Adversarial Risk Analysis for First-Price Sealed-Bid Auctions

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by

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Abstract

First-price sealed-bid (FPSB) auctions have mostly been modelled in auction theory using the decision-theoretic and Bayesian game-theoretic approaches. From a practical point of view, each approach has its limitations. To overcome those limitations, Ríos Insua et al. (2009) introduced an approach, called adversarial risk analysis (ARA) that provides an optimal solution for one of the intervening agents, based on a decision making problem in hand and treating the intelligent adversaries' decisions as uncertainties. ARA solutions for FPSB auctions have previously been found but only under strong assumptions which make the model somewhat unrealistic.

In this thesis, we use ARA methodology and model bidders' behaviours in FPSB auctions using more realistic assumptions. First, we model bidders' behaviours by defining a new utility function that considers bidders' wealth which is assumed to be different for each bidder. We consider bidders' wealth since it is a significant determinant of their bidding behaviour in these auctions. Also, we define new risk behaviour parameters that change with the relative change in circumstances of bidders' wealth. In our modelling, we assume that the auctioned item is normal and has a reserve price which is known in advance to each bidder. We find ARA solutions not only for risk-neutral but also for risk-averse as well as risk-seeking bidders. We model these auctions by ARA framework using non-strategic play, level-k thinking, mirror equilibrium (ME) and Bayes Nash equilibrium (BNE) solution concepts. Finding ARA solutions using non-strategic play and level-k thinking, ME and BNE (asymmetric case) solution concepts, we assume two bidders. Whereas, we assume n bidders while finding ARA solutions using BNE symmetric case.

Second, we use ARA methodology and model bidders' behaviours using the utility function that takes into account bidders' winning and losing regret for the auctioned item. We define new winning and losing regret parameters and a modified utility function in order to take into account the effect of bidders' wealth on their bidding behaviours. Using the modified utility function, we find ARA solutions using non-strategic play and level-k thinking solution concepts assuming n bidders participating in these auctions.

We give numerical examples to illustrate our methodology.

Note on Publications

This thesis is by publications and the details of the publications are as follows:

Chapter 3: Published Online

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Note that there may be small inconsistencies in the notations because the Chapters have been reproduced as accepted/submitted for publications. Also, there is a reference list at the end of each Chapter.

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Contents

List of Figures

List of Tables

Chapter 1

Introduction

An *auction* is a sale in which goods (items), property or services are sold to the highest bidder. Most of auction theory revolves around the four basic auction types that are used for the sale of a single item:

1. *First-price sealed-bid (FPSB) auctions* in which bidders place their bids in sealed envelopes and hand over them to the auctioneer. The envelopes are opened and the individual with the highest bid wins and pays the amount equal to the bid.

2. *Second-price sealed-bid (SPSB) auctions* (Vickrey auctions) in which bidders place their bids in sealed envelopes and hand over them to the auctioneer. The envelopes are opened and the individual with the highest bid wins but pays a price equal to the second-highest bid.

3. *Open ascending-bid auctions* (English auctions) in which participants make increasingly higher bids, each stopping bidding when they are not prepared to pay more than the current highest bid. This continues until no participant

is prepared to make a higher bid; the highest bidder wins the auction at the final amount bid. In the case when the auctioned item has reserve price, it is only sold if the bidding reaches a reserve price set by the auctioneer.

4. *Open descending-bid auctions* (Dutch auctions) in which the price is set by the auctioneer at a level sufficiently high to deter all bidders, and is progressively lowered until a bidder is prepared to buy at the current price, winning the auction.

Many other types of auctions also exist in which one or more than one items are sold such as sequential auctions, multi-unit auctions, combinatorial auctions, general mth-price sealed-bid auctions, random mth-price sealed-bid auctions and all-pay auctions etc. For example, a *sequential auction* is an auction in which several items are sold, one after the other, to the same group of potential buyers. However, in this thesis, we consider only FPSB auctions and find solutions from one of the bidders' (the decision maker) view point.

1.1 Motivation

FPSB auctions have extensively been analysed using decision-theoretic and Bayesian game-theoretic approaches in the literature. However, both of these approaches have certain limitations that make these approaches unrealistic in practical competitive situations. Also, there has been considerable debate over the relative merits and demerits of these two approaches to model the real life competitive situations. This debate includes Kadane and Larkey (1982a,b) in favour of decision theory and Harsanyi (1982a,b) in favour of Bayesian game theory. For more comments on this debate, see, e.g., Rothkopf (1983); Roth and Schoumaker (1983) and Kahan (1983). Myerson (1991) also made a concise criticism on decision theory.

Ríos Insua et al. (2009) developed an approach called adversarial risk analysis (ARA) that has advantages over both the decision-theoretic and Bayesian game-theoretic approaches and has been applied in many real life competitive situations. Banks et al. (2015) used ARA for FPSB auctions assuming only risk-neutral bidders and found solutions using non-strategic play, minimax perspective, Bayes Nash Equilibrium (BNE), level-k thinking and mirror equilibrium (ME) solution concepts. However, they did not consider bidders' wealth in their modelling. Also, they did not consider that the auctioned item has a reserve price which is typically what occurs in FPSB auctions. In practice, bidders' wealth and reserve price of the auctioned item, both play a significant role in determining the decision maker's optimal bid. Also, it has been observed in many experiments performed under a variety of environments by different researchers that bidders in FPSB auctions bid more aggressively than the risk-neutral types. Therefore, it is important to model bidders' overbidding behaviours in these auctions while using an ARA framework. This has not been done previously.

1.2 Contributions in this Thesis

The main contributions contained in this thesis are as follows:

We consider a reserve price for the auctioned item (typically, known to each bidder in advance); we take into account bidders' wealth that is assumed to be different for each bidder; we assume that the auctioned item is *normal*[1]. Within this scenario, we

1. Extend Banks et al. (2015) by developing ARA solutions for non-strategic play and level-k thinking solution concepts assuming two bidders for a realistic case, wherein,

   a) we find solutions not only when the bidders are assumed to be risk-neutral but also when they are assumed to be risk-averse and risk-seeking bidders,

   b) we define a new utility function for the bidders,

   c) we define a new CRRA parameter to incorporate the effect of increase in wealth on the bidders' risk behaviours.

2. Extend Banks et al. (2015) and (Ejaz et al., 2021, or Chapter 3) by developing ARA solutions for ME and BNE (symmetric and asymmetric case) solution concepts, wherein,

   a) we find solutions not only for risk-neutral bidders but also for risk-averse and risk-seeking bidders,

   b) we extend Hubbard and Paarsch (2014) and find numerical solutions using the shooting algorithm for risk-averse and risk-seeking bidders.

[1]An item with positive income elasticity is defined as a *normal item* in economic theory, i.e., the demand for a normal item rises with an increase in income and falls with a decrease in income (see, e.g., Fisher, 1990; Goeree et al., 2002; Piros and Pinto, 2013; Perloff, 2015; Baisa, 2017, for more details).

3. Find ARA solutions using a utility function that incorporates bidders' winning and losing regret using non-strategic play and level-k thinking solution concepts assuming n bidders, wherein,

    a) we modify the utility function used by Engelbrecht-Wiggans and Katok (2007),

    b) we define new winning and losing regret parameters to incorporate the effect of increase in wealth on bidders' bidding behaviours.

1.3 Structure of the Thesis

The rest of the thesis is organised as follows. In Chapter 2, we briefly provide some literature on modelling FPSB auctions and bidders' bidding behaviours in these auctions. Chapter 2 also explains how an ARA approach works and the computational issues in finding the decision maker's optimal bids using certain solution concepts. In Chapter 3, we propose our new utility function and the new CRRA parameters. We assume two bidders, Brenda and Charles in Chapter 3 and derive the ARA solutions from Brenda's perspective using non-strategic play and level-k thinking solution concepts. In Chapter 4, we find ARA solutions from Brenda's perspective using BNE solution concept assuming n bidders and ME solution concept assuming two bidders. In Chapter 5, we assume n bidders and find ARA solutions using a risk-neutral utility function that incorporates bidders' winning and losing regret. Chapter 5 contains ARA solutions using non-strategic play and level-k thinking solution concepts. In Chapter 6, we provide the conclusion of our research work and some further work that could be done.

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Chapter 2

First-Price Sealed-Bid Auctions and Adversarial Risk Analysis

In this Chapter, we briefly provide some literature review on the approaches used to model first-price sealed-bid (FPSB) auctions and bidders' bidding behaviours in these auctions. This Chapter also explains how an adversarial risk analysis (ARA) approach works and discusses the computational issues while finding the solutions.

2.1 First-Price Sealed-Bid Auctions

2.1.1 Frameworks to Model FPSB Auctions

The *Bayesian game-theoretic* and *decision-theoretic* frameworks have widely been applied to model FPSB auctions in the literature. The latter framework considers the decision making problem from the decision maker's point of view where she wants to maximise her expected utility and believes that she is bidding against non-strategic bidders (playing against Nature) who might bid according to their true values for the item only. Friedman (1956) presented the first academic paper using a decision-theoretic approach on FPSB auctions. Further work using a decision-theoretic approach for auctions includes, competitive bidding in high risk situations (Capen et al., 1971), allocating bidding capital (Keefer et al., 1991), modelling competitive bidding (Rothkopf and Harstad, 1994), the right tool for auctions (Rothkopf, 2007), behavioural models for FPSB auctions (Wang and Guo, 2017) among others. In practice, bidders use decision-theoretic models rather than the game-theoretic models for auctions (Rothkopf, 2007).

In contrast, a Bayesian game-theoretic model assumes that each bidder strategically draws the value for the auctioned item from the same (symmetric auctions) or different (asymmetric auctions) but known distribution(s) to all bidders and all bidders seek a Bayes Nash Equilibrium (BNE) bidding strategy. Vickrey (1961) assumed risk-neutral (indifferent to risk when making a decision) bidders who draw their values independently and privately from a uniform distribution and found a bidding strategy using a Bayesian game-theoretic approach. That bidding strategy in auction theory is known as a risk-neutral Nash Equilibrium (RNNE) bidding strategy and is given by

$$b(v_i) = \frac{n-1}{n} v_i, \quad i = 1 \ldots, n,$$

where n is the number of bidders and v_i is the ith bidder's true value. Vickery's (1961) paper is considered to be a significant breakthrough in auction theory. Further work for auctions using a Bayesian game-theoretic model includes, study of bidding with unknown casts (Criesmer et al., 1967), competitive bidding with disparate information (Wilson, 1969), optimal auctions (Riley and Samuelson, 1981), theory and behaviour of single object auctions (Cox et al., 1982a), auction theory of heterogeneous bidders (Cox et al., 1982b), optimal auctions with risk-averse buyers (Maskin and Riley, 1984), asymmetric auctions (Maskin and Riley, 2000), semiparametric estimation of FPSB auctions (Campo et al., 2011), existence of monotone equilibrium in FPSB auctions (Gentry et al., 2015), hidden reserve price with risk-averse bidders (Li and Tan, 2017) among others.

However, in practice, bidders draw their valuations for the auctioned item from their own distributions which are not commonly known to all bidders and also the other bidders may be strategic. Therefore, the "common knowledge" and "non-strategic opponents" assumptions in Bayesian game theory and decision theory, respectively, make these two approaches unrealistic in real life competitive situations.

2.1.2 Modelling Bidding Behaviour in FPSB Auctions

Bidding behaviours in FPSB auctions have extensively been studied using experimental techniques. In these experiments, a consistent outcome found is that the bidders consistently bid above the RNNE bidding strategy (see e.g., Dorsey and Razzolini, 2003; Engelbrecht-Wiggans, 1989; Engelbrecht-Wiggans and Katok, 2007, among others). The reasons for this overbidding remain an unsolved puzzle (Engelbrecht-Wiggans and Katok, 2007). Several explanations have been given in economic literature to describe the overbidding behaviours of the bidders in these auctions. These explanations can generally be classified into four main categories: risk-aversion, interpersonal interaction, learning direction, and regret.

Risk-averse bidders are bidders who bid more aggressively than risk-neutral bidders. They have an aversion to losing the auctioned item to another bidder. Therefore, the bidders' overbidding behaviours have been explained by assuming that the bidders have a risk-averse utility function. A constant relative risk-averse (CRRA) utility function is the most commonly used function to explain this overbidding behaviour (Holt and Laury, 2002). A basic CRRA utility function for the ith bidder having wealth w_i is defined as

$$u_i(w_i) = w_i^{r_i}, \quad w_i > 0, \tag{2.1}$$

where, $(1 - r_i) = -w_i u_i''(w_i)/u_i'(w_i)$ is the coefficient of CRRA or Arrow-Pratt measure of relative risk-aversion (Pratt, 1964; Arrow, 1965). The utility function (2.1) is the utility function for a risk-neutral bidder when $r_i = 1$, it is the utility function for a risk-averse bidder when $0 < r_i < 1$ and it is the utility function for a risk-seeking bidder when $r_i > 1$, where risk-seeking bidders bid less aggressively than those of risk-neutral bidders (Holt and Laury, 2002). In principle, r_i can take any value greater than 1 i.e., $r_i > 1$ for risk-seeking. However, it has been argued that, in practice, r_i does not take values greater than 2 (Holt and Laury, 2002). Therefore, in this thesis, we consider $1 < r_i \leq 2$ when the ith bidder is assumed to be risk-seeking.

Lu and Perrigne (2008); Gentry et al. (2015); Li and Tan (2017) among others used (2.1) and defined the ith bidder's utility function having wealth w_i as

$$u_i(b_i, v_i, w_i) = \begin{cases} (w_i + v_i - b_i)^{r_i}, & \text{if she wins the bid,} \\ w_i^{r_i}, & \text{if she loses the bid,} \end{cases} \tag{2.2}$$

where, v_i is the ith bidder's true value for the auctioned item and b_i is her amount of bid. Cox et al. (1982a, 1985); Maréchal and Morand (2011), among others, also used (2.1) without considering bidders' wealth and defined the ith bidder's utility function as

$$u_i(b_i, v_i) = \begin{cases} (v_i - b_i)^{r_i}, & \text{if she wins the bid,} \\ 0, & \text{if she loses the bid.} \end{cases} \tag{2.3}$$

Note that (2.3) is the special case of (2.2) when $w_i = 0$. However, it has been shown that the utility function (2.2) is unrealistic because it will yield a positive utility even when $b_i > w_i$ (as long as $w_i + v_i \geq b_i$) in which case the bidder does not have the ability to buy the auctioned item (see, Ejaz et al., 2021a, or Chapter 3 for more details). Using (2.3), the expected utility of the ith bidder can be defined as

$$\Psi_i = (v_i - b_i)^{r_i} F_{ij}(b_i), \tag{2.4}$$

where, F_{ij} is the ith bidder's subjective distribution on the jth bidder's bid and the jth bidder is assumed to the bidder who has the maximum bid among the other $(n-1)$ bidders. The optimal bid b_i^* for the ith bidder using a decision-theoretic approach may be found by finding the turning point of Ψ_i in (2.4). We have

$$\begin{aligned} \frac{d\Psi_i}{db_i} =& - r_i(v_i - b_i)^{r_i - 1} F_{ij}(b_i) + (v_i - b_i)^{r_i} \frac{d}{db_i}(F_{ij}(b_i)) \\ =&(v_i - b_i)^{r_i - 1}\Big[(v_i - b_i)\frac{d}{db_i}(F_{ij}(b_i)) - r_i F_{ij}(b_i)\Big] \\ =&(v_i - b_i)^{r_i - 1}\Big[(v_i - b_i) f_{ij}(b_i) - r_i F_{ij}(b_i)\Big], \end{aligned}$$

where f_{ij} is the pdf of b_i. Equating the above equation to zero, we get

$$(v_i - b_i)f_{ij}(b_i) - r_i F_{ij}(b_i) = 0. \tag{2.5}$$

The above equation may need numerical methods to solve it for b_i.

Isaac and Walker (1985); Dufwenberg and Gneezy (2002); Morgan et al. (2003) among others gave another explanation that bidders bid over RNNE bidding strategy due to inter-personal interactions and comparisons in FPSB auctions.

Neugebauer and Selten (2006) found that most of the bidders adjust their bids in the same way as that of learning direction theory proposed by Selten and Stoecker (1986). This theory leads to the direction in which the bids are likely to be adjusted based on the feedback over time. However, this theory does not give any explanation of bidders' initial overbidding.

Engelbrecht-Wiggans (1989) provided another possible explanation of overbidding in these auctions and defined bidders' utilities by a linear combination of their profit, winning and losing regrets. Engelbrecht-Wiggans and Katok (2007) also modelled the ith bidder's bidding behaviour by taking into account her winning and losing regrets while assuming that she has a risk-neutral utility. They defined the ith bidder's utility function from a bid b_i while having true value v_i as

$$u_i(b_i, v_i) = \begin{cases} (v_i - b_i) - \int_{b_j : \underline{v}_j < b_j \leq b_i} [\zeta_i + \eta_i(b_i - b_j)] f_{ij}(b_j | b_j \leq b_i) db_j, & \text{if she wins,} \\ - \int_{b_j : b_i < b_j \leq v_i} [\vartheta_i + \theta_i(v_i - b_j)] f_{ij}(b_j | b_j > b_i) db_j, & \text{if she loses,} \end{cases} \tag{2.6}$$

where b_j is the jth bidder's bid and is the maximum of the other $(n-1)$ bidders' bid and $f_{ij}(b_j)$ is the probability distribution on b_j that the ith bidder believes. In (2.6), $b_i - b_j$ is the excess amount of money if the ith bidder wins and her utility suffers by an amount $\zeta_i + \eta_i(b_i - b_j)$ where $\eta_i \geq 0$. Larger values of η_i means a higher winning regret to the ith bidder. Negative values of ζ_i allows some pleasure to the ith bidder in case of winning. On the other hand, if the ith bidder loses and the highest bid satisfies the inequality $b_i \leq b_j \leq v_i$, then the ith bidder misses an opportunity to win at a favourable price and her utility suffers by an amount $\vartheta_i + \theta_i(v_i - b_j)$ where $\vartheta_i, \theta_i \geq 0$. Engelbrecht-Wiggans and Katok (2007) defined the expected utility for the ith bidder as

$$\begin{aligned}\Psi_i = \big[(v_i - b_i) - \int_{b_j:\underline{v}_j < b_j \leq b_i} [\zeta_i + \eta_i(b_i - b_j)] f_{ij}(b_j | b_j \leq b_i) db_j\big] F_{ij}(b_i) \\ -\big[\int_{b_j:b_i < b_j \leq v_i} [\vartheta_i + \theta_i(v_i - b_j)] f_{ij}(b_j | b_j > b_i) db_j\big][1 - F_{ij}(b_i)].\end{aligned}$$

As, $f_{ij}(b_j|b_j \leq b_i) = f_{ij}(b_j)/F_{ij}(b_i)$ and $f_{ij}(b_j|b_j > b_i) = f_{ij}(b_j)/[1 - F_{ij}(b_i)]$, the above equation simplifies to

$$\begin{aligned}\Psi_i = (v_i - b_i)F_{ij}(b_i) - \int_{b_j:\underline{v}_j < b_j \leq b_i} [\zeta_i + \eta_i(b_i - b_j)] f_{ij}(b_j) db_j \\ - \int_{b_j:b_i < b_j \leq v_i} [\vartheta_i + \theta_i(v_i - b_j)] f_{ij}(b_j) db_j.\end{aligned} \quad (2.7)$$

Now, the optimal bid b_i^* for the ith bidder may be found by finding the turning point of Ψ_i in (2.7). Therefore, by applying Leibniz's rule to (2.7), we have

$$
\begin{aligned}
\frac{d\Psi_i}{db_i} =&(v_i - b_i)f_{ij}(b_i) - F_{ij}(b_i) - \Big[[\zeta_i + \eta_i(b_i - b_i)]f_{ij}(b_i)\frac{d}{db_i}(b_i) \\
&- [\zeta_i + \eta_i(b_i - \underline{v}_j)]f_{ij}(\underline{v}_j)\frac{d}{db_i}(\underline{v}_j) + \int_{\underline{v}_j}^{b_i} \eta_i f_{ij}(b_j)db_j\Big] \\
&- \Big[[\vartheta_i + \theta_i(v_i - v_i)]f_{ij}(v_i)\frac{d}{db_i}(v_i) - [\vartheta_i + \theta_i(v_i - b_i)]f_{ij}(b_i)\frac{d}{db_i}(b_i) + 0\Big] \\
=&(v_i - b_i)f_{ij}(b_i) - F_{ij}(b_i) - \Big[[\zeta_i f_{ij}(b_i) + \eta_i F_{ij}(b_i)]\Big] \\
&- \Big[- \vartheta_i f_{ij}(b_i) - \theta_i(v_i - b_i)f_{ij}(b_i)\Big] \\
=&\big[(-\zeta_i + \vartheta_i) + (1 + \theta_i)(v_i - b_i)\big] f_{ij}(b_i) - (1 + \eta_i)F_{ij}(b_i).
\end{aligned}
$$

Now, dividing both sides of the above equation by $1 + \theta_i$ and substituting $\varphi_i = (-\zeta_i + \vartheta_i)/(1 + \theta_i)$ and $\rho_i = (1 + \eta_i)/(1 + \theta_i)$ gives

$$
\frac{d\Psi_i}{db_i}/(1 + \theta_i) = \big[\varphi_i + (v_i - b_i)\big] f_{ij}(b_i) - \rho_i F_{ij}(b_i).
$$

Equating the above equation to zero, gives us the first order condition

$$
\big[\varphi_i + (v_i - b_i)\big] f_{ij}(b_i) - \rho_i F_{ij}(b_i) = 0. \tag{2.8}
$$

Now, if $-\zeta_i + \vartheta_i = 0$ (i.e., $\varphi_i = 0$), then (2.8) would be exactly the same as that of (2.5), the first order condition for a CRRA bidder with risk aversion parameter $r_i = \rho_i$, which leads to the following observations:

- If $\eta_i = \theta_i$, i.e., $r_i = \rho_i = 1$, then b^* would be the optimum bid of a risk-neutral bidder.

- If $\eta_i < \theta_i$, i.e., $0 < r_i = \rho_i < 1$, then b^* would be the optimum bid of a risk-averse bidder. The level of risk-aversion would increase as $\rho_i \to 0$.
- If $\theta_i < \eta_i \leq 1 + 2\theta_i$, i.e., $1 < r_i = \rho_i \leq 2$, then b^* would be the optimum bid of a risk-seeking bidder. The level of risk-seeking behaviour would increase as $\rho_i \to 2$.

However, Engelbrecht-Wiggans and Katok (2007) in (2.6) did not take into account the ith bidder's wealth and therefore assumed $w_i = 0$. Whereas, using (2.6), the ith bidder can find her optimal bid $b_i^* > 0$, which is greater than her assumed wealth $w_i = 0$. This is unrealistic because, in practice, $b_i^* \leq w_i$. Therefore, considering bidders' wealth is a significant determinant of bidders' bidding behaviour in these auctions (see e.g., Gentry et al., 2015; Ejaz et al., 2021a, among others). Moreover, assuming that the ith bidder have wealth w_i such that $w_i \geq b_i^*$ is a realistic assumption to be made. Also, using such type of utility function, FPSB auctions have not been modelled by an ARA framework previously.

2.2 Adversarial Risk Analysis

To overcome the limitations of the Bayesian game-theoretic and decision-theoretic approaches that we have discussed in the previous section, Ríos Insua et al. (2009) introduced an approach called *adversarial risk analysis* (ARA). This approach models competitive decision-making problems such as war, politics, cyber-security, counter-terrorism, auctions etc., from one of the players' (the decision maker) point of view only in the presence of intelligent adversaries. Using an ARA framework, the decision maker may believe that her adversaries are strategic and finds her optimal action by placing her subjective distributions to take into account the unknown preferences, beliefs and utilities of her intelligent adversaries. Unlike Bayesian game theory, ARA does not assume that these subjective distributions are commonly known to each player. Since, in ARA, the decision maker uses her subjective distributions to model the uncertainties around her adversaries' beliefs and preferences etc., it is a Bayesian approach. Including auctions, this approach has been applied to model a variety of real life competitive situations such as network routing for insurgency (Wang and Banks, 2011), counter-terrorism risk management (Merrick and Parnell, 2011), Borel games (Banks et al., 2011), the Somali pirates case (Sevillano et al., 2012), counter-terrorism modelling (Rios and Ríos Insua, 2012), autonomous social agents (Esteban and Ríos Insua, 2014), pricing strategies with remanufacturing (Deng and Ma, 2015), urban security resource allocation (Gil et al., 2016), adversarial classification (Naveiro et al., 2019), counter-terrorist online surveillance (Gil and Parra-Arnau, 2019), cyber-security (Rios Insua et al., 2021), insider threat (Joshi et al., 2020) and FPSB auctions (Banks et al., 2015), (Ejaz et al., 2021a,b, or Chapter 3,4 respectivley) among others.

In particular, ARA addresses three different kinds of uncertainties:

1. *Aleatory uncertainty*, which is about the randomness of outcomes conditional on the actions taken by all the opponents of the decision maker including herself. However, for auctions, aleatory uncertainty arises when the decision maker does not have a full knowledge about the condition of the auctioned item. Therefore, she could have an uncertainty on her own true value. In

that case, her true value would be a random variable that could follow a certain distribution elicited by the decision maker.

2. *Epistemic uncertainty*, which is about the strategic choices of intelligent adversaries, that may include unknown preferences, beliefs, and capabilities of each of the opponents. For auctions, epistemic uncertainty arises when the decision maker is uncertain about her opponents' private values, wealth, risk behaviour parameters or regret parameters etc.

3. *Concept uncertainty*, which is about the decision maker's belief how her adversaries frame the decision making problem. It is about the assessment of solution concept (non-strategic play, level-k thinking, BNE or mirror equilibrium (ME) etc.) that each of her adversaries will use. For auctions, the concept uncertainty is the same as that for other real life applications, i.e., it is about the solution concept.

A brief description of some common solution concepts is as follows:

- *Non-strategic play*, in which the decision maker believes that her adversaries' actions are independent of their opponents' actions (including the decision maker's action). For auctions, when the decision maker believes that her adversaries' bids are independent of their opponents' bids (including the decision maker's bid), she finds her optimal bid using this solution concept.

- *Level-k thinking*, when the decision maker believes that her adversaries are intelligent who might make their decisions by taking into account their opponents' preferences, utilities and beliefs. Then she could use a level-k thinking solution concept to find her optimal decision for a decision making problem in hand. In a level-k thinking solution concept, the decision maker being a level-k thinker believes that her adversaries are level-$(k-1)$ thinkers and each level-$(k-1)$ thinker believes that her/his adversaries are level-$(k-2)$ thinkers and so on. For instance, when $k = 1$, she believes that her opponent is a level-0 thinker, that is, a non-strategic player. A level-2 analysis means that the decision maker assumes that her opponent is a level-1 thinker, who believes that she is a level-0 thinker. A level-3 analysis means that the decision maker assumes that her opponent is a level-2 thinker who models her as a level-1 thinker and so on. Thus, the decision maker always thinks one level higher than her rivals in this solution concept.

- *Bayes Nash Equilibrium*, in which the decision maker believes that her adversaries are assuming that there is a great deal of common knowledge among all the participants of the decision making problem.

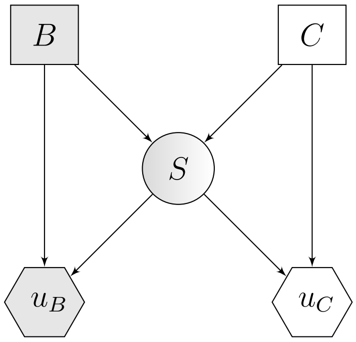
- *Mirror Equilibrium*, when the decision maker believes that her adversaries are modelling their opponents' (including the decision maker) actions in the same way as she is modelling their actions, then she could use a mirror equilibrium (ME) solution concept. In this solution concept, all players use their subjective distributions over the probabilities and utilities of their adversaries and seek an equilibrium.

A detailed explanation about non-strategic play and level-k thinking solution concepts is provided in Chapter 3 and those of BNE and ME in Chapter 4.

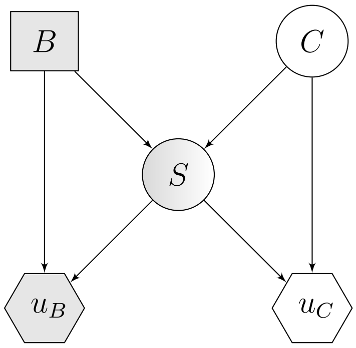
In order to solve a given decision making problem by an ARA approach, each of aleatory, epistemic and concept uncertainty needs to be modelled. At the first step, the decision maker has to handle the concept uncertainty. Epistemic uncertainty is dealt in the second step and aleatory uncertainty is considered to be modelled in the end. Each step bears its own challenges.

Now, we illustrate the ARA methodology by considering a two player game from one of the players' point of view named Brenda (B) who is playing against her opponent named Charles (C). Let b be the Brenda's choice that she makes from a set of choices $\mathcal{B}$. She believes that c is Charles's choice that he could make from his set of choices $\mathcal{C}$. From the choices, b and c, let S be the resulting random outcome that takes a value s from a set of possible outcomes $\mathcal{S}$. Let $u_B(b, c, s)$ and $u_C(b, c, s)$ be the utilities received by Brenda and Charles respectively, given the choices b, c and outcome s. A typical objective could be to find Brenda's optimal choice b^* that maximises her expected utility. This game is represented in the form of a bi-agent influence diagram (BAID) in Figure 2.1. In this diagram, the decision, chance and utility nodes are represented by rectangles, circles and hexagons, respectively. Figure 2.1a represents this game from both Brenda and Charles's perspective while Figure 2.1b represents this game from Brenda's perspective only where, she is uncertain about Charles's decision and thus his decision node is represented by a chance node.

Using an ARA approach, Brenda has to *integrate out* her concept, epistemic and aleatory uncertainties at the chance nodes and then to *maximise* her expected utility at the decision node. Typically, Brenda does not know about which solution concept Charles will use. But, she can place her subjective distribution over possible solution concepts that Charles will use based upon her past experience with Charles or some information from other sources. She can solve the decision making problem using each solution concept and can find her optimal decision for the respective solution concept. Then, she can find her optimal decision that



(a) The BAID for simultaneous game (game-theoretic perspective).



(b) The BAID from Brenda's point of view (ARA perspective).

Figure 2.1: The BAID for a two players' game.

maximises her expected utility using the weighted mixture of her optimal decisions found against each solution concept. Suppose that Brenda believes that Charles is an expected utility maximiser and therefore finds her optimal decision for the given game as follows.

She takes into account her uncertainty around c by placing her subjective distribution $p_B(c)$ and finds her expected utility as

$$\Psi_B(b) = \int \Psi_B(b, c) p_B(c)\, dc, \tag{2.9}$$

where $\Psi_B(b,c)$ is Brenda's expected utility while taking into account her uncertainty about the random outcome S given b and c and is defined as

$$\Psi_B(b,c) = \int u_B(b,c,s) p_B(s|b,c)\, ds,$$

and $p_B(s|b,c)$ is Brenda's subjective distribution that takes into account her aleatory uncertainty around the chance outcome S given b and c. Then, she finds her optimal decision b^* that maximises her expected utility in (2.9) as

$$b^* = \arg\max_{b \in \mathcal{B}} \Psi_B(b).$$

The major challenge using an ARA approach is to deal with the epistemic uncertainty, i.e., the uncertainty around $p_B(c)$. This uncertainty depends upon the solution concept that Brenda believes that Charles might use. In general, the epistemic uncertainties are complex to describe, even for simple solution concepts. In this illustration, we assume that Brenda believes that Charles might choose his action c^* that maximises his expected utility, just like her. Therefore, she may aim to find c^*, if she knows about Charles's utility $u_C(b,c,s)$, his probabilities $p_C(b)$ and $p_C(s|b,c)$. However, typically she does not know about these quantities and can model her uncertainty about Charles's utility through eliciting a random utility $U_C(b,c,s)$ and those of her uncertainties about his probabilities through random probabilities $P_C(b)$ and $P_C(s|b,c)$. Then, she can find Charles's random optimal choice C^* by using a backward induction approach similar to her own, i.e., first she will find Charles's random expected utility as

$$\begin{aligned}\Psi_C(c) &= \int \Psi_C(b,c)P_C(b)\,db \\ &= \int \left[\int U_C(b,c,s)P_C(s|b,c)\,ds \right] P_C(b)\,db.\end{aligned} \tag{2.10}$$

Then, she will find Charles's random optimal choice C^* that maximises his random expected utility in (2.10) as

$$C^* = \arg\max_{c\in\mathcal{C}} \Psi_C(c).$$

Finally, she will find her required distribution $p_B(c)$ on Charles's choice c as

$$\int_{-\infty}^{c} p_B(\varphi)d\varphi = \Pr(C^* \leq c).$$

In the above illustration, we have assumed that sets $\mathcal{B}$, $\mathcal{C}$ and $\mathcal{S}$ are sets that contain infinitely many points. If any of these sets is assumed to be of finite cardinality, then the corresponding integrals would be replaced by sums. This is how Brenda can find her optimal choice when she believes that her opponent is an expected utility maximiser.

2.3 Computational Issues

In this section, we describe computational challenges that arise using the Bayesian game theory, decision theory and the ARA approaches.

2.3.1 Bayesian Game Theory

To illustrate the computational challenges while finding a BNE solution, suppose that the bidders use the CRRA utility function (2.3). Then the BNE bid functions $b_i(v_i)$, $i = 1, \ldots, n$ for the asymmetric case (valuation distributions are assumed to be different for each bidder) could be found by maximizing each expected utility function with respect to its argument b_i. This yields the following system of *ordinary differential equations* (ODEs) (see e.g., Hubbard and Paarsch, 2014)

$$v_i'(b) = \frac{G_i[v_i(b)]}{g_i[v_i(b)]} \left\{ \left[\frac{1}{(n-1)} \sum_{s=1}^{n} \frac{r_s}{v_s(b) - b} \right] - \frac{r_i}{v_i(b) - b} \right\}, \quad i = 1, 2, \ldots, n, \quad (2.11)$$

where $g_i[v_i(b)]$ is the probability density function (pdf) of the inverse bid function $v_i(b)$. Let $\bar{b}$ be the common highest possible bid and $\underline{b}$ be the common lowest possible bid. Also, suppose that $\underline{v}$ and $\bar{v}$ are the common lower and upper support of $G_i[v_i(b)]$ respectively. Then, as $v_i(\underline{b}) = b(\underline{v}) = \underline{v}$, the following are the boundary conditions on the equilibrium inverse bid functions:

$$\begin{aligned} &v_i(\underline{b}) = \underline{v} \text{ (or equivalently } v_i(\underline{v}) = \underline{v}), && \text{Left boundary condition,} \\ &v_i(\bar{b}) = \bar{v}, && \text{Right boundary condition,} \end{aligned} \quad (2.12)$$

for all $i = 1, \ldots, n$. The solution to the system of ODEs (2.11) is needed which satisfies both of the boundary conditions on the inverse bid functions. In general, no closed-form solution of (2.11) exists and numerical methods are thus required. Marshall et al. (1994) found numerical solutions for asymmetric auctions by adopting the shooting algorithm assuming risk-neutral bidders which could be used to find a numerical solution for (2.11). Further work on numerical methods includes, comparing competition and collusion (Bajari, 2001), perturbation approach (Fibich and Gavious, 2003), asymmetric first-price auctions (Gayle and Richard, 2008), investigating bid preferences (Hubbard and Paarsch, 2009), asymmetric first-price auctions (Fibich and Gavish, 2011), equilibria in auction models with asymmetries (Hubbard and Paarsch, 2014) among others.

However, these works did not include numerical solutions when the bidders are risk-averse or risk-seeking. Moreover, the utility function (2.3) does not take into account bidders' wealth and also has other limitations (see Ejaz et al., 2021a, or Chapter 3). Further, Hubbard and Paarsch (2014) and others found the BNE solution under the common prior assumption, i.e., each bidder knows the valuation distributions of the other bidders and that this is common knowledge, which is also unrealistic.

2.3.2 Decision Theory

Finding the solutions using a decision-theoretic approach, the decision maker do not need to find any kind of equilibrium solution like BNE or ME, yet she may need to find solutions numerically, as in general, no close form solutions exist for these solution concepts also. For example, using a decision-theoretic approach, the decision maker can obtain (2.5) but this equation, in general, needs numerical solutions to solve for b_i. She can find solutions by using any of numerical methods such as bisection method, Newton's method or fixed-point iteration method etc. (see e.g., McNamee and Pan, 2013).

2.3.3 ARA

Using the BNE (asymmetric case) solution concept, the ARA approach assumes that the distributions $G_i[v_i(b)]$ on the inverse bid functions are not commonly known to each bidder (unlike Bayesian game theory). Even under this assumption, the decision maker essentially needs to solve (2.11) and have to face the same computational challenges as using Bayesian game theory. However, using the ARA approach, the decision maker eventually finds the distribution of her competitive bid after solving (2.11). Then she uses her own value and risk aversion parameter and finds the her optimal bid or expected optimal bid that maximises her expected utility.

Also, in general, no close form solution exists while finding ARA solutions using a ME solution concept because this solution concept seeks an equilibrium (equilibria in case of more than two bidders) too and numerical methods as described earlier in this section are thus required. For two bidders, the key calculations are same for both BNE asymmetric case and ME solution concepts but the assumptions and perspective are different (see e.g., Wang and Banks, 2011). However, using the ME solution concept in case of $n > 2$ bidders, the decision maker needs to find a numerical equilibrium solution by placing subjective distributions over the unknown quantities of the other $(n-1)$ bidders from each of her opponents' point of view. Thus, for each opponent, she needs to find $(n-1)$ numerical equilibrium solutions. Then she finds each of her opponents' belief about the distribution of each of his/her rivals' bids and finds their optimal bids. Then the decision maker finds the distribution of the maximum of her $(n-1)$ opponents' bid. Finally, she uses her own true value and risk behaviour parameter and finds her optimal bid that maximises her expected utility.

Also, finding ARA solutions using a non-strategic play or level-k thinking solution concepts, the decision maker has to bear essentially similar computational challenges as that of finding solutions using a decision-theoretic approach. For example, finding ARA solutions using a level-k thinking solution concept for a CRRA utility function of the form (2.3), the decision maker has to solve the similar equations such as (2.5) at level $(k-1)$ from each of her adversaries' perspective to find their optimal bids by taking into account the uncertainties such as their valuations and risk parameters. Then she finds the distribution of the maximum bid among other $(n-1)$ bidders from her level-$(k-1)$ analysis. Finally, she uses her own value and risk aversion and solves a computationally similar equation as (2.5) to find her optimal bid that maximises her expected utility. However, finding the ARA solutions for non-strategic play or level-k thinking solution concepts using a utility function (2.6) with bidders' regret are computationally more challenging than finding these solutions using a CRRA utility function (2.3).

In this thesis, we also consider the effect of bidders' wealth on their bidding behaviour by defining a new utility function. Therefore, the decision maker needs to take into account the uncertainties around her adversaries' wealth too in addition to their values and risk behaviours while using a certain solution concept (non-strategic play, level-k thinking, BNE or ME etc). Then, she uses her own true value, wealth and risk behaviour parameter and finds her optimal bid that maximises her expected utility. Also, we consider the effect of bidders' wealth on their bidding behaviour by defining a utility function that takes into account bidders' winning and losing regrets. Thus, the decision maker also needs to place her subjective distributions on her adversaries' wealth in addition to their values and regret parameters while using such type of utility function. Finally, she uses her own true value, wealth and regret parameters and finds her optimal bid that maximises her expected utility.

Moreover, as ARA takes into account the uncertainties such as bidders' valuations, wealth, risk behaviours parameters or regret parameters etc., Monte Carlo methods may be used to find solutions using a given solution concept.

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Chapter 3

Adversarial Risk Analysis for First-Price Sealed-Bid Auctions

In this chapter, we use ARA methodology to model FPSB auctions using more realistic assumptions. We define a new utility function that considers bidders' wealth, we assume a reserve price and find solutions not only for risk-neutral but also for risk-averse as well as risk-seeking bidders. We model the problem using ARA for non-strategic play and level-k thinking solution concepts.

3.1 Introduction

3.1.1 Decision vs Game-Theoretic Approaches

The first modern academic paper on *first-price sealed-bid* (FPSB) auctions was presented by Friedman (1956) and used a *decision-theoretic* approach. Further work using decision-theoretic models followed. Capen et al. (1971); Keefer et al. (1991) and Rothkopf and Harstad (1994) concluded that in practice, bidders use decision-theoretic models to decide upon their bids. Rothkopf (2007) and Wang and Guo (2017) provided several arguments as to why decision theory dominates game theory. Engelbrecht-Wiggans and Katok (2007) also used a decision-theoretic model for FPSB auctions assuming risk-neutral bidders while taking into account bidders' winning and losing regret.

In contrast, a large number of *Bayesian game-theoretic* approaches have been proposed for FPSB auctions as well. Vickrey (1961) analysed n bidders in FPSB auctions using Bayesian game theory with the values drawn from a uniform distribution with common support. This work was further extended by Criesmer et al. (1967) and Riley and Samuelson (1981). Wilson (1969) developed the first closed form equilibrium analysis with the common value model where, the value of the auctioned item is the same for all bidders. Cox et al. (1982a) and Cox et al. (1982b) generalised Vickrey's model to the case of risk-averse bidders. Maskin and Riley (2000) analysed FPSB auctions assuming the valuations of each bidder are drawn from commonly known different distributions. Myerson (1981); Milgrom and Weber (1982); Goeree et al. (2002); Bajari and Hortacsu (2005); Campo et al. (2011); Gentry et al. (2015); Li and Tan (2017) among others also used Bayesian game-theoretic models for auctions.

However, both these approaches have their drawbacks. While a decision-theoretic approach does not require the *common knowledge* assumption, it does assume that the other bidders are non-strategic. Assuming non-strategic bidders may be unrealistic because bidders may often be strategic. In contrast, a Bayesian game-theoretic model requires a strong common knowledge assumption that all bidders (who are considered to be strategic) draw their valuations for the auctioned item from commonly known distributions. The common knowledge assumption can also be unrealistic because the distribution used by one bidder is usually not commonly known to others. In fact, often the bidders try to keep their information secret so as to gain competitive advantage. Also, finding a Bayes Nash Equilibrium (BNE) becomes increasingly difficult as the games get more realistic (complex) and often it may be that a unique BNE solution to a given game does not exist.

3.1.2 Adversarial Risk Analysis

To overcome the shortcomings of both the decision-theoretic and the Bayesian game-theoretic approaches, Ríos Insua et al. (2009) introduced an approach called *adversarial risk analysis* (ARA) to model the decision making problems in the presence of an intelligent adversary such as those encountered in cyber-security, counter-terrorism, war, politics, auctions, etc. ARA is a Bayesian approach because subjective distributions are used to model the uncertainties about the outcomes and about the unknown preferences, beliefs and capabilities of intelligent adversaries. However, unlike Bayesian game theory, ARA does not require that these subjective distributions be commonly known. Another important difference to Bayesian game theory is that ARA aims to solve the decision making problem for just one of the players and does not aim to find an equilibrium solution for all the players. For this reason, finding an ARA solution is relatively less difficult, even for complex problems.

Since its introduction, it has been used to model a variety of problems such as network routing for insurgency (Wang and Banks, 2011), international piracy (Sevillano et al., 2012), counter-terrorism (Rios and Ríos Insua, 2012), autonomous social agents (Esteban and Ríos Insua, 2014), urban security resource allocation (Gil et al., 2016), adversarial classification (Naveiro et al., 2019), computational advancements (Ekin et al., 2019), reinforcement learning (Gallego et al., 2019), counter-terrorist online surveillance (Gil and Parra-Arnau, 2019), cyber-security (Ríos Insua et al., 2021), autonomous agents (Esteban et al., 2020), insider threat (Joshi et al., 2020) and military command and control decision making (Caballero et al., 2021).

We will illustrate how ARA works by considering a two player game between Brenda (B), and Charles (C). Let b and c be the choices they make from their respective sets of actions $\mathcal{B}$ and $\mathcal{C}$. The resulting outcome S is a chance variable which takes a value s from a set of possible outcomes $\mathcal{S}$. Let $u_B(b, c, s)$ and $u_C(b, c, s)$ be the utility functions that determine the utilities received by Brenda and Charles respectively, given a pair of actions (b, c) and outcome s. Suppose that we are solving the problem for Brenda. Then, a typical objective will be to find the optimal action b^* that maximises her expected utility. Figure 3.1 represents this game in the form of a bi-agent influence diagram (BAID), where rectangles, circles and hexagons represent decision, chance and utility nodes, respectively. Figure 3.1a represents this game from both Brenda and Charles's point of view while Figure 3.1b represents this game from Brenda's point of view only where, Charles's decision node is now a chance node for Brenda because she is uncertain about his actions.

The basic idea behind finding an ARA solution is to *integrate out* the uncertainties at the chance nodes and then to *maximise* the expected utility at the decision nodes. When solving the problem for Brenda, we have two chance nodes, namely C and S, and the sole decision node B. ARA can be solved using backward

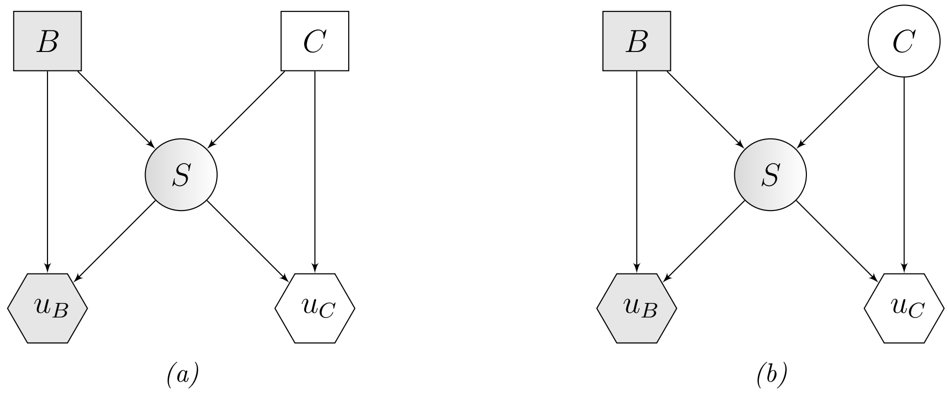


Figure 3.1: (a) The BAID for simultaneous game. (b) The BAID from Brenda's perspective.

induction for Brenda as follows.

We first find her expected utility by taking into account her uncertainty about the outcomes S

$$\Psi_B(b, c) = \int u_B(b, c, s) p_B(s|b, c)\, ds,$$

where $p_B(s|b, c)$ denotes Brenda's uncertainty in S given actions b and c. Next, we can find her expected utility by taking into account her uncertainty about Charles's actions as

$$\Psi_B(b) = \int \Psi_B(b, c) p_B(c)\, dc, \tag{3.1}$$

where $p_B(c)$ is Brenda's uncertainty in C. Then, we can find the optimal action b^* that maximises her expected utility as

$$b^* = \arg\max_{b \in \mathcal{B}} \Psi_B(b).$$

The main challenge in the above modelling is to determine $p_B(c)$. Brenda may elicit $p_B(c)$ either by using her subjective beliefs or by using data on the past auction bids by Charles on similar items or by using an expert opinion. But, she could also choose to elicit $p_B(c)$ by modelling Charles's strategic thinking process. For example, she may believe that Charles is an expected utility maximiser, just like her, and would choose the action c^* that maximises his expected utility. She can aim to find c^*, if Charles's utility function $u_C(b, c, s)$ and his probabilities $p_C(b)$ and $p_C(s|b, c)$ are available to her. However, these quantities are typically not available to her. She can model her uncertainty about Charles's utility and his probabilities through eliciting a random utility $U_C(b, c, s)$ and random probabilities $P_C(b)$ and $P_C(s|b, c)$. Then, following a backward induction approach similar to her own, she can find Charles's random optimal action C^* as

$$C^* = \arg\max_{c \in \mathcal{C}} \Psi_C(c),$$

where $\Psi_C(c)$ is his random expected utility. Then, she could find her required predictive distribution $p_B(c)$ about Charles's action c as

$$\int_{-\infty}^{c} p_B(\varphi) d\varphi = \Pr(C^* \leq c). \tag{3.2}$$

We have assumed that the sets $\mathcal{B}$, $\mathcal{C}$ and $\mathcal{S}$ are the sets that contain infinitely many points. If any of these sets were of finite cardinality then the corresponding integrals would be replaced by sums. Also, we have assumed that Charles is an expected utility maximiser when assessing $p_B(c)$. However, this probability can be assessed by assuming a number of alternative solution concepts. For example, Brenda can solve this problem by assuming that Charles is a non-strategic player or that he is a level-k thinker or that he uses BNE to determine his optimal action and so on. An ARA solution can be found for each of these solution concepts (Banks et al., 2015).

Banks et al. (2015) modelled FPSB auctions assuming that each bidder is risk-neutral (see below). They derived ARA solutions assuming that the opponent has different solution concepts such as non-strategic play, minimax perspective, level-k thinking, mirror equilibrium and BNE. However, they did not consider bidders' wealth, while defining their utility function. Also, they did not model risk-averse or risk-seeking behaviour of the bidders. They also did not consider reserve price for the auctioned item which is a common practice in FPSB auctions. In reality, it is well known that bidders' wealth, their risk appetite and whether the item had a reserve price or not, all play a role in determining the bid the item may attract.

3.1.3 Utility Function

In the context of auctions, *risk* relates to the risk of not winning the item. A *risk-neutral* bidder is defined as the one who is indifferent to risk when making a bidding decision. A risk-neutral behaviour could be due to a rational decision making process and a person taking a calculated balanced approach. *Risk-averse* bidders are the bidders who do not want to lose the item. Thus, they bid more aggressively than risk-neutral bidders. Whereas, *risk-seeking* bidders are the bidders who are keen to get the item at a low price. Thus, they bid less aggressively than risk-neutral bidders. Previous literature reveals that risk aversion is an important determinant of bidders' bidding behaviour in auctions (Milgrom and Weber, 1982; Maskin and Riley, 1984; Gentry et al., 2015). Specifically in FPSB auctions, a bidder does not really know about other bidders' behaviours. Therefore, risk aversion is relevant to these auctions (Cox et al., 1988; Kagel, 1995; Dorsey and Razzolini, 2003).

A commonly used utility function in the literature of auctions for risk-averse bidders is the constant relative risk-averse (CRRA) utility function and it is so because of its computational ease (Holt and Laury, 2002). A basic CRRA utility function for a bidder having wealth w is defined as

$$u(w) = w^r, \qquad w > 0, \tag{3.3}$$

where, $(1 - r) = -wu''(w)/u'(w)$ is the coefficient of CRRA or Arrow-Pratt measure of relative risk-aversion (Pratt, 1964; Arrow, 1965), which measures the proportion of wealth an individual will choose to hold on a risky asset, for a given level of wealth w. The utility function (3.3) is strictly convex for $1 < r \leq 2$, which represents the risk-seeking behaviour, it is linear for $r = 1$, which represents the risk-neutral behaviour and it is strictly concave for $0 < r < 1$, which represents the risk-averse behaviour (Holt and Laury, 2002).

Cox et al. (1982a, 1985); Maréchal and Morand (2011), among others, used (3.3) without considering bidders' wealth and defined their utility function as

$$u(b, v) = (v - b)^r,$$

where, v is a bidder's true value and b is the successful bid. Others, for example, Lu and Perrigne (2008); Li and Tan (2017) used (3.3) while also considering bidders' wealth and defined their utility function in case of their successful bid as

$$u(b, v, w) = (w + v - b)^r, \tag{3.4}$$

where, they assumed that all bidders have the same wealth $w \geq 0$. However, in Section 3.2 we show that this utility function is unrealistic for multiple reasons. We propose a new utility function that is more realistic and does not have the drawbacks that (3.4) has.

Normal good: In Economic theory a good that has positive income elasticity is defined as a *normal good*. That is, demand for a normal good rises when income increases and falls when the income falls (see, e.g. Fisher, 1990; Goeree et al., 2002; Piros and Pinto, 2013; Perloff, 2015). It is an item for which a person's demand increases with increase in her wealth (Baisa, 2017). The items that are typically considered to be normal include consumables, but also items such as collectibles, houses, cars, jewellery, etc. Thus, when bidding on a normal item, a bidder could be more risk-averse (or less risk-seeking) with increase in their wealth. That is, relative wealth and risk behaviour are linked and it is therefore not enough to model the risk behaviour by using a CRRA parameter that is not directly linked to the relative wealth. However, except for Baisa (2017), the type of goods auctioned has not specifically been taken into account when defining utility functions and CRRA parameters.

3.1.4 Contributions in this Chapter

The main contributions contained in this chapter are as follows:

- We extend Banks et al. (2015) by developing ARA solutions for non-strategic play and level-k thinking solution concepts for a realistic case, wherein,
    - we consider a reserve price for the auctioned item (typically, known to each bidder in advance),
    - we take into account bidders' wealth,
    - we find solutions not only for risk-neutral bidders but also for risk-averse and risk-seeking bidders.
- we assume that the auctioned item is normal and define a new utility function for the bidders.
- unlike the utility function (3.4), where it is assumed that all bidders have the same wealth, we assume that the bidders may have different wealths.
- we use the CRRA parameter r and also define a new CRRA parameter a to incorporate the effect of increase in wealth on the bidders' risk behaviour.

3.1.5 Structure of the Chapter

In this chapter, we assume that we are finding ARA solutions for Brenda (B) against her opponent Charles (C) in an FPSB auction. The remainder of the chapter is organised as follows. In Section 3.2, we propose our new utility function and the new CRRA parameters. In Section 3.3, we derive the ARA solution for the FPSB game, where Brenda assumes that Charles is a non-strategic player. In Section 3.4, we derive the solution, where Brenda assumes that Charles is a level-k thinker. In both Sections 3.3 and 3.4, we also illustrate the ARA solutions with detailed numerical examples. Finally, in Section 3.5, we discuss the results obtained in this chapter and sketch ideas for further work.

3.2 New Utility Function and New CRRA Parameters

3.2.1 Drawbacks of Utility Function (3.4)

For the utility function (3.4), the wealth w has been defined in many ways in the auction literature. For example, $w \geq 0$ (Lu and Perrigne, 2008; Li and Tan, 2017), $0 \leq w \leq \bar{w}$, where $\bar{w}$ is the upper support of wealth among n bidders (Gentry et al., 2015) and $w > \varrho$, where ϱ is the entry cost of auction (Li et al., 2015). However, none of these constraints incorporate the bid value which is rather important because the bidder can pay the amount of her successful bid b only when her wealth is greater than or equal to her bid i.e., $w \geq b$. Indeed, the utility function (3.4) will yield a positive utility even when $b > w$ (as long as $w + v \geq b$) in which case the bidder does not have the ability to buy the item. This is unrealistic.

For the sake of simplicity of the notation, we assume that the auction has no entry cost. Alternatively, we could also assume that w is the wealth after discounting the entry cost. Then, using (3.4), Brenda's expected utility would be of the form

$$\Psi_1 = (w + v - b)^r F(b) + w^r [1 - F(b)], \tag{3.5}$$

where $F(b)$ is Brenda's probability of winning the auctioned item. The expected utility function (3.5) has been used by Gentry et al. (2015). Li and Tan (2017), and others using a game-theoretic perspective. Having w unconstrained with respect to b can lead to unrealistic solutions. For example, not only can the optimal bid value b^* be greater than w, but also, b^* could decrease as w increases, which is inconsistent with the item being normal. We illustrate this using a simple example.

Example 3.1 *Suppose that Brenda has true value* $v = \$150$ *for the auctioned item which has no reserve price. From her subjective belief about Charles, lets assume that she elicits the distribution* $F(c) = 9c/(8 \times 200) - c^9/(8 \times 200^9)$ *on Charles's bid c (Banks et al., 2015). To find her optimal bid, she can replace c by b to obtain* $F(b)$*, her probability of winning the item. Using the expected utility function* (3.5)*, Brenda can find her optimal bid by finding*

$$b^* = \underset{b \in \mathbb{R}^+}{\arg\max} \; [\{w + v - b\}^r F(b) + w^r \{1 - F(b)\}].$$

Some numerical results are shown in Table 3.1. The table shows that b^* *increases with increase in her risk-aversion, which is realistic. However, it also shows that* b^* *decreases as her wealth* w *increases, which is unrealistic when the item is normal. Further, it shows that* b^* *can be much higher than her wealth* w *i.e.* $b^* > w$ *especially for higher risk-aversion levels (*$r = 0.10$ *or* 0.05*).*

Now, if we assume that $w = 0$, then the expected utility function (3.5) would be

$$\Psi_1 = (v - b)^r F(b). \tag{3.6}$$

Table 3.1: Brenda's optimal bids b^* using expected utility function (3.5) with different wealth and risk aversion levels.

r	0.90	0.50	0.10	0.05
$w = 0$	78.93	99.88	135.84	148.38
$w = 50$	76.44	82.14	87.51	88.15
$w = 150$	75.69	78.46	81.12	81.44

This function has been used for risk-averse bidders by Cox et al. (1982a) and Cox et al. (1985), among others, and by Banks et al. (2015) for risk-neutral bidders. The first row of Table 3.1 shows Brenda's optimal bids for this special case where $w = 0$. Finding b^* using (3.6) is also unrealistic again because $b^* > w$.

3.2.2 New CRRA Parameters

When an item is normal, the bidders' willingness to pay for it increases with their wealth (Baisa, 2017). We propose to introduce an additional risk behaviour parameter a that will change with the relative change in circumstances of each bidder's wealth. The original risk behaviour parameter r will remain unchanged and will denote the baseline risk behaviour level of the bidder. This relative change in circumstances could occur in two specific cases; firstly, when a bidder's wealth changes and secondly, when a bidder attempts to model the risk behaviour of an opponent. We describe how the new parameter a can be defined in each of these cases.

Firstly, to model Brenda's risk behaviour at an increase level of her wealth compared with her own lower level of wealth, we modify the CRRA parameter as follows: we define r_B to be Brenda's baseline risk behaviour parameter, which represents her natural risk appetite at her wealth, say w_1. Note that r_B is the same as the r we defined earlier in Subsection 3.1.3. Lets assume that her circumstances change (e.g. she gains an inheritance) and her wealth is increased to w_2 ($w_2 > w_1$). At this increased wealth level, we expect her to be more risk-averse (or less risk-seeking) for the same auctioned item (since the item is assumed normal). So, we modify her risk behaviour parameter having wealth w_2 relative to wealth w_1 as

$$a_B = \begin{cases} r_B^{\frac{1}{h}} & \text{if} \quad 0 < r_B < 1, \\ r_B & \text{if} \quad r_B = 1, \\ r_B^h & \text{if} \quad 1 < r_B \leq 2, \end{cases} \tag{3.7}$$

where, we define $0 < h = w_1/w_2 < 1$, when $w_1 < w_2$. Note that here, for $0 < r_B \leq 2$ ($r_B \neq 1$), $a_B < r_B$, i.e. if Brenda was risk-averse (risk-seeking) at wealth level w_1, then she is even more risk-averse (less risk-seeking) at wealth level w_2. Also, when $r_B = 1$, $a_B = r_B$, i.e. if Brenda was risk-neutral at wealth level w_1, she is also risk-neutral at wealth level w_2. There is no change to her risk behaviour (that is, $a_B = r_B$) if her wealth decreases.

Secondly, when Brenda is bidding against Charles who has wealth w_C, we modify their CRRA parameters as follows: if Brenda believes that Charles has wealth w_C, we define R_C as Brenda's belief about Charles's natural risk appetite for the auctioned item. In this case, Brenda could draw R_C from a uniform distribution U with support $(0, 1)$ if she believes that Charles is a risk-averse bidder. She could draw R_C from a uniform distribution U with support $(1, 2]$ if she believes that Charles is a risk-seeking bidder. In this case, we assume that Brenda has wealth w_B, where $w_B > w_C$, i.e. Brenda has more wealth than Charles. Thus, Brenda's risk behaviour parameter in this case would be same as defined in (3.7) with $0 < h = w_C/w_B < 1$. However, if Brenda believes that Charles has more wealth than her, i.e. $w_C > w_B$, she can modify Charles's risk-behaviour parameter as

$$A_C = \begin{cases} R_C^{\frac{1}{h}} & \text{if} \quad 0 < R_C < 1, \\ R_C & \text{if} \quad R_C = 1, \\ R_C^{h} & \text{if} \quad 1 < R_C \leq 2, \end{cases} \tag{3.8}$$

where $0 < h = w_B/w_C < 1$ and $A_C < R_C$ if Brenda believes that Charles is risk-averse (risk-seeking) bidder in this case. Thus, A_C may take values in the interval $(0, 1)$ if she believes that he is a risk-averse bidder. On the other hand if she believes that he is a risk-seeking bidder, A_C may take values in the interval $(1, 2^h]$ and $A_C = 1$, if she believes that he is a risk-neutral bidder. In this case, Brenda's risk behaviour parameter would remain unchanged i.e. it is $a_B = r_B$ for $0 < r_B \leq 2$.

3.2.3 New Utility Function

We propose to modify the utility function so that $w \geq b$, i.e. a bidder's bid value cannot be greater than that bidder's wealth. Additionally, since in equilibrium, bidders never bid above their true values (Gentry et al., 2015), we in fact have $b \leq v \leq w$, because a bidder can bid and pay an amount $b \leq v$, only if the bidder's wealth is at least equal to the true value. Further, we assume that a bidder's wealth remains unchanged if the bid is unsuccessful.

We propose to use the following utility function for Brenda:

$$u(b, v, w) = \begin{cases} w + (v - b)^{a_B} & \text{if she wins the bid,} \\ w & \text{if she loses the bid,} \end{cases} \tag{3.9}$$

where, a_B is a modified CRRA parameter as defined in Section 3.2.2. Thus, for our proposed utility function (3.9), Brenda's expected utility would be of the form

$$\Psi_2 = \{w + (v - b)^{a_B}\}F(b) + w\{1 - F(b)\}. \tag{3.10}$$

The above equation simplifies to

$$\Psi_2 = w + (v - b)^{a_B} F(b). \tag{3.11}$$

Thus, using (3.11), Brenda can find her optimal bid by solving the following equation

$$b^* = \underset{b \in \mathbb{R}^+}{\arg\max}\, [w + (v - b)^{a_B} F(b)]. \tag{3.12}$$

The utility function (3.9) is more realistic in the sense that it allows Brenda to add up her profit to her wealth after the successful bid. It also allows Brenda to bid strictly less than or equal to her true value v and consequently to bid less than or equal to her wealth at any assumed level of her risk-aversion. Lets assume that Brenda has wealth $w = \$150$, true value $v = \$150$ and $F(b)$ as considered earlier in this section. Then, using (3.12), she could get her optimal bids as shown in the first row of Table 3.1 for different assumed risk-aversion levels. Note that she will get these values since her optimal bid is not affected by her wealth in (3.12). Thus, this function gives the leverage to assume any wealth $w \geq v$ for Brenda and also her optimal bid to be bounded above by her wealth for any assumed risk-aversion level, i.e. $b^* \leq v \leq w$.

We define (3.9) as Brenda's wealth plus her profit where she could be risk-neutral, risk-averse or risk-seeking in her profit. Thus, by letting her profit $v - b = x$, we can show that she has a CRRA profit in (3.9) since

$$\frac{d}{dx}[u(x, w)] = u'(x, w) = a_B x^{a_B - 1},$$
$$\frac{d}{dx}[u'(x, w)] = u''(x, w) = a_B(a_B - 1)x^{a_B - 2}.$$

So

$$\frac{u''(x, w)}{u'(x, w)} = \frac{a_B(a_B - 1)x^{a_B - 2}}{a_B x^{a_B - 1}},$$
$$\implies 1 - a_B = -x\frac{u''(x, w)}{u'(x, w)},$$

where $1 - a_B$ is the coefficient of CRRA that incorporates the effect of increase in wealth on bidders' risk behaviour and is defined in Section 3.2.2.

3.3 Non-Strategic Play

In this section, we show how an ARA solution for Brenda's optimal bid can be found when she assumes that Charles is a non-strategic opponent and will bid an amount that is independent of Brenda's bid. We assume that Brenda bids an amount b, having wealth w_B and true value v_B for the auctioned item. We assume that the auctioned item is normal and it has a reserve price τ such that $\tau < b \leq v_B \leq w_B$. We assume that τ is known in advance to each bidder. We define Brenda's wealth w_B, as the money she has at her disposal. She does not know about Charles's wealth W_C and places a distribution H_{BC} on his wealth. She also does not know about Charles's true value V_C and his bid C for the auctioned item. So, she places a distribution G_{BC} on his true value according to her belief and then finds the distribution of his bid C, as defined in (3.14) below. For Charles, she also believes that $\tau < c \leq v_C \leq w_C$ holds, where v_C and w_C are chosen by Brenda from the distributions G_{BC} and H_{BC}, respectively. Brenda's probability of winning from a bid of amount b is given by

$$F_{BC}(b) = \Pr(C \leq b),$$

where, F_{BC} is the distribution over Charles's bid with support $(\underline{v}_C, \bar{v}_C] \subseteq \mathbb{R}^+$, $\underline{v}_C \geq \tau$, that Brenda believes and C is the random variable of her belief about Charles's bid. To obtain F_{BC}, Brenda divides her introspection into two parts as G_{BC}, the cumulative distribution function (CDF) that quantifies her uncertainty of Charles's true value and T_{BC}, the CDF that quantifies her uncertainty for the fraction of Charles's true value $p = c/v_C$ that he bids. Note that the supports for G_{BC} and T_{BC} are $(\underline{v}_C, \bar{v}_C]$ and $(\underline{v}_C/v_C, 1]$, respectively. She can then derive her subjective distribution function over $C = PV_C$, the amount of Charles's bid as

$$\begin{aligned} F_{BC}(c) = \Pr[\underline{v}_C < PV_C \leq c] = &\int_{\underline{v}_C}^{c} \int_{\underline{v}_C/v_C}^{1} g_{BC}(v_C) t_{BC}(p)\, dp\, dv_C \\ &+ \int_{c}^{\bar{v}_C} \int_{\underline{v}_C/v_C}^{c/v_C} g_{BC}(v_C) t_{BC}(p)\, dp\, dv_C. \end{aligned} \tag{3.13}$$

As $\int_{\underline{v}_C/v_C}^{1} t_{BC}(p)dp = 1$, the above equation simplifies to

$$F_{BC}(c) = G_{BC}(c) + \int_{c}^{\bar{v}_C} g_{BC}(v_C) T_{BC}(c/v_C)\, dv_C, \tag{3.14}$$

where, $g_{BC}(v_C)$ is the probability density function for Charles's true value that Brenda elicits and $t_{BC}(p)$ is the probability density function for the fraction of Charles's true value that Brenda believes he will bid. Equation (3.13) assumes that Charles's true value V_C and fraction of his true value P are independent. In (3.13), the whole region of integration has been divided into two regions for the random variables V_C and P in order to find the distribution of C. Figure 3.2 shows the division of the integration region into these two regions. The area between the two curves shows the integration region between $\underline{v}_C$ and $\bar{v}_C$. The area A corresponds to the first integral part whereas area B corresponds to the second integral part of (3.13).

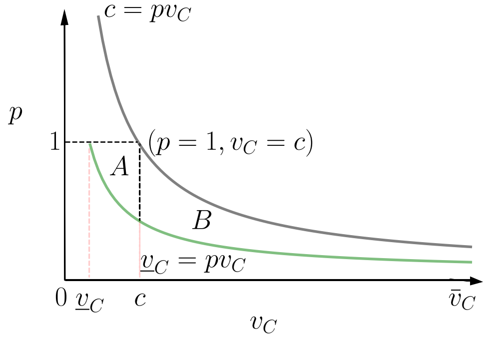


Figure 3.2: Region of integration when the bid is a proportion of the true value with reserve price τ.

Then, using (3.9), we rewrite Brenda's expected utility function (3.10) as

$$\Psi_B(b) = \{w_B + (v_B - b)^{a_B}\}F_{BC}(b) + w_B\{1 - F_{BC}(b)\}.$$

This equation simplifies to

$$\Psi_B(b) = w_B + (v_B - b)^{a_B} F_{BC}(b), \tag{3.15}$$

where $F_{BC}(b)$ is her probability that her bid b will be successful. Finally, Brenda's optimal bid b^* may be found by solving the following equation

$$b^* = \underset{\underline{v}_C \leq b \leq v_B}{\arg\max}\, [w_B + (v_B - b)^{a_B} F_{BC}(b)]. \tag{3.16}$$

Numerical methods may often be needed to solve (3.16) for b^*. Note that b^* is, in fact, a function of a_B and therefore, a function of r_B and w_C. Then r_B, being the baseline risk behaviour for Brenda, is constant. Therefore, we can re-write (3.16) as

$$b^*(w_C) = \underset{\underline{v}_C \leq b \leq v_B}{\arg\max}\, [w_B + (v_B - b)^{a_B} F_{BC}(b)]. \tag{3.17}$$

If Brenda has information on Charles's wealth, she can use (3.17) to find her optimal bid amount. Alternatively, she can take into consideration her uncertainty around w_C and find the expected value of her optimal bid as

$$\mathrm{E}(b^*) = \int b^*(w_C)\, dH_{BC}(w_C). \tag{3.18}$$

Comparing the derivation above with the ARA sketch provided in Section 3.1.2, the reader can note that $F_{BC}(c)$ of (3.14) is the $p_B(c)$ in (3.2) obtained by assuming that the opponent is a non-strategic player and that (3.15) provides the expected utility $\Psi_B(b)$ given in (3.1) for this particular problem. Also, note that in the above analysis, we have assumed that all the probability distributions considered are continuous. If any of the distributions are discrete then the corresponding integrals would be replaced by summations.

Example 3.2 *Suppose Brenda's true value for the item* $v_B = \$150$*, her wealth* $w_B = \$150$ *and the auctioned item has a reserve price* τ*. She could elicit her uncertainty on Charles's true value using a uniform distribution* $G_{BC}(v_C) = (v_C - \underline{v}_C)/(\bar{v}_C - \underline{v}_C)$ *with support* $(\underline{v}_C, \bar{v}_C], \underline{v}_C \geq \tau$*. Since Brenda believes that Charles is a non-strategic player whose bid amount will be independent of her bid, she could assume a distribution on* p*, the proportion of Charles's true value that Brenda believes he will bid, given by* $T_{BC}(p) = (p^8 - (\underline{v}_C/v_C)^8)/(1-(\underline{v}_C/v_C)^8)$ *with support* $(\underline{v}_C/v_C, 1]$*. Finally, suppose that she has a uniform distribution* $H_{BC}(w_C) = (w_C - \underline{w}_C)/(\bar{w}_C - \underline{w}_C)$ *on Charles's wealth* W_C *with support* $(\underline{w}_C, \bar{w}_C]$ *where* $\underline{w}_C \geq \underline{v}_C$*. Then we have* $g_{BC}(v_C) = 1/(\bar{v}_C - \underline{v}_C)$*. Note that the distributions* $T_{BC}(p)$ *and* H_{BC} *elicited here are taken from Banks et al. (2015) but with the reserve price* τ *that constrains* $\underline{v}_C$*. Using* (3.14) *where* c *is replaced by* b*, we get*

$$
\begin{aligned}
F_{BC}(b) =& \frac{b-\underline{v}_C}{\bar{v}_C-\underline{v}_C} + \frac{b^8-\underline{v}_C^8}{\bar{v}_C-\underline{v}_C}\Bigg[-\frac{\sqrt{2}}{16\times\underline{v}_C^7}\ln\Big(\frac{\bar{v}_C^2+\bar{v}_C\underline{v}_C\sqrt{2}+\underline{v}_C^2}{\bar{v}_C^2-\bar{v}_C\underline{v}_C\sqrt{2}+\underline{v}_C^2}\Big) \\
&+\frac{\sqrt{2}}{16\times\underline{v}_C^7}\ln\Big(\frac{b^2+b\underline{v}_C\sqrt{2}+\underline{v}_C^2}{b^2-b\underline{v}_C\sqrt{2}+\underline{v}_C^2}\Big)-\frac{\sqrt{2}}{8\times\underline{v}_C^7}\tan^{-1}\Big(\frac{\bar{v}_C\sqrt{2}}{\underline{v}_C}+1\Big) \\
&+\frac{\sqrt{2}}{8\times\underline{v}_C^7}\tan^{-1}\Big(\frac{b\sqrt{2}}{\underline{v}_C}+1\Big)-\frac{\sqrt{2}}{8\times\underline{v}_C^7}\tan^{-1}\Big(\frac{\bar{v}_C\sqrt{2}}{\underline{v}_C}-1\Big) \\
&+\frac{\sqrt{2}}{8\times\underline{v}_C^7}\tan^{-1}\Big(\frac{b\sqrt{2}}{\underline{v}_C}-1\Big)-\frac{1}{4\times\underline{v}_C^7}\tan^{-1}\Big(\frac{\bar{v}_C}{\underline{v}_C}\Big)+\frac{1}{4\times\underline{v}_C^7}\tan^{-1}\Big(\frac{b}{\underline{v}_C}\Big) \\
&+\frac{\ln(\bar{v}_C-\underline{v}_C)}{8\times\underline{v}^7}-\frac{\ln(b-\underline{v}_C)}{8\times\underline{v}_C^7}-\frac{\ln(\bar{v}_C+\underline{v}_C)}{8\times\underline{v}_C^7}+\frac{\ln(b+\underline{v}_C)}{8\times\underline{v}_C^7}\Bigg].
\end{aligned}
\tag{3.19}
$$

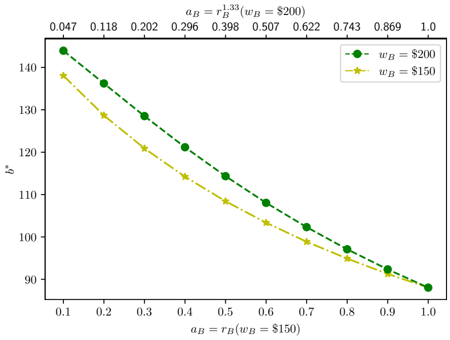
Suppose $\tau = \$30$ *and Brenda believes that* G_{BC} *has support* $(30, 200]$. *Then substituting these in* (3.19), *she can get* $F_{BC}(b)$ *which she can then use to find her optimal bid by solving* (3.16) *which also takes into consideration her risk appetite. Now, assuming that Brenda is a risk-neutral bidder i.e.* $a_B = 1$, *her optimal bid by solving* (3.16) *for* $w_B = v_B = \$150$ *turns out to be* \$88.05 *with probability of winning of* 0.415 *and with the expected utility of* 175.72.

Next, we assume that Brenda is a risk-averse bidder. For the sake of simplicity, we assume that Brenda chooses Charles's wealth w_C *to be* \$150 *from the distribution* H_{BC}. *We assume that Brenda's baseline risk behaviour parameter is* r_B *when her wealth* $w_B = \$150$. *As her and Charles's wealth are assumed to be the same, it is natural to take* $a_B = r_B$. *If Brenda's wealth was to increase to* $w_B = \$200$, *then, we expect her to be more risk-averse than Charles because she is able (and willing since the item is normal) to pay more to increase her chance of winning the bid. By using* (3.7), *we model Brenda's risk-aversion when* $w_B = \$200$ *relative* *to* $w_B = \$150$ *as* $a_B = r_B^{1/h} = r_B^{200/150} = r_B^{1.33}$. *In Table 3.2, we show how Brenda's optimal bids, her probabilities of winning and her expected utilities change with change in her wealth and also how they change with the change in* r_B. *It shows that with increase in her wealth to* $w_B = \$200$, *she is more risk-averse and consequently bids higher than when* $w_B = \$150$. *In general, it shows that an increase in risk aversion leads to higher optimum bid (resulting in a higher probability of winning that bid) but a lower expected utility nonetheless. We also plot these in Figure 3.3.*

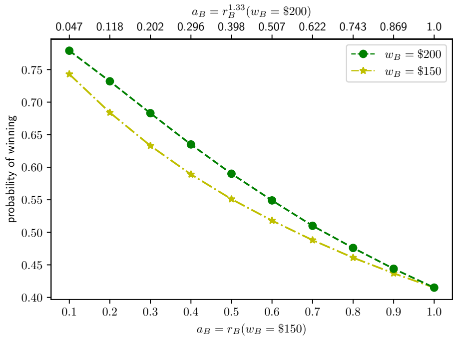
Table 3.2: Brenda's optimal bids, probabilities of winning and her expected utilities when she is risk-averse and risk-neutral bidder and she thinks that $w_C = \$150$.

$a_B = r_B$ ($w_B = \$150$)	0.10	0.20	0.30	0.40	0.50	0.60	0.70	0.80	0.90	1.00
$b^*(w_C)$	138.06	128.69	120.89	114.22	108.43	103.36	98.87	94.86	91.28	88.05
$F_{BC}(b^*)$	0.743	0.684	0.633	0.589	0.551	0.518	0.488	0.461	0.437	0.415
Ψ_B	150.95	151.26	151.74	152.46	153.55	155.19	157.66	161.40	167.08	175.72
$a_B = r_B^{1.33}$ ($w_B = \$200$)	0.047	0.118	0.202	0.296	0.398	0.507	0.622	0.743	0.869	1.00
$b^*(w_C)$	143.95	136.22	128.52	121.17	114.34	108.05	102.32	97.09	92.35	88.05
$F_{BC}(b^*)$	0.779	0.732	0.683	0.635	0.590	0.549	0.511	0.476	0.444	0.415
Ψ_B	200.85	201.00	201.27	201.72	202.45	203.65	205.65	209.08	215.05	225.72

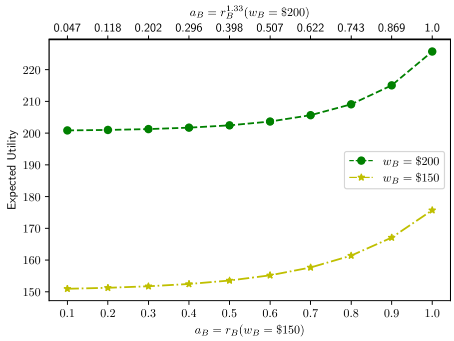
But often, Brenda may instead be uncertain about W_C *and may prefer to take into account her uncertainty and obtain her expected optimal value after doing so. To do this, she can first elicit* H_{BC} *and then solve* (3.18) *using Monte Carlo methods. Suppose she believes that Charles's wealth is uniformly distributed between* \$100 *and* \$300. *She will derive* a_B *as described in Section 3.2.2, whereby,* $a_B = r_B^{1/h}$, *for* $h = w_C/w_B$, *when* $w_B > w_C$ *and* $a_B = r_B$ *otherwise. Since* W_C *is*



(a) Brenda's optimal bids.



(b) Brenda's probabilities of winning.



(c) Brenda's expected utilities.

Figure 3.3: Comparison of Brenda's (risk-averse and risk-neutral) optimal bids, her probabilities of winning and her expected utilities when she has $w_B = \$150$ and $\$200$.

a random variable, so will be A_B. *Then, her expected optimal bids, probabilities of winning, and expected utilities are given in Table 3.3.*

Table 3.3: Brenda's expected optimal bids, probabilities of winning and her expected utilities when she is risk-averse and risk-neutral bidder and draws W_C *from* H_{BC}.

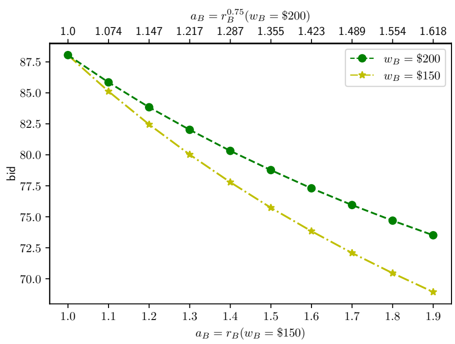
r_B ($w_B = \$200$)	0.10	0.20	0.30	0.40	0.50	0.60	0.70	0.80	0.90	1.00
$\mathrm{E}(a_B)$	0.07	0.16	0.25	0.35	0.44	0.55	0.66	0.77	0.88	1.00
$\mathrm{E}(b^*)$	140.91	132.52	124.73	117.94	111.84	106.19	100.89	96.21	91.89	88.05
$F_{BC}[\mathrm{E}(b^*)]$	0.761	0.708	0.658	0.614	0.574	0.536	0.501	0.470	0.441	0.415
$\Psi_B[\mathrm{E}(b^*)]$	200.89	201.12	201.48	202.07	202.85	204.29	206.55	210.11	215.74	225.72

Finally, we find Brenda's optimal bids, probabilities of winning and expected utilities when she is assumed to be a risk-seeking bidder and $w_B = \$150$. *These are listed in Table 3.4. These show that as the risk seeking behaviour intensifies, the optimal bids get lower and so does the probabilities of winning the bids. Again, we can model how her bids will change if her wealth was to increase, to say* \$200. *By using* (3.7), *we model Brenda's risk-seeking behaviour when* $w_B = \$200$ *as* $a_B = r_B^h = r_B^{150/200} = r_B^{0.75}$. *It shows that with increase in her wealth, i.e.* $w_B = \$200$, *she is less risk-seeking and consequently she bids higher than when* $w_B = \$150$. *In general, it shows that an increase in risk seeking behaviour leads to a lower optimum bid (resulting in a lower probability of winning that bid) but a higher expected utility nonetheless. We also plot these results in Figure 3.4.*

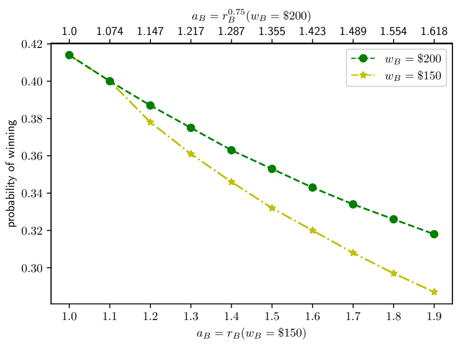
Once again, Brenda may prefer to take into account her uncertainty in W_C *and obtain her expected optimal value after doing so. To do this, she can solve* (3.18) *using Monte Carlo methods. Her expected optimal bids, probabilities of winning, and expected utilities are given in Table 3.5.*

Table 3.4: Brenda's optimal bids, probabilities of winning and her expected utilities when she is risk-seeking and risk-neutral bidder and she thinks that $w_C = \$150$.

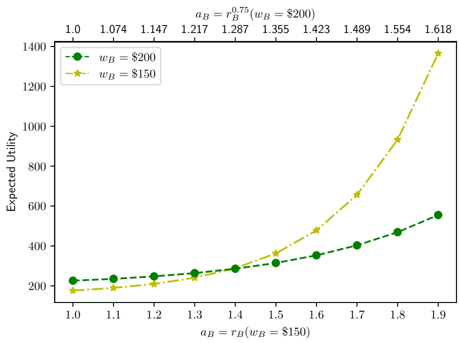
$a_B = r_B$ ($w_B = \$150$)	1.00	1.10	1.20	1.30	1.40	1.50	1.60	1.70	1.80	1.90
$b^*(w_C)$	88.05	85.12	82.45	80.02	77.79	75.73	73.84	72.08	70.45	68.93
$F_{BC}(b^*)$	0.415	0.400	0.378	0.361	0.346	0.332	0.320	0.308	0.297	0.287
Ψ_B	175.72	189.39	209.30	240.36	288.39	362.50	478.03	656.24	933.26	1365.39
$a_B = r_B^{0.75}$ ($w_B = \$200$)	1.000	1.074	1.147	1.217	1.287	1.355	1.423	1.489	1.554	1.618
$b^*(w_C)$	88.05	85.85	83.84	82.02	80.32	78.77	77.30	75.95	74.69	73.51
$F_{BC}(b^*)$	0.415	0.400	0.387	0.375	0.363	0.353	0.343	0.334	0.326	0.318
Ψ_B	225.72	234.91	247.42	263.68	285.50	314.32	352.85	402.99	469.06	554.90



(a) Brenda's optimal bids.



(b) Brenda's probabilities of winning.



(c) Brenda's expected utilities.

Figure 3.4: Comparison of Brenda's (risk-seeking and risk-neutral) optimal bids, her probabilities of winning and her expected utilities when she has $w_B = \$150$ and $\$200$.

Table 3.5: Brenda's expected optimal bids, probabilities of winning and her expected utilities when she is risk-seeking and risk-neutral bidder and draws W_C from H_{BC}.

r_B ($w_B = \$200$)	1.00	1.10	1.20	1.30	1.40	1.50	1.60	1.70	1.80	1.90
$\mathrm{E}(a_B)$	1.00	1.09	1.17	1.26	1.34	1.42	1.51	1.60	1.68	1.76
$\mathrm{E}(b^*)$	88.05	85.49	83.18	81.01	79.11	77.39	75.62	74.00	72.54	71.30
$F_{BC}[\mathrm{E}(b^*)]$	0.415	0.398	0.383	0.368	0.355	0.344	0.332	0.320	0.311	0.301
$\Psi_B[\mathrm{E}(b^*)]$	225.72	237.36	252.22	276.33	307.23	350.94	422.22	527.87	664.06	857.73

3.4 Level-k Thinking

We will now show how an ARA solution for a two player FPSB auction can be derived when Brenda believes that Charles is a level-k thinker. In a level-k analysis, we model how deeply the opponent thinks about the problem (Stahl and Wilson, 1995). It is not only an important solution concept to be considered for a strategic adversary (Banks et al., 2015) but also provides the flexibility to model the opponent at different levels of strategic thinking including being a non-strategic player (when $k = 0$). If the decision maker performs a level-k analysis, she believes that her opponent is a level-$(k-1)$ thinker who would model her as a level-$(k-2)$ player. For instance, when $k = 1$, she believes that her opponent is a level-0 thinker, that is, a non-strategic player. A level-2 analysis means that the decision maker assumes that her opponent is a level-1 thinker, who believes that she is a level-0 thinker. A level-3 analysis means that the decision maker assumes that her opponent is a level-2 thinker who models her as a level-1 thinker and so on. So, in such type of modelling, the decision maker attempts to think one level deeper than her opponent.

The key question is how large should the k be? In principle, k could take higher values. One could argue that players could choose a higher k when playing highly structured games such as Chess or Go. However, Ho et al. (1998) and Lee and Wolpert (2012) presented experimental evidence that in practice people do not usually think higher than level 2 or 3. Therefore, for FPSB auctions, it makes sense to solve the level-k problem for k being $1, 2$ or 3.

As described earlier, we extend the work by Banks et al. (2015). We derive Brenda's optimal bid for the level-k thinking solution concept assuming not only that she and Charles are risk-neutral bidders but also when they are risk-averse or risk-seeking bidders. Also, we consider their wealth as well as assume that the auctioned item has a reserve price τ when deriving the optimal bid.

For $k = 1$, where Brenda believes herself to be a level-1 thinker and models Charles as a level-0 (non-strategic) thinker, the problem is identical to the non-strategic thinking problem modeled in Section 3.3.
Here, we derive ARA solution for the case where $k = 2$. In this case, Brenda models herself as a level-2 thinker and believes that Charles is a level-1 thinker, who (Charles) models Brenda as a level-0 thinker. Brenda performs ARA by placing a subjective distribution G_{BCB} with support $(\underline{v}_{CB}, \bar{v}_{CB}]$ on her true value V_{CB} that Charles might elicit, a distribution T_{BCB} with support $(\underline{v}_{CB}/V_{CB}, 1]$ on the fraction of her true value that she thinks Charles would think that she would bid. This allows her to derive F_{BCB} using (3.14) (but with roles reversed) with support $(\underline{b}_{CB}, \bar{b}_{CB}]$ that she believes is the distribution of her bid that Charles might elicit. She would then elicit a distribution H_{BCB} on her wealth with support $(\underline{w}_{CB}, \bar{w}_{CB}]$, where $\underline{w}_{CB} \geq \underline{v}_{CB}$ and $\bar{w}_{CB} \geq \bar{v}_{CB}$, that she thinks Charles would elicit, a distribution G_{BC} with support $(\underline{v}_C, \bar{v}_C]$ on Charles's true value V_C, and

H_{BC} with support $(\underline{w}_C, \bar{w}_C]$ on Charles's wealth W_C where $\underline{w}_C \geq \underline{v}_C$ and $\bar{w}_C \geq \bar{v}_C$. She can then find his optimal bid for given w_B, w_C, v_C and r_C as

$$C^*(w_B, w_C, v_C, r_C) = \underset{c > \underline{v}_{CB}}{\arg\max}\, [w_C + (v_C - c)^{a_C} F_{BCB}(c)],$$

where $\underline{v}_{CB} \geq \tau$ and A_C is defined in (3.8). Here, $h = W_B/W_C$. As described in Section 3.2.2, Brenda could elicit a distribution on R_C. This distribution, along with H_{BCB} and H_{BC} would allow her to derive the distribution for A_C which we shall denote by S_{BC}. She can then find the expected value of Charles's optimal bid that she thinks he will derive as a level-1 thinker as

$$\mathrm{E}(C^*) = \int\int C^*(w_B, w_C, v_C, r_C)\, dG_{BC}(v_C)\, dS_{BC}(a_C).$$

Using $\mathrm{E}(C^*)$ and a representative value of $V_C \sim G_{BC}$ (say, $\mathrm{E}(V_C)$) she can find $\mathrm{E}(C^*)/\,\mathrm{E}(V_C) = q$, the fraction of Charles's true value that Brenda believes he may bid. Now, she can find the distribution F_{BC} of Charles's bid C using the change of variable formula as

$$f_{BC} = |J| g_{BC} = \frac{g_{BC}}{q}. \tag{3.20}$$

Finally, she would obtain her optimal bid for a given value of w_C by solving

$$b^*(w_C) = \underset{b > \delta}{\arg\max}\, [w_B + (v_B - b)^{a_B} F_{BC}(b)], \tag{3.21}$$

where $\delta = q\underline{v}_C$ and a_B is Brenda's risk-behaviour parameter defined in (4.4). If Brenda has information on Charles's wealth, she can use (3.21) to find her optimal bid amount. Alternatively, she can take into consideration her uncertainty around w_C and find the expected value of her optimal bid using (3.18).

Comparing the derivation above with the ARA sketch provided in Section 4.1.1, the reader can note that $f_{BC}(c)$ in (3.20) gives the $p_B(c)$ in (3.2), obtained by assuming that the opponent is a level-1 thinker and that (3.15) provides the expected utility $\Psi_B(b)$ defined in (3.1) for this particular problem. Also, note that in the above analysis, we have assumed that all the probability distributions considered are continuous. If any of the distributions are discrete then the corresponding integrals would be replaced by summations.

Example 3.3 *Suppose Brenda's true value for the item* $v_B = \$150$*, her wealth* $w_B = \$200$ *and the auctioned item has a reserve price* $\tau = \$30$*. She, as a level-*2 *thinker, thinks that Charles is a level-*1 *thinker who would model her as a non-strategic player (level-*0 *thinker). She believes that:*

- *Charles would model her true value for the auctioned item as being uniformly distributed with support* $(\$30, \$200]$*, i.e.,* $G_{BCB} = (v_{CB} - 30)/(200 - 30)$.
- *Charles would elicit his uncertainty around the fraction of her true value that she would bid as* $T_{BCB} = (p^8 - (30/v_{CB})^8)/(1 - (30/v_{CB})^8)$ *with support* $(30/v_{CB}, 1]$.
- *Charles would elicit his uncertainty around her wealth to be uniformly distributed on* $(\$150, \$250]$*, i.e.,* $H_{BCB} = (w_{CB} - 150)/(250 - 150)$.
- *Charles's true value is uniformly distributed as* $G_{BC} = (v_C - 100)/(200 - 100)$ *with support* $(\$100, \$200]$.

- *Charles's wealth is uniformly distributed as* $H_{BC} = (w_C - 100)/(300 - 100)$ *with support* $(\$100, \$300]$.

Once she has elicited these, then following the algorithm below e.g., when she and Charles are risk-averse bidders, she can perform Monte Carlo simulations to derive her expected optimal bid $\mathrm{E}(b^*)$, *her probability of winning using that expected optimal bid,* $F_{BC}[\mathrm{E}(b^*)]$, *and her expected utility at that expected optimal bid* $\Psi_B[\mathrm{E}(b^*)]$ *for various levels of her risk behaviour as well as for the various levels of risk behaviour of Charles. These are summarised in Tables 3.6 to 3.9 below, where both in Tables 3.6 and 3.7, we find* $\mathrm{E}(C^*) = 112.45$ *and* $\mathrm{E}(V_C) = 150.00$. *So,*

$$q = \frac{112.45}{150.00} = 0.7497.$$

Thus,

$$F_{BC}(c) = \frac{c - q \times 100}{q(200 - 100)} = \frac{c - 74.97}{149.93 - 74.97} = \frac{c - 74.97}{74.97},$$

and the mean of this distribution is 112.45. *While both in Tables 3.8 and 3.9, we find* $\mathrm{E}(C^*) = 77.36$ *and* $\mathrm{E}(V_C) = 150.00$. *So,*

$$q = \frac{77.36}{150.00} = 0.5157.$$

Thus,

$$F_{BC}(c) = \frac{c - q \times 100}{q(200 - 100)} = \frac{c - 51.57}{103.15 - 51.57} = \frac{c - 51.57}{51.57},$$

and the mean of this distribution is 77.36.

Level*-2 *Thinking: Algorithm when Brenda and Charles are Risk-Averse

1. *Sample* $w_{BC} \sim H_{BCB}$
2. *Sample* $w_C \sim H_{BC}$
3. *Sample* $v_C \sim G_{BC}$
4. *Sample* $r_C \sim U_{BC}(0,1)$
5. *Define* $h = \min(1, \frac{w_{BC}}{w_C})$
6. *Calculate* $a_C = r_C^{1/h}$
7. *Find* F_{BCB} *from (3.14) using* G_{BCB} *and* T_{BCB}
8. *Solve numerically*

$$C^*(w_B, w_C, v_C, r_C) = \underset{\underline{v}_{CB} < c \leq v_C}{\arg\max} \; [w_C + (v_C - c)^{a_C} F_{BCB}(c)]$$

9. *Repeat* (1) *to* (8) N *times (number of simulations)*
10. *Calculate* $q = \mathrm{E}(C^*)/\,\mathrm{E}(V_C)$
11. *Find* $f_{BC} = |J|g_{BC} = g_{BC}/q$ *and thus find* F_{BC}
12. *Set* $w_B = 200$ *and* $v_B = 150$
13. *Sample* $w_C \sim H_{BC}$
14. *Let* $r_B = 0.1$
15. *Define* $h = \min(1, \frac{w_C}{w_B})$
16. *Calculate* $a_B = r_B^{1/h}$
17. *Using* F_{BC} *found in* (11)*, solve numerically*

$$b^*(w_C) = \underset{b \leq v_B}{\arg\max} \; [w_B + (v_B - b)^{a_B} F_{BC}(b)]$$

18. *Repeat* (12) *to* (17) N *times*
19. *Find* $\mathrm{E}(b^*)$*,* $\mathrm{E}(a_B)$*,* $F_{BC}[\mathrm{E}(b^*)]$ *and* $\Psi_B[\mathrm{E}(b^*)]$

Table 3.6: Brenda's expected optimal bids, probabilities of winning and her expected utilities when she is a risk-averse bidder and she assumes that Charles is also a risk-averse bidder.

r_B	0.10	0.20	0.30	0.40	0.50	0.60	0.70	0.80	0.90	1.00
$\mathrm{E}(a_B)$	0.07	0.16	0.25	0.35	0.44	0.55	0.66	0.77	0.88	1.00
$\mathrm{E}(b^*)$	144.88	139.76	135.08	130.85	127.05	123.56	120.31	117.42	114.83	112.48
$F_{BC}[\mathrm{E}(b^*)]$	0.933	0.864	0.802	0.746	0.695	0.648	0.605	0.566	0.532	0.500
$\Psi_B[\mathrm{E}(b^*)]$	201.05	201.25	201.58	202.10	202.76	203.52	205.67	208.28	212.20	218.78

Table 3.7: Brenda's expected optimal bids, probabilities of winning and her expected utilities when she is a risk-seeking bidder and she assumes that Charles is a risk-averse bidder.

r_B	1.00	1.10	1.20	1.30	1.40	1.50	1.60	1.70	1.80	1.90
$\mathrm{E}(a_B)$	1.00	1.09	1.17	1.26	1.34	1.43	1.52	1.60	1.68	1.76
$\mathrm{E}(b^*)$	112.48	110.92	109.50	108.18	107.01	105.91	104.83	103.88	103.03	102.22
$F_{BC}[\mathrm{E}(b^*)]$	0.500	0.480	0.461	0.443	0.427	0.413	0.398	0.386	0.374	0.363
$\Psi_B[\mathrm{E}(b^*)]$	218.78	226.07	235.00	248.91	266.01	292.70	330.51	377.20	440.95	528.11

Table 3.8: Brenda's expected optimal bids, probabilities of winning and her expected utilities when she is a risk-seeking bidder and she assumes that Charles is also a risk-seeking bidder.

r_B	1.00	1.10	1.20	1.30	1.40	1.50	1.60	1.70	1.80	1.90
$E(a_B)$	1.00	1.09	1.17	1.26	1.34	1.43	1.52	1.60	1.68	1.76
$E(b^*)$	100.79	98.73	96.90	95.19	93.63	92.13	90.76	89.53	88.38	87.38
$F_{BC}[E(b^*)]$	0.954	0.914	0.879	0.846	0.815	0.786	0.760	0.736	0.714	0.694
$\Psi_B[E(b^*)]$	246.96	266.81	291.68	331.28	381.06	460.58	575.91	721.59	924.83	1208.68

Table 3.9: Brenda's expected optimal bids, probabilities of winning and her expected utilities when she is a risk-averse bidder and she assumes that Charles is a risk-seeking bidder.

r_B	0.10	0.20	0.30	0.40	0.50	0.60	0.70	0.80	0.90	1.00
$E(a_B)$	0.07	0.16	0.25	0.35	0.44	0.55	0.66	0.77	0.88	1.00
$E(b^*)$	143.27	136.65	130.75	125.00	120.00	115.14	111.06	107.36	103.85	100.79
$F_{BC}[E(b^*)]$	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	0.954
$\Psi_B[E(b^*)]$	201.14	201.51	202.09	203.08	204.47	207.05	211.21	217.99	229.14	246.96

Now, we provide a brief sketch of how Brenda would find the ARA solution when she wants to perform a level-3 analysis. In this case, $k = 3$ and Brenda assumes that Charles is a level-2 thinker who would model Brenda as a level-1 thinker and believes that Brenda would model Charles as level-0 thinker. To find the ARA solution in this case, Brenda would perform the level-2 analysis detailed above for Charles and obtain his optimal bid $C^*(W_B)$ using (3.21). Using (3.21) and using Monte Carlo methods to incorporate the uncertainty in W_B, she would get her belief about F_{BC}, the distribution of Charles's (level-2 thinker) bid. She can then obtain her optimal bid $b^*(w_C)$ or the expected value of her optimal bid $E(b^*)$ using similar process as in (3.17) and (3.18), respectively.

3.5 Conclusion and Further Work

In this chapter, we propose a better way to model the FPSB auctions than what has previously been done. Specifically, we assume that the item being auctioned is a normal item. First, we propose a new utility function that is realistic and constrains the bid value and true value in consideration with the wealth of the bidder, and a new CRRA parameter that models the change in risk behaviour of the bidder with increase in their wealth, as would be expected for a normal item. Secondly, we model the problem using the ARA approach and here we extend the ARA solution developed by Banks et al. (2015) to consider the reserve price, to include not only risk-neutral but also, risk-averse and risk-seeking bidders and to allow each bidder to have different wealth.

In this chapter, we show how an ARA solution for an FPSB auction problem can be derived when assuming that the adversary is a non-strategic player and when assuming that they are a level-k thinker instead. We provide numerical examples to illustrate these solutions. The solutions are very easy to be found using a basic Monte Carlo approach. The example shows that overall, the optimal bids for risk-averse bidders are higher than risk-neutral bidders and that the optimal bids increase as the level of risk aversion increases. In contrast, but as expected, the optimal bids for the risk-seeking bidders are lower than risk-neutral bidders and the optimal bids decrease as the level of risk-seeking behaviour intensifies. The examples also highlight that the probability of winning the bid increases with risk aversion and decreases with risk-seeking behaviour. Further, it also shows that a bidder would typically bid higher with an increase in their relative wealth.

While it is possible for the defender to model the problem using ARA by assuming various solution concepts for the adversary, it may be possible that the defender does not know how their adversary may solve the problem and therefore needs to incorporate concept (model) uncertainty into their solutions. This can be easily done (Banks et al., 2015).

The practical challenge in adopting a Bayesian approach of incorporating uncertainties using prior distributions is the elicitation of these distributions. Ríos Insua et al. (2016) provide an outline for a robustness analysis for ARA. It is important to investigate the sensitivity of the optimal bid to any errors or mis-specifications in the utilities and the probabilities elicited for the analysis. Robustness analysis of ARA to these elicitations is necessary, but has yet to be developed.

ARA solutions for FPSB auctions for other solution concepts such as the BNE, mirror equilibrium or minimax approach have yet to be derived. Also, we have derived solutions for a two person game. General solutions for n-player games have to be derived too. Finally, other utility functions have also been proposed to model FPSB auctions, for example, the utility function that incorporates bidders' winning and losing regret such as used by Engelbrecht-Wiggans and Katok (2007). ARA solutions for a different utility function such as this one have to be derived and sensitivity of the ARA solution to the choice of the utility function needs to be studied as well.

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Chapter 4

Adversarial Risk Analysis for Auctions using Mirror Equilibrium and Bayes Nash Equilibrium

In this chapter, we use the adversarial risk analysis (ARA) methodology to model first-price sealed-bid auctions under quite realistic assumptions. We find ARA solutions for mirror equilibrium and Bayes Nash equilibrium solution concepts, not only for risk-neutral but also for risk-averse and risk-seeking bidders. We also consider bidders having different wealth and assume that the auctioned item has a reserve price.

4.1 Introduction

4.1.1 Modelling FPSB Auctions

A decision theorist typically uses her personal probabilities in a way as described by Savage (1954) and these probabilities describe her subjective beliefs about the possible actions that her rivals may take. For auctions, a decision theorist aims to find her optimal bid by assessing the probability distribution of the best competitive bid while she assumes that the other bidders are non-strategic. Friedman (1956) presented the first modern academic paper that used a *decision-theoretic* model for *first-price sealed-bid* (FPSB) *auctions*[1]. Decision theorists argue that, in practice, bidders make use of a decision-theoretic approach rather than a game-theoretic approach for auctions (see e.g. Capen et al., 1971; Keefer et al., 1991; Rothkopf and Harstad, 1994; Rothkopf, 2007; Wang and Guo, 2017; Engelbrecht-Wiggans and Katok, 2007, among others).

On the other hand, a *Bayesian game theorist* assumes that her opponents are strategic, they draw their valuations for the auctioned item from the commonly known distributions to all bidders and then they all find a Bayes Nash equilibrium (BNE) bidding strategy. Using a Bayesian game-theoretic approach, Vickrey (1961) modelled FPSB auctions assuming that n bidders draw their valuations from a commonly known uniform distribution. Criesmer et al. (1967); Wilson (1969); Riley and Samuelson (1981); Cox et al. (1982a,b); Maskin and Riley (2000); Goeree et al. (2002); Bajari (2001); Campo et al. (2011); Gentry et al. (2015); Li

[1]Bidders place their bids in sealed envelopes and simultaneously hand over them to the auctioneer. Those envelopes are then opened and the bidder with the highest bid wins and pays the amount equal to the bid.

and Tan (2017) among others also analysed these auctions using a Bayesian game-theoretic approach.

Consider the FPSB auction in which $n \geq 2$ bidders compete to buy a single auctioned item. Let the ith bidder's value and bid for the auctioned item be v_i and b_i, respectively, for $i = 1, 2, \ldots, n$. The ith bidder will win the auctioned item if $b_i = b^*$, the highest bid, and in that case, will realize an economic profit of $v_i - b^*$. As one would expect, the probability of winning would increase with an increase in bid but the profit would then decrease. Thus, the bidder has to balance between the desirable increase in winning probability and the undesirable decrease in profit.

We follow Hubbard and Paarsch (2014) and sketch how the Bayesian game-theoretic solution for an asymmetric FPSB auction has been found in the literature of auctions. It is assumed that the bidders randomly draw their valuations for the item from the distributions $G_i[v_i]$ for $i = 1, \ldots, n$ which are commonly known to all bidders with common support $[\underline{v}, \bar{v}]$. We are interested in finding the BNE bid functions. Let $b_i(v_i)$ be the bid function that typically maps the ith bidder's values to her bids and gives her optimum bid given her true value v_i. It is assumed that the bid functions are strictly monotonically increasing in bidders' valuations; i.e., bidders' bids increase with increase in their true values. This monotone nature of $b_i(v_i)$ allows us to write v_i as a function of b_i. The function that maps the ith bidder's bids to her values is called the inverse bid function and denoted by $v_i(b_i)$. Suppose that the bidders use the constant relative risk aversion (CRRA) utility function

$$u(b; v) = (v - b)^r, \tag{4.1}$$

where $(1 - r) = -wu''(w)/u'(w)$ is the coefficient of CRRA or Arrow-Pratt measure of relative risk aversion (Pratt, 1964; Arrow, 1965). Then, the BNE bid functions $b_i(v_i)$ could be found by maximizing each expected utility function with respect to its argument b_i. In most cases, no closed-form solution exists and numerical methods are thus required to find $b_i(v_i)$. Marshall et al. (1994) found numerical solutions for asymmetric auctions with risk-neutral bidders by adopting the shooting algorithm for solving the differential equation formulation of the problem. Bajari (2001); Fibich and Gavious (2003); Gayle and Richard (2008); Hubbard and Paarsch (2009); Fibich and Gavish (2011); Hubbard and Paarsch (2014) among others also gave numerical solutions for these auctions when the bidders are assumed to be risk-neutral.

These works did not include numerical solutions when the bidders are risk-averse or risk-seeking. Moreover, the utility function (4.1) does not take into account bidders' wealth and also has other limitations (Ejaz et al., 2021, or Chapter 3). Further, Hubbard and Paarsch (2014) and others found the BNE solution under the common prior assumption; i.e., each bidder knows the valuation distributions of the other bidders and that this is common knowledge, which is also not realistic as discussed below.

4.1.2 Adversarial Risk Analysis (ARA)

Both the decision and game-theoretic approaches have their drawbacks in practice. It may be unrealistic to assume other bidders are non-strategic as in a decision-theoretic approach because bidders may often be strategic. On the other hand, the common knowledge assumption about the valuation distributions in the Bayesian game-theoretic model can also be unrealistic because the distribution used by one bidder is usually not commonly known to others. In fact, often the bidders try to keep their bids secret in order to win the auctioned item and to gain a competitive advantage. Also, it becomes more difficult to find a BNE as the games get more complex and thus it may be possible that a unique BNE solution for such a game does not exist.

Ríos Insua et al. (2009) introduced an approach called adversarial risk analysis (ARA) to overcome the drawbacks of both decision-theoretic and Bayesian game-theoretic approaches. ARA takes into account the presence of an intelligent adversary and models strategic decision-making problems for which some specific examples are given below. ARA adopts a Bayesian approach and uses subjective prior distributions to elicit the uncertainty around the unknown beliefs, capabilities and preferences of the intelligent adversary as well as around the possible outcomes. However, ARA differs from Bayesian game theory in a number of important ways. Unlike Bayesian game theory, (i) ARA does not assume that these subjective distributions are commonly known to all players; (ii) ARA allows for different assumptions about the strategy of the intelligent adversary and these result in different solution concepts such as non-strategic play, level-k thinking, mirror equillibria (ME) or BNE; and finally, (iii) ARA solves the decision-making problem from the perspective of just one of the players and does not need to find an equilibrium solution for all players as in Bayesian game theory. Thus, an ARA solution is comparatively easy to find even for more complex decision-making problems.

ARA has been used to model a variety of real life decision-making problems such as network routing for insurgency (Wang and Banks, 2011), international piracy

(Sevillano et al., 2012), counter-terrorism (Rios and Ríos Insua, 2012), autonomous social agents (Esteban and Ríos Insua, 2014), urban security resource allocation (Gil et al., 2016), adversarial classification (Naveiro et al., 2019), counter-terrorist online surveillance (Gil and Parra-Arnau, 2019), cyber-security (Ríos Insua et al., 2021), insider threat (Joshi et al., 2021) and FPSB auctions (Ejaz et al., 2021, or Chapter 3) among others.

ARA solutions for FPSB auctions have been found previously. Banks et al. (2015) modelled these auctions assuming that each bidder is risk-neutral. They did not consider the bidder's wealth and also did not take into account the reserve price for the auctioned item. Ejaz et al. (2021) extended this work by using a more realistic utility function (4.2) (defined in the following Section 4.1.3) and by incorporating the risk behavior of bidders who may have different wealth. However, they only derived ARA solutions for the 'non-strategic play' and the 'level-k thinking' solution concepts.

4.1.3 Utility Function for FPSB Auctions

In economic theory, a good that has positive income elasticity is defined as a *normal good*. That is, demand for a normal good rises when income increases and falls when the income falls (see, for example, Fisher, 1990; Goeree et al., 2002; Piros and Pinto, 2013; Perloff, 2015; Baisa, 2017, for more details).

In the context of auctions, it is important to note that *risk* relates to the risk of not winning the item. A *risk-neutral* bidder is one who is indifferent to risk when making a bidding decision. A *risk-averse* bidder is a bidder who does not want to lose the item and bids more aggressively than a risk-neutral bidder. Whereas, a *risk-seeking* bidder is keen to get the item at a low price and bids less aggressively than a risk-neutral one.

Baisa (2017) stated that when bidding on a normal item, the bidders' willingness to pay for the auctioned item increases with their wealth. Thus, bidding on a normal item, bidders could be more risk-averse (or less risk-seeking) with increase in their wealth. Assuming two bidders, Ejaz et al. (2021) proposed a modified risk behavior parameter that changes with the relative change in circumstances of the bidders' wealth. This modified risk behavior parameter is defined later in Section 4.2.2 for the general case of n bidders.

With n bidders, we define the utility function for the ith bidder as

$$u(b_i, v_i, w_i) = \begin{cases} w_i + (v_i - b_i)^{a_i} & \text{if she wins the bid,} \\ w_i & \text{if she loses the bid,} \end{cases} \tag{4.2}$$

where w_i is the wealth, a_i is a modified CRRA parameter and v_i is her true value of the item. This utility function takes into account the bidders' wealth, risk appetite and change in risk behavior due to perceived relative wealth in relation to other bidders' wealth. Therefore, as explained in more detail in Ejaz et al. (2021) or Chapter 3 that the utility function given in (4.2) is considered to be a more realistic one for auctions compared to the standard CRRA utility function of the form $(w_i + v_i - b_i)^{r_i}$ used previously, for instance, by Lu and Perrigne (2008); Gentry et al. (2015); Li and Tan (2017), etc. Using the utility function (4.2), we make the natural assumption that $b_i \leq v_i \leq w_i$, because the ith bidder does not want to bid more than the true value and can pay an amount $b_i \leq v_i$ only if the bidder's wealth is at least equal to the true value. Thus, using the utility function

(4.2), the ith bidder's expected utility for a bid b_i would be of the form

$$\Psi_i(b_i) = \{w_i + (v_i - b_i)^{a_i}\}F(b_i) + w_i\{1 - F(b_i)\},$$

where $F(b_i)$ is her probability of winning the item from a bid b_i. The above equation simplifies to

$$\Psi_i(b_i) = w_i + (v_i - b_i)^{a_i} F(b_i). \tag{4.3}$$

Thus, using (4.3), the ith bidder can find her optimal bid by solving the equation

$$\max_{b_i}[w_i + (v_i - b_i)^{a_i} F(b_i)].$$

4.1.4 Contributions in this Chapter

The main contributions of this chapter are as follows:

- We assume that the auctioned item is normal and extend Banks et al. (2015) and Ejaz et al. (2021) or Chapter 3 by developing ARA solutions for mirror equilibrium and BNE solution concepts, wherein,
  - we assume that the auctioned item has a reserve price τ (typically, known to each bidder in advance),
  - we take into account bidders' wealth,
  - we find solutions not only for risk-neutral bidders but also for risk-averse and risk-seeking bidders.
- We extend the work of Hubbard and Paarsch (2014), wherein,
  - we solve the problem using ARA,

- we take into account bidders' wealth,
- we use the utility function (4.2) and find numerical solutions using the shooting algorithm for risk-averse and risk-seeking bidders.

4.1.5 Structure of the Chapter

The structure of the chapter is as follows. Section 4.2 briefly describes how ARA works and presents an asymmetric FPSB auction model along with its BNE solution using the utility function (4.2). We also generalize the shooting algorithm found in Hubbard and Paarsch (2014) to allow risk-averse and risk-seeking behavior of bidders while assuming that the bidders have different wealths. Section 4.3 explains how an ARA solution can be found for the mirror equilibrium solution concept for two bidders. In Section 4.4, ARA solutions for the BNE solution concept for FPSB auctions are developed. Finally, Section 4.5 contains a discussion of the results obtained in this chapter and areas for future research.

4.2 Asymmetric FPSB Auctions Model

In this Section, we present the general ARA framework for FPSB auctions. We also derive the BNE solution for these auctions using the utility function (4.2) and discuss the shooting algorithm to find the numerical solution.

4.2.1 ARA Framework for FPSB Auctions

We consider a two player game between Brenda (B) and Charles (C) in order to give a general insight into the ARA framework. Let the choices that B and C make be denoted by b and c from their respective set of actions $\mathcal{B}$ and $\mathcal{C}$, respectively. The resulting outcome, denoted by S, is a chance variable and takes the values from a set of possible outcomes $\mathcal{S}$. Let the utilities received by B and C from a pair of actions (b, c) and outcome s be denoted by the utility functions $u_B(b, c, s)$ and $u_C(b, c, s)$, respectively. Suppose we solve the problem from Brenda's perspective. Among other objectives, of course, a typical objective could be to find her optimal choice b^* that maximizes her expected utility. This game could be represented by a bi-agent influence diagram (BAID) as shown in Figure 4.1 where decision, chance and utility nodes are denoted by rectangles, circles and hexagons, respectively. Figure 4.1a represents this game from both B and C's point of view while Figure 4.1b represents this game from only B's point of view where she is uncertain about C's choices and thus, C's decision node is now a chance node for B and therefore represented by a circle instead of a rectangle.

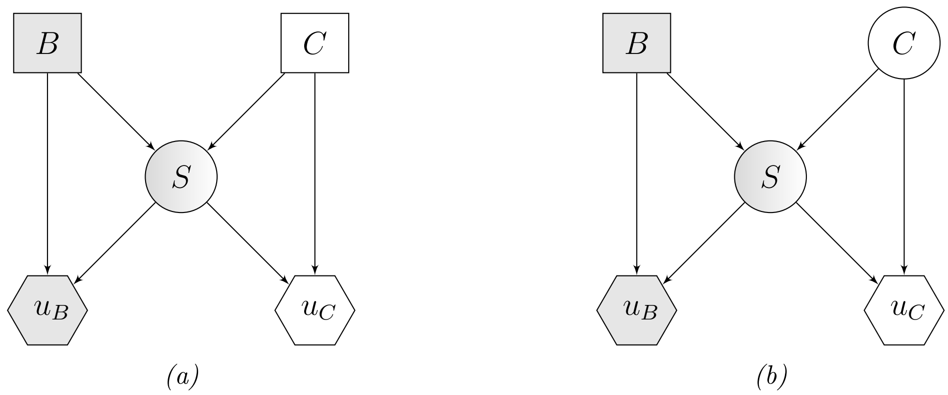


Figure 4.1: (a) The BAID for the two player game (b) The BAID for the two players from Brenda's perspective

ARA is typically solved using backward induction, where, one first considers the very last node and then solves for each node in the reverse order until one reaches the starting node. To find an ARA solution of the game from Brenda's perspective, we need to integrate out the uncertainties at chance nodes C and S to reach the decision node B that maximise her expected utility. Thus, an ARA solution of this game could be found as follows.

We start with Brenda's expected utility while taking into account her uncertainty about the outcomes S

$$\Psi_B(b,c) = \int u_B(b,c,s) p_B(s|b,c) ds,$$

where $p_B(s|b,c)$ is B's uncertainty in S given b and c. Then, we take into account her uncertainty about C's choices and find her expected utility as

$$\Psi_B(b) = \int \Psi_B(b,c) p_B(c) dc,$$

where $p_B(c)$ is Brenda's uncertainty in C. Finally, we find the optimal choice b^* that maximises her expected utility as

$$b^* = \arg\max_{b \in \mathcal{B}} \Psi_B(b).$$

The determination of $p_B(c)$ in the above modelling is the main challenge. Brenda may elicit $p_B(c)$ either by using an expert opinion or by using data on Charles's past choices for a similar problem or by her subjective beliefs. On the other hand, Brenda could also elicit $p_B(c)$ by modelling Charles's strategic thinking process. For instance, she may believe that Charles is also an expected utility maximizer like she is and may make the choice c^* to maximise his expected utility. She can aim to find c^*, if she knows about Charles's utility function $u_C(b,c,s)$ and his probabilities $p_C(b)$ and $p_C(s|b,c)$. However, $u_C(b,c,s)$, $p_C(b)$ and $p_C(s|b,c)$ are usually unknown to her. She can model her uncertainty about Charles's utility and his probabilities by eliciting a random utility $U_C(b,c,s)$ and random probabilities $P_C(b)$ and $P_C(s|b,c)$. If Brenda thinks that Charles is an expected utility maximizer, she can find Charles's random optimal choice C^* similar to how she finds her own as described above. However, Charles could be assumed to use a number of alternative solution concepts and thus $p_B(c)$ could be determined accordingly. Note that in this chapter, we elicit $p_B(c)$ by eliciting $G_B[v_B(c)]$, the cumulative distribution on the equilibrium inverse bid function that Charles will bid c.

Ejaz et al. (2021) or Chapter 3 found ARA solutions for FPSB auctions from Brenda's perspective using the utility function (4.2) for the cases where she assumes that Charles is a non-strategic player or that he is a level-k thinker. In Section 4.3, we find ARA solutions from Brenda's perspective, when she thinks that Charles will find his optimal bid using a mirror equilibrium solution concept. Then in Section 4.4, we find an ARA solution when she believes that her opponents use BNE to find their optimal bids. The ARA solutions we derive in Sections 4.3 and 4.4 turn out to be based on the BNE solution for asymmetric FPSB auctions using the utility function (4.2). Therefore, we first derive the BNE solution in the next Section.

4.2.2 BNE Solution using Utility Function (4.2)

Here we derive the BNE solution for our asymmetric FPSB auction model which is based upon the utility function (4.2). Suppose that we have n bidders and they draw their valuations independently from other bidders. We assume that the auctioned item has a reserve price τ (typically, known to each bidder in advance).The bidder who bids the largest among n bidders wins the auction, and pays the amount equal to the bid.

We assume three types of asymmetries in our model. The first type of asymmetry is that the valuation distribution for each bidder is different; i.e., we assume that the bidders draw their valuations from the distributions $G_i(v_i)$ with common support $[\underline{v}, \bar{v}]$ for $i = 1, 2, \ldots, n$ such that $\underline{v} \geq \tau$. The second type of asymmetry is that the bidders are assumed to be heterogeneous in their risk behavior (risk-neutral, risk-averse or risk-seeking). Finally, the third type of asymmetry is that the bidders draw their wealth from different distributions $H_i(w_i)$ with support $[\underline{w}_i, \bar{w}_i]$ such that $\underline{w}_i \geq \underline{v}$ for $i = 1, 2, \ldots, n$. Note that those distributions may have different supports (not necessarily common). This third type of asymmetry, in particular, has not been considered previously in the literature of asymmetric FPSB auctions.

Ejaz et al. (2021) or Chapter 3 introduced a modified risk behavior parameter assuming two bidders. This modification is based on the theory of *normal* goods and assumes that a bidder will be willing to pay more for an item (that is, become more risk averse or less risk seeking) if they think that they have relatively more wealth compared to other bidders. We generalize it for the case when we have n bidders. We assume that each bidder has a baseline risk behavior parameter r_i, $i = 1, \ldots, n$, which represents the ith bidder's natural risk appetite. This r_i can then be used to derive a modified risk parameter a_i which will be dependent on the ith bidder's wealth relative to other bidders. This modified risk parameter a_s will be different from r_s only if the sth player is considered to be the wealthiest among all bidders, else $a_i = r_i$, for all $i \neq s$. In the rest of this Section, we treat the situation as being from the perspective of the ith bidder.

Suppose that w_i is the ith bidder's wealth and she believes that she is the wealthiest among all n bidders. She also believes that the jth bidder having wealth w_j is the wealthiest among the other $(n-1)$ bidders. In this case, we define the modified risk behavior parameter a_i for the ith bidder as

$$a_i = \begin{cases} r_i^{\frac{1}{h}} & \text{if} \quad 0 < r_i < 1, \\ r_i & \text{if} \quad r_i = 1, \\ r_i^h & \text{if} \quad 1 < r_i \leq 2, \end{cases} \tag{4.4}$$

where $0 < h = w_j/w_i < 1$. Here, the ith bidder becomes more risk-averse (or less risk-seeking) as she believes she is wealthiest among all n bidders. She would also believe that the other $(n-1)$ bidders' (including the jth bidder) modified risk behavior parameter would be equal to their baseline risk behavior parameter; i.e., $a_s = r_s$ for $s = 1, 2, \ldots, n$, $s \neq i$. On the other hand, if she believes that the jth bidder is the wealthiest bidder among all n bidders, she can find a_j, the jth bidder's modified risk behavior parameter from (4.4) by replacing i with j and taking $h = w_i/w_j < 1$. In this situation, we also assume that she would then model the $(n-1)$ bidders' (including her own) risk behavior as being unchanged so that $a_s = r_s$ for $s = 1, 2, \ldots, n$, $s \neq j$.

Now, using the utility function (4.2), the expected utility of the ith bidder from making a bid b_i is

$$\Psi_i = w_i + (v_i - b_i)^{a_i} Pr(\text{win}|b_i). \tag{4.5}$$

Following Hubbard and Paarsch (2014), let B_s and V_s, $s = 1, \dots, n$, $s \neq i$, be the random (unknown) bids and true values of the other $(n-1)$ bidders, respectively. Assuming that the bidders are independent in making their bids, the ith bidder would win the auction if

$$\begin{aligned} Pr[\text{win}|b_i] &= Pr[B_1 < b_i, B_2 < b_i, \dots, B_{i-1} < b_i, B_{i+1} < b_i, \dots, B_n < b_i] \\ &= \prod_{s \neq i} G_s[v_s(b_i)], \end{aligned}$$

where $G_s[v_s(b_i)]$ are the distribution functions on the inverse bid function for the other $(n-1)$ bidders. Thus, Equation (4.5) becomes

$$\Psi_i = w_i + (v_i - b_i)^{a_i} \prod_{s \neq i} G_s[v_s(b_i)]. \tag{4.6}$$

Our motive is to find the BNE bid functions $b_i(v_i)$. This could be done by maximizing each expected utility function with respect to its argument b_i. Thus, the first order condition for (4.6) is

$$\begin{aligned} \frac{\mathrm{d}\Psi_i}{\mathrm{d}b_i} &= -a_i(v_i - b_i)^{a_i - 1} \prod_{s \neq i} G_s[v_s(b_i)] + \\ &\qquad (v_i - b_i)^{a_i} \sum_{s \neq i} g_s[v_s(b_i)] \frac{\mathrm{d}v_s(b_i)}{\mathrm{d}b_i} \prod_{l \neq s,i} G_l[v_l(b_i)] = 0 \\ \implies \frac{a_i}{(v_i - b_i)} \prod_{s \neq i} G_s[v_s(b_i)] &= \sum_{s \neq i} g_s[v_s(b_i)] \frac{\mathrm{d}v_s(b_i)}{\mathrm{d}b_i} \prod_{l \neq s,i} G_l[v_l(b_i)], \end{aligned}$$

where, $g_s[v_s(b_i)]$ are the probability distributions on the inverse bid function for the other $(n-1)$ bidders. Now, to find the BNE inverse bid functions, we can replace b_i with a general bid b and v_i with $v_i(b)$ because in the asymmetric setting

$v_i(b) \neq v_s(b)$ for $i \neq s$ (Hubbard and Paarsch, 2014). Thus, the above equation simplifies to

$$\frac{a_i}{(v_i(b) - b)} \prod_{s \neq i} G_s[v_s(b)] = \sum_{s \neq i} g_s[v_s(b)] \frac{\mathrm{d}v_s(b)}{\mathrm{d}b} \prod_{l \neq s,i} G_l[v_l(b)].$$

This further simplifies to

$$\frac{a_i}{v_i(b) - b} = \sum_{s \neq i} \frac{g_s[v_s(b)]}{G_s[v_s(b)]} v_s'(b). \tag{4.7}$$

This can be summed over all n bidders to yield

$$\sum_{s=1}^{n} \frac{a_s}{v_s(b) - b} = (n-1) \sum_{s=1}^{n} \frac{g_s[v_s(b)]}{G_s[v_s(b)]} v_s'(b),$$

or

$$\frac{1}{(n-1)} \sum_{s=1}^{n} \frac{a_s}{v_s(b) - b} = \sum_{s=1}^{n} \frac{g_s[v_s(b)]}{G_s[v_s(b)]} v_s'(b).$$

Now, subtracting Equation (4.7) from this expression yields

$$\left[\frac{1}{(n-1)} \sum_{s=1}^{n} \frac{a_s}{v_s(b) - b}\right] - \frac{a_i}{v_i(b) - b} = \frac{g_i[v_i(b)]}{G_i[v_i(b)]} v_i'(b).$$

This expression then leads to the following system of ODE's formulation

$$v_i'(b) = \frac{G_i[v_i(b)]}{g_i[v_i(b)]} \left\{ \left[\frac{1}{(n-1)} \sum_{s=1}^{n} \frac{a_s}{v_s(b) - b}\right] - \frac{a_i}{v_i(b) - b} \right\}, \quad i = 1, \ldots, n. \tag{4.8}$$

Let $\bar{b}$ be the common highest possible bid and $\underline{b}$ be the common lowest possible bid. Then, as $v_i(\underline{b}) = b(\underline{v}) = \underline{v}$, the following are the boundary conditions on the equilibrium inverse bid functions:

$$\begin{aligned} &v_i(\underline{b}) = \underline{v} \text{ (or equivalently } v_i(\underline{v}) = \underline{v}), && \text{Left boundary condition,} \\ &v_i(\bar{b}) = \bar{v}, && \text{Right boundary condition,} \end{aligned} \tag{4.9}$$

for all $i = 1, \ldots, n$. The solution to the system of ODEs (4.8) is needed which satisfy both of the boundary conditions on the inverse bid functions. Note that Equation (4.8) based on the utility function (4.2) turns out to be the same as found by Hubbard and Paarsch (2014) and others based on the utility function (4.1). Whereas, in (4.8), we have incorporated the effect of bidders' wealth on their bidding behavior in the form of a modified risk behavior parameter a_i as defined in (4.4) rather than just the baseline risk behavior parameter r_i which has been used by Hubbard and Paarsch (2014) and others. In general, no closed-form solution of (4.8) exists and numerical methods are thus required.

Boundary-value problems may be solved numerically by repeatedly solving initial value problems until the boundary conditions are satisfied. The algorithms that use this approach are known as shooting algorithms. In these shooting algorithms, one of the boundary conditions is treated as an initial value and the other as the target value. For the left boundary condition $v_i(\underline{v}) = \underline{v}$, we a priori know both the valuation and the bid while for the right boundary condition $v_i(\bar{b}) = \bar{v}$, we only know the valuation $\bar{v}$, but not the common highest possible bid $\bar{b}$. Because of this, we treat the right boundary condition as the initial value with $\bar{b}$ to be found iteratively and the left boundary condition as the target value. The shooting algorithm using this approach is given below. If the guess for $\bar{b}$ (denoted by $\bar{b}_k$ in the given algorithm) is too low at a particular iteration, then the approximate solutions will not satisfy the target value at the left boundary condition. In this case, $\bar{b}_k$ needs to be increased for the next iteration. If $\bar{b}_k$ is too high, then divergence occurs in which case the guess needs to be decreased for the next iteration.

The Shooting Algorithm

For a pre-specified tolerance ϵ, the shooting algorithm can be summarised as follows:

1. Find the values of the modified risk behavior parameters a_s which are a function of wealth w_s and the baseline risk behavior parameter r_s for the sth bidder, $s = 1, \ldots, n$.

2. Set $\delta = 0.01$ and $k = 1$.

3. Make a guess for the common high bid (initial value) $\bar{b}_1 \in [\underline{v}, \bar{v}]$.

4. Solve the system of ODE's (4.8) backwards on the interval $[\underline{v}, \bar{b}_k]$ to obtain approximate equilibrium inverse bid functions $v_{s,k}(b)$, $s = 1, \ldots, n$.

5. Evaluate $v_{s,k}(b)$ at $b = \underline{v}$ for $s = 1, \ldots, n$ to decide whether to decrease, increase or stop at the guessed value $\bar{b}_k$. Specifically:

   If the solution at $\underline{v}$ diverges, decrease the guessed value $\bar{b}_k$ by setting $\bar{b}_{k+1} = (1 - \delta)\bar{b}_k$. Then set $k = k + 1$ and go back to Step 4;

   else if the solution at $\underline{v}$ is in $[\underline{v}, \bar{v}]$, but does not meet the pre-specified tolerance criteria for at least one bidder (that is, $|v_{s,k}(\underline{v}) - \underline{v}| > \epsilon$ for some s), increase the guessed value $\bar{b}_k$ by setting $\bar{b}_{k+1} = (1+\delta)\bar{b}_k$. Then set $k = k + 1$ and go back to Step 4;

   else stop if the solution at $\underline{v}$ converges, that is,

   $$|v_{s,k}(\underline{v}) - \underline{v}| \leq \epsilon \quad \text{for all} \quad s = 1, \ldots, n,$$

   and take $v_{s,k}(b)$ to be $\hat{v}_s(b)$, the approximate solution of (4.8).

4.3 ARA for FPSB Auctions using Mirror Equilibrium (ME)

In a level-k analysis, the decision maker believes that her opponent is a level-$(k-1)$ thinker who models her as a level-$(k-2)$ player and so on where $k = 1, 2, \ldots$. Thus, it is possible that the decision maker may have to do infinite regress in a level-k analysis. ME is an alternative solution concept which avoids that regress. In ME, the decision maker believes that her opponent is modelling her actions in the same way as she is modelling his actions. Both of them use their subjective distributions over the probabilities and utilities of the opponent and seek an equilibrium. In this Section, we consider two bidders, Brenda and Charles, and find ARA solutions from Brenda's perspective using a ME solution concept.

ARA for FPSB using ME solves the problem from Brenda's perspective where she believes that Charles will randomly draw the value of the item for Brenda from a distribution which is different from his own distribution and it assumes that the distributions are not commonly known (unlike in BNE) to both bidders. So, Brenda does not know about Charles's true value v_C for the auctioned item and his wealth w_C and places distributions G_{BC} on v_C and H_{BC} on w_C. Likewise, she believes that Charles also does not know about her true value v_B for the auctioned item and her wealth w_B and believes that he places distributions G_{CB} on v_B and H_{CB} on w_B.

Thus, Brenda would solve

$$\begin{aligned} &\max_b \{W_B + (V_B - b)^{a_B} G_{BC}(v_C(b))\}, \\ &\max_c \{W_C + (V_C - c)^{a_C} G_{CB}(v_B(c))\}, \end{aligned} \tag{4.10}$$

where b is Brenda's bid that Charles might elicit, c is Brenda's belief about Charles's bid, $W_B \sim H_{CB}$ is Brenda's wealth that Charles might elicit, $V_B \sim G_{CB}$ is Brenda's true value that Charles might elicit, $W_C \sim H_{BC}$ is Brenda's belief about Charles's wealth, $V_C \sim G_{BC}$ is Brenda's belief about Charles's true value, a_C is Brenda's belief about Charles's modified risk behavior parameter and a_B is Brenda's modified risk-behavior parameter that Charles might elicit. In the case $w_B > w_C$, a_B could be found from (4.4) by allowing an abuse of notation in replacing i with B and j with C. If $w_B < w_C$, then it would be the other way round. Figure 4.2 is the BAID that represents the ME solution concept for two bidders.

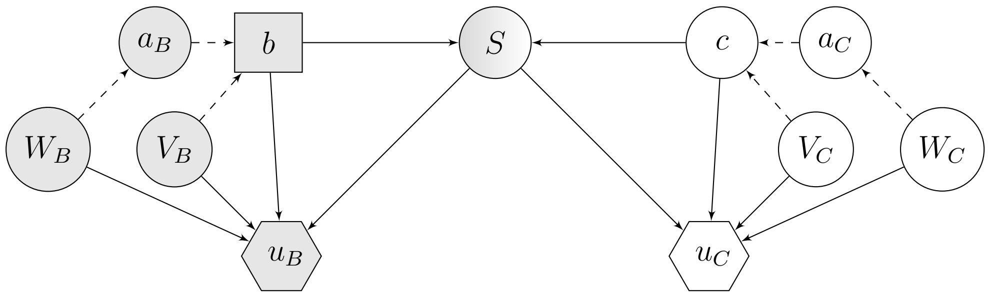


Figure 4.2: The BAID showing decision (rectangles), chance (circle) and utility (hexagons) nodes for the ME solution concept for two bidders from Brenda's perspective.

By taking the first derivative of each of the functions in (4.10) and equating to zero, Brenda obtains the following pair of ODEs for equilibrium:

$$\begin{aligned} v_B'(b) &= \frac{G_{CB}(v_B(b))}{g_{CB}(v_B(b))} \times \frac{a_C}{\{v_C(b) - b\}}, \\ v_C'(b) &= \frac{G_{BC}(v_C(b))}{g_{BC}(v_C(b))} \times \frac{a_B}{\{v_B(b) - b\}}, \end{aligned} \tag{4.11}$$

where $v_C(b) = V_C$ and $v_B(b) = V_B$ are the inverse bid functions for Charles and Brenda, respectively. Note that (4.11) is the special case of (4.8) with only two bidders but from an ARA perspective using the ME solution concept. Also, this pair of ODEs has the same boundary conditions on the inverse bid functions as given in (4.9). In general, no closed-form solution exists even for the pair of differential equations (4.11) and numerical methods are required to find the solutions. In order to find equilibrium bid functions $b(v_B)$ and $b(v_C)$ from the pair of ODEs (4.11), we use the shooting algorithm as described in Section 4.2.2. The solution obtained from any numerical method would not be an exact solution and thus the equilibrium bid functions obtained for Brenda and Charles are approximate equilibrium bid functions denoted here by $\hat{b}(v_B)$ and $\hat{b}(v_C)$, respectively. As Brenda is uncertain about Charles's risk behavior, his wealth, his valuation and believes that Charles is also uncertain in these quantities about her, she can use Monte Carlo methods to take into account these uncertainties.

Example 4.1 *Let $\Gamma(v;\kappa,\lambda)$ be the truncated Gamma distribution on the interval $(30,160]$ with parameters κ and λ. Suppose Brenda believes that Charles's valuation is a truncated Gamma distribution $G_{BC}(v_C) = \Gamma(v_C;5,25)$ and she has the belief that Charles believes that her valuation is a truncated Exponential distribution $G_{CB}(v_B) = \Gamma(v_B;1,80)$. The auctioned item has a reserve price $\tau = 25$ which is known in advance to both bidders. Suppose Brenda has a uniform distribution $H_{BC} = U[30,170]$ on Charles's wealth. Moreover, suppose that Brenda believes that Charles might elicit her wealth as being uniformly distributed with $H_{CB} = U[40,200]$. Figure 4.3 shows the assumed valuation distributions plot for Brenda and Charles.*

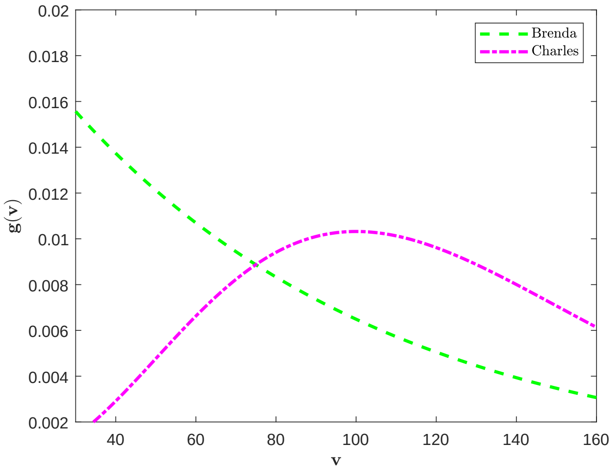
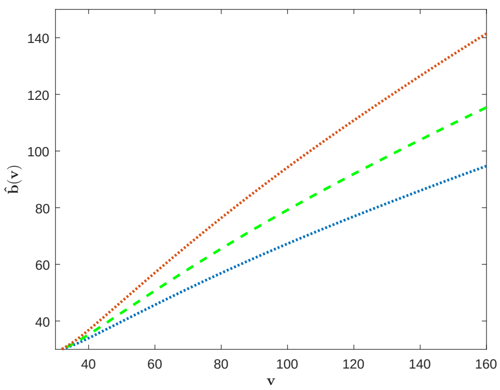
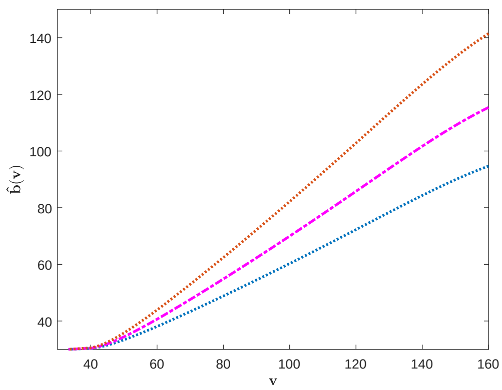


Figure 4.3: Plot of assumed valuation distributions for Brenda and Charles

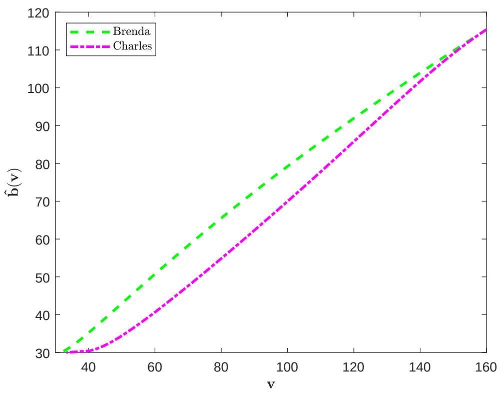
Firstly, we assume that Brenda believes that Charles is a risk-averse bidder and she believes that Charles believes that she is also a risk-averse bidder. She believes that they both draw their baseline risk-aversion parameters from the uniform distribution $U_{BC} = U_{CB} = U(0,1)$. *Assuming these distributions, she uses Monte Carlo methods to take into account their risk-behavior, wealth, and valuation uncertainties and performs* $N = 1000$ *simulations. The simulated approximate equilibrium bid functions* $\hat{b}(v_B)$ *and* $\hat{b}(v_C)$ *with 95% probability (credible) intervals for Brenda and Charles, respectively, are shown in Figure 4.4 for this case.*



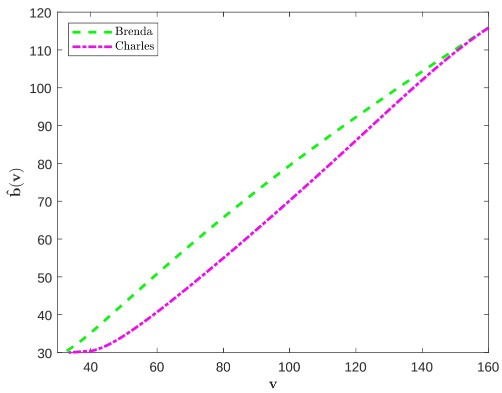
(a) Median and 95% probability interval for Brenda's approximate equilibrium bid function



(b) Median and 95% probability interval for Charles's approximate equilibrium bid function



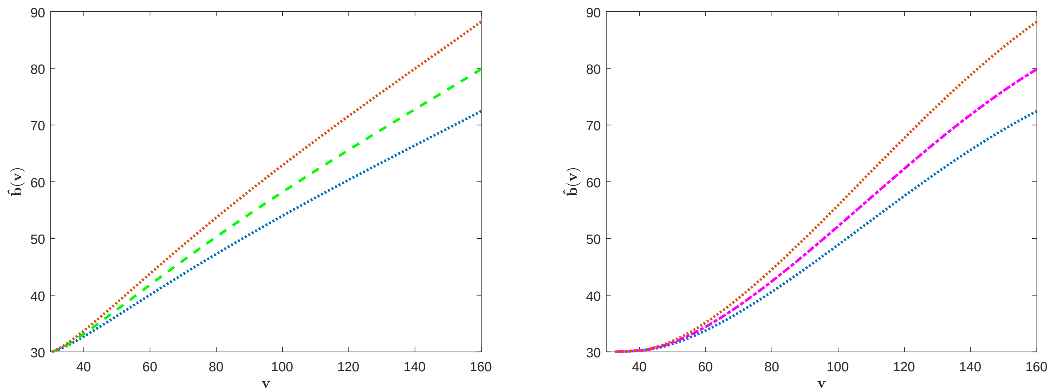
(c) Median of approximate equilibrium bid functions for Brenda and Charles



(d) Mean of approximate equilibrium bid functions for Brenda and Charles

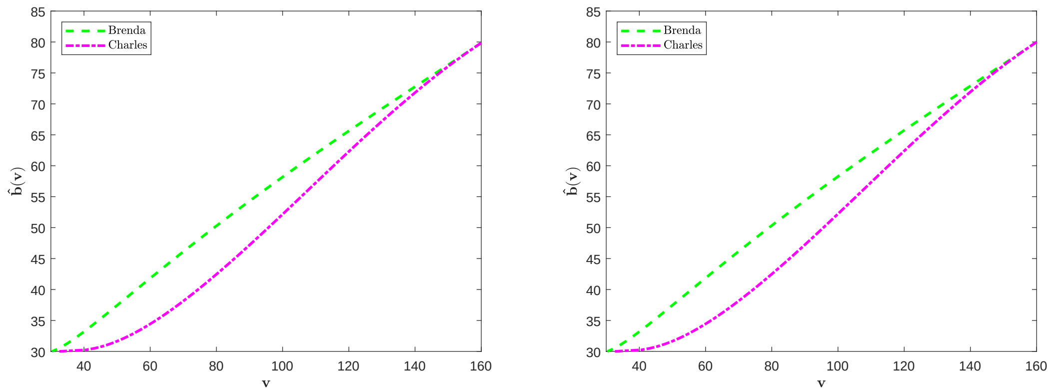
Figure 4.4: Approximate equilibrium bid functions for Brenda and Charles with 95% probability intervals when both are risk-averse bidders

Secondly, we assume that Brenda believes that Charles is a risk-seeking bidder and she believes that Charles believes that she is also a risk-seeking bidder. She believes that they both draw their baseline risk-seeking parameters from the uniform distribution $U_{BC} = U_{CB} = U(1, 2]$. *Assuming these distributions on their risk* *behavior, wealth and valuations, the results for* $N = 1000$ *simulations are shown in Figure 4.5 for this case.*



(a) Median and 95% probability interval for Brenda's approximate equilibrium bid function

(b) Median and 95% probability interval for Charles's approximate equilibrium bid function



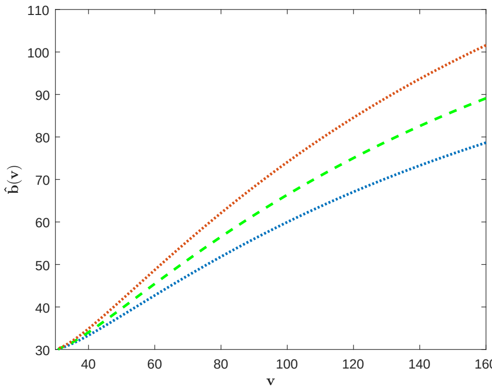
(c) Median of approximate equilibrium bid functions for Brenda and Charles

(d) Mean of approximate equilibrium bid functions for Brenda and Charles

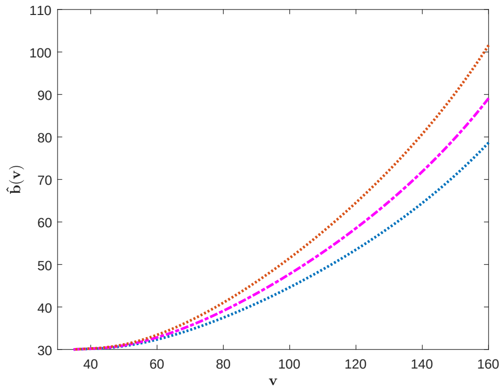
Figure 4.5: Approximate equilibrium bid functions for Brenda and Charles with 95% probability intervals when both are risk-seeking bidders

Thirdly, we assume that Brenda believes that Charles is a risk-seeking bidder and she believes that Charles believes that she is a risk-averse bidder. She draws Charles's baseline risk seeking parameter from the uniform distribution $U_{BC} =$

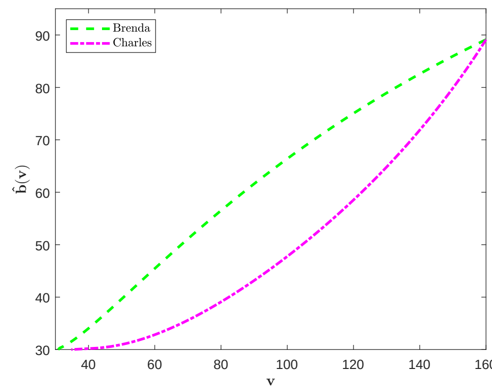
$U(1,2]$ and she believes that Charles draws her baseline risk aversion parameter from the uniform distribution $U_{CB} = U(0,1)$. The results of Monte Carlo simulations for this case are shown in Figure 4.6.



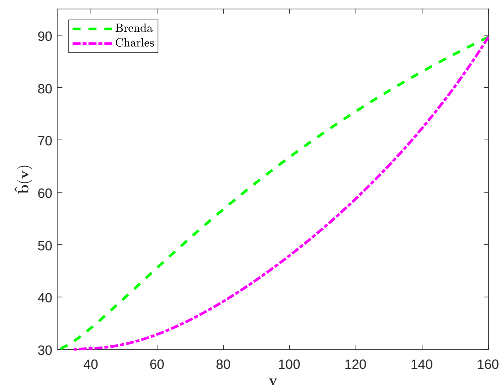
(a) Median and 95% probability interval for Brenda's approximate equilibrium bid function



(b) Median and 95% probability interval for Charles's approximate equilibrium bid function



(c) Median of approximate equilibrium bid functions for Brenda and Charles



(d) Mean of approximate equilibrium bid functions for Brenda and Charles

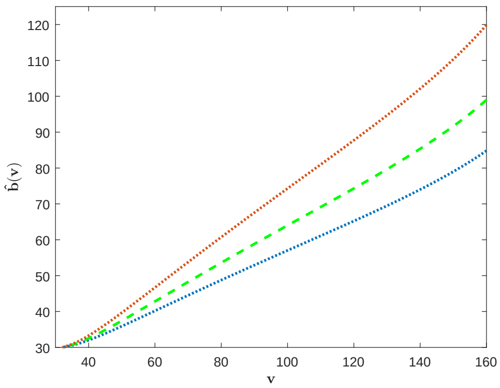
Figure 4.6: Approximate equilibrium bid functions for Brenda and Charles with 95% probability intervals when Brenda is a risk-averse bidder and Charles is a risk-seeking bidder

Finally, we assume that Brenda believes that Charles is a risk-averse bidder and she believes that Charles believes that she is a risk-seeking bidder. She draws

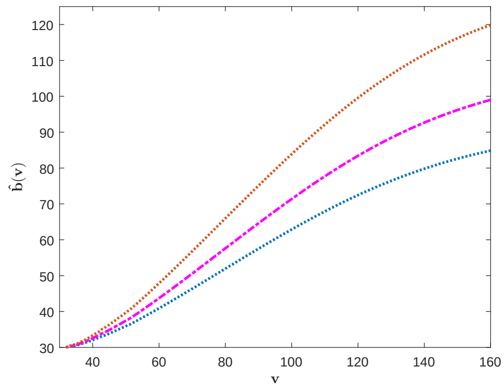
Charles's baseline risk aversion parameter from the uniform distribution $U_{BC} = U(0,1)$ *and she believes that Charles draws her baseline risk seeking parameter from the uniform distribution* $U_{CB} = U(1,2]$. *Figure 4.7 shows the results obtained from Monte Carlo simulations for this case.*

Note that in all four cases presented above, the plots of the probability intervals converge to the lower boundary condition as the shooting algorithm has used this as the 'target' for the system of ODEs given in (4.11). *Whereas, these probability intervals are apart at the upper boundary condition. This is because at this end, the shooting algorithm yields different values of* $\bar{b}$ *for the (different) simulated values of wealth and modified risk behavior parameters. Overall, the simulation results show that Brenda would most likely win the auction in the scenarios considered except where she is a risk-seeking bidder and Charles is a risk-averse bidder (see Figure 4.7).*

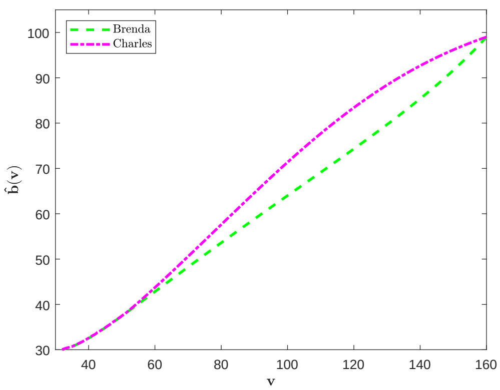
Now, Brenda needs to find her optimal bid. For illustrative purposes, we consider the first case which is shown in Figure 4.4 where both Brenda and Charles are assumed to be risk-averse bidders. She could use the mode or the mean of Charles's valuation distribution as an estimate of his true value v_C. *Suppose she decides to use the mode of Charles's valuation distribution as an estimate of* v_C. *In our Example, she has a truncated Gamma distribution on Charles's value with* $\kappa = 5$ *and* $\lambda = 25$. *Thus, the mode of this distribution is* $(\kappa - 1)\lambda = 4 \times 25 = 100$. *Now, suppose she uses the mean of the approximated equilibrium bid functions presented in Figure 4.4d to obtain an estimate of her optimal bid (alternatively, she could use the median of the approximated equilibrium bid functions as presented in Figure 4.4c). Thus, assuming that Brenda has her true value* $v_B = 100$, *her approximate mean optimal bid is* 78.24 *against Charles's approximate mean optimal* *bid of* 70.20*. Moreover, the data for Figure 4.4d shows that Brenda's bid would be a winning bid (that is, greater than* 70.20*) for all values of* $v = v_B \gtrsim 86.33$.



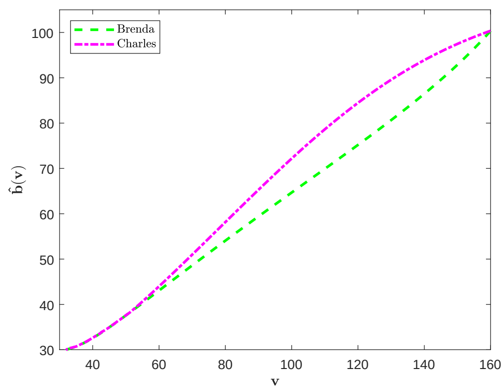
(a) Median and 95% probability interval for Brenda's approximate equilibrium bid function



(b) Median and 95% probability interval for Charles's approximate equilibrium bid function



(c) Median of approximate equilibrium bid functions for Brenda and Charles



(d) Mean of approximate equilibrium bid functions for Brenda and Charles

Figure 4.7: Approximate equilibrium bid functions for Brenda and Charles with 95% probability intervals when Brenda is a risk-seeking bidder and Charles is a risk-averse bidder

4.4 ARA for FPSB Auctions using BNE

In this section, we develop ARA solutions from Brenda's perspective where she assumes that the other bidders use a BNE to find their optimal bids. There are the symmetric and asymmetric cases to consider.

4.4.1 Symmetric Case

From a Bayesian game-theoretic perspective, the symmetric case with n bidders has the assumption that bidders draw their valuations independently from a common distribution G that is known to all the bidders and who are aware that each bidder knows G. This is the strong common knowledge assumption.

Here, we derive an ARA solution from the ith bidder's perspective assuming n bidders when she believes that all of her opponents will solve the problem using a symmetric BNE approach. For consistency and simplicity, we assume that the ith bidder is Brenda and she assumes that all of the other $(n-1)$ bidders will draw their true values from the distribution G with support $(\underline{v}, \bar{v}]$ such that $\underline{v} \geq \tau$, where τ is the reserve price for the item. Note that here, unlike in BNE, we do not assume that Brenda and her $(n-1)$ opponents know the common distribution G. Moreover, we take into account bidders' wealth where Brenda believes that all of her rivals draw their wealth from a distribution H with support $(\underline{w}, \bar{w}]$ such that $\underline{w} \geq \underline{v}$. We assume that Brenda being the ith bidder has true value v_i, modified risk behavior parameter a_i as defined in (4.4) and wealth w_i for the item and consequently bids $b(v_i)$. For simplicity, we relabel w_i to w, a_i to a, v_i to v and thus $b(v_i)$ to $b(v)$.

Now, using the modified risk behavior parameter (4.4) and the utility function (4.2), we find the equilibrium bid function $b(v)$. Thus, Brenda's expected utility having bid function $b(v)$ would be

$$[w + \{v - b(v)\}^a]Pr[\text{Brenda wins}] + wPr[\text{Brenda loses}].$$

Since the bidding function $b(v)$ is a strictly increasing function of v, then $b(v)$ would be the winning bid if and only if all other $(n-1)$ bidders bid $b(v_s) < b(v)$ for $s = 1, \ldots, n,\ s \neq i$. Now, as each bidder independently draws their value, the ith bidder will win with probability $G(v)^{n-1}$ So, Brenda's expected utility by bidding $b(v)$ would be

$$[w + \{v - b(v)\}^a]G(v)^{n-1} + w[1 - G(v)^{n-1}],$$

which simplifies to

$$w + [v - b(v)]^a G(v)^{n-1}. \tag{4.12}$$

For all the other $(n-1)$ bidders with their own wealth, own modified risk behavior parameters and own true values, the expression (4.12) would hold. The equilibrium bid would be $b(v)$ and could be found by solving

$$\max_x \{w + [v - b(x)]^a G(x)^{n-1}\},$$

for some x around v. At that value of x, the derivative of the above expression must be zero:

$$\frac{d}{dx}[w + \{v - b(x)\}^a G(x)^{n-1}] = 0,$$
$$b'(v) + \frac{n-1}{a} g(v)G(v)^{-1} b(v) = \frac{v(n-1)}{a} g(v)G(v)^{-1}.$$

This is a first order linear differential equation. Using the integrating factor method to solve it for any v in the support $(\underline{v}, \bar{v}]$ of G yields

$$b(v) = \frac{\frac{n-1}{a}\int_{\underline{v}}^{v} zg(z)G(z)^{\frac{n-1}{a}-1}dz}{G(v)^{\frac{n-1}{a}}}.$$

By using integration by parts, this expression may be simplified to

$$b(v) = v - \frac{1}{G(v)^{\frac{n-1}{a}}}\int_{\underline{v}}^{v} G(z)^{\frac{n-1}{a}}dz. \tag{4.13}$$

In general, the integral in this expression needs to be approximated numerically. Note that Equation (4.13) based on the utility function (4.2) turns out to be the same as given by Josheski (2017) and others based on the utility function (4.1). However, in (4.13), we have incorporated the effect of bidders' wealth on their bidding behavior in the form of a modified risk behavior parameter as defined in (4.4) rather than just the baseline risk behavior parameter which has been used by Josheski (2017) and others. Also, they consider risk-neutral and risk-averse bidders in their model. Whereas, we model these auctions not only for risk-neutral and risk-averse bidders but also for risk-seeking bidders.

Now, suppose that each bidder draws her/his value v from a uniform distribution with support $(\underline{v}, \bar{v}]$. Then Equation (4.13) would simplify to the equilibrium bid function

$$b(v) = \frac{a\underline{v} + (n-1)v}{n-1+a}. \tag{4.14}$$

Let W^* be the highest wealth among the other $(n-1)$ bidders. If Brenda's wealth $w = w_i$ was such that $w > W^*$, then her value of $a = a_i$ would follow from (4.4) with $0 < h = W^*/w < 1$. On the other hand, if $w \leq W^*$, then $a = r$, her baseline risk behavior parameter. To obtain W^*, Brenda could, for example, use the mean of the highest order statistic of H of the other $(n-1)$ bidders.

Example 4.2 *Suppose that the value each bidder holds is an independent draw from the uniform distribution $G(v) = \frac{v-30}{170}$ with support $(30, 200]$ and the auctioned item has reserve price $\tau = 30$. It is also assumed that each bidder draws other bidders' wealth from the uniform distribution $H(x) = \frac{x-50}{200}$ with support $[50, 250]$. Assuming that Brenda and all other bidders are risk-neutral; i.e., $a = 1$, Equation* (4.14) *becomes*

$$b(v) = \frac{30 + (n-1)v}{n},$$

which is an increasing function of v and n.

Now, we assume that Brenda and all other bidders are risk-averse. We take two cases. In the first case, we assume that $w > W^*$ and in the second case, we assume that $w \leq W^*$.

1. *Suppose Brenda finds the highest wealth W^* among the other $(n-1)$ bidders by using order statistics. It is well-known that if there are m iid continuous random variables $X_1, \ldots, X_m$ with a common probability density function q*

and corresponding distribution function Q, then the probability density of the mth order statistic $X_{(m)}$ is given by mqQ^{m-1}. In the case when q has a uniform distribution on $[\alpha, \beta]$, the probability density of $X_{(m)}$ is given by

$$\frac{m(x-\alpha)^{m-1}}{(\beta-\alpha)^m}, \quad \alpha \leq x \leq \beta.$$

Simple integration then yields

$$\mathbb{E}\big(X_{(m)}\big) = \frac{m\beta + \alpha}{m+1}.$$

Brenda then takes $W^* = \mathbb{E}\big(W_{(n-1)}\big)$. Recalling that the other bidders' wealth are distributed as a uniform distribution on $[50, 250]$, then substituting $\beta = 250$, $\alpha = 50$, and $m = n-1$ into this last equation yields

$$W^* = \frac{(n-1) \times 250 + 50}{n} = 250 - \frac{200}{n}.$$

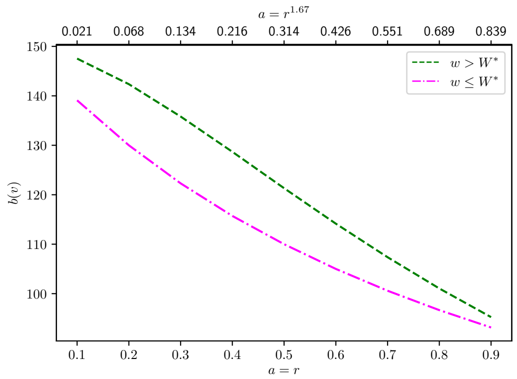
We find Brenda's optimal bids for the choices $n = 2$, 5 and 10. For these values of n, Brenda would have $W^* = 150, 210$, and 230, respectively. Let Brenda have wealth $w = 250$; i.e., $w > W^*$. In this case, her modified risk-aversion parameter would be $a = r^{\frac{1}{h}} = r^{\frac{w}{W^*}}$ and r is assumed to take any value in the interval $(0, 1)$.

2. *Let Brenda have wealth $w = 150$ whereas $W^* = 150, 210, 230$ for $n = 2, 5, 10$, respectively, as we assumed above. Thus, $w \leq W^*$ and her modified risk-aversion parameter would be $a = r$ in this case.*

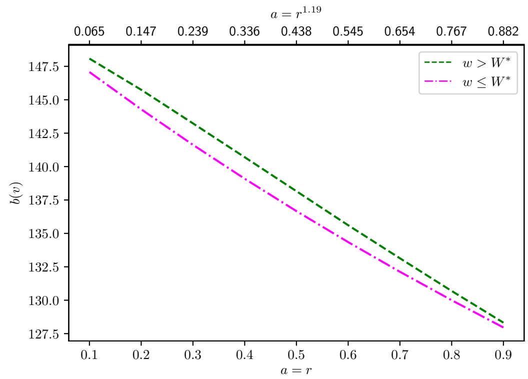
From (4.14), we would have for both cases (with an appropriate a as explained above)

$$b(v) = \frac{30a + (n-1)v}{n-1+a}. \tag{4.15}$$

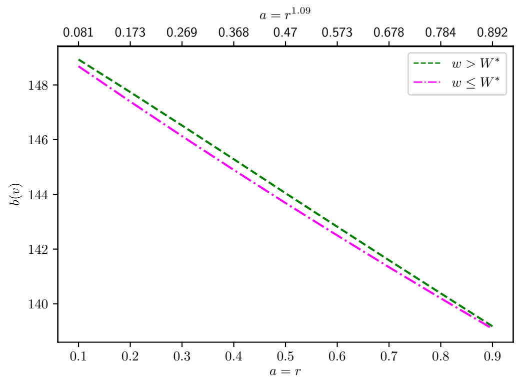
Figure 4.8 shows the plots of Brenda's bid function for different values of n *where all bidders are risk-averse and her true value* $v = 150$. *The upper curve in each plot shows how Brenda's bid function changes at different risk-aversion levels for the first case; i.e.,* $w > W^*$ *while the lower curve shows how her bid function changes at different risk-aversion levels for the second case; i.e.,* $w \leq W^*$. *It shows that being wealthier than the other* $(n-1)$ *bidders, she would bid more aggressively and could have a greater chance to win the auctioned item. It also shows that with increase in* n, *Brenda would expect an increase in* W^* *and consequently her bid would increase.*



(a) $n = 2, W^* = 150$



(b) $n = 5, W^* = 210$



(c) $n = 10, W^* = 230$

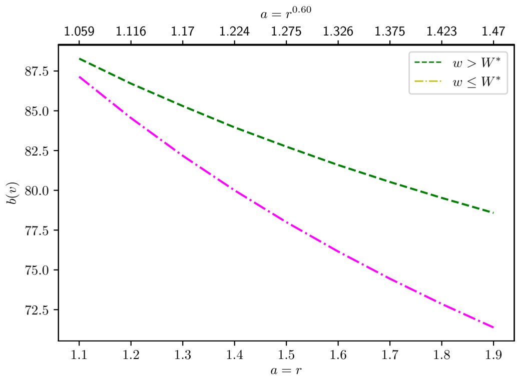
Figure 4.8: Brenda's optimal bids for different values of n *and* a *when she and other bidders are risk-averse*

Now, we assume that Brenda and all other bidders are risk-seeking and again take those two cases. In the first case, we assume that $w > W^*$ *and in the second case, we assume that* $w \leq W^*$.

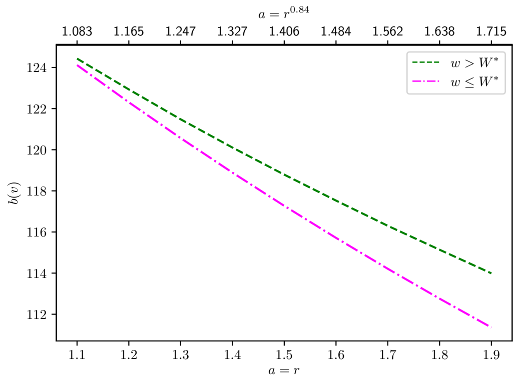
1. *Let Brenda have wealth* $w = 250$ *so that as before,* $w > W^*$ *for* $n = 2$, 5 *and* 10. *Thus, in this case, her modified risk-seeking parameter would be* $a = r^h = r^{\frac{W^*}{w}}$ *and* r *is her baseline risk-seeking parameter that can take any value in the interval* $(1, 2]$ *that she believes.*

2. *Let Brenda have wealth* $w = 150$ *so that* $w \leq W^*$. *Thus, her modified risk-seeking parameter in this case would be* $a = r$.

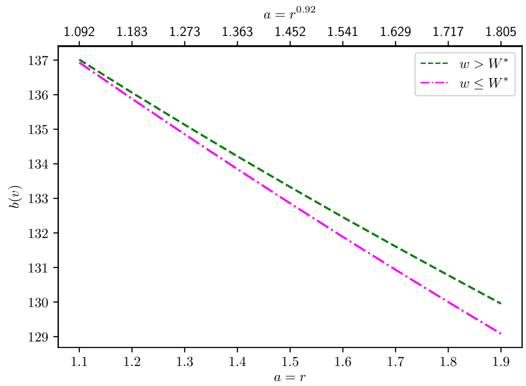
For these two cases, we would again have (4.15) *with an appropriate choice of* a *as explained above. Figure 4.9 shows the plots of Brenda's bid function for different values of* n *where all bidders are risk-seeking and her true value* $v = 150$. *The upper curve in each plot shows how Brenda's bid function changes at different risk-seeking levels for the first case; i.e.,* $w > W^*$ *while the lower curve shows how her bid function changes at different risk-seeking levels for the second case; i.e.,* $w \leq W^*$. *It shows that being wealthier than the other* $(n - 1)$ *bidders, she could be less risk-seeking, could bid higher than the other bidders and so could have a greater chance to win the auctioned item. It also shows that with increase in* n, *Brenda would expect an increase in* W^* *and consequently her bid would increase.*



(a) $n = 2, W^* = 150$



(b) $n = 5, W^* = 210$



(c) $n = 10, W^* = 230$

Figure 4.9: Brenda's optimal bids for different values of n and a when she and other bidders are risk-seeking

4.4.2 Asymmetric Case

From a BNE perspective, an asymmetric case has the assumption that the bidders draw their valuations for the item from different distributions and that all of these distributions are commonly known to each bidder.

Here, we derive an ARA solution from Brenda's perspective for a 2-player game where, she assumes that Charles will solve his problem using an asymmetric BNE perspective. Brenda thinks that Charles's true value v_C has a distribution G_{BC} and his wealth w_C has a distribution H_{BC}. Also, Brenda believes that Charles might elicit that her true value v_B has a distribution G_{CB} and her wealth w_B has a distribution H_{CB}. As neither knows the true value and wealth of their opponent, the ARA approach solves the following system of equations from Brenda's perspective:

$$\begin{aligned}&\max_b\{W_B + (V_B - b)^{a_B} G_{BC}(v_C(b))\},\\&\max_c\{W_C + (V_C - c)^{a_C} G_{CB}(v_B(c))\},\end{aligned} \tag{4.16}$$

where b is Brenda's bid that Charles might elicit, c is Brenda's belief about Charles's bid, a_C is Brenda's belief about Charles's modified risk behavior parameter and a_B is Brenda's modified risk-behavior parameter that Charles might elicit. These values of a_B and a_C will depend on whether $w_B > w_C$ or $w_B \leq w_C$ (see (4.4) and the text following).

Note that (4.16) turns out to be the same as (4.10) and therefore, its ARA solution is the same as that derived for the ME solution concept in Section 4.3. Thus, for two bidders, the ARA solutions for the BNE asymmetric case and ME turn out to be the same, but of course the assumptions made are different. When there are more than two bidders, the ARA perspective opens a larger class of equilibrium problems. We discuss this further in Section 4.5.

4.5 Discussion and Further Work

In this chapter, we provide a more realistic approach to modelling FPSB auctions than what has previously been done. We assume that the auctioned item is *normal* which is typically the nature of many items sold using these auctions. We extend Banks et al. (2015) and Ejaz et al. (2021) by developing ARA solutions for ME and BNE solution concepts, where we take into account uncertainties such as bidders' wealth and their heterogeneous risk behavior (risk-neutral, risk-averse or risk-seeking).

We provide a general framework on how to find ARA solutions assuming n bidders for the BNE solution concept for the symmetric case, as well as for the ME solution concept when there are two bidders. We show that the ARA solutions for the BNE solution concept with two bidders for the asymmetric case turn out to be the same as the ARA solutions derived for the ME solution concept. We provide numerical examples to illustrate ME and the BNE solutions for the symmetric case. To take into account uncertainties such as bidders' wealth and heterogeneous risk behavior, we use a Monte Carlo method to find ME solutions under different scenarios of bidders' risk behavior. The bidding behavior of bidders has been illustrated by approximate equilibrium bid functions along with 95% probability intervals. The results show that the bidder having more wealth is more risk-averse (less risk-seeking) and thus bids more aggressively than the bidder having less wealth. Similar results have been found for the BNE symmetric case. Thus, for the *normal* items, it shows that a bidder would typically bid higher with an increase in her relative wealth. This is consistent with the definition of *normal* items.

Finding the ARA solution for the ME solution concept for $n \geq 3$ bidders could be quite challenging. The decision maker needs to model not only what she believes are the valuation distributions of other bidders, but also what she thinks are the valuation distributions each bidder has for the other bidders. She would have to solve the problem from each of her opponents' perspectives and thus would get the optimal strategy for each of them. Then, from the distribution of optimal strategies of her opponents, she can find her own optimal strategy.

Also, if the decision maker is uncertain about the solution concept that her adversaries are going to use, she needs to incorporate concept (model) uncertainty into her solutions. This could be done as follows. She first finds the optimal solution $\mathbb{E}(b^*|\mathcal{M})$ conditional on the given model $\mathcal{M}$ (non-strategic play, level-k thinking, BNE, ME etc.) she wants to consider. Then she has to elicit a probability distribution $p(\mathcal{M})$ that reflects her uncertainty on $\mathcal{M}$. The optimal solution taking into account the model uncertainty is given by

$$\mathbb{E}(b^*) = \sum_{\mathcal{M}} \mathbb{E}(b^*|\mathcal{M})p(\mathcal{M}).$$

Moreover, the practical challenge in adopting a Bayesian approach is elicitation of prior distributions. Ríos Insua et al. (2016) highlight the significance and provide an outline for the robustness analysis for ARA. The sensitivity of the optimal solution to any errors or mis-specifications in the utilities and probabilities elicited for the analysis is important to investigate but has yet to be developed in an ARA framework.

Finally, ARA solutions are specific to the choice of the utility function. Here, we have illustrated how ARA solutions can be developed for a utility function (4.2) which is considered to be more realistic than other CRRA utility functions previously proposed. However, there are other utility functions such as, for instance, the one used by Engelbrecht-Wiggans and Katok (2007) which takes into account winning and losing regrets, that could be more appropriate for a given problem. ARA solutions for these other utility functions will also need to be found as necessary. Importantly, sensitivity of the ARA solutions to the choice of utility function should be studied too.

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Chapter 5

Adversarial Risk Analysis for Auctions using Non-Strategic Play and Level-k Thinking: A General Case of n Bidders with Regret

In this chapter, we apply an adversarial risk analysis approach for first-price sealed-bid auctions and find solutions using non-strategic play and level-k thinking solution concepts assuming n bidders. We define new regret parameters and a modified utility function to incorporate the effect of bidders' wealth on their bidding behavior. In our modelling, we assume that the auctioned item is a normal item and has a reserve price. We give numerical examples to illustrate our methodology for each solution concept.

5.1 Introduction

5.1.1 Models for First-Price Sealed-Bid Auctions

Considerable progress has been made during the past half century in modelling of various auctions formats and the bidders' behaviors. *First-price sealed-bid* (FPSB) auctions[1] are one of the commonly used auction formats. Bidders' behaviors for FPSB auctions have commonly been modelled using the *decision-theoretic* and *Bayesian game-theoretic* approaches. Using a decision-theoretic approach, the decision maker believes that all other bidders are non-strategic and finds their optimal bid by placing subjective distributions on each of the other bidders' valuations for the auctioned item. It has been argued that the bidders use decision-theoretic models for auctions since they typically assess the probability distribution of the best competitive bid and then optimize their decision about the bid against that assessed probability distribution (see e.g. Capen et al., 1971; Keefer et al., 1991; Rothkopf and Harstad, 1994; Rothkopf, 2007; Wang and Guo, 2017, among others). However, decision theory models non-strategic adversaries, whereas bidders can often be strategic. A Bayesian game-theoretic model on the other hand, models bidders to be strategic adversaries. In this model, bidders draw their valuations for the auctioned item from a distribution that is considered to be commonly known to all bidders and find their optimal bids using a Bayes Nash Equilibrium (BNE) bidding strategy. Vickrey (1961) made a significant breakthrough in understanding bidders' behaviors using a Bayesian game-theoretic model. Further work using a Bayesian game-theoretic approach includes Criesmer et al. (1967); Wilson (1969);

[1] In FPSB auctions, bidders submit their bids in sealed envelopes to the auctioneer. Those envelopes are then opened and the bidder with the highest bid wins the auction and pays the amount equal to the bid.

Riley and Samuelson (1981); Cox et al. (1982b,a); Maskin and Riley (1984, 2000); Campo et al. (2011); Gentry et al. (2015); Li and Tan (2017) among others. The assumptions of “non-strategic opponents" in decision theory and of the “common knowledge" in Bayesian game theory make these two approaches unrealistic in practical situations in which the opponents may be strategic and may wish to keep their bidding distribution a secret to gain competitive advantage. See Joshi et al. (2020) for further discussion on the limitations of a Bayesian game theoretic approach including on games with incomplete information.

Ríos Insua et al. (2009) introduced an approach called *adversarial risk analysis* (ARA) that is considered to be more realistic to model real life situations involving strategic adversaries than the traditional decision theory and game theoretic approaches. While ARA allows the decision maker to model strategic opponents as in game theory, it models the problem from the decision maker’s point of view alone, thus eliminating the need to assume distributions that are commonly known by all players. ARA is a Bayesian approach because the decision maker uses her subjective distributions to model the unknown preferences, capabilities and beliefs of her strategic opponents. However, unlike game theory, these subjective distributions do not need to be commonly known to all players. Also, ARA allows the decision maker to model how her intelligent adversaries frame the problem which includes different solution concepts such as non-strategic play, level-k thinking, mirror equilibrium (ME) or indeed the BNE. While using a non-strategic play solution concept, the decision maker models their rivals to be non-strategic - whose bids are independent of their rivals’ bids; when using a BNE solution concept, the decision maker believes that their adversaries will use a BNE strategy to find their optimal bids. In a level-k thinking solution concept, the decision maker being a level-k thinker believes that her rivals are level-$(k-1)$ thinkers and each level-$(k-1)$ thinker believes that her/his rivals are level-$(k-2)$ thinkers and so on. Thus, the decision maker always thinks one level higher than her rivals in this solution concept. While using a ME solution concept, the decision maker believes that her rivals are modelling their opponents' (including the decision maker's) actions in the same way as she is modelling their actions. All use their subjective distributions over the probabilities and utilities of their rivals and seek an equilibrium.

Banks et al. (2015) modelled FPSB auctions assuming that each bidder is risk-neutral and found ARA solutions for non-strategic play, minimax perspective, level-k thinking, ME and BNE solution concepts. Ejaz et al. (2021a) or Chapter 3 extended this work by using a more realistic utility function for these auctions and assuming that the bidders have different risk behaviors and different wealth. They derived ARA solutions using non-strategic play and the level-k thinking solution concepts assuming two bidders. Then, Ejaz et al. (2021b) or Chapter 4 modelled these auctions using ME and BNE solution concepts under ARA framework where they assumed n bidders, all of whom have different risk behaviors and different wealth.

In all of these models, bidders' bidding behavior is determined using a utility function that depends only upon each bidder's profit. However, there may be other potentially important components than profit such as institutional framework (Fox and Tversky, 1998), misperception of probabilities of winning (Dorsey and Razzolini, 2003) and bidders' regret (Engelbrecht-Wiggans, 1989; Engelbrecht-Wiggans and Katok, 2007) etc., that affect bidders' bidding behavior.

5.1.2 Utility Functions for FPSB Auctions

In FPSB auctions, suppose that the bidders are risk-neutral who draw their values independently and privately. Then, a bidding strategy that is found using a game-theoretic approach is known as a risk-neutral Nash Equilibrium (RNNE) bidding strategy (see Vickrey, 1961). The FPSB auctions have extensively been analysed using experimental techniques. A consistent outcome found in the experiments is that bidders consistently bid above the RNNE bidding strategy (see e.g., Dorsey and Razzolini, 2003; Engelbrecht-Wiggans, 1989; Engelbrecht-Wiggans and Katok, 2007, among others). Different types of explanations such as risk-aversion, interpersonal interaction, learning direction, and regret etc. have been given in economic literature for this overbidding behavior of the bidders.

Risk aversion is relevant to take into account in FPSB auctions as one explanation of bidders' overbidding behavior (see e.g., Milgrom and Weber, 1982; Maskin and Riley, 1984; Cox et al., 1988; Gentry et al., 2015, among others). In such modelling, a constant relative risk-averse (CRRA) utility function is the most commonly used function (Holt and Laury, 2002). A CRRA utility function for the ith bidder having wealth w_i is typically defined as

$$u_i(b_i, v_i, w_i) = \begin{cases} (w_i + v_i - b_i)^{r_i}, & \text{if she wins the bid,} \\ w_i^{r_i}, & \text{if she loses the bid,} \end{cases} \tag{5.1}$$

where, $(1 - r_i)$ is the coefficient of CRRA, v_i is the ith bidder's true value for the auctioned item and b_i is her amount of bid. However, it has been shown (Ejaz et al., 2021a, or Chapter 3) that the utility function (5.1) is unrealistic because it can yield a positive utility even when the bidder does not have the ability to buy the auctioned item. Ejaz et al. (2021a) or Chapter 3 assumed that the auctioned item is *normal*[2] and defined a new utility function for the ith bidder participating in FPSB auctions as

$$u(b_i, v_i, w_i) = \begin{cases} w_i + (v_i - b_i)^{a_i}, & \text{if she wins the bid,} \\ w_i, & \text{if she loses the bid,} \end{cases} \tag{5.2}$$

where a_i is a modified CRRA parameter that changes with the relative change in circumstances of the bidders' wealth. However, Kagel (1995, p. 525) suggested that risk aversion may not be the only factor that generates bidding above the RNNE in FPSB auctions. Inter-personal interactions and comparisons is another explanation of bidders' overbidding behavior in FPSB auctions (see e.g., Isaac and Walker, 1985; Dufwenberg and Gneezy, 2002; Morgan et al., 2003). Another explanation of this overbidding behavior is using learning direction theory proposed by Selten and Stoecker (1986). Based on *feedback over time* this theory leads to the direction in which the bids are likely to be adjusted. Neugebauer and Selten (2006) found that most of the bidders adjust their bids in the same way as that of learning direction theory. However, this theory does not give any explanation for bidders' initial overbidding.

Engelbrecht-Wiggans (1989) gave another possible explanation of bidders' overbidding behavior and modelled their utility by a linear combination of their profit and two types of regret. They stated that in FPSB auctions, the winner pays more than the second highest bid and therefore could realize a winning regret by

[2]An item with positive income elasticity is defined as a *normal item* in economic theory, i.e., the demand for a normal item rises with an increase in income and falls with a decrease in income (see, e.g., Fisher, 1990; Goeree et al., 2002; Piros and Pinto, 2013; Perloff, 2015; Baisa, 2017, for more details)

paying much more than the second highest bid. On the other hand, some winner's bid could be less than that of the losing bidder's valuation (willingness to pay). Thus, in this case the loser could regret bidding too low because she has missed an opportunity to win the item at a favourable price. So, the bidder could not only be sensitive to her expected profit but also to the expected amount of her winning and losing regret when deciding on her bid amount. The weight on each type of regret potentially determines her bidding behavior. Engelbrecht-Wiggans and Katok (2007) modelled the ith bidder's bidding behavior by taking into account her winning and losing regrets while assuming that she has a risk-neutral utility. They defined the ith bidder's utility function from a bid b_i while having true value v_i as

$$u_i(b_i, v_i) = \begin{cases} (v_i - b_i) - \int_{b_j:\underline{v}_j < b_j \leq b_i} [\zeta_i + \eta_i(b_i - b_j)] f_{ij}(b_j | b_j \leq b_i) db_j, & \text{if she wins the bid,} \\ -\int_{b_j:b_i < b_j \leq v_i} [\vartheta_i + \theta_i(v_i - b_j)] f_{ij}(b_j | b_j > b_i) db_j, & \text{if she loses the bid,} \end{cases} \tag{5.3}$$

where b_j is the jth bidder's bid and is the maximum of the bids from the ith bidder's $(n-1)$ opponents and $f_{ij}(b_j)$ is the probability distribution on b_j that the ith bidder believes. Also, $b_i - b_j$ is the excess amount of money paid if the ith bidder wins and her utility suffers by an amount $\zeta_i + \eta_i(b_i - b_j)$ where $\eta_i \geq 0$. Larger values of η_i means a higher winning regret for the ith bidder. Negative values of ζ_i allows some pleasure to the ith bidder in case of winning. On the other hand, if the ith bidder loses and the highest bid satisfies the inequality $b_i \leq b_j \leq v_i$, then the ith bidder misses an opportunity to win at a favourable price and her utility suffers by an amount $\vartheta_i + \theta_i(v_i - b_j)$, where $\vartheta_i, \theta_i \geq 0$.

However, the model proposed by Engelbrecht-Wiggans and Katok (2007) does have a few limitations. Firstly, the utility function (5.3) does not take into account the bidder's wealth and therefore assumes that all bidders have zero wealth, that is $w_i = 0$, $\forall i$. It is unrealistic to assume that the ith bidder can place their optimal bid $b_i^* > 0$ found using this utility function with zero wealth. In fact, considering bidders' wealth is a significant determinant of bidders' bidding behavior in these auctions (see e.g., Gentry et al., 2015; Ejaz et al., 2021a, among others). Secondly, Engelbrecht-Wiggans and Katok (2007) found their optimal bid using a decision theoretic approach, the limitations of which have been discussed above. ARA solutions for FPSB auctions using a utility function that takes into account the winning and losing regret have not been found yet. Since this type of utility function contains integrals whose limits vary with change of decision maker's opponent valuation at each draw, therefore, it turns out that finding ARA solutions for such a type of utility function is methodologically and computationally more challenging than finding ARA solutions using a utility function such as the one given in (5.2).

5.1.3 Contributions in this Chapter

The main contributions contained in this Chapter are as follows:

- We modify the utility function (5.3) used by Engelbrecht-Wiggans and Katok (2007) wherein,
    - we consider the wealth of each bidder and assume that a bidder can only bid an amount less than or equal to their wealth,
    - we assume that the bidders may have different wealth,

- we assume that the auctioned item is normal and define new winning and losing regret parameters to incorporate the effect of increase in wealth on bidders' bidding behavior,
- we assumed that the auctioned item has a reserve price (typically, known to each bidder in advance).

- We find ARA solutions for non-strategic play and level-k thinking solution concepts using the modified utility function.

5.1.4 Structure of the Chapter

The rest of this Chapter is organised as follows. In Section 5.2, we define new regret parameters and the modified utility function. In Section 5.3, we derive ARA solutions using the non-strategic play solution concept assuming n bidders participating in a FPSB auction. In Section 5.4, we derive ARA solutions using the level-k thinking solution concept. Finally, in Section 5.5, we discuss the results obtained in this Chapter and present some ideas for future work.

5.2 New Regret Parameters and Modified Utility Function

The winning regret parameters ζ_i and η_i in (5.3) yield a linear function where ζ_i is allowed to take both the negative and positive values and $\eta_i \geq 0$. Positive values of ζ_i result in an increase in bidder's winning regret and consequently her bid would decrease. Whereas, with the negative values of ζ_i, the bidder can realize some pleasure from winning. But, with sufficiently negative values of ζ_i, the bidder could bid in excess of her true value. On the other hand, the losing regret parameters ϑ_i and θ_i also yield a linear function where ϑ_i, $\theta_i \geq 0$. The greater the values of ϑ_i and θ_i, the more the losing regret to the bidder and as a result she would bid higher.

Since in equilibrium, bidders never bid above their true values (Gentry et al., 2015), we make a realistic assumption that the ith bidder will bid b_i such that $b_i \leq v_i \leq w_i$. Therefore, we do not allow negative values of ζ_i because negative values can result in the ith bidder bidding more than her true value. Also, with an increase in positive values of ζ_i, the ith bidder's winning regret increases which can be modelled by having larger values of η_i only in the utility function. Therefore, we model the ith bidder's winning regret only with η_i and set $\zeta_i = 0$. Similarly, the ith bidder's losing regret can be modelled by having just θ_i and therefore, we set $\vartheta_i = 0$.

When the auctioned item is normal, bidders' winning and losing regret could change with the relative change in the circumstances of their wealth. For the ith bidder, suppose her original winning and losing regret parameters are η_i and θ_i, respectively. These are fixed and we call them her baseline regret parameters. Note that η_i and θ_i are the same as defined in (5.3). The relative change in circumstances could occur in two possible cases; firstly, when a bidder's wealth changes and secondly, when a bidder attempts to model the winning and losing regrets of her rivals.

Firstly, we model the ith bidder's regret at an increased level of her wealth compared with her original level of wealth and thus modify her regret parameters. As mentioned above, η_i is the ith bidder's baseline winning regret parameter and θ_i is her baseline losing regret parameter, which represent her natural regret appetite at her wealth, say $w_{i(1)}$. Lets assume that the circumstances of the ith bidder change (e.g., she gains an inheritance) and her wealth is increased to $w_{i(2)}$. At this increased wealth level, we expect an increase in her losing regret and a decrease in her winning regret for the same auctioned item (since the item is assumed to be a normal item). So, we modify her winning regret parameter having wealth $w_{i(2)}$ relative to wealth $w_{i(1)}$ as

$$\hat{\eta}_i = h \times \eta_i, \quad \eta_i > 0, \tag{5.4}$$

and we modify her losing regret parameter having wealth $w_{i(2)}$ relative to wealth $w_{i(1)}$ as

$$\hat{\theta}_i = \frac{1}{h} \times \theta_i, \quad \theta_i > 0, \tag{5.5}$$

where, we define $0 < h = w_{i(1)}/w_{i(2)} \leq 1$. Note that when $h = 1$, that means no change in the ith bidder's wealth and thus she would have $\hat{\eta}_i = \eta_i$ and $\hat{\theta}_i = \theta_i$. Secondly, when the ith bidder is bidding against her $(n-1)$ rivals in FPSB auctions and believes that she is the wealthiest among all n bidders, then she would have more losing regret and less winning regret compared to her baseline. Suppose that w_i is the wealth of the ith bidder and w_j is the wealth of the jth bidder who the ith bidder believes is the wealthiest among the other $(n-1)$ bidders. Thus, for $w_i > w_j$, we modify the ith bidder's winning and losing regret by using (5.4) and (5.5), respectively, where $0 < h = w_j/w_i \leq 1$. In this case, she would model the other $(n-1)$ bidders' (including the jth bidder's) modified regret parameters as being equal to their baseline regret parameters, i.e., $\hat{\eta}_s = \eta_s$ and $\hat{\theta}_s = \theta_s$ for $s = 1, 2, \ldots, n,\ s \neq i$. In contrast, if the ith bidder believes that the jth bidder is the wealthiest bidder among all n bidders, she can find the jth bidder's modified regret parameters from (5.4) and (5.5) by replacing i with j and taking

$0 < h = w_i/w_j \leq 1$. In this case, we assume that she would model the regret parameters of the other $(n-1)$ bidders (including herself) as being unchanged, i.e., $\hat{\eta}_s = \eta_s$ and $\hat{\theta}_s = \theta_s$ for $s = 1, 2, \ldots, n$, $s \neq j$.

In the utility function (5.3), $(v_i - b_j)$ yields the opportunity loss by the ith bidder at a favourable price if she loses, i.e., the extent of the difference between her true value v_i for the item and the winning bid b_j if $b_i < b_j \leq v_i$. However, the ith bidder could have regret on just the extent of the difference between her losing bid b_i and the winning bid b_j, i.e., $(b_j - b_i)$ provided that $b_i < b_j \leq v_i$ which is what tends to happen in practice. Thus, using (5.4) and (5.5), we define our modified utility function for the ith bidder as

$$u_i(b_i, v_i, w_i) = \begin{cases} (w_i + v_i - b_i) - \int_{b_j : b_j \leq b_i} \hat{\eta}_i (b_i - b_j) f_{ij}(b_j | b_j \leq b_i) db_j, & \text{if she wins the bid,} \\ w_i - \int_{b_j : b_i < b_j \leq v_i} \hat{\theta}_i (b_j - b_i) f_{ij}(b_j | b_j > b_i) db_j, & \text{if she loses the bid,} \end{cases} \tag{5.6}$$

where the effect of bidders' wealth has been incorporated in the form of modified regret parameters in (5.6) which has not been encountered yet for such type of utility function.

5.3 Non-Strategic Play

In this section, we assume n bidders and find ARA solutions for one of the bidders named Brenda. Using a non-strategic play solution concept, Brenda believes that all the other $(n-1)$ bidders are non-strategic, i.e., their analyses do not depend upon the situation of Brenda and the other bidders. We assume that each of Brenda's opponents will bid an amount that is independent of her bid. We assume that Brenda is the first bidder among n bidders and she bids an amount b_1, having wealth w_1 and true value v_1 for the auctioned item. We assume that the auctioned item has a reserve price τ such that $\tau < b_1 \leq v_1 \leq w_1$ and it is a normal item. Brenda believes that the other $(n-1)$ bidders have wealth W_i for $i = 2, \ldots, n$ that are unknown to her and therefore, she places the distributions H_{1i} with support $[\underline{w}_i, \bar{w}_i]$ on their wealth according to her belief such that $\underline{w}_i > \tau$, $i = 2, \ldots, n$. She also does not know about their true values V_i and their bids B_i for the auctioned item. Therefore, she places distributions G_{1i} with support $[\underline{v}_i, \bar{v}_i]$ on their true values according to her belief such that $\tau < \underline{v}_i \leq \underline{w}_i$, $\bar{v}_i \leq \bar{w}_i$, $i = 2, \ldots, n$. Then she finds the distributions F_{1i} (defined in Equation (5.8) later in this section) of their bids with support $[\underline{b}_i, \bar{b}_i]$ such that $\tau < \underline{b}_i \leq \underline{v}_i$ and $\bar{b}_i \leq \bar{v}_i$. Assuming that the bids are continuous, Brenda can find her probability of winning from a bid of amount b_1 against the ith bidder as

$$F_{1i}(b_1) = \Pr(B_i < b_1), \quad i = 2, \ldots, n,$$

where, F_{1i} is the distribution over the ith bidder's bid that Brenda believes. To obtain F_{1i}, Brenda divides her introspection into two parts as G_{1i}, the cumulative distribution function (CDF) that quantifies her uncertainty for the ith bidder's true value and T_{1i}, the CDF that quantifies her uncertainty for the fraction of the true value $p_i = b_i/v_i$ that the ith bidder bids, where b_i and v_i are the respective bids and true values of the ith bidder, $i = 2, \ldots, n$. The support for T_{1i} is $(\underline{v}_i/v_i, 1]$. She can then derive her subjective distribution function over $B_i = P_i V_i$, the amount of the ith bidder's random bid, as

$$F_{1i}(b_i) = \Pr[\underline{v}_i < P_i V_i \le b_i] = \int_{\underline{v}_i}^{b_i} \int_{\underline{v}_i/v_i}^{1} g_{1i}(v_i) t_{1i}(p_i)\, dp_i\, dv_i + \int_{b_i}^{\bar{v}_i} \int_{\underline{v}_i/v_i}^{b_i/v_i} g_{1i}(v_i) t_{1i}(p_i)\, dp_i\, dv_i. \tag{5.7}$$

As $\int_{\underline{v}_i/v_i}^{1} t_{1i}(p_i)dp_i = 1$, the above equation simplifies to

$$F_{1i}(b_i) = G_{1i}(b_i) + \int_{b_i}^{\bar{v}_i} g_{1i}(v_i) T_{1i}(b_i/v_i)\, dv_i, \tag{5.8}$$

where, $g_{1i}(v_i)$ is the probability density function for the ith bidder's true value that Brenda elicits and $t_{1i}(p_i)$ is the probability density function for the fraction of the ith bidder's true value that Brenda believes he will bid. Equation (5.7) assumes that the ith bidder's true value V_i and fraction of his true value P_i are independent.

Now, based upon the assumption that the bidder having more wealth tends to bid higher than the bidder having less wealth when the auctioned item is normal, Brenda believes that the jth bidder having the highest wealth w_j among her $(n-1)$ opponents will bid b_j which is the maximum bid among the other $(n-1)$ bidders. So, Brenda will consider the jth bidder as her competitor bidder. Thus, using (5.4), (5.5) and (5.6), Brenda's expected utility is

$$\Psi_1(b_1) = \left[(w_1 + v_1 - b_1) - \int_{b_j:\underline{v}_j < b_j \le b_1} \hat{\eta}_1(b_1 - b_j) f_{1j}(b_j | b_j \le b_1) db_j\right] F_{1j}(b_1) + \left[w_1 - \int_{b_j: b_1 < b_j \le v_1} \hat{\theta}_1(b_j - b_1) f_{1j}(b_j | b_j > b_1) db_j\right] [1 - F_{1j}(b_1)],$$

where F_{1j} is the probability distribution on b_j that Brenda believes which can be found by using (5.8) and $\hat{\eta}_1$ and $\hat{\theta}_1$ are her modified winning and losing regret parameters, respectively. As, $f_{1j}(b_j|b_j \leq b_1) = f_{1j}(b_j)/F_{1j}(b_1)$ and $f_{1j}(b_j|b_j > b_1) = f_{1j}(b_j)/[1 - F_{1j}(b_1)]$, the above equation simplifies to

$$\begin{aligned}\Psi_1(b_1) = w_1 + (v_1 - b_1)F_{1j}(b_1) - \int_{b_j:\underline{v}_j<b_j<b_1} &\hat{\eta}_1(b_1 - b_j)f_{1j}(b_j)db_j \\ - \int_{b_j:b_1<b_j\leq v_1} &\hat{\theta}_1(b_j - b_1)f_{1j}(b_j)db_j.\end{aligned}$$

Finally, Brenda can find her optimal bid b_1^* by solving the following

$$\begin{aligned}b_1^* = \underset{\underline{v}_j<b_1\leq v_1}{\arg\max}\ [w_1 + (v_1 - b_1)F_{1j}(b_1) - \int_{b_j:\underline{v}_j<b_j<b_1} &\hat{\eta}_1(b_1 - b_j)f_{1j}(b_j)db_j \\ - \int_{b_j:b_1<b_j\leq v_1} &\hat{\theta}_1(b_j - b_1)f_{1j}(b_j)db_j].\end{aligned} \tag{5.9}$$

In fact, b_1^* is a function of $\hat{\eta}_i$ and $\hat{\theta}_i$ and therefore, a function of η_i, θ_i and w_j. As η_i and θ_i are Brenda's fixed baseline regret parameters, we can treat b_1^* as a function of w_j and so re-write (5.9) as

$$\begin{aligned}b_1^*(w_j) = \underset{b_1>\tau}{\arg\max}\ [w_1 + (v_1 - b_1)F_{1j}(b_1) - \int_{b_j:b_j<b_1} &\hat{\eta}_1(b_1 - b_j)f_{1j}(b_j)db_j \\ - \int_{b_j:b_1<b_j\leq v_1} &\hat{\theta}_1(b_j - b_1)f_{1j}(b_j)db_j].\end{aligned} \tag{5.10}$$

Brenda can use (5.10) to find her optimal bid by taking into account her uncertainty around w_j and find the expected value of her optimal bid amount as

$$\mathrm{E}(b_1^*) = \int b_1^*(w_j)dH_{1j}(w_j), \tag{5.11}$$

where H_{1j} is the distribution on w_j that Brenda believes. Numerical methods may often be needed to solve (5.11) for $\mathrm{E}(b_1^*)$.

Example 5.1 *Suppose Brenda is bidding against her opponents Alex and Charles. Brenda believes that Alex and Charles are non-strategic bidders. Let Brenda, Alex and Charles be the first, second and the third bidders, respectively. Let Brenda's true value for the item be* $v_1 = \$200$ *and the auctioned item have a reserve price* $\tau = \$25$*. Also,*

- *Brenda has a uniform distribution on Alex's true value with* $\underline{v}_2 = \$30, \bar{v}_2 = \$200$*, i.e.,* $G_{12} = \frac{(v_2-30)}{200-30}$.
- *Brenda has a uniform distribution on Alex's wealth with* $\underline{w}_2 = \$100, \bar{w}_2 = \$250$*, i.e.,* $H_{12} = \frac{(w_2-100)}{250-100}$.
- *Brenda has a uniform distribution on Charles's true value with* $\underline{v}_3 = \$30, \bar{v}_3 = \$250$*, i.e.,* $G_{13} = \frac{(v_3-30)}{250-30}$.
- *Brenda has a uniform distribution on Charles's wealth with* $\underline{w}_3 = \$150, \bar{w}_3 = \$300$*, i.e.,* $H_{13} = \frac{(w_3-150)}{300-150}$.
- *Brenda elicit her uncertainties around the fraction of her opponents true values that they would bid as* $T_{1i} = \frac{p_i^8-(\underline{v}_i/v_i)^8}{1-(\underline{v}_i/v_i)^8}$ *with support* $(\underline{v}_i/v_i < p_i \leq 1]$ *for* $i = 2, 3$.

First, for the sake of illustration and simplicity, we assume that Brenda chooses Alex and Charles's wealth to be $w_2 = \$170$ *and* $w_3 = \$210$*, respectively, according to her belief. Thus, she would believe Charles as her competitor as he has more wealth than Alex and will find her optimal bid against him. Thus, using* (5.8) *where* b_i *is replaced by* b_1*, she will find the distribution of her bid against Charles as*

$$
\begin{aligned}
F(b_1) =& \frac{b_1 - \underline{v}_3}{\bar{v}_3 - \underline{v}_3} + \frac{b_1^8 - \underline{v}_3^8}{\bar{v}_3 - \underline{v}_3}\Big[- \frac{\sqrt{2}}{16 \times \underline{v}_3^7} \ln\Big(\frac{\bar{v}_3^2 + \bar{v}_3\underline{v}_3\sqrt{2} + \underline{v}_3^2}{\bar{v}_3^2 - \bar{v}_3\underline{v}_3\sqrt{2} + \underline{v}_3^2}\Big) \\
&+ \frac{\sqrt{2}}{16 \times \underline{v}_3^7} \ln\Big(\frac{b_1^2 + b_1\underline{v}_3\sqrt{2} + \underline{v}_3^2}{b_1^2 - b_1\underline{v}_3\sqrt{2} + \underline{v}_3^2}\Big) - \frac{\sqrt{2}}{8 \times \underline{v}_3^7} \tan^{-1}\Big(\frac{\bar{v}_3\sqrt{2}}{\underline{v}_3} + 1\Big) \\
&+ \frac{\sqrt{2}}{8 \times \underline{v}_3^7} \tan^{-1}\Big(\frac{b_1\sqrt{2}}{\underline{v}_3} + 1\Big) - \frac{\sqrt{2}}{8 \times \underline{v}_3^7} \tan^{-1}\Big(\frac{\bar{v}_3\sqrt{2}}{\underline{v}_3} - 1\Big) \\
&+ \frac{\sqrt{2}}{8 \times \underline{v}_3^7} \tan^{-1}\Big(\frac{b_1\sqrt{2}}{\underline{v}_3} - 1\Big) - \frac{1}{4 \times \underline{v}_3^7} \tan^{-1}\Big(\frac{\bar{v}_3}{\underline{v}_3}\Big) + \frac{1}{4 \times \underline{v}_3^7} \tan^{-1}\Big(\frac{b_1}{\underline{v}_3}\Big) \\
&+ \frac{\ln(\bar{v}_3 - \underline{v}_3)}{8 \times \underline{v}_3^7} - \frac{\ln(b_1 - \underline{v}_3)}{8 \times \underline{v}_3^7} - \frac{\ln(\bar{v}_3 + \underline{v}_3)}{8 \times \underline{v}_3^7} + \frac{\ln(b_1 + \underline{v}_3)}{8 \times \underline{v}_3^7}\Big].
\end{aligned}
$$

We assume that Brenda's baseline winning and losing regret parameters are η_1 *and* θ_1*, respectively, and suppose that her wealth is* $w_1 = \$200$*. As her wealth* $w_1 = \$200$ *is less than that of Charles' wealth of* $w_3 = \$210$*, we take* $\hat{\eta}_1 = \eta_1$ *and* $\hat{\theta}_1 = \theta_1$ *as defined in Section 5.2. Table 5.1 shows Brenda's optimal bids, her probabilities of winning and her expected utilities for various assumed levels of her winning and losing regret parameters for this case.*

Now, we assume Brenda's wealth increases to $w_1 = \$250$*, while Charles's wealth is unchanged at* $w_3 = \$210$*. In this case, we expect her to have more losing regret and less winning regret than Charles because she is able (since the item is normal) to pay more to increase her chance of winning the bid. By using* (5.4)*, we model Brenda's reduced winning regret when* $w_1 = \$250$ *relative to* $w_3 = \$210$ *as* $\hat{\eta}_1 = h \times \eta_1 = (210/250) \times \eta_1 = 0.84\eta_1$*. Her increased losing regret is modelled by using* (5.5) *when* $w_1 = \$250$ *relative to* $w_3 = \$210$ *as* $\hat{\theta}_1 = (1/h) \times \theta_1 = (250/210) \times \theta_1 = 1.19\theta_1$*. Table 5.2 shows how Brenda's optimal bids, her probabilities of winning*

Table 5.1: Brenda's optimal bids, her probabilities of winning and expected utilities when $w_1 = \$200$ *and* $w_3 = \$210$.

$\hat{\eta}_1 = \eta_1$	$\hat{\theta}_1 = \theta_1$	0.50	1.00	2.00	3.00	4.00
	b_1^*	108.98	119.12	133.23	142.81	149.84
0.50	$F(b_1^*)$	0.43	0.48	0.54	0.59	0.62
	$\Psi_1(b_1^*)$	222.35	215.66	206.13	199.60	194.80
	b_1^*	97.39	107.63	122.34	132.57	140.21
1.00	$F(b_1^*)$	0.37	0.42	0.49	0.54	0.58
	$\Psi_1(b_1^*)$	214.78	205.93	192.92	183.70	176.77
	b_1^*	81.75	91.49	106.24	117.00	125.27
2.00	$F(b_1^*)$	0.29	0.34	0.41	0.47	0.51
	$\Psi_1(b_1^*)$	204.63	192.26	173.13	158.89	147.80
	b_1^*	71.73	80.68	94.82	105.54	114.00
3.00	$F(b_1^*)$	0.24	0.28	0.36	0.41	0.45
	$\Psi_1(b_1^*)$	198.18	183.14	159.01	140.37	125.44
	b_1^*	64.77	72.94	86.26	96.68	105.11
4.00	$F(b_1^*)$	0.20	0.24	0.31	0.36	0.41
	$\Psi_1(b_1^*)$	193.73	176.65	148.44	125.99	107.62

and her expected utilities change with change in η_1 *and* θ_1 *for this case. In general, it shows that an increase in losing regret leads Brenda to a higher optimum bid, resulting in a higher probability of winning at that bid, but with a lower expected utility. Whereas, an increase in winning regret leads Brenda to a lower optimum bid, resulting in a lower probability of winning at that bid, and a lower expected utility.*

However, in practice, Brenda may instead be uncertain about w_2 *and* w_3*, Alex and Charles's wealth, respectively, and thus would have to take into account her uncertainty around* w_2 *and* w_3*. Under such a situation, she can elicit* H_{12} *and* H_{13}*, the distributions on Alex and Charles's wealth, respectively. In this case, she can use the expected value of her opponents' wealth as an estimate to find the wealthiest bidder among her opponents. Thus,* $\max\{\mathrm{E}(W_2), \mathrm{E}(W_3)\} = \max\{175, 225\} = 225 = \mathrm{E}(W_3)$*. So, Brenda would believe Charles (the 3rd bidder) is her competitor*

Table 5.2: Brenda's optimal bids, her probabilities of winning and expected utilities when $w_1 = \$250$ *and* $w_3 = \$210$.

		θ_1	0.50	1.00	2.00	3.00	4.00
η_1	$\hat{\eta}_1$	$\hat{\theta}_1$	0.60	1.19	2.38	3.57	4.76
		b_1^*	113.34	124.44	139.21	148.84	155.73
0.50	0.42	$F(b_1^*)$	0.44	0.51	0.57	0.61	0.64
		$\Psi_1(b_1^*)$	272.46	265.45	255.95	249.72	245.29
		b_1^*	102.90	114.26	129.79	140.14	147.65
1.00	0.84	$F(b_1^*)$	0.40	0.45	0.52	0.57	0.61
		$\Psi_1(b_1^*)$	265.24	256.11	243.34	234.71	228.42
		b_1^*	88.03	99.19	115.20	126.32	134.61
2.00	1.68	$F(b_1^*)$	0.32	0.38	0.46	0.51	0.55
		$\Psi_1(b_1^*)$	255.01	242.21	223.44	210.13	200.11
		b_1^*	77.96	88.52	104.31	115.68	124.35
3.00	2.52	$F(b_1^*)$	0.27	0.32	0.40	0.46	0.50
		$\Psi_1(b_1^*)$	248.12	232.41	208.42	190.78	177.16
		b_1^*	70.71	80.57	95.83	107.16	115.97
4.00	3.36	$F(b_1^*)$	0.23	0.28	0.36	0.42	0.46
		$\Psi_1(b_1^*)$	243.19	225.12	196.68	175.13	158.13

because he has maximum expected wealth and thus tends to bid higher than Alex. Now, using (5.11) *and Monte Carlo simulations, she can derive her expected optimal bid* $\mathrm{E}(b_1^*)$, *her probability of winning,* $F[\mathrm{E}(b_1^*)]$ *at that expected optimal bid and her expected utility,* $\Psi_1[\mathrm{E}(b_1^*)]$ *for various assumed values of her winning and losing regret parameters. We assume* $w_1 = \$250$ *and perform* $N = 500$ *simulations and summarise these results in Table 5.3.*

5.4 Level-k Thinking

A level-k analysis is the modelling of how deeply the opponent of a decision maker (Brenda) thinks about the problem (Stahl and Wilson, 1995). Thus, a level-k analysis for an n-player FPSB auction is when Brenda, a level-k thinker believes

Table 5.3: Brenda's expected optimal bids, her probabilities of winning and expected utilities.

		θ_1	0.50	1.00	2.00	3.00	4.00
η_1	$\mathrm{E}(\hat{\eta}_1)$	$\mathrm{E}(\hat{\theta}_1)$	0.69	1.38	2.76	4.14	5.52
		$\mathrm{E}(b_1^*)$	113.07	124.09	138.75	148.32	155.18
0.50	0.38	$F[\mathrm{E}(b_1^*)]$	0.45	0.50	0.57	0.61	0.64
		$\Psi_1[\mathrm{E}(b_1^*)]$	272.23	265.16	255.56	249.28	244.80
		$\mathrm{E}(b_1^*)$	102.56	113.82	129.20	139.47	146.95
1.00	0.75	$F[\mathrm{E}(b_1^*)]$	0.39	0.45	0.52	0.57	0.61
		$\Psi_1[\mathrm{E}(b_1^*)]$	264.90	255.65	242.71	233.95	227.58
		$\mathrm{E}(b_1^*)$	87.74	98.77	114.58	125.57	133.78
2.00	1.50	$F[\mathrm{E}(b_1^*)]$	0.32	0.38	0.45	0.51	0.55
		$\Psi_1[\mathrm{E}(b_1^*)]$	254.56	241.59	222.50	208.96	198.76
		$\mathrm{E}(b_1^*)$	77.76	88.21	103.76	114.96	123.52
3.00	2.26	$F[\mathrm{E}(b_1^*)]$	0.27	0.32	0.40	0.46	0.50
		$\Psi_1[\mathrm{E}(b_1^*)]$	247.65	231.71	207.33	189.37	175.48
		$\mathrm{E}(b_1^*)$	70.59	80.36	95.38	106.52	115.19
4.00	3.01	$F[\mathrm{E}(b_1^*)]$	0.23	0.28	0.36	0.41	0.46
		$\Psi_1[\mathrm{E}(b_1^*)]$	242.71	224.39	195.50	173.57	156.24

that the other $(n-1)$ bidders are level-(k-1) thinkers and each of them believes that the other $(n-1)$ bidders are level-(k-2) thinkers, and so on.

Non-strategic play which we have discussed in Section 5.3 is basically a level-1 analysis where Brenda believes herself to be a level-1 thinker and models the other $(n-1)$ bidders as level-0 (non-strategic) thinkers.

Here, we derive ARA solutions for Brenda for the case where $k = 2$. In a level-2 analysis, Brenda models herself as a level-2 thinker and believes that the other $(n-1)$ bidders are level-1 thinkers with each of them modelling the other $(n-1)$ bidders as level-0 thinkers. Here, again we assume that Brenda is the 1st bidder among the n bidders and believes that G_{1il} and H_{1il} are the distributions of the lth bidder's true value and wealth that the ith bidder might elicit with supports $[\underline{v}_{il}, \bar{v}_{il}]$ and $[\underline{w}_{il}, \bar{w}_{il}]$, $i = 2, \ldots, n$, $l = 1, \ldots, n$, $i \neq l$, respectively, such that $\tau < \underline{v}_{il} \leq \underline{w}_{il}$ and $\bar{v}_{il} \leq \bar{w}_{il}$. Also, Brenda believes that F_{1il} with supports $[\underline{b}_{il}, \bar{b}_{il}]$ is the distribution of the lth bidder's bid that the ith bidder might elicit such that $\tau < \underline{b}_{il} \leq \underline{v}_{il}$ and $\bar{b}_{il} \leq \bar{v}_{il}$. Brenda believes that the ith bidder will find F_{1il} using (5.8) as

$$F_{1il}(b_{il}) = G_{1il}(b_{il}) + \int_{b_{il}}^{\bar{v}_{il}} g_{1il}(v_{il}) T_{1il}(b_{il}/v_{il})\, dv_{il}, \tag{5.12}$$

where, $g_{1il}(v_{il})$ is the probability density function for the lth bidder's true value that the ith bidder might elicit that Brenda believes and $t_{1il}(p_{il})$ is Brenda's belief about the probability density function for the fraction of the lth bidder's true value this bidder will bid that the ith bidder might elicit.

Now, suppose that $w_{1i}, v_{1i}, \eta_{1i}$ and θ_{1i} are the wealth, true value, winning regret parameter and losing regret parameter, respectively, of the ith (level-1) bidder that Brenda believes. Also, let w_{1ij} be the wealth of the wealthiest bidder bidding b_{ij}. This is assumed by Brenda to be the maximum bid among the other $(n-1)$ bidders that the ith (level-1) bidder might believe. (We remark that the value of j will depend on the value of i, but for simplicity of notation, we do not formally show this dependency.) So, Brenda believes that the ith (level-1) bidder would find his optimal bid against the wealthiest among the other $(n-1)$ bidders for given $w_{1i}, w_{1ij}, v_{1i}, \eta_{1i}$ and θ_{1i} as

$$B^*_{1i}(w_{1i}, w_{1ij}, v_{1i}, \eta_{1i}, \theta_{1i}) = \arg\max_{b_{1i} > \tau} [W_{1i} + (V_{1i} - b_{1i}) F_{1ij}(b_{1i})$$
$$- \int_{b_{ij}: b_{ij} < b_{1i}} \hat{\eta}_{1i}(b_{1i} - b_{ij}) f_{1ij}(b_{ij}) db_{ij} - \int_{b_{ij}: b_{1i} < b_{ij} \leq V_{1i}} \hat{\theta}_{1i}(b_{ij} - b_{1i}) f_{1ij}(b_{ij}) db_{ij}],$$
$$i = 2, \ldots, n, \quad j \neq i,$$

where W_{1i} and V_{1i} are the ith bidder's wealth and true value, respectively, that Brenda believes. Also, F_{1ij} is the probability distribution on b_{ij} that the ith bidder can find against the jth bidder using (5.12) that Brenda believes. In practice, Brenda is uncertain about η_{1i} and θ_{1i} and could elicit distributions on these parameters. The distributions on η_{1i} and θ_{1i} along with the distributions on w_{1i} and w_{1ij} would allow her to derive the distributions for $\hat{\eta}_{1i}$ and $\hat{\theta}_{1i}$, respectively. We denote the distributions on $\hat{\eta}_{1i}$ and $\hat{\theta}_{1i}$ by Q_{1i} and S_{1i}, respectively. Then, she can find the ith bidder's expected optimal bid that she believe the ith bidder will derive against the wealthiest among other $(n-1)$ bidders as

$$\mathrm{E}(B^*_{1i}) = \int\int\int B^*_{1i}(w_{1i}, w_{1ij}, v_{1i}, \eta_{1i}, \theta_{1i})\, dG_{1ij}(v_{ij})\, dQ_{1i}(\hat{\eta}_{1i})\, dS_{1i}(\hat{\theta}_{1i}), \quad i = 2, \ldots, n, \quad i \neq j, \tag{5.13}$$

where G_{1ij} is the distribution on the jth bidder's true value that the ith bidder might elicit that Brenda believes. Now, using (5.13), Brenda will find j such that $b_j = \max\{\mathrm{E}(B^*_{1i}), i = 2, \ldots, n\}$. This bidder is the one that Brenda considers as her competitor bidder. Then, she will find F_{1j}, the distribution of her bid using the change of variable formula as

$$f_{1j} = |J_1| \times g_{1j} = \frac{1}{q_1} \times g_{1j},$$

where $q_1 = b_j/v_j$, v_j is the expected true value of the bidder bidding b_j and g_{1j} is the probability distribution of the true value of the bidder bidding b_j. Finally, after having F_{1j}, Brenda would find her optimal bid for a given value of w_j, the wealth of the bidder bidding b_j as

$$b_1^*(w_j) = \underset{b_1 > \tau}{\arg\max}\ [w_1 + (v_1 - b_1)F_{1j}(b_1) - \int_{b_j : b_j < b_1} \hat{\eta}_1(b_1 - b_j) f_{1j}(b_j) db_j - \int_{b_j : b_1 < b_j \leq v_1} \hat{\theta}_1(b_j - b_1) f_{1j}(b_j) db_j], \quad (5.14)$$

where $\hat{\eta}_1$ and $\hat{\theta}_1$ are Brenda's modified winning and losing regret parameters, respectively. Brenda can use (5.14) to find her optimal bid by taking into account her uncertainty around w_j and can find her expected optimal bid using (5.11).

Example 5.2 *Suppose Brenda is a level-*2 *thinker and believes that her opponents Alex and Charles are level-*1 *thinkers. Brenda believes that Alex thinks that Charles and Brenda are level-*0 *thinkers. Similarly, Brenda believes that Charles thinks that Alex and Brenda are level-*0 *thinkers. Let Brenda, Alex and Charles be the first, second and third bidders, respectively. Let Brenda's true value for the item be* $v_1 = \$200$*, she has her wealth* $w_1 = \$250$ *and the auctioned item has a reserve price* $\tau = \$25$*. Then Brenda believes that:*

- *Alex has a uniform distribution on Brenda's true value with* $\underline{v}_{21} = \$30$ *and* $\bar{v}_{21} = \$200$*, i.e.,* $G_{121} = \frac{(v_{21}-30)}{200-30}$.
- *Alex has a uniform distribution on Brenda's wealth with* $\underline{w}_{21} = \$100$ *and* $\bar{w}_{21} = \$250$*, i.e.,* $H_{121} = \frac{(w_{21}-100)}{250-100}$.
- *Alex has a uniform distribution on Charles's true value with* $\underline{v}_{23} = \$30$ *and* $\bar{v}_{23} = \$250$*, i.e.,* $G_{123} = \frac{(v_{23}-30)}{250-30}$.
- *Alex has a uniform distribution on Charles's wealth with* $\underline{w}_{23} = \$150$ *and* $\bar{w}_{23} = \$300$*, i.e.,* $H_{123} = \frac{(w_{23}-150)}{300-150}$.

- *Charles has a uniform distribution on Brenda's true value with* $\underline{v}_{31} = \$30$ *and* $\bar{v}_{31} = \$230$*, i.e.,* $G_{131} = \frac{(v_{31}-30)}{230-30}$.

- *Charles has a uniform distribution on Brenda's wealth with* $\underline{w}_{31} = \$125$ *and* $\bar{w}_{31} = \$275$*, i.e.,* $H_{131} = \frac{(w_{31}-125)}{275-125}$.

- *Charles has a uniform distribution on Alex's true value with* $\underline{v}_{32} = \$50$ *and* $\bar{v}_{32} = \$220$*, i.e.,* $G_{132} = \frac{(v_{32}-50)}{220-50}$.

- *Charles has a uniform distribution on Alex's wealth with* $\underline{w}_{32} = \$150$ *and* $\bar{w}_{32} = \$350$*, i.e.,* $H_{132} = \frac{(w_{32}-150)}{350-150}$.

- *Alex's valuation distribution is uniform with* $\underline{v}_2 = \$100$ *and* $\bar{v}_2 = \$200$*, i.e.,* $G_{12} = \frac{(v_2-100)}{200-100}$.

- *Alex's wealth distribution is uniform with* $\underline{w}_2 = \$100$ *and* $\bar{w}_2 = \$300$*, i.e.,* $H_{12} = \frac{(w_2-100)}{300-100}$.

- *Charles's valuation distribution is uniform with* $\underline{v}_3 = \$50$ *and* $\bar{v}_3 = \$300$*, i.e.,* $G_{13} = \frac{(v_3-50)}{300-50}$.

- *Charles's wealth distribution is uniform with* $\underline{w}_3 = \$100$ *and* $\bar{w}_3 = \$350$*, i.e.,* $H_{13} = \frac{(w_3-100)}{350-100}$.

- *Alex's winning regret is distributed as Gamma with shape parameter* 1 *and scale parameter* 2 *and his losing regret is also distributed as Gamma with shape parameter* 3 *and scale parameter* 1.

- *Charles's winning regret is distributed as Gamma with shape parameter* 1.5 *and scale parameter* 1 *and his losing regret is also distributed as Gamma with shape parameter* 3 *and scale parameter* 1.5.

- *Alex and Charles elicit their uncertainties around the fraction of their opponents true values that they would bid as* $T_{1il} = \frac{p_{il}^8 - (\underline{v}_{il}/v_{il})^8}{1-(\underline{v}_{il}/v_{il})^8}$ *with support* $(\underline{v}_{il}/v_{il} < p_{il} \leq 1]$ *for* $i = 2, 3$ *and* $l = 1, 2, 3,\ i \neq l$.

*Brenda believes that each of her opponents will use the expected value of their opponents' wealth as an estimate to find the wealthiest among their opponents. Thus, Brenda believes that Alex being a level-*1 *thinker will find his wealthiest opponent as* $\max\{\mathrm{E}(W_{21}), \mathrm{E}(W_{23})\} = \max\{175, 225\} = 225 = \mathrm{E}(W_{23})$, *i.e., he would believe Charles as his competitor bidder and will find his optimal bid* $\mathrm{E}(B^*_{12})$ *against him. Also, Brenda believes that Charles being a level-*1 *thinker will find his wealthiest opponent as* $\max\{\mathrm{E}(W_{31}), \mathrm{E}(W_{32})\} = \max\{200, 250\} = 250 = \mathrm{E}(W_{32})$, *i.e., he would believe Alex as his competitor bidder and will find his optimal bid* $\mathrm{E}(B^*_{13})$ *against him. Using* (5.8), (5.13) *and from Monte Carlo simulations (*$N = 500$*), she will find Alex's expected optimal bid* $\mathrm{E}(B^*_{12}) = 98.58$ *against Charles by taking into account Alex's winning and losing regret. Also, from Monte Carlo simulations, she will find Charles's expected optimal bid* $\mathrm{E}(B^*_{13}) = 115.04$ *against Alex. Then, she will find* j *such that* $b_j = \max\{\mathrm{E}(B^*_{12}), \mathrm{E}(B^*_{13})\} = \max\{98.58, 115.04\} = 115.04 = \mathrm{E}(B^*_{13})$. *So* $j = 3$ *and hence Charles's (3rd bidder) is her competitor bidder with expected optimal bid of* 115.04. *Finally, she would find*

$$q_1 = \frac{115.04}{175} = 0.6573,$$

and

$$F_{1j}(b_1) = \frac{b_1 - q_1 \times \underline{v}_3}{q_1 \times (\bar{v}_3 - \underline{v}_3)} = \frac{b_1 - q_1 \times 50}{q_1 \times (300 - 50)} = \frac{b_1 - 32.87}{164.32}.$$

Now, using (5.11) *and Monte Carlo simulations, she can derive her expected optimal bid* $\mathrm{E}(b_1^*)$*, her probability of winning* $F[\mathrm{E}(b_1^*)]$ *for that expected optimal bid and her expected utility* $\Psi_1[\mathrm{E}(b_1^*)]$ *for that expected optimal bid for various levels of her regret parameters. These are summarized in Table 5.4.*

Table 5.4: Brenda's (level-2 thinker) expected optimal bids, her probabilities of winning and expected utilities.

		θ_1	0.50	1.00	2.00	3.00	4.00
η_1	$\mathrm{E}(\hat{\eta}_1)$	$\mathrm{E}(\hat{\theta}_1)$	0.59	1.18	2.36	3.55	4.73
		$\mathrm{E}(b_1^*)$	120.70	133.50	149.63	159.42	166.02
0.50	0.44	$F[\mathrm{E}(b_1^*)]$	0.53	0.61	0.71	0.77	0.81
		$\Psi_1[\mathrm{E}(b_1^*)]$	270.86	261.41	249.50	242.29	237.46
		$\mathrm{E}(b_1^*)$	109.78	122.77	139.92	150.79	158.30
1.00	0.87	$F[\mathrm{E}(b_1^*)]$	0.47	0.55	0.65	0.72	0.76
		$\Psi_1[\mathrm{E}(b_1^*)]$	261.93	249.41	232.84	222.38	215.18
		$\mathrm{E}(b_1^*)$	94.54	107.09	124.80	136.74	145.39
2.00	1.74	$F[\mathrm{E}(b_1^*)]$	0.38	0.45	0.56	0.63	0.68
		$\Psi_1[\mathrm{E}(b_1^*)]$	249.44	231.75	206.66	189.70	177.48
		$\mathrm{E}(b_1^*)$	84.40	96.15	113.51	125.79	134.97
3.00	2.61	$F[\mathrm{E}(b_1^*)]$	0.31	0.38	0.49	0.56	0.62
		$\Psi_1[\mathrm{E}(b_1^*)]$	241.13	219.40	187.00	163.99	146.80
		$\mathrm{E}(b_1^*)$	77.15	88.05	104.74	116.99	126.39
4.00	3.48	$F[\mathrm{E}(b_1^*)]$	0.27	0.34	0.44	0.51	0.57
		$\Psi_1[\mathrm{E}(b_1^*)]$	235.20	210.27	171.70	143.23	121.35

Now, we provide a brief sketch on how to find the ARA solution when Brenda wants to perform a level-3 analysis. In this case, Brenda models herself as a level-3 thinker and believes that her $(n-1)$ opponents are level-2 thinkers. Each of these level-2 thinkers would model their $(n-1)$ opponents as level-1 thinkers and each of these level-1 thinkers would model their $(n-1)$ rivals as level-0 thinkers. To find the ARA solution in this case, Brenda would perform the level-2 analysis detailed above for each of the other $(n-1)$ bidders and will obtain their optimal bids using (5.14) where $b_1^*(w_i)$, b_1, w_1, v_1, F_{1j}, $\hat{\eta}_1$, and $\hat{\theta}_1$, b_j are replaced by $B_{1i}^*(w_{ij})$, b_{1i}, W_{1i},

V_{1i}, F_{1ij}, $\hat{\eta}_{1i}$, and $\hat{\theta}_{1i}$ and b_{ij} respectively and w_{ij} be wealth of the level-1 thinker bidder bidding maximum among other $(n-1)$ bidders that the ith bidder believes for $i = 2, \ldots, n$, $i \neq j$. Then, she will find b_j, the maximum bid among other $(n-1)$ bidders using (5.14) (with the replaced quantities) and would get her belief about F_{1j}, the distribution of the jth bidder's (level-2 thinker) bid. Then, she can obtain her optimal bid $b_1^*(w_j)$ or the expected optimal bid $\mathrm{E}(b_1^*)$ using similar process as in (5.10) and (5.11), respectively.

Using a level-k analysis, the key question is how large should the k be? Players could choose a higher k while playing games such as Chess or Go, which are highly structured games. However, Ho et al. (1998) and Lee and Wolpert (2012) based on experimental evidences stated that, people typically do not think higher than level 2 or 3. Therefore, it makes sense to solve the level-k problem for k being 1, 2 or 3 for FPSB auctions.

5.5 Conclusion and Future Work

In this Chapter, we assume that a single (*normal*) item is being auctioned in a FPSB auction that has a reserve price which is typically known to each bidder in advance. We define new regret parameters to take into account the effect of bidder's wealth on their bidding behavior and modify the utility function as used by Engelbrecht-Wiggans and Katok (2007) that incorporates bidders' winning and losing regret. We find ARA solutions not only using non-strategic play but also using the level-k thinking solution concept assuming n bidders participating in these auctions. For this type of utility function, we take into account the uncertainties in bidders' winning and losing regret in addition to their valuations and wealth. We model how an increase in the decision maker's wealth will affect their winning and losing regrets. We also provide numerical examples in which we use Monte Carlo methods to illustrate our methodology to find ARA solutions for each solution concept. Finding ARA solutions for a utility function that considers winning and losing regret while taking into account the uncertainties regarding bidders' valuations, wealth and their regrets was methodologically and computationally more challenging than that for a CRRA utility function. ARA solutions for the other solution concepts such as ME and BNE for the utility function that we developed in this Chapter are yet to be found.

Without loss of generality, we have made an assumption that the bidder with the maximum wealth among other $(n-1)$ bidders could also have more valuation for the auctioned item (because the item is normal) and therefore would bid maximum among other $(n-1)$ bidders. Therefore, the decision maker will consider the bidder with the maximum wealth among the other $(n-1)$ bidders as her main competitor and will find her optimal bid against him. But, in practice, it could be possible that a bidder with relatively low wealth could have a higher value for the auctioned item and therefore could bid higher than the bidder having relatively higher wealth but low valuation. Thus, it is important to model the maximum bid among other $(n-1)$ bidders in a more general way that considers both the wealth and valuation of the bidders.

We have assumed that the decision maker believes that each of her rivals are of the same type, i.e., either non-strategic players or level-k thinkers. However, it might be possible that the decision maker is uncertain about her rivals' solution concepts. This is called *concept (model) uncertainty.* In this case, she needs to take into account her concept uncertainty in order to find her optimal decision. Under concept uncertainty, first she needs to find her optimal decision $\mathrm{E}(b^*|\mathcal{M})$ conditional on the given solution concept $\mathcal{M}$ (for example, non-strategic play, level-k thinking, BNE and ME etc.). Second, she needs to elicit her subjective distribution $\Pr(\mathcal{M})$ that reflects her uncertainty on $\mathcal{M}$. Then, she can find her optimal decision by taking into account concept uncertainty as

$$\mathrm{E}(b^*) = \sum_{\mathcal{M}} \mathrm{E}(b^*|\mathcal{M}) \Pr(\mathcal{M}).$$

One of the main challenges in implementing an ARA solution is the elicitation of prior distributions. The elicitation of prior distributions is considered to be a practical challenge while adopting almost any Bayesian approach, however, this challenge is further accentuated in ARA where the decision maker not only needs to elicit their own uncertainties, but also, the uncertainties of their adversaries. Analysis of the robustness of the ARA solution to the choice of the prior distributions is therefore necessary. Ríos Insua et al. (2016) provide an outline for a robustness analysis using an ARA framework. However, mathematical framework for implementing a prior robustness analysis for an ARA model has not yet been developed. This is an important direction for further work.

Finally, many other types of auctions have been studied and used in practice. These include other variations on the sealed bid auction process, such as the *second-price* (Vickrey, 1961), the *third-price* (Kagel and Levin, 1993), or in general, the m^{th}*−price* (Cason, 1995; Shogren et al., 2001) sealed bid auctions have been proposed. Shogren et al. (2001) showed that a random m^{th}*−price* sealed-bid auction[3] can induce sincere bidding in theory and practice than a second-price sealed-bid

[3] In a random m^{th}*−price* sealed-bid auction, each bid is rank-ordered from highest to lowest; the auctioneer selects a random number, the m in the m^{th}−price sealed-bid auction, uniformly-

auction when bidders' values are far below or above the market-clearing price[4] of the auctioned items. ARA solutions for these variants on the sealed bid auctions process need to be developed.

Finally, ARA methodology can further be applied to the auctions of a sequential paradigm where the decision maker and her opponents' decisions alternatively evolve over time such as in English[5] or Dutch[6] auctions etc. In such auctions, there could be short or long term interactions among the bidders over time. An approach developed by González-Ortega et al. (2019) could be used for short term interactions with changing dynamics while a Markov decision process could be used for long term interactions, with a fixed structure as argued by Joshi et al. (2020). Thus, developing ARA solutions to sequential auctions is a challenging research problem too.

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distributed between 2 and n (n bidders); and the auctioneer sells one auctioned item to each of the $(m-1)$ highest bidders at the $m^{th}-$price.

[4] The price at which the quantity demanded of an item or service is equal to the quantity supplied and no surplus or shortage exists in the market.

[5] The participants make increasingly higher bids and stop bidding when they are not prepared to pay more than the current highest bid. This continues until no participant is prepared to make a higher bid; the highest bidder wins the auction at the final amount bid.

[6] In Dutch auctions, the price is set by the auctioneer at a level sufficiently high to deter all bidders, and is progressively lowered until a bidder is prepared to buy at the current price auctioned item.

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Chapter 6

Conclusion, Limitations and Further Work

6.1 Conclusion

In this thesis, we use adversarial risk analysis (ARA) methodology and provide a more realistic approach to model bidders' behaviours in first-price sealed-bid (FPSB) auctions than what has previously been done. Specifically, we assume that a single item is being auctioned and that it is a normal item which has a reserve price that is typically known to each bidder in advance.

First, we assume two bidders in FPSB auctions and propose a new utility function that takes into account bidders' wealth which is an important determinant of their bidding behaviour in these auctions. The new utility function is realistic in a sense that it constrains the bidders' bids and their true values in consideration with their wealth. For our new proposed utility function, we define a new CRRA parameter that models the change in bidders' risk behaviour with the relative change in their wealth, as could be expected in case of a normal item. Then, we extend the ARA solutions developed by Banks et al. (2015) using non-strategic play and level-k thinking solution concepts, wherein we consider that the auctioned item has reserve price, we assume not only risk-neutral bidders but also, risk-averse and risk-seeking bidders and we assume each bidder has different wealth. We give numerical examples to illustrate our methodology for each solution concept. A Monte Carlo approach has been used to find the solutions in the given examples. The examples show that when the bidders are assumed to be risk-averse, they bid more aggressively than that when they are assumed to be risk-neutral. Whereas, the risk-seeking bidders bid lower than that of risk-neutral bidders as expected.

Second, we find ARA solutions using mirror equilibrium (ME) and Bayes Nash Equilibrium (BNE) solution concepts by taking into account the uncertainties around bidders' risk behaviour (risk-neutral, risk-averse or risk-seeking) and their wealth that is assumed to be different for each bidder. For the symmetric case of BNE solution concept, we assume n bidders and provide a general framework to find ARA solutions. Whereas, for two bidders using the asymmetric case of BNE solution concept, we show that ARA solutions turn out to be the same as that of ARA solutions derived using ME solution concept. To illustrate our methodology for each solution concept, we provide numerical examples. The uncertainties around bidders' wealth and their heterogeneous risk behaviour have been incorporated using a Monte Carlo approach for the ME and BNE solution concepts. The bidders' bidding behaviour has been illustrated by approximate equilibrium bid functions along with 95% probability intervals. The results show that the bidders become more risk-averse (less risk-seeking) with an increase in their wealth and thus bid higher as expected.

Third, we find ARA solutions using non-strategic play and level-k thinking solution concepts assuming n bidders participating in FPSB auctions for another type of utility function that incorporate bidders' winning and losing regret. We consider this type of utility function because CRRA utility function only explains bidders' observed behaviour and does not explain the inherent reasons why bidders bid more than risk-neutral Nash Equilibrium (RNNE) bid in FPSB auctions. Among other reasons, one of the reasons explained in the economic literature of this overbidding is that the bidders could have more losing regret than their winning regret, therefore they bid more aggressively than RNNE bid. We propose a new utility function by improving the utility function defined by Engelbrecht-Wiggans and Katok (2007). We consider that bidders have different wealth that has not been considered yet for such type of utility function. In order to take into account the effect of bidders' wealth on their bidding behaviour, we define new regret parameters. We take into account the uncertainties on bidders' winning and losing regret in addition to their valuations and wealth. We provide numerical examples in which we use Monte Carlo method to show how bidders' wealth affect their bidding behaviours. Finding ARA solutions while taking into account the uncertainties such as bidders' valuation, wealth and their regrets by Monte Carlo methods were methodologically and computationally more challenging than that for the utility function with CRRA profit.

6.2 Limitations

Using an ARA approach, the decision maker's optimal decision depends upon the elicitation of her prior distributions to model her rivals actions for a certain solution concept. She places those prior distributions according to her belief based upon her past experience against her adversaries or some information from her informant etc. However, in practice, it might possible that she either does not have any experience with some of her adversaries ($n > 2$) or she could not get any information about those adversaries from any other source. In that case, there is a chance of misspecification of prior distributions and that could result the decision maker making a losing bid or a winning bid that is far from the second highest bid.

Also, in this thesis, we have assumed that the decision maker believes that each of her rivals use a certain solution concept (non-strategic play, level-k thinking, ME or BNE) and models her actions using that solution concept. However, it might be possible that the decision maker is uncertain about which solution concept her rivals are going to be used. This type of uncertainty is called concept (model) uncertainty and needs to be modelled as well.

For two bidders, our model is realistic. However, assuming n bidders, we have made an assumption without loss of generality that the bidder with the maximum wealth among other ($n-1$) bidders could also have more valuation for the auctioned item (because the item is normal) and therefore would bid the maximum amount among the other ($n - 1$) bidders. As a result, the decision maker will consider the bidder with the maximum wealth among other ($n - 1$) bidders as her competitor and find her optimal bid against him. But, in practice, it could be possible that a bidder with relatively low wealth could relatively have a high value for the auctioned item and therefore could bid high than the bidder having relatively high wealth.

6.3 Further Work

As mentioned in Section 6.2, a critical issue while adopting an ARA approach is the prior elicitation. Therefore, a robustness analysis of ARA solutions to the choice of prior distributions is necessary to be carried out. Ríos Insua et al. (2016) provided an outline how a robustness analysis using an ARA approach can be performed. However, a mathematical framework for a prior robustness analysis using an ARA approach has not been developed yet. It could be an important area for further work.

Also, it is important for the decision maker to take into account her concept (model) uncertainty. In order to do this, first she needs to find her expected optimal decision conditional on the given solution concept $\mathcal{M}$ i.e., she needs to find $\mathrm{E}(b^*|\mathcal{M})$. Second, she needs to elicit her subjective distribution $\Pr(\mathcal{M})$ that reflects her uncertainty on $\mathcal{M}$. Then, she can find her expected optimal decision as

$$\mathrm{E}(b^*) = \sum_{\mathcal{M}} \mathrm{E}(b^*|\mathcal{M}) \Pr(\mathcal{M}).$$

Finding the decision maker's optimal bid in case of n bidders, we have assumed that she believes that the bidder with the maximum wealth among the other $(n-1)$ bidders could also have more valuation for the auctioned item. Therefore, this bidder would make the maximum bid among the other $(n-1)$ bidders. This could be unrealistic in practice. Thus, it is important to find a realistic way to model the maximum bid among other $(n-1)$ bidders while taking into account their wealth and valuations.

Also, using the CRRA utility function and the risk-neutral utility function with bidders' regret that we have used in this thesis, the ARA solutions for other type of auctions such as general $m^{th}-price$ sealed-bid auctions (second-price sealed-bid auctions for $m = 2$ by Vickrey (1961), third-price sealed-bid auction for $m = 3$ studied by Kagel and Levin (1993)) and a random $m^{th}-price$ sealed-bid auctions studied by Cason (1995); Shogren et al. (2001) could be interesting to find.

Moreover, ARA solutions can further be found for the other type of auctions such as in English, Dutch or sequential auctions etc., where the decision maker and her rivals' decisions alternatively evolve over time. In such auctions, there could develop short or long term interactions among the bidders over time. González-Ortega et al. (2019) developed an approach which could be applied to cater for short term interactions with changing dynamics while a Markov decision process (MDP) with a fixed structure could be used for long term interactions as suggested by Joshi et al. (2020).

Further, ARA solutions using ME and BNE solution concepts for the utility function that considers bidders' regret are yet to be found and could be challenging.

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Appendix

Co-Authorship Forms

The co-authorship forms related to three articles included in this thesis are provided on the following pages

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Chapter 3: Adversarial Risk Analysis for First-Price Sealed-Bid Auctions

Accepted for Publication by Australian and New Zealand Journal of Statistics, doi: 10.1111/ANZS.12315

Nature of contribution by PhD candidate	Modelling, Computation, Writing and Reviewing
Extent of contribution by PhD candidate (%)	70

CO-AUTHORS

Name	Nature of Contribution
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Stephen Joe	Conceptualisation and reviewing

Certification by Co-Authors

The undersigned hereby certify that:

- the above statement correctly reflects the nature and extent of the PhD candidate's contribution to this work, and the nature of the contribution of each of the co-authors; and

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Chapter 4: Adversarial Risk Analysis for Auctions using Mirror Equilibrium and Bays Nash Equilibrium
Accepted for Publication by Decision Analysis

Nature of contribution by PhD candidate	Modelling, Computation, Writing and Reviewing
Extent of contribution by PhD candidate (%)	70

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Chaitanya Joshi	Conceptualisation and reviewing

Certification by Co-Authors

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- the above statement correctly reflects the nature and extent of the PhD candidate's contribution to this work, and the nature of the contribution of each of the co-authors; and

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Chapter 5: Adversarial Risk Analysis for Auctions using Non-Strategic Play and Level-k Thinking: A General Case of n Bidders with Regret

Submitted for Publication in Communications in Statistics-Theory and Methods

Nature of contribution by PhD candidate	Conceptualisation, Modelling, Computation, Writing and Reviewing
Extent of contribution by PhD candidate (%)	70

CO-AUTHORS

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Stephen Joe	Conceptualisation and reviewing

Certification by Co-Authors

The undersigned hereby certify that:

- the above statement correctly reflects the nature and extent of the PhD candidate's contribution to this work, and the nature of the contribution of each of the co-authors; and

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