Finite-time singularity formation at a magnetic neutral line in Hall magnetohydrodynamics

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Abstract

The formation of a current sheet in a weakly collisional plasma can be modelled as a finite-time singularity solution of magnetohydrodynamic equations. We use an exact self-similar solution to confirm and generalise a previous finding that, in sharp contrast to two-dimensional solutions in standard MHD, a finite-time collapse to a current sheet can occur in Hall MHD. We derive a criterion for the finite-time singularity in terms of initial conditions, and we use an intermediate asymptotic solution for the evolution of an axial magnetic field to obtain a general expression for the singularity formation time. We illustrate the analytical results by numerical solutions.

Keywords: Finite-time singularities, Hall MHD, Magnetic reconnection
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1. Introduction

The formation of finite-time singularities in fluids and plasmas has been a subject of much recent research interest [1, 2]. The research is motivated by potential applications to laboratory and astrophysical plasmas. In particular, Hall magnetohydrodynamics (MHD) is believed to provide a model for fast magnetic reconnection in weakly collisional plasmas [3, 4], and finite-time singularity solutions of Hall MHD equations can be used to describe the formation of reconnecting current sheets [5, 6].

Singularity formation models, which identify the current sheet formation with an explosive growth of the electric current density at a magnetic neutral line, have been repeatedly used to describe the sheet formation in standard MHD [7, 8, 9]. Exact analytical self-similar MHD solutions exhibit exponential growth of the electric current density, and the exponential behaviour was confirmed by numerical simulations [9, 10]. Analytical arguments [11] also show that ideal incompressible MHD solutions near the neutral line should evolve exponentially unless a singularity is driven by an imposed pressure.

In this paper we investigate a self-similar solution for current sheet formation in Hall MHD. We consider a general set of initial conditions and derive a criterion for the formation of a finite-time singularity. The new solution reduces to an exponentially evolving MHD solution in two dimensions upon setting the Hall term to zero.

2. Formulation of the problem and self-similar solutions

The incompressible Hall MHD equations [12] in dimensionless form are given by the generalised Ohm's law

$$\mathbf{E} + \mathbf{v} \times \mathbf{B} = \eta \mathbf{J} + d_i(\mathbf{J} \times \mathbf{B} - \nabla p_e), \tag{1}$$

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the momentum equation

$$\partial_t \mathbf{v} + (\mathbf{v} \cdot \nabla)\mathbf{v} = -\nabla p + \mathbf{J} \times \mathbf{B} + \nu \nabla^2 \mathbf{v}, \tag{2}$$

the incompressibility equation

$$\nabla \cdot \mathbf{v} = 0, \tag{3}$$

and electromagnetic equations

$$\nabla \cdot \mathbf{B} = 0, \tag{4}$$

$$\mathbf{J} = \nabla \times \mathbf{B}, \tag{5}$$

$$\nabla \times \mathbf{E} = -\partial_t \mathbf{B}. \tag{6}$$

Here $\mathbf{v}$ is the plasma velocity, $\mathbf{B}$ is the magnetic field, $\mathbf{J}$ is the electric current density, $\mathbf{E}$ is the electric field, p and p_e are the total plasma pressure and electron pressure. The pressures p and p_e are assumed to be scalar [13]. The length and magnetic field are normalised by some reference values L and B_0. The velocity $\mathbf{v}$ is normalised by the Alfvén speed $v_A = B_0/\sqrt{4\pi\rho}$, and time is normalised by the Alfvén time $t_A = L/v_A$. The resistivity η and viscosity ν are assumed to be constant and are normalised by $4\pi L v_A/c^2$ and $L v_A$ respectively. The dimensionless parameter that quantifies the role of collisionless effects is the ion skin depth $d_i = c/(L\omega_{pi})$, where the ion plasma frequency is $\omega_{pi} = (4\pi n e^2/m_i)^{1/2}$. Here n is the number density, e is the ion electric charge, m_i is the ion mass, $\rho = m_i n$ is the mass density and c is the speed of light.

In what follows, we solve the Hall MHD equations via similarity reduction and show that some initial conditions lead to a finite time singularity. The Hall effect is manifested as the non-linear term $d_i \mathbf{J} \times \mathbf{B}$ in equation (1), which vanishes in standard MHD. Our self-similar solution reduces to an exponentially evolving MHD solution in the limit $d_i = 0$.

We assume a $2\frac{1}{2}$D model, in which all quantities are considered in three dimensions but there is no dependence on the z co-ordinate ($\partial_z = 0$). The incompressibility equation (3) then dictates that

$$\mathbf{v}(x, y, t) = \nabla\phi \times \hat{\mathbf{z}} + W\hat{\mathbf{z}}. \tag{7}$$

Similarly, to satisfy equation (4), we use the flux function ψ to represent the magnetic field:

$$\mathbf{B}(x, y, t) = \nabla\psi \times \hat{\mathbf{z}} + Z\hat{\mathbf{z}}. \tag{8}$$

The pressure terms do not contribute to the z-components of equations (1) and (2). To eliminate the pressure terms in the x and y components, we take the curl of those equations. Equations (1)-(6) simplify to the following system:

$$\partial_t \psi + [\psi, \phi] = \eta \nabla^2 \psi + d_i [\psi, Z], \tag{9}$$

$$\partial_t Z + [Z, \phi] = [W, \psi] + \eta \nabla^2 Z + d_i [\nabla^2 \psi, \psi], \tag{10}$$

$$\partial_t W + [W, \phi] = [Z, \psi] + \nu \nabla^2 W, \tag{11}$$

$$\partial_t (\nabla^2 \phi) + [\nabla^2 \phi, \phi] = [\nabla^2 \psi, \psi] + \nu \nabla^2 (\nabla^2 \phi), \tag{12}$$

where the Poisson bracket notation is typified by $[\psi, \phi] = \partial_x \psi \partial_y \phi - \partial_y \psi \partial_x \phi$.

We reduce the system of equations (9)-(12) to a system of ordinary differential equations that describe a hyperbolic (X-point) planar magnetic field, driven by a stagnation-point flow:

$$\psi = \alpha(t) x^2 - \beta(t) y^2 + 2\eta \int (\alpha - \beta) dt, \tag{13}$$

$$\phi = -\gamma(t) x y. \tag{14}$$

For the axial velocity W and magnetic field Z we assume

$$W = f(t) x^2 + g(t) y^2 + 2\nu \int (f + g) dt, \tag{15}$$

$$Z = h(t)xy, \tag{16}$$

where the functional form of the axial magnetic field corresponds to the well-known quadrupolar structure in Hall magnetic reconnection [13, 14]. On substituting equations (13)-(16) into the system (9)-(12) we get

$$\dot{\alpha} - 2\alpha(\gamma + d_i h) = 0, \tag{17}$$

$$\dot{\beta} + 2\beta(\gamma + d_i h) = 0, \tag{18}$$

$$\dot{f} - 2\gamma f + 2\alpha h = 0, \tag{19}$$

$$\dot{g} + 2\gamma g + 2\beta h = 0, \tag{20}$$

$$\dot{h} + 4\alpha g + 4\beta f = 0, \tag{21}$$

where the dot represents differentiation with respect to dimensionless time. We recover a similarity reduction in 2D MHD [9] by setting $f = g = h = 0$.

3. Collapse to a Current Sheet in Hall MHD

For a general set of initial conditions, $\alpha(0) = \alpha_0$, $\beta(0) = \beta_0$, $\gamma(0) = \gamma_0$, $f(0) = f_0$, $g(0) = g_0$ and $h(0) = h_0$, integration of equations (17)-(21) yields

$$\alpha\beta = \alpha_0\beta_0, \tag{22}$$

$$\alpha + d_i f = (\alpha_0 + d_i f_0)\exp(2\Gamma), \tag{23}$$

$$\beta - d_i g = (\beta_0 - d_i g_0)\exp(-2\Gamma), \tag{24}$$

$$h^2 - 4fg = h_0^2 - 4f_0 g_0, \tag{25}$$

where $\Gamma = \int_0^t \gamma(t')\,\mathrm{d}t'$. The solution of the system is known to exhibit a finite-time singularity for the following initial conditions: $\alpha_0 = \beta_0 = 1$, $f_0 = g_0 = 0$ [6]. Here we strengthen and generalise that result by considering arbitrary initial conditions.

A finite-time collapse to a current sheet occurs if a finite-time singularity is present in the solution. Specifically, if $h(t) \to \infty$ as $t \to t_s$, then it follows from equations (17), (18) and (22) that, depending on the sign of h, either $\alpha(t) \to \infty$, $\beta(t) \to 0$ or $\alpha(t) \to 0$, $\beta(t) \to \infty$ as $t \to t_s$, which constitutes the collapse of an initial magnetic X-point.

We obtain an equation for $h(t)$ by differentiating equation (21) and using equations (17)-(20) and (22)-(25). We get, after some algebra,

$$\ddot{h} - 2d_i^2 h^3 - a^2 h = 0, \tag{26}$$

where a^2 is defined as

$$a^2 = -2[4d_i(\alpha_0 g_0 - \beta_0 f_0) - 8\alpha_0\beta_0 + d_i^2 h_0^2]. \tag{27}$$

A singularity criterion can be obtained by using a mechanical analogy. Integration of equation (26) yields an analogue of energy conservation:

$$\frac{1}{2}\dot{h}^2 = -U(h), \tag{28}$$

where the quartic function

$$U(h) = -\frac{1}{2}(d_i^2 h^4 + a^2 h^2) + \frac{1}{2}(d_i^2 h_0^4 + a^2 h_0^2) - 8(\alpha_0 g_0 + \beta_0 f_0)^2 \tag{29}$$

is analogous to potential energy. Hence we can interpret the solution of equation (26) as the position of a particle in this potential.

The particle motion is bounded, and thus $h(t)$ remains finite, if the following three conditions are satisfied. First, $U(h)$ has a local minimum. Second, $h(t)$ does not reach the local maxima $\pm h_{max}$ of $U(h)$, where $h_{max}^2 = -a^2/2d_i^2$. Third, at $t = 0$, $h(0) = h_0$ lies between the maxima $\pm h_{max}$. In other words, $h(t)$ does not escape the local potential well if these conditions are satisfied. Near the origin, we have $U(h) \approx \text{const} - a^2h^2/2$. To satisfy the first condition we must have $a^2 < 0$. To satisfy the second condition we require that $\dot{h}^2 \leq 0$ at the maxima, or equivalently that $U(h_{max}) \geq 0$. After some algebra, we have

$$\alpha_0\beta_0(\alpha_0 + d_i f_0)(\beta_0 - d_i g_0) \geq 0. \tag{30}$$

The third condition means that $h_0^2 \leq h_{max}^2$, and so

$$d_i(\alpha_0 g_0 - \beta_0 f_0) - 2\alpha_0\beta_0 \geq 0. \tag{31}$$

Equation (31) is in fact a stronger condition than $a^2 < 0$, and so we only have the last two conditions for the solution $h(t)$ to remain near the origin. If either equation (30) or equation (31) is not satisfied and $d_i > 0$, the solution develops a finite-time singularity (or evolves exponentially if $d_i = 0$).

Thus the self-similar solution will not contain a finite-time singularity if the initial conditions α_0, β_0, f_0 and g_0 are such that equations (30) and (31) are satisfied. It is worth emphasising that these equations put strong constraints on the initial conditions, needed to prevent a collapse. For an initial large-scale X-point geometry of the planar magnetic field ($\alpha_0 \simeq \beta_0 \simeq 1$ and $d_i \ll 1$ in our dimensionless units), equations (30) and (31) require the initial axial speed to be strongly super-Alfvénic, $W \simeq d_i^{-1} \gg 1$ for $x \simeq y \simeq 1$, which makes the collapse virtually inevitable for any physically plausible initial condition.

Although equation (26) can be solved in terms of Jacobi elliptic functions, it is useful to approximate the solution in terms of elementary functions. Assuming $a^2 > 0$, we let each variable depend on a power of $\tau = (t_s - t)$ near the singularity, then let $\tau \to 0$. For large h, $\ddot{h} \approx 2d_i^2 h^3$, and so h is proportional to $\pm\tau^{-1}$. It is then straightforward [6] to derive the singularity scalings for the other variables by balancing the leading-order terms in equations (22)-(25): for instance, $\alpha \sim \tau^{-2}$ and $\beta \sim \tau^2$ if $h \sim \tau^{-1}$, describing the behaviour of the self-similar solution near the singularity.

We now use asymptotic analysis to determine the singularity time t_s, assuming $a^2 > 0$. For small time, $(d_i h)^2 \ll 1$, and equation (26) simplifies to $\ddot{h} \approx a^2 h$, with a solution given by

$$h(t) \approx h_0 \cosh(at) + \frac{\dot{h}_0}{a} \sinh(at), \tag{32}$$

where $h_0 = h(0)$ and $\dot{h}_0 = -4(\alpha_0 g_0 + \beta_0 f_0)$. Near the singularity, we integrate equation (26) and neglect the integration constant since $h \to \infty$:

$$\dot{h}^2 \approx d_i^2 h^4 + a^2 h^2. \tag{33}$$

Consequently, near the singularity

$$h(t) \approx \frac{2a^2 k \exp(at)}{1 - (d_i a k)^2 \exp(2at)}, \tag{34}$$

where k is an integration constant. An intermediate asymptotic solution follows from equations (32) and (34) by requiring that they coincide in the range $a^{-1} < t < t_s$, which leads to the intermediate asymptotic solution

$$h(t) \approx \left[h_0 \cosh(at) + \frac{\dot{h}_0}{a} \sinh(at)\right] \left[1 - \frac{d_i^2}{16a^2}\left(h_0 + \frac{\dot{h}_0}{a}\right)^2 \exp(2at)\right]^{-1}, \tag{35}$$

and so the singularity time t_s in terms of the initial values is

$$t_s = \frac{1}{2a} \ln\left[\frac{16a^2}{d_i^2}\left(h_0 + \frac{\dot{h}_0}{a}\right)^{-2}\right]. \tag{36}$$

We illustrate the analytical results by plotting the numerical solutions of the system (17)-(21). There are six variables in the system but only five equations, so we have to make an assumption for one of the variables in order to solve the system. We choose $\gamma(t) = \text{const}$ for consistency with previous studies [9, 10]. Figure 1 gives examples of both nonsingular and singular behaviour, determined by the initial conditions.

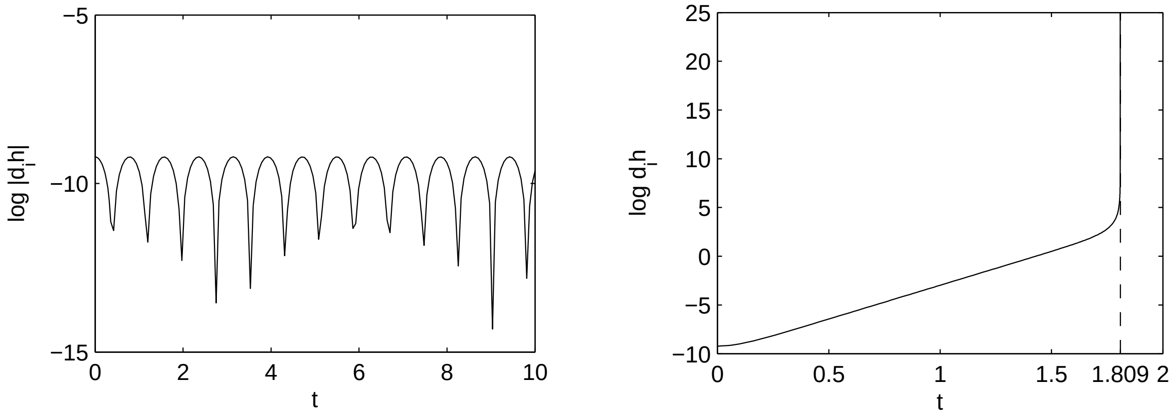

Figure 1: Left: Plot of $h(t)$ for the initial conditions $\alpha_0 = \beta_0 = 1$, $\gamma_0 = 0.5$, $d_i f_0 = -2$, $d_i g_0 = 2$ and $d_i h_0 = 10^{-4}$. Because the initial conditions satisfy equations (30) and (31), no finite-time singularity is present, and $h(t)$ oscillates about $h = 0$. Right: Plot of $h(t)$ for the initial conditions $\alpha_0 = \beta_0 = 1$, $\gamma_0 = 0.5$, $d_i f_0 = 2$, $d_i g_0 = -2$ and $d_i h_0 = 10^{-4}$. Equation (36) predicts the singularity time $t_s = 1.809$. The numerical solution also confirms that $\alpha(t) \to \infty$, $\beta(t) \to 0$ as $t \to t_s$, which constitutes the collapse of an initial magnetic X-point.

4. Discussion

In this paper we have presented a self-similar solution for current sheet formation at a magnetic neutral line in incompressible Hall MHD. The solution complements the available exact steady solution for magnetic merging in Hall MHD [14] and exhibits a finite-time singularity that describes the collapse to a current sheet. While such collapse has been previously demonstrated for a particular choice of initial conditions [6], we have significantly extended the earlier result by considering general initial conditions. Specifically, equations (30) and (31) provide a new criterion for the finite-time singularity formation in terms of the initial conditions, and equation (35) gives a new intermediate asymptotic solution for the evolution of an axial magnetic field. The intermediate asymptotic solution yields an expression for the singularity formation time (equation (36)), which generalises equation (46) in Ref. [6].

Equation (36) shows that the collapse time increases if the strength of the Hall term, quantified by the ion skin depth d_i, decreases. In the limit $d_i \to 0$, the singularity formation time $t_s \to \infty$, corresponding to the absence of finite-time singularities in 2D MHD evolution [9, 10, 11]. We also illustrated both the finite-time singularity criterion and the predicted collapse time numerically.

In the context of a general initial and boundary value problem, our solution can be considered as a low-order Taylor expansion of the flux and stream functions at the origin. This implies that, for general initial and boundary conditions, the solution only holds locally and breaks down before the singularity is reached. Despite the limitations of the self-similar solution, the value of our calculation is that the formula for the singularity formation time quantifies the role of the Hall effect and initial conditions in the current sheet formation.

Our solution may be applicable in a weakly collisional plasma of the solar corona, where the reference values of $L = 10^{9.5}\,\mathrm{cm}$, $B_0 = 10^2\,\mathrm{G}$ and $n = 10^9\,\mathrm{cm}^{-3}$ yield the dimensionless ion skin depth $d_i \approx 10^{-6.5}$. An explosive character of energy release in solar flares [3] can be explained by a rapid transition from slow resistive reconnection to fast Hall reconnection in an evolving current sheet. Our solution models such rapid transition as a singularity formation at time t_s. Assuming $a \sim h_0 \sim 1$, our solution predicts the transition time $t_s \sim 10\, t_A$, where the Alfvén time $t_A = L/v_A = 10^{0.5}\,\mathrm{s}$ in the corona. This estimate is consistent with typical flare onset times and simulation results [15].

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