On class operators for the lower radical class and semisimple closure constructions.

N. R. McConnell, R. G. McDougall, T. Stokes and L. K. Thornton.

October 2023

Abstract. In this work, we construct the lower radical class and the semisimple closure for a given class using class operators and detail some of the properties of these operators and their interplay with the operators already used in radical theory. The setting is the class of algebras introduced by Puczyłowski which ensures the results hold in groups, multi-operator groups such as rings, as well as loops and hoops.

Keywords: lower radical class, semisimple closure, class operator.
Classification: 16N80, 17A65.

1 Introduction

Research in radical theory has been under way for more than a century, begun by Wedderburn investigating hypercomplex numbers [17], explored further in classes of associative rings through key examples by Köthe [7] and Jacobson [6], before its formal grounding in this setting being undertaken independently by Kurosh [8] and Amitsur [2]. This platform allowed for its rapid development (see the reference list in the text by Gardner and Wiegandt [4] for a useful record of the way the theory unfolded for rings up to the date of publication) with key notions of the theory extended to other mathematical objects having suitable internal structure (for example groups [9],[3] and topological spaces [19]) to ensure the related object induced through a mapping is more accessible for analysis and description. While some object properties satisfy the Kurosh-Amitsur framework, a central problem has been to identify a smallest suitable (radical) subobject acting as the kernel for a mapping so that the image induced is free from radical subobjects (and called semisimple). The collection of radical objects forms a radical class and the collection of semisimple objects forms a semisimple class. Class operators, introduced early as part of the theory's development [18], under slight adjustment were shown to be a suitable vehicle to re-introduce the theory [11] in terms of base radical [13] and base semisimple classes [14], and were used effectively to determine the radical theory possible in certain universal classes containing a finite number of class objects [16]. The cost of this simpler construction is that the class subobject so constructed may no longer be the smallest radical subobject to contain the object property under consideration. The class operators used to date are not generally able to construct the suitable radical subobject and so the aim of this paper is to introduce appropriate additional operators to close this gap and then to go on and detail some of the properties of these operators in their own right as well as their interplay with the familiar operators. The work is set in the class of algebras introduced by Puczyłowski [15] (here called P-algebras) which includes ideal-determined varieties such as (multi-operator) groups, rings, loops and hoops. Some of the results require us to work in an ideal-determined variety [12].

2 Preliminaries

Classes are denoted by calligraphic font such as $\mathcal{X}$ and elements of a class by uppercase letters such as A. We denote by $\mathcal{A}$ a class of objects called *P-algebras* with $0 \in \mathcal{A}$ the *zero P-algebra* and $\cong$ an equivalence relation on $\mathcal{A}$ called *isomorphism* [15]. All subclasses of $\mathcal{A}$ are assumed to be *abstract*, meaning closed under isomorphisms and containing 0 (as in [11]). A class $\mathcal{A}$ of P-algebras satisfies the following axioms:

1. For each $A \in \mathcal{A}$ there is associated a complete lattice $(L_A, \leq)$ such that $L_A \subseteq \mathcal{A}$, and L_A has top element A and bottom element $0_A \cong 0$. For all $I \in L_A$ we call I an *ideal* of A, denoted $I \triangleleft A$.

2. For every $I \in L_A$, the interval $[0, I]_{L_A} = \{B \in L_A \mid 0 \leq B \leq I\}$ is a sublattice of L_I.

3. With every $I \triangleleft A$ there exists $A/I \in \mathcal{A}$, with $L_{A/I} = \{K/I \mid K \triangleleft A, I \leq K \leq A\}$, and the

map given by $K \mapsto K/I$ is an isomorphism $[I, A]_{L_A} \to L_{A/I}$.

4. If $A \cong B$ then there is an isomorphism $f : L_A \to L_B$ such that for each $I \in L_A$, $I \cong f(I)$ and $A/I \cong B/f(I)$.

5. For each $A \in \mathcal{A}$ and $I, J \lhd A$ with $J \leq I, (A/J)/(I/J) \cong A/I$, and for any $I, J \lhd A, (I \vee J)/I \cong J/(I \wedge J)$.

By axiom 3, for any $A \in \mathcal{A}$, $A/A = 0$, and by axiom 4, $A \cong 0$ if and only if $A = 0_A$. If $I \lhd A$ then A/I is a *factor* of A. We say $I \lhd A$ is a *proper ideal* if $I \neq A$ and similarly, A/I is *proper factor* if $A/I \neq A$. For the remainder of this paper we will consider those algebras satisfying the condition $A/0 \cong A$ (this could easily have been added as an additional axiom but is included here to retain the axiom structure in its original form).

Any ideal-determined variety in the sense of [5] gives an example of the abstract set-up above. Let $\mathcal{V}$ be a variety with distinguished nullary 0. Suppose that the ρ-class containing 0 determines ρ then $\mathcal{V}$ is a *0-regular* variety. If $\sigma(0, 0, \ldots, 0) = 0$ is an identity for each operation σ in the signature of $\mathcal{V}$, then we call $\mathcal{V}$ a *0-normal* variety [12]. It is *ideal-determined* in the sense of [5] if moreover any two congruences permute at zero, meaning that for any two congruences ρ, θ on A, $(a, 0) \in \rho \circ \theta$ if and only if $(a, 0) \in \theta \circ \rho$; equivalently, $(a, 0) \in \rho \vee \theta$ implies the existence of $b \in A$ for which $(b, 0) \in \rho$ and $(a, b) \in \theta$. (In fact the definition of "ideal-determined" given in [5] is shown there to be equivalent to the one given here.) A variation on the famous result due to Malcev linking congruence permutable varieties to the existence of a certain ternary term, indirectly shown in [5], and demonstrated explicitly in [1] (see Theorem 2.4), says that this permuting at zero property is equivalent to the existence of a binary term $s(x, y)$ in the variety for which $s(x, x) = 0$ and $s(x, 0) = x$. It follows that the ideals of an algebra in an ideal-determined variety are nothing but its congruence classes that contain 0 (exactly as for rings, and indeed all the familiar examples if one replaces "0" by "1" as appropriate).

In an ideal-determined variety $\mathcal{A}$, for $A \in \mathcal{A}$ and $I \in L_A$, A/I is the factor algebra in $\mathcal{A}$ obtained by factoring out I from A in the usual way: I determines a unique congruence ρ_I for which I is the ρ_I-class containing 0, and then A/I is nothing but A/ρ_I. All of the P-algebra axioms hold, in particular the last (which is a version of the second isomorphism theorem). The algebra 0 in an ideal-determined variety stands for the trivial algebra $\{0\}$; we let context determine the meaning.

In particular, in the ideal-determined variety $\mathcal{A}$, we can form the direct sum $A \oplus B$ of $A, B \in \mathcal{A}$, which is nothing but their usual Cartesian product $A \times B$ as algebras, so copies of A, B are factors of it, but they also embed as ideals. Thus the congruence π_1 on $A \times B$ given by $(a_1, a_2)\pi_1(b_1, b_2)$ if and only if $a_2 = b_2$ is such that $A \oplus B/\pi_1 \cong B$, and the π_1-class containing $(0, 0) = 0$ in $A \oplus B$ is the ideal $\bar{A} = \{(a, 0) \mid a \in A\}$ which is isomorphic to A. Then $A \oplus B/\bar{A} \cong B$ and vice versa.

A class $\mathcal{X} \subseteq \mathcal{A}$ is *hereditary* if, for all $A \in \mathcal{X}$, $I \lhd A$ implies $I \in \mathcal{X}$, and $\mathcal{X}$ is *factor closed* if, for all $A \in \mathcal{X}$, $I \lhd A$ implies $A/I \in \mathcal{X}$. A class $\mathcal{X}$ is *closed under extensions* if $\mathcal{X}$ has the property that whenever I is an ideal of A and both I and A/I are elements of $\mathcal{X}$, then $A \in \mathcal{X}$. A universal class is abstract, hereditary and factor closed.

Define J to be an *accessible* of $A \in \mathcal{A}$ if there exists a finite chain $J = J_0 \lhd J_1 \lhd \ldots \lhd J_n = A$. If J is nonzero and has no nonzero proper ideals we say J is *simple*.

Theorem 2.1. (Lemma 4.1 [11]) *If A is a P-algebra, then every accessible of a factor of A is a factor of an accessible of A. Specifically, if I is an accessible of A/J then $I = K/J$ for some accessible K of A.*

The class operators $\mathbf{U}$, $\mathbf{S}$ and $\neg$ act on subclasses $\mathcal{X}$ of $\mathcal{A}$ and are defined by

$\mathbf{U}(\mathcal{X}) = \{A \in \mathcal{A} \mid A \text{ has no nonzero factor in } \mathcal{X}\}$,

$\mathbf{S}(\mathcal{X}) = \{A \in \mathcal{A} \mid A \text{ has no nonzero accessible in } \mathcal{X}\}$, and

$\neg(\mathcal{X}) = \{A \in \mathcal{A} \mid A \text{ is not in } \mathcal{X}\}$.

It follows from $A/0 \cong A$ that $\mathbf{U}(\mathcal{X}) \cap \mathcal{X} = \{0\}$.

Remark 1. *A subclass $\mathcal{X}$ of $\mathcal{A}$ is hereditary if and only if $\mathcal{X} = \mathbf{S}\neg(\mathcal{X})$ and $\mathcal{X}$ is factor closed if and only if $\mathcal{X} = \mathbf{U}\neg(\mathcal{X})$.*

Theorem 2.2. (Theorem 2.1 [11]) *For all subclasses $\mathcal{X}$ and $\mathcal{Y}$ of $\mathcal{A}$,* *(i)* *If* $\mathcal{X} \subseteq \mathcal{Y}$ *then* $\mathbf{U}(\mathcal{Y}) \subseteq \mathbf{U}(\mathcal{X})$ *and* $\mathbf{S}(\mathcal{Y}) \subseteq \mathbf{S}(\mathcal{X})$,

(ii) $\mathbf{U}(\mathcal{X}) \subseteq \neg(\mathcal{X})$ *and* $\mathbf{S}(\mathcal{X}) \subseteq \neg(\mathcal{X})$,

(iii) $\mathbf{U}\neg\mathbf{U}(\mathcal{X}) = \mathbf{U}(\mathcal{X})$ *and* $\mathbf{S}\neg\mathbf{S}(\mathcal{X}) = \mathbf{S}(\mathcal{X})$,

(iv) $\mathbf{UUUU}(\mathcal{X}) = \mathbf{UU}(\mathcal{X})$ *and* $\mathbf{SSSS}(\mathcal{X}) = \mathbf{SS}(\mathcal{X})$, *and*

(v) $\mathbf{USUS}(\mathcal{X}) = \mathbf{US}(\mathcal{X})$ *and* $\mathbf{SUSU}(\mathcal{X}) = \mathbf{SU}(\mathcal{X})$.

Theorem 2.3. (Lemma 4.5 [11]) *For all subclasses* $\mathcal{X} \subseteq \mathcal{A}$, $\mathbf{S}\neg\mathbf{U}(\mathcal{X})$ *is factor closed.*

The primary compositions for these operators to establish radical and semisimple classes are then

$\mathbf{US}(\mathcal{X}) = \{A \in \mathcal{A} \,|\, \text{every nonzero factor of } A \text{ has a nonzero accessible in } \mathcal{X}\}$, and
$\mathbf{SU}(\mathcal{X}) = \{A \in \mathcal{A} \,|\, \text{every nonzero accessible of } A \text{ has a nonzero factor in } \mathcal{X}\}$.

Definition 1. *For any subclass* $\mathcal{X} \subseteq \mathcal{A}$, *we call*

(i) $\mathcal{X}$ *a* base radical class *whenever* $\mathcal{X} = \mathbf{US}(\mathcal{X})$, *and*

(ii) $\mathcal{X}$ *a* base semisimple class *whenever* $\mathcal{X} = \mathbf{SU}(\mathcal{X})$.

Remark 2. *For any subclass* $\mathcal{X}$, *the class* $\mathbf{US}(\mathcal{X})$ *is always a base radical class and the class* $\mathbf{SU}(\mathcal{X})$ *is always a base semisimple class, although in both cases not necessarily equal to the class* $\mathcal{X}$ *itself.*

Remark 3. *When* $\mathbf{U}(\mathcal{X})$ *is a radical class it is called* the upper radical class of $\mathcal{X}$ [4] *and from the definition above* $\mathbf{U}(\mathcal{X}) = \mathbf{USU}(\mathcal{X})$.

For the remainder of this work, the term 'radical class' will mean base radical class and 'semisimple class' will mean base semisimple class. The smallest radical class to contain a class $\mathcal{X}$ is called the *lower radical class* usually denoted $\mathcal{L}(\mathcal{X})$, and the smallest semisimple class containing $\mathcal{X}$ is called the *semisimple closure* of $\mathcal{X}$ usually denoted $\overline{\mathcal{X}}$.

Theorem 2.4. (Theorem 3.3 [11]) *For all subclasses $\mathcal{X} \subseteq \mathcal{A}$,*

(i) $\mathbf{US}(\mathcal{X})$ *is a radical class; if $\mathcal{X}$ is factor closed, then the lower radical class containing $\mathcal{X}$, $\mathcal{L}(\mathcal{X}) = \mathbf{US}(\mathcal{X})$,*

(ii) $\mathbf{SU}(\mathcal{X})$ *is a semisimple class; if $\mathcal{X}$ is hereditary, then the semisimple closure of $\mathcal{X}$, $\overline{\mathcal{X}} = \mathbf{SU}(\mathcal{X})$.*

The lower radical class and semisimple closure constructions for a class $\mathcal{X}$ are generally unreachable under composition of $\mathbf{U}, \mathbf{S}$ and $\neg$ alone. For example, consider the ring A on four non-associative elements introduced by Leavitt and Watters [10] (and detailed in [4], page 305) with its one proper ideal I which itself has one simple proper ideal J. The homomorphic images A/I and I/J are simple and $A/I, I/J$ and J are non-isomorphic. The subuniversal class generated is $\mathcal{A}_1 = \{0, A, I, J, A/I, I/J\}$. The smallest radical class to contain $\{I\}$ is $\mathcal{L}(\{I\}) = \{0, I, I/J\}$, which cannot be generated through $\mathbf{U}, \mathbf{S}$ and $\neg$ since the possible compositions generating radicals classes [16] are

$\mathbf{US}(\{I\}) = \mathbf{USSS}(\{I\}) = \mathbf{USU}\neg(\{I\}) = \mathbf{UU}(\{I\}) = \{0\}$ and

$\mathbf{USS}(\{I\}) = \mathbf{USU}(\{I\}) = \mathbf{US}\neg(\{I\}) = \mathbf{UUU}(\{I\}) = \mathcal{A}_1$.

Therein lies the motivation for introducing new operators; to find an operator description of the lower radical class as well as the semisimple closure of a subclass $\mathcal{X}$ of $\mathcal{A}$. In addition, these operators have many interesting properties.

Let A and B be any two P-algebras. If B is a factor of A then we call A a *preimage* of B, and if B is an accessible of A then A is called a *cover* of B. The properties 'is a preimage of' and 'is a cover of' form preorders. We define two class operators $\mathbf{V}$ and $\mathbf{T}$ that can act on any subclass $\mathcal{X} \subseteq \mathcal{A}$ as

$\mathbf{T}(\mathcal{X}) = \{A \in \mathcal{A} \mid A$ has no nonzero cover in $\mathcal{X}\}$ and

$\mathbf{V}(\mathcal{X}) = \{A \in \mathcal{A} \mid A$ has no nonzero preimage in $\mathcal{X}\}$.

In the following section we make extensive use of the *duality* between particular operators (like $\mathbf{U}$ and $\mathbf{S}$ and between pairs of operators like $\mathbf{U}$ and $\mathbf{S}$, with $\mathbf{V}$ and $\mathbf{T}$). By duality we mean that in certain contexts, operators can be exchanged to generate concepts and statements which are related but different in their meaning. We see this for example in Definition 1 and we will see much more of it in both the actions of the class operators and the proof technique it offers when appropriate. For the class of P-algebras $\mathcal{A}$, the duality is not perfect (both for $\mathbf{U}$ and $\mathbf{S}$ [11] and as we will see for $\mathbf{V}$ and $\mathbf{T}$ in Proposition 3.13).

3 Main results for classes of P-algebras

3.1 The lower radical and semisimple constructions

The operators $\mathbf{V}$ and $\mathbf{T}$ can be combined with $\mathbf{U}, \mathbf{S}$ and $\neg$ to obtain new classes of interest. In particular, for any subclass $\mathcal{X} \subseteq \mathcal{A}$,

- $\mathbf{UV}(\mathcal{X}) = \{A \in \mathcal{A} |$ every nonzero factor of A has a nonzero preimage in $\mathcal{X}\}$ which we will soon show is the smallest factor closed class to contain $\mathcal{X}$,
- $\mathbf{VU}(\mathcal{X}) = \{A \in \mathcal{A} |$ every nonzero preimage of A has a nonzero factor in $\mathcal{X}\}$, the preimage closure of $\mathcal{X}$,
- $\mathbf{ST}(\mathcal{X}) = \{A \in \mathcal{A} |$ every nonzero accessible of A has a nonzero cover in $\mathcal{X}\}$, the smallest hereditary class to contain $\mathcal{X}$, and
- $\mathbf{TS}(\mathcal{X}) = \{A \in \mathcal{A} |$ every nonzero cover of A has a nonzero accessible in $\mathcal{X}\}$, the smallest factor closed class to contain $\mathcal{X}$.

We now verify these claims.

Proposition 3.1. *For all subclasses $\mathcal{X} \subseteq \mathcal{A}$,*

(i) $\mathbf{UV}(\mathcal{X})$ *is the smallest factor closed class to contain $\mathcal{X}$ and* $\mathbf{UV}(\mathcal{X}) = \neg\mathbf{V}(\mathcal{X})$*,*

(ii) $\mathbf{VU}(\mathcal{X})$ *is the preimage closure of $\mathcal{X}$ and* $\mathbf{VU}(\mathcal{X}) = \neg\mathbf{U}(\mathcal{X})$*,*

(iii) $\mathbf{ST}(\mathcal{X})$ *is the smallest hereditary class to contain $\mathcal{X}$ and* $\mathbf{ST}(\mathcal{X}) = \neg\mathbf{T}(\mathcal{X})$*, and*

(iv) $\mathbf{TS}(\mathcal{X})$ *is the cover closure of $\mathcal{X}$ and* $\mathbf{TS}(\mathcal{X}) = \neg\mathbf{S}(\mathcal{X})$*.*

Proof. The proof in all cases for $\mathcal{X} = \{0\}$ is straightforward since $\neg\mathbf{V}(\mathcal{X}) = \neg\mathbf{U}(\mathcal{X}) = \neg\mathbf{T}(\mathcal{X}) = \neg\mathbf{S}(\mathcal{X}) = \{0\}$. Now to prove (i) and (iii) for a nonzero class $\mathcal{X}$: If $A \in \neg\mathbf{V}(\mathcal{X})$ then

A has a nonzero preimage in $\mathcal{X}$ and A is in the smallest factor closed class to contain $\mathcal{X}$. Conversely, if A is in the smallest factor closed class to contain $\mathcal{X}$ then A has a nonzero preimage in $\mathcal{X}$ and so $A \in \neg\mathbf{V}(\mathcal{X})$. For all $\mathcal{X} \subseteq \mathcal{A}$, $\mathbf{UV}(\mathcal{X}) \subseteq \neg\mathbf{V}(\mathcal{X})$. Suppose $\mathcal{A} \in \neg\mathbf{V}(\mathcal{X})$. Then A has a nonzero preimage $C \in \mathcal{X}$ and for every nonzero factor B of A, B has C as a nonzero preimage. Therefore every nonzero factor of A has a nonzero preimage in $\mathcal{X}$, hence $A \in \mathbf{UV}(\mathcal{X})$ and $\mathbf{UV}(\mathcal{X}) = \neg\mathbf{V}(\mathcal{X})$. The dual argument exchanging $\mathbf{U}$ with $\mathbf{S}$ and $\mathbf{V}$ with $\mathbf{T}$ proves (iii).
(ii) and (iv): Let $\mathcal{X} \neq \{0\}$ and $A \in \neg\mathbf{U}(\mathcal{X})$. Then A has a nonzero factor B in $\mathcal{X}$ so A is in the nonzero preimage closure of $\mathcal{X}$. Conversely, if A is in the preimage closure of $\mathcal{X}$ then A has a nonzero factor in $\mathcal{X}$, hence $A \in \neg\mathbf{U}(\mathcal{X})$. For all $\mathcal{X} \subseteq \mathcal{A}$, $\mathbf{VU}(\mathcal{X}) \subseteq \neg\mathbf{U}(\mathcal{X})$. Let $A \in \neg\mathbf{U}(\mathcal{X})$. Then A has a nonzero factor B in $\mathcal{X}$ and for every nonzero preimage C of A, C has B as a nonzero factor and so every nonzero preimage of A has a nonzero factor in $\mathcal{X}$. Hence $A \in \mathbf{VU}(\mathcal{X})$ and $\mathbf{VU}(\mathcal{X}) = \neg\mathbf{U}(\mathcal{X})$. Similary, the dual argument proves (iv). $\square$

When $\mathcal{X}$ is factor closed, $\mathcal{X} = \mathbf{U}\neg(\mathcal{X}) = \neg\mathbf{V}(\mathcal{X})$ and, when $\mathcal{X}$ is hereditary, $\mathcal{X} = \mathbf{S}\neg(\mathcal{X}) = \neg\mathbf{T}(\mathcal{X})$. The smallest radical class to contain any factor closed class $\mathcal{X}$ is $\mathbf{US}(\mathcal{X})$ and the smallest semisimple class to contain any hereditary $\mathcal{X}$ is $\mathbf{SU}(\mathcal{X})$ (Theorem 2.4). The inclusion of $\mathbf{V}$ and $\mathbf{T}$ allows removal of the requirement for existing conditions on $\mathcal{X}$ and provides an operator description of the smallest radical class and the smallest semisimple class to contain any $\mathcal{X}$, as our next result shows.

Theorem 3.2. *For all $\mathcal{X} \subseteq \mathcal{A}$, the lower radical class $\mathcal{L}(\mathcal{X}) = \mathbf{US}\neg\mathbf{V}(\mathcal{X}) = \mathbf{USUV}(\mathcal{X})$. Dually, the semisimple closure of $\mathcal{X}$, $\overline{\mathcal{X}} = \mathbf{SU}\neg\mathbf{T}(\mathcal{X}) = \mathbf{SUST}(\mathcal{X})$.*

Proof. For all subclasses $\mathcal{X} \subseteq \mathcal{A}$, any radical class which contains $\mathcal{X}$ will also contain the factor closure of $\mathcal{X}$, that is $\neg\mathbf{V}(\mathcal{X})$. Since $\neg\mathbf{V}(\mathcal{X})$ is factor closed, Theorem 2.4(i) shows $\mathbf{US}(\neg\mathbf{V}(\mathcal{X}))$ is the smallest radical class to contain $\neg\mathbf{V}(\mathcal{X})$, and hence it is the smallest radical class to contain $\mathcal{X}$. By Proposition 3.1(i), $\mathcal{L}(\mathcal{X}) = \mathbf{US}\neg\mathbf{V}(\mathcal{X}) = \mathbf{USUV}(\mathcal{X})$. The dual argument using Theorem 2.4(ii) proves the second part. $\square$

This allows us to constructively describe the lower radical class for any class $\mathcal{X}$ as the class for which every nonzero factor of a P-algebra has a nonzero accessible which has a nonzero preimage in $\mathcal{X}$ and its dual, the semsimple closure for any class $\mathcal{X}$, as the class for which every nonzero accessible of a P-algebra has a nonzero factor which has a nonzero cover in $\mathcal{X}$.

Corollary 3.3. *For all subclasses $\mathcal{X} \subseteq \mathcal{A}$ and $A \in \mathcal{X}$, A is $\mathbf{U}\neg\mathbf{T}(\mathcal{X})$-semisimple.*

Proof. That $\mathbf{U}\neg\mathbf{T}(\mathcal{X}) = \mathbf{UST}(\mathcal{X})$ follows from Proposition 3.1 *(iii)* so $\mathbf{U}\neg\mathbf{T}(\mathcal{X})$ is a radical class for all subclasses $\mathcal{X}$ in $\mathcal{A}$. Then $\mathbf{S}(\mathbf{U}\neg\mathbf{T}(\mathcal{X}))$ is the semisimple class of $\mathbf{U}\neg\mathbf{T}(\mathcal{X})$ and Theorem 3.2 shows it is the smallest semisimple class to contain $\mathcal{X}$. Therefore $\mathcal{X} \subseteq \mathbf{S}(\mathbf{U}\neg\mathbf{T}(\mathcal{X}))$ and hence for $A \in \mathcal{X}$, A is $\mathbf{U}\neg\mathbf{T}(\mathcal{X})$-semisimple. □

3.2 Elementary properties for operators V and T

The following theorem outlines six fundamental relationships between classes generated by the operators, while highlighting the duality between $\mathbf{U}$ and $\mathbf{S}$, and $\mathbf{V}$ and $\mathbf{T}$.

Theorem 3.4. *For all subclasses $\mathcal{X} \subseteq \mathcal{A}$,*

(i) $\mathbf{UVU}(\mathcal{X}) = \mathbf{U}(\mathcal{X})$ *and* $\mathbf{STS}(\mathcal{X}) = \mathbf{S}(\mathcal{X})$,

(ii) $\mathbf{VUV}(\mathcal{X}) = \mathbf{V}(\mathcal{X})$ *and* $\mathbf{TST}(\mathcal{X}) = \mathbf{T}(\mathcal{X})$, *and*

(iii) $\mathbf{V}\neg\mathbf{V}(\mathcal{X}) = \mathbf{V}(\mathcal{X})$ *and* $\mathbf{T}\neg\mathbf{T}(\mathcal{X}) = \mathbf{T}(\mathcal{X})$.

Proof. The following arguments make extensive use of Proposition 3.1.
(i): Since $\mathbf{U}(\mathbf{VU}(\mathcal{X})) = \mathbf{U}(\neg\mathbf{U}(\mathcal{X}))$, it follows from Theorem 2.2*(iii)* that $\mathbf{UVU}(\mathcal{X}) = \mathbf{U}(\mathcal{X})$. The dual argument using shows $\mathbf{S}(\mathbf{TS}(\mathcal{X})) = \mathbf{S}(\neg\mathbf{S}(\mathcal{X})) = \mathbf{S}(\mathcal{X})$, proving *(i)*.
(ii): As $\mathbf{VUV}(\mathcal{X}) = \neg\mathbf{UV}(\mathcal{X}) = \neg\neg\mathbf{V}(\mathcal{X}) = \mathbf{V}(\mathcal{X})$, this proves the first part. Dually, $\mathbf{TST}(\mathcal{X}) = \neg\mathbf{ST}(\mathcal{X}) = \neg\neg\mathbf{T}(\mathcal{X}) = \mathbf{T}(\mathcal{X})$, which proves *(ii)*.
(iii): Now, $\mathbf{V}\neg\mathbf{V}(\mathcal{X}) = \mathbf{VUV}(\mathcal{X})$ and $\mathbf{T}\neg\mathbf{T}(\mathcal{X}) = \mathbf{TST}(\mathcal{X})$ so it follows from *(ii)* that $\mathbf{V}\neg\mathbf{V}(\mathcal{X}) = \mathbf{V}(\mathcal{X})$ and $\mathbf{T}\neg\mathbf{T}(\mathcal{X}) = \mathbf{T}(\mathcal{X})$, showing *(iii)*. □

From Theorem 2.2*(iv)* and *(v)* we know the operators $\mathbf{UU}, \mathbf{US}, \mathbf{SS}$ and $\mathbf{SU}$ are all idempotent. Our next result shows we now have six more idempotent operators.

Theorem 3.5. *For all subclasses $\mathcal{X} \subseteq \mathcal{A}$,*

(i) $\mathbf{UVUV}(\mathcal{X}) = \mathbf{UV}(\mathcal{X})$*,*

(ii) $\mathbf{STST}(\mathcal{X}) = \mathbf{ST}(\mathcal{X})$*,*

(iii) $\mathbf{VUVU}(\mathcal{X}) = \mathbf{VU}(\mathcal{X})$*,*

(iv) $\mathbf{TSTS}(\mathcal{X}) = \mathbf{TS}(\mathcal{X})$*,*

(v) $\mathbf{VVVV}(\mathcal{X}) = \mathbf{VV}(\mathcal{X})$*, and*

(vi) $\mathbf{TTTT}(\mathcal{X}) = \mathbf{TT}(\mathcal{X})$*.*

Proof. (i) - (iv): It follows from Theorem 3.4 that $\mathbf{U}(\mathbf{VUV}(\mathcal{X})) = \mathbf{U}(\mathbf{V}(\mathcal{X}))$*(i)*, $\mathbf{S}(\mathbf{TST}(\mathcal{X})) = \mathbf{S}(\mathbf{T}(\mathcal{X}))$ *(ii)*, $\mathbf{V}(\mathbf{UVU}(\mathcal{X})) = \mathbf{VU}(\mathcal{X})$ *(iii)* and $\mathbf{T}(\mathbf{STS}(\mathcal{X})) = \mathbf{TS}(\mathcal{X})$ *(iv)*. *(v) and (vi)*: For all subclasses $\mathcal{X} \subseteq \mathcal{A}$, $\mathbf{VV}(\mathcal{X}) \subseteq \neg\mathbf{V}(\mathcal{X})$, since $\neg\mathbf{V}(\mathcal{X}) \supseteq \mathcal{X}$ for all $\mathcal{X}$ and so $\neg\mathbf{VV}(\mathcal{X}) \supseteq \mathbf{V}(\mathcal{X})$, whence $\mathbf{VV}(\mathcal{X}) \subseteq \neg\mathbf{V}(\mathcal{X})$. It follows that $\mathbf{VV}(\mathbf{VV}(\mathcal{X})) \subseteq \mathbf{VV}(\neg\mathbf{V}(\mathcal{X})) = \mathbf{V}(\mathbf{V}\neg\mathbf{V}(\mathcal{X}))$ which, by Theorem 3.4*(iii)*, is $\mathbf{VV}(\mathcal{X})$. Conversely, $\mathbf{V}\neg(\mathcal{X}) \subseteq \mathbf{VV}(\mathcal{X})$, since $\mathbf{V}\neg(\mathcal{X}) \subseteq \mathcal{X}$ for all $\mathcal{X}$, and so $\mathbf{VV}\neg(\mathcal{X}) \supseteq \mathbf{V}(\mathcal{X})$, whence $\mathbf{VV}(\mathcal{X}) = \mathbf{VV}\neg\neg(\mathcal{X}) \supseteq \mathbf{V}\neg(\mathcal{X})$ we have $\mathbf{V}(\mathbf{V}(\mathcal{X})) = \mathbf{V}\neg\mathbf{V}(\mathbf{V}(\mathcal{X})) = \mathbf{V}\neg(\mathbf{VV}(\mathcal{X})) \subseteq \mathbf{VV}(\mathbf{VV}(\mathcal{X}))$ so both classes are equal. Dualising shows *(vi)*. □

Proposition 3.6. *For all subclasses $\mathcal{X} \subseteq \mathcal{A}$, the largest factor closed class contained in $\mathbf{T}(\mathcal{X})$ is a radical class and the largest hereditary class contained in $\mathbf{V}(\mathcal{X})$ is a semisimple class.*

Proof. The largest factor closed subclass of $\mathbf{T}(\mathcal{X})$ is $\mathbf{U}\neg\mathbf{T}(\mathcal{X})$ which, by Proposition 3.1*(iii)*, is $\mathbf{UST}(\mathcal{X})$, a radical class by Remark 2. The dual argument using Proposition 3.1*(i)* proves the second part. □

It follows that

Corollary 3.7. *For all subclasses $\mathcal{X} \subseteq \mathcal{A}$, the class $\mathbf{V}(\mathcal{X})$ is hereditary if and only if $\mathbf{V}(\mathcal{X})$ is a semisimple class. Dually, $\mathbf{T}(\mathcal{X})$ is factor closed if and only if $\mathbf{T}(\mathcal{X})$ is a radical class.*

Proposition 3.8. *For all subclasses $\mathcal{X} \subseteq \mathcal{A}$, $\mathbf{V}(\mathcal{X}) = \mathcal{A}$ and $\mathbf{T}(\mathcal{X}) = \mathcal{A}$ if and only if $\mathcal{X} = \{0\}$.*

Proof. If $\mathcal{X} = \{0\}$, it is clear that $\mathbf{V}(\mathcal{X}) = \mathcal{A}$ and $\mathbf{T}(\mathcal{X}) = \mathcal{A}$. Conversely, suppose $\mathbf{V}(\mathcal{X}) = \mathcal{A}$ and $\mathbf{T}(\mathcal{X}) = \mathcal{A}$ with $\mathcal{X} \neq \{0\}$. Then there exists nonzero $A \in \mathcal{X}$, so $A \notin \mathbf{V}(\mathcal{X})$ and $A \notin \mathbf{T}(\mathcal{X})$, a contradiction. Therefore $\mathcal{X}$ must be the class $\{0\}$, which completes the proof. □

The class $\mathbf{U}(\mathcal{X})$ can be a nontrivial radical class. For example, when $\mathcal{X}$ is any hereditary class, $\mathcal{X} = \mathbf{S}\neg\mathcal{X}$ and so $\mathbf{U}(\mathcal{X}) = \mathbf{US}\neg(\mathcal{X})$ which is a radical class by Remark 2. Dually, $\mathbf{S}(\mathcal{X})$ can be a semisimple class. When $\mathcal{X}$ is any factor closed class, $\mathcal{X} = \mathbf{U}\neg(\mathcal{X})$ and $\mathbf{S}(\mathcal{X}) = \mathbf{SU}\neg(\mathcal{X})$ which is a semisimple class by the same Remark. For any class $\mathcal{X}$, the classes $\mathbf{V}(\mathcal{X})$ and $\mathbf{T}(\mathcal{X})$ are only ever trivially radical or semismple classes. These classes do however share some of the properties of radical and semisimple classes as demonstrated in the following result.

Proposition 3.9. *For all subclasses $\mathcal{X} \subseteq \mathcal{A}$, $\mathbf{V}(\mathcal{X})$ and $\mathbf{T}(\mathcal{X})$ are closed under extensions.*

Proof. Let $A \in \mathbf{V}(\mathcal{X})$ where A is an ideal of $B \in \mathcal{A}$. Suppose $B/A \in \mathbf{V}(\mathcal{X})$. Then, since $\mathbf{V}(\mathcal{X})$ is closed under preimages, $B \in \mathbf{V}(\mathcal{X})$ showing $\mathbf{V}(\mathcal{X})$ is closed under extensions. Dually, suppose $A \in \mathbf{T}(\mathcal{X})$ where A is an ideal of $B \in \mathcal{A}$, and let $B/A \in \mathbf{T}(\mathcal{X})$. Then, since $\mathbf{T}(\mathcal{X})$ is closed under covers and $A \in \mathbf{T}(\mathcal{X})$, it follows that $B \in \mathbf{T}(\mathcal{X})$ showing $\mathbf{T}(\mathcal{X})$ is also closed under extensions. □

Proposition 3.10. *For all subclasses $\mathcal{X} \subseteq \mathcal{A}$, the largest hereditary subclass of $\mathbf{V}(\mathcal{X})$ is the semisimple class of the smallest radical class containing $\mathcal{X}$.*

Proof. By Theorem 3.2, the smallest radical class to contain $\mathcal{X}$ is $\mathbf{US}\neg\mathbf{V}(\mathcal{X})$. Its semisimple class is $\mathbf{S}(\mathbf{US}\neg\mathbf{V}(\mathcal{X}))$ which, from Proposition 3.1*(i)*, is $\mathbf{SUSUV}(\mathcal{X})$. Theorem 2.2*(v)* shows $\mathbf{SUSU}(\mathbf{V}(\mathcal{X})) = \mathbf{SU}(\mathbf{V}(\mathcal{X}))$ and, again by Proposition 3.1*(i)*, we know $\mathbf{SUV}(\mathcal{X}) = \mathbf{S}\neg\mathbf{V}(\mathcal{X})$, which is the largest hereditary subclass of $\mathbf{V}(\mathcal{X})$. □

The proof of Proposition 3.10 can be dualised for the operators $\mathbf{U}$ with $\mathbf{S}$, and $\mathbf{V}$ with $\mathbf{T}$ and noting Remark 3 to show

Proposition 3.11. *For all subclasses $\mathcal{X} \subseteq \mathcal{A}$, the largest factor closed subclass of $\mathbf{T}(\mathcal{X})$ is the upper radical class of the smallest semisimple class containing $\mathcal{X}$.*

Hence, from Propositions 3.10 and 3.11, we can see that $\mathbf{S}\neg\mathbf{V}(\mathcal{X})$ is the largest class to be free of $\mathcal{X}$-accessibles and $\mathbf{U}\neg\mathbf{T}(\mathcal{X})$ is the largest class to be free of $\mathcal{X}$-factors.

Proposition 3.12. *For all subclasses $\mathcal{X} \subseteq \mathcal{A}$, $\neg\mathbf{S}\neg\mathbf{U}(\mathcal{X})$ is the smallest cover-closed, preimage-closed class containing $\mathcal{X}$.*

Proof. It is clear, for all subclasses $\mathcal{X} \subseteq \mathcal{A}$, that $\mathcal{X} \subseteq \neg\mathbf{S}\neg\mathbf{U}(\mathcal{X})$. By Proposition 3.1*(iv)*, $\neg\mathbf{S}\neg\mathbf{U}(\mathcal{X}) = \mathbf{TS}\neg\mathbf{U}(\mathcal{X})$ which is cover-closed. Suppose $A \in \neg\mathbf{S}\neg\mathbf{U}(\mathcal{X})$ and let A^* be a preimage of A such that $A^* \in \mathbf{S}\neg\mathbf{U}(\mathcal{X})$. By Theorem 2.3, $\mathbf{S}\neg\mathbf{U}(\mathcal{X})$ is factor closed so it follows that $A \in \mathbf{S}\neg\mathbf{U}(\mathcal{X})$, a contradiction. Hence $\neg\mathbf{S}\neg\mathbf{U}(\mathcal{X})$ is also preimage-closed. If $\mathcal{X} \subseteq \mathcal{Y}$ such that $\mathcal{Y} \subseteq \neg\mathbf{S}\neg\mathbf{U}(\mathcal{X})$ and $\mathcal{Y}$ is cover-closed and preimage-closed, then Proposition 3.1*(ii)* and *(iv)* shows $\mathcal{Y} = \neg\mathbf{U}(\mathcal{Y}) = \neg\mathbf{S}(\mathcal{Y})$. Therefore $\neg\mathbf{S}\neg\mathbf{U}(\mathcal{X}) \subseteq \neg\mathbf{S}\neg\mathbf{U}(\mathcal{Y}) = \neg\mathbf{S}(\mathcal{Y}) = \mathcal{Y} \subseteq \neg\mathbf{S}\neg\mathbf{U}(\mathcal{X})$, so all classes are equal and $\neg\mathbf{S}\neg\mathbf{U}(\mathcal{X})$ is the smallest cover-closed, preimage-closed class to contain $\mathcal{X}$. □

Proposition 3.13. *For all subclasses $\mathcal{X} \subseteq \mathcal{A}$, $\mathbf{V}\neg\mathbf{T}(\mathcal{X}) \subseteq \mathbf{T}\neg\mathbf{V}(\mathcal{X})$.*

Proof. If nonzero $A \in \neg\mathbf{T}\neg\mathbf{V}(\mathcal{X})$ then A has a cover B such that B has a preimage C in $\mathcal{X}$. Since A is an accessible of B, A is the accessible of a factor of C. By Theorem 2.1, A must be the factor of an accessible, say D, of C which means A has a preimage D with a cover C in $\mathcal{X}$, showing $A \in \neg\mathbf{V}\neg\mathbf{T}(\mathcal{X})$. Therefore $\neg\mathbf{T}\neg\mathbf{V}(\mathcal{X}) \subseteq \neg\mathbf{V}\neg\mathbf{T}(\mathcal{X})$ and taking complements gives $\mathbf{V}\neg\mathbf{T}(\mathcal{X}) \subseteq \mathbf{T}\neg\mathbf{V}(\mathcal{X})$. □

In general we cannot dualise Proposition 3.13 since factors of accessibles may not necessarily be accessibles of factors (see the comment after Lemma 4.3 in [11] for an example of when this is not the case). Lemma 4.4 of [11] shows that when the property of being an ideal is transitive, every factor of an accessible of $A \in \mathcal{A}$ is an accessible of a factor of A so when ideals are transitive the two classes in Proposition 3.13 will be equal.

Proposition 3.14. *For all subclasses $\mathcal{X} \subseteq \mathcal{A}$, $\mathbf{T}\neg\mathbf{V}\neg\mathbf{T}(\mathcal{X}) = \mathbf{V}\neg\mathbf{T}(\mathcal{X})$.*

Proof. By Theorem 3.4*(iii)*, $\mathbf{V}\neg\mathbf{T}(\mathcal{X}) = \mathbf{V}\neg(\mathbf{T}\neg\mathbf{T}(\mathcal{X}))$ and, from Proposition 3.13, it follows that $(\mathbf{V}\neg\mathbf{T})\neg\mathbf{T}(\mathcal{X}) \subseteq (\mathbf{T}\neg\mathbf{V})\neg\mathbf{T}(\mathcal{X})$. Since $\mathbf{T}\neg\mathbf{V}\neg\mathbf{T}(\mathcal{X}) \subseteq \neg\neg\mathbf{V}\neg\mathbf{T}(\mathcal{X}) = \mathbf{V}\neg\mathbf{T}(\mathcal{X})$, we see that all classes are equal. □

Proposition 3.15. *For all subclasses $\mathcal{X} \subseteq \mathcal{A}$, $\neg\mathbf{V}\neg\mathbf{T}(\mathcal{X})$ is the smallest hereditary, factor closed class containing $\mathcal{X}$.*

Proof. By Proposition 3.14, $\neg\mathbf{V}\neg\mathbf{T}(\mathcal{X}) = \neg\mathbf{T}\neg\mathbf{V}\neg\mathbf{T}(\mathcal{X})$ which is hereditary by Proposition 3.1*(iii)*. Now, suppose $\mathcal{X} \subseteq \mathcal{Y}$ and $\mathcal{Y} \subseteq \neg\mathbf{V}\neg\mathbf{T}(\mathcal{X})$ such that $\mathcal{Y}$ is hereditary and factor closed. Then $\neg\mathbf{V}\neg\mathbf{T}(\mathcal{X}) \subseteq \neg\mathbf{V}\neg\mathbf{T}(\mathcal{Y})$ and, by Proposition 3.1*(i)* and *(iii)*, $\neg\mathbf{V}\neg\mathbf{T}(\mathcal{Y}) = \neg\mathbf{V}(\mathcal{Y}) = \mathcal{Y} \subseteq \neg\mathbf{V}\neg\mathbf{T}(\mathcal{X})$. Therefore all classes are equal and $\neg\mathbf{V}\neg\mathbf{T}(\mathcal{X})$ is the smallest hereditary, factor closed class containing $\mathcal{X}$. □

Proposition 3.16. *For all subclasses $\mathcal{X} \subseteq \mathcal{A}$, $\neg\mathbf{V}\neg\mathbf{T}(\mathcal{X}) = \mathbf{UV}\neg\mathbf{T}(\mathcal{X}) = \mathbf{UVST}(\mathcal{X}) = \mathbf{SV}\neg\mathbf{T}(\mathcal{X}) = \mathbf{SVST}(\mathcal{X})$.*

Proof. By Proposition 3.1*(i)* and *(iii)*, for all $\mathcal{X} \subseteq \mathcal{A}$ we have $\neg\mathbf{V}\neg\mathbf{T}(\mathcal{X}) = \mathbf{UV}\neg\mathbf{T}(\mathcal{X}) = \mathbf{UVST}(\mathcal{X})$. The class $\neg\mathbf{V}\neg\mathbf{T}(\mathcal{X})$ is hereditary by Proposition 3.15, so $\neg\mathbf{V}\neg\mathbf{T}(\mathcal{X}) = \mathbf{S}\neg(\neg\mathbf{V}\neg\mathbf{T}(\mathcal{X})) = \mathbf{SV}\neg\mathbf{T}(\mathcal{X})$ and finally, again by Proposition 3.1*(iii)*, $\mathbf{SV}\neg\mathbf{T}(\mathcal{X}) = \mathbf{SVST}(\mathcal{X})$. □

Proposition 3.17. *If $\mathcal{X} \subseteq \mathcal{A}$ and $\mathcal{X}$ is factor closed, then $\mathbf{SU}(\mathcal{X}) \subseteq \mathbf{SUUT}(\mathcal{X})$. If $\mathcal{X}$ is hereditary then $\mathbf{US}(\mathcal{X}) \subseteq \mathbf{USSV}(\mathcal{X})$.*

Proof. Suppose a subclass $\mathcal{X} \subseteq \mathcal{A}$ is factor closed and let $A \in \mathbf{SU}(\mathcal{X})$. Then every nonzero accessible B of A has a nonzero factor C in $\mathcal{X}$. Since $\mathcal{X}$ is factor closed, $\mathcal{X} \subseteq \mathbf{UT}(\mathcal{X})$ so $C \in \mathbf{UT}(\mathcal{X})$, showing B has a nonzero factor in $\mathbf{UT}(\mathcal{X})$. Hence every nonzero accessible of A has a nonzero factor in $\mathbf{UT}(\mathcal{X})$ and so $A \in \mathbf{SUUT}(\mathcal{X})$. The dual argument proves $\mathbf{US}(\mathcal{X}) \subseteq \mathbf{USSV}(\mathcal{X})$ when $\mathcal{X}$ is hereditary. □

It is apparent that the operators $\mathbf{U}, \mathbf{S}, \mathbf{T}$ and $\mathbf{V}$ are special cases of a more general operator type which is the subject of further investigation.

3.3 The ideal-determined case

For the remainder we assume $\mathcal{A}$ is an ideal-determined variety.

Proposition 3.18. *For all subclasses $\mathcal{X} \subseteq \mathcal{A}$, $\mathbf{V}(\mathcal{X})$ is either factor closed or hereditary if and only if $\mathbf{V}(\mathcal{X}) = \{0\}$ or $\mathbf{V}(\mathcal{X}) = \mathcal{A}$.*

Proof. Let $\mathcal{X}$ be a subclass of $\mathcal{A}$. If $\mathbf{V}(\mathcal{X}) = \{0\}$ or $\mathbf{V}(\mathcal{X}) = \mathcal{A}$ then it is clear that $\mathbf{V}(\mathcal{X})$ is factor closed or hereditary (in fact both). If $\mathbf{V}(\mathcal{X}) \neq \{0\}$ and $\mathbf{V}(\mathcal{X}) \neq \mathcal{A}$, then $\neg\mathbf{V}(\mathcal{X}) \neq \{0\}$. Let A be a nonzero element of $\mathbf{V}(\mathcal{X})$ and B be a nonzero element of $\neg\mathbf{V}(\mathcal{X})$ and consider $A \oplus B$. $A \oplus B$ has factor A and hence $A \oplus B \in \mathbf{V}(\mathcal{X})$ since $A \oplus B$ is a preimage of A and $\mathbf{V}(\mathcal{X})$ is closed under preimages. But $A \oplus B$ has B both as a factor and an accessible, therefore $\mathbf{V}(\mathcal{X})$ is neither factor closed nor hereditary. $\square$

Dualising this argument shows the following.

Proposition 3.19. *For all subclasses $\mathcal{X} \subseteq \mathcal{A}$, $\mathbf{T}(\mathcal{X})$ is either factor closed or hereditary if and only if $\mathbf{T}(\mathcal{X}) = \{0\}$ or $\mathbf{T}(\mathcal{X}) = \mathcal{A}$.*

As a consequence of Propositions 3.8 and 3.18 we have the following.

Corollary 3.20. *For all subclasses $\mathcal{X} \subseteq \mathcal{A}$, the following are equivalent:*

(i) $\mathbf{V}(\mathcal{X})$ *is hereditary,*

(ii) $\mathbf{V}(\mathcal{X})$ *is factor closed,*

(iii) $\mathbf{V}(\mathcal{X})$ *is a radical class,*

(iv) $\mathbf{V}(\mathcal{X})$ *is a semisimple class,*

(v) $\mathbf{V}(\mathcal{X})$ *is a radical-semisimple class, and*

(vi) $\mathbf{V}(\mathcal{X}) = \{0\}$ *or* $\mathbf{V}(\mathcal{X}) = \mathcal{A}$.

Proof. *(i) implies (ii) - (vi)*: Given *(i)* that $\mathbf{V}(\mathcal{X})$ is hereditary then, by Proposition 3.18, $\mathbf{V}(\mathcal{X}) = \mathcal{A}$ or $\mathbf{V}(\mathcal{X}) = \{0\}$ *(vi)*. It is straightforward to see that *(ii)* - *(vi)* follow.
(ii) implies (i): If $\mathbf{V}(\mathcal{X})$ is factor closed *(ii)* then, by the same argument, $\mathbf{V}(\mathcal{X}) = \{0\}$ or $\mathbf{V}(\mathcal{X}) = \mathcal{A}$ and hence $\mathbf{V}(\mathcal{X})$ is hereditary *(i)*.
It is clear that *(iii)* implies *(ii)*, and that *(iv), (v)* and *(vi)* imply *(i)*, which completes the proof. □

The dual result, using Propositions 3.8 and 3.19, also holds true. That is,

Corollary 3.21. *For all subclasses $\mathcal{X} \subseteq \mathcal{A}$, the following are equivalent:*

(i) $\mathbf{T}(\mathcal{X})$ *is hereditary,*

(ii) $\mathbf{T}(\mathcal{X})$ *is factor closed,*

(iii) $\mathbf{T}(\mathcal{X})$ *is a radical class,*

(iv) $\mathbf{T}(\mathcal{X})$ *is a semisimple class,*

(v) $\mathbf{T}(\mathcal{X})$ *is a radical-semisimple class, and*

(vi) $\mathbf{T}(\mathcal{X}) = \{0\}$ *or* $\mathbf{T}(\mathcal{X}) = \mathcal{A}$.

In the ideal-determined case, the operators $\mathbf{T}$ and $\mathbf{V}$ tend to generate the trivial class containing only the zero algebra.

Proposition 3.22. *If* $\mathbf{T}(\mathcal{X})$ *is a proper nonzero subclass of* $\mathcal{A}$*, then* $\mathbf{TT}(\mathcal{X}) = \{0\}$*. Dually, if* $\mathbf{V}(\mathcal{X})$ *is a proper nonzero subclass of* $\mathcal{A}$*, then* $\mathbf{VV}(\mathcal{X}) = \{0\}$.

Proof. Let $\mathcal{X}$ be such that $\mathbf{T}(\mathcal{X})$ is a proper nonzero subclass of $\mathcal{A}$. Then $\neg\mathbf{T}(\mathcal{X}) \neq \{0\}$ and consider nonzero A and nonzero B such that $A \in \mathbf{T}(\mathcal{X})$ and $B \in \neg\mathbf{T}(\mathcal{X})$. Then $A \oplus B$ is a nonzero cover of A, hence $A \oplus B \in \mathbf{T}(\mathcal{X})$, but $A \oplus B$ is also a nonzero cover of B, so $A \in \neg\mathbf{T}(\mathbf{T}(\mathcal{X}))$. Since A and B were arbitrary it follows that both $\neg\mathbf{T}(\mathcal{X}) \subseteq \neg\mathbf{TT}(\mathcal{X})$ and $\mathbf{T}(\mathcal{X}) \subseteq \neg\mathbf{TT}(\mathcal{X})$. Therefore $\neg\mathbf{TT}(\mathcal{X}) = \mathcal{A}$ and so $\mathbf{TT}(\mathcal{X}) = \{0\}$. The dual argument proves the second part. □

Proposition 3.23. *For all nonzero subclasses $\mathcal{X} \subseteq \mathcal{A}$,*

$$\mathbf{T}\neg\mathbf{S}(\mathcal{X}) = \mathbf{V}\neg\mathbf{U}(\mathcal{X}) = \mathbf{V}\neg\mathbf{S}(\mathcal{X}) = \mathbf{T}\neg\mathbf{U}(\mathcal{X}) = \{0\}.$$

Proof. Let $\mathcal{X}$ be nonzero and suppose there exists nonzero $A \in \mathbf{T}\neg\mathbf{S}(\mathcal{X})$. Then A has no nonzero cover in $\neg\mathbf{S}(\mathcal{X})$. Since $\mathcal{X}$ is nonzero, there exists nonzero $B \in \mathcal{X}$ so $A \oplus B \in \neg\mathbf{S}(\mathcal{X})$. As $A \oplus B$ is a cover of A, this contradicts the fact $A \in \mathbf{T}\neg\mathbf{S}(\mathcal{X})$. Hence $A = 0$ and $\mathbf{T}\neg\mathbf{S}(\mathcal{X}) = \{0\}$. The dual argument proves $\mathbf{V}\neg\mathbf{U}(\mathcal{X}) = \{0\}$.
Now, let A and B be nonzero with $B \in \mathcal{X}$. Then $A \oplus B \in \neg\mathbf{S}(\mathcal{X})$ since $B \triangleleft A \oplus B$. Since A is a factor of $A \oplus B$, it follows that A has a nonzero preimage in $\neg\mathbf{S}(\mathcal{X})$ so $A \in \neg\mathbf{V}\neg\mathbf{S}(\mathcal{X})$. As A was an arbitrary nonzero element in $\mathcal{A}$, we have that $\neg\mathbf{V}\neg\mathbf{S}(\mathcal{X}) = \mathcal{A}$ and $\mathbf{V}\neg\mathbf{S}(\mathcal{X}) = \{0\}$. The dual argument shows $\mathbf{T}\neg\mathbf{U}(\mathcal{X}) = \{0\}$ and hence $\mathbf{T}\neg\mathbf{S}(\mathcal{X}) = \mathbf{V}\neg\mathbf{U}(\mathcal{X}) = \mathbf{V}\neg\mathbf{S}(\mathcal{X}) = \mathbf{T}\neg\mathbf{U}(\mathcal{X}) = \{0\}$. □

Proposition 3.24. *For all subclasses $\mathcal{X} \subseteq \mathcal{A}$, the class $\mathbf{TV}(\mathcal{X}) = \{0\}$ or $\mathbf{TV}(\mathcal{X}) = \mathcal{A}$.*

Proof. Let $A \in \mathcal{A}$ such that A is nonzero and $A \in \mathbf{TV}(\mathcal{X})$. Now, assume $\neg\mathbf{TV}(\mathcal{X})$ is nonzero and let nonzero $B \in \neg\mathbf{TV}(\mathcal{X})$. Then there exists B^* such that B^* is a nonzero cover of B and $B^* \in \mathbf{V}(\mathcal{X})$. Consider $A \oplus B^*$, a cover of A. Then $A \oplus B^* \in \mathbf{TV}(\mathcal{X})$ so $A \oplus B^*$ has a nonzero preimage in $\mathcal{X}$ and B^* has a nonzero preimage in $\mathcal{X}$ since B^* is a factor of $A \oplus B^*$. Therefore $B^* \notin \mathbf{V}(\mathcal{X})$, a contradiction, showing $\neg\mathbf{TV}(\mathcal{X}) = \{0\}$ and $\mathbf{TV}(\mathcal{X}) = \mathcal{A}$. □

Using the dual argument,

Proposition 3.25. *For all subclasses $\mathcal{X} \subseteq \mathcal{A}$, the class $\mathbf{VT}(\mathcal{X}) = \{0\}$ or $\mathbf{VT}(\mathcal{X}) = \mathcal{A}$.*

Hence, for all subclasses $\mathcal{X} \subseteq \mathcal{A}$, the classes $\mathbf{TV}(\mathcal{X})$ and $\mathbf{VT}(\mathcal{X})$ are trivially radical-semisimple classes.

Corollary 3.26. *For all subclasses $\mathcal{X} \subseteq \mathcal{A}$, $\neg\mathbf{TV}(\mathcal{X}) = \mathbf{STV}(\mathcal{X}) = \mathbf{UTV}(\mathcal{X})$ and $\neg\mathbf{VT}(\mathcal{X}) = \mathbf{SVT}(\mathcal{X}) = \mathbf{UVT}(\mathcal{X})$.*

Proof. From Proposition 3.24, the class $\mathbf{TV}(\mathcal{X}) = \{0\}$ or $\mathcal{A}$ for all subclasses $\mathcal{X} \subseteq \mathcal{A}$. If $\mathbf{TV}(\mathcal{X}) = \{0\}$ then $\neg(\mathbf{TV}(\mathcal{X})) = \mathbf{S}(\mathbf{TV}(\mathcal{X})) = \mathbf{U}(\mathbf{TV}(\mathcal{X})) = \mathcal{A}$. Similarly, if $\mathbf{TV}(\mathcal{X}) = \mathcal{A}$ then $\neg(\mathbf{TV}(\mathcal{X})) = \mathbf{S}(\mathbf{TV}(\mathcal{X})) = \mathbf{U}(\mathbf{TV}(\mathcal{X})) = \{0\}$. Interchanging $\mathbf{V}$ and $\mathbf{T}$ and using Proposition 3.25 proves the second part. □

The development of radical theory has centred on the element properties of the class objects, but it is clear there is much to be gleaned about its nature using this class operator approach.

References

[1] P. Agliano and A. Ursini, Ideals and other generalizations of congruence classes, *J. Austral. Math. Soc.* (Ser. A) **53**,103-115 (1992).

[2] S. A. Amitsur, A general theory of radicals I, Radicals in complete lattices, *Amer. J. Math.*, **74**, 774-786 (1952).

[3] B. J. Gardner, Kurosh-Amitsur radical theory for groups, *An. Stiint. Univ. Ovidius. Constanta.*, **18(2)**, 73–90 (2010).

[4] B. J. Gardner and R. Wiegandt, *Radical theory of rings*, Marcel Dekker, (2004).

[5] H. P. Gumm and A. Ursini, Ideals in universal algebras, *Alg. Univ.* **19**, 45–54 (1984).

[6] N. Jacobson, The radical and semi-simplicity for arbitrary rings, *Amer. J. Math.*, 300-320 (1945).

[7] G. Köthe, Die struktur der ringe, deren restklassenring nach dem radikal vollständig reduzibel ist, *Math. Zeitschr*, **32**, 161–186 (1930).

[8] A. G. Kurosh, Radicals of rings and algebras, *Mat. Sb.*, **75**, 13–26 (1953).

[9] A. G. Kurosh, Radicals in the theory of groups, *Sibirsk. Mat. Zh.*, **3**, 912–931 (1962).

[10] W. G. Leavitt and J. F. Watters, On smallest radical and semisimple classes, *Glasgow Math. J.*, **12**, 98–104 (1971).

[11] N. R. McConnell, R. G. McDougall and T. Stokes, On base radical and semisimple classes defined by class operators, *Acta Math. Hungar.*, **138(4)**, 307–328 (2013).

[12] N. R. McConnell and T. E. Stokes, Radicals of 0-regular algebras, *Acta Math. Hungar.*, **113** (2006), 19–37.

[13] R. G. McDougall, A generalisation of the lower radical class, *Bull. Aust. Math. Soc.*, **59**, 139–146 (1999).

[14] R. G. McDougall, The base semisimple class, *Comm. Alg.*, **28**, 4269–4283 (2000).

[15] E. R. Puczyłowski, On general theory of radicals, *Alg. Univ.*, **30**, 53–60 (1993).

[16] L. K. Thornton, On base radical and semisimple operators for a class of finite algebras, *Beitr. Algebra Geom.*, **59(2)**, 361–374 (2018).

[17] J. H. Wedderburn, On hypercomplex numbers, *Proc. London Math. Soc.*, **(2)** 6, 77–118 (1908).

[18] R. Wiegandt, *Radical and semisimple classes of rings*, Queen's Papers in Pure and Appl. Math., Kingston, Ontario, (1974).

[19] R. Wiegandt, A condition in general radical theory and its meaning for rings, topological spaces and graphs', *Acta Math. Acad. Sci. H.*, **26**, 233–240 (1975).

N. R. McConnell, Defence Science and Technology Group, Department of Defence, Australia

R. G. McDougall, School of Science, Technology and Engineering, University of the Sunshine Coast,

T. Stokes, Department of Mathematics, University of Waikato, and

L. K. Thornton, School of Science, Technology and Engineering, University of the Sunshine Coast.