

Awhi: *OK. Now you try a number.*
Ben: *My lucky number 19.*
Awhi: *That'll be one thousand, nine hundred, and nineteen.*
Ben: *Equals. So we need to think of a rule.*
Awhi: *Its like double the number. Its nineteen, nineteen.*
Ben: *What about 20? Oh you'll get 2020.*

They appear to have predicted what the product would be when nineteen was multiplied by one hundred and one by utilizing the patterns that were beginning to emerge for them. They confirmed their prediction ("Equals") before attending to a more generalised account of the situation. Ben then used their emerging informal conjecture ("double the number") to pose and confirm a further prediction. The ability to predict, form a conjecture then test it is indicative of a robust generalisation process. In this case, and with others in the study, the children chose a particular path because they were using the spreadsheet. The shape of their investigation was determined by the particular pedagogical approach. They were also able to quickly move beyond the constraints of the

prescribed task, forming a fresh generalisation. They had reset the sub-goal of the investigation and were exploring the effect on a 4-digit number.

Awhi: *Oh try 1919.*

Ben: *I just have to move that little number there, 1919.*

The following output is produced:

193819

Interestingly, they seemed to disregard this output and form a prediction based on their fore-conceptions. Their interpretation, underpinned by their prevailing discourses in the associated domains superseded the output, or influenced their noticing: what they brought to the foreground.

Awhi: *Now make that 1818, and see if its 1818.*

Ben: *Oh look eighteen, three, six, eighteen.*

There was an unexpected output, which made them re-engage in the activity, reflect on the output and attempt to reconcile it with their current perspective. It caused them to reshape their emerging conjecture.

Awhi: *Before it was 193619: write that number down somewhere (183618) and then we'll try 1919 again.*

Ben: *Yeah, see nineteen, three, eight, nineteen. Oh that's an eight.*

Awhi: *What's the pattern for two digits? It puts the number down first then doubles the number. This is four digits. It puts the number down first then doubles, and then repeats the number.*

The data indicated that the pupils engaged a local hermeneutic circle as they familiarised themselves with the task then moved between their broader perspectives and engagement with the task. They interpreted the task from their

fore-structures in the associated domains, then influenced by the affordances of the pedagogical medium, engaged with the task. This engagement shifted their perspective in varying degrees: their viewpoint was modified; they set fresh sub-goals in the investigative process, and re-interpreted the task from these fresh perspectives. Each re-engagement transformed their underlying discourse to some extent. In this way their understanding was an ongoing process that emerged from evolving interpretations through this iterative process.

The data also suggested they were using a visual referent for the theory that was evolving. They were looking at the actual visual sequence itself that was producing the number patterns. The third to sixth lines from the above transcript illustrate that interpretation through their naming of the products as, for example, eighteen, three, six, eighteen. They were seeing the number as three or four discrete visual elements, rather than thinking of a consequence of an operation. Their concluding generalisation confirmed this also in the seventh line of dialogue. It could well be with appropriate scaffolding the pattern may be investigated in a more fulsome manner, exploring the processes that produced that visual pattern. Meanwhile, once more the data implied that the spreadsheet environment influenced their approach to the investigation. It filtered the path to, and the nature of their conjectures, with their subsequent interpretations shaped in visual rather than procedural terms. Their understanding emerged from these interpretations as they engaged with the task through their various underlying perspectives.

It is also noteworthy that the characteristic of spreadsheets to produce immediate responses to inputted data assisted the further development of their emerging theory; it facilitated the risk taking aspect of the investigative process (Beare, 1993; Calder, 2002). As well, it allowed them promptly to pose and test notions within their emerging theory, set new sub-goals in the investigation, engage with the activity, then reorganise their existing frame. The understanding is the learner's interpretation through these evolving perspectives.

The spreadsheet environment has enabled the pupils to process the mathematical phenomena in particular ways. The setting of the sub-goals was influenced by the

visual tabular structure of the spreadsheet output and appeared to organise their thinking so that their generalisations and understandings were shaped in a manner that was specific to this environment. This is consistent with the broader notion addressed in the research that the learning trajectory will evolve differently through the spreadsheet medium, and that the spreadsheet activity facilitates the reorganisation of the thinking and as a consequence the interpretations and understanding.

Sub-goals emerging from transforming perspectives

Another group engaged in a different but comparable manner. This data is likewise illustrative of a localised hermeneutic circle, manifest through the settling on sub-goals that were influenced by, and influencing of, the preconceptions of the pupils and the spreadsheet pedagogical medium through which they were filtered. In this episode the sub-goals led to the forming and testing of informal conjectures, with an associated reorganisation of interpretation and explanation. Interestingly, the engagement with the task and the evolving learning trajectory seemed to provide evidence about how the nature of the conjectures and the way the pupils interpreted their engagement, was constrained by the examples they chose as well as their underpinning preconceptions and the pedagogical environment. While this selection of examples to explore was drawn from those underpinning preconceptions, the nature of it led to a different interpretative version.

This episode illustrates how the immediate generation of visual tabular output either confronted or enhanced the pupils' emerging informal theory. As their perceptions and underlying discourses were modified to some extent, they re-envisaged the situation and reset their immediate investigative sub-goal. With each engagement, interpretation, and reflection the informal conjectures associated with the situation were refined. It appeared that this, in turn, modified and was modified by the accompanying evolutions of their conceptual frame. The investigative sub-goals emerged from their transforming perspective.

For example, Adam's present understandings, borne of his underlying discourses, stimulated activity derived from his particular interpretation of the situation. His understandings within the spreadsheet environment, symbiotically meshed with the particular attributes and affordances of that digital medium, influenced the immediate sub-goal and activity. These various perspectives, filtered through the medium with which it was engaged, framed the interpretation of the mathematical aspects of the task. A particular, culturally- and historically-situated response was evoked. Further complicating an already complex situation of interrelationships was the social positioning within the group allied with Adam's predispositions. The group's initial response was Adam's; hence what was brought forth for the other group members may have been repositioned beyond their personal preconceptions. This is further considered in the concluding discussion of the chapter. Meanwhile:

> *Adam:* *101 and then…Now 2 digit numbers. So we've got…. in the A column we have 101, in the B we have 1 to 15, and in the third column we have a formula.*

Adam was articulating his actions as he entered 101 into cell A1 then filled down, followed by the integers one to fifteen in consecutive B column cells starting from one in cell B1. He next entered the formula =A1*B1 into cell C1 to make the product of 101 x 1 in cell C1. When he filled this column down, the spreadsheet worksheet looked as below:

A	B	C
101	1	101
101	2	202
101	3	303
101	4	404
101	5	505
101	6	606
101	7	707
101	8	808
101	9	909
101	10	1010

101	11	1111
101	12	1212
101	13	1313
101	14	1414
101	15	1515

Beth: *Oh that's interesting – look at that. The numbers just repeat themselves.*

Adam: *Oh, ok yeah.*

The structure of the table of related values revealed the visual pattern of the products as an explicit, almost immediate response to the input. The pupils appeared to have an expectation that creating the table through the affordances of the spreadsheet environment would give insights into a pattern. This was perhaps indicative of an intuitive initial attempt at generalisation. Given that predisposition with the pupils, evoked by the pedagogical medium, it was the particular affordances of the spreadsheet to create a visual, tabular structure that opened up the opportunity to observe the relationship between the factor that was varying and the product. The engagement with the task through this particular pedagogical medium, had modified their perspectives and allowed them to set new sub-goals from these repositioned angles. Their interpretations of the output allowed them to test and confirm their emerging informal generalisation.

Beth: *So you can predict.*

Adam: *Shall we do 20 now?*

Beth: *That'll be 2020. Let's try some others.*

Their repositioned perspectives had conditioned their interpretation and explanation, allowing a confident prediction beyond the output. They had reset their investigative sub-goal with which to re-engage with the task. They explored how the product changes when 3-digit numbers are multiplied by one hundred and one. The following output was produced:

A	B	C
101	101	10201
101	102	10302
101	103	10403
101	104	10504

Adam: *101, 102, 103, 104, so there's a pattern you've got your 101 and in the middle you've got 20, 30, 40, 50.*

Beth: *Quite right.*

Although, on the surface their observation was not quite right (the final digit is not 1 in each case), there was an apparent consensus of interpretation. There are several viewpoints that might be occupied in the discussion of this, one being that the pupils were indicating the B column in the observation rather than the A, that is "you've got your 101" implied 101, 102, 103, 104 as opposed to 101 in each instance. This seemed to be reasonable, given the following dialogue, but illustrates the complexities involved with interpretation and how a range of possible interpretations might be employed. The clarity of the output in its visual structured form may have contributed to Adam and Beth's mutual interpretive accord. Nevertheless, fresh impetus was given to their investigative path from the evolving perspectives. They reset their sub-goals by adjusting the type of 3-digit number to enable further insights into the pattern.

Carl: *So what would 126 be?*

Adam: *Would it be 10706, 120706, 12706?*

These predictions emerged from the evolving perspectives borne of the previous engagements. There was some uncertainty perhaps due to the variation in the factor, but the pupils' confidence and willingness to attempt variations and refine their generalisation was evident. They then tried 126:

101	126	12726

Adam: *12726.*

Carl: *So it's the same thing.*

Beth: *The first two and the last stay the same and then the outside numbers are added together.*

Adam: *Let's predict 135, it's going to be 13635.*

Beth: *Let's check.*

101	135	13635

Their generalisation while articulated with visual referents (i.e. the nature and position of the digits) was consistent with the output. Their ongoing informal conjecture emerged through cycles of setting of investigative sub-goals based on their underlying discourses, engagement with the task, and the interpretation and reflection on the output and associated dialogue. The ensuing shifts in perception allowed them to reset sub-goals from fresh perspectives.

Beth: *Yes we cracked it, now shall we do a three digit?*

Carl: *We've got three digits.*

Beth: *Yes we have.*

Adam: *Make some rules that help you predict when you have a one-, two- or three-digit number, do they work.*

Beth: *Okay, a one-digit number is – it is just the first and last number multiplied by…*

Adam: *So its 2x1 is 2, 2x1 is 2 and the 0; the zero stays the same.*

Beth: *Zero is constant and you are just adding 1 on to the outside numbers.*

Adam: *Zero is constant in the middle.*

Beth: *When using one-digit numbers.*

[Note the visual lens that they apply to their evolving conjecture].

Adam: *So with the numbers on the outside you just add one more on.*

Beth: *No we're not.*

Carl: *It's the same as the one-digit number.*

Adam: *The two-digit number is just double – the same number written twice.*

Beth: *Very good. Double the digits.*

There was a shared understanding of the descriptions employed to explain their interpretation of the informal conjectures associated with the patterns formed by multiplying with one- and two-digit numbers. However, further negotiation and interaction with the activity, the medium, the group and their underlying perspectives occurred before a shared interpretation of the three-digit pattern could be articulated.

Beth: *Three-digit number is we add the numbers on the outside.*

Adam: *We had 10 then we got 02. So we've got those numbers at the front and back [Referring to 102 becoming 10302].*

Beth: *Or you could go the other way and say let's do plus one.*

Carl: *We do 135; 135 + 1 in the middle [135 became 13635].*

Adam: *The middle one minus one is the outside number 5-1=4, 4-1=3, 6-1=5.*

Adam seemed to be drawing on some mathematical preconceptions that encouraged him to investigate simple computational links between the numbers. Beth sought further clarification, which drew Adam back to situating his ideas upon their earlier perspective.

Beth: *So we relate it to that how?*

Adam: *So what you could do is; the first digit is… There's our number 126, our number plus one [referring to 127 from the output 12726].*

Beth: *Isn't there an easier way of looking at it?*

Carl: *I would say its 35+1, 36+1 [referring to 135 becoming 13635, and then appearing to predict that 136 would become 137..].*

Beth: *It's just that on here, 101+1=102, 102+1=103 [referring to 101 becoming 10201, 102 becoming 10302]. The middle two digits are just them two digits + 1. You've got the 1 from 102 there.*

Carl: *By that theory you get 36 and by our theory, it should have been 33.*

There was a perturbation between the output, and the version of interpretation that various group members were trying to ascribe. Adam's discourse in the investigative domain, framed the resetting of their sub-goal in the form of a new informal conjecture.

Adam: *That one went up by one, maybe try 200 and something and see if it goes up by one.*

They entered 235 to obtain the following output:

101	235	23735

This output reorganised their perspective, allowing them to reframe their interpretation

Carl: *235 it goes up a 2 instead of one. So it is related to that first digit.*

Through the investigation of the latest sub-goal they have gained further insights into a more encompassing generalisation. Previously the interpretation of the situation included the notion of adding one as this was consistent with the output to that point. As their fresh sub-goal led them beyond that into a revised

interpretive account, they refined their informal conjecture. They re-engaged with the task to test further their newer emerging interpretation:

101	335	33835

Adam: *Okay 338 so it's gone up by 3. So 435 will be 43935.*

They entered 435 with the following output:

101	435	43935

They tried to reconcile this output with their interpretation as they moved between their evolving perspective and engagement with the task.

Beth: *The first digit is the same as the… is constant. The next two digits are the sum of the first number and the last two numbers.*

Adam: *3+5=8, 5 and 4 are 9, 5 and 3 are 8 [referring to 435 becoming 43935 (5 and 4 are 9) and 335 becoming 33835 (5 and 3 are 8)].*

Beth: *So the middle digit is addition of the first one and the last one.*

Adam: *And the second digit is the same number carried on from the first number.*

Beth: *The first digit is the same as in the 3-digit number; second digit is the sum of the first and last digit.*

Adam: *No, the third number, the second one just carries over from the first one.*

Beth: *So the second digit remains constant of [the same as] the number in column B [e.g., with 335 becoming 33835, the 33 is the same at the front, and the 35 at the end, while the third digit is the sum of the first and last digits (8 comes from 3 + 5)].*

Adam: *The fourth is the same. The first, fourth and fifth are the same as the first number. The second and third come from adding the others together.*

While the group's generalised conjecture was relatively simplistic in its construction, and was considered in visual terms such as the positioning and patterning of digits, it had nevertheless emerged from mathematical investigation and thinking. This process involved mathematical reasoning and collective argumentation as the pupils interpreted the data, negotiated its meaning, and justified their interpretations. As these interpretations in turn modified their underlying perceptions, they reset their sub-goals as their version of the mathematical generalisation evolved, and was described in a manner they had a shared understanding of. Their continuing dialogue gave other personalised insights into the learning process involved. They reflected on the pedagogical processes evoked as well as the mathematical ideas and process.

Carl: *One thing as the numbers got higher and higher it got easier to see the pattern.*
Beth: *Yeah, we found patterns by investigation.*
Carl: *Using trial and error also.*

They continued with a 'what if' consideration that was part of the investigation. They used the faculty of the spreadsheet to multiply three-digit decimal numbers by one hundred and one. They continued with their most recent example 435, but explored the consequences of relocating the decimal point.

Adam: *What if we used decimals?*
Beth: *Okay do a few with decimals 4.35.*

101	4.35	439.35

Adam: *Try a higher one 43.5.*

101	43.5	4393.5

Adam: *4393.50, a whole new can of worms here.*

Beth: *Although the numbers look the same.*

They considered the output as it appeared on the screen:

101	435	43935
101	4.35	439.35
101	43.5	4393.5

They inputted another:

101	0.435	43.935

Beth: *They are the same numbers but just with the decimal.*

Adam: *Let's see 25.4 then. Should be 25654.*

Beth: *And the point.*

Adam: *Oh yes 2565.4*

Beth: *Yes it is. So it's the same but with the point.*

It appeared Beth was implying that the digits in the product remain the same, but their place value was governed by the position of the decimal point in the factor(s). The pupils had concluded that multiplication with decimal numbers produced the same digits as with the corresponding whole numbers, but the positioning of the decimal point and hence the magnitude of the product was dependent on the positioning of the decimal point in the factor(s).

The pupils' earlier, evolved interpretation with integers had premised their interpretation of the application with decimals. This interpretation had become an element of the preconceptions that they privileged in their interpretation of the decimal situation, illustrating how their underpinning discourse in this domain had been modified through a localised hermeneutic process. Described simplistically, this process involved cyclical iterations of interpretation, setting of sub-goals, engagement with the task that modified their perspectives,

interpretation from the fresh perspective, and then resetting of sub-goals, etcetera. Interestingly, the engagement and evolving learning trajectory also seemed to give evidence of how the nature of the conjectures and the way they interpreted their engagement, was constrained by the examples they chose, as well as their underpinning preconceptions and the pedagogical environment. While those underpinning preconceptions influenced their choice, the type of examples they chose led to a version of interpretation, shaped in a particular manner.

The particular numbers pupils used in their investigation with the 3-digit aspect meant that there wasn't the complicating feature of the first and last digit summing to more than ten; that is, for their generalisation, they decided the second digit stayed the same as in the original number, while the third digit was the sum of the first and last digits. When the first and last digits summed to more than ten, it affected the second digit and may have instigated a perturbation causing the nature of the investigation to move in a slightly different direction. This alternative trajectory might potentially indicate the computational reasoning that underpinned the interpretation they articulated in visual terms. This occurred because the digit in the hundreds column multiplied by one, plus the digit in the ones column times a hundred, add to give the total number of hundreds in the product e.g., 438 multiplied by 101 is 44238. The number of hundreds in the product is 400 x 1 + 8 x 100 = 400 + 800 = 1200 which being 1000 or greater 'carries' 1000 into the thousands place, making 43,000 into 44,000. In some situations, as with the following one, this stimulates the negotiation of a more refined interpretation, including aspects of the computational processes that affect the patterning of the digits.

This is also indicative of the complexity of influences entailed in a local hermeneutic circle. While the learner, the mathematical task, the pedagogical medium, and the learner's discourses in those and related domains have primacy in the evolution of interpretation and understanding, discourses to do with power, advocacy, and expectation were pervasive. The particular examples employed, the inter-relationships of the group, and the manner in which their contributions are fashioned and expressed, all influenced the interpretations of and within the process in subtle ways. While in the broader picture even these understated

flavourings are borne of underlying discourses; that is, everything is brought forth from its interpretative lineage, in the local situation there seems a fortuitous element allied to their intervention. In the relatively brief illustrative situation described above, the choice of numbers to explore the 3-digit patterns as ones where the sum of the first and third were less than ten, shaped the interpretation in a particular way. One pupil did allude to the more generalised case with Adam's comment regarding that pattern: "The fourth is the same. The first, fourth and fifth are the same as the first number. The second and third come from adding the others together" but they didn't incorporate that into this interpretative version of their generalisation. The following situation while illustrative of the broader hermeneutic principles was also illuminating regarding those more subtle influences that might flavour the localised interpretation.

Negotiating shared meanings

The following data was interesting for the way in which the two pupils focussed on (Jo & Sam) drew different interpretations of the same data, when it was engaged through the same pedagogical medium. It illustrates how approaching it from varying individual conceptual positions differentiated their interpretation. They disagreed with each other's generalisation, but through further iterations of the hermeneutic circle, each interpretation was folded into the other's evolving perspective. Through investigation with the spreadsheet, and the subsequent discourse, they found a common appropriate interpretation. Their disagreement, the tension generated by the other's approach, followed by the moderation of their personal perspective, led to the accommodation of a shared interpretation of the generalisation, that was facilitated by the exploratory medium.

They had already negotiated the sense of the task through initially entering a formula to represent the situation, then interpreting the table of values that was generated. They drew on the preconceptions borne of their prevailing discourses. At this stage they had used the table to make generalisations regarding multiplying two-digit numbers by 101, and were now investigating multiplying

three digit numbers by 101. They had adapted their approach to entering a three-digit number, then interpreting the output through their existing frame.

They had tried 943, which generated: 95243

Then 983, which generated: 99283

The pupils' initial generalisations centred about viewing their data through a visual lens. They looked at the situation of the digits and changes to particular digits in those positions. They both interpreted the situation from their personal perspectives, but as they oscillated between these broader perspectives and engaging with the task, their interpretations diverged.

Jo: *First digit of the 3-digit number is the starting number of the final number.*

Sam: *First digit equals first digit; second digit equals second digit plus one. Get it? Third digit equals third digit minus one.*

Jo: *Let's try another. 18584.*

Sam: *What's your other number?*

Jo: *184. The middle one is the last one added to the first, or is it plus one.*

Jo seemed intent on building a more rigorous generalisation by exploring other inputs, whereas Sam had found a pattern that fits the first two outputs generated and was keen to formalise that in some way. She seemed to anticipate that there was a generalisation that would give her a methodology to predict. She was motivated by what she saw in front of her in a visual sense, her experience with the table from the two-digit exploration, and the direction in which the medium and its underlying discourses led her initial conjecture. Once again, their personal perspectives influenced their interpretation of the situation in diverging ways, as evidenced by what they said and what they did, by their dialogue and actions.

Sam: *My rule is first digit equals first digit; second digit equals second digit plus one. Third digit equals third digit minus one. Fourth digit equals second digit; fifth digit equals third digit.*

Jo: *Let's try… no, 3 digits now.*

Sam: *Let's try 175.*

1 7 6 7 5

was generated on screen.

Sam: *That'll be 18475 (Using her version of the rule: from 175, first digit =1; second digit = 7+1; third digit = 5-1; fourth digit = 7; fifth digit = 5, that is 18475). Which is exactly the same.*

Jo: *But it's not right. I got that answer (rule), she got that, but when we tried it, the rule didn't work again.*

Sam: *It's not right for this, though it does work for the others.*

This provided quite a dilemma. Each pupil had a rule that worked in a particular situation, but not in another. They reflected on the situation through their broader fore-structures, wondering if there were different rules for different situations. They debated some possibilities, coloured by their personal perspectives. They then posed a conjecture; moving from predicting and generalising, to a more rigorous process that included testing a broad range of examples, including looking for counter-examples, then the refining of the conjecture. Their focus moved between the interpretation of the task through broader, personal, mathematical frames, and interpreting and reflecting on these frames through the output as they engaged in the task. Part of that process included the redefinition of sub-goals in their investigation.

Sam: *Put 943 in and I get 95243; so its right and that's the rule.*

Jo: *But it doesn't work for this other one. Try 348.*

This produced the output of: 35148

Sam: *Both rules don't work!!*

Their perspective, the space from which their interpretations were made, evolved once more. They then re-engaged with the task. Jo considered another aspect, which seemed to link her approach to that of Sam.

Jo: *But if we add 3 + 8, you get eleven.*

She had introduced a procedural aspect but it was instigated by a visual cue, indicating that they still seemed to be interpreting the generalisation with a visual lens.

Jo: *So that's the 1 for the third digit [from the 11 above].*

Sam: *And if you add 1 to the second digit from the third digit, that's the second digit + 1. Yes. Lets check 943 again. [943 became 95243]. 9+3 is 12 so we get first digit = first digit; second digit = second digit plus the one from the twelve, when the first and last are added together [9+3=12]; then the third digit is the 2 from the twelve; and the fourth and fifth stay the same [as the second and third i.e. 4 and 3].*

Jo: *So the middle one is the first digit and the last digit added together, but when it goes over ten, the one gets added to the second digit.*

They had reached a consensual rule, mediated by the pedagogical lens and the generated dialogue. It was still very much in terms of the position and visual change in the particular digits, but with some computational reasoning involved (adding two single digit numbers and place value). Throughout, their focus and interpretations oscillated between their broader perspectives, then engagement with the tasks, in a manner illustrative of a localised hermeneutic circle. While initially divergent in their perspectives and interpretations, the generated dialogue, being formative of each underlying discourse, mediated a consensual viewpoint. Not that this was an uncomplicated, unencumbered generalisable 'result', but in those particular contextual circumstances there was a shared, mutual interpretive stance that would fore-structure subsequent mathematical, and other, experiences.

Generalising through a hermeneutic process

In this episode, the pupils' gaze oscillated between engagement with the mathematical task and reflection on the underlying broader perspectives that influenced that engagement. A local hermeneutic circle was evoked as their preconceptions, yielded from the prevailing discourses in the constituent mathematical, pedagogical, and other socio-cultural domains, framed those engagements in a specific individual manner. The interpretations, borne of those engagements, subsequently influenced those perceptions and their prevailing perspectives evolved. The pupils set fresh sub-goals as the data were viewed from these new, modifying perspectives. The local hermeneutic circle initiated this establishment of temporary sub-goals, while the investigation of the sub-goals, as manifest in the pupils' actions and dialogue, influenced the way the hermeneutic circle transpired. The episode demonstrated instances of the interplay between iterations of the hermeneutic circle and the emergence of the investigative sub-goals.

The data were produced when a group of pupils were investigating the task "Dividing one by the counting numbers" (see Figure 4).

In the first case pupils negotiated to gain some initial familiarisation of the task.

Sara: *One divided by one is one - it should be lower than one.*

Jay: *Try putting one divided by two, and that should be 0.5.*

They then entered 1 to 5 in column A and =A1/1 in column B to get:

A	B
1	1
2	2
3	3
4	4
5	5

The cells in column B were dividing the corresponding column A cell by one, hence producing the same sequence. The pupils were aware that the output should be less than one. This posed an immediate tension with their initial thoughts and fostered the resetting of their sub-goal. This was also the beginning of the hermeneutic circle. Sara's pervading school mathematics discourse suggested one output, that it should be less than one, while at the micro level of the investigation the output was greater than one. The output didn't mesh with their expectation. This oscillation between the macro perspective (the pervading discourse) and the micro (the actual investigation), and the interpretive response that this elicited, occurred within the particular social frame, instigating a distinctive response to the investigation. They continued:

Sara: *Is it other numbers divided by one or one divided by other numbers?*

Jay: *Lets recheck.*

She entered =A1/4 and got the following output:

1	0.25
2	2

Jay: *Umm, we're not going to get change… - we'll have to change each one.*

They appeared to feel intuitively there should be a way to easily produce a table of values to explore. The spreadsheet environment was shaping the sense making of the task and the setting of their sub-goals. Critically, it was enabling them to immediately generalise, produce output, and then explore this visually.

They explored other formula e.g., =A1/4 and =B1/(4+1), eventually settling on one that they perceived as modelling the situation.

Jay: *Oh now I see =1/A1.*

They generated the following output:

A	B
1	1
2	0.5
3	0.33333…
4	0.25
5	0.2
6	0.16161616…
7	0.1428514285…
8	0.125 etc.

The cells in column B, now contained the output when one was divided by the number in the corresponding cell of column A. For example, one divided by one is one, one divided by two is 0.5, etcetera. They considered the numbers that produced terminating decimals and the consequences for their emerging perspective. This engagement with the task influenced their overall perception of the situation. They re-interpreted their broader mathematical lens through the engagement with the task before reflecting on this output from their re-organised perspective.

Sara: *So that's the pattern. When the number doubles, it's terminating. Like 1, 2, 4, 8 gives 1, 0.5, 0.25, 0.125.*

Jay: *So the answer is terminating and is in half lots. Lets try that = 0.125/2; gives 0.0625-which is there.*

[She finds it on the generated output from above].

The structured, visual nature of the spreadsheet had prompted the pupils to pose a new conjecture, reset their sub-goal, and then allowed them to easily investigate the idea of doubling the numbers. The table gave them some other information however.

Jay: *1 divided by 5 goes 0.2, which is terminating too.*

[Long pause].

This created a tension with their most recent conjecture. It required them to reconcile, through an interpretive lens, the output produced with their underlying discourse. After further exploration, they reshaped their conjecture, incorporating their earlier idea.

Sara: *If you take these numbers out they double and the answer halves.*

Jay: *That makes sense though, if you're doubling one, the other must be half.*

Like 125 0.008; 250 0.004.

Sara: *What's next. Let's check 500.*

Jay: *Let's just go on forever.*

The pupils generated a huge list of output, down to over 4260. The nature and structure of the spreadsheet enabled them to seamlessly, yet intentionally, generate large amounts of relevant data, thus fashioning their emerging theory in a particular way. They weren't shackled by the repetitive task of dividing one by hundreds of numbers individually, and the possible errors that might result. The affordance of the spreadsheet to undertake many calculations simultaneously allowed them to investigate the task in a unique manner. It reorganised the ways that they might have approached the task, and extended the type of mathematical thinking they would have usually done at this level. They confirmed that when

the divisor was doubled from 250 to 500, the quotient halved from 0.004 to 0.002, and likewise when the divisor doubled again to 1000, the quotient similarly halved to 0.001. Jay read off the table of output:

Jay: *500 0.002; 1000 0.001.*

This indicated the relationship between the numbers that gave terminating decimals and the powers of ten. It led to a conjecture couched in visual terms:

Sara: *When you add a zero* [to the divisor]*, a zero gets added after the point* [decimal point].

Sara was articulating an interpretation of the situation as envisaged through a particular school mathematics lens; for example, 5 gave an output of 0.2, 50 gave an output of 0.02, and 500 gave an output of 0.002. It was the cyclical shifting of their focus between the conceptual frame and the mathematical task that initiated the sub-goals in their investigative trajectory, stimulating the refinements to their emerging interpretations. This evolution of interpretation was also filtered by the affordances of the spreadsheet pedagogical medium and the particular interactions this medium evoked. Their conjecture and conceptual understanding evolved through a series of interpretive fixes as the output and subsequent dialogue influenced the setting of their sub-goals. Through the interpretive lens they evoked, their dialogue reflected the oscillation between the ascendant school discourse and the generated output. The following was also recorded on a piece of working paper, as a list of the numbers that produced terminating decimals:

1, ~~2~~, ~~5~~, 10, ~~20~~, 100, 1000

After recording two and five, it appeared they noticed that these were factors of ten and subsequently crossed them out. This observation also occurred with the twenty and one hundred. This interpretation was later verified with the pupils. They had made sense of, explored, and generalised aspects of the investigation, culminating in the indication of a relatively complex notion of factors and the generalisation process. The pedagogical medium through which they had

engaged in the task seemed to have influenced the contextualisation and approach they had taken. As their attention focused on alternating emphases between their prevailing discourses and interactions with the mathematical tasks, their interpretations and explanations evolved: they set sub-goals from their prevailing discourses, and engaged in activity motivated by that sub-goal, with the activity and the dialogue it evoked reshaping their discourse in that domain. The re-setting of the succeeding sub-goal from this reshaped perspective led to a developing interpretation. The consequence of each engagement and reflection was an evolution of perspective and interpretation. Each re-setting of a sub-goal, with its subsequent associated action and interpretation, was an enactment of the hermeneutic circle.

Conclusions

This chapter has attempted to enrich the evolving picture of how pupils using spreadsheets to investigate mathematical situations, might have shaped their investigation in particular ways. The ways in which the affordances of the spreadsheet environment, interact with broader socio-cultural frames in the organisation of the learning trajectory, and related thinking, were also considered. While broader underlying discourses shaped the interpretation, the learning trajectories, as evidenced through the sub-goals traversed, were also influenced by the affordances of the pedagogical medium. In what ways did the spreadsheet environment influence the engagement with the mathematical tasks through the generation of sub-goals, and the understanding that this engagement facilitated? Specifically, it was concerned with how using spreadsheets as an investigative tool, might have influenced the understanding of the problem, thus shaping the emerging perspectives of the learner. As pupils engaged with the task through the setting and exploration of sub-goals, the approach taken to investigate it, was influenced by the spreadsheet environment.

The data supported the supposition that the availability of the spreadsheet led to the pupils familiarising themselves with, then framing the problem through a visual, tabular lens. The data episodes illustrated how the pupils' interpretations

of the situation they encountered were influenced by the creation and subsequent reflection on these tables of numerical output. These tables were typically generated by a formula; they were a function of the formula that modelled the situation as interpreted by the pupils. This particular affordance of the spreadsheet as a pedagogical medium influenced the way the pupils interpreted and explained their emerging generalisations and informal theories. While the spreadsheet environment in conjunction with other socio-cultural influences filtered their thinking, it seemed that the visual structure of the tables was influential in the process. It permitted more direct comparisons between adjacent values or columns, highlighting aspects of the patterns the pupils were attempting to examine. At times, this was between outputs, and at other times between inputs. The pupils could manipulate a whole column of values that, coupled with other affordances such as the almost immediate response to the input of data, allowed them to interpret and make decisions far more readily. They might more quickly and easily have perceived relationships between numerical outputs within the tables on which to base their informal conjectures. They may have noticed relational aspects that would have eluded them in a slower, more atomised examination. This facility to compare so readily might also have left space for other influences: for instance, personal value judgments might have been more accessible and influential in the investigative process.

The data also indicated that the medium evoked an immediate response of generalisation, either explicitly through deriving formulas to model the situation, or implicitly by looking to *Fill Down*, or develop simple iterative procedures. Tension, arising from differences between expected and actual output, and opportunities, arising from possibilities emerging from these distinctive processes, led to the setting and resetting of sub-goals. These, in turn, further shaped the understanding of the investigative situation, and the interpretation of mathematical conjectures. Researchers into the use of other digital media have likewise reported how the tension, evoked when expectation of output conflicted with preconceptions, has promoted a productive form of learning (e.g., Kieren & Drijvers, 2006, in their study involving CAS).

The data demonstrated that varying the pedagogical media provide distinctive responses in the social interaction that contextualise the mathematical ideas, hence framing the construction of informal mathematical conjectures in particular ways. It also supported the contention that this subsequently conditioned the negotiation of the mathematical understanding. Not only were there negotiated shared meanings associated with technical language such as *Fill Down* that were particular to the medium, but the interactions were informed by the particular shape and manner by which output had been processed and was presented. The pupils' conversation related to the particular form of their activity in the spreadsheet environment. These aspects were influential in the setting of the sub-goals and consequentially the learning trajectory and the organisation of their thinking. As Brown (1996) has argued, the mathematical understanding is a function of the social frame within which it is immersed, and the social frame evolves uniquely in each environment. The data backed the supposition that the availability of the spreadsheet led the pupils to familiarise themselves with, then frame the problem through a visual, tabular lens. It appears that it also evoked an immediate response of generalisation, either explicitly through deriving formulas to model the situation, or implicitly by looking to *Fill Down*, or to develop simple iterative procedures. The first two situations highlighted this, where those pupils using the spreadsheets produced a table of output quickly, then analysed it for visual patterns. Their dialogue indicated this visual approach to interpretation and it echoes of visual reasoning (Borba & Villarreal, 2005; Smart, 1995).

The episodes provided an illustration of how the actual investigative trajectory evolved. The pupils almost immediately entered formula to generate data to help make sense of the problem, as well as to generate possible solutions. They indicated that the spreadsheet environment evoked that response. The data revealed a story of the pupils using the spreadsheet to obtain a broad perspective of the situation, as they immediately looked to generalise a formula that they could enter. They frequently initially engaged with the tasks by employing formulae and the *Fill Down* functions to generate numerical tables that might subsequently be analysed for patterns. Their language reflected this, but the interactions also contained more language of generalisation, and it took them generally fewer interactions to start a more formal generalisation process. Using a

hermeneutic process, their pervading mathematical discourse in this domain enabled them to interact with the mathematical activity. They produced output that was interpreted visually. Tension, arising from differences between expected and actual output, and opportunities, arising from possibilities emerging from these distinctive processes, led to the setting and resetting of sub-goals. These, in turn, further shaped the understanding of the investigative situation, and the interpretation of mathematical conjectures. Their interpretations, from each engagement with the task, influenced their understanding, and enabled them to re-engage with it from a modified perspective. These 'fixes' were expressed discursively, and were illustrated in the pupils' dialogue and output.

A complicating influence on an already complex situation of interrelationships was the social positioning within the group allied with individual group members' predispositions. In the second situation, it was apparent that one of the group, Adam, took the initiative by engaging with the task in the first instance from his own perspective. The group's initial response was Adam's; hence what was brought forth for the other group members may have been repositioned beyond their personal preconceptions. His leadership in this regard drew on discourses related to power, as did all engagements involving the inter-relationships of the groups, and the manner in which their contributions were fashioned and expressed. Relationships and confidence associated with advocacy and expectation were influential with the interpretations of, and within, the hermeneutic process. Likewise the engagement with the tasks also influenced the perspectives of power and expectation of contribution within the group. These social discourses were influential, and were influenced in an ongoing formative manner by the engagement at the localised level, in the same way as the mathematical and pedagogical ones were. They were all inextricably linked and persuasive of each other.

The data also demonstrated how the intuitive beginnings of the mathematical conjecture, were enhanced by deductive reasoning. They were mutually reinforcing (Fischbein, 1994; Schoenfeld, 1985). The pupil exchanges, relating to their mathematical explanations, negotiated the resetting of sub-goals, and refining of the emerging conjectures. The sub-goals were a function of the

ongoing familiarisation of the situation as much as the resolving of mathematical aspects. In fact, the data illustrated that the two are inextricably linked. The learner's understanding of the situation framed the settlement of sub-goals, while the sub-goals decided on conditioned the understanding of the situation (Nunokawa, 2002). The emergence and engagement with the sub-goals facilitated interaction. The collective argumentation, in conjunction with the visual output, led to the formation of generalisations (Yackel, 2002). There was a distinct pathway to mathematical thinking and understanding, induced through the particular pedagogical medium.

The pupils also identified speed of response, the structured format, ease of editing and reviewing responses to generalisation, linking symbolic and visual forms, and the interactive nature as being conducive to the investigative process. While this particular medium has unfastened unique avenues of exploration, it has as a consequence fashioned the investigation in a way that for some, may have constrained their understanding. All pedagogical media have opportunities and constraints associated with the learning experience. This research was concerned with those particular to the spreadsheet environment and how they might have influenced the learning trajectories and the facilitation of understanding. The influence of those affordances varies for individuals as their learning experience evolves from the interplay of a broad range of perspectives filtered through the pedagogical medium. One feature of the experience that retained an element of commonality was the setting of sub-goals within the iterations of a local hermeneutic circle. The setting and exploration of the sub-goals, as evidenced by the pupils' activity and dialogue was, in my opinion, allied to the medium through which it was engaged. The data were illustrative of this, while also supportive of the contention that the setting and exploration of sub-goals was constituent of the learning trajectory, and by implication the understanding.

CHAPTER NINE: Visualisation

Ano ko te maramakua ngaro
Kua ara ano
Just like the moon that disappears
and then rises again

Introduction

Spreadsheets have been found to offer an accessible medium for young children tackling numerical methods. With the potential to simultaneously link symbolic, numeric, and visual forms, they have been shown to enhance the conceptualisation of some numerical processes (Baker & Biesel, 2001; Calder, 2002). When pupils encountered numerical ideas within the construction and transformation of spreadsheet workbooks, the experience of translating situations into spreadsheet models that encapsulate symbolic, numerical and visual versions of the situation appeared to enhance their understanding of the mathematical ideas. Associated with this affordance is the notion of visualisation. Here visualisation bridged the concrete and abstract manifestations of mathematical experiences.

Visualisation in mathematics and mathematics education takes various guises. It has often been associated with geometrical representations and spatial ability, but the creation and interpretation of visual images is also considered an aspect of broader definitions. A visual image has been considered as a mental scheme encapsulating spatial or visual phenomena (Presmeg, 1986, 2006). This deliberately broad perspective not only attends to representation and the transformation of shapes or models, but as well incorporates mental imaging including, according to Presmeg's version, the spatial arrangement of numerical, verbal, or mathematical symbols to form an image. She has identified and described five different types of imagery: concrete pictorial (pictures in the

mind); dynamic imagery (moving or manipulating images); memory images of formulae (seeing a formula in the mind); kinaesthetic imagery (images of movement or activity); and pattern imagery (visual-spatial representations of relations). It is the notion of pattern imagery that this research draws on, when discussing the ways students used visual elements in their mathematical thinking, as they approached the numerical and symbolic patterns evident within the relationships they modelled with the spreadsheet.

Presmeg's account of the visual image is premised on mental interpretations of external representations, while others contend that the external representation emerges from mathematical interpretation as manifest in the mind (e.g., Eisenburg & Dreyfus, 1989). These perspectives are variously labelled external (outside the mind) and internal (inside the mind) respectively. Meanwhile, Borba and Villarreal (2005) ascribe to a view of visualisation as an ongoing process where the two perspectives are inextricably linked and inter-dependent. They contend that technology, in this case digital, interact with other influences in humans-with-media collectives. These collectives, whose interpretations are persuaded by visual images, also work through visual imaging to shape cognition. The images are also affected by this interaction, hence the visual images and the mathematical thinking evolve as the world inhabited by our experiences and activities likewise evolves.

While some mathematicians contend that mathematics itself is transforming through its interaction with computers (Devlin, 1997; Francis, 1996), there is no consensus amongst them regarding this point. Borba and Villarreal (2005) argue that ICT emphasises the visual aspect of mathematics, and changes the status of visualisation in mathematics education. The positive role visualisation plays in supporting conceptual understanding is frequently advocated (Bishop, 1989; Dreyfus, 1991; Dubinsky & Tall, 1991), but visualisation has often been considered as secondary, or supportive, of a symbolic, analytical, or algebraic conceptualisation. However, there is growing evidence that visual reasoning is itself legitimate mathematical reasoning (Borba & Villarreal, 2005). In studies involving students using graphic calculators and computer software (e.g., Julie, 1993; Smart, 1995; Villarreal, 2000), ICT mediated the mathematical

understanding, and a visual approach to reasoning was identified. The researchers also contend that this visual reasoning, initiated by interacting with the mathematics through an ICT medium, extended students' mathematical conceptualisation, "…they employed their visual knowledge to help make generalisations and solve any new problems. In doing so, they extended their mathematics beyond what was expected by the teacher and the textbook" (Smart, 1995, p. 203).

Geometry, with its visual and construction elements lending themselves to an interactive approach, was one of the first mathematical areas to embrace digital technology. Software developed specifically for geometrical reasoning and exploration also opened alternative avenues for engaging in relational mathematics. In various studies involving DGS, the dynamic, visual representations enhanced the understanding of functions (e.g., Mariotti, Laborde, & Façade, 2003), while in a study of students' understanding of key aspects of geometrical transformations when engaged with *The Geometer's Sketchpad,* Hollebrands (2003) reported the development of deeper understandings of transformations as functions. Tall (2000), while acknowledging that the use of ICT and its effect on mathematics is at a very early stage of its evolution, found that a graphic approach to calculus, developed in the right way, led to understanding of the most subtle of formal concepts. He reported that a graphic approach to calculus offered insights into far deeper ideas about differentiability. Others have maintained that visual-graphical representations available in software such as spreadsheets have the potential to allow students to develop mathematical concepts and relationships (e.g., Lemke, 1996).

Visual influences on the learning process

Giving the learner the potential to visualise both in tabular and graphical form permits them the opportunity to re-envisage their approach to mathematical engagement and process ideas in alternative ways. This contributes a distinctive flavour to the learning experience when the tasks were engaged with through the spreadsheet. In one research setting for this study, where the pre-service teachers

employed different media in the investigation of the same task, their perspectives of emerging situations and hence the learning trajectories differed. Significantly, the social interactions appeared to shape the analysis of the patterns in distinctive ways. Given that the path to, and manifestation of, the patterns differed, the conversations indicated a different approach once the patterns were viewed. Those using the spreadsheet used a more visual approach. They were observing and discussing visual aspects eg the situation of digits or zeros. For example:

> *Rita:* *You take the zero out. What about when you get to the three digits? Was that 22? So the middle number is still a double? Okay, so when you've got three digits, you get two, two, five, two, three.*

Those using pencil and paper were more concerned with the operation aspects that generated the patterns. For example:

> *Justin:* *Basically if you times your number by a hundred and then by one you would add them together and get your answer.*

The pre-service teachers working in the spreadsheet environment employed visual referents in their analysis and explanations. To generalise a pattern in terms of the sequence of digits is significantly different from generalising in terms of an operation. In this aspect, the different settings had filtered the conversation and approach, and by inference the understanding. As Brown (1996) has argued, mathematical understanding is a function of the social frame within which it is immersed, and the social frame evolves uniquely in each environment. In the following episodes, the numerical output positioned in tabular form presented a visual version of the output to the pre-service teachers. This shaped their interpretations of the situation in particular ways. This visual version also influenced and was evident in their explanations, as manifested in their dialogue and actions. They negotiated the refinement of their informal conjecture through alternating their focus from the "part" (the task) and the "whole" (their prevailing discourses in the associated domains). As their attention brought each of these

alternatively to the fore, that evolving perspective framed their interpretations. A local hermeneutic circle was evoked. Interestingly, the informal conjectures that emerged were articulated in terms of the type and position of digits or the patterns in their arrangement. They used visual referents in their descriptions and explanations rather than mathematical processes. They appeared to set a learning trajectory couched in visual terms. This led to a reorganisation of their thinking and interpretation, in a manner particular to the spreadsheet environment.

The groups working in the spreadsheet environment, tended to initially perceive that the bigger picture was most easily accessed through entering a sequential, formulaic structure into the spreadsheet and then visually analysing for patterns. For example:

Kyle: *I haven't predicted. I was just going to put in A1 times 101 and drag it down.*

After he did this, the following output was produced:

A	B
1	101
2	202
3	303
4	404
…	…
…	…
14	1414
15	1515
16	1616

Josie: *So we're investigating the pattern of 1 to 16 times 101.*

The tabulated visual form of the output gave them immediate access to lists of comparative, numerical data that indicated a pattern to the relationship between the inputted values and the consequent output, while also providing emphasis to a relationship between the output values.

Kyle:	*It goes up in hundreds plus a one.*
Josie:	*It's because one doesn't change the multiplying.*
Kyle:	*Its always a one that you're adding on.*
Josie:	*So 2 times 101 is 202?*
Kyle:	*101 times 2 is 202? When you multiply numbers by 101, you also notice 3 times is 303 and 4 times is 404. So if you went 20 times 101 it would be 2020.*

Kyle was considering the pattern in the output and seemed to be noticing the incremental change by one in the hundreds and ones column. This might have been interpreted as him seeing multiplication as repeated addition, but his second comment in this segment appears to indicate that he recognised the counting sequence in the output, rather than the accumulated multiples of 101. He seemed to be interpreting the visual representation of a familiar pattern. His last comment was also indicative of interpreting a pattern with visual referents. Josie appeared to be focusing more on the multiplicative process as she reflected on the possible reasons for Kyle's preliminary interpretation. However, it seemed that she was still noticing the visual pattern predominantly, but looking to reconcile her preconceptions of multiplication with the output. Her comment "So 2 times 101 is 202?" indicated that the explanation in terms of the multiplicative process followed the visual recognition of the pattern. As well, there was an element of uncertainty with the process undertaken inherent with that comment. This gave the impression she wasn't absolutely sure that 2 times 101 was 202 from the operation only, but that the structure and the pattern indicated it, with her preconceptions of the computational process confirming this interpretation. Interestingly, Kyle had also incorporated Josie's perspective into his emerging informal conjecture. He verbally rationalised the multiplicative aspect to his visually referenced pattern, then articulated his prediction in the newly acquired terms. Their dialogue, and the accompanying reflection and emergent explanations, had repositioned their perspective, and Josie's interpretation of Kyle's view allowed her to set a new sub-goal and extended their engagement further, beyond the immediate input, to a 3-digit number. She had recognised Kyle's pattern, and explored further based on that evolving rule for their pattern.

Josie: *If you did a huge number like five hundred times 101 it would be 500500 wouldn't it?*

Kyle: *Let's have a look. It's 50500 and its just shown it doesn't do that.*

Josie: *Let's try a hundred times 101.*

Kyle: *10100. If you put in 800 it would be 80800.*

They now had the following additional output:

100	10100
500	50500
800	80800

Josie: *Yes, that works.*

The further making and testing of their predictions enabled them to establish a pattern to this type of number; that is, inputting a multiple of hundred up to one thousand. Their discussion seemed to focus more on the pattern through a visual lens rather than an operational one; that is, the pattern of the digits in the outcome, rather than how the numerical operation affected the structure of the outcome. They continued:

Josie: *Try using other numbers like 440 and see what it does.*

Kyle: *Yes, that is what I was saying 44440. Its like double the number but with the zero added on.*

Josie: *Take out the zero and it's the same.*

Kyle: *566 [The output is 57166]. Where does the seven one come from? [The 71 in 57166].*

Josie: *Maybe it's a decimal point thing. It's easy to see the pattern but what is the rule that will give you that? What's causing it to do it?*

They referred back to their original table with the counting numbers multiplied by 101. Column B output (the products) were:

101
202
303
…
…
909
1010
1111
1212
…

Kyle: *It's going up a hundred plus a one.*

Josie: *The two digit numbers are when you get into 10, 11, 12.*

Kyle: *It's still a hundred plus a one. I wonder what the three-digit number will do? But why is this 566 like that? It's gone to 57166.*

The first three-digit numbers they tried (e.g., 440) produced output (e.g., 44440) that had similar visual features to the explanation for the two-digit pattern they had settled on. They were able to reconcile it with their theory by removing the zero. From their perspective, this seemed a reasonable approach to take as they had used visual referents in this preliminary theory and visually it maintained some tenuous consistency. If they had been developing their theory through referring to the multiplicative process, just removing the zero might not have been so simply reconciled. Nevertheless the output from 566 (i.e. 57166) disturbed that initial theory when applied to other types of three-digit numbers. They reset their approach and returned their attention to the 440 input. They began to investigate the adjacent values and reflected on the output.

A	B
440	44440
441	44541
442	44642
443	44743

Josie: *A break happens between those ones and the zero.*

She referred to the 44 and the 0 from 440 in column A, the input. Josie seemed to be viewing the corresponding output in column B as having the first two digits (44) and the last digit (0) split with a related number positioned between them. This was consistent with the subsequent values and the articulation of the conjecture's ongoing development. Josie continued:

Josie: *If you go 441, the break happens between those two fours and that one [they then entered 433 to produce 43733]. The same happens between the four and the double threes. Thirty-seven has been added in there [the 37 in 43733].*

Josie had altered the way she noticed the pattern, placing the split between the first and second digits (4 and 33) rather than the second and third (44 and 1) as she had previously. It was hard to determine her motivation for this change, but perhaps she was still giving emphasis to the noticing of double numbers from the earlier theory concerning the two-digit input. She might have held on to this notion as being a key influence and carried it through to her current interpretation. She was postulating and explaining in terms of the patterns she perceived in the positioning and value of the digits. The visual image was concerned with the spatial arrangement of the numerical symbols. Viewing from her modified perspective, she suggested a further value and prediction. She split the 5 and 33 and inserted 37.

Josie: *Try like 533 and does it come up as 53733? Now it's 53833.*

The students in her group reflected on this unexpected output, and then used it to amend their evolving conjecture. Their attention was oscillating between the task and their emerging theory, with each iteration of this recursive process causing a

reorganisation of their thinking and the subsequent refinement of their conjecture. They continue to examine the situation:

Josie: *Try 633 and does it come off as 63933?*

Kyle: *Yes.*

They have produced the following output in their worksheet:

433	43733
533	53833
633	63933

Josie: *If you go 733, you are going to get…*

Kyle: *74000?*

733	74033

Kyle: *It's 74033.*

Kyle has noticed and incorporated the 37, 38, 39 pattern with the second and third digits into his prediction. Significantly for the purposes of this discussion he noticed this known sequence through the affordances the visual representation of the output in tabular form offered. This variation evoked another disturbance of their existing position.

Josie: *Okay, why did it do that? It has gone to the 10 there and then 33.*

Kyle: *That is why it's a four.*

Josie: *It's rounded one up to the ten [39 has become 40 - researcher's interpretation].*

Kyle: *Why isn't there a one?*

Josie: *Because it goes up in tens. Try 833 and it should be 89133. See. Now try something different, try 325. What do you reckon is going to happen?*

Kyle: *3373.*

325	32825

Kyle: *No, its 32825.*

Josie: *33825 if you try 335.*

325	32825
335	33835
345	34845

Kyle: *See, that is changing because we are not keeping it the same. It's not 25, it's 35, then 45 [the last two digits].*

Josie: *Take that back to 325. That just goes up in hundreds.*

Kyle: *Hundreds and thousands. That has gone up a thousand and a ten.*

Josie: *Is that because you are using a freakish number?*

Although they had made some accurate predictions through the patterns they saw in the tabulated output, they appeared to be struggling to articulate a generalised conjecture that might be reconciled with the output produced by all 3-digit numbers. They reconsidered some of their earlier engagements as they grappled with an encompassing theory.

100	10100
200	20200
300	30300
400	40400
500	50500

They compared this to their earlier informal conjecture with one-digit numbers; that you add a hundred plus a one. That is for:

1	101
2	202
3	303
4	404

Josie: *You just put two zeros on the end. When you go up in a three-digit number, those zeros are added. Like here, it only goes up in hundreds. For a three-digit number, it goes up one ten thousand and one hundred.*

At this stage through various constraints, they had run out of impetus with the investigation of the task. They had reconciled various aspects of the task and to a point had offered logically formed and articulated explanations for their interpretations. The conjectures that emerged from their generalisations enabled them to predict the output for one- and two-digit numbers, but their theorising with 3-digit numbers was constrained by several factors, including the numbers they had chosen, time, dwindling focus, and their approach. These and other opportunities and constraints to their learning trajectory were influenced by their underlying discourses in the associated domains. The thinking, interpretation and subsequent decision-making was framed by the preconceptions they brought to the activities, which in turn were modified as they engaged with the task. Central to this cycling, recursive process was the pedagogical medium through which the activity was filtered. The spreadsheet environment constrained or offered opportunities to the engagement that were distinctive; that is, it offered particular affordances. It was noticeable that the pre-service teachers, while engaging and interpreting their activity through the gaze of the spreadsheet, couched their tentative, then emerging conjectures and explanations in terms of the positions and type of digits that were generated in the output. They used visual referents in their accounts. They not only fashioned their accounts in terms of these spatial arrangements, the patterns that the spreadsheet environment afforded through sequential, visual, tabulated output were influential in the way they negotiated their learning trajectory. It evoked a particular response.

A second episode illustrated similar propensities to the learning pathways with regards to visualisation. It involved a group of pupils engaging with the same **101 X table** task. This group based their predictions and informal generalisation on the visual aspects of the table, but found it was influencing their interpretation

more than they anticipated and thought desirable. One of the pupils, Nat drew on his prior knowledge to initially engage with the task.

Nat: *1 times, 101; 2 times, 202; 3 times, 303. We use that little cell for our formula and drag it down.*

After some exploration and discussion they produced the output below:

A	B
1	101
2	202
3	303
…	…
…	…
43	4343
44	4444
45	4545
46	4646

Kim: *Now we have our two-digit number.*

Nat: *What's our prediction? We forgot to predict on that one.*

Kim: *What do you mean when you say prediction?*

Nat: *What will the answer be? Like what would 23 times 101 be or 24.*

Kim: *Well its 2323, so its double the number. 2424- see.*

Nat: *OK, double the number.*

Their generalisation was based on the visual interpretation of the table of values. They could see the visual pattern of the two-digit number repeating quite clearly within the table structure and had used this to interpret the pattern and form their generalisation. They also articulated their generalisation in terms of the visual attributes. They both understood "double the number" to be the number repeated side by side i.e. 2424 rather than double the number 24, 24 multiplied by 2 = 48. They had interpreted and explained their informal conjecture in terms of the type and position of the digits, rather than the computational process that produced it or another computational procedure. The recursive evolutionary process of

developing and refining informal conjectures through ongoing engagements of the task through a modifying interpretive lens resumed. Nat continued:

Nat: *What do we predict the answer to be if we times 100 times 101, 200 times 101, 300 by 101. What is our prediction as to what the answer will be?*

Kim: *100 times 101 is going to be 201.*

Kim reverted to applying a school mathematics lens to the situation in what appeared to be a procedural manner. Surprisingly, he had used the wrong procedure, however, and had found the sum of the two numbers rather than the product. Nat had a different prediction.

Nat: *A thousand and one.*

They entered 100 into their spreadsheet workbook and the following output was produced:

100	10100

Kim: *No, one thousand ten.*

Although this was an incorrect reading of the output, the output was nevertheless quite different from both their predictions, causing them surprise and prompting them to address their approach. They self-identified their use of the visual structure of the tabular output to frame their sense-making of the pattern in their predictions.

Nat: *Wow! We've been thinking more about what it looks like on here [spreadsheet] than what the answer is going to be.*

Kim: *We're watching the columns as opposed to the maths.*

Nat: *We're worried about the technical side of it instead of the actual maths.*

It seemed to me that they were engaging with the mathematics, but via a pedagogical medium that they were less familiar with investigating through, with the subsequent engagement structured differently to their usual medium of engagement (most likely paper and pencil). The alternative medium appeared to have re-organised their perspective and seemed to be initiating reflection on their thinking and method. The table structure facilitated their interpretation and generalisation with the pattern formed through using two-digit numbers, but caused perturbation, when they tried a three-digit number. It may have been that their mathematical understandings in this particular area, as well as the pedagogical medium, constrained their interpretation, but their self-identification of the influence of the visually-interpreted, tabular structure suggested that it was at least one of the contributing influences to their investigative trajectory. Nat commented further:

Nat: *Make some rules that help you predict when you have a one-, two-, or 3-digit number. Do they work? You can see it.*

They examined the output when corresponding one, two, and 3-digit numbers are used.

1	101
10	1010
100	10100

They considered the first one.

Nat: *There is no zero after the answer. There is no zero after the one [the one in the ones place value column – researcher's comment]. When it's 10 times 101 there is one zero. When you get to a 100 there is two zeros.*

Kim: *If we kept going down the numbers, they're going to get bigger and more zeros on the end?*

Nat: *Yes, because that is the pattern.*

Kim: *Do they work?*

Nat: *It seems to work. The theory sounds good. We'll try another one.*

Kim: *Try ten thousand.*

The following output was produced:

10000	1010000

Nat: *Are we going to go up in ten thousands? We're getting really big numbers this time.*

Kim: *Can you explain to me what the pattern is.*

Nat: *In the ones column, you've got no zeros because it is the one digits column then you times it by 101. When you are doing it with bigger numbers, double digits like this 10, 20, 40, 50, 60, you have one zero because you are in the tens column. After the 101, all you have to do is put one zero to represent the ones column. As it gets bigger, you put zeros there to represent the ones and the tens. That is how I see it.*

Kim: *Ones, tens and the thousands with the zeros. I still don't get it like this. Shall we use some normal numbers? What about 111 or 383?*

They entered those numbers generating the following output:

111	11211
383	38683

Kim: *See, I can see that. That is much easier to see than all these zeros. The two end numbers add to give the middle one.*

Their explanations involved visual referents; the position and type of numbers. The manner in which the output was aligned in the spreadsheet format gestured towards the visual aspects of the pattern, affecting what the students noticed. They had perceived the numerical patterns and articulated their explanations in terms of the number and position of the digits, the zeros in particular. Their preconceptions in the number domain have suggested the type of input that might be productively used, but it was the particular nature of the output as framed by the spreadsheet format that influenced the re-structuring of their thinking and approach, as they reset their sub-goals in the investigation process. Hence, the pedagogical medium was influential in the emergence of their learning trajectory and the evolving understanding associated with this. In essence, understanding is the learner's interpretation of the situation as envisioned through the interplay of the task, the pedagogical medium, the learner's prevailing discourses, and the dialogue this interplay elicits. Nat rationalised this new perspective within his existing frame:

Nat: *It has done exactly the same thing as the others.*
Kim: *So what is 116 times 101?*
Nat: *It becomes 11716.*

The output confirmed their emerging conjecture but within the constraints of the input they have chosen. They appeared comfortable with this theory, although perhaps teacher intervention would have facilitated further, more in-depth analysis. However from the standpoint of this research, the data were illustrative of the tabular structure of the spreadsheet filtering the pupils' visual imaging. They have referred to visual elements. Their ongoing dialogue prompted by activity questions gave further insights:

Kim: *How did we solve the problem?*

Nat: *We predicted the first one because it was pretty easy. We did one to ten based on what we knew. When we tried the two digit numbers, we focussed on the technology.*

Kim: *What mathematics did we use to solve the problem? I think we used the formulas again.*

Nat: *For our predicting, we would've used multiplication mostly. I don't know what maths I used because I focussed on the spreadsheet.*

Kim: *When we were predicting, we were problem solving but we're doing that in our heads.*

Summary

These comments gave further confirmation of my contention that the spreadsheet environment influenced the negotiation of the investigative pathway, the engagement with the mathematical phenomena, and the consequential understanding that emerged. The comments indicated that the students' interaction with the spreadsheet framed the activity and shaped the ongoing interpretation and conceptualisation in specific ways. From these and other episodes and excerpts, one element of this filtering of perspectives was visual imaging, both in the interpretation of the mathematical phenomena and the articulation of their explanations. Kim's final comment indicated a complementary form of visualisation too, as he signalled that they were processing the mathematics "in their heads". This mental imaging was indicative of picturing external images and the processing of these and other aspects internally. The visual tabular structure of the spreadsheet output and the manner in which it managed the data had an effect on the ongoing engagement.

These, and other opportunities and constraints to their learning trajectory, were influenced by their prevailing discourses in the associated domains. The thinking, interpretation and subsequent decision-making were shaped by the preconceptions they entered into the activities with, which in turn were modified

as they engaged with the task. Central to this cycling, iterative process was the pedagogical medium through which the activity was filtered, and its propensity to evoke a visual lens. The activity and interpretations were as much a function of the pedagogical medium as they were of the mathematical and socio-cultural influences that each student brought to the task. They were meshed and influential in the evolution of each other. The manner in which they engaged with the spreadsheet was framed by the students' prevailing discourses, while they in turn were shaped by the engagement through the spreadsheet environment. The particular pedagogical medium constrained the engagement or offered opportunities distinctive to the environment: it had specific affordances, one of which was the version of visualisation employed in this study. Hence the pedagogical medium, and in this case the associated affordance to visualise, was influential in the way the pupils' thinking and approach were organised. Their learning trajectories and subsequent understanding were a function of that medium acting in conjunction with other stimuli. The students interpreted the output visually and used visual referents in their explanations of the patterns they perceived. Fundamentally, the pedagogical medium, the mathematical phenomena, the students' preconceptions, and the dialogue evoked were inextricably linked. The learner's evolving understanding was influenced by their visual interpretation of the situation through the interplay of those various filters.

CHAPTER TEN: Visual perturbations in digital pedagogical media

Na te moa

I takahi te rata

The moa which is trodden on by a moa when young

Will never grow straight.

So early influences cannot be altered.

Introduction

Several studies have investigated how the formation of informal conjectures, and the dialogue they evoke, might influence young children's learning trajectories, and enhance their mathematical thinking (e.g., Ponte, 2001; Ruthven, 2001; Carpenter et al., 2003). In a digital environment, the visual output and its distinctive qualities can lead to interpretation and response of a particular nature. In this chapter, the notion of visual perturbations is explored, and situated within the data obtained, when the students engaged with number investigative tasks in a spreadsheet environment. When learners engage in mathematical investigation, they interpret the task, their responses to it, and the output of their deliberations through the lens of their preconceptions; their emerging mathematical discourse in that perceived area. Social and cultural experiences always condition our situation (Gallagher, 1992), and thus the perspective from which our interpretations are made. Learners enter such engagement with preconceptions of the mathematics, and the pedagogical medium through which it is encountered. Their understandings are filtered by means of a variety of cultural forms (Cole, 1996), with particular pedagogical media seen as cultural forms that model different ways of knowing (Povey, 1997). The engagement with the task likewise alters the learner's conceptualisation, which then allows the learner to re-engage with the task from a fresh perspective. This cyclical process of interpretation, engagement, reflection and re-interpretation continues until some resolution occurs. This echoes Borba and Villarreal's notion of humans-with-media (2005),

where they see understanding emerging from an iterative process of re-engagements of collectives of learners, media and environmental aspects, with the mathematical phenomena. Other researchers (e.g., Drijvers, 2003), emphasise the eminence of mental schemes, which develop in social interaction (Kieren & Drijvers, 2006).

When learners investigate in a digital environment, some input, borne of the students' engagement with, or reflection on the task, is entered. The subsequent output is produced visually, almost instantaneously (Calder, 2004b) and can initiate dialogue and reflection, perhaps internally for the student working individually. This leads to a repositioning of their perspective, even if only slight, and they re-engage with the task. They either temporarily reconcile their interpretation of the task with their present understanding (i.e. find a solution) or they engage in an iterative process, oscillating between the task and their emerging understanding. This allows for a type of learning trajectory that can occur in various media (Gallagher, 1992), but is evident in many learning situations that involve a digital pedagogical medium (Borba & Villarreal, 2005).

There are, however, affordances associated with the process. This chapter is concerned with one aspect that might be perceived as a constraint, visual perturbations, but which can offer opportunities for enhanced mathematical understanding. When the students' preconceptions suggest an output that is different from that produced, a tension arises. There is a gap between the expected and the actual visual output. It is this visual perturbation that can either evoke, or alternatively scaffold, further reflection that might lead to the reshaping of the learner's perspective, their emerging understanding. It shifts their conceptual position from the space they occupied prior to that engagement. The learner's reaction to the visual output, if it emerges as a conceptual tension, is what I have defined as a visual perturbation. It is the tension for the learner between what their preconceptions indicated would visually appear, and the actual visual output the pedagogical medium produced.

As learners re-engage with the task, informal mathematical conjectures often have their speculative beginnings (Calder, Brown, Hanley & Darby, 2006). Other

researchers have noted that the development of mathematical conjecture and reasoning can be derived from intuitive beginnings (Bergqvist, 2005; Dreyfus, 1999; Jones, 2000). This intuitive, emerging mathematical reasoning can be of a visual nature. In both algebraic and geometric contexts learners have used visual reasoning to underpin the approach taken to conjecturing and generalisation (Calder, 2004a; Hershkowitz, 1998). Meanwhile, Lin (2005) claims that generating and refuting conjectures is an effective learning strategy, while argumentation can be used constructively for the emergence of new mathematical conceptualisation (Yackel, 2002). Visual perturbations, and the dialogue they evoke, can generate informal conjectures and mathematical reasoning as the learners negotiate their interpretation of the unexpected situation. Research into students learning in a CAS environment, likewise revealed that probably the most valuable learning occurred when the CAS techniques provided a conflict with the students' expectations (Kieran & Drijvers, 2006). If the visual perturbation induced by investigating in a digital medium meant the learner framed their informal conjectures in a particular way, it is reasonable to assume that their understanding will likewise emerge from a different perspective.

Discussion

The data in this study illustrate the notion of visual perturbation. When the output generated differed to the expected output that the pupils' preconceptions had suggested would be created, a sense of unease was evident. This tension disturbed the pupils' perceptions of the situation leading them re-engage with the task from a modified position. It influenced their interpretations and decision-making and consequently transformed their learning trajectory. The output, in visual form, flavoured the pupils' reactions, interpretations, articulated accounts, and their subsequent re-engagement with the task. They posed and tested informal conjectures, incorporating their interactions from within the visual tabular form. They settled on a common interpretation through dialogue, shaping their explanations in visual terms. The conceptual perceptions to which they subscribed prior to that engagement were revised, and they re-engaged with the tasks. This facilitated the ongoing evolution of their mathematical thinking. The visual perturbations invoked at various junctures through the engagement with mathematical phenomena in the spreadsheet environment shaped the learning

trajectories, and by inference the understanding, in particular ways. Within the notion of visual perturbation several variations emerged, often interacting in complementary ways. The data were evidence of visual perturbations leading to changes in prediction. These changes in prediction often caused an unsettling and reorganisation of the mathematical preconceptions or approach taken, but the re-engagement was of an exploratory nature.

In other instances the visual perturbation caused a reshaping of the conjecture or generalisation. This was similar to a change of prediction, but the re-engagement was more reflective and encompassed a broader perspective compared to a specific example. This was more often accompanied by a significant amount of dialogue and negotiation of shared understanding. Elsewhere, the data illustrated the visual perturbation enabling the pupils to re-negotiate their understanding of the intentions of the task itself. This was interwoven with the investigative process, with interplay between the two during the investigative trajectory. At times the visual perturbation was associated with an idea or area of which they had no previous conceptual recognition. The tension this evoked often led them towards seeking further intervention and clarification, frequently with the assistance of the teacher. Finally, the data were, on occasion, indicative of visual perturbations that led the pupils to investigate further a technical or formatting aspect associated with their exploration. This also was frequently linked to conceptual exploration. For example, the rethinking of their approach to formatting an actual formula due to a visual perturbation was a structural aspect, but involves a change to their mathematical thinking as well.

The episodes show that the particular pedagogical medium of the spreadsheet, at times induced a particular approach to mathematical investigation. This occurred through the tension that arose from the learner's engagement with the task, when the actual output differed from that which their preconceptions led them to expect. This output being in visual form, led to the term visual perturbations, and it appeared this was a particular characteristic of the learning trajectory when using spreadsheets. Some of the episodes in the data that illustrate the different types of visual perturbation are examined, and the ways in which they influence the learners' interpretation and learning trajectories discussed. Interestingly, these

various forms of visual perturbation don't necessarily emerge discretely; an episode can illustrate several types of visual perturbation in an interrelated manner.

Influencing the learning trajectory

This relates to an activity set in a scenario that allowed the pupils to explore different ways that they could get a pocket money allowance (see Figure 2).

This particular dialogue and output related to investigating the second option. The pupils initially began to enter the counting number sequence into the spreadsheet.

1
2
3

Mike, using his current understandings, his prevailing mathematical discourse in number operation, immediately had a conflict between what he saw, and what his more global perspective was telling him it should be. This created the visual perturbation.

Mike: *Hey there's a bit of a twist, look, third week he gets 4 cents. We'll have to change it.*

His mathematical preconceptions and understanding of the situation allowed him to predict with confidence the outcome of 4 cents for the third week, yet the screen displayed 3. Hence he recognised the tension and articulated the need to reconcile this. This facilitated a change in the process by which the output was produced. It also lead them to re-negotiate their sense making of the task, in that it suggested a process of re-negotiation of what the task was about; their interpretation of the task rather than the engagement in its investigation. His partner Jay started to enter input into Cell A3.

Mike: *No, no, no we'll have to be in C (column C of the spreadsheet).*

This was another visual perturbation, but of a different nature. It seemed to be primarily due to his current understandings of the structure and processes of the spreadsheet environment, rather than his mathematical preconceptions. Thus they were addressing a technical or formatting aspect associated with their investigation. Mike was also perhaps looking to show in some way the relationship between the counting sequence, in this case illustrating the number of weeks, and the amount of money received each week. The pedagogical medium through which he engaged the mathematical phenomena was beginning to structure his approach to the task and his thinking. It was this informal indication of a relationship, and the possibility of a pattern for the amount of money received, that is the beginning of the mathematical thinking, however.

Jay entered 1 into cell C1 to represent the cent for the first week. He began to enter a formula into C2, which he simultaneously verbalised:

Jay: *= A1 + 1.*
Mike: *No, no.*

Again there was a tension between what Mike is seeing, and what he thought it should be. This did not seem to be related to any mathematical preconception however, but rather was due to his understanding of the spreadsheet structure. He realised the formula should relate to cell C1. Also it was not a tension created by a difference in expected and actual output, and so differed from the notion of a visual perturbation. Jay continued:

Jay: *= C1 + 1 + 0.*

The output in C column was now:

1
2

Mike suggested the next entry:

Mike: *Equals C2 + 2.*

The output was now:

1
2
4

Jay: *Goes up by two. We have to double each week.*

He pondered on the input to the next cell (Cell C4).

Jay: *Equals C3 +...*

He considered which number to add to C3 to continue the doubling pattern. Mike meantime, addressed the same output, but his preconceptions were different, so clearly his thinking was too. His interpretation of the question, the spreadsheet, and his mathematical understanding of the processes involved also influenced his thinking.

Mike: *According to this it doubles each week.*
Jay: *How do you make it double?*
Mike: *Times by two, and star is times.*

Mike took over the keyboard and entered =C3*2 into cell C4 then filled down in the cells below.

Jay: *Look at the amount of cash you get on double though.*
Mike: *That's the biggest one.*
Jay: *See that huge amount of cash.*

The spreadsheet had enabled them to quickly process the large amounts of data with the particular medium shaping their investigation in a distinct, structured manner. It afforded them the opportunity to engage with the investigation in a particular way, as there was a difference in what they expected from option 2, and the size of the actual output. Their surprise with this difference was

illustrative of a visual perturbation. Throughout the process, the visual perturbations, the difference between what their prevailing discourse suggested and the actual output, influenced their decisions, and hence their learning trajectory. Their mathematical reflection was a function of their interaction with the task filtered by the pedagogical medium through which it was encountered, and their prevailing mathematical discourse. As their perspective was also repositioned through each interaction, the spreadsheet environment had also influenced this aspect.

Unexpected versions of mathematics

The next scenario illustrates a different type of visual perturbation. Tension evoked from the difference between the expected and actual output was evident, but in this situation the visual perturbation arose when the actual output was beyond the scope of the pupils' current conceptualisation. This involved the scientific form of very large numbers. The pupils sought teacher intervention, for reconciliation of their mathematical preconceptions with the output.

This episode related to a traditional investigation based around the story of the Grand Vizier Ben Dahir choosing his reward for inventing the game of chess (see Figure 3).

This investigation was initiated after the pupils had already had some experience of using the spreadsheet. They were less tentative regarding the operational aspects of using them; for example, they were more comfortable generating formulas, and had an expectation of what output they might get based on some accumulated experience.

Ana:	*It goes 1,2,4,8,16 …., so its doubling.*
Lucy:	*=A1 times 2.*
Ana:	*Is that fill down.*
Lucy:	*Go down to 64.*
Ana:	*Right go to fill, then down.*

They made an initial interpretation of the problem, and immediately saw a way the spreadsheet would help them explore the problem. However, there was some unexpected output in a visual form they couldn't recognise.

Lucy: *What the ....*
Ana: *Eh…*
Lucy: *What you…*
Ana: *9.22337 E+18.*

The unexpected outcome produced a significant disturbance as they attempted to reconcile it through their prevailing discourse. This was a visual perturbation that was associated with an idea or area of which they had no previous knowledge, that is, scientific notation. They quickly decided it was beyond their conceptual scope and sought the teacher's input. The teacher gave some explanation about scientific form related to place value. They made sense of this within their current conceptualisation.

Lucy: *So that would be the decimal space up 18 numbers.*

They wrote it out on paper to get a picture of it within their current frame:

9223370000000000000

They re-engaged with the activity from their repositioned perspective.

Lucy: *We have to add it all up.*
Ana: *Wow it's big.*
Lucy: *= A1+A2+A3 ...*
Ana: *Takes a long time, because it's 64.*

Lucy was using a simple adding notation with the spreadsheet, to sum the column of spreadsheet cells A1, A2, A3 etc. Ana realised, and articulated, that there were 64 cells from A1 to A64, so it would take a long time to enter them individually. They acknowledged the scope of this particular task, and intuitively felt the

medium offered possibilities for a more efficient approach. They reflected on prior knowledge and earlier experiences, and between themselves negotiated a way to undertake their decided trajectory more easily.

Lucy:	*Sum.*
Ana:	*= sum (A1:A64).*
Lucy:	*1.84467E+19.*
Ana:	*How long will that feed?*
Lucy:	*1.84467E+19 divided by 2000.*

The sum of the values in cells A1 to A64 was 1.84467 x 10^{19} that is 18446700000000000000. There was no reaction to the scientific form of the output at all this time, and they were almost seamlessly moving into the next phase of their investigation with the newly reconciled concept. Their prevailing discourse in this area had been repositioned through the reconciliation of their preconceptions with the unexpected output. This reconciliation and subsequent repositioning was initiated by the visual perturbation they encountered as a result of investigating using this particular pedagogical medium.

Reconciling technical aspects and alternative forms

The following episode arose from another group's engagement with the Rice Mate task. While the learning pathway evolved differently from the previous group, there were comparable visual perturbations evident in the data. Their initial engagement was constrained by their memory of technical aspects, but the unexpected output that was generated from engaging with the task, permitted alternative approaches to be considered and explored. This re-envisioning fashioned their understanding in this regard. The tension that arose when there was a gap between their expected output and the actual output promoted the restructuring of their perspective and they approached the task in a slightly modified manner. The recursion of their attending to the task, and interpretation through modified perspectives, allowed the evolution of understanding of technical and conceptual elements of their activity. They began by considering the first square of the chessboard and negotiating a way to double the number of grains of rice in subsequent squares:

Tony: *Ok, just type in 1.*

Fran: *Oh, what about =.*

Tony: *Right.*

Fran: *A1 um becomes 2.*

Tony: *You have to write in the number first.*

Fran: *Oh yeah, I knew that.*

Tony: *A1 times 2. Where is the times button.*

Fran: *Times is the star button.*

Tony: *OK, A1 * 2.*

They entered this and the output in cell A1 changed from 1 to A1*1. This was not the output that they were expecting, causing them to re-consider their technical approach.

Fran: *Don't you push fill down.*

Tony: *Something like that.*

Fran: *You have to go like this……oh.*

The following output was generated:

A1*2
A1*2
A1*2
A1*2
…etc.

Again, the output was unexpected and related to a technical or formatting aspect. Their mathematical preconceptions probably enabled them to envisage a sequence of numbers doubling from one in some form, but the screen output being different and unexpected led them to re-evaluate the manner in which they engaged the exploration of the task.

Tony: *This could take a while …*

Fran: *What about we do one, two, to sixty-four. Then you just push OK.*

Fran appeared to be referring back to her earlier spreadsheet experiences and considering an alternative approach with the intention of generating the counting numbers from one to sixty-four in the A column. She perhaps was anticipating relating the positional number of each chessboard square to a sequence modelling the amount of rice (i.e. two to the power of n, where n is the chessboard square number). Again they were temporarily thwarted by their technical expertise with the spreadsheet, but they maintained a level of interest and confidence necessary to continue the investigative process. Tony continued, articulating his perception of their desired approach. Their alternating engagements with the task, then reflection on the output through their mathematical and spreadsheet preconceptions was facilitating the evolution of their approach to the task, and the emergence of the technical aspects required to enable that approach.

Tony: *In A1 we want 1 and then you go something like =A1*2 then you go fill down and it times everything by 2. So 1 by 2, then 2 by 2, then 4 by 2, then 8 by 2, 16 by 2.*

Fran: *To double it? Times 2 more than the one before.*

They continued after a brief interaction with the researcher:

Tony: *The amount of rice for each year will be in each cell.*

Fran: *What's the first thing we need to start off with? =A1*2.*

Tony: *We have 1 in cell 1 [for one grain of rice], and then we add the formula in cell A2 now.*

Fran: *And then fill down.*

Tony: *Got it. Go right down to find out.*

They have now entered:

	A
1	1
2	=A1*2

They *Fill Down* from cell A2 to produce the sequence of numbers they anticipated would give them the number of grains of rice for each square of the chessboard. They encountered something unexpected with the following output generated:

	A
1	1
2	2
3	4
4	8
…	…
…	…
26	33554432
27	67108864
28	1.34E+08
29	2.68E+08
…	…
…	…

Fran: *Ok that isn't suppose to happen.*

Tony: *9.22E +18 that makes a lot of sense.*

Fran: *Oh yeah, it comes with all up to here, but then it gets too far.*

The output was unforeseen and in a form they weren't familiar with (scientific form). There was a tension between the expected and actual output causing them to reflect, adjust their position, and re-interpret. These pupils initially sought a technical solution to resolve their visual perturbation. They looked for a way to reformat the spreadsheet to alleviate their dubious perceptual position.

Fran: *Oh, make bigger cells.*

Tony: *You can make the cell bigger. Pick it up and move it over.*

Fran: *That should be enough.*

Tony: *It still doesn't work.*

Still perturbed by what the spreadsheet displayed, they sought my intervention, so the notion of scientific form was discussed with them. They indicated that they had a better perception of the idea and proceeded with the task. Tony considered the output 2.25E+15:

Tony: *When you get past the 5 you will need a lot of zeros. We'll need thirteen more.*

Fran: *We'll have to write out the whole answer.*

NC: *You can leave it how it is as long as you understand; you know what it means.*

Tony: *Oh, that's OK then.*

Fran: *You can still just do it from here where it is.*

They continued with the task, maintaining the numbers in scientific form as they negotiated a way to sum the column of numerical values. This they managed, drawing on their prior understanding of the technical process required. This generated:

1.84467E+19

Tony: *Yeah!!!! It worked.*

Fran: *We got it!*

Tony: *Wow. It's a really, really big number.*

Drawing on their freshly modified perspective, they considered how it might appear in decimal notation. Their shared understanding required further negotiation, however.

Tony: *How many zeros.*

Fran: *19.*

Tony: *Did you count these numbers here?*

Fran: *No.*

Tony: *You need to count from the decimal point to the end and then add the zeros.*

Fran: *Those numbers count as well.*

Tony: *184467. How do you think we say that number?*

Fran: *A bagilliganzillan!*

They continued with the task, but carried forward their modified perspective; a perspective moderated through iterations of engagement and interpretation, but initiated by the visual perturbation. The sum of the sixty-four numbers that corresponded to the amount of rice on each of the squares was entered into cell A66, and the size and form of the numbers appeared to lead them towards another way of engaging with the task that involved a formatting aspect.

Fran: *So 1.84467E+19.*

Tony: *It's a lot to type in.*

Fran: *Go down to the bottom, we could use the cell.*

Tony: *Divide A66 by…*

Fran: *Which one is for division? = A66.*

Tony: *Would it be this one?*

Fran: *=A66/6221409060. That's the world population.*

Tony: *You could see how many times 365 goes into 29650279.*

Fran: *You could also see the rice on the chest board can feed the whole world for that many days and how many years it would do.*

Tony: *=365/A67 is that right.*

Fran: *A67/365.*

They entered =A67/365 to find the number of years the rice would approximately feed the population of the world for. Cell A67 contained the quotient after cell A66 (the total number of grains of rice) was divided by the population of the world. They still had some mathematical thinking and interpretation to undertake related to how long the rice would feed the world, but for this analysis, the

intention was to consider how engaging through the medium of the spreadsheet might have influenced their engagement and the emergence of their understanding in particular ways. Their learning trajectory was shaped, via interpretation and engagement, by the various associated socio-cultural filters including the spreadsheet environment. Their preconceptions were mediated by the pedagogical medium and their understanding and explanations as evidenced by their subsequent interactions had incorporated those modified perceptions. It appeared to be the visual perturbations that instigated, then influenced the nature of those interactions.

Influencing the posing of informal conjectures

The next two scenarios relate to the **101 X** activity, a task in which the pupils investigated the pattern formed by the 101 times table. The two pupils (Awhi & Ben) had entered the counting numbers into column A and were exploring the pattern formed when multiplying by 101 in column B:

1
2
3
4 etc

Awhi: *=A2 * 101. Enter.*

Ben: *202.*

Contemplating the output produced from their unique conceptual perspective, they postulated an informal, rudimentary conjecture through prediction.

Awhi: *Now let us try this again with three. Ok, what number do you think that will equal? 302?*

Ben: *No, 3003, (they copy the formula down to produce the output below).*

1	101
2	202
3	303
4	404 etc.

Ben [continues]: 303.

The actual output was different from the output the pupils expected. This had created a visual perturbation, which in this case was easily reconciled with their prevailing mathematical discourse. The visual perturbation had caused a reshaping of their prediction that allowed them to reposition their conceptualisation. It also initiated the beginnings of a conjecture or informal generalisation.

Awhi: *If you go by 3, it goes 3 times 100 and zero and 3 times 1; 303.*

They then explored a range of two and three digit numbers, before extending the investigation beyond the constraints of the task.

Awhi: *Oh try 1919.*
Ben: *I just have to move that little number there, 1919.*

The following output was produced:

193819

Interestingly, they seemed to disregard this output and formed a prediction based on their preconceptions.

Awhi: *Now make that 1818, and see if it's 1818 [the output].*
Ben: *Oh look, eighteen 3, 6, eighteen.*

There was a visual perturbation, which made them re-engage in the activity, reflect on the output, and attempt to reconcile it with their current perspective. It caused them to reshape their emerging conjecture.

Awhi: *Before it was 193619; write that number down somewhere (183618) and then we'll try 1919 again.*

Ben: *Yeah see nineteen, 3, 8, nineteen. Oh that's an eight.*

Awhi: *What's the pattern for two digits? It puts the number down first then doubles the number. This is four digits. It puts the number down first then doubles, then repeats the number.*

The visual perturbation made them reflect on their original conjecture and reposition their perspective on the initial, intuitive generalisation. It stimulated their mathematical thinking, as they reconciled the difference between what they expected and the actual output, and rationalised it as a new generalisation. This new generalisation was couched in visual terms. They used visual reasoning, referring to the type and position of the digits as they related to the input.

Reshaping generalisations

The next episode was part of the same investigation, but with a different pair of pupils, as they began to explore what happens to decimals. Ant predicted that if they multiplied 1.4 by 101, they would get 14.14

Bev: *I get it, cos if you go 14 you'll get fourteen, fourteen.*

Ant: *We'll just make sure.*

They entered 1.4, expecting to get 14.14 as the output.

Bev: *141.4, it should be 1, 4 (after the decimal point, that is 14.14).*

This created a visual perturbation. They began to rationalise this gap between the expected output (14.14) and the actual output (141.4). This visual perturbation caused a reshaping of their conjecture or informal generalisation. In doing so they drew on their current understandings of decimals and multiplication, but also had

to amend that position to reconcile the visual perturbation the pedagogical medium has evoked. Again they used a visual lens to do so.

Ant: *We're doing decimals so its 141.4.*

Bev: *So it puts down the decimal [point] with the first number then it puts the 1 on, then it puts in the point single number whatever.*

Ant: *It takes away the decimal to make the number a teen. Fourteen.*

Bev: *141.*

Ant: *Yeah. It takes away the decimal [14 – researcher's insertions] and then it adds a one to the end [141], and then it puts the decimal in with the four [141.4].*

Bev recognised that this was more a visual description of this particular case rather than a generalisation. There was still a tension with her prevailing discourse.

Bev: *No it doesn't, not always, maybe. It might depend which number it is.*

Ant: *Try 21 or 2.1. See what that does.*

According to Ant's conjecture from above, they would be expecting to take away the decimal point (21), add a one to the end (211), and then re-insert the decimal point and the one (211.1). However the output is 212.1, which created another visual perturbation to be reconciled.

Bev: *No it doesn't.*

Ant: *Two, where's the point? One two point one.*

Bev: *Oh yeah, so its like, the first number equals…*

They tried to formulate a more generalised conjecture.

Ant: *Takes away the decimal and puts that number down, then puts the first number behind the second number. Aw, how are we going to write this?*

Bev proffered a definition that they negotiated the meaning of, then situated within their emerging conjecture.

Bev: *It doubles the first numbers.*
Ant: *Takes away the decimal, doubles the first number, then puts the decimal back in.*
Bev: *How does it get here?*

They then entered 2.4 and made predictions regarding the output in light of their newer conjecture.

Ant: *Twenty-four, twenty-four with the decimal in here.*
Bev: *It will be doubled; twenty four, twenty four but the last number has a point in it, a decimal.*

The pupils' predictions were confirmed, and they negotiated the final form of their generalisation. They were still generalising in visual rather than procedural terms, and Bev suggested a name for their theory, double number decimals, one that they both have a shared sense of understanding. This mutual comprehension had emerged through the process; the investigative trajectory through which they had negotiated their way. As with an example of a group in the previous chapter, the pupils had associated the term "double numbers" with the visual repetition of the digits e.g., 2424, rather than an operational meaning of actually doubling the number e.g., 24 X 2 = 48. This accentuated the visual interpretation they were applying in their dialogue. The investigative trajectory was influenced by the pedagogical medium through which the pupils engaged with the mathematical activity. More specifically, the questions evoked, the path they took, and the conjectures they formed and tested were fashioned by visual perturbations: the tension arising in their prevailing discourse by the difference between the expected and actual output. The process shouldn't necessarily have stopped just

there, however. An intervention, perhaps in the form of a teacher's scaffolding question, might have initiated the investigation of why this visual pattern occurs. The pupils meantime had initiated a 'what if' question of their own, with a rhetorical suggestion for the possible outcome, framed in the new negotiated conceptualisation.

Bev: *What if we just go point four.*

Ant: *Double number decimals?*

Conclusions

Each of the above episodes illustrates how the learning trajectory was influenced by the learners' encounter with some unexpected visual output as they engaged in tasks in this particular domain, through the pedagogical medium of the spreadsheet. The perturbation, and the dialogue that ensued as the learners reconciled their existing perspective with this unexpected output, seemed to create opportunities for the re-positioning of their prevailing discourse, as they negotiated possible solutions to the situations.

The engagement with the task, and with the medium, often evoked dialogue. This was an inherent part of the negotiation of understanding. When the learners' preconceptions suggested an output that differed to that produced, a tension arose. This output, in visual form, initiated the learners' reactions, reflections and subsequent re-engagement with the task. The learners posed and tested informal conjectures, and negotiated a common interpretation through dialogue. This facilitated mathematical thinking. It shifted them from the conceptual space they occupied prior to that engagement.

The data in this study illustrated the notion of visual perturbation. Within this notion there seemed to be several manifestations or variations.

1. When the visual perturbation led to a change in prediction. It caused an unsettling and repositioning of the prevailing discourse, and the re-engagement was of an exploratory nature.

2. When the visual perturbation caused a reshaping of the conjecture or generalisation. This was similar to that above, but the re-engagement was more reflective and global in nature as compared to a specific example. This was more often accompanied by a significant amount of dialogue and negotiation of meaning.
3. When the visual perturbation made the students re-negotiate their sense making of the task itself. This was not a distinct process from the investigative trajectory, but interwoven, with each influencing the other.
4. When the visual perturbation was associated with an idea or area with which the students had no previous knowledge. The tension this evoked often led them towards seeking further intervention, frequently in the form of teacher-led scaffolding.
5. When the visual perturbation led them to further investigate and reconcile their understanding of a technical or formatting aspect associated with their exploration. This was often also symbiotically linked to the conceptual exploration, but sometimes in unexpected ways. For instance, the rethinking of their approach to formatting an actual formula due to a visual perturbation was a structural aspect, but they were simultaneously re-engaging with a mathematical process while negotiating their understanding of the format. They might at times also have been engaging with more abstract conceptualisations associated, but not directly related to the process (e.g., in this case some form of algebraic thinking).

These episodes show that the particular pedagogical medium of the spreadsheet, at times induced a particular approach to mathematical investigation. This occurred through the tension that arose from the learner's engagement with the task, when the actual output differed from that which their preconceptions led them to expect. This output being in visual form, led to the term visual perturbations, and it appeared this was a particular characteristic of the learning trajectory when using spreadsheets. It may be that this is a generic characteristic of learning trajectories in digital media. Certainly the literature suggests that with CAS software, unexpected outcomes that arose while engaging with algebraic tasks through that medium, influenced the learning trajectories and provided rich

opportunities for learning (Kieren & Drijvers, 2006). Discussing mathematical thinking when using digital images, Mason (2005) also suggested that when response to a particular action does not meet the expectation, the tension arising might provoke further reflective engagement. The data also supported my contention that engaging with the mathematical phenomena through the spreadsheet fashioned the learners' approach; their learning trajectory spiralled through a pathway, influenced by particular aspects such as the visual, interactive nature of the engagement and the structuring of the output. This trajectory was partially due to the distinctive characteristics of the digital medium, its associated affordances, and their interplay with other influences. In the next chapter, amongst other conclusions, the ways this environment might enhance the propensity of learners to become risk-takers in their engagement is discussed, an aspect that appeared to facilitate the investigative process.

CHAPTER ELEVEN: Conclusions and Implications

Nau te rourou, naku te rourou,
Kaora ki te manuhiri.

Your basket and my basket,
Each contributes to a greater whole.

Conclusions

The word conclusion is derived from the Latin *concludere*, to close, and even with the signification of being a summary, there is the connotation of an ending, or termination associated with it. Yet the interpretation of the data in this study and the methodology ascribed privilege an ongoing, fermentative process to research, and the understandings revealed with regards to the research questions. As a consequence, the conclusions cannot be depicted as precise reality that has been derived conclusively from the data, but are more tentative assertions that the data were illustrative of, and that informed broader discourses. They are part of ongoing cycles of analysis, and might be considered as interpretations subject to a range of perspectives that are historically and contextually situated. These in turn contribute to other perspectives and analyses. The stories and conclusions that emerged were a consensus between the data and the underlying discourses of the researcher, participants, supervisors, presentation audiences, and literature in the related domains. Nevertheless, there were commonalities and patterns that did emerge from the data and resonance between the data and conclusions with the findings of other researchers in alternative settings.

In what ways is mathematical understanding reorganised when mathematical phenomena are engaged through digital pedagogical media, the spreadsheet, in particular? The following sections address the research questions through the various filters that constitute the research process.

How the learning experience differs

A key aspect examined by this research study was the nature of the learning experience when mathematical phenomena were engaged with through the pedagogical medium of the spreadsheet. Coupled with this, was the consideration of ways learning trajectories might differ in a spreadsheet environment from the investigation of mathematical phenomena through other pedagogical media such as paper-and-pencil. The thesis looked at what the characteristics of the learning encounters were and what opportunities the medium afforded that were particular to the spreadsheet. Affordances are the opportunities for activity and interaction that arise within particular settings. They are a potential for action, the facility of an environment or artefact to enable the intentions of the student within a particular problem situation (Tanner & Jones, 2000). We might consider them as perceived opportunities offered through the pedagogical medium, in relationship with the propensities and intentions of the user. The data in the research were illustrative of various affordances offered by the spreadsheet medium, some of which were consistent with those affordances attributed to other digital media. These are discussed first with their ramifications for the shaping of the learning trajectories addressed in the following section.

One characteristic of the spreadsheet environment that the data indicated was influential in the learning process was the visual, tabular structure of the output produced. It allowed for clearer comparisons to be made between adjacent cells or columns, and more direct links to be drawn between input and output. The students were able to transform easily a column or table of values, a process that facilitated the perception and confirmation of relationships and emerging informal conjectures. This, coupled with other affordances such as the immediate feedback, enhanced their opportunities to interpret and make decisions more readily. The facility to compare output more easily left space in the investigative process for other influences such as personal value judgments and experimentation. These tables were typically generated by formulas; they were a function of the formulas engendered by the students' interpretations and intentions to model the situations. As such, there was interplay between the two

representations that highlighted their relationship and that of the corresponding numerical form. This aspect was also illustrative of the interactive nature of the engagement of the tasks through the spreadsheet medium. The data were also indicative of how this characteristic shaped the subsequent interpretation and explanations while also influencing the evolution of the learning trajectory in particular ways. The students used visual referents when forming and explaining their emerging generalisations and theories.

Viewing the visual representation simultaneously with a symbolic form enabled the students to alter the symbolic and observe the effect on the table structure, and the numerical data within it. This helps them to see the connections between those forms. The visual representations of the output in either tabular or graphical form were an affordance of the spreadsheet environment, as was the facility to view and interact with multi-representations of the data. While this interactive faculty was more constrained than the manipulation of geometrical figures in DGS software, it did nevertheless proffer a dynamic experience, an aspect the data confirmed. Associated with this was the affordance of the spreadsheet and ICT in general to give immediate feedback. The students were able to change formulas or numerical values and get a relatively instant response to their input. This appeared to foster a more experimental, exploratory approach, as the students were willing to pose informal conjectures, immediately test them and reflect on the output.

This speed of response enabled large amounts of data to be easily transformed perhaps by computational operations. The data were indicative of this affordance, and allied with the accuracy complicit to this, it removed the computational fetters of doing many repetitive known computations, giving access to investigating situations that might otherwise not be possible in the school situation. Rich mathematical tasks such as *Dividing one by the counting numbers* would not have been as accessible without the spreadsheet or another digital technology facilitating the accurate management of the large number of computations required to generalise and test the patterns. It also meant that realistic data from more meaningful contexts that don't use 'tidy' numbers could be investigated at earlier levels, without the complexity of computation clogging

the thinking and mathematising processes. These affordances, in conjunction with the visual structure, opened opportunity for patterns to be recognised and explained more readily. The students were able to assess promptly their emerging formative conjectures and more easily model situations. The students were confident of accuracy when applying procedural processes, but the corollary to this was that they were required to be more explicit when entering formulas for mathematical manipulation. While the previously discussed attributes enhanced their capacity and willingness to experiment in this regard, eventually the precision necessary for their ongoing investigation of the situations assumed there was meaningful interpretation and translation between the symbolic and numerical or visual representations.

Significant in this discussion is the relationship between the pedagogical medium and the learner. While the digital medium influenced the learner's approach in particular ways, and hence the understanding that emerged, it was his/her underlying preconceptions that guided the manner in which the digital medium was employed. In this sense the learner shaped the technology. The student's engagement is shaped by the medium, but also influences the medium (Hoyles & Noss, 2003). There is a symbiotic element to this bilateral liaison, for while the learner's existing knowledge and understanding guides the way the technology is used, the affordances of the technology influence the approaches and strategies the learner uses and as a consequence the understanding. In the next section, the ways in which the affordances of the spreadsheet environment condition actual learning trajectories are discussed.

Learning trajectories

The notion of learning trajectory has been defined by two differentiated meanings. There is a distinction made between the intended (*hypothetical*) learning trajectory and the *actual* learning trajectory. The *hypothetical* learning trajectory is perceived, in conjunction with curricula and planning, as the identification and characterisation of potential pathways to develop mathematical thinking (Sacristan, Calder, Rojano, & Santos-Trigo, in press). On the other hand,

the *actual* learning trajectory is the actual pathway the learner negotiates as his/her mathematical thinking evolves from working on activities or tasks. Digital technologies, if used appropriately, enable mathematical phenomena to be presented and explored in ways that provide opportunities to initiate and enhance mathematical thinking, and make sense of what is happening. When the learning experience differs with digital technology (as compared to the experience in traditional settings), we can assume that *actual* learning trajectories and understanding will also differ.

Hoyles & Noss (1987) considered the learning situation in which the interaction in a microworld takes place, by taking into account the learner, the teacher, the setting and the activity which, by inference, will reflect the past experiences and intuitions of the learner together with the experiences and intentions of the teacher. They envisioned a microworld as being composed of four elements: the pupil component (concerned with the existing understandings that the student brings to the learning situation); the technical component (consisting of the software or programming language and the associated representational system); the pedagogical component (the medium through which the mathematical phenomena is engaged and the interventions that take place); and the contextual component (the social setting of the activities).

The diagram below (Sacristan, Calder, Rojano, & Santos-Trigo, in press) illustrates the interplay of some of the broader key influences on learning trajectories, and the complementariness and commonalities that might exist. These influences are positioned within a representation framed by the components identified by Hoyles and Noss (1987).

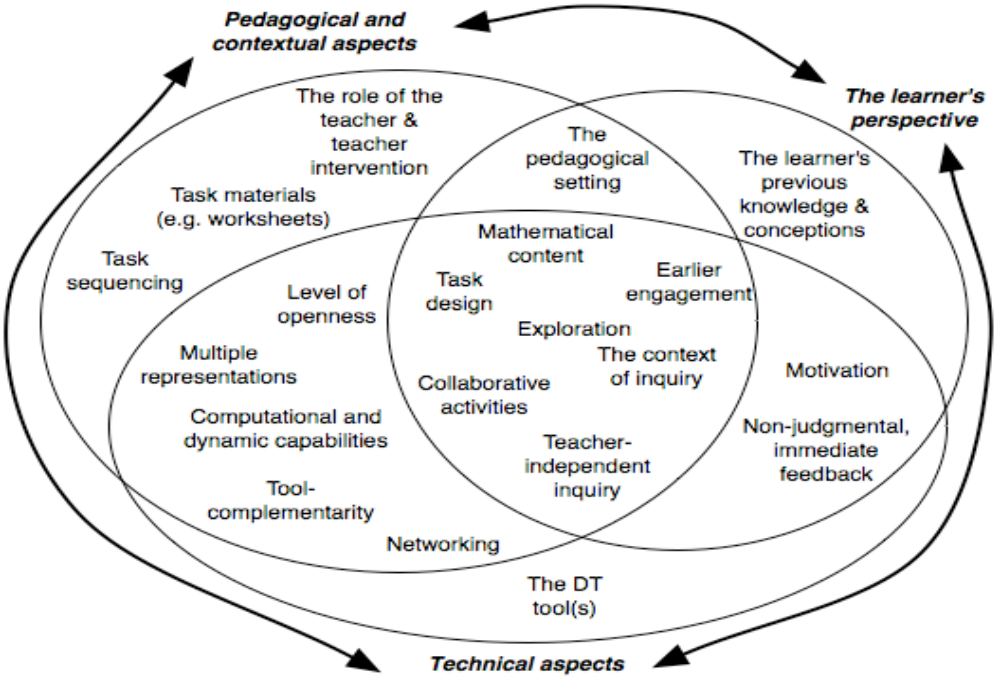

Figure 7: Some affordances and aspects of digital technologies to be considered in the design and analysis of a digital technology-based learning trajectory, (Sacristan, Calder, Rojano, & Santos-Trigo, in press).

The researcher developed activities, programmes, and a structure for the interactions with the students; that is, a *hypothetical* learning trajectory. However, in the context of the research questions, this study was concerned with the *actual* learning trajectories traversed by the students. In particular, it is how the pedagogical medium of the spreadsheet influenced the *actual* learning trajectories, and by implication the learning and understanding, which is central to this thesis.

The particular ways actual learning trajectories might evolve

One of the key aspects of the engagement that was influenced by the spreadsheet as pedagogical medium was the initial engagement with the tasks. Across a range of activities the students, sometimes after a brief familiarisation of the problem, moved immediately to engagement within the spreadsheet environment. Usually this was to generate tables or columns of data, often through the use of formulas and the *Fill Down* function. This emerged from both the observational and interview data and was influential in the episodes considered through the gaze of the hermeneutic circle. This initial engagement allowed them to experiment with

the intentions of the tasks and to familiarise themselves with the situation. They more readily moved from initial exploration, through prediction and verification, to the generalisation phase. Often, they immediately looked to generalise a formula to model the situation. The visual, tabular structure coupled with the speed of response facilitated their observation of patterns. Their language reflected this and frequently contained the language of generalisation.

The data illustrated several versions of initial engagement. At times, the engagement involved familiarisation of the task (this could be ongoing), at other times, the purpose was the exploration of formula to produce an anticipated output, while in other instances it was to begin immediately the prediction or generalisation phases. The influence of this initial engagement permeated the subsequent ongoing interaction. The distinctive nature of this engagement framed the ongoing interactions, interpretations and explanations as the students envisioned their investigation through that particular lens. The actual learning trajectories were shaped by that initial engagement of creating formulas or columns and tables of data to model the mathematical situation. Digital technologies are generally more conducive to the modelling of mathematical situations than pencil-and-paper media, and the data were illustrative of the spreadsheet enhancing this aspect. The capacity to manipulate large amounts of data quickly, coupled with the potential for symbolic, numerical, and visual representations enabled the students to produce models that could be observed simultaneously, with the links and relationships between them explored in an interactive manner. Also consistent with the findings of other researchers (e.g., Ainsworth et al., 1998), the students' interaction with alternative representations promoted learning through the comparison or combination of representations, enabling broader perceptions than what might have been gained from a single representation. As well, when the students were required to relate different representations to each other, they had to engage in activity such as dialogue, interpretation, and explanation that enhanced understanding.

The spreadsheet environment was also influential in the generation of sub-goals as the students' learning trajectories unfolded. As they alternated between attending to the activities from the perspective of their underlying perceptions,

and then reflecting on this engagement with consequential modification of their evolving perspectives, they set sub-goals that plotted their ongoing interaction. These were frequently reset in response to the output generated within the spreadsheet environment. Sub-goals were generated at times because of opportunities afforded by the particular pedagogical medium. As well as those attributes that facilitated the modelling process, the facility to test immediately and reflect on emerging informal conjectures gave potential for the sub-goals the students set being shaped by the medium. The data, discussed in Chapter Eight, indicated this. The data demonstrated how the students' interpretations of the situations they encountered were influenced by the visual, tabular structure. It allowed more direct comparison of adjacent columns and enabled them more easily to perceive relationships between numerical values on which to base their new sub-goal, often linked to an emerging informal conjecture. It enhanced their ability to perceive relationships and recognise patterns in the data. Seeing the pattern evoked questions. On occasion the students pondered why the pattern was there, and what was underpinning a particular visual sequence.

While investigating in this environment, the students learnt to pose questions and sub-goals but also were encouraged to create personal explanations, explanations that were often visually referenced probably due to the pedagogical medium. It also gave opportunity through its various affordances for the students to explore powerful ideas and to explore concepts that they might not otherwise be exposed to. At times the learning trajectory evolved in unexpected ways. When the output varied, sometimes markedly, from what was expected, it caused tension that often led to the resetting of the sub-goal and substantial shifts in the way the student interpreted or engaged the situation. This is considered further in the section examining the propensities of the spreadsheet in the reorganisation of thinking. This aspect and other affordances including the interactive nature of the environment also appeared to stimulate discussion. The students wanted to verbally articulate the rapidly generated output and discuss the connections they could see, not least when it was unexpected. This aspect of surprise provoked curiosity and intrigue, which allied with the interactive and visual nature of the experience, in the students' general view made the learning 'more fun and interesting'. This, in turn, enhanced the motivational aspects of working through

the spreadsheet medium, a feature that emerged in the interview, survey, and observational data.

For some of the students, the pure novelty of the learning experience in a fresh context, seemed to allow them to break the fetters of their previous accumulation of mathematics learning, some or all of which may have been negative. For others, there was the intrinsic motivation that was fostered by the capabilities the spreadsheet allowed the learner, that is, the potential to investigate complex problems in a reflective manner, to see visual representations of data simultaneously with symbolic forms, and the interactive nature of computer usage per se. Other researchers (e.g., Drier, 2000; Manouchehri, 1997) also identified this motivational aspect for students in their research with spreadsheets in mathematics programmes. Caution is needed where the data might have indicated the motivation was based superficially on novelty, as clearly the sustainability of this advantage would be limited if the spreadsheet, as advocated, was always available as a tool for problem solving.

Engaging the mathematical phenomena through a pedagogical medium that allowed the students to test informal conjectures, link the symbolic to the visual, and see the general through the specific, while being interactive and giving immediate feedback, enhanced the students' willingness and propensity to employ an investigative approach. They appeared to be more willing to take risks. This aspect, which was evidenced by various versions of the data, will be considered in the next section.

Risk taking

The learner's propensity and comfort to move beyond known procedures in recognisable situations, is indicative of their willingness to try fresh strategies in their approach to investigation and problem solving, By implication, problem solving contains an element of the unknown that requires unravelling and addressing through the application of strategies in new situations or in an unfamiliar manner. This requires a degree of creativity and a willingness to take conceptual or procedural risks of a mathematical nature. It is risk taking in a

positive, creative sense as compared to risky behaviour. The data were indicative of the spreadsheet environment affording learning behaviours and responses that facilitated the learner's willingness to take risks while operating within an investigative cycle. This seemed to allow the students to pose informal conjectures, to explore then reflect on them, before, perhaps after several investigative iterations, either validating or rejecting them. The offering and investigation of informal conjectures fostered mathematical thinking. These affordances were evident in the spreadsheet environment, but in some instances were characteristic of other digital pedagogical media.

The speed of response to input, when using the spreadsheet, indicated their suitability for facilitating mathematical reasoning. When the students observed a pattern or graph rapidly, they developed the freedom to explore variations and, perhaps with teacher intervention, learned to make conjectures, and then pose questions themselves. This facility to immediately test predictions, reflect on outcomes, then make further conjectures, not only enhanced the students' ability to solve problems and communicate mathematically, but developed their logic and reasoning as the students investigated variations, or the application of procedures. Chance et al. (2000) also found that this aspect, coupled with the speed of computation, allowed the learner to concentrate more on conceptual understanding. Baker et al. (1993) and Sandholtz et al. (1997) also reported improved high-level reasoning and problem solving linked with this capability. The data from the research study were similar to previous studies in this regard. The teacher needs to create an environment where mathematical ideas are discussed, and freely explored. Healy and Sutherland (1991) in their research into the ways students used spreadsheets to investigate number problems, found that the students became very engrossed in the problems and needed less support. This would, however, depend on the suitability of the problems and the hypothetical or intended learning trajectory.

Martin Neyland, (1994) has suggested that students are more likely to take ownership of the problem, the solution and the strategies involved, if they are actively involved with the discovery and the formation of the mathematical generalisations as well. The data indicated that the spreadsheet environment gave

an element of control to the learner that also seemed to enhance their willingness to take risks. While the facilitation role for the teacher is a critical aspect in the emergence of an environment that encourages risk taking, there was also a need for intervention to promote students' reflection on the processes involved, and an emphasis on the significance of the syntax from the initial engagement of their work through the spreadsheets medium, if they were to move beyond a superficial syntactical level. As well, the students using the spreadsheets progressed more quickly into exploring larger numbers and decimals. This appeared to indicate a greater propensity for exploration and risk taking engendered by the spreadsheet environment. Yet, although that is consistent with other findings (Beare, 1993; Calder, 2001; Sandholtz et al., 1997), certainly no correlation between using spreadsheets and greater mathematical risk taking can be drawn from this study.

Both the visual (e.g., Borba & Villarreal, 2005) and interactive (e.g., Mackrell, 2006) nature of the medium has contributed to the shaping of mathematical understanding in a distinct manner that is different from pencil-and-paper approaches. The research also indicated that the propensity to see and engage with multi-representations of data (numerical, symbolic and visual), to manipulate and transform large amounts of realistic data, and to foster the links between content areas, promoted the learner's use of prediction, conjecture making, and problem posing. The speed of response to inputted data, allowing the results of the prediction or conjecture to be considered more rapidly, stimulated discussion and encouraged risk-taking and experimentation.

Aspects related to investigating in the spreadsheet environment such as the tabular format for output, the immediacy of the response to input, the facility to compute large amounts of data simultaneously, and to modify various elements quickly and easily, all appeared to engender confidence in the students to try things and take risks. Confidence is a very personal condition though and is borne of a layering of interactions and interpretations, some seemingly unrelated to the situation in which confidence or lack of confidence might be noticed. Two people given the identical spreadsheet experience would probably

have distinctive responses invoked. One student might feel very confident to try new approaches, and another not at all confident. Nevertheless, from my perspective, the environment had the potential to enhance the students' willingness to take risks. It was also a relatively non-threatening, easily managed environment, conducive to making predictions, testing conjectures, and exploration without inhibition.

The following comments were indicative of responses illustrating a generic benefit of the spreadsheet as an investigative tool and indicated the propensity that is engendered by the medium for a confident investigational approach and a willingness to take risks.

> Beth: I looked at how it was written down and look at all the patterns; then I sorted it out in my head then put it down, and if it wasn't right then try another one. Experiment.
>
> Ella: I found it helpful that it could calculate itself, and I had more time to work on the problem.
>
> Fran: Using a spreadsheet made it more likely to have a go at something new because it does many things for you. You have unlimited room. You can delete, wipe stuff out.

While the data were illustrative of the spreadsheet, acting as a pedagogical medium, offering a distinctive environment for the students to investigate mathematical phenomena and influencing their actual learning trajectories, they also gestured towards it having transformative qualities; of it behaving as a conduit for the reshaping or reorganisation of their thinking. The next section, which will preface the examination of this position, considers an interpretation of the learning process that emerged from the research.

The nature of the learning process

One of the research questions addressed the nature of the learning process when students used spreadsheets to investigate mathematical phenomena. Both the literature and the data were accommodating of the notion that learning is a recursive, interpretive process with understanding emerging through cycles of interpretation, engagement, reflection (usually with accompanying explanation) then re-interpretation from a modified, evolving perspective. A distinctive feature of this research project was that this interpretative frame was applied to learning situations involving digital technology rather than an instrumental approach (e.g., Artigue, 1997) frequently employed in the analysis of learning through digital media. There were, however, several conditioning elements associated with this position. Firstly, the selection of literature was guided by both my predispositions and those of the supervisors, and secondly, the noticing that occurred in the data and the literature both by myself and other influential entities, such as the supervisors, audiences and reviewers, were orchestrated by their prevailing discourses in the associated domains. These two aspects will be given primacy in the discussion of the limitations of the findings, but it is important to have an awareness of their influence at this preliminary phase. As well, the data were drawn from particular students in specific settings so the findings are conditioned by that historical contextualisation.

Drawing from contemporary social science perspectives, and their application to education and mathematics education in particular, a moderate hermeneutic theoretical framework was settled upon and a version of learning, that was privileged by that perspective. This evolved as I underwent a hermeneutic circle in the research process itself; an influential, formative aspect that shaped the approaches taken and the spaces occupied as the data were interpreted. As the aspects attended to alternated between underlying discourses, and interaction with the literature, the data, or other participants (including supervisors and reviewers) my perspective was transformed. These historically situated spaces were central to the interpretations made at each particular juncture and as such

were constitutive of the data and the ongoing interpretation. Likewise, the initial analysis framed the subsequent viewing of the data through the moderate hermeneutic lens, hence comprising the data yet shaping the way it was viewed. The subsequent hermeneutic analysis is borne of that initial interpretation. It was an iteration of that interpretive process and central to the discussion and the conclusions drawn.

Hermeneutics is manifest in many forms and has as yet unresolved issues, but a common theme in its various versions is reference to the interpretation of meaning, and understanding. Central also is that each fundamental element of the interpretive process whether listener, text, or meanings bring to bear their own historically situated socio-cultural discourses that influence the interpretation. A moderate hermeneutic perspective also acknowledges that language, the vehicle for interpretation, is imbued with the conditioning preconceptions that permeate its character. The interpretations emerged through social activity and dialogue, with all the historical, political and cultural influences that implies. Understanding can't deny or elude the influences of its history. It also recognises the notion that understanding is filtered through the learning community, phenomena, the pedagogical medium, and active participation, including the language evoked. Understanding in mathematics can be seen as an evolution of historically positioned meanings dependent on the spaces from which they are observed and the media through which they are encountered.

As well, the moderate hermeneutic perspective subscribes to the transformative influence of the educative process, as opposed to an emancipatory one invoked when education might be employed to break the fetters of political and institutional power and authority. The notion of conceptual cognition as an evolving process is also emphasised by this perspective, with concepts seen as formative, developing progressions that emerge from iterations of engagement, reflection, and explanation, rather than set actualities. A moderate hermeneutic frame also has potential to reconcile elements of acquisitional (as per Piaget's thesis) and participatory (as underpinned by Vygotsky's thesis) theories hence accentuating its significance as a theoretical frame on which to hinge the learning theory employed in this project.

Fundamental in the application of this interpretive position to the learning situation is the principle of the hermeneutic circle. While the interpretation of the hermeneutic circle ascribed in this thesis was well rehearsed in Chapter Three, just how this was manifest in the analysis of data requires further elucidation. Hermeneutics is perceived as a theory of the interpretation of meaning (Gallagher, 1992). In the complex social milieu that constitutes the classroom and interaction, there is a range of interpretation occurring; often by people of events, but also of other people, of tasks, pedagogical approaches and media, curriculum and schemes, environment etc. In fact, all occurrences involve a complex array of interpretation nested within the influences from which they arise and the layers of interpretation associated with each of those influences. While giving recognition to that complexity, and taking those influences into account, my gaze was narrowed to a localised hermeneutic circle in which the situation of groups of learners engaging mathematical phenomena through the pedagogical medium of the spreadsheet was considered, and how individual understanding might emerge from the interplay of those various filters. This recursive process involved the shifting of the learner's focus from the "whole" (the learner's preconceptions or broader prevailing discourses in the associated domains) to the "part" (the mathematical phenomenon), with interpretation and explanation occurring at ensuing iterations. In this version, the one privileged by this research, the learner interpreted the mathematical phenomena from the perspective ascribed by their underlying discourses and preconceptions, then engaged with it through the pedagogical medium of the spreadsheet. The ensuing activity and dialogue modified the students' perceptual frame to varying degrees, and they re-engaged with the phenomenon from this modified perspective. The layering of the interpretation of these temporary fixes led to an emerging understanding. Hermeneutics can be understood as the manifestation and restoration of meaning that a person makes sense of in a personal way. In essence, the learner's prevailing discourses, the pedagogical medium, and the mathematical phenomenon are inextricably meshed. It is from their interplay with the students that the understanding develops.

When this perspective was settled upon from the personal transformative process that had been undergone (see Chapter Seven), the data were then re-examined through this re-envisioned lens. The next section draws on the findings from that analysis as the manner in which the spreadsheet environment fashioned a transformation of the students' approach to investigating mathematical phenomena is discussed. The associated corollary of the potential for the reorganisation of the students' thinking and understanding through offering alternative affordances and learning trajectories to other pedagogical media is central to that discussion.

The reorganisation of mathematical thinking and understanding

The spreadsheet environment reshaped the students' approaches and the manner in which they traversed their actual learning trajectories, by the particular nature of their experiences while working within that environment. It allowed them to engage in alternative processes and to envisage their interpretations and explanations from fresh perspectives. The mathematising facilitated by the medium was transformed by the visual, interactive nature of the investigative process. They used visual elements in their reasoning, while their explanations were punctuated with visual referents, such as the position and visual pattern of the digits. As such, the generalisations that emerged were couched in visual terms. They interpreted and explained their reasoning in alternative ways. There was a visual perspective to their mathematical thinking, while the visual tabular structure enhanced the possibility of seeing relationships in ways that might otherwise have been unattainable or inaccessible. Coupled with other affordances, such as the increased speed of the feedback, this visual dimension expanded the boundaries of what constituted mathematical knowledge, and gave students access to ideas earlier than teachers' usual expectation. It allowed a shift in focus from calculation techniques to a focus on mathematical thinking and understanding. Modelling the situations with various representations, and the capacity to think mathematically and generalise enhanced by the simultaneous viewing and translation between these alternative forms, also fostered the reorganisation of the learners' thinking.

The type of tasks that could be engaged with was expanded by the spreadsheet's facility to calculate large amounts of data accurately (e.g., with the Rice Mate activity), enhancing the opportunity for alternative forms of investigation to emerge, with different interpretations. The designing of software with the deliberate intention of shaping the learning in particular ways (e.g., Fathom or Cabri 3D) also recognises the influence that digital pedagogical media exert on the interpretation and organisation of meaning. The novelty of the experience, and this expansion of the learning situation allowed learners the opportunity to unshackle their thinking from the fetters of their previous accumulation of mathematical learning, and re-envision their interpretation from fresh perspectives, hence offering the possibility for the reorganisation of their thinking.

Another aspect the data highlighted regarding the reorganisation of thinking, was the nature of the students' initial engagement. Their approach was distinctive from the students in the classroom situation in that they immediately explored symbolic and tabular models of the situation - frequently with multiple, structured output, rather than a single numerical example. This framed the subsequent investigation of the mathematical activities, flavouring the investigative process and the explanations with this distinguishing perspective. Their dialogue also contained phrases and meanings particular to the medium. Investigating by processes such as *Fill Down* or using a spreadsheet formula, offered an alternative exploratory landscape with potential for the understanding to emerge in restructured ways. The speed and varying representations of feedback were also influential in the rearrangement of the students' methods and restructuring of the manner in which their learning trajectories and understandings evolved.

A particular element of this reorganisation of thinking and understanding that the research revealed was concerned with the notion of visual perturbation. While cognitive conflicts have been discussed in previous research (e.g., Kieren & Drijvers, 2006), the initiation of cognitive tension through the actual visual output differing from that which the students expected doesn't appear to have

been documented. When the students anticipated an output suggested by their preconceptions, and the actual output produced differed, a tension arose. There was a gap between the expected output indicated by the learner's preconceptions, and the actual visual output produced by the pedagogical medium. The data were indicative of this visual perturbation evoking dialogue, and mathematical conjecture and reasoning of a distinctive nature, hence permitting a reshaping of the students' perspective, and the consequential potential for the reorganisation of their thinking and understanding. Analysis of the data through a local hermeneutic circle (see Chapter Ten) differentiated between the versions of this notion of visual perturbation, revealing the varying features and illustration by the data of five distinctive types. These were when the visual perturbation: led to a change of prediction; caused a reshaping of the conjecture or generalisation; made students re-negotiate the sense-making of the task; was associated with an idea or area students hadn't experienced previously; or led students to further investigate and reconcile their understanding of a technical or formatting aspect associated with their exploration. While these types of visual perturbations could be distinguished, they weren't mutually exclusive, and often occurred in interrelated and mutually influential ways.

The study also gave insights into how envisioning the data through a hermeneutic lens illuminates the linkage between the emergence of mathematical understanding for the individual and the associated evolution of mathematics. By conceiving the cultural formation of mathematics as a hermeneutic process, the individual engagement and interpretation inform this broader interpretive cycle as the mathematical discourses evolve. Extending the boundaries of mathematics through the filtering of alternative pedagogical media also influenced the individual research trajectory, as methods were sought to give insights into the various interpretations. When a hermeneutic perspective frames the research process, the engagement of individual research practices, and the interpretations they induce, influence the evolution of mathematics education research. These elements are mutually constitutive of each other and develop in an inter-related manner.

In summary, the research project undertaken for the purposes of this thesis revealed several fresh approaches or perspectives to knowledge production, while introducing some new knowledge in domains associated or contributory to the research. They were:

- Envisaging the learning process, when mathematical phenomena were engaged through the pedagogical medium of the spreadsheet, through a moderate hermeneutic frame.
- Application of a localised hermeneutic circle to situations where the digital technology of the spreadsheet acted as the pedagogical medium.
- Examination and identification of the affordances of spreadsheets for learners investigating number phenomena and how they influenced actual learning trajectories and understanding.
- Examination and identification of commonalities between the affordances and associated learning trajectories of spreadsheets as a pedagogical medium, as compared to other digital media.
- Investigation of the ways primary school-aged children used spreadsheets to engage with mathematical phenomena, number investigations in particular.
- Discussion of the interplay and relationships between the digital media and the learner: The way this symbiotic relationship emerged within the spreadsheet environment.
- Application of a localised hermeneutic circle to research as a transformative process.
- Discussion of the connectedness between the emergence of personal mathematics understanding, the cultural formation of mathematics, the transforming of an individual's research trajectory, and the evolution of mathematics education research, when they are perceived as hermeneutic processes.
- Examination of the manner in which investigative sub-goals emerged in the spreadsheet environment, and the influence of this on the actual learning trajectories and consequential interpretations and understanding.
- Evaluation of the ways cognitive tension emerged in the spreadsheet environment, identification of visual perturbations, and differentiation

(with description and illustration) of various versions of visual perturbation.

The aims and research questions of the research project were not to isolate causal variables in order to predict behaviour, nor to establish direct relationships between the influences of spreadsheets acting as a pedagogical medium, and the understandings that might emerge. In my considered opinion, this is would not be desirable and would be most difficult, if not impossible, given the complex relational environment of the classroom, with its plethora of associated, constitutive discourses. The purpose was to further inform how utilising digital technologies, in this case the spreadsheet, as pedagogical media might influence the learning process and understanding in mathematics education. There were limitations associated with the methodology and methods employed, which are discussed in the following section.

Limitations

With any perspective employed, and with any methodology used to frame the research process, there are opportunities enabled, but accompanying constraints that are associated with the framework utilised and the approaches engaged. The researcher's noticing and selection with regard to these overriding influences are governed by their desire to maximise the opportunities to best examine the research questions and the research situation. It is important, however, to acknowledge the manner in which these decisions might constrain the research process and limit, or reveal nuances in, the articulated findings. In this section, the constraining influences of the interpretive methodology will be considered, followed by a discussion of the limitations related to particular approaches to data collection and analysis, and some acknowledgment of the justification and rationale for researching within those constraints.

A key element to the limitations associated with research undertaken through an interpretive lens is the subjectivity of the researcher. Implicit to any interpretation is the researcher's perspective. Any interpretation will be the researcher's version of events, and as such framed by his/her prevailing discourses in the related contributing areas. There are multiple versions and interpretations of situations with the researcher's evolving perspective crucial to the viewpoint from which any explanations and perceptions are framed. While striving to employ methods that best examine the research questions, the researcher nonetheless determines the research questions, the aims, and methodology, the research settings and participants. The whole underlying structure of the research was imbued with the flavour of the researcher's perspective. To some extent the research and analysis are self-fulfilling. They are the filtering of personally-held perspectives on practice through prevailing discourses. The selection of the literature was guided by the predispositions of the researcher, supervisors and other influential participants in the research process, such as colleagues or the presenters and participants at conference papers that the researcher has chosen to attend. Everything the researcher engaged in the research process was historically positioned; a function of how he thought in past renditions of the research process. As Mason (2002) has suggested, given that data are a construction of the researcher and that their perspective is constantly evolving, they may never compose an identical version of any situation again.

The researcher and his/her underlying prevailing discourses has determined what was noticed in all facets of the research process. From the literature, to the observations, to the interpretation and explanations, the researcher is selective about what s/he privileges; the features s/he brings to the foreground. What the researcher held on to as he sought to make sense of situations was determined to some degree by his underlying constitutional influences, and shaped the explanations and analysis. What is noticed is also influenced by the supervisors' predispositions, and those of others who interact with the researcher such as reviewers, audiences etcetera, with their perspectives likewise orchestrated by their prevailing discourses in the associated domains. By using unstructured observations and interviews, the

selectivity arises to a certain extent from the situation rather than being predominantly determined by the researcher. With an interpretive frame, the interpretations of the situation are constrained by the participants' accounts and viewpoints, with an accompanying potential for partiality or lack of completeness through the limitations of the emergence of their positions.

As well, language is the vehicle of the interpretation and explanations, and language is inherently partial and dependent on the perspectives of all the participants in the dialogical encounters. Language is imbued with the connotations ascribed to it by both the actors and the audience. There is a constant interplay of interpretation between the speaker/writer and listener/reader so a limitation of the data that comprised spoken or written phenomena was that the meanings of the language may not have been the same as those intended. The influence of the presence of the researcher would also have had bearing on the interactions in the form of the action and dialogue that occurred. Although the participants were familiar with me from previous encounters, there would still have been some expectations about the engagement associated with my presence. The fact that they had worked with spreadsheets previously with me, and the availability of computers in the settings involved, may have been suggestive in the nature of the interaction with the mathematical phenomena. The types of tasks selected were suitable for exploration by the spreadsheet and the students may have intuitively recognised that suitability, or recognised the similarities in the design of the tasks to others they had done. They were new tasks to the students, however, and whatever the nature of the tasks they would have interpreted them through the lens of their preconceptions.

There were limitations associated with influences involving the differentiation of power in classroom situations not only between the participants and myself, but also between the members of participant groups. In several instances with the data, one member of the group took the lead, with the interactions observed and articulation of ideas being primarily the interpretation of those individuals rather than a negotiated consensus of the group. This may have limited the completeness of the data, or caused bias.

There were also constraints with the accessibility to the medium associated with this as a dominant individual may have controlled the computer and constrained the approach intended by other group members.

There were other limitations associated specifically with some of the methods employed. Limitations of the rating scales-style questions in the questionnaire, for instance, include the tendency for respondents to avoid extreme responses, the questions being promulgated by the researcher and therefore possibly not giving full scope to the participants' views, and the connotations the participants gave to the language used and the intervals ascribed. Another perceived limitation might have been that the participants were determined by convenience and availability rather than attempting to obtain representativeness. In terms of the intentions of the research and the nature of the research questions, this was not considered unduly problematic however. The Otago Problem Challenge data had limitations in terms of the problem-solving work had generally being done in collaborative settings while the challenge was an individual task. The questions selected were constrained by the perspectives and organisational constraints imposed by the administrators, while for the spreadsheet work involving these activities I had again made considered but subjective decisions on the suitability of the tasks, perhaps influenced by the type of activities I had used with the students previously. This offered potential for the students to recognise the style of the tasks, perhaps indicating the type of approach they might have chosen to pursue. There were constraints associated with attempting to reconcile quantitative data within an interpretive frame, but while no attribution of causality was attributed to the outcomes of the analysis of this data, the data were nonetheless informative of emerging perspectives that the varied accounts rendered. It was supportive of a tentative emerging picture that informed the examination of the research questions.

While there were limitations associated with the approaches taken to investigate the research questions, it is important to consider them in light of the intentions of the research. The research was situated in classrooms and recognition was given to the nature and complexity of these educational

settings. The intention of the research was to seek further understanding and insights into the ways learning emerges rather than to isolate variables in the pursuit of the production of predictive generalisations. While generalisations and patterns did emerge in the data they were historically situated within the settings they occurred. The data couldn't be displaced from the context and culture in which they were generated without compromising the integrity of the generalisations described. The understanding of the interconnected features and the understandings that emerge from the interrelationships between students, teacher, researcher, pedagogical medium, mathematical phenomena, and the learning environment are impoverished if the learning situations are fragmented and removed from the social reality in which they exist. The purpose of the research was to allow some sense making and clearer understanding of the ways students learn when engaging mathematical tasks through the pedagogical medium of the spreadsheet. The research questions were concerned with explanation and clarification through the patterns and generalisations that emerged rather than causal relationships between perceived realities.

Implications

The data indicated that the learning experience was different when the students investigated mathematical phenomena through the pedagogical medium of the spreadsheet. The particular characteristics of the experience and the opportunities afforded by the medium gave scope for alternative learning trajectories to emerge. In conjunction with other contributory aspects that influenced the learning process, this offered opportunities for the re-envisioning of ideas and thinking, allowing students to approach the tasks and think in alternative ways. Also evidenced by the data was the propensity of the medium to expand the boundaries of what constitutes school mathematics. With one perceived aim of the classroom teacher, and the education system in general, being to optimise learning opportunities for the students, an implication of the research would be to make spreadsheets and other digital technology (e.g., *Tinkerplots*) available for all students for the investigation

of mathematical phenomena. This would be to complement the engagement through other media, rather than replacing them. It would have implications in terms of software, accessibility, and professional development.

One of the reasons for examining the spreadsheet rather than other digital technologies was accessibility. It is part of the general software bundle that is available on most computers, especially with Microsoft office being the pre-eminent generalist software offered. Other software, including *Tinkerplots*, for exploratory data analysis, and *Cabri-geometry*, for dynamic geometry exploration are relatively expensive for site licences and within the context of tight financial budgets, the expense for the amount of use might not be as beneficial comparative to other school investment. This does not detract from the educational benefits associated with such software, but reflects the reality of school boards making financial decisions on limited budgets. At both the school and Ministry of Education level, there is scope for decisions to be made about the utilisation of appropriate software and the ways centralised approaches to expenditure might augment the opportunities for schools to purchase the optimal resources, including digital technologies, for enhancing the learning and understanding of their pupils. Linked to this would be the necessity for accompanying professional development for teachers to enable them to recognise and optimise the learning associated with the affordances of the software. This would need to be well-resourced ongoing, co-constructed professional development situated within the contexts of the classrooms and mathematics education, if it were to make sustainable transformations.

Modification or perhaps revolution of the nature of school mathematics tasks and hypothetical learning trajectories would need to occur to make better use of and reflect these alternative learning opportunities. There would also be ramifications for assessment, both formative and summative. Another implication of the research might be the development of further software, or the evolution of existing ones, that give recognition to the affordances identified using digital technologies as learning media. For example, the way *Cabri-3D* enhances the learning situation through utilising the visualisation

and dynamic manipulation affordances within the software. Perhaps the evolution of spreadsheets so they contain hexagonal rather than rectangular cells (Mason, 2005) would permit the iterative process inherent to *Fill Down/Fill Right* to be applied in more than two directions or to other mathematical processes. Further use of the internet including the growth and promotion of dynamic applets and more global learning communities, beyond the confines of the traditional classroom, are further opportunities that might evolve with the infusion of more digital technology into school mathematics programmes.

On a broader level, the impact of digital technologies on society and investigative processes in general offers scope for the changing of the nature of some elements of mathematics and mathematical thinking per se. While there is recognition in some quarters of the mathematics community, that some evolution has already occurred (for instance, the emergence of visual reasoning as a 'legitimate' form of mathematising) there is certainly no consensus within that community regarding this aspect, nor orchestrated intention to explore the boundaries of such possibilities. In the domain of mathematics education, digital technologies are given greater privilege, although their potential use in the classroom is still only partially realised. Modelling is one aspect of mathematics education that might be given greater primacy in both the content and pedagogical areas. The nature and immediacy of feedback, which was featured in the analysis, enables the successive refinement of informal conjectures and solutions. Perhaps there will be an emergence of mathematical thinking more centred about refined guess-and-check approaches.

This research project addressed the research questions and further informed the web of knowledge regarding the use of digital technologies as pedagogical media in the learning of mathematics. It provided insights into this relatively recent, yet emerging element of mathematics education. It also invoked possibilities for research that might expand this domain further. More research into the use of digital technologies in primary school settings would extend the understanding of their influence on the learning process. As

well, research needs to be undertaken with students whose mathematical experience has always had ICT available as a pedagogical medium. The impact of technologies that are used outside school settings; for example, for games, on mathematics understanding in various areas would also extend existing knowledge in this domain.

The data were illuminating with regards to the reshaping of learning trajectories and the reorganisation of mathematical thinking and understanding, which offers potential to further enhance or expand the mathematical experience as per the graphical approaches used to develop alternative visual understandings in calculus. Amidst the optimism engendered by the affordance of visual representation offered by digital medium, some researchers warn of the noise created by the rapid increase in pre-fabricated and learner-generated visual images (e.g., Mason, 2005). They fear it may constrain the human faculty to mentally image, to think in terms of images. How the images are used, and are connected to pedagogical intentions is a consideration to monitor in this regard. Just how the learning environment containing digital media might mediate the social structures within the classroom and how this might influence the ways teachers maintain participation in the classroom (the social knowledge web referred to by Sinclair and Jackiw, 2005) are implications that also require consideration.

The manner, in which mathematics education research evolves from cyclical interactions with individual research processes, as mathematics is engaged through digital pedagogical media, also requires attention. As mathematics emerges from alternative frames, and research processes transform under the gaze of those various perspectives, the ways we examine the formations of mathematical knowledge and mathematics education research, will widen through ongoing engagement, interpretation, and evaluation. Research underpinned by a hermeneutic frame, in its various manifestations, or involving the application of the hermeneutic circle to learning or the research process, would further enhance knowledge of the learning process and the ways understanding emerges for the individual student or researcher. The ways in which digital technologies permeate our interaction with our world is

predicted to grow exponentially. The interpretations we make of situations will be influenced by the interplay of this, and other, filters. At present, there is a relatively open horizon of opportunity for the evolution of both understanding and the ways we produce knowledge. Ongoing research allows the opportunity to feed-forward into this evolution in an informed manner.

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APPENDICES

APPENDIX A: Example of a spreadsheet activity.

Dividing 1 by the Counting Numbers

When we divide 1 by 2, we get 0.5, a terminating decimal.
When we divide 1 by 3, we get 0.33333…, a recurring decimal.

Investigate which numbers, when we divide the number 1 by them, give terminating, and which give recurring decimals.

Figure 4: Dividing one by the counting numbers task.

APPENDIX B: Example of Otago Problem Challenge SET.

Problem Challenge 2002

SET TWO

Time allowed – 30 minutes

Administration Day – May 21

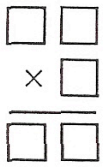
1. Helen has had her cat since it was a kitten. She said, "If you take half my cat's age and add 3 you get 10."

   How old is her cat?

2. I have $9.25 in my pocket in 5c, 10c, 20c, 50c, and $1 coins. I have an equal number of each type of coin.

   How many coins do I have in my pocket?

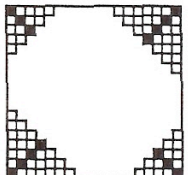
3. Place the numbers 1, 2, 3, 4, 5 into the squares shown on the right, so that each number goes into a square, and the multiplication shown is correct.



4. The area of a square is 64 square centimetres. A rectangle has the same perimeter as the square. The length of the rectangle is three times its width.

   What is the area of the rectangle in square centimetres?

5. A square floor was covered with square tiles of the same size. Black tiles were used on the diagonals, and white tiles were used everywhere else. The picture on the right (which is not drawn to scale) shows the floor after some of the tiles were laid. When the floor was finished 41 black tiles were used altogether.



APPENDIX C: Interview questions.

The interviews will be semi-structured and therefore the questions below are initial questions only, but are indicative of the types and tones of questions that might be used. Prompts will also be used to follow up the students' responses.

- Have you used spreadsheets for doing maths before or seen anyone else using them?
- When you saw the problem, how did you think you would start?
- What were the maths ideas the spreadsheet helped you with most?
- What type of activity did you find them most useful for?
- Did it make any work harder? If so, what did using the spreadsheet make harder?
- Did you find using the spreadsheets more enjoyable than doing number problem solving in class? In what ways was it more enjoyable?
- Could you make up questions or activities for others that would be good for using spreadsheets? What type would they be?
- Is there anything else about using the spreadsheets that you found interesting or would like to share?

APPENDIX D: Questionnaire.

Questionnaire for Year 6 Students

1 **Could you do the activities with the spreadsheets?**

- **without any help** ☐
- **with a little help** ☐
- **with a lot of help** ☐
- **could not do them** ☐

2 **If you needed help to do them, what was it you needed help with?**

_____

3 **Did the activities with the spreadsheets help you understand some of the maths?**

Yes ☐ **No** ☐

If Yes, what maths did they help you with?

_____

4 **How did you use the spreadsheets to solve the problems?**

_____

5 **Which parts did you find most useful?**

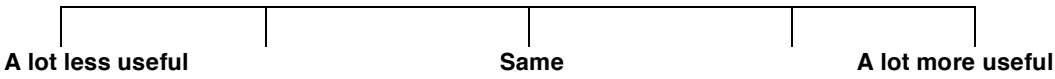
_____

6 **Did you enjoy working with the spreadsheets?**

Yes ☐ **No** ☐

7 **Which parts did you enjoy the most?**

_____

8 **How did you find the spreadsheet activities compared to the other problem solving activities?**

Mark where you think you were compared to your other maths problem solving lessons.