SQUEEZED STATES

AND

QUANTUM NONDEMOLITION MEASUREMENTS

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ABSTRACT

The first four chapters of this thesis are concerned with the detailed mathematical and physical properties of special minimum uncertainty states of the electromagnetic field: the "squeezed states". These are states with reduced quantum fluctuations in certain observables (the quadrature phase amplitudes).

In chapter two it is shown that the one mode and two mode squeezed states are obtained by acting on the vacuum state with elements of the sympletic group in one and two dimensions respectively.

In chapter three properties of squeezed states in a quantum optical context are investigated. The complex P representation of Drummond and Gardiner is shown to be especially useful for discussing the non-classical statistics of these states, and the complex P-representation for a squeezed state is obtained. An analysis of the effect of irreversible dynamics on squeezed states is also given.

In chapter four a detailed analysis of three devices proposed to produce squeezed states; degenerate parametric oscillation, two photon absorption and dispersive optical bistability, is presented. It is shown in an exact, fully quantum treatment that the reduction of fluctuations obtainable in all these devices is limited, when operated in a steady state regime. The greatest reduction of fluctuations (a factor of two) occurs in the parametric oscillator operated at threshold. The addition of a second driving field in this device enables the direction of squeezing to be changed.

The final two chapters are given over to a discussion of Quantum Non Demolition (QND) measurements. In chapter five a particular QND scheme (back action evading) is analysed. This is based on a parametric amplification interaction. The complex P representation is used to discuss the non-unitary effects of damping and state reduction in a unified way. It is shown that the parametric amplifier, despite being non back action evading, is useful for making QND measurements.

Finally in chapter six, two quantum counting QND schemes, based on quantum optical four wave mixing interactions, are analysed. The effect of state reduction is included, thus enabling an analysis of a measurement sequence.

PREFACE

Quantum Optics is one of the few fields that provide realizable schemes for the experimental verification of some of the more unusual features of non-relativistic quantum mechanics. Two recent examples are the observation of the non classical statistical effect of photon antibunching [Carmichael and Walls, 1976; Kimble et.al., 1977] and, more recently, the experimental verification of the violation of Bells inequality [Aspect et.al., 1982].

The major points of the divergence of classical and quantum physics occur firstly in the discussion of the statistics of observable quantities, and secondly in the treatment of the measurement process itself. The former point has particular relevance in Quantum Optics where many measurements must confront the manifestly quantum nature of the electromagnetic field. The first part of this thesis (chapters one to four) is addressed to discussions of this point.

Recently a class of measurements designed to detect gravitational radiation has forced a new consideration of how quantum mechanics really does affect the results of very accurate measurements. Such considerations have brought about an interaction between the sometimes remote subject of measurement theory and the very real problems of detection electronics. In the second part of this thesis (chapters five and six) we discuss these and related issues in some detail.

Chapter one is of an introductory nature and is concerned with developing definitions and outlining techniques used throughout the thesis. Section 1.1 in particular is an attempt to define precisely what is meant by "Quantum fluctuations" and "Quantum statistics".

The central concept of a minimum uncertainty state is introduced in chapter one and is taken up in greater detail in chapter two, where a special class of states, the so-called "Squeezed States", are defined. These are states in which the quantum uncertainty of certain variables may be arbitrarily small. The formal theory of squeezed states in one and two dimensions is shown to be intimately connected with the mathematical structure of the sympletic group.

In chapter three discussion of squeezed states becomes a little less abstract as we discuss squeezed states of the electromagnetic field; how they may be detected, how they could be produced and how their properties are affected by irreversible dynamics (such as damping).

In chapter four we analyse in detail a number of proposed schemes to generate squeezed states. Specifically, these are; Degenerate Parametric Oscillation, Non-Linear Absorption (two photon absorption) and Dispersive Optical Bistability. In all these devices the reduction in fluctuations is shown to be limited, with the parametric oscillator providing the greatest reduction in fluctuations.

Chapters five and six are concerned with Quantum Non Demolition (QND) measurements. The subject is introduced in chapter five where a specific QND scheme (a back action evasion measurement) based on parametric amplification, is presented. It is shown that the Fokker-Planck equation techniques of quantum optics are very useful in QND analyses and enable a complete discussion of the measurement process including the non-unitary effects of damping and quantum state reduction. It is shown that despite some earlier suggestions, the parametric amplifier does enable true QND measurements to be performed.

Finally in chapter six two quantum counting QND proposals are presented and analysed in detail, including the effect of state reduction. The possible use of these schemes to detect weak classical forces (e.g. gravitational radiation) is suggested.

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CHAPTER 1

INTRODUCTION

1.1 Quantum Fluctuations

In this section we wish to elucidate precisely what we mean by "Quantum Fluctuations" and, in particular, by quantum fluctuations of the electromagnetic field.

In classical physics a physical system is represented mathematically by the points of a $2n$ dimensional real manifold called the "phase space" of the system. A coordinate system for this manifold is given by selecting a set of canonically conjugate variables $\{\underset{\sim}{p}, \underset{\sim}{q}\}$. The possible "states" of the physical system are represented in a one to one manner by a set of points in the phase space [Scheibe, 1973].

An observable in classical physics is any real function $f(\underset{\sim}{p}, \underset{\sim}{q})$ on the phase space of the system. If we know with certainty the values of $\underset{\sim}{q}$ and $\underset{\sim}{p}$ lie in a certain subset B , of phase space, we also know with certainty the value of the observable f lies within a subset $\Delta = f(B)$ of the real line.

In classical mechanics we also define a "statistical ensemble", as a series (preferably infinite) of individual systems, identical in that they all have the same phase space. If we do not know with certainty the state of a system, i.e. we do not know for certain that $\{q, p\}$ lie in a particular subset B , of phase space, then we do not know with certainty that the observable $f(p, q)$ lies in the subset $f(B)$ of the real line. It is then meaningful to ask; given a sequence of measurements to determine $f(q, p)$, with what frequency will the result lie in the subset $\Delta = f(B)$?

We may resort to the concept of the statistical ensemble to speak of the probabilities introduced in classical mechanics. We consider an identical measurement of the same quantity $\mathcal{f}$ made on each member of the ensemble, as constituting a measurement of $\mathcal{f}$ on the ensemble. If each member in the ensemble is in a different state, each measurement of $\mathcal{f}$ will lie in a different subset of the real line. We can then ask; with what frequency will a measurement of $\mathcal{f}$ on each member of the ensemble lie in a particular subset of the real line.

By the "classical fluctuations" in $\mathcal{f}$ we mean the "width" of the probability distribution associated with these frequencies. The "magnitude of the fluctuations" is defined as the variance of this distribution.

The two interpretations, (time sequence of measurements on a single system and a single ensemble measurement) are formally linked by an ergodic hypothesis [Jancel, 1969].

The process of measurement in classical physics is not accorded any special status and in particular may be carried out without disturbing the system in any way. Thus the fluctuations in the measured results of an observable $\mathcal{f}$ are taken to be objective fluctuations of the observable itself.

In the description of physical systems by quantum mechanics we take a completely different direction. The concept of the state of a system is no longer mathematically characterised by a phase space and the measurement process itself is given a special status in the postulates of the theory. We now outline the mathematical structures used in a quantum theoretical description of a physical system. This outline is based on the discussion of Beltrametti and Cassinelli [1981] and we refer the reader to this book for further details [also Fano, 1971].

The fundamental divergence of quantum mechanics from classical mechanics is expressed by the following proposition. If we are certain of the state of a physical system, we cannot be certain that the result of a measurement of an observable $\mathcal{A}$ will yield a result in a particular subset of the real line. It is only meaningful to ask; "what is the probability $p(\mathcal{A}, \Delta)$ that when we make a measurement of $\mathcal{A}$ on the system the value obtained will belong to a subset Δ of $\mathbb{R}$?" [Fano, 1971].

We see that probabilities enter a quantum mechanical description of reality in a completely different way to the probabilities of a classical description. A comparison of probability propositions in classical and quantum descriptions might lead one to suppose that in quantum mechanics the "state" of a system is fundamentally unknowable in complete certainty. This however would require imposing a classical description of "state" on quantum mechanics (i.e. the "phase space description"). The point is that such a description of state is not adequate for a physical system with manifestly quantum behaviour. We need a new concept of state which at least in principal, may be completely knowable, but which nevertheless requires probability propositions.

The ensemble concept is still useful in understanding the quantum mechanical probability propositions. Consider an infinite sequence of identical systems all prepared in precisely the same state. An identical measurement (i.e. with the same accuracy) of the observable

$\mathcal{A}$ is made on each member of the ensemble. Due to the nature of the quantum mechanical state we will obtain a sequence of possibly different results. We then ask, "with what frequency does a measurement of $\mathcal{A}$ on each member of the ensemble yield a result in a subset Δ of $\mathbb{R}$?". This frequency defines $p(\mathcal{A}, \Delta)$.

To attempt to interpret $p(\mathcal{A}, \Delta)$ by a time sequence of measurements one is forced to take into account another major point of departure of quantum mechanics from classical mechanics. Quantum mechanics contains as a fundamental postulate a proposition which specifies exactly how a system is changed upon a measurement. A measurement may leave the system in a state different from the state prior to measurement. Thus a sequence of measurements on a single system could be a sequence of measurements on different states. However in the ensemble description we specified that a measurement be made on a sequence of IDENTICAL systems. Thus for a sequence of measurements on a single system to be equivalent to a single measurement on a sequence of identical systems, one must impose the restriction that after each measurement the system be returned to its state prior to the measurement. This is entirely unnecessary in classical mechanics as each measurement is assumed to leave the state of the system unchanged.

Thus even if we know the state of the system with certainty we will obtain "fluctuations" in the measurement of an observable on the ensemble defined by the system. This is what we mean by the term "quantum fluctuations".

We can, of course, extend the above analysis to include the possibility that we do not know with certainty the state of the system. This extension is made in the same manner as the corresponding classical case. The members of the ensemble are now no longer required to be in the same state, but in some distribution of states. This introduces uncertainties in the outcome of measurements on ensemble members, over and above the intrinsic quantum uncertainties. This prescription of state must not be confused with the quantum mechanical concept of a mixed state, which does not admit the classical "ignorance" interpretation given above for many mixed state density operators [see Beltrametti and Casinelli, 1981].

We now outline some of the essential elements of the mathematical structures of quantum theory, and make more precise the heuristic considerations of previous paragraphs.

To every physical system we associate a separable Hilbert space $\mathcal{H}$, and to every observable we associate a self-adjoint, not necessarily bounded operator $\hat{A}$ defined in $\mathcal{H}$.

The <u>states</u> of the system are in one to one correspondence with the positive definite trace class operators ρ, of trace one defined on $\mathcal{H}$. These are known as "Density Operators".

Let $\hat{P}_A(\Delta)$ denote the projection operator determined by the spectral decomposition of $\hat{A}$, then $p(\mathcal{A}, \Delta)$, the probability that a measurement of the observable $\mathcal{A}$ has a result lying in a subset Δ of the real line, is given by

$$p(\mathcal{A}, \Delta) = \mathrm{Tr}[\rho \hat{P}_A(\Delta)] \tag{1.1}$$

where Tr signifies the trace operation.

A few comments as to the form of the density operator for pure states and mixtures is in order.

If $\{\rho^i\}$ are a sequence of density operators and $\{w_i\}$ are a sequence of positive real numbers such that $\sum_i w_i = 1$, then $\Sigma_i w_i \rho^i$ is also a density operator. If a density operator cannot be decomposed in this fashion we say it represents a pure state. In this case $\rho^2 = \rho$, and ρ is a projector onto a one dimensional subspace. In general for every state ρ there exists an orthonormal sequence of vectors $\{\psi_i\}$ of $\mathcal{H}$ and a corresponding sequence of numbers $\{w_i\}$ defined above, such that

$$\rho = \Sigma_i w_i \hat{P}^{\psi_i} \tag{1.2}$$

where $\hat{P}^{\psi_i}$ is a projector onto the one dimensional subspace spanned by ψ_i. If all the w_i except one is zero then ρ is a pure state, otherwise it is mixed. Furthermore since the linear superposition of a pure state is itself a pure state, the decomposition (1.2) via the set $\{w_i\}$ may not be unique. This is a unique feature of the quantum mechanical concept of state.

We now turn to a discussion of the statistics of an observable as defined by formal quantum mechanics.

The "spectrum" of a physical quantity $\mathcal{A}$ represented by the operator $\hat{A}$, in the state ρ is the subset S of the real line on which the probability $\mathrm{Tr}(\rho \hat{P}_A(\Delta))$ is concentrated, i.e. the possible values of the observable $\mathcal{A}$.

The mean value of $\mathcal{A}$ is defined as the mean of the distribution $\mathrm{Tr}(\rho \hat{P}_A(\Delta))$. This is given by the spectral theorem as,

$$\langle \hat{A} \rangle = \mathrm{Tr}(\rho \hat{A}) \tag{1.3}$$

If we expand ρ in terms of pure states by

$$\rho = \Sigma_i w_i \hat{P}^{\psi_i} \tag{1.4}$$

then

$$\langle \hat{A} \rangle = \Sigma_i w_i \langle \psi_i , \hat{A} \psi_i \rangle \tag{1.5}$$

where $\langle \psi , \varphi \rangle$ represents the scalar product in the Hilbert space. In terms of the Dirac notation $\langle \psi , \varphi \rangle = \langle \psi | \varphi \rangle$, and $\langle \psi_i , \hat{A} \psi_i \rangle = \langle \psi_i | \hat{A} | \psi_i \rangle$.

The variance of $\mathcal{A}$, indicated by $V(\hat{A}, \rho)$, is the variance of the distribution $Tr(\rho \hat{P}_A(\Delta))$.

Consider the case in which ρ represents a pure state, i.e. $\rho = \hat{P}^{\psi}$, where $\hat{P}^{\psi}$ is the projector onto the one dimensional subspace spanned by ψ . Then $V(\hat{A}, \rho)$ exists and is finite if and only if $\psi \in \mathcal{D}(\hat{A})$ (the domain of $\hat{A}$). In this case

$$V(\hat{A}, \rho) \equiv V(\hat{A}, \psi) = \langle \psi, \hat{A}^2 \psi \rangle - \langle \hat{A} \rangle^2 \tag{1.6}$$

When ρ is a non-pure state given by (1.4) then

$$V(\hat{A}, \rho) = \Sigma_i w_i \langle \psi_i , \hat{A}^2 \psi_i \rangle - \langle \hat{A} \rangle^2 \tag{1.7}$$

if and only if $\psi_i \in \mathcal{D}(A)$ for every ψ_i .

Given $\hat{A}$, it is possible to prove that, for every $\varepsilon > 0$, there exists a unit vector ψ such that $V(\hat{A}, \psi) \leqslant \varepsilon$ [Beltrametti and Casinelli, 1981]. This simply means, that there is at least one state in which the distribution around the mean is sharp. In the case in which every state in the ensemble is known to be in just such a "sharp" state, identical measurements of $\hat{A}$ on each member of the ensemble will produce a spectrum of results concentrated on an arbitrarily small interval centred on the mean. We will say that in this case the quantum fluctuations of $\hat{A}$ are small. The variance $V(\hat{A}, \psi)$, then, can be used to characterise the size of the quantum fluctuations of $\hat{A}$ in the state P^{ψ}. In fact we will use $V(\hat{A}, \rho)$ to characterise the size of the quantum fluctuations of $\hat{A}$ in any state ρ, mixed or pure.

Consider now two incompatible observables represented by the two non-commuting operators $\hat{A}$, $\hat{B}$. We wish to determine how the product $V(\hat{A},\psi).V(\hat{B},\psi)$, $\psi \in \mathcal{D}(\hat{A}) \cap \mathcal{D}(\hat{B})$ varies with ψ.

There are two mutually exclusive alternatives;

1. $\forall\, \varepsilon > 0, \exists\, \psi ; \quad V(\hat{A},\psi).V(\hat{B},\psi) < \varepsilon$
2. $\forall\, \psi, \exists\, \varepsilon > 0 ; \quad V(\hat{A},\psi).V(\hat{B},\psi) > \varepsilon$

It can be proved [Beltrametti and Casinelli, 1981] that alternative (2) will only occur if

(a) $\hat{A}$ and $\hat{B}$ do not commute

(b) neither $\hat{A}$ nor $\hat{B}$ is bounded

(c) $\hat{A}$ and $\hat{B}$ have no point spectrum or if either has a non-empty point spectrum, its eigenvectors lie outside the domain of the other.

Alternative (2) is known as the "Heisenberg Uncertainty Relation", and an example is provided by the position and momentum operators of a free particle in all space.

We now present some simple examples in which either of the conditions a, b, c is relaxed and show that alternative (2) fails to occur.

Let us assume $\hat{A}$ and $\hat{B}$ commute, then they can both be written as the function of a single self adjoint operator $\hat{C}$. For example let $\hat{A} = \hat{q}^2$, $\hat{B} = \hat{q}^3$, where $\hat{q}$ is the position operator for a free particle. It suffices to choose the vector $\psi(x)$ corresponding to the spectral representation of $\hat{q}$ in which $\hat{q}$ is simply a multiplication operator. Then $V(\hat{A},\psi) = V(\hat{B},\psi) = 0$

We now turn to condition (b). Consider a free particle in an infinitely deep well with edges at $x = 0$ and $x = 1$. The Hilbert space appropriate to this problem may be taken as $L^2([0,1])$ i.e. the space of square integrable functions with respect to the measure dx , on the closed interval $[0,1]$.

We now consider the position and momentum operators $\hat{q}$, $\hat{p}$. Clearly $\hat{q}$ is a bounded operator and $\mathcal{D}(\hat{q}) = \mathcal{H}$. We define $\hat{p}$ with respect to the domain

$$\mathcal{D}(\hat{p}) = \left\{ \psi \in L^2;\ \psi \text{ is continuous};\ \frac{d\psi}{dx} \in L^2 \right. \\ \left. ;\ \psi(1) = \psi(0)\, e^{i\varphi}.\ 0 \leq \varphi \leq 2\pi \right\}$$

We then have,

$$[\hat{q}, \hat{p}] = i\hbar$$
$$\hat{q}\ \psi(x) = x\ \psi(x)$$
$$\hat{p}\ \psi(x) = \frac{-i}{\hbar} \cdot \frac{d\psi}{dx}(x)$$

Consider the state $\psi_n(x) = \exp\{(2\pi i n + i\varphi)x\}$. It is easily verified that $V(\hat{p}, \psi_n(x)) = 0$ while $V(\hat{q}\ \psi_n(x)) = \frac{1}{12}$.
Thus alternative (1) occurs.

The violation of condition (c) obviously leads to alternative (1). We need only choose an eigenvector from $\mathcal{D}(\hat{A})$ that also lies in $\mathcal{D}(\hat{B})$ then $V(\hat{A},\psi) \, . \, V(\hat{B},\psi) = 0$.

We now obtain a more precise expression of alternative two.

Consider two self adjoint operators $\hat{A}$, $\hat{B}$ (not necessarily bounded which may or may not commute) suitably displaced to have zero mean.

$$\begin{aligned} V(\hat{A},\psi) \, . \, V(\hat{B},\psi) &= \langle \psi, \hat{A}^2\psi \rangle \langle \psi, \hat{B}^2\psi \rangle \\ &= \langle \hat{A}\psi . \hat{A}\psi \rangle \langle \hat{B}\psi . \hat{B}\psi \rangle \\ &\geqslant | \langle \hat{A}\psi . \hat{B}\psi \rangle |^2 \end{aligned} \tag{1.8a}$$

which follows from the Schwartz inequality. However.

$$| \langle \hat{A}\psi, \hat{B}\psi \rangle |^2 \geqslant 1/4 . | \langle \hat{A}\psi, \hat{B}\psi \rangle - \langle \hat{B}\psi, \hat{A}\psi \rangle |^2 \tag{1.8b}$$

thus

$$V(\hat{A},\psi) . \, V(\hat{B},\psi) \geqslant 1/4 . | \langle \hat{A}\psi . \hat{B}\psi \rangle - \langle \hat{B}\psi . \hat{A}\psi \rangle |^2 \tag{1.9}$$

A number of difficulties can arise from (1.9) if care is not taken with the Domain of the operators $\hat{A}$, $\hat{B}$.

As an example consider the case in which both $\hat{A}$ and $\hat{B}$ are bounded. Thus $\mathcal{D}(\hat{A}) = \mathcal{D}(\hat{B}) = \mathcal{H}$. The right hand side of (1.9) may be written as

$$1/4 . | \langle \hat{B}\hat{A}\psi . \psi \rangle - \langle \hat{A}\hat{B}\psi . \psi \rangle |^2$$

$$= 1/4 . | \langle [\hat{B}, \hat{A}]\psi . \psi \rangle |^2 \tag{1.10}$$

If $[\hat{B},\hat{A}]$ is a non-zero C – number substitution of (1.10) into (1.9) would appear to contradict the fact that for bounded operators we expect alternative (1) to hold. However if

$$[\hat{B},\hat{A}] = \lambda I \qquad \lambda \neq 0$$

then $\hat{A}$ and $\hat{B}$ cannot both be bounded without a contradiction occuring [Reed and Simon, 1972]. The contradiction is only avoided if $\lambda = 0$, in which case alternative (1) does indeed occur as expected. If $[\hat{B},\hat{A}]$ is not a c-number then (1.10) may not be defined, if it is $[\hat{B},\hat{A}]$ is bounded and alternative (1) occurs.

Let us now consider the more interesting case in which neither $\hat{A}$ or $\hat{B}$ is bounded for which the purely quantum effect of alternative (2) occurs.

If we assume,

$$\hat{B}\psi,\ \hat{A}\psi \in \mathcal{D}(\hat{A}) \cap \mathcal{D}(\hat{B}) \qquad (1.11)$$

then

$$V(\hat{A},\psi).V(\hat{B},\psi) \geq 1/4.|\langle [\hat{B},\hat{A}]\psi,\psi\rangle|^2$$

if $[\hat{B},\hat{A}] = \lambda$, $\lambda \in \mathbb{C}$

$$V(\hat{A},\psi).V(\hat{B},\psi) \geq 1/4.|\lambda|^2 \qquad (1.12)$$

As the left hand side of (1.12) is independent of ψ, alternative (2) occurs.

An example of (1.12) is provided by the position and momentum observables of a single particle in a harmonic potential.

The domain of the position operator $\mathcal{D}(\hat{p})$ is formed by functions, $\psi(x)$, absolutely continuous on every finite interval of the real axis and such that $\frac{d\psi}{dx} \in L^2(-\infty,\infty)$. The domain $\mathcal{D}(\hat{q})$ is formed by functions such that $x\psi(x) \in L^2(-\infty,\infty)$, which we indicate by $\mathcal{L}$. We see that condition (1.11) will be satisfied providing we only consider solutions $\psi(x) \in \mathcal{L}$. It can easily be shown that in fact this is not a restriction on solutions of the Harmonic oscillator in the Hilbert space $L^2(-\infty,\infty)$ [G. Fano, 1971].

The canonical commutation relation $[\hat{q},\hat{p}] = i\hbar$ is thus well defined in $\mathcal{L}$ and equation (1.12) becomes

$$V(\hat{p}).V(\hat{q}) \geqslant \frac{\hbar^2}{4}. \tag{1.13}$$

which is the usual statement of the uncertainty relation.

1.2 Deterministic and Stochastic Dynamic Evolution

Given the state of an isolated quantum system $\rho(t_1)$ at time t_1, the state of the system at another time is deterministically given by

$$\rho(t_2) = U(t_2 - t_1)\,\rho(t_1)\,U^{\dagger}(t_2 - t_1) \tag{1.4}$$

providing no measurement has been made, and where $U(t)$ is a continuous one parameter group of unitary operators on $\mathcal{H}$. This operator is referred to as the evolution operator.

Using Stones theorem $U(\tau)$ is of the form $e^{-i\hat{H}\tau}$, for some self adjoint, time independent operator $\hat{H}$, the Hamiltonian. If one then considers $\hat{H}$ as the infinitesimal group generator equation (1.14) may be written as

$$\frac{d}{dt}\rho(t) = -\frac{i}{\hbar}[\hat{H},\rho(t)] \tag{1.15}$$

Equation (1.15) readily admits a generalization to the case where $\hat{H}$ is not time independent.

The probability distribution $Tr(\rho(t)\hat{P}_A(a))$ of some physical quantity $\hat{A}$ is a continuous function of time. The quantum statistical properties of any observable are thus fully determined at any time, provided $U(t)$ is known.

Rarely, however, is the situation of interest quite as simple as an isolated quantum system. Usually the system with which we are interested is coupled to another system or possibly many other systems. In particular we have the important case in which a quantum system interacting with a macroscopic system suffers disturbances due to thermal and quantum fluctuations, which make the evolution irreversible, and lead to damping and diffusion of the observables of interest. Furthermore the system may be open to external driving forces of a classical nature.

For most situations of practical importance we are thus forced to consider the interesting situation where quantum statistical and classical statistical concepts are inextricably intertwined.

In situations such as these equation (1.15) is no longer adequate, since to fully take account of the coupled dynamics of the system of interest and all the systems to which it is coupled is an intractable problem.

However there do exist techniques, the so called "master equation techniques" which proceed from the Liouville equation for the total coupled system and result in an evolution equation for the density operator of the system of interest. Such techniques are thoroughly discussed by Louisell [1973] . We summarise here the essential elements of this treatment.

Consider the entire coupled system to be described by a Hamiltonian of the form;

$$\hat{H} = \hat{H}_S + \hat{H}_R + \Sigma_i \hat{\Gamma}_i \hat{A}_i \tag{1.6}$$

where H_S is the Hamiltonian for the sub-system of interest, $\hat{H}_R$ is the Hamiltonian for the larger system to which it is coupled, referred to as the "reservoir", $\hat{\Gamma}_i$ are reservoir operators and $\hat{A}_i$ are system operators.

We let $\rho(t)$ be the total density operator in the interaction picture, thus

$$\frac{d}{dt}\rho(t) = -\frac{i}{\hbar}[\Sigma_i \Gamma_i A_i, \rho(t)] \tag{1.17}$$

Since we are only interested in the system dynamics we define the system density operator by tracing $\rho(t)$ over the reservoir variables:

$$\rho_s(t) = Tr_R(\rho(t))$$

If we now assume

1. The total density operator factorises at $t = 0$, that is, the system and reservoir are initially uncorrelated,

2. The coupling to the reservoir is a small perturbation,

3. The system is Markovian, that is $\rho_s(t)$ does not depend on t at previous times,

we then obtain [P. Drummond, 1979]

$$\frac{d}{dt}\rho_s(t) = \Sigma_{ij}\ \delta(\omega_i, -\omega_j)\{[\hat{A}_i\hat{A}_j\rho_s(t) - \hat{A}_i\rho_s(t)\hat{A}_j]\omega^+_{ij}$$

$$-[\hat{A}_i\rho_s(t)\hat{A}_j - \rho_s(t)\hat{A}_j\hat{A}_i]\omega^-_{ji}\}$$

where ω_i determine the time dependence of A_i under free evolution in the Heisenberg picture. The terms ω_{ij} are correlation functions of the reservoir variables and contain the statistical properties (classical and quantum) of the reservoir:

$$\omega^{+}_{ij} = \frac{1}{\hbar}\int_0^\infty e^{i\omega_i t}\, Tr[f_R(0)\, \hat{\Gamma}_i(t).\hat{\Gamma}_j(0)]\, dt$$

$$\omega^{-}_{ji} = \frac{1}{\hbar}\int_0^\infty e^{i\omega_j t}\, Tr_R[f_R(0)\, \hat{\Gamma}_j(0)\, \hat{\Gamma}_i(t)]\, dt.$$

where $f_R(0)$ is the initial density operator for the reservoir.

If the Hamiltonian (1.16) contains an additional term due to a reversible interaction (e.g. an external driving) and the coupling to the reservoir is of the form $\hat{O}\hat{\Gamma}^{\dagger} + \hat{O}^{\dagger}\hat{\Gamma}$ the master equation is approximately given as, [Drummond et.al., 1981b]

$$\frac{\partial}{\partial t}\rho(t) = \frac{1}{\hbar}[\rho(t), \hat{H}_{REV}] + \lambda\left([O\rho, O^{\dagger}] + [\hat{O}, \rho\hat{O}^{\dagger}]\right)$$

$$+\mu\left([\hat{O}, \hat{O}^{\dagger}\rho] + [\rho\hat{O}, O^{\dagger}]\right) \qquad (1.18)$$

where λ, μ are determined by the reservoir expectation values alone, and $\hat{H}_{REV}$ is the reversible part of the Hamiltonian.

1.3 Quantum Electrodynamics

In the standard treatment of quantum electrodynamics the free electromagnetic field $\hat{A}_\mu$ is expanded as [S. Gupta, 1977]

$$A_\mu = \frac{1}{\sqrt{V}} \Sigma_{\underset{\sim}{k}} \left(\frac{c\hbar}{2k_o}\right)^{1/2} [\hat{a}_\mu(k) e^{i(\underset{\sim}{k}.\underset{\sim}{r} - k_o x_o)} + h.c.] \qquad (1.19)$$

where $k_o = |\underset{\sim}{k}| = \frac{\omega_k}{c}$

If we take the supplementary condition

$$< \partial_\mu \hat{A}_\mu > = 0$$

then for all practical purposes the longitudinal and temporal photons ($\hat{a}(o)$) are not present. Equation (1.19) then may be written as

$$A_i = \frac{1}{\sqrt{V}} \Sigma_k \left(\frac{\hbar}{2\omega_k \varepsilon_o}\right) \{ a_i(k) e^{i(k.r - \omega_k t)} + h.c. \} \qquad (1.20)$$

where $i = 1, 2$ corresponds to two orthogonal polarizations, perpendicular to $\underset{\sim}{k}$.

The canonical commutation relations for the electromagnetic field imply

$$[\hat{a}_i(k), \hat{a}_j^\dagger(k')] = \delta_{ij} \cdot \delta_{kk'} \qquad (1.21a)$$

$$[\hat{a}_i(k),_j \hat{a}(k')] = [\hat{a}_i^\dagger(k), \hat{a}_j^\dagger(k')] = 0 \qquad (1.21b)$$

the usual bose commutation relations.

The energy of the photon field is given by

$$\hat{H} = \hbar \sum_{\underset{\sim}{k},i} \omega_k \left(\hat{a}_i^\dagger(\underset{\sim}{k}).\hat{a}_i(\underset{\sim}{k}) + \hat{a}_i(\underset{\sim}{k}).\hat{a}_i^\dagger(\underset{\sim}{k}) \right) \qquad (1.22)$$

In view of the relations (1.21a,b) and (1.22) it is clear that the $\hat{a}_i(\underset{\sim}{k})$ are in the form of harmonic oscillator creation and annhilation operators.

Corresponding to the electromagnetic field $\hat{A}_\mu$ defined by (1.20) we have the electric field operator

$$\hat{E}_i(r,t) = \frac{i}{\sqrt{V}} \Sigma_k \left(\frac{\hbar\omega_k}{2\varepsilon_0}\right)^{1/2} \left\{ \hat{a}_i(k)\, e^{i(k.r - \omega_k t)} - h.c \right\} \qquad (1.23)$$

The electric field operator may be written as,

$$\hat{E}_i(r,t) = \frac{1}{\sqrt{V}} \Sigma_k \left(\frac{2\hbar\omega_k}{\varepsilon_0}\right)^{1/2} \left\{ \hat{X}_1^{i,k} \operatorname{Sin}(\omega_k t - k.r) - \hat{X}_2^{i,k} \operatorname{Cos}(\omega_k t - k.r) \right\} \qquad (1.24)$$

where

$$\hat{X}_1^{i,k} \equiv 1/2.\left(\hat{a}_i(k) + \hat{a}^\dagger_i(k)\right) \qquad (1.25a)$$

$$\hat{X}_2^{i,k} \equiv (1/2i).\left(\hat{a}_i(k) - \hat{a}^\dagger_i(k)\right) \qquad (1.25b)$$

$\hat{X}_1^i$, $\hat{X}_2^i$ are referred to as the "quadrature phases" of the electric field.

The form of (1.25a,b) suggest that $\hat{X}_1^\mu$, $\hat{X}_2^\mu$ may also be viewed as dimensionless position and momentum operators respectively, for the harmonic oscillator corresponding to the μ'th mode of the electromagnetic field. This identification is further born out in that the commutation relations for the quadrature phase operators given by

$$[\hat{X}_1^i , \hat{X}_2^k] = \frac{i}{2}\delta_{ik} \qquad (1.26)$$

are the usual canonical commutation relations.

Since the electric field operator is unbounded the quadrature phase operators are likewise unbounded. Thus, in view of the discussion of selection (1.2) the Heisenberg uncertainty relation is appropriate and takes the form

$$V(\hat{X}_1^i) \cdot V(\hat{X}_2^i) \geq 1/16 \tag{1.27}$$

The Hilbert space of the multi-mode field may be considered as the tensor product of the Hilbert space of the Harmonic oscillator for each mode;

$$\mathcal{H} = \prod_\mu \otimes \mathcal{H}_\mu \tag{1.28}$$

In each Hilbert space we have the complete orthonormal sequence of vectors $|n_k\rangle$ where $n_k = 0, 1, 2, \ldots$ corresponding to the diagonal representation of the Hamiltonian

$$\hat{H}_k = \hbar \cdot \omega_k \, \hat{a}^\dagger_k \, \hat{a}_k$$

that is,

$$a^\dagger_\mu a_\mu \, |n_\mu\rangle = n_\mu \, |n_\mu\rangle \tag{1.29}$$

This orthonormal set is known as the "number state" representation. It follows then from (1.28) that a complete orthonormal basis for the electromagnetic field is given by the set $\{ |\{n_k\}\rangle \}$ where $\{ |\{n_k\}\rangle \} = \{ \prod_k |n_k\rangle ; \; n_k \in \mathcal{H}_k \}$.

We will often refer to the state $|n\rangle$ as an n-photon state, in particular we have the zero photon state $|0\rangle$ also known as the vacuum state. The vacuum state is such that $a|0\rangle = 0$.

In addition to the number state basis it is convenient to define a "coherent state" basis;

$$\{|\alpha_k\rangle\} = \{ \prod_k |\alpha_k\rangle ; \; a_k|\alpha_k\rangle = \alpha_k|\alpha_k\rangle , \; \alpha_k \in \mathbb{C} \} \tag{1.30}$$

[Glauber, 1963b; Klauder and Sudarshan, 1968]. Such a basis is easily seen to be non-orthogonal and over-complete [Louisell, 1973]. Clearly the coherent state basis contains $|0\rangle$, the vacuum state.

We may expand the coherent states in the number state representation as,

$$|\alpha_k\rangle = \exp(-1/2.|\alpha_k|^2 + \alpha_k a_k^\dagger) |0\rangle \tag{1.31}$$

The means of the quadrature phase operators in a coherent state are given by

$$\langle\alpha_k| \hat{X}_1^k |\alpha_k\rangle = \mathcal{R}e(\alpha_k)$$
$$\langle\alpha_k| \hat{X}_2^k |\alpha_k\rangle = \mathcal{I}m(\alpha_k)$$

The quadrature phases are thus seen to represent the real and imaginary parts of the complex amplitude. We shall see in chapter two that the coherent states may be viewed as harmonic oscillator ground states displaced to $x = \mathcal{R}e(\alpha)$ and viewed from a frame moving with momentum $p = \mathcal{I}m(\alpha)$. For now we simply note that the states $|\alpha_k\rangle$ minimise the Heisenberg relation; they are "minimum uncertainty states",

$$V(\hat{q}_k, \alpha_k).\ V(\hat{p}_k, \alpha_k) = \hbar^2/4. \tag{1.32}$$

or in terms of the field quadrature phases

$$V(\hat{X}_1^k, \alpha_k).V(\hat{X}_2^k, \alpha_k) = 1/16. \tag{1.33}$$

The quantum statistics of the free electromagnetic field is essentially equivalent to the quantum statistics of the Harmonic oscillator.

A number of quantities may be defined to characterise the quantum fluctuations of the electromagnetic field. Let ρ be the density operator for a single mode field. Following Glauber [R.J. Glauber 1963a,b] we characterise the properties of the electromagnetic field by defining a normally ordered correlation function

$$G^{n,m} = Tr(\rho\, a^{\dagger n} a^m) \tag{1.34}$$

Experiments involving photoelectric detection correspond to measurements of even ordered correlation functions $G^{n,n}$. Such measurements however do not yield information on phase dependent properties which are characterised by odd ordered correlation functions.

Another perspective is provided by evaluating (1.34) in the number state basis.

$$G^{n,m} = \sum_{n=0}^{\infty} \langle n|\rho (a^{\dagger})^{n} a^{m}|n\rangle$$

$$= \sum_{k=0}^{\infty} \frac{1}{k!}\left\{(k+m)!\,(k+n)!\right\}^{1/2} \langle k+m|\rho|k+n\rangle \tag{1.35}$$

For the even ordered moments we have,

$$G^{n,n} = \sum_{k=0}^{\infty} \frac{(k+n)!}{k!} \langle k+n|\rho|k+n\rangle \tag{1.36}$$

Thus even ordered moments depend only on the diagonal elements of the density matrix in the number state representation. Any phase dependent properties will be determined by the off diagonal elements.

An example of an even ordered correlation of great importance is the photon-count correlation, defined by;

$$g^{2}(0) \equiv \frac{G^{2,2}}{(G^{1,1})^{2}} = \frac{\langle a^{\dagger}a^{\dagger}\,a\,a\rangle}{\langle a^{\dagger}a\rangle^{2}} \tag{1.37}$$

If we define the photon-number probability distribution $P(n)$ by

$$P(n) \equiv \mathrm{Tr}\,(\rho\,|n\rangle\langle n|) \tag{1.38}$$

$$= \langle n|\rho|n\rangle$$

then $g^2(o)$ may be written as [Walls and Milburn, 1981]

$$g^2(o) = 1 + \frac{\sigma^2 - \bar{n}}{\bar{n}^2} \qquad (1.39)$$

where $\bar{n}$ and σ^2 are the mean and variance respectively, of the distribution $P(n)$.

Examples of $g^2(o)$ for various fields are given by Walls [1980]. A laser has a poisson number distribution $(\rho = |\alpha\rangle\langle\alpha|)$ and thus $g^2(o) = 1$. Two other situations may arise. The field may be characterised by super-poissonian statistics (i.e. $\sigma^2 > \bar{n}$) in which case $g^2(o) > 1$. An example is provided by thermal light where $g^2(o) = 2$. We refer to such a situation as "photon bunching". On the other hand the field may be characterised by sub-poissonian statistics $\sigma^2 < \bar{n}$ in which case $g^2(o) < 1$ e.g. a pure number state $\rho = |\ell\rangle\langle\ell|$,

$$g^2(o) = 1 - 1/\ell$$

A value of $g^2(o) < 1$ is described as photon antibunching. Such a situation is a manifestation of the non-classical statistics that can occur for pure quantum systems.

Photon count correlations are clearly not the only way in which the quantum fluctuations of the electromagnetic field may be characterised. In particular we may choose the variances in the quadrature phases $\hat{X}_1$, $\hat{X}_2$ to characterise the statistics.

It is easily seen from (1.25a,b) that

$$V(\hat{X}_1) = 1/4\left\{ V(a) + V(a^\dagger) + 2(\langle a^\dagger a\rangle - \langle a^\dagger\rangle\langle a\rangle) + 1\right\} \qquad (1.40a)$$

$$V(\hat{X}_2) = -1/4.\left\{V(a) + V(a^\dagger) - 2(\langle a^\dagger a\rangle - \langle a^\dagger\rangle\langle a\rangle) - 1\right\} \qquad (1.40b)$$

It is clear that the variances in $\hat{X}_1$ and $\hat{X}_2$ depend on odd ordered correlations and thus are not completely characterised by $P(n)$, the diagonal elements of ρ. For some fields however there is a close connection as we shall see.

The variances in the quadrature phases may be conveniently characterised by the correlation matrix $C(\hat{X}_1, \hat{X}_2)$ defined by

$$C(\hat{X}_1, \hat{X}_2)_{p,q} = 1/2 \langle \hat{X}_p \hat{X}_q + \hat{X}_q \hat{X}_p \rangle - \langle \hat{X}_p \rangle \langle \hat{X}_q \rangle \quad (p, q = 1, 2) \tag{1.41}$$

If we also define the co-variance matrix $C(a, a^{+})$ by

$$C(a, a^{+}) = \begin{pmatrix} \langle a^2 \rangle - \langle a \rangle^2 & 1/2 \langle a a^{+} + a^{+} a \rangle - \langle a^{+} \rangle \langle a \rangle \\ 1/2 \langle a a^{+} + a^{+} a \rangle - \langle a^{+} \rangle \langle a \rangle & \langle a^{+2} \rangle - \langle a^{+} \rangle^2 \end{pmatrix} \tag{1.42}$$

then

$$C(X_1, X_2) = \Omega \, C(a, a^{+}) \, \Omega^T \tag{1.43}$$

where

$$\Omega = \begin{pmatrix} 1 & 1 \\ -i & i \end{pmatrix} \tag{1.44}$$

If the field is in a coherent state we have

$$C(\hat{X}_1, \hat{X}_2) = 1/4 \; I \tag{1.45}$$

while for a pure number state we have,

$$C(\hat{X}_1, \hat{X}_2) = 1/4 . (2n + 1) \; I \tag{1.46}$$

A number state clearly is not a minimum uncertainty state.

A useful pictorial representation of the statistical properties of the states of a harmonic oscillator is made using a "complex amplitude" diagram (Figure 1.1). This is a real manifold whose points represent possible values for the quantities $\hat{X}_1$ and $\hat{X}_2$. An orthogonal co-ordinate system is provided by the values of $\hat{X}_1$ and $\hat{X}_2$, i.e. by the possible values for the real and imaginary parts of the complex amplitude a. The state of the system is then characterised by a set of points in this manifold. The "most likely" values for the observables $\hat{X}_1$ and $\hat{X}_2$ are contained within an "error volume" which is taken to be determined by the variance in $\hat{X}_1$ and $\hat{X}_2$ for a given state. For example if the system were in a coherent state $|\alpha\rangle$ for which the variances in $\hat{X}_1$ and $\hat{X}_2$ are equal, the error volume is a circle, centred on a point $(\mathrm{Re}(\alpha), \mathrm{Im}(\alpha))$ representing the mean of a. The radius of the error circle is given by $V(\hat{X}_1)^{1/2} = 1/2$. Such a diagram contains information on both the amplitude and phase of the state and is also known as an "amplitude and phase" diagram. Another example, the number state, is shown in Figure 1.1b. As a number state has random phase in both its amplitude and its variance, it is represented by an error annulus centred on zero, and width of one. Then, noting that $\langle a^{+}a\rangle = \langle \hat{X}_1^2 + \hat{X}_2^2\rangle - 1/2$ the area of the annulus is π which is four times the minimum allowable area.

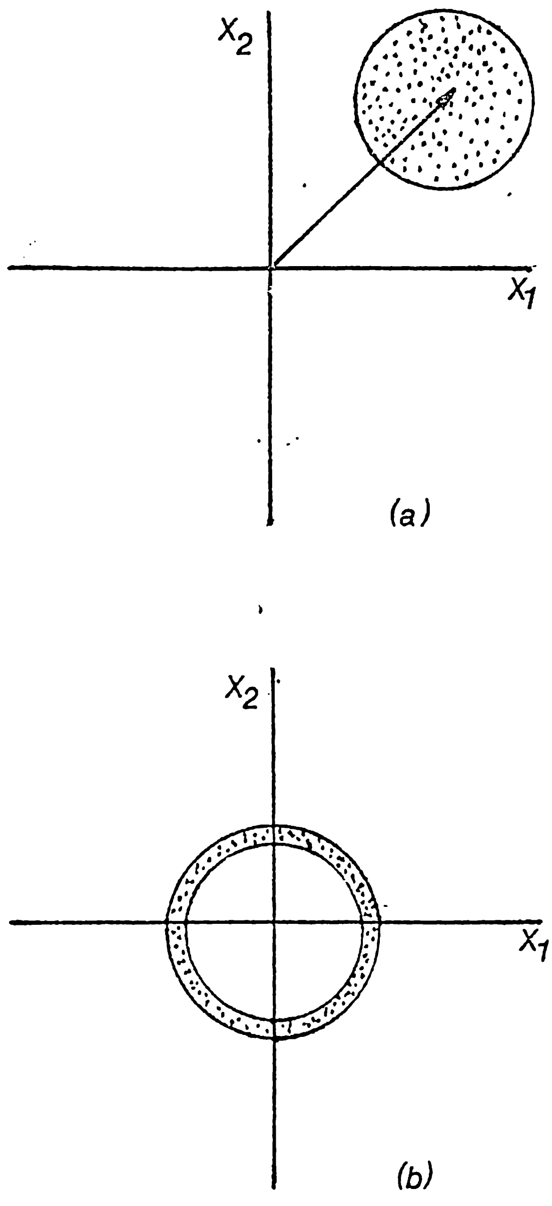

Figure 1.1: Complex amplitude diagram for a coherent state (a) and a number state (b)

1.4 P-Representations

To study the statistics of the quantum field in cases where the non-diagonal number state matrix elements are significant, several researchers have found it convenient to represent ρ using a distribution over a c-number phase space. Such a distribution was developed by Sudarshan [1963] and Glauber [1963b] as an expansion in diagonal coherent state projection operators

$$\rho \quad = \quad \int d^2\alpha \; P(\alpha,\alpha^*)\, |\alpha\rangle\langle\alpha| \tag{1.47}$$

However for fields in which uniquely quantum statistics are manifest, the Glauber-Sudarshan P-function often does not exist as a well behaved function.

Other distribution functions have been developed such as; the Wigner function, obtained from the P-function as,

$$W(\alpha,\alpha^*) \;=\; 2/\pi \;.\; \int P(\beta,\beta^*)\; e^{-2|\beta-\alpha|^2}\, d^2\beta .$$

the Q function,

$$Q(\alpha,\alpha^*) = \langle\alpha|\rho|\alpha\rangle$$

and the Glauber R-representation

$$R(\alpha^*,\beta) \quad = \langle\alpha|\rho|\beta\rangle \exp\{1/2\,(|\alpha|^2 + |\beta|^2)\}$$

$$\rho \quad = \quad \frac{1}{\pi^2} \int d^2\alpha\, d^2\beta\; |\alpha\rangle\, R(\alpha^*,\beta)\, \exp\{-1/2(|\alpha|^2+|\beta|^2)\}\langle\beta|$$

All these representations have various unsatisfactory features [Walls and Milburn, 1981] and are not generally useful in quantum optics.

In order to treat problems where non-classical quantum statistics are important a class of generalised P-representations were introduced by Drummond and Gardiner [1980]. This class of representation is defined by,

$$\rho = \int_{\mathcal{D}} P(\alpha\ \beta)\, \hat{\Lambda}(\alpha,\beta)\, d\mu(\alpha,\beta) \tag{1.48}$$

where

$$\hat{\Lambda}(\alpha,\beta) = \frac{|\alpha\rangle\langle\beta^*|}{\langle\beta^*|\alpha\rangle}$$

and $d\mu(\alpha,\beta)$ is an integration measure which is chosen to generate members of the class. Some examples of the integration measure are

(a) Glauber representation

$$d\mu(\alpha,\beta) = \delta^2(\alpha^* - \beta)\ d^2\alpha\ d^2\beta. \tag{1.49}$$

(b) Complex P-representation

$$d\mu(\alpha,\beta) = d\alpha\ d\beta. \tag{1.50}$$

where (α,β) are treated as independent complex variables to be integrated on individual contours C, C'. Existence theorems for this representation have been given by Drummond and Gardiner [1980]. In particular the complex P-representation is known to exist for a density operator expanded in a finite set of number states, a characteristic situation where non-classical statistics may arise, as noted earlier. This representation gives rise to a Fokker-Planck equation which under certain circumstances may be integrated exactly and the solution normalised on appropriate contours. An example is provided by the density operator of a pure number state $\rho = |n\rangle\langle n|$, where

$$P(\alpha,\beta) = -1/4\pi^2 . \; e^{\alpha\beta} . \; \frac{n!}{(\alpha\beta)^{n+1}} \tag{1.51}$$

which may be normalised on a contour which encloses the origin.

(c) Positive P-representation

$$d\mu(\alpha,\beta) = d^2\alpha \; d^2\beta .$$

This representation allows α , β to vary independently over the entire complex phase space. Drummond and Gardiner have shown that this representation exists for a physical realizable density operator and may always be chosen positive. These authors have also shown that provided a Fokker-Planck equation exists for the Glauber-Sudarshan representation a corresponding Fokker-Planck equation exists, with a positive semi-definite diffusion matrix, for the positive P-representation. This enables corresponding stochastic differential equations to be obtained.

These generalised representations may be used to obtain c-number time development equations from the Liouville or Master equation for ρ , using certain operator correspondences [Drummond and Gardiner, 1980]. These are:

$$a \hat{\Lambda}(\alpha,\beta) = \alpha \hat{\Lambda}(\alpha,\beta) \tag{1.52a}$$

$$a^{+} \hat{\Lambda}(\alpha,\beta) = \left(\frac{\partial}{\partial\alpha} + \beta \right) \hat{\Lambda}(\alpha,\beta) \tag{1.52b}$$

$$\hat{\Lambda}(\alpha,\beta)\, a = \left(\alpha + \frac{\partial}{\partial\beta} \right) \hat{\Lambda}(\alpha,\beta) \tag{1.52c}$$

$$\hat{\Lambda}(\alpha,\beta) = \beta \hat{\Lambda}(\alpha,\beta) \tag{1.52d}$$

The normally ordered moments of bose operators are given by

$$\langle (a^\dagger)^n a^m \rangle = \langle \alpha^m \beta^n \rangle = \int_{\mathcal{D}} P(\alpha,\beta)\, \beta^n \alpha^m \, d\mu(\alpha,\beta) \tag{1.53}$$

where $d\mu$ is the appropriate measure and $\mathcal{D}$ is the domain of integration.

The form of equation (1.53) suggests we interpret $P(\alpha,\beta)$ as a quasi-probability distribution for the independent variables α , β . If we define the covariance matrix in these variables by

$$C(\alpha,\beta) = \begin{pmatrix} \langle \alpha^2 \rangle - \langle \alpha \rangle^2 & \langle \alpha\beta \rangle - \langle \alpha \rangle\langle \beta \rangle \\ \langle \alpha\beta \rangle - \langle \alpha \rangle\langle \beta \rangle & \langle \beta^2 \rangle - \langle \beta \rangle^2 \end{pmatrix}$$

then

$$C(\hat{X}_1,\hat{X}_2) = \Omega\, C(\alpha,\beta)\, \Omega^T + 1/4\ I \tag{1.54}$$

The complex P-representation may be used to derive a relationship between photon number and quadrature phase statistics.

Defining $\hat{N} \equiv \hat{a}^\dagger \hat{a}$, it follows that

$$V(\hat{N}) - \langle \hat{N} \rangle = \langle \hat{a}^\dagger \hat{a}^\dagger \hat{a}\, \hat{a} \rangle - \langle \hat{N} \rangle^2 \tag{1.54b}$$

We then define the displaced operators by

$$\begin{aligned} \hat{A} &= \hat{a} - z \\ \hat{A}^\dagger &= \hat{a}^\dagger - z^* \end{aligned} \tag{1.55}$$

where $\quad z = \langle \hat{a} \rangle$

The complex P-representation for the $\hat{A}$, $\hat{A}^\dagger$ variables gives the normally ordered moments by,

$$\langle A^{\dagger n} A^m \rangle = \int \mu^n \nu^m P(\mu,\nu)\, d\mu\, d\nu. \tag{1.56}$$

It is easily seen that $P(\mu,\nu)$ defines a distribution with zero mean, i.e.

$$\langle \hat{A} \rangle = \langle \mu \rangle = 0$$
$$\langle \hat{A}^{\dagger} \rangle = \langle \nu \rangle = 0$$

Furthermore we have

$$C(\mu,\nu) = C(\alpha,\beta) \tag{1.57}$$

that is the covariance matrix associated with the μ, ν variables is the same as the covariance matrix associated with the complex P-representation in the original variables α, β (Eq. 1.53).

We now define the normally ordered variances in the quadrature phases by,

$$\left.\begin{aligned} V(:\hat{X}_1:) &\equiv \langle :\hat{X}_1^2: \rangle - \langle \hat{X}_1 \rangle^2 \\ V(:\hat{X}_2:) &\equiv \langle :\hat{X}_2: \rangle - \langle \hat{X}_2 \rangle^2 \end{aligned}\right] \tag{1.58}$$

Then, in terms of the elements of $C(\alpha,\beta)$ (or $C(\mu,\nu)$) we find

$$\left.\begin{aligned} V(:\hat{X}_1:) &= 1/4.(C_{11} + C_{22} + 2C_{12}) = V(\hat{X}_1) - 1/4 \\ V(:\hat{X}_2:) &= -1/4.(C_{11} + C_{22} - 2C_{12}) = V(\hat{X}_2) - 1/4 \end{aligned}\right] \tag{1.59}$$

We now assume that the $P(\mu,\nu)$ representation defines a Gaussian distribution for the moments, then [Gardiner, 1982],

$$\langle \mu^2 \nu^2 \rangle = \langle \mu^2 \rangle \langle \nu^2 \rangle + 2\langle \mu\nu \rangle^2 \tag{1.60}$$

and note that all odd moments are zero.

Thus, using (1.54b), (1.56) and (1.60) we have

$$V(\hat{N}) - \langle \hat{N} \rangle = |z|^2.\ \Big\{(C_{11}C_{22} + C_{12}^2)/|z|^2 + \frac{z^2}{|z|^2} C_{11} + \frac{(z^*)^2}{|z|^2} C_{22} + 2C_{12}\Big\} \tag{1.61}$$

If we assume $|\gamma|$ is larger than any term in the covariance matrix, then, using (1.59) we have,

$$V(\hat{N}) - \langle \hat{N} \rangle \approx 4|\gamma|^2 \times \begin{cases} V(:\hat{X}_1:) & \mathcal{I}m(\gamma) = 0 \\ V(:\hat{X}_2:) & \mathcal{R}e(\gamma) = 0 \end{cases} \qquad (1.62)$$

We further note that when $\mathcal{I}m(\gamma) = 0$, $\langle \hat{X}_1 \rangle = |\gamma|$ and $\langle \hat{X}_2 \rangle = 0$, similarly when $\mathcal{R}e(\gamma) = 0$, $\langle \hat{X}_1 \rangle = 0$ and $\langle \hat{X}_2 \rangle = |\gamma|$.
We thus conclude that subpoissonian statistics implies that the quadrature carrying the coherent excitation has a negative normally ordered variance. This implies that the fluctuations in that quadrature are less than would occur for a coherent state. We shall discuss this situation more fully in chapter 3, in relation to a special class of such states known as "squeezed states".

1.5 Measurement in Quantum Mechanics

The quantum mechanical concept of an observable and fluctuations of an observable are only fully expressed by reference to specific measurement propositions.

One has the intuitive physical picture of a small quantum system upon which measurements are to be made coupled to a macroscopic readout/amplifier whose complete description may be given entirely within a classical picture. While a complete description of the measurement process may be possible entirely within the framework of quantum mechanics, such a description would be enormously difficult for two reasons,

i) the amplifier system by definition is a macroscopic object with many degrees of freedom and a complete deterministic treatment of its evolution is intractable;

ii) by definition we are dealing with a system described both by quantum and classical concepts; concepts which are so radically different in their formulation that the formal derivation of correspondence rules would be very difficult.

Various theoretical proposals to obtain a "complete" measurement theory despite the above difficulties have been given [for a review see D'Espagnat, 1971]. In fact the second difficulty above may be completely circumvented by the introduction of so called "super selection rules" [Beltrametti and Cassinelli, 1981]. In this thesis we will not be addressing any of these questions, and turn now to the Von Neuman measurement postulates which will provide the framework for our discussion of measurements.

An intuitive realization of a measurement procedure would be one in which the physical system to be measured is not destroyed as a result of measurement. It then becomes meaningful to speak of the "final state" of the system. A typical counter example is provided by a photo-electric determination of the energy of a single photon, whereby the photon is destroyed by the very act of measurement.

A system may admit many measurements of the non-destrictive kind and to further specify the class of measuring instruments we will be concerned with we adopt Von Neumans projection postulate. This says that if a physical quantity is measured twice in succession then the same value is obtained each time. This amounts to assuming that after a measurement of an observable $\hat{A}$ with a point spectrum $\{\lambda_i\}$ with result λ_i the final state is an eigenstate corresponding to the eigenvalue λ_i . We further assume that the measurement is instantaneous and arbitrarily accurate.

A restricted version of Von Neumans projection postulate is the following. If ρ represents the density operator of the system, and the measurement of a quantity $\hat{A}$ yields a result in a subset E of the spectrum of $\hat{A}$, the density operator of the system after measurement is given by the transformation,

$$\rho(t) \longrightarrow \frac{\hat{P}_A(E)\rho(t)\,\hat{P}_A(E)}{Tr(\rho\,\hat{P}_A(E))} \tag{1.55}$$

where $\hat{P}_A(E)$ is a projection operator.

Equation (1.53) allows us to meaningfully speak of the probability distribution of a physical quantity given the result of a previous measurement of some other physical quantity.

A specific example of the above discussion is provided by a measurement scheme which will be of great interest to us. Let $\mathcal{S}$ represent the system upon which we wish to make a measurement of some quantity $\hat{A}_S$. Let $\mathcal{M}$ represent a system coupled to $\mathcal{S}$ upon which we actually make a measurement of some quantity B_M . The coupled evolution of $\mathcal{S}$ and $\mathcal{M}$ lead to correlations between the observables of the two systems and allow results for $\hat{A}_S$ to be inferred from direct measurements of B_M .

We will describe both systems entirely within a quantum framework and thus the appropriate Hilbert space for a description of the coupled evolution is $\mathcal{H}_S \otimes \mathcal{H}_M$ where $\mathcal{H}_{S(M)}$ represent the Hilbert spaces of the two subsystems.

If the systems, initially independent, are described by the density operators $\rho_S(0)$ and $\rho_M(0)$, then the state of the coupled system at time τ after the interaction was turned on, is given by

$$\rho_{S\otimes M}(\tau) = U_{S\otimes M}(\tau)\,\rho_S(0) \otimes \rho_m(0)\, U^{\dagger}_{S\otimes m}(\tau) \tag{1.56}$$

where $U_{S\otimes M}(\tau)$ is the unitary evolution operator for the coupled system.

At time τ we make a measurement of $\hat{B}_M$ and obtain a result in a subset E of the spectrum of $\hat{B}_M$. If $\hat{P}_{B_M}(E)$ is the corresponding projection operator the state of the coupled system after readout is

$$\bar{\rho}_{S\otimes M}(\tau) = \mathcal{N}(I_S \otimes \hat{P}_{B_m}(E))\,\rho_{S\otimes m}(\tau)\,(I_S \otimes \hat{P}_{B_m}(E)) \tag{1.57}$$

where $\qquad \mathcal{N}^{-1} = \mathrm{Tr}\,(\rho_{S\otimes m}(\tau)\,.\,I \otimes \hat{P}_{B_m}(E))$

and I_S is the identity operator in $\mathcal{S}$ that is, $I_S\,\rho_s(\tau)\,I_s = \rho_s(\tau)$. The state of the system after readout may be obtained by tracing out over meter states.

$$\begin{aligned} \bar{\rho}_S(\tau; E) &= \mathrm{Tr}_m\left\{\hat{P}_B(E)\,\rho_{S\otimes m}(\tau)\,\hat{P}_B(E)\right\} \\ &= \mathrm{Tr}_m\left\{P_B(E)\,\rho_{S\otimes m}(\tau)\right\} \end{aligned} \tag{1.58}$$

where we have written $\bar{\rho}_S$ as a function of E to indicate it is conditional on the results of measurements on M .

If we let $|E_B\rangle_m$ indicate the one dimensional subspace in the range of $\hat{P}_B(E)$

$$\bar{\rho}_S(\tau : E) = {}_m\langle E_B|\,\rho_{S\otimes m}(\tau)\,|E_B\rangle_m \tag{1.59}$$

A more complete discussion of this measurement scheme with specific examples will be made in chapter five.

CHAPTER 2

MINIMUM UNCERTAINTY STATES

2.1 Introduction

It would seem that if one continually refined the accuracy of a sequence of measurements of an observable on any system (macroscopic or microscopic) one would ultimately confront the essential quantum nature of the physical system. At some point the classical concept of the state of the system as being represented by a point or set of points, in a real phase space, would have to be abandoned.

For example, suppose one set out to determine the position and momentum of a particle moving in a harmonic potential. Imagine that this will be done by preparing an ensemble of IDENTICAL systems (i.e. in the same "state") and making a perfectly accurate measurement of momentum on some members of the ensemble and a perfectly accurate measurement of position on other ensemble members. If the actual state of the system was truly described by the classical concept of state, one would obtain a series of identical values for the momentum measurements and a series of identical values for the position measurements. However, the quantum mechanical treatment of this idealised experiment suggests that such a result would not occur. Instead what would result is a distribution of values for momentum and position the variances of which would satisfy equation (1.13). While it is possible that either the momentum distribution or position distribution will be concentrated at a point, such distributions cannot occur simultaneously. The best that can be achieved if one seeks to describe the system by both its position and momentum occurs when the state of each member of the ensemble was such that

$$V(\hat{p},\psi).V(\hat{q},\psi) \quad = \quad \hbar^2/4 \tag{2.1}$$

If this situation occurs we speak of the state of the system ψ as being a "minimum uncertainty state" which we shall abbreviate to m.u.s.

Of course this is not a property of harmonic potentials but as we saw in section 1.1, it is a fundamental property of any pair of non-commuting unbounded operators.

We now determine under what conditions a state $|\psi\rangle$ will be a m.u.s.

The equality in equation (1.12) will only occur when an equality occurs in the Schwartz relation (1.8a) and in the relation (1.8b). This requires two conditions on the state $|\psi\rangle$

1. $$\hat{A}\,|\psi\rangle \quad = \quad \lambda\ \hat{B}\,|\psi\rangle \tag{2.2a}$$

2. $$\langle\psi|\,\hat{A}\hat{B} + \hat{B}\hat{A}\,|\psi\rangle \quad = \quad 0 \tag{2.2b}$$

And of course the minimum uncertainty relation Eq. (1.12) is well defined only if the relation (1.11) is satisfied. Condition (2.2b) simply requires that the quantity $\langle\hat{A}\psi, \hat{B}\psi\rangle$ is purely imaginary.

Given a set of unbounded canonical co-ordinate and momenta $\{\hat{q}_i, \hat{p}_i\}$ $(i = 1, n)$ and a state $|\psi\rangle$ we define

$$\hat{Q}_i \quad \equiv \quad \hat{q}_i \quad - \quad \langle\psi|\,\hat{q}_i\,|\psi\rangle \tag{2.3}$$

$$\hat{P}_i \quad \equiv \quad \hat{p}_i \quad - \quad \langle\psi|\,\hat{p}_i\,|\psi\rangle \tag{2.4}$$

We then define the state $|\psi\rangle$ to be a m.u.s. if

$$\hat{Q}_i |\psi\rangle = \lambda_i \hat{P}_i |\psi\rangle \quad , \quad \lambda_i \in \mathbb{C} \tag{2.5a}$$

$$\langle\psi| \hat{P}_i \hat{Q}_i + \hat{Q}_i \hat{P}_i |\psi\rangle = 0 \tag{2.5b}$$

for all $i = 1, n$. We then find upon using the canonical commutation relations

$$[\hat{q}_i, \hat{p}_j] = i\hbar\, \delta_{ij} \tag{2.6}$$

that

$$V(\hat{q}_i, \psi) = i\hbar \lambda_i / 2 \tag{2.7}$$

$$V(\hat{p}_i, \psi) = -i\hbar / 2\lambda_i \tag{2.8}$$

$$\frac{V(\hat{q}\ \ \psi)}{V(\hat{p}, \psi)} = -\lambda_i^2 \tag{2.9}$$

The m.u.s. $|\psi\rangle$ refers to no particular system, however we shall be specifically concerned with the m.u.s. of the n dimensional harmonic oscillator. It is convenient to define bose operators by a complex extension of this lie algebra;

$$\begin{aligned} a_i &= \frac{1}{\sqrt{2\hbar}}\left(\mu_i \hat{q}_i + i\hat{p}_i/\mu_i\right) \\ a_i^{+} &= \frac{1}{\sqrt{2\hbar}}\left(\mu_i \hat{q}_i - i\hat{p}_i/\mu_i\right) \end{aligned} \tag{2.11}$$

Using (2.6) we find,

$$[a_i, a_j^{\dagger}] = \delta_{ij} \tag{2.12}$$

The usual harmonic oscillator annihilation and creation operators have $\mu_i = \sqrt{\omega_i}$ where ω_i is the frequency of the i'th mode.

By the analogy with section (1.3) we are lead to define the operators $\hat{X}_1^i$, $\hat{X}_2^i$ by,

$$\hat{X}_1^i \equiv 1/2\,(a_i + a_i^\dagger) = \frac{\mu_i}{\sqrt{2\hbar}}\,\hat{q}_i \tag{2.13a}$$

$$X_2^i \equiv 1/(2i)(a_i - a^\dagger_i) = \frac{1}{\sqrt{2\hbar}}\cdot \hat{p}_i/\mu_i \tag{2.13b}$$

If we now consider the ground state of the harmonic oscillator $|0\rangle$ for which $a_i|0\rangle = 0$

we see that the state $|0\rangle$ is a m.u.s. with $\lambda_i = -i/\mu_i^2$. Thus,

$$V(\hat{X}_1^i, 0) = 1/4 \tag{2.14a}$$

$$V/\hat{X}_2^i, 0) = 1/4 \tag{2.14b}$$

Since the state $|0\rangle$ is in fact a "coherent state" in the sense of section (1.3) we take the relations (2.14a,b) as the defining relations for a coherent state. That is the state $|\psi\rangle$ is coherent if

$$V(\hat{X}_1^i, \psi) = 1/4 \tag{2.15a}$$

$$V(\hat{X}_2^i, \psi) = 1/4 \tag{2.15b}$$

Given the (coherent) m.u.s. $|0\rangle$ do there exist other states $|\psi\rangle$ unitarily related to $|0\rangle$ which are minimum uncertainty states for the chosen canonical variables? (It should be noted that a state $|\psi\rangle$ is only a m.u.s. with respect to a particular set of canonical variables. Of course such a set need not be unique. In what follows determination of m.u.s. will always be made for a given set of canonical co-ordinates. It may then transpire that the m.u.s.

state obtained is also a m.u.s. in various other sets of canonical coordinates.) Answering this question will be the chief concern of this chapter.

We shall approach this question by making an alternative interpretation of a unitary operator, we consider a unitary operator as effecting a transformation of the canonical coordinates themselves while the state of the harmonic system remains the ground state. Clearly this is just a generalization of the equivalence of the Schrodinger and Heisenberg pictures, where the unitary operator is the time development operator.

We wish to consider states $|\gamma_j\rangle$

$$|\gamma_j\rangle = U(\gamma_j)\,|0\rangle \tag{2.16}$$

where the γ_j are real parameters. If $\hat{A}$ is any self adjoint operator corresponding to an observable, then

$$\langle\gamma_j|\,\hat{A}\,|\gamma_j\rangle = \langle 0|\,\bar{\hat{A}}\,|0\rangle \tag{2.17}$$

where

$$\bar{\hat{A}} \equiv U^{\dagger}(\gamma_j)\,\hat{A}\,U(\gamma_j) \tag{2.18}$$

Equations (2.17) and (2.18) formally provide for the alternative interpretation of $U(\gamma_j)$ discussed in the previous paragraph.

In this thesis we will restrict the $U(\gamma_j)$ to be those which effect only linear transformations of the canonical variables. Of course every state unitarily related to a m.u.s. is itself a m.u.s. in the transformed canonical variables, however we consider only those states which under a unitary transformation remain m.u.s. in the original, given, variables. Later we consider the possibility that the state so generated is a m.u.s. with respect to other canonical variables.

In classical mechanics canonical transformations leave invariant the poisson brackets and thus leave form invariant the equations of motion in either the Hamiltonian or Newtonian formulation [Mariwalla, 1975]. They may be said to transform "solutions into solutions". Such transformations are also referred to as Dynamic Symmetry transformations, the symmetry referring to the form invariance of the equations of motion. These symmetries may be easily discussed in terms of the global properties of the corresponding lie group. As a particular example of such symmetries we refer the reader to the treatment of the dynamic symmetry of the harmonic oscillator by Wulfman and Wybourne [1976]. Such symmetries may also be associated with constants of motion [Lutzky, 1978]. For a review of dynamic symmetry considerations see Mariwalla [1975].

In quantum mechanics all canonical transformations correspond to a unitary operator, which is an element of a continuous one parameter unitary group. Stones theorem then states [Reed and Simon, 1972] that for the unitary operator $U(\gamma)$ of such a group there is a self adjoint operator $\hat{A}$ (not necessarily bounded) on the Hilbert space such that $U(\gamma) = e^{i\gamma\hat{A}}$. We refer to $\hat{A}$ as the generator of the group. Since commutation relations are invariant under unitary transformations the Heisenberg equations of motion remain form invariant. This is analogous to the form invariance of the corresponding classical equations of motion.

The simplest linear canonical transformations corresponds to translations, i.e.

$$\left.\begin{aligned} \hat{q}_i \rightarrow \bar{\hat{q}}_i &= \hat{q}_i + A_i \\ \hat{p}_i \rightarrow \bar{\hat{p}}_i &= \hat{p}_i + B_i \end{aligned}\right\} \qquad (2.19)$$

This corresponds to describing the system in terms of a new spatial origin and from a frame moving with momentum B with respect to the original frame with momentum $\langle \hat{\underset{\sim}{p}} \rangle$.

The infinitesimal generators of these transformations are the elements of the Heisenberg-Weyl lie algebra $N(n)$, the $\hat{q}$, and $\hat{p}$ themselves. The unitary operator which produces the transformation (2.19) may be written as

$$U(A, B) \equiv D(\alpha_i) = \exp\left\{ \sum_{i=1}^{n} (\alpha_i a_i^{\dagger} - \alpha_i^{*} a_i) \right\} \tag{2.20}$$

where

$$\alpha_i = \frac{1}{\sqrt{2\hbar}} (\mu_i A_i + i B_i / \mu_i)$$

The harmonic oscillator state

$$|\{\alpha_i\}\rangle = D(\alpha_i)|0\rangle \tag{2.21}$$

is a m.u.s. state with

$$\langle \hat{q}_i \rangle = A_i$$
$$\langle \hat{p}_i \rangle = B_i$$

Furthermore it is easily verified that

$$C(\bar{q}_i, \bar{p}_i) = C(\hat{q}_i, \hat{p}_i)$$

where C is covariance matrix. Thus the state $|\{\alpha_i\}\rangle$ is a coherent state. We see that the members of the family of coherent states are in one-to-one correspondence with the ground states of a family of harmonic oscillators related by the transformations (2.19) [Biedenharn and Louck, 1981].

It follows from (2.20),(2.21) and (2.12) that

$$|\{\alpha_i\}\rangle \quad = \quad \prod_{i=1}^{m} |\alpha_i\rangle \tag{2.22}$$

that is, the multimode coherent states are formed by the direct product of the coherent states of each mode. Thus the states $|\{\alpha_i\}\rangle$ are simply the coherent states discussed in section (1.3).

The simple displacement transformation (2.19) has given all the usual coherent m.u.s. Clearly there are many other linear canonical transformations possible. What other transformations will lead to m.u.s.? This would seem to open a veritable "pandoras box" of states once the number of dimensions exceeds one or two. However one may see at the outset that there are at least two general classes of m.u.s. The first of these are the coherent m.u.s. defined by relations (2.15a,b). Secondly, one must admit the possibility of states for which equations (2.15a,b) do not hold yet which still satisfy equation (2.10). These states would have reduced fluctuations in at least one canonical coordinate and increased fluctuations in the corresponding conjugate coordinate.

2.2 The Sympletic Group and Minimum Uncertainty States

We define the row vector $\hat{\underset{\sim}{z}}$ by

$$\hat{\underset{\sim}{z}} \quad \equiv \quad \sqrt{2\hbar}\,.\,(\,\hat{\underset{\sim}{X}}_1, \hat{\underset{\sim}{X}}_2\,) \tag{2.23}$$

where the coordinates of $\hat{\underset{\sim}{X}}_1$ and $\hat{\underset{\sim}{X}}_2$ are defined by equations (2.15a) and (2.15b). The commutation relations may be written

$$\hat{\underset{\sim}{z}}^T \cdot \hat{\underset{\sim}{z}} \quad - (\,\hat{z}^T \cdot \hat{z}\,)^T \quad = \quad i\hbar\, K$$

Then a linear transformation $\mathcal{S}$, where

$$\hat{\underset{\sim}{z}}' \quad = \quad \hat{\underset{\sim}{z}}\,\mathcal{S} \tag{2.24}$$

will be canonical if [Moshinsky, 1973]

$$S K S^T = K \qquad (2.25)$$

where

$$K = \begin{bmatrix} 0 & I \\ -I & 0 \end{bmatrix}$$

and T signifies transpose.

The matrices S form a group which is the linear sympletic group $Sp(2n)$. It is shown by Moshinsky that if we write

$$S = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \qquad (2.26)$$

where A, B, C, D are $n \times n$ real matrices, then for S to be canonical we require

$$C B^T = D A^T - I \qquad (2.27a)$$

$$B A^T = A B^T \qquad (2.27b)$$

$$B^T D = D^T B \qquad (2.27c)$$

If we further restrict S to be orthogonal we require in addition, that

$$A = D \qquad (2.28a)$$

$$B = -C \qquad (2.28b)$$

$$(A^T - i B^T)(A + i B) = I \qquad (2.28c)$$

This subgroup of $Sp(2n)$ is the unitary group $U(n)$, it is the invariance group of the n dimensional isotropic harmonic oscillator with the Hamiltonian

$$H = \frac{\omega}{2} \hat{\underset{\sim}{z}} \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix} \hat{\underset{\sim}{z}}^T \tag{2.29}$$

where $\omega = \mu^2$, is the fundamental frequency.

The generators of $Sp(2n)$ are the following $n(2n+1)$ bilinear operators [Moshinsky, 1973]

$$H_i \equiv 1/2\,(a_i^\dagger a_i + a_i a_i^\dagger) \equiv C_{ii} + 1/2 \qquad (i = 1, \dots n) \tag{2.30a}$$

$$a_i^\dagger a_j \equiv C_{ij} \qquad (i \neq j\,;\, i,j = 1,\dots n) \tag{2.30b}$$

$$a_i^\dagger a_j^\dagger \;,\; a_i a_j \qquad (i \leq j\,;\, i = 1,\dots n) \tag{2.30c}$$

The n^2 operators C_{ij} are the generators of the orthogonal subgroup $U(n)$.

Let $U(\gamma_j)$ be the unitary operator which defines the linear sympletic transformation S by,

$$\hat{\underset{\sim}{z}}' \equiv U^\dagger \hat{\underset{\sim}{z}} U \equiv \hat{\underset{\sim}{z}} S \tag{2.31}$$

We will indicate this by $U \in Sp(2n)$.

We wish to find $U \in Sp(2n)$ such that the state $|\psi\rangle = U|0\rangle$ is a m.u.s. in $\hat{\underset{\sim}{z}}$. Since the elements of $Sp(2n)$ produce linear transformations only, the state $|\psi\rangle$ has the same mean values as the ground state, i.e. $\langle\psi|\hat{q}_i|\psi\rangle = \langle\psi|\hat{p}|\psi\rangle = 0$. The first minimum uncertainty condition, equation (2.5a) may then be written in vector form as,

$$(\hat{\underset{\sim}{X}}_1 + i\,\hat{\underset{\sim}{X}}_2 \Lambda)\,|\psi\rangle = 0 \tag{2.32}$$

where Λ is a possibly complex diagonal matrix; $\Lambda = Diag(\lambda_1, \dots, \lambda_n)$.

Using (2.32) the second minimum uncertainty condition (2.56) becomes

$$-i\,(\lambda_i - \lambda_i^*)\quad V(\hat{X}_2^i) \quad = \quad 0 \tag{2.33}$$

Since $\hat{X}_2^i$ is a self adjoint operator $V(\hat{X}_2^i) > 0$, thus equation (2.33) requires that Λ be a REAL diagonal matrix.

The m.u.s. are determined by the following theorem.

<u>THEOREM ONE</u>: The state $|\Psi\rangle = U|0\rangle$, $U \in Sp(2n)$ is a m.u.s. if and only if a real diagonal matrix Λ exists such that

$$\begin{aligned} A &= D\Lambda \\ C &= -B\Lambda \end{aligned} \tag{2.34}$$

where A, B, C, D are real $n \times n$ matrices defined in (2.26)

<u>PROOF</u>: Consider

$$|\varphi\rangle = (\hat{\underset{\sim}{X}}_1 + i\,\hat{\underset{\sim}{X}}_2\,\Lambda)\,|\Psi\rangle$$

where Λ is a real diagonal matrix. Thus

$$\begin{aligned} U|\varphi\rangle &= U^{\dagger}(\hat{X}_1 + i\,\hat{X}_2\Lambda)U\,|0\rangle \\ &= \{\hat{\underset{\sim}{X}}_1(A + iB\Lambda) + i\,\hat{\underset{\sim}{X}}_2(D\Lambda - iC)\}\,|0\rangle \\ &= \{\hat{X}_1 + i\,\hat{X}_2(D\Lambda - iC)(A + iB\Lambda)^{-1}\}|0\rangle\,(A + iB\Lambda) \end{aligned}$$

If Λ satisfies (2.34) we have

$$U|\varphi\rangle = (\hat{X}_1 + i\hat{X}_2)\,|0\rangle\,(A + iB\Lambda) = 0$$

$$\therefore \qquad (\hat{X}_1 + i\hat{X}_2\Lambda)\,|\Psi\rangle = 0$$

Thus since Λ is real both minimum uncertainty conditions are satisfied and $|\Psi\rangle$ is a m.u.s.

Conversely if $|\psi\rangle$ is a m.u.s.

$$(\hat{X}_1 + i \hat{X}_2 \Lambda) |\psi\rangle = 0$$

$$\therefore \quad \{ \hat{X}_1 + i \hat{X}_2 (D\Lambda - iC)(A + iB\Lambda)^{-1} \} |0\rangle = 0$$

which is true if

$$(D\Lambda - iC) = (A + iB\Lambda)$$

and thus equations (2.34) follow if Λ is real. Again since Λ is real both minimum uncertainty conditions are satisfied.

<u>COROLLARY 1:</u> If $|\psi\rangle$ is a m.u.s. for which $\Lambda = D_i (\lambda_1, \ldots \lambda_n)$ then

$$\left. \begin{aligned} V(\hat{X}_1^i , \psi) &= 1/4 \, . \, \lambda_i \\ V(\hat{X}_2^i \psi) &= 1/4 \, . \, \frac{1}{\lambda_i} \end{aligned} \right\} \tag{2.35}$$

<u>PROOF:</u> Since $|\psi\rangle$ is a m.u.s. state

$$(\hat{X}_1 + i \hat{X}_2 \Lambda) |\psi\rangle = 0$$

which may be written as

$$X_1^i \, |\psi\rangle = - i \lambda_i \, \hat{X}_2^i \, |\psi\rangle \tag{2.36}$$

Furthermore,

$$\langle \psi | \hat{X}_1^i \hat{X}_2^i + \hat{X}_2^i \hat{X}_1^i | \psi \rangle = 0 \tag{2.37}$$

Using the commutation relations on (2.37) we have

$$\langle \psi | \hat{X}_2^i \hat{X}_1^i | \psi \rangle = - i / 4$$

whereupon using (2.36) and noting that $|\psi\rangle$ has zero mean, equations (2.35) follow.

COROLLARY 2: If $|\psi\rangle = U|0\rangle$ is a m.u.s. for which $\Lambda = I$, $|\psi\rangle$ is a coherent state and $U \in U(n)$.

The first part clearly follows from corollary one and the definitions (2.15a) and (2.15b). To prove the second part we note that if $U \in U(n)$ the unitary subgroup, S is orthogonal and equations (2.28a,b,c) are satisfied. Thus $\Lambda = I$. To demonstrate this more clearly we define the covariance matrix Σ by,

$$\Sigma \equiv 1/2 \langle\psi| \hat{\underset{\sim}{z}}^T \cdot \hat{\underset{\sim}{z}} + (\hat{\underset{\sim}{z}}^T \cdot \hat{\underset{\sim}{z}})^T |\psi\rangle \tag{2.38}$$

then

$$\Sigma = 1/2 \; S \langle 0| \hat{\underset{\sim}{z}}^T \cdot \hat{\underset{\sim}{z}} + (\hat{\underset{\sim}{z}}^T \cdot \hat{\underset{\sim}{z}})^T |0\rangle S = \frac{\hbar}{2} S^T S$$

If S is orthogonal, $\Sigma = (\hbar/2) . I$ and $|\psi\rangle$, like the state $|0\rangle$ is coherent. We are thus led to the following theorem.

THEOREM TWO: $|\psi\rangle = U|\varphi\rangle$; $U \in U(n)$ is a m.u.s. if and only if $|\varphi\rangle$ is coherent, in which case $|\psi\rangle$ is itself coherent.

PROOF: Assume $|\varphi\rangle$ is coherent, then

$$(\hat{\underset{\sim}{X}}_1 + i \hat{\underset{\sim}{X}}_2) |\varphi\rangle = 0$$
$$U (\hat{\underset{\sim}{X}}_1 + i \hat{\underset{\sim}{X}}_2) U^{\dagger} |\psi\rangle = 0$$

Now $$U \hat{\underset{\sim}{z}} U^{\dagger} = \hat{\underset{\sim}{z}} S^{-1}$$

but since $U \in U(n)$, $S^{-1} = S^T$, then using equations (2.28a,b) we have

$$(\hat{\underset{\sim}{X}}_1 + i \hat{\underset{\sim}{X}}_2)|\psi\rangle = 0$$

thus $|\psi\rangle$ is a m.u.s. with $\Lambda = I$.

Conversely if $|\psi\rangle$ is a m.u.s.

$$(\hat{\underset{\sim}{X}}_1 + i\hat{\underset{\sim}{X}}_2\Lambda)|\psi\rangle = 0$$

$$U^{\dagger}(\hat{\underset{\sim}{X}}_1 + i\hat{\underset{\sim}{X}}_2\Lambda)U|\varphi\rangle = 0$$

from which it follows that $\Lambda = I$

and
$$(\hat{\underset{\sim}{X}}_1 + i\hat{\underset{\sim}{X}}_2)|\varphi\rangle = 0$$

Thus $|\varphi\rangle$ is coherent. Note that since $|0\rangle$ has zero complex amplitude the state $U|0\rangle$ with $U \in U(n)$ is equivalent to the vacuum state $|0\rangle$ in all its moments.

If $|\psi\rangle$ is a m.u.s. for which $\Lambda \neq I$ we define the state $|\psi\rangle$ to be a "Squeezed State". Squeezed states are minimum uncertainty states for which the fluctuations in a particular canonical variable are reduced below the fluctuations of that variable that would result were the system in a coherent state.

2.3: Phase Insensitive Noise

An insight into the nature of a squeezed state may by obtained by considering the phase dependence of the noise (i.e. fluctuations).

A state with phase insensitive noise may be defined as one in which the associated noise is randomly distributed in phase [Caves, 1982]. A more precise definition may be obtained as follows.

States with phase insensitive noise are defined as those states for which the covariance matrix Σ is proportional to the identity. This requires

$$V(\hat{X}_1^i, \psi) = V(\hat{X}_2^i, \psi) = \text{CONSTANT} \tag{2.39a}$$

and

$$\langle \psi | \hat{X}_1^i \hat{X}_2^i + \hat{X}_2^i \hat{X}_1^i | \psi \rangle = 0 \tag{2.39b}$$

Clearly all coherent states have phase insensitive noise. Furthermore coherent states remain coherent under unitary transformations which are elements of $U(n)$. Thus phase insensitive noise is an invariant property of the $U(n)$ subgroup of $Sp(2n)$.

It is then clear that all squeezed states will have phase sensitive noise.

It should be noted that a state with phase insensitive noise does not have to be a m.u.s. A number state, for example has phase insensitive noise (equation (1.46)) but is not a m.u.s.

The generator H_i (equation (2.30a)) is proportional to the Hamiltonian for a harmonic oscillator of frequency μ_i^2 . The unitary time evolution operator for this mode $U(t)$, given by

$$U(t) = \exp\left(-\frac{i}{\hbar}\hat{H}t\right)$$

(where $\hat{H} = \hbar\mu_i^2 H_i$), is in fact an element of the $U(n)$ subgroup of $Sp(2n)$. Then by theorem two we have that a harmonic oscillator initially in a coherent state will remain in a coherent state under free evolution. Furthermore since the noise phase insensitivity of coherent states is invariant under the action of elements of $U(n)$, the quantum fluctuations of a harmonic oscillator in a coherent state are time independent. We may then speak of the harmonic oscillator coherent states as having "time stationary noise" [Caves, 1982]. Clearly squeezed states do not have time stationary noise.

2.4 Single Mode Squeezed States

We now treat in some detail the m.u.s. of a one dimensional harmonic oscillator (i.e. a single mode).

The infinitesimal generators of $Sp(2)$ may be written in the following Hermitian form.

$$\hat{T}_1 = 1/2\,(a\,a^\dagger + a^\dagger a) = (\hat{X}_1^2 + \hat{X}_2^2) \tag{2.40a}$$

$$\hat{T}_2 = 1/2\,(a^2 + a^{\dagger 2}) = (\hat{X}_1^2 - \hat{X}_2^2) \tag{2.40b}$$

$$\hat{T}_3 = -i/2\,(a^2 - a^{\dagger 2}) = (\hat{X}_1\hat{X}_2 + \hat{X}_2\hat{X}_1) \tag{2.40c}$$

The generators form the lie algebra.

$$[\hat{T}_1, \hat{T}_2] = -2i\,\hat{T}_3 \tag{2.41a}$$

$$[\hat{T}_2, \hat{T}_3] = 2i\,\hat{T}_1 \tag{2.41b}$$

$$[\hat{T}_3, \hat{T}_1] = -2i\,\hat{T}_2 \tag{2.41c}$$

which is clearly non-compact, a manifestation of the unbounded nature of $\hat{X}_1$ and $\hat{X}_2$.

The following simple commutators are easily verified

$$[\hat{T}_1, \hat{X}_1] = -i\,\hat{X}_2 \tag{2.42a}$$

$$[\hat{T}_2, \hat{X}_1] = i\,\hat{X}_2 \tag{2.42b}$$

$$[\hat{T}_3, \hat{X}_1] = -i\,\hat{X}_1 \tag{2.42c}$$

$$[\hat{T}_1, \hat{X}_2] = i\,\hat{X}_1 \tag{2.42d}$$

$$[\hat{T}_2, \hat{X}_2] = i\,\hat{X}_1 \tag{2.42e}$$

$$[\hat{T}_3, \hat{X}_2] = i\,\hat{X}_2 \tag{2.42f}$$

The unitary operator elements of the corresponding lie group are defined by

$$U_i(\gamma_i) \equiv exp\,(-i\gamma_i \hat{T}_i) \qquad (\gamma_i \in \mathbb{R}) \tag{2.43}$$

The Stone-Von Neuman theorem shows that each $U_i(\theta)$ is itself an element of a continuous one parameter group and thus

$$U_i(\theta + \varphi) = U_i(\theta) \cdot U_i(\varphi) \tag{2.44a}$$

$$U_i^{-1}(\theta) = U(-\theta) = U_i^{\dagger}(\theta) \tag{2.44b}$$

Associated with each U_i we have the matrices S_i defined by

$$\begin{aligned} \hat{\underset{\sim}{z}}' &= U_i^{\dagger}\, \hat{\underset{\sim}{z}}\, U_i \\ &\equiv \hat{\underset{\sim}{z}}\, S_i \end{aligned} \tag{2.45}$$

Using the relation [Louisell, 1973]

$$\begin{aligned} & exp(\xi \hat{A})\, \hat{B}\, exp(-\xi \hat{A}) \\ & \qquad = \hat{B} + \xi[\hat{A}, \hat{B}] + \frac{\xi^2}{2}[\hat{A}, [\hat{A}, \hat{B}]] \end{aligned} \tag{2.46}$$

together with equations (2.42) we find

$$S_1(\gamma_1) = \begin{bmatrix} Cos\,\gamma_1 & -Sin\,\gamma_1 \\ Sin\,\gamma_1 & Cos\,\gamma_1 \end{bmatrix} \tag{2.47a}$$

$$S_2(\gamma_2) = \begin{bmatrix} Cosh\,\gamma_2 & -Sinh\,\gamma_2 \\ -Sinh\,\gamma_2 & Cosh\,\gamma_2 \end{bmatrix} \tag{2.47b}$$

$$S_3(\gamma_3) = \begin{bmatrix} e^{\gamma_3} & 0 \\ 0 & e^{-\gamma_3} \end{bmatrix} \tag{2.47c}$$

We note that in fact S_2 is not an independent transformation but may be written in terms of S_1 and S_3. As

$$S_2(r_2) = S_1(\pi/4)\, S_3(-r_2)\, S_1(-\pi/4) \tag{2.48}$$

Or equivalently,

$$U_2(r_2) = U_1(-\pi/4)\, U_3(-r_2)\, U_1(\pi/4) \tag{2.49}$$

Thus the most general linear canonical transformation in one dimension is produced by the unitary operator;

$$U(r, \theta, \varphi) = U_1(\theta) . U_3(r) . U_1(\varphi) \tag{2.50}$$

Clearly U_1 simply corresponds to rotations in the complex amplitude plane and gives the orthogonal subgroup of $Sp(2)$. In fact it is easily verified that S_1 satisfies equations (2.34) for $\Lambda = I$. By theorem two we then conclude that U_1 generates coherent states from coherent states and in particular the state $U_1|0\rangle$ is coherent.

The transformation produced by U_3 corresponds to a change of scale in the canonical coordinates that leaves the bi-linear form $\hat{X}_1 . \hat{X}_2$ invariant. S_3 satisfies equations (2.34) for

$$\Lambda = e^{2r_3} \tag{2.51}$$

and thus the state $U_3|0\rangle$ is a m.u.s. Then by corrolary one of theorem one the state $|\psi\rangle = U_3|0\rangle$ has the following variances in the canonical variables;

$$V(\hat{X}_1, \psi) = 1/4 . e^{2r_3}$$

$$V(\hat{X}_2, \psi) = 1/4\, e^{-2r_3} \tag{2.52}$$

The state $|\psi\rangle$ is then, by definition, a squeezed state. In figure 2.1a we have plotted the state $|\psi\rangle$ on a complex amplitude diagram. We see that the error volume is in fact an ellipse with its principal axes parallel to $\hat{X}_1$ and $\hat{X}_2$ axes. The projection of the error ellipse onto the $\hat{X}_1$ and $\hat{X}_2$ axes gives directly the variances in $\hat{X}_1$ and $\hat{X}_2$ respectively. The error ellipse itself is centered at the origin indicating that $|\psi\rangle$ is a zero amplitude state.

To produce a squeezed state with non-zero amplitude, we simply apply the translation operator $D(\alpha)$ to the state $|\psi\rangle$. Since $D(\alpha)$ cannot change the variances of the state $|\psi\rangle$ but only the mean values $\langle \hat{X}_1\rangle$ and $\langle \hat{X}_2\rangle$, it is clear that the resulting state $D(\alpha)|\psi\rangle$ is also a squeezed state. We may imagine that every point on and within the error ellipse originally centered at the origin is displaced a distance r at an angle θ, where $\alpha = r e^{i\theta}$, Figure (2.1b).

In summary, the single mode squeezed states of non-zero amplitude, designated $|\alpha; r\rangle$, are given by

$$|\alpha; r\rangle = D(\alpha)\, U_3(r)\, |0\rangle \tag{2.53}$$

where

$$\begin{aligned}
\langle \hat{X}_1\rangle &= \mathrm{Re}(\alpha) \\
\langle \hat{X}_2\rangle &= \mathrm{Im}(\alpha) \\
V(\hat{X}_1) &= 1/4 \,.\, e^{2r} \\
V(\hat{X}_2) &= 1/4 \,.\, e^{-2r}
\end{aligned}$$

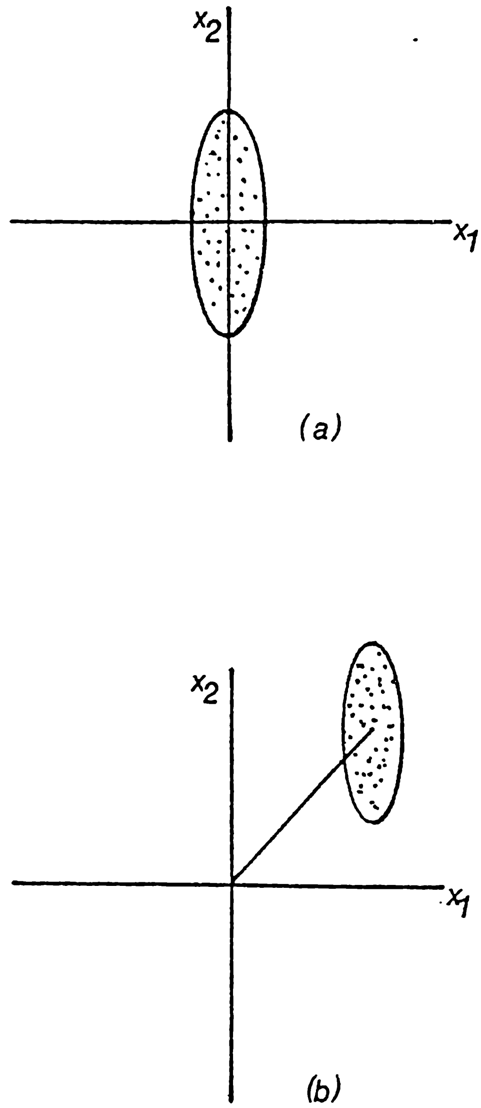

Figure 2.1 (a) Complex amplitude diagram for a Squeezed Vacuum state; (b) for a displaced squeezed state.

Let us now consider the state $|r\rangle = U_L(r)|0\rangle$. It is easily verified that there does not exist any real diagonal matrix that will satisfy (2.34) for $S_L(r)$. We conclude that $|r\rangle$ is not a m.u.s. in the chosen canonical coordinates.

In fact equation (2.49) provides the fundamental reason why $|\gamma\rangle$ fails to be a m.u.s. Using equation (2.49) we may write

$$
\begin{aligned}
|r\rangle &= U_1(-\pi/4)\, U_3(-r)\, U_1(\pi/4)\, |0\rangle \\
&= U_1(-\pi/4)\, |0;-r\rangle
\end{aligned}
\tag{2.54}
$$

We see that $|r\rangle$ is generated from a squeezed state, by an element of $U(n)$, the orthogonal subgroup of $Sp(2)$, and thus, by theorem two, fails to be a m.u.s.

Another illustration of why $|r\rangle$ fails to be a m.u.s. in the chosen variables is provided by reference to figure 2.2. Consider the vacuum state represented in figure 2.2 (a) We now rotate all the points in this frame by $-\pi/4$ by application of $U_1(-\pi/4)$ and obtain a new coordinate frame $\hat{X}_1'$, $\hat{X}_2'$ (b) The error circle, of course, remains a circle. We now "squeeze" the error circle along principal axes which are parallel to the original coordinate frame, by an application of $U_3(-r)$ (the scale changing transformation). (c) (Of course in the "squeezed" coordinate frame itself the circle is still a circle, but we are not interested in the variances of these variables). Finally, we rotate all the points in the plane through $\pi/4$ with respect to the original frame, by the application of $U_1(\pi/4)$ (d) We see that we are left with an error ellipse whose principal axes no longer coincide with the coordinate axes, and thus the final variances in $\hat{X}_1$ and $\hat{X}_2$ are greater than for the vacuum state.

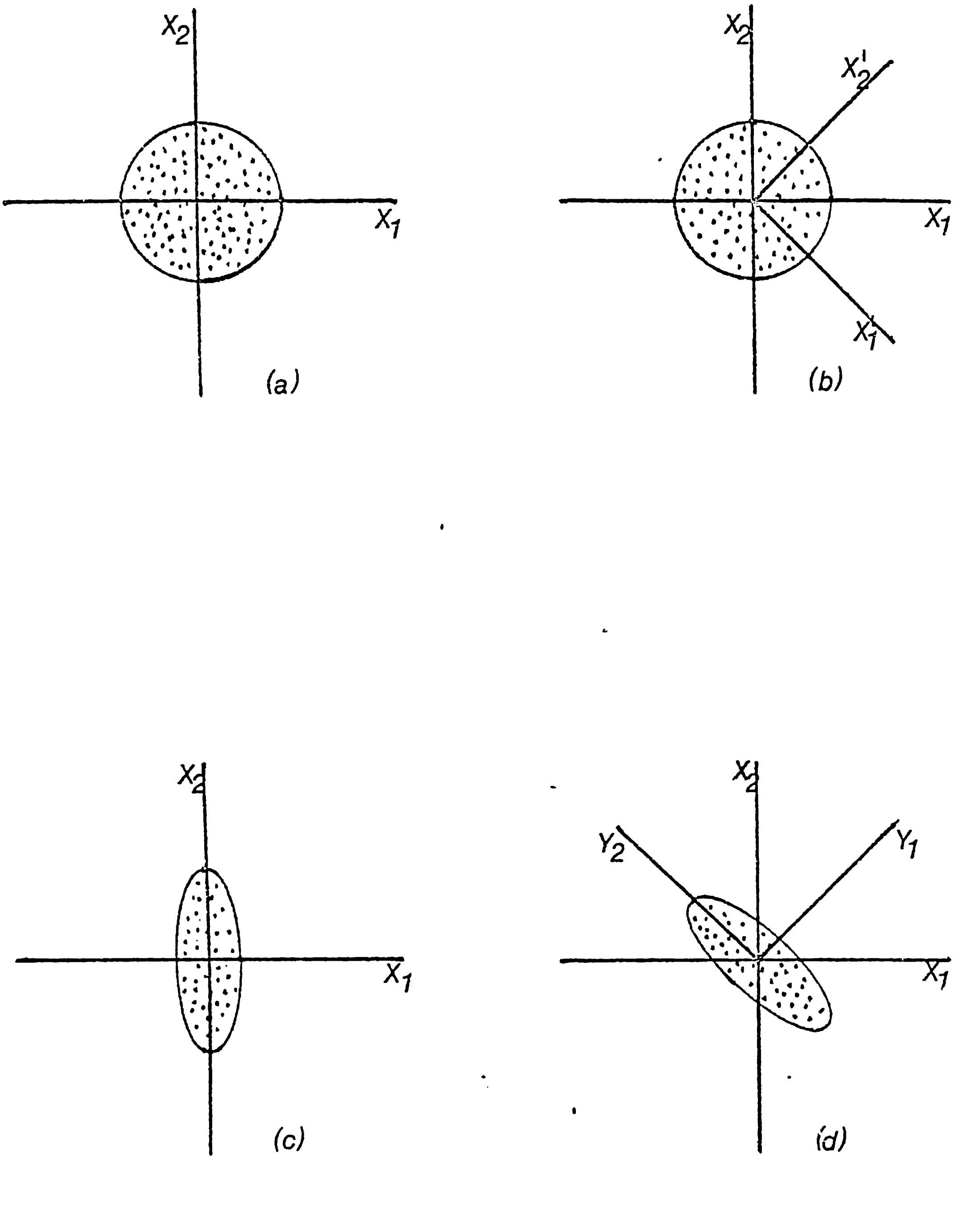

Figure 2.2: A pictorial representation of the action of the $Sp(2)$ group element $U_2(\gamma)$ on the vacuum state. See text for details.

The above geometrical construction shows clearly that the state $|\gamma\rangle$ is a m.u.s. and in fact a squeezed state in canonical variables other than those we were initially interested in. These new variables are a simple linear combination of the original variables, as is suggested by equation (2.48).

Consider the variables $\hat{\underset{\sim}{y}}$ defined by

$$\hat{\underset{\sim}{y}} \equiv \sqrt{2\hbar}\,(\hat{Y}_1, \hat{Y}_2) = U_1^{\dagger}(\pi/4)\,\hat{\underset{\sim}{z}}\,U_1(\pi/4) = \hat{\underset{\sim}{z}}\,S_1(\pi/4) \tag{2.55}$$

We now evaluate the covariance matrix Σ for $\hat{\underset{\sim}{y}}$ in the state $|\gamma\rangle$ given by equation (2.54).

$$\Sigma = 1/2 \langle 0|\, U_2^{T}(\gamma_2)\,(\hat{\underset{\sim}{y}}^{T}.\hat{\underset{\sim}{y}} + (\hat{\underset{\sim}{y}}^{T}.\hat{\underset{\sim}{y}})^{T})\,U_2(\gamma_2)\,|0\rangle$$

$$(2.56)$$

$$= 1/2 \langle 0|\, U_1^{T}(\pi/4)\, U_3^{\dagger}(-\gamma_2)\, U_1^{\dagger}(-\pi/4) \left\{ \hat{\underset{\sim}{y}}^{T}\hat{\underset{\sim}{y}} + (\hat{\underset{\sim}{y}}^{T}\hat{\underset{\sim}{y}})^{T} \right\} U_1(-\pi/4)\, U_3(-\gamma_2)\, U_1(\pi/4)\, |0\rangle$$

$$= 1/2 \langle 0|\, U_1^{T}(\pi/4)\, U_3^{T}(-\gamma_2) \left\{ \hat{\underset{\sim}{z}}^{T}.\hat{\underset{\sim}{z}} + (\hat{\underset{\sim}{z}}^{T}.\hat{\underset{\sim}{z}})^{T} \right\} U_3(-\gamma_2)\, U_1(\pi/4)\, |0\rangle$$

where we have used equation (2.55) together with $U_1(\pi/4)\, U_1(-\pi/4) = I$

Then we have

$$\Sigma = 1/2 .\, S_3^{T}(-\gamma_2)\, S_1^{T}(\pi/4) \langle 0| \left\{ \hat{\underset{\sim}{z}}^{T}.\hat{\underset{\sim}{z}} + (\hat{\underset{\sim}{z}}^{T}.\hat{\underset{\sim}{z}})^{T} \right\} |0\rangle\, S_1(\pi/4)\, S_3(-\gamma_2)$$

$$= \hbar/2 \quad S_3^{T}(-\gamma_2) . S_3(-\gamma_2)$$

$$= \frac{\hbar}{2} \begin{pmatrix} e^{-2\gamma_2} & 0 \\ 0 & e^{2\gamma_2} \end{pmatrix} \tag{2.57}$$

Thus

$$V(\hat{Y}_1) = 1/4 .\, e^{-2\gamma_2}$$

$$(2.58)$$

$$V(\hat{Y}_2) = 1/4 .\, e^{2\gamma_2}$$

We conclude that the state $U_2(\gamma_2)|0\rangle$ is a squeezed state in the variables $\hat{Y}_1$ and $\hat{Y}_2$ where (from equation (2.55))

$$\hat{Y}_1 = \frac{1}{\sqrt{2}} (\hat{X}_1 + \hat{X}_2)$$

$$\hat{Y}_2 = \frac{1}{\sqrt{2}} . (\hat{X}_2 - \hat{X}_1) \tag{2.59}$$

This may be conveniently written as

$$(\hat{Y}_1 + i \hat{Y}_2) = (\hat{X}_1 + i \hat{X}_2) e^{-i\pi/4} . \tag{2.60}$$

which clearly indicates that the $\hat{Y}_1$, $\hat{Y}_2$ complex amplitude diagram is rotated by $\pi/4$ with respect to the $\hat{X}_1$, $\hat{X}_2$ diagram. See figure (2.3a).

If we define bose operators in the ($\hat{Y}_1$, $\hat{Y}_2$) frame by

$$b = (\hat{Y}_1 + i \hat{Y}_2)$$

$$b^{\dagger} = (\hat{Y}_1 - i \hat{Y}_2) \tag{2.61}$$

we have that

$$b = a . e^{-i\pi/4} \tag{2.62}$$

Using equation (6.2) we may write the generator $\hat{T}_2$ given by equation (2.40b) in terms of $b, b^{\dagger}$, then

$$\hat{T}_2 = i/2 (b^2 - b^{\dagger 2})$$

which is clearly proportional to the generator $\hat{T}_3{}'$ of the scale changing transformation in the new frame.

The state $U_2(\gamma_2)|0\rangle$ is a zero amplitude state in both coordinate frames. We may obtain a state with non-zero amplitude by application of the displacement operator. With respect to the ($\hat{X}_1$, $\hat{X}_2$) frame this takes the form

$$D(\alpha) = \exp\{\alpha a^{\dagger} - \alpha^{*} a\}$$

Then, using (2.62) we have

$$D(\alpha) = \exp\{\beta b^{\dagger} - \beta^{*} b\}$$

where

$$\beta = \alpha\, e^{-i\pi/4}$$

Thus the state $|\psi\rangle = D(\alpha)\, U_2(r)\, |0\rangle$ has average complex amplitudes given by

$$\langle \hat{X}_1 + i\hat{X}_2 \rangle = \langle \hat{Y}_1 + i\hat{Y}_2 \rangle\, e^{i\pi/4} = \alpha \qquad (2.63)$$

and variances

$$V(\hat{Y}_1, \psi) = 1/4 .\, e^{-2r}$$

$$(2.64)$$

$$V(\hat{Y}_2, \psi) = 1/4 .\, e^{2r}$$

The preceeding analysis admits an obvious extension to states produced by the general unitary operator of equation (2.50). Thus the state $|\psi\rangle$ given by

$$|\psi\rangle = D(\alpha)\, U_1(-\theta)\, U_3(r)\, U_1(\varphi)\, |0\rangle \qquad (2.65)$$

is a m.u.s. squeezed state in the variables $\hat{\underset{\sim}{y}}$ where

$$\hat{\underset{\sim}{y}} = U_1^{\dagger}(\theta)\, \hat{\underset{\sim}{z}}\, U_1(\theta)$$
$$= \hat{\underset{\sim}{z}} . S_1(\theta)$$

which may be written more conveniently as

$$(\hat{Y}_1 + i\hat{Y}_2) = (\hat{X}_1 + i\hat{X}_2)\, e^{-i\theta} \qquad (2.66)$$

$$\text{i.e.} \qquad b = a\, e^{-i\theta}$$

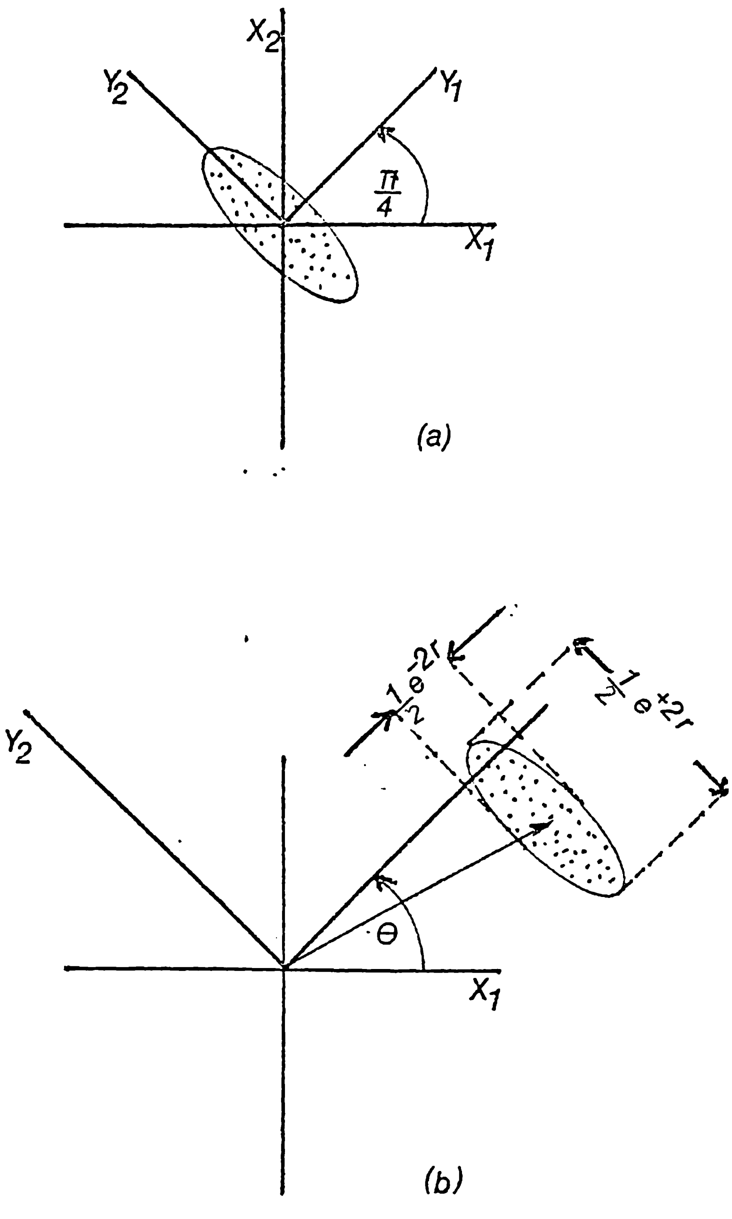

Figure 2.3 Complex amplitude diagram for a rotated squeezed state (a) and a squeezed state with the direction of squeezing out of phase with the coherent excitation(b).

Since $|\psi\rangle$ is squeezed directly in the b mode variables, it is equivalently given by

$$|\psi\rangle = \bar{D}(\beta).\ \bar{U}_3(r)\,|0\rangle \qquad (2.69)$$

where

$$\bar{D}(\beta) = \exp(\beta b^{\dagger} - \beta^* b) \qquad (2.71)$$

$$\bar{U}_3(r) = \exp\left(-r/2\ (b^2 - b^{\dagger 2})\right) \qquad (2.72)$$

and $\beta = \alpha\, e^{-i\theta}$

Then by using equation (2.66) we may write (2.72) as

$$\bar{U}_3(-r) \equiv \hat{S}(\xi) = \exp\left(1/2\ (\xi a^2 - \xi a^{\dagger 2})\right) \qquad (2.73)$$

where

$$\xi = -r\, e^{2i\theta} \qquad (2.74)$$

Finally then we have that the state $|\psi\rangle$ is equivalent to the state $|\alpha, \xi\rangle$ where

$$|\alpha, \xi\rangle = D(\alpha)\ \hat{S}(\xi)\,|0\rangle \quad ; \quad \xi = -r\, e^{2i\theta} \qquad (2.75)$$

and

$$\langle \hat{X}_1 + i\hat{X}_2 \rangle = \langle \hat{Y}_1 + i\,\hat{Y}_2 \rangle e^{i\theta} = \alpha \qquad (2.76)$$

$$V(\hat{Y}_1) = 1/4.\ e^{2r} \qquad (2.77a)$$

$$V(\hat{Y}_2) = 1/4.\ e^{-2r} \qquad (2.77b)$$

The state $|\alpha, \xi\rangle$ is the general single mode squeezed state defined by Caves [1981d]. We see it is generated from the vacuum by the general unitary operator representation of $Sp(2)$. The operator $\hat{S}(\xi)$ is known as the "squeeze operator" [Caves, 1981d].

We conclude that the most general single mode squeezed states, for a given set of canonical variables are generated from the vacuum by the unitary operator corresponding to complementary scale changes in those variables. (see figure 2.3(b)).

The transformations produced by $\hat{S}(\xi)$ in the bose variables are given by

$$\hat{S}^{\dagger}(\xi)\; a\; \hat{S}(\xi) \;=\; a \operatorname{Cosh} r \;+\; a^{\dagger} e^{2i\theta} \operatorname{Sinh} r \qquad (2.78a)$$

$$\hat{S}^{\dagger}(\xi)\; a\; \hat{S}(\xi) \;=\; a^{\dagger} \operatorname{Cosh} r \;+\; a\, e^{-2i\theta} \operatorname{Sinh} r \qquad (2.78b)$$

Single mode squeezed states have been considered by a number of authors under a variety of names. Stoler [1970, 1971] first introduced them under the name of "minimum uncertainty packets". They have also been considered by Lu [1972] and Yuen [1975] who referred to them as "Two photon coherent states (TPCS). (Also see Canivell and Segler, 1977).

While the family of TPCS is equivalent to the family of squeezed states, the precise definition of a TPCS is somewhat different to the definition of squeezed states given by equation (2.75).

A TPCS is defined by

$$|\beta\rangle_g \;\equiv\; U_L(\mu,\nu)\; D(\beta)\; |0\rangle \qquad (2.79)$$

where

$$D(\beta) \;=\; \exp(\beta a^{\dagger} - \beta^{*} a) \qquad (2.80)$$

and U_L is defined by

$$U_L\, a\, U_L^{\dagger} \;=\; \mu a \;+\; \nu a^{\dagger} \;\equiv\; b \qquad (2.81)$$

Comparing equations (2.81) and (2.78a) we see that $U_L(\mu,\nu) \equiv S(-\xi)$ where $\xi = r\, e^{i\theta}$ and

$$\mu = \cosh r \quad ; \quad \nu = e^{i\theta} \sinh r \tag{2.82}$$

Thus a TPCS is obtained by displacing the vacuum, then applying the squeeze operator. This is the opposite procedure for defining a squeezed, but is clearly equivalent on geometrical grounds.

The formal identification of the parametrization of the squeezed state $|\alpha, \xi\rangle$ and the TPCS $|\beta\rangle_g$ may be found by writing (2.79) as

$$|\beta\rangle_g = U_L(\mu,\nu)\, D(\beta)\, U_L^{\dagger}(\mu,\nu)\, U_L(\mu,\nu)\, |0\rangle$$

$$= \bar{D}(\beta)\, \hat{S}(-\xi)\, |0\rangle \tag{2.82}$$

whereupon using equations (2.81) we have

$$|\beta\rangle_g = D(\alpha)\, S(-\xi)\, |0\rangle = |\alpha ; -\xi\rangle \tag{2.83}$$

where

$$\alpha = \beta\mu - \beta^{*}\nu \tag{2.84}$$

We shall discuss the properties of single mode squeezed states in more detail in Chapter three.

2.5 Two Mode Squeezed States

We now proceed to a generalization of the one mode m.u.s. to the two mode case using the representations of $Sp(4)$.

A set of states for the two dimensional harmonic oscillator using the semi-direct product $N(2) \circledS Sp(4)$ of the Heisenberg and Sympletic groups have been discussed by Gulshani and Volkov [1980]. However the states obtained were not restricted to be m.u.s. A comparison between the states of Gulshani and Volkov and the m.u.s. obtained in this section may be found in Milburn [1982].

The ten generators of $Sp(4)$ may be written in the form [Milburn, 1982],

$$\hat{T}_1 = 1/2(aa^\dagger + a^\dagger a) = (\hat{X}_1^2 + \hat{X}_2^2) \tag{2.85a}$$

$$\hat{T}_2 = 1/2(bb^\dagger + b^\dagger b) = (\hat{Y}_1^2 + \hat{Y}_2^2) \tag{2.85b}$$

$$\hat{T}_3 = (a^\dagger b + b^\dagger a) = 2(\hat{X}_1 \hat{Y}_1 + \hat{X}_2 \hat{Y}_2) \tag{2.85c}$$

$$\hat{T}_4 = -i(a^\dagger b - b^\dagger a) = 2(\hat{X}_1 \hat{Y}_2 - \hat{X}_2 \hat{Y}_1) \tag{2.85d}$$

$$\hat{T}_5 = (ab + a^\dagger b^\dagger) = 2(\hat{X}_1 \hat{Y}_1 - \hat{X}_2 \hat{Y}_2) \tag{2.85e}$$

$$\hat{T}_6 = -i(ab - a^\dagger b^\dagger) = 2(\hat{X}_2 \hat{Y}_1 + \hat{X}_1 \hat{Y}_2) \tag{2.85f}$$

$$\hat{T}_7 = 1/2(a^2 + a^{\dagger 2}) = (\hat{X}_1^2 - \hat{X}_2^2) \tag{2.85g}$$

$$\hat{T}_8 = -i/2(a^2 - a^{\dagger 2}) = (\hat{X}_1 \hat{X}_2 + \hat{X}_2 \hat{X}_1) \tag{2.85h}$$

$$\hat{T}_9 = 1/2(b^2 + b^{\dagger 2}) = (\hat{Y}_1^2 - \hat{Y}_2^2) \tag{2.85i}$$

$$\hat{T}_{10} = -i/2(b^2 - b^{\dagger 2}) = (\hat{Y}_1 \hat{Y}_2 + \hat{Y}_2 \hat{Y}_1) \tag{2.85j}$$

where a , b are the bose operators for the two modes.

The generators $\hat{T}_1$ to $\hat{T}_4$ form a closed lie algebra and are the generators of the $U(n)$ subgroup of $Sp(4)$. In particular if $\mu_1 = \mu_2$.

$$\hat{T}_4 = \hat{L}_3 = 1/\hbar\,(\hat{q}_1 \hat{p}_2 - \hat{p}_1 \hat{q}_2)$$

where $\hat{L}_3$ is the generator for $SO(2)$, the group of rotations in the two dimensional plane. When $\mu_1 = \mu_2$ the operators a, b may be envisaged as the annihilation operators for a two dimensional isotropic oscillator, which clearly has $SO(2)$ invariance.

The generators $\hat{T}_1$ and $\hat{T}_2$ are proportional to the free Hamiltonians for two harmonic oscillators of frequencies μ_1^2 and μ_2^2 respectively. As for the one dimensional case $\hat{T}_1$ and $\hat{T}_2$ are generators of rotations in the respective complex amplitude planes for the two modes.

$\hat{T}_8$ and $\hat{T}_{10}$ are the generators of the scale changing transformations in the two modes. The operators $\hat{T}_7$ and $\hat{T}_9$ do not generate independent transformations, as was discussed in the one dimensional case, but may be decomposed into scale changes and complex amplitude rotations for each mode.

The unitary operators generated by each $\hat{T}_i$ are defined by

$$U_i(\gamma_i) = \exp(-i\gamma_i \hat{T}_i) \tag{2.86}$$

The corresponding matrix transformations are

$$S_1 = \begin{pmatrix} \cos\gamma_1 & 0 & -\sin\gamma_1 & 0 \\ 0 & 1 & 0 & 0 \\ \sin\gamma_1 & 0 & \cos\gamma_1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{2.87a}$$

$$S_2 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\gamma_2 & 0 & -\sin\gamma_2 \\ 0 & 0 & 1 & 0 \\ 0 & \sin\gamma_2 & 0 & \cos\gamma_2 \end{pmatrix} \tag{2.87b}$$

$$S_3 = \begin{pmatrix} \cos\gamma_3 & 0 & 0 & -\sin\gamma_3 \\ 0 & \cos\gamma_3 & -\sin\gamma_3 & 0 \\ 0 & \sin\gamma_3 & \cos\gamma_3 & 0 \\ \sin\gamma_3 & 0 & 0 & \cos\gamma_3 \end{pmatrix} \tag{2.87c}$$

$$S_4 = \begin{pmatrix} \cos\gamma_4 & \sin\gamma_4 & 0 & 0 \\ -\sin\gamma_4 & \cos\gamma_4 & 0 & 0 \\ 0 & 0 & \cos\gamma_4 & \sin\gamma_4 \\ 0 & 0 & -\sin\gamma_4 & \cos\gamma_4 \end{pmatrix} \tag{2.87d}$$

$$S_5 = \begin{pmatrix} \cosh\gamma_5 & 0 & 0 & -\sinh\gamma_5 \\ 0 & \cosh\gamma_5 & -\sinh\gamma_5 & 0 \\ 0 & -\sinh\gamma_5 & \cosh\gamma_5 & 0 \\ -\sinh\gamma_5 & 0 & 0 & \cosh\gamma_5 \end{pmatrix} \tag{2.87e}$$

$$S_6 = \begin{pmatrix} \cosh\gamma_6 & \sinh\gamma_6 & 0 & 0 \\ \sinh\gamma_6 & \cosh\gamma_6 & 0 & 0 \\ 0 & 0 & \cosh\gamma_6 & -\sinh\gamma_6 \\ 0 & 0 & -\sinh\gamma_6 & \cosh\gamma_6 \end{pmatrix} \qquad (2.87f)$$

$$S_7 = \begin{pmatrix} \cosh\gamma_7 & 0 & -\sinh\gamma_7 & 0 \\ 0 & 1 & 0 & 0 \\ -\sinh\gamma_7 & 0 & \cosh\gamma_7 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \qquad (2.87g)$$

$$S_8 = \begin{pmatrix} e^{\gamma_8} & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & e^{-\gamma_8} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \qquad (2.87h)$$

$$S_9 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cosh\gamma_9 & 0 & -\sinh\gamma_9 \\ 0 & 0 & 1 & 0 \\ 0 & -\sinh\gamma_9 & 0 & \cosh\gamma_9 \end{pmatrix} \qquad (2.87i)$$

$$S_{10} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & e^{\gamma_{10}} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & e^{\gamma_{10}} \end{pmatrix} \tag{2.87j}$$

It is then easily verified that

$$S_7 = S_1(\pi/4) . S_8(-\gamma_7) . S_1(-\pi/4) \tag{2.88a}$$

$$S_9 = S_2(\pi/4) . S_{10}(-\gamma_9) . S_2(-\pi/4) \tag{2.89a}$$

$$S_5 = S_3(\pi/4) . S_8(-\gamma_5) . S_{10}(-\gamma_5) . S_3(-\pi/4) \tag{2.90a}$$

$$S_6 = S_3(\pi/4) . S_7(\gamma_6) . S_9(\gamma_6) . S_3(-\pi/4) \tag{2.91a}$$

$$S_4 = S_3(-\pi/4) . S_2(\gamma_4) . S_1(-\gamma_4) . S_3(\pi/4) \tag{2.92a}$$

or

$$U_7 = U_1(-\pi/4) . U_8(-\gamma_7) . U_1(\pi/4) \tag{2.88a}$$

$$U_9 = U_2(-\pi/4) . U_{10}(-\gamma_9) . U_2(\pi/4) \tag{2.89b}$$

$$U_5 = U_3(-\pi/4) . U_{10}(-\gamma_5) . U_8(-\gamma_5)\ U_3(\pi/4) \tag{2.90b}$$

$$U_6 = U_3(-\pi/4) . U_9(\gamma_6) . U_7(\gamma_6) . U_3(\pi/4) \tag{2.91b}$$

$$U_4 = U_3(\pi/4) . U_2(\gamma_4) . U_1(-\gamma_4) . U_3(-\pi/4). \tag{2.92b}$$

The most general unitary operator of $Sp(4)$ is clearly quite involved, we shall restrict our discussion to the simpler important cases.

The operators U_1 through to U_4 are elements of the orthogonal subgroup of $Sp(4)$ and thus by theorem two generate coherent states.

Consider the operator

$$U(r_1, r_2) = U_8(r_1) \cdot U_{10}(r_2) \tag{2.93}$$

which defines the following transformation matrix

$$S(r_1, r_2) = \begin{pmatrix} e^{r_1} & 0 & 0 & 0 \\ 0 & e^{r_2} & 0 & 0 \\ 0 & 0 & e^{-r_1} & 0 \\ 0 & 0 & 0 & e^{-r_2} \end{pmatrix} \tag{2.94}$$

$S(r_1, r_2)$ clearly satisfies the minimum uncertainty conditions for the real diagonal matrix Λ .

$$\Lambda = \begin{pmatrix} e^{2r_1} & 0 \\ 0 & e^{2r_2} \end{pmatrix} \tag{2.95}$$

Then by corollary one of theorem one, the state $|\psi\rangle = U(r_1\ r_2)|0\rangle$ has the following quadrature phase variances,

$$\begin{aligned} V(\hat{X}_1) &= 1/4 \, . \, e^{2r_1} \\ V(\hat{X}_2) &= 1/4 \, . \, e^{-2r_1} \\ V(\hat{Y}_1) &= 1/4 \, . \, e^{2r_2} \\ V(\hat{Y}_2) &= 1/4 \, . \, e^{-2r_2} \end{aligned} \tag{2.96}$$

The state $|\psi\rangle$ is thus a squeezed state. Clearly

$$|\psi\rangle = |0;r_1\rangle . |0;r_2\rangle \equiv |0,0;r_1,r_2\rangle \tag{2.97}$$

that is the two mode squeezed state $|\Psi\rangle$ is the direct product of two single mode squeezed states. We may then define the two mode squeezed state with non-zero amplitudes by

$$
\begin{aligned}
|\alpha_1, \alpha_2 ; r_1, r_2\rangle &\equiv D(\alpha_1) . D(\alpha_2) \, |0, 0 ; r_1, r_2\rangle \\
&= |\alpha_1 ; r_1\rangle . |\alpha_2 , r_2\rangle
\end{aligned}
\tag{2.98}
$$

where $D(\alpha_1)$ ($D(\alpha_2)$) is the displacement operator for the a (b) mode, and $|\alpha_i ; r_i\rangle$ are defined in equation (2.53).

The operators U_5 , U_6 , U_7 , U_9 do not generate m.u.s. from the vacuum since there does not exist any real Λ to satisfy the minimum uncertainty conditions (2.34). For the corresponding matrix representations of each of these operators.

However, as we said in the single mode case, there do exist variables which are linear functions of the original canonical co-ordinates, which do carry reduced fluctuations in these states.

For the state $U_7 . U_9 |0\rangle$ the variables which carry the reduced fluctuations are given by

$$
\hat{\underset{\sim}{z}}' = \hat{\underset{\sim}{z}} \; S_7(\pi/4) . \; S_2(\pi/4) \tag{2.99}
$$

which is clearly a simple extension of the single mode case (equation (2.55)).

The state $U_5 |0\rangle$ carries reduced fluctuations in the variables

$$
\hat{\underset{\sim}{z}}' = \hat{\underset{\sim}{z}} \; . \; S_3(\pi/4) \tag{2.98}
$$

as can be seen from (2.90b). This transformation may be written as,

$$
\hat{X}_1' = \frac{1}{\sqrt{2}} . (\hat{X}_1 + \hat{Y}_2) \tag{2.99a}
$$

$$
\hat{X}_2' = \frac{1}{\sqrt{2}} . (- \hat{Y}_1 + \hat{X}_2) \tag{2.99b}
$$

$$\hat{Y}_1' = \frac{1}{\sqrt{2}} (\hat{Y}_1 + \hat{X}_2) \qquad (2.99c)$$

$$\hat{Y}_2' = \frac{1}{\sqrt{2}} \cdot (-\hat{X}_1 + \hat{Y}_2) \qquad (2.99d)$$

We define new boson variables in the transformed frame by

$$c \equiv (\hat{X}_1' + i\hat{X}_2')$$
$$d = (\hat{Y}_1' + i\hat{Y}_2')$$

then

$$c = \frac{1}{\sqrt{2}} (a - ib) \qquad (2.100a)$$

$$d = \frac{1}{\sqrt{2}} \cdot (b - ia) \qquad (2.100b)$$

In the new variables the generator T_5 becomes,

$$i/2 \, (d^2 - d^{\dagger 2} + c^2 - c^{\dagger 2})$$

which is clearly proportional to the generator of the scale changing transformation in the new variables.

Similarly the state $U_6|0\rangle$ carries reduced fluctuations in the variables

$$\hat{\underline{z}}' = \hat{\underline{z}} \cdot S_3(\pi/4) \cdot S_1(\pi/4) \, S_2(\pi/4) \qquad (2.102)$$

The new variables define the new boson operators c , d which may be written in terms of a , b as

$$c = \frac{1}{\sqrt{2}} \cdot (a e^{-i\pi/4} - b e^{i\pi/4})$$

$$(2.101)$$

$$d = \frac{1}{\sqrt{2}} (b e^{-i\pi/4} - a e^{i\pi/4})$$

or

$$a = \frac{1}{\sqrt{2}} . (c e^{i\pi/4} - d e^{-i\pi/4})$$

$$b = \frac{1}{\sqrt{2}} (d e^{i\pi/4} - c e^{-i\pi/4})$$

In the new variables the generator $\hat{T}_6$ becomes

$$i/2 . (c^2 - c^{+2} + d - d^{+2})$$

which is once again proportional to the generator of scale transformations in the new variables.

C.M. Caves [1982] has defined a set of states generated from the vacuum by an operator which in the two mode case becomes

$$S(\xi) \equiv exp (- \xi a^{+} b^{+} + \xi^{*} a b) \tag{2.103}$$

Using the generators of $Sp(4)$ this may be written as

$$S(\xi) = exp \Big(i (p_1 \hat{T}_5 + p_2 \hat{T}_6) \Big) \tag{2.104}$$

where

$$p_1 = - \mathscr{I}m(\xi)$$

$$p_2 = \mathscr{R}e(\xi)$$

Clearly $S(\xi)$ will not generate m.u.s. in the modes a , b . However there are linear combinations of these modes which do carry reduced fluctuations.

We first note that

$$S(\xi) = R^{+}(\theta) \; U_6(r) . R(\theta) \tag{2.105}$$

where $\xi = r e^{2i\theta}$ and

$$R(\theta) = U_1(\theta) . U_2(\theta) \tag{2.106}$$

Then using (2.91b) together with (2.105) and (2.106) we find that

$$S(\xi) = R^T(\theta)\, U_3(-\pi/4).\, R(-\pi/4).\, U_8(r).\, U_{10}(r).\, R(\pi/4).\, U_3(\pi/4).\, R(\theta) \qquad (2.107)$$

thus the state $S(\xi)|0\rangle$ carries reduced fluctuations in the variables $\hat{\underset{\sim}{z}}'$ where

$$\hat{\underset{\sim}{z}}' = \hat{\underset{\sim}{z}}\, S_1(\theta)\, S_2(\theta)\, S_3(\pi/4)\, S_1(\pi/4)\, S_2(\pi/4) \qquad (2.108)$$

Once again, we define boson operators c , d in the new variables and thus

$$c = 1/2\,(a\, e^{-i(\pi/4+\theta)} - b\, e^{i(\pi/4-\theta)})$$

$$d = 1/2\,(b\, e^{-i(\pi/4+\theta)} - a\, e^{i(\pi/4-\theta)}) \qquad (2.109)$$

or

$$a = (c\, e^{i(\pi/4+\theta)} - d\, e^{-i(\pi/4-\theta)})$$

$$(2.110)$$

$$b = (d\, e^{i(\pi/4+\theta)} - c\, e^{-i(\pi/4-\theta)})$$

In the new bose variables $S(\xi)$ may be written

$$S(\xi) = \exp\left\{ -ir[-i/2(c^2 - c^{\dagger 2} + d^2 - d^{\dagger 2})]\right\} \qquad (2.111)$$

which is clearly the unitary scale changing operator in the new variables.

We conclude that the two mode squeezed states for a given set of canonical variables are generated from the vacuum by the unitary operator effecting scale transformations in those canonical variables. Since each mode transforms independently under such transformations ($[\hat{T}_8, \hat{T}_{10}] = 0$) , the general two mode squeezed states are given by the direct product of the squeezed states for each mode independently.

CHAPTER 3

SQUEEZED STATES IN QUANTUM OPTICS

In the previous chapter a detailed discussion of the definition of squeezed states in terms of the representations of the symplectic group, $Sp(2n)$, was presented. In explaining the abstract structure of squeezed states, we have neglected some of their more detailed properties. It will be the object of this chapter to collect under one heading such properties.

The emphasis in this chapter will shift from the mathematics of squeezed states to a more physical description within the context of quantum optics.

The prototypical quantum optical measurement is photon counting, which further defines the particular moments which are of interest. We are thus lead to consider such things as the mean photon number and $g^2(0)$ for an optical field in a squeezed state.

The quadrature phase variables themselves, so important to an understanding of squeezed states, must, at least in simple measurement schemes, be related back to photon counting experiments.

One question of particular importance concerns the possible light mode interactions which will generate squeezed states. We touch upon this question only briefly in this chapter and reserve a detailed discussion of particular squeezed state generation schemes to the next chapter.

3.1 Squeezed States and Photon Antibunching

Consider the squeezed state $|\alpha; r\rangle$ defined by $|\alpha, r\rangle = D(\alpha) S(r) |0\rangle$. There is no loss of generality here as the phase of α is arbitrary and we are free to align our canonical coordinate frame with the principal axes of the error ellipse (note, $r > 0$) (see figure 3.1).

Unless otherwise stated, we will adopt the "phase" convention that the minor axis of the ellipse is parallel to the $\hat{X}_1$ direction. Thus

$$r \geq 0 \qquad\qquad 0 \leq \theta \leq \pi/2$$

We immediately have the following moments for the state depicted in figure 3.1.

$$\langle \hat{X}_1 \rangle = \mathrm{Re}(\alpha) \tag{3.1}$$

$$\langle \hat{X}_2 \rangle = \mathrm{Im}(\alpha) \tag{3.2}$$

$$V(\hat{X}_1) = 1/4 . e^{-2r} \tag{3.3}$$

$$V(\hat{X}_2) = 1/4 . e^{2r} \tag{3.4}$$

$$\left.\begin{aligned} \langle b^2 \rangle &= \alpha^2 - \mu . \nu . \\ \langle b^{\dagger 2} b \rangle &= -\alpha \mu \nu + 2\alpha^* \nu^2 + \alpha^* \alpha^2 \\ \langle b^{\dagger 2} b^2 \rangle &= 2\nu^4 + \mu^2 \nu^2 - \mu \nu (\alpha^2 + \alpha^{*2}) + 4|\alpha|^2 \nu^2 + |\alpha|^4 \end{aligned}\right\} \tag{3.5}$$

where $\mu = \cosh r \;;\; \nu = \sinh r$

The mean photon number of a squeezed state is given by [Caves, 1981]

$$\langle \alpha, r | \hat{N} | \alpha, r \rangle = |\alpha|^2 + \sinh^2 r \tag{3.5}$$

where $\hat{N} = a^\dagger a$. The variance of $\hat{N}$ is

$$V(\hat{N}) = 1/2 \sinh^2 2r + |\alpha|^2 \left[\cosh 2r - \sinh 2r . \cos 2\theta\right]$$

and

$$V(\hat{N}) - \langle \hat{N} \rangle = |\alpha|^2 \left[\cosh 2r - \sinh 2r \cos 2\theta - 1\right] + \sinh^2 r . \cosh 2r \tag{3.7}$$

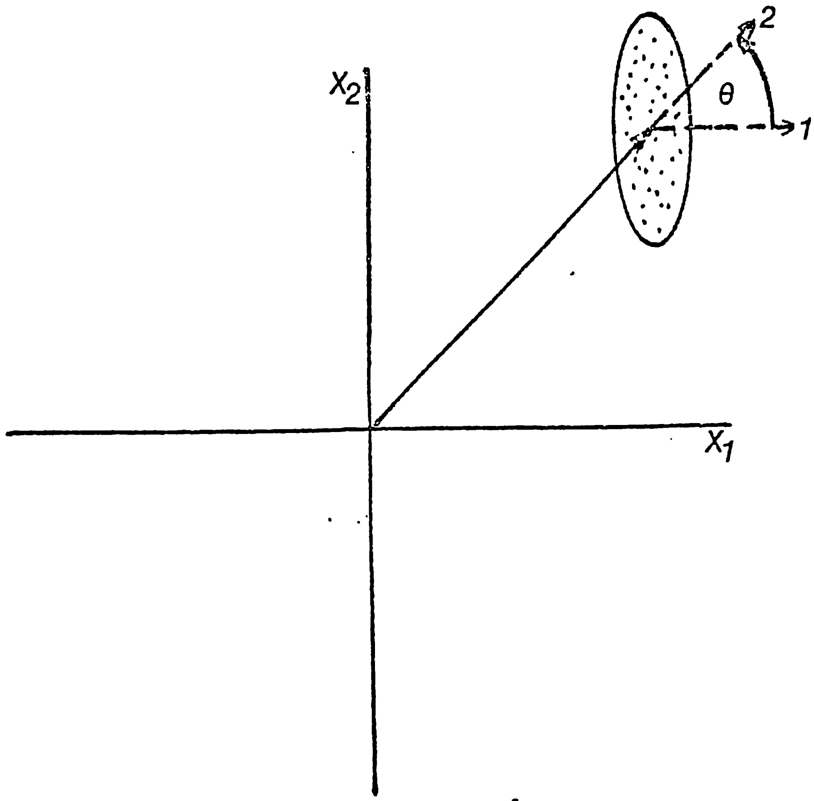

Figure 3.1 We align our axes such that $\hat{X}_1$ direction is parallel to the minor axis of the error ellipse. The direction (1) is referred to as the direction of squeezing and direction (2) as the direction of coherent excitation. We then define α by $\alpha = \langle\hat{X}_1\rangle + i\langle\hat{X}_2\rangle$ with

$$\theta = \tan^{-1}\left(\frac{\langle X_2\rangle}{\langle X_1\rangle}\right)$$

and thus

$$g^2(0) = 1 + \frac{|\alpha|^2\left(\text{Cosh}\,2r - \text{Sinh}\,2r\,\text{Cos}\,2\theta - 1\right) + \text{Sinh}^2 r\,\text{Cosh}(2r)}{\left(|\alpha|^2 + \text{Sinh}^2 r\right)^2} \tag{3.8}$$

When $|\alpha| >> \text{Sinh}^2 r\,\text{Cosh}\,2r$ (that is when the intensity is predominantly due to the coherent excitation)

$$\begin{aligned} V(\hat{N}) - \langle\hat{N}\rangle &\approx |\alpha| \left\{\text{Cosh}\,2r - \text{Sinh}\,2r\,\text{Cos}\,2\theta - 1\right\} \\ &= 4|\alpha|^2 \begin{cases} V(:\hat{X}_1:) & , \theta = 0 \\ V(:\hat{X}_2:) & , \theta = \pi/2 \end{cases} \end{aligned} \tag{3.9}$$

and

$$g^2(0) \approx 1 + \frac{1}{|\alpha|^2}\left(\text{Cosh}\,2r - \text{Sinh}\,2r\,\text{Cos}\,2\theta - 1\right) \tag{3.10}$$

$V(:\hat{X}_1:)$ and $V(:\hat{X}_2:)$ are the normally ordered variances defined in equation (1.58). It is clear that subpoissonian statistics and thus photon antibunching will arise whenever the direction of squeezing is parallel to the direction of coherent excitation, that is, when the squeezing is in phase with the complex amplitude. In fact we have

$$g^2(0) = \begin{cases} 1 + (e^{-2r} - 1)/|\alpha|^2 & , \theta = 0 \\ 1 + (e^{2r} - 1)/|\alpha|^2 & , \theta = \pi/2 \end{cases} \tag{3.11}$$

thus $g^2(0) < 1$ when $\theta = 0$ and $|\alpha|^2$ is large (see figure 3.2a,b).

We now discuss a number of interesting cases.

When $\cos 2\theta = \tanh r$ we have that $V(\hat{N}) - \langle \hat{N} \rangle$ is always positive and $r = \ln(\cot \theta)$. This is only possible if $\theta \neq 0$. In this case photon bunching will occur $(g^{2}(0) > 1)$.

When $r = 0$, the error ellipse is a circle and θ is not defined, in fact it is arbitrary. In this case $g^{2}(0) = 1$ and the statistics is poissonian. Of course this is just the usual coherent state result.

In some situations of physical interest it may be possible to change the sign of the squeeze parameter. In the next chapter we will consider just such a situation. In terms of the phase convention used here a variation of r from positive to negative is not possible as r is defined as positive. In terms of this phase convention a discontinuous change of $\pi/2$ in the value of θ, through the region it is not defined (i.e. $r = 0$), corresponds to a change in the sign of the squeeze parameter. If there is a fixed reference phase in the problem, say of some driving field, a phase convention is not necessary, all phase differences being calculated with respect to the reference phase.

3.2 Quasiprobability Representations for Squeezed States

Considerable insight into the nature of a squeezed state may be obtained by expanding the pure state density operator $\rho = |\alpha; r\rangle\langle\alpha; r|$, in terms of either the complex P-representation (section 1.4) or the Q-representation. We consider the Q-representation first which has the attraction of being always positive. The treatment is similar to that of Yuen [1976].

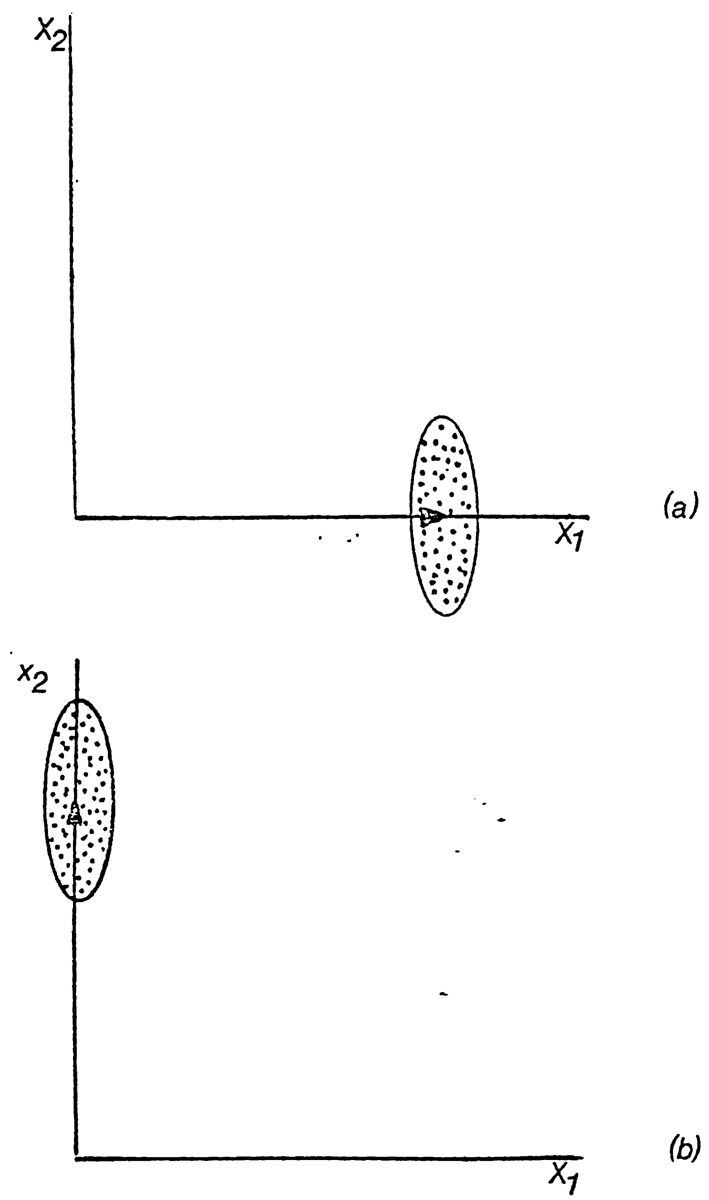

Figure 3.2a,b In figure 3.2a $\theta = 0$ and we expect photon antibunching to occur $g^2(0) < 1$.

In figure 3.2b $\theta = \pi/2$ and we expect photon bunching to occur $g^2(0) > 1$.

The Q-representation of ρ is defined by

$$Q(\alpha,\alpha^*) \equiv \langle\alpha|\rho|\alpha\rangle \tag{3.12}$$

Antinormally ordered moments are given by

$$\langle a^n a^{\dagger m}\rangle = \int \alpha^n \alpha^{*m} . Q(\alpha,\alpha^*) \, d^2\alpha \equiv \langle\alpha^n \alpha^{*m}\rangle \tag{3.13}$$

(for details on the Q-representation see [Drummond et.al., 1981].

If $C(\alpha,\alpha^*)$ is the covariance matrix defined by moments of α and α^* over $Q(\alpha,\alpha^*)$ we have that the covariance matrix $C(a,a^\dagger)$ for the operators, defined by equation (1.42), is given by

$$C(a,a^\dagger) = C(\alpha,\alpha^*) - 1/2 . \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \tag{3.14}$$

Thus the covariance matrix for the $\hat{X}_1$, $\hat{X}_2$ variables (equation (1.41)) is given by

$$C(\hat{X}_1, \hat{X}_2) = \Omega \, C(\alpha,\alpha^*) \, \Omega^T - 1/4 \, I \tag{3.15}$$

(equation (3.15) shows clearly that δ function Q representations cannot exist).

If $\rho = |\alpha;r\rangle\langle\alpha;r|$ then

$$\begin{aligned} Q(\beta,\beta^*) &= |\langle\beta|\, D(\alpha)\, S(r)\,|0\rangle|^2 \\ &= |\langle\beta'|0,r\rangle|^2 \end{aligned} \tag{3.16}$$

where $\beta' = \beta-\alpha$ and $|0,r\rangle = S(r)|0\rangle$. It is shown by Yuen [1976, equation (3.20)] that

$$\langle\beta|0,r\rangle = \frac{1}{\pi \cosh r} . exp\left\{-1/2\,|\beta|^2 - \frac{\tanh r}{2} . \beta^{*2}\right\} \tag{3.17}$$

and thus

$$Q(\beta, \beta^*) = \frac{1}{\pi \cosh r} \exp\left\{ -|\beta - \alpha|^2 - \frac{\tanh r}{2}\left[(\beta^* - \alpha^*)^2 + (\beta - \alpha)^2\right]\right\} \tag{3.18}$$

This is a multivariate Gaussian distribution and may be written as,

$$Q(\beta, \beta^*) = \frac{1}{\pi \cosh r} \exp\left\{ -1/2\, (\underset{\sim}{\beta} - \underset{\sim}{\alpha})^T\, C^{-1}(\beta, \beta^*)\, (\underset{\sim}{\beta} - \underset{\sim}{\alpha})\right\} \tag{3.19}$$

where $(\underset{\sim}{\beta} - \underset{\sim}{\alpha})^T = (\beta - \alpha,\ \beta^* - \alpha^*)$

and

$$C(\beta, \beta^*) = 1/2 \cdot \begin{pmatrix} -\sinh 2r & \cosh 2r + 1 \\ \cosh 2r + 1 & -\sinh 2r \end{pmatrix} \tag{3.20}$$

with $\langle a \rangle = \langle \beta \rangle = \alpha$

$\langle a^\dagger \rangle = \langle \beta^* \rangle = \langle \alpha^* \rangle$

It is easily verified using (3.15) and (3.20) that

$$C(\hat{X}_1, \hat{X}_2) = 1/4 \cdot \begin{pmatrix} e^{-2r} & 0 \\ 0 & e^{2r} \end{pmatrix} \tag{3.21}$$

We now consider the complex P-representation of a single mode squeezed state.

Consider a system in a pure squeezed state, i.e. $|\gamma, r\rangle\langle\gamma, r|$
Using equation (1.48) we expand ρ in terms of the complex P-representation

$$\rho = \int_{C_1}\int_{C_2} P(\alpha, \beta) \frac{|\alpha\rangle\langle\beta^*|}{\langle\beta^*|\alpha\rangle}\, d\alpha\, d\beta \tag{3.22}$$

Using equation (2.75) this becomes

$$|0;r\rangle\langle 0;r| = \int_{C_1}\int_{C_2} P(\alpha,\beta)\ \frac{|\alpha-\gamma\rangle\langle\beta^*-\gamma|}{\langle\beta^*|\alpha\rangle}\,d\alpha.d\beta$$

$$= \int_{C_1}\int_{C_2} P(\mu,\nu)\ \frac{|\mu\rangle\langle\nu|}{\langle\nu^*+\gamma|\mu+\gamma\rangle}\,d\mu.d\nu$$

$$= \int_{C_1}\int_{C_2} P(\mu,\nu:r)\ \hat{\Lambda}(\mu,\nu)\,d\mu\,d\nu \tag{3.33}$$

where $\mu = \alpha - \gamma \qquad \nu = \beta - \gamma^*$

and
$$\hat{\Lambda}(\mu,\nu) = \frac{|\mu\rangle\langle\nu^*|}{\langle\nu^*+\gamma|\mu+\gamma\rangle} = \frac{|\mu\rangle\langle\nu^*|}{\langle\nu^*|\mu\rangle}$$

Using equations (2.73) and (2.75) we write

$$|0,r\rangle = exp(ir\hat{T}_3)\,|0\rangle \tag{3.34}$$

where $\hat{T}_3 = -i/2\,(a^2 - a^{\dagger 2})$

(equation (2.40c)).

If we now define $\rho' \equiv |0;r\rangle\langle 0;r|$, using (3.34), we may express the derivative of ρ' with respect to r as,

$$\frac{\partial\rho'}{\partial r} = i\,[\hat{T}_3,\rho'] \tag{3.35}$$

We now substitute the right hand side of equation (3.33) for ρ' and using the identities of equations (1.52a-d) we find

$$\int_{C_1}\int_{C_2}\frac{\partial P}{\partial r}\,\hat{\Lambda}(\mu,\nu)\,d\mu.d\nu = -1/2\int\int P\left\{\frac{\partial^2}{\partial\mu^2} + \frac{\partial^2}{\partial\nu^2} + 2\nu\frac{\partial}{\partial\mu} + 2\mu\frac{\partial}{\partial\nu}\right\}\Lambda(\mu,\nu)\,d\mu.d\nu \tag{3.36}$$

If the contours C_1, C_2 are such that $P(\mu,\nu)$ vanishes at the boundaries or has the same value at the beginning and end of the contour, partial integration of (3.36) may be performed to yield

$$\frac{\partial P}{\partial r} = \left\{\left(\frac{\partial}{\partial \nu}\mu + \frac{\partial}{\partial \mu}\cdot\nu\right) - 1/2\left(\frac{\partial^2}{\partial \mu^2} + \frac{\partial^2}{\partial \nu^2}\right)\right\} P(\mu,\nu;r) \tag{3.37}$$

which may be written in the form,

$$\frac{\partial P}{\partial r} = \left\{-\underset{\sim}{z}'^T M \underset{\sim}{z} + 1/2\ \underset{\sim}{z}'^T N \underset{\sim}{z}'\right\} P(\underset{\sim}{z};r) \tag{3.38}$$

where

$$\underset{\sim}{z}'^T = \left(\frac{\partial}{\partial \mu}, \frac{\partial}{\partial \nu}\right) \quad ; \quad \underset{\sim}{z}^T = (\mu, \nu)$$

$$M = -\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \tag{3.39}$$

$$N = -\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \tag{3.40}$$

Equation (3.38) is in the form of the Fokker-Planck equation for an Orstein-Uhlenbeck stochastic process [C.W. Gardiner, 1982].

The diffusion matrix N is of course not positive definite. This is characteristic of situations where non-classical photon statistics are known to arise. [Drummond, Gardiner and Walls, 1981].

Equation (3.38) may be converted to a Fokker-Planck equation with a positive definite diffusion matrix by the transformation

$$\underset{\sim}{\omega} = \begin{pmatrix} \mu' \\ \nu' \end{pmatrix} = \begin{pmatrix} i & 0 \\ 0 & i \end{pmatrix}\begin{pmatrix} \mu \\ \nu \end{pmatrix} \tag{3.41}$$

Equation (3.38) becomes

$$\frac{\partial P(\underset{\sim}{\omega};r)}{\partial r} = (- \underset{\sim}{\omega}'^{T} M \underset{\sim}{\omega} + 1/2\, \underset{\sim}{\omega}'^{T} S \underset{\sim}{\omega})\, P(\underset{\sim}{\omega};r) \qquad (3.42)$$

where M is given by equation (3.39) and $S = -N$. Thus S is positive definite.

Equation (3.42) may be solved by similar methods to those used to solve the classical Orstein-Uhlenbeck equation (see appendix one).

The "initial" condition (i.e. the solution for $r = 0$) is the complex P-representation for a coherent state.

$$P(\alpha, \beta ; 0) = \delta(\alpha - \gamma)\, \delta(\beta - \gamma^{*}) \qquad (3.43)$$

The solution of equation (3.38) with these initial conditions is the Gaussian

$$P(\underset{\sim}{z} ; r) = \frac{-1}{2\pi \sinh r} \cdot \exp\left(-1/2\, \underset{\sim}{z}^{T} C^{-1}(\underset{\sim}{z})\, \underset{\sim}{z} \right) \qquad (3.44)$$

where

$$C(\underset{\sim}{z}) = 1/2 \begin{pmatrix} -\sinh 2r & \cosh 2r - 1 \\ \cosh 2r - 1 & -\sinh 2r \end{pmatrix}$$

The negative normalization is due to the jacobian of the transformation in equation (3.41).

Using $\mu = \alpha - \gamma$, $\nu = \beta - \gamma^{*}$ we may write

$$P(\alpha, \beta) = \frac{-1}{2\pi \sinh r} \exp\left\{ (\alpha - \gamma)(\beta - \gamma^{*}) + \frac{\coth r}{2}\left[(\alpha - \gamma)^{2} + (\beta - \gamma^{*})^{2} \right] \right\} \qquad (3.45)$$

This simple displacement transformation only changes the mean but not the covariance matrix thus

$$\langle \alpha \rangle = \gamma \quad ; \quad \langle \beta \rangle = \gamma^{*}$$

$$C(\alpha, \beta) = C(\underset{\sim}{z})$$

Equation (3.43) is thus the complex P-representation for the squeezed state $|\alpha, r\rangle$. The contours on which it is normalised are the imaginary axes in both α and β space. Note that for these values of α, β $P(\alpha,\beta)$ is in fact negative. If the distribution of (3.43) were normalized on the Glauber-Sudarshan contour $P(\alpha,\beta)$ would diverge.

Since $P(\alpha,\beta)$ is in Gaussian form we expect the discussion in section 1.4 concerning the connection between photon number statistics and the normally ordered variances of $\hat{X}_1$ and $\hat{X}_2$, to be applicable. As we saw explicitly in section 1.1 (equation (3.9)) this is indeed the case.

The complex P-representation for a squeezed state may be used to derive an expression for the photon number probability distribution.

The photon number distribution for a system in a state described by the density operator ρ is defined by

$$P(n) = \langle n| \rho |n\rangle$$

In terms of the complex P-representation of ρ we have [Walls et.al., 1982]

$$P(n) = \int_C P(\alpha,\beta) \frac{(\alpha\beta)^n}{n!} \exp(-\alpha\beta)\, d\alpha\, d\beta \qquad (3.46)$$

Upon substitution of $P(\alpha,\beta)$ in equation (3.45) into equation (3.46) and integration along the imaginary axis in α, β space, we find (with γ real)

$$P(n) = (\cosh r)^{-1} \exp\left\{-\gamma(1+\tanh r)\right\}(2^n n!)^{-1}.(\tanh r)^n \times \left\{ H_n\left(\frac{\gamma e^r}{\sqrt{\sinh 2r}}\right)\right\}^2 \qquad (3.47)$$

where $H_n(x)$ are Hermite polynomials.

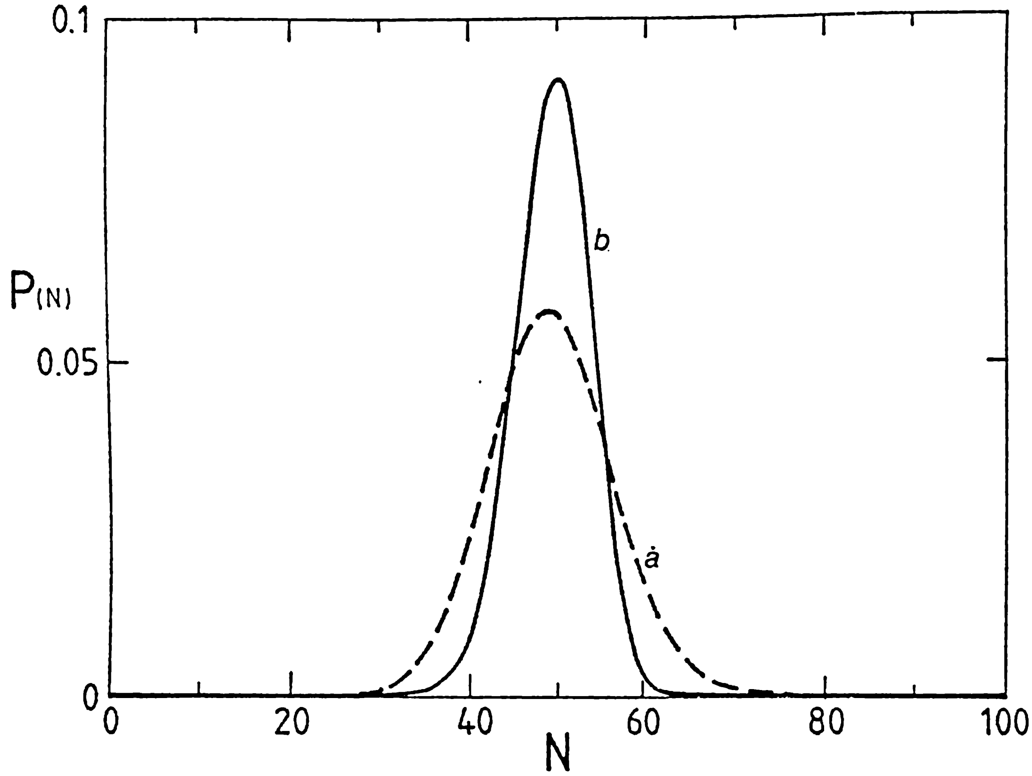

Figure 3.3 Photon number probability $P(n)$ against photon number for (a) coherent state, $\alpha = 7.0$ (b) squeezed state with $\theta = 0$. $r = 0.5$. [Milburn and Walls, 1982a].

This expression for $P(n)$ agrees with the result of a derivation given by Yuen [Yuen, 1976] who directly evaluated $|\langle n|\beta\rangle_g|^2$ where $|\beta\rangle_g$ is a two photon coherent state [see also Neumann and Haug, 1979].

In figure 3.3a,b we have plotted $P(n)$ for

(a) coherent state, (b) squeezed state with $\theta = 0$.

The subpoissonian nature of the squeezed state with squeezing in phase with the excitation is clearly evident.

3.3 Detection of Squeezed States

It is clear from the discussion of chapter two that the formal structure of non-relativistic quantum mechanics admits the possibility of squeezed states for the electromagnetic field. The experimental verification of this obviously requires some method of detecting squeezed states.

The quadrature phase operators themselves are self-adjoint thus, in principal at least, they are measurable. However we will not discuss in this thesis any scheme to make direct quadrature phase measurements.

Such schemes have been discussed by Yuen and Shapiro [1978, 1980] and Shapiro et.al. [1979] who prove the full equivalence of homodyne detection to single quadrature field measurements and that of heterodyne detection to two quadrature field measurements. They also propose a new receiver configuration called Two Photon Coherent State heterodyning to realise all measurements described by TPCS.

The essential property of a squeezed state is that it has phase dependent noise, furthermore the noise at a particular phase is less than it would be for a coherent state. It is the possible observation of this phase dependent noise which we will consider as the basis for a squeezed state detection scheme.

The simplest quantum optical measurement schemes are based on photo-electron detection, for example, the determination of the photon number statistics of an electromagnetic field [Mandel, 1958; Mandel and Sudarshan, 1964; Kelly and Kleiner, 1964; Glauber, 1963]. However squeezing is a phase sensitive property which suggests that some form of interference system may be required, in addition to photo detection.

The clue to a realizable squeezed state detection scheme is provided by the relation between photon number statistics and the normally ordered variances (equation (1.62)). A detection scheme which exploits this connection has been proposed by Mandel [1982]. We now present an analysis of such a scheme which differs from Mandel's analysis but which yields similar predictions.

Consider two fields $E_1(r,t)$ and $E_2(r,t)$ of the same frequency, combined on a 50/50 beam splitter (figure 3.4).

This configuration is essentially identical to the single field quadrature homodyne detection scheme of Yuen and Shapiro [1980].

We expand the two incident fields into the usual positive and negative frequency components

$$\hat{E}_1(r,t) = i\left(\frac{\hbar\omega}{2V\varepsilon_0}\right)^{1/2}\left\{a\,e^{i(k.r-\omega t)} - a^{\dagger}\,e^{-i(k.r-\omega t)}\right\} \qquad (3.48)$$

$$\hat{E}_2(r,t) = i\left(\frac{\hbar\omega}{2V\varepsilon_0}\right)^{1/2}\left\{b\,e^{i(k.r-\omega t)} - b^{\dagger}\,e^{-i(k.r-\omega t)}\right\} \qquad (3.49)$$

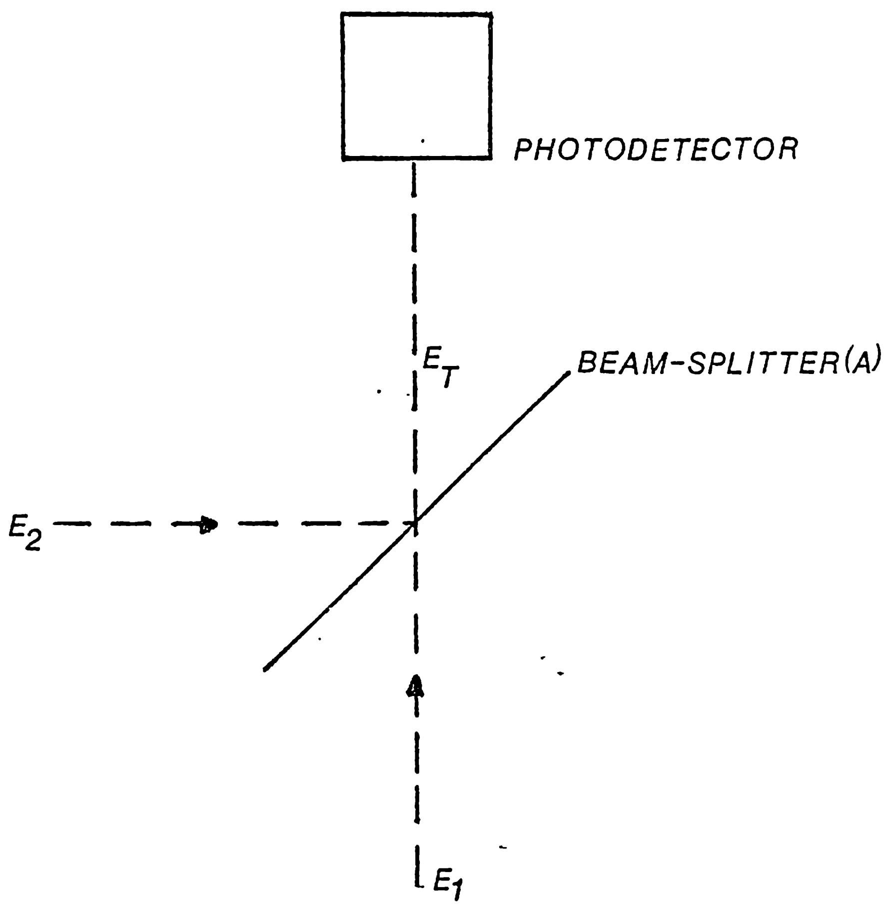

Figure 3.4 Homodyne detection of squeezed states. A is a 50/50 beam splitter.

where a, b are boson operators which characterise the two modes E_1 and E_2 respectively. Both fields are taken to have the same sense of polarization, and are phase locked.

The total field after combination is given by

$$\hat{E}_T(r,t) = i.\left(\frac{\hbar\omega}{2V\varepsilon_0}\right)^{1/2}.\left\{c\, e^{i(kr-\omega t)} - c^{\dagger} e^{-i(k.r-\omega t)}\right\} \tag{3.50}$$

where

$$c = \frac{1}{\sqrt{2}}.(ia + b) \tag{3.51}$$

and we have included a 90° phase shift between the reflected and transmitted beams at the beam splitter.

The photo detector of course, responds to the moments of $c^{\dagger}c$ We thus define the number operator $\hat{N} \equiv c^{\dagger}c$ Equation (1.54b) then gives,

$$V(\hat{N}) - \langle\hat{N}\rangle = \langle c^{\dagger 2}c^{2}\rangle - \langle\hat{N}\rangle^{2}$$

Substitution of (3.50) yields

$$c^{\dagger}.c = 1/2\,(a^{\dagger}a + b^{\dagger}b + i(b^{\dagger}a - a^{\dagger}b)) \tag{3.52}$$

$$\begin{aligned} c^{\dagger 2}c^{2} = 1/4.(\, & a^{\dagger 2}a^{2} + b^{\dagger 2}b^{2} + 4\,a^{\dagger}a\,b^{\dagger}b \\ & - a^{\dagger 2}b^{2} - a^{2}b^{\dagger 2} - 2i\,a^{\dagger 2}a\,b \\ & + 2i\,a^{\dagger}a^{2}b^{\dagger} + 2i\,a\,b^{\dagger 2}b - 2i\,a^{\dagger}b^{\dagger}b^{2}) \end{aligned} \tag{3.53}$$

We now assume that the a mode is in a coherent state $|\alpha\rangle$, while the b mode is in a pure squeezed state $|\beta, r\rangle$. Using equations (3.5) one is lead to consider two cases.

CASE ONE: Choose α, β in phase, that is

$$\alpha = A e^{i\theta}$$
$$\beta = B e^{i\theta}$$

Then,

$$V(\hat{N}) - \langle \hat{N} \rangle = 1/4 . (\nu^4 + \mu^2 \nu^2 + 2 (A^2 + B^2) \nu^2 + 2 (A^2 - B^2) \mu \nu) \quad (3.54)$$

If we further assume that the coherent field is sufficiently intense so that $A >> B$ and $A^2 >> \mu^2 \nu^2 + \nu^4$ we find that

$$V(\hat{N}) - \langle \hat{N} \rangle = A^2/4 . (e^{2r} - 1)$$
$$= 4 A^2 . V (: \hat{X}_2 :) \quad (3.55)$$

where $$\hat{X}_2 = 1/2i (b - b^\dagger)$$

CASE TWO: We now choose α, β 90° out of phase. In this case,

$$V(\hat{N}) - \langle \hat{N} \rangle = 1/4 . (\nu^4 + \mu^2 \nu^2 + 2 \nu^2 (A - B)^2 - 2 (A - B)^2 \mu \nu) \quad (3.56)$$

If $A = B$, we have complete interference whence $V(\hat{N}) - \langle \hat{N} \rangle = 1/4 \, Sinh^2 r . Cosh(2r)$ which is always positive. However if as in case one, we choose A sufficiently large we find

$$V(\hat{N}) - \langle \hat{N} \rangle \approx 4 A^2 \; V (: \hat{X}_1 :) \quad (3.57)$$

where $$\hat{X}_1 = 1/2 (b + b^\dagger)$$

We thus see that if by varying the phase difference between the signal field E_1 and the intense coherent field, the photo electron statistics varied between superpoissonian and subpoissonian, one would conclude that the signal field carried reduced fluctuations in one quadrature.

Equations (3.55) and (3.57) show a close similarity to equations (3.9) for a pure squeezed state. The essential difference is that the amplitude A appearing in equations (3.55) and (3.56) is not the amplitude of the squeezed state, but the amplitude of the homodyning field.

In the above analysis we have made use of the two external beam splitter modes a, b. In fact the state of these modes is quite arbitrary, one may even be taken as the vacuum state. Such an approach has been used to discuss laser interferometer gravitational wave detectors [C.M. Caves, 1980; Walls and Milburn, 1981]. It is shown in these schemes that simple single input interferometers are limited in accuracy by vacuum fluctuations entering via the b mode input. Greater accuracy is possible if the b mode input is taken to be squeezed.

Mandel's analysis of the homodyne detection scheme does not make use of beam splitter modes. Instead he models the coherent superposition of a light beam with a coherent beam of amplitude α, by displacing the states of a multimode light beam with the displacement operator $D(\alpha)$. It seems that for normally ordered variances the results obtained are the same in the two treatments.

3.4 Generation of Squeezed States

The initial motivation for a consideration of squeezed states came from researchers in the field of optical communications theory [Yuen et.al., 1978, 1979, 1980] where the practical implications of quantum noise limited optical communications system are only too apparent.

Squeezed states are also assuming practical importance in certain measurement schemes (quantum non-demolition measurements) to detect extremely weak forces, such as gravitational waves, where the displacement caused by the force is less than the uncertainty in a squeezed state (see section 3 of this thesis for more details and references). In this context a novel application of squeezed states to reduce the effects of fluctuations due to quantum noise in a Michelson interferometer gravitational wave detector has recently been proposed by Caves [1980, 1981].

In the light of even these few applications the question of how to generate squeezed states in a practical device is obviously of great importance. On a more fundamental level, the possibility of generating and detecting light with such highly non-classical statistics, is in itself, sufficient motivation for proposing practical generation schemes.

In this section then, we wish to make a few preliminary statements on how squeezed states may be generated. A number of possible practical schemes will be analysed in far greater depth in the next chapter.

There are, at least, two ways in which a system can be prepared in a specified quantum mechanical state.

Firstly, one may use the non-unitary effect of a measurement. Detailed examples of this will be provided in section three of this thesis, however a simple example is easily provided. Given a real one dimensional harmonic oscillator, how might one prepare it in a squeezed state? The answer is immediately provided when we realise that $\hat{X}_1$ is in fact, a dimensionless position operator. Thus whenever a position measurement is made on the particle in the harmonic potential, it will be placed in a state with small variance in $\hat{X}_1$ depending on the accuracy of the measurement. It will thus be in a "squeezed state" after the measurement.

Secondly, one may use the unitary evolution of coupled systems to drive the state of a subsystem into a specified state. For example one may prepare a harmonic oscillator in a coherent state by coupling its position operator to a classical driving force. The interaction Hamiltonian is then of the form

$$H_I = \hbar \varepsilon (a + a^{\dagger}) \tag{3.58}$$

which is clearly the generator of the $N(1)$ group of displacements in the complex plane. The interaction Hamiltonian in the interaction picture is the generator of the unitary time evolution operator, which in this case, is also the unitary representation of an element of $N(1)$ with time as the group parameter. Clearly then to generate a squeezed state one must at least require the interaction Hamiltonian to contain the generator of the scale changing transformation. In quantum optics the initial state of the field prior to the interaction is normally the vacuum state or a coherent state, and thus squeezed states will be excited in that mode whenever it undergoes an interaction with a Hamiltonian corresponding to the scale changing transformation for that mode.

For example in the one mode case an interaction Hamiltonian of the form

$$H_I = -i\hbar/2 (a^2 - a^{\dagger 2}) \tag{3.59}$$

will result in a squeezed state being generated in this mode.

Such light mode interactions may arise in quantum optics when light passes through a non-linear crystal, i.e. a crystal whose polarizability is a non-linear function of the applied electric field. Usually the polarizability is expanded in powers of the instantaneous electric field,

$$P_i = \chi_{ij} E_j + \chi^{(2)}_{ijk} E_j \cdot E_k + \chi^{(3)}_{ijkl} E_j E_k E_l + \cdots \tag{3.60}$$

where $\chi^{(i)}$ is the i'th order susceptibility tensor and summation over repeated indices is assumed.

The interaction Hamiltonian may be written in the form [Drummond, 1979]

$$H = \int d^3r . \hat{E}_i \left\{ \chi^{(2)}_{ijk} \hat{E}_j \hat{E}_k + \chi^{(3)}_{ijkl} \hat{E}_j \hat{E}_k \hat{E}_l + \cdots \right\} \tag{3.61}$$

In cases where phase matching is possible the quadratic term in the polarizability $\chi^{(2)}$ is significant. If we write the electric field as

$$E(r,t) = i \sum_{\underset{\sim}{k}} \left(\frac{\hbar \omega}{2V\varepsilon_k} \right)^{1/2} \hat{\underset{\sim}{e}}_k \left(e^{i\underset{\sim}{k}.\underset{\sim}{r}} a_{\underset{\sim}{k}}(t) - h.c. \right)$$

where $\hat{\underset{\sim}{e}}_k$ is a polarization vector, the interaction Hamiltonian becomes [Tucker and Walls, 1969]

$$H_I = \sum_{k,k',k''} K \, a^{\dagger}_{k}(t) \, a_{k'}(t) \, a_{k''}(t) + h.c. \tag{3.62}$$

where K is a, possibly complex, coupling constant given by

$$K = -i \left(\frac{\hbar \omega' \omega' \omega''}{2V\varepsilon'\varepsilon''\varepsilon''} \right)^{1/2} \hat{\underset{\sim}{e}}_k . \chi^{(2)} : \hat{\underset{\sim}{e}}_{k'} \hat{\underset{\sim}{e}}_{k'} \frac{1}{V} \int d^3\underset{\sim}{r} \, e^{i(k'-k'-k'')}$$

Non negligible coupling arises when the phase matching condition occurs, i.e. $\underset{\sim}{k} = \underset{\sim}{k}' + \underset{\sim}{k}''$. Two situations may then arise.

1. Frequency Conversion: We assume that there is an intense monochromatic laser field of frequency ω_L present in the medium which is the source of all non-negligible coupling. The appropriate Hamiltonian is obtained by setting k' or k'' equal to k_L in equation (3.62) [Tucker and Walls, 1969]. Thus the interaction Hamiltonian may be written

$$H_I = (\hbar/2)\, \kappa\, \mathcal{E}_L\, (a_1^\dagger a_2 + a_2^\dagger a_1) \tag{3.63}$$

where we have neglected all terms which are non-energy conserving, that is the condition $\omega_1 = \omega_2 + \omega_L$ is taken to hold, and we have chosen the phase of the driving field $\mathcal{E}_L$ so that $\kappa \cdot \mathcal{E}_L$ is real. By comparison of equation (3.63) and equation (2.85c) we see that H_I is proportional to the two mode element of $Sp(4),\ \hat{T}_3$. Since this is an element of the $U(n)$ subalgebra, we do not expect this interaction to produce squeezed states.

2. Parametric Amplification: If we now assume that $k = k_L$ the interaction Hamiltonian becomes (ignoring non resonant terms)

$$H_I = i\hbar/2.\, |\kappa\, \mathcal{E}_L^*|\, (a_1 a_2 - a_2^\dagger a_1^\dagger) \tag{3.64}$$

where $\omega_1 = \omega_2 = \omega_L/2$ and we have chosen the phase of the driving field so that $\kappa\, \mathcal{E}^*$ is pure imaginary. Equation (3.64) gives the interaction Hamiltonian for a parametric amplifier in a semiclassical approximation. If the a_1, a_2 modes coincide so that $a_1 \equiv a_2 \equiv a$ we have the degenerate case in which

$$H_I = i\hbar/2\, |\kappa\, \mathcal{E}^*|.\, (a^2 - a^{\dagger 2}) \tag{3.65}$$

Using equation (2.85h) we see that the degenerate interaction Hamiltonian is in fact proportional to the scale changing transformation generator $\hat{T}_8$. We thus expect the degenerate parametric amplifier to be a possible squeezed state generator, for the mode a itself.

In the non-degenerate situation, represented by the interaction Hamiltonian in equation (3.64), the reduction of fluctuations is not obtained in the a_1, a_2 modes directly. This interaction Hamiltonian is proportional to the generator $\hat{T}_6$ and thus produces states with reduced fluctuations in the variables b_1, b_2 where

$$b_1 = \frac{1}{\sqrt{2}} (a_1 e^{-i\pi/4} - a_2 e^{i\pi/4})$$

$$b_2 = \frac{1}{\sqrt{2}} (a_2 e^{-i\pi/4} - a_1 e^{i\pi/4}) \qquad (3.67)$$

(see equations (2.101).

In order to determine whether the non-degenerate parametric amplifier does indeed produce squeezed states, we must combine the output modes with suitable phase shifts so that the resultant field, prior to detection, is represented by the boson modes in equations (3.67). Such recombination schemes will be a characteristic feature of any squeezed state generation and detection process.

We may write equations (3.67) as

$$b_1 e^{i\pi/4} = \frac{1}{\sqrt{2}} (a_1 - i a_2)$$

$$b_2 e^{i\pi/4} = \frac{1}{\sqrt{2}} (a_2 - i a_1)$$

We thus see that the a_1, a_2 modes should be combined with a 90° phase difference between them. The squeezed direction will then be 45° out of phase with the excitation in the a_1 and a_2 modes themselves.

In media with inversion symmetry the lowest order non-vanishing nonlinearity is a polarization cubic in the electric field amplitudes [Bloembergen, 1977]. Such a nonlinearity couples four modes of the electromagnetic field and is known as a four wave mixing interaction. To approximate quadratic Hamiltonians using $\chi^{(3)}$ interactions it is then necessary to treat two modes classically. Examples of $\chi^{(3)}$ processes are two photon absorption and stimulated raman effect, [Bloembergen, 1977] and optical bistability [Drummond and Walls, 1980]. We will consider some of these processes within a fully quantum mechanical framework in the next chapter.

3.5 Squeezed States and Irreversible Dynamics

No system in nature is ever completely isolated from the rest of the universe. If we wish to model the true dynamics of a particular system we must take into account the effect of interactions with the environment. These interactions give rise to fluctuating forces on the system of interest and lead to damping. Such processes are essentially irreversible and co-exist with any coherent reversible dynamics exhibited by the system.

In the previous section we gave heuristic consideration to possible reversible processes which may produce squeezed states. However as the essential property of a squeezed state is its reduced quantum noise, it must be considered whether any irreversible process will limit the extent of this reduction or indeed whether it will preclude it altogether. In this section we wish to address this question, using the master equation techniques outlined in section 1.2, together with the c-number techniques of section 1.4.

We consider first the situation of a Harmonic oscillator initially in a squeezed state, undergoing damping, As in section 1.2 we model this through the Hamiltonian.

$$H = \hbar\omega a^{\dagger}a + a\Gamma^{\dagger} + a^{\dagger}\Gamma \tag{3.68}$$

where $[a, a^{\dagger}] = 1$ and Γ, $\Gamma^{\dagger}$ are reservoir operators. This leads to the master equation

$$\frac{\partial\rho}{\partial t} = \gamma/2(2a\rho a^{\dagger} - a^{\dagger}a\rho - \rho a^{\dagger}a) + \bar{n}\gamma(a^{\dagger}\rho a + a\rho a^{\dagger} - a^{\dagger}a\rho - \rho a a^{\dagger}) \tag{3.69}$$

where γ is the damping constant and $\bar{n} = (\exp(\hbar\omega/kT) - 1)^{-1}$ with T the temperature of the reservoir. If we now expand ρ using the complex P-representation we obtain from the master equation the Fokker-Planck equation

$$\frac{\partial P(\alpha,\beta;t)}{\partial t} = \left\{\gamma/2\left(\frac{\partial}{\partial\alpha}\alpha + \frac{\partial}{\partial\beta}\beta\right) + \gamma\bar{n}\frac{\partial^2}{\partial\alpha\partial\beta}\right\} P(\alpha,\beta) \tag{3.70}$$

Equation (3.70) is easily solved by the method of Wang and Uhlenbeck (appendix one) subject to the initial condition

$$P(\alpha,\beta;0) = \frac{-1}{2\pi\sinh r}\cdot\exp\left\{(\alpha-\gamma)(\beta-\gamma^{*}) + \frac{\coth r}{2}\left[(\alpha-\gamma)^2 + (\beta-\gamma^{*})^2\right]\right\}$$

corresponding to the initial squeezed state $|\gamma; r\rangle$.

The solution is

$$P(\alpha,\beta) = \frac{-1}{2\pi\sqrt{\mathrm{Det}\,C}} \exp\left\{-1/2\,(\underset{\sim}{z}-\langle \underset{\sim}{z}(t)\rangle)^T C^{-1}(\underset{\sim}{z}-\langle \underset{\sim}{z}(t)\rangle)\right\} \tag{3.71}$$

where

$$\begin{aligned} \underset{\sim}{z}^T &= (\alpha, \beta) \\ \langle \underset{\sim}{z}(t)\rangle &= \begin{pmatrix} \gamma\, e^{-\gamma t/2} \\ \gamma^*\, e^{-\gamma t/2} \end{pmatrix} \end{aligned} \tag{3.72}$$

$$C = \begin{pmatrix} -\sinh 2r & \cosh 2r - 1 \\ \cosh 2r - 1 & -\sinh 2r \end{pmatrix} \frac{e^{-\gamma t}}{2} + \begin{pmatrix} 0 & \bar{n} \\ \bar{n} & 0 \end{pmatrix} (1 - e^{-\gamma t}) \tag{3.73}$$

Using equation (1.54) we then obtain for the quadrature phase variances

$$\begin{aligned} V(\hat{X}_1(t)) &= 1/4\left\{(e^{-2r}-1)\,e^{-\gamma t} + 2\bar{n}(1-e^{-\gamma t}) + 1\right\} \\ V(\hat{X}_2(t)) &= 1/4\left\{(e^{2r}-1)\,e^{-\gamma t} + 2\bar{n}(1-e^{-\gamma t}) + 1\right\} \end{aligned} \tag{3.74}$$

Equations (3.74) may also be obtained using the Q-function [Milburn and Walls, 1982b; Yuen, 1976] and similar results have been obtained by Hillery and Scully [1982a] using quantum Langevin equations.

From equation (3.74) we see that even if the bath is at absolute zero $\bar{n} = 0$, the effect of spontaneous quantum noise arising from the interaction with the vacuum, is to increase the fluctuations in the original squeezed state. This is illustrated in figure (3.5) where we depict the evolution of an initial squeezed state coupled to a reservoir at zero temperature. Similar results have been obtained by Caves [1981], by considering losses at the end mirrors of a cavity.

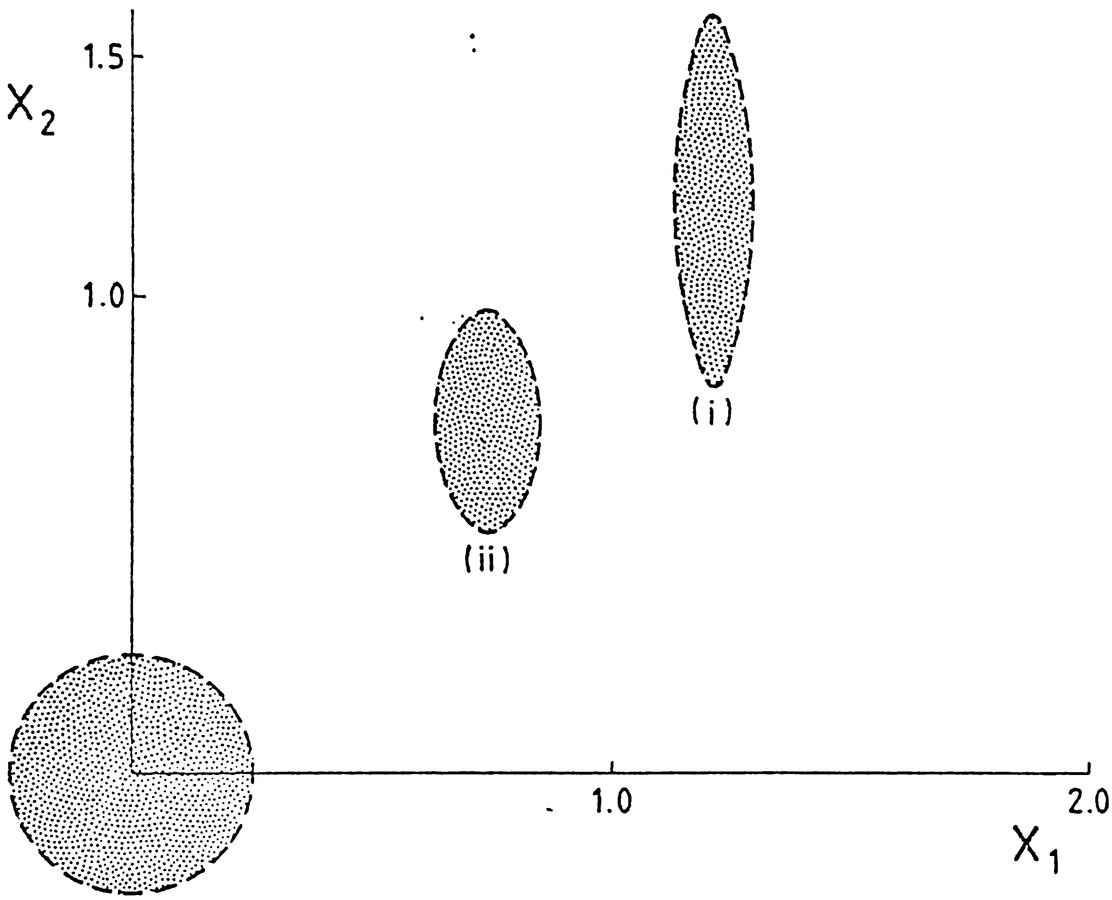

Figure 3.5 Complex amplitude diagram of an initially squeezed state undergoing damping. Figures i), ii) and iii) are for increasing increments of time.

One may view this phenomenon as "equilibration" between the squeezed state of the oscillator and the vacuum.

In the absence of a coherent quadratic term in the Hamiltonian we have just seen that an initially unsqueezed state will decay to the usual vacuum state. We now wish to consider what will happen if the Hamiltonian contains a quadratic term proportional to the generation of the scale changing transformation. We would expect to see competition arising between the irreversible effect of damping and the coherent effect of "squeezing". We are thus lead to consider the Hamiltonian

$$H = i\hbar |\chi \mathcal{E}^*| (a^2 - a^{\dagger 2}) + \hbar\omega a^\dagger a + a^\dagger \Gamma + a \Gamma^\dagger \qquad (3.75)$$

where we have chosen the phase of the driving field so that $\chi \mathcal{E}^*$ is purely imaginary. This Hamiltonian could represent three modes interacting via a $\chi^{(2)}$ susceptibility inside a cavity with two modes degenerate and one mode highly excited and thus treated clasically. The external driving has been set zero. The coupling to the heat both operators Γ , may be taken to model the damping of the intra-cavity mode. Any external coherent driving will not change the variances.

Equation (3.75) leads to the master equation

$$\frac{\partial \rho}{\partial t} = \kappa [a^2 - a^{\dagger 2}, \rho] + \gamma/2 (2a\rho a^\dagger - a^\dagger a\rho - \rho a^\dagger a) \qquad (3.76)$$

where $\kappa = |\chi \mathcal{E}^*|$ and we have assumed the reservoir to be at absolute zero.

Using the complex P-representation we then obtain the following equation

$$\frac{\partial P(\alpha,\beta)}{\partial t} = \left\{ \frac{\partial}{\partial \alpha} \left(\frac{\gamma \alpha}{2} + 2\kappa\beta \right) + \frac{\partial}{\partial \beta} \left(\frac{\gamma\beta}{2} + 2\kappa\alpha \right) - \kappa \left(\frac{\partial^2}{\partial \alpha^2} + \frac{\partial^2}{\partial \beta^2} \right) \right\} P(\alpha,\beta;t) \qquad (3.77)$$

Once again this equation may be solved by the method of Wang and Uhlenbeck. We assume the state of the system is initially coherent. That is,

$$P(\alpha, \beta; 0) = \delta(\alpha - \gamma)\,\delta(\beta - \gamma^*)$$

The solution is in the Gaussian form given in equation (3.71) where

$$\langle z(t)\rangle = \begin{pmatrix} \mathrm{Re}(\gamma)\, e^{\lambda_1 t} + i\,\mathrm{Im}(\gamma)\, e^{\lambda_2 t} \\ \mathrm{Re}(\gamma)\, e^{\lambda_1 t} - i\,\mathrm{Im}(\gamma)\, e^{\lambda_2 t} \end{pmatrix} \tag{3.78}$$

$$C(z) = -\kappa \begin{pmatrix} \frac{e^{\lambda_1 t}}{\lambda_1}\sinh\lambda_1 t + \frac{e^{\lambda_2 t}}{\lambda_2}\sinh\lambda_2 t & \frac{e^{\lambda_1 t}}{\lambda_1}\sinh\lambda_1 t - \frac{e^{\lambda_2 t}}{\lambda_2}\sinh\lambda_2 t \\ \frac{e^{\lambda_1 t}}{\lambda_1}\sinh\lambda_1 t - \frac{e^{\lambda_2 t}}{\lambda_2}\sinh\lambda_2 t & \frac{e^{\lambda_1 t}}{\lambda_1}\sinh\lambda_1 t + \frac{e^{\lambda_2 t}}{\lambda_2}\sinh\lambda_2 t \end{pmatrix} \tag{3.79}$$

and

$$\lambda_{1,2} = -\gamma/2 \mp 2\kappa \tag{3.80}$$

Using equation (1.54) we then obtain

$$V(\hat{X}_1(t)) = 1/4.\left\{\frac{2\kappa}{\lambda_1}(1 - e^{2\lambda_1 t}) + 1\right\} \tag{3.81}$$

$$V(\hat{X}_2(t)) = 1/4.\left\{-\frac{2\kappa}{\lambda_2}(1 - e^{2\lambda_2 t}) + 1\right\} \tag{3.82}$$

If $2\kappa \gg \gamma/2$ then equations (3.81) and (3.82) reduce to

$$V(\hat{X}_1(t)) = 1/4.\ e^{-2r}$$

$$V(\hat{X}_2(t)) = 1/4\ e^{2r} \tag{3.83}$$

where $r = 2\kappa t$. As expected the oscillator is driven into a squeezed state. From equations (3.78) we also have

$$
\begin{aligned}
\langle \hat{X}_1(t) \rangle &= \langle \hat{X}_1(0) \rangle e^{-r} \\
\langle \hat{X}_2(t) \rangle &= \langle \hat{X}_2(0) \rangle e^{r}
\end{aligned}
\tag{3.84}
$$

indicating that the quadrature with increased fluctuations is in fact amplified.

In the case where $\gamma/2 = 2\kappa$ we find that in the long time limit

$$
\begin{aligned}
\lim_{t\to\infty} V(\hat{X}_1(t)) &= 1/8 \\
V(\hat{X}_2(t)) &\to \infty
\end{aligned}
\tag{3.85}
$$

We see that there is a minimum value for the reduction of fluctuations in $\hat{X}_1$ while the fluctuations in $\hat{X}_2$ diverge.

Finally in the case $\gamma/2 \gg 2\kappa$ we find also in the long time limit that

$$
\begin{aligned}
\lim_{t\to\infty} V(X_1(t)) &= 1/4 - \kappa/\gamma \quad (\gamma \gg \kappa) \\
\lim_{t\to\infty} V(\hat{X}_2(t)) &= 1/4 + \kappa/\gamma \quad (\gamma \gg \kappa)
\end{aligned}
\tag{3.86}
$$

Once again there is a limit to the possible reduction in fluctuations.

We conclude that whenever a mode is damped there may be a limit to the size of the reduction of fluctuations. Alternatively, whenever the mode of interest is coupled to modes in their vacuum state there is a possible limitation to the reduction of fluctuations, depending on the relative magnitude of the size of this coupling and the size of the squeeze parameter.

Since the squeeze parameter is competing with the spontaneous damping of the system, the reduction of fluctuations is necessarily limited if the system is to reach a steady state as this in itself requires that the damping be greater than the squeeze parameter. At least in this simple case, the greatest reduction in fluctuations possible, given that the system reaches a steady state, is by a factor of two, which occurs when the squeezing effect and damping are precisely balanced. To obtain greater squeezing from a simple quadratic Hamiltonian such as that considered here we must necessarily look to a transient regime. We will not consider this point further in this thesis. In the next chapter we consider more realistic models for systems capable of producing light with reduced quadrature fluctuations in a steady state regime.

CHAPTER 4

GENERATION OF SQUEEZED STATES

In the previous chapter we made a number of heuristic comments concerning the production of squeezed states. These simple calculations were all obtained in a semiclassical limit where only the mode of interest was quantised, all other modes being treated classically. In the case of the parametric amplifier the results obtained predict that such a device, in the absence of damping will produce squeezed states, in agreement with other authors [Stoler, 1971; E.Y.C. Lu, 1972; U. Takahashi, 1965; M.T. Raiford, 1974; L. Mista et.al., 1977]. We also saw that in the presence of damping the degree of reduction in fluctuations is limited if the device reaches a steady state.

In this chapter we wish to present a fully quantum mechanical treatment of the degenerate parametric amplifier including the quantization of the pump and signal modes. We once more consider the device to be included in a cavity and thus capable of reaching a steady state. In this configuration the model is known as a degenerate parametric oscillator [Milburn and Walls, 1981].

We then include in the model an additional driving field at the idler frequency. The addition of a second driving field allows the amplified quadrature to be squeezed and, in addition provides a parameter which allows the direction of squeezing to be altered [Milburn and Walls, 1982a].

Finally we present an analysis of the squeezing obtainable in two other quantum optical devices (1) the non-linear (two-photon) absorber and (2) the optical bistable device. Both of these processes occur through a third order susceptibility in contrast to the second order effect of parametric amplification.

4.1 Quantum Analysis of a Degenerate Parametric Oscillator

As discussed in chapter three a parametric oscillator is an intra-cavity device involving the coupling of three modes of the electromagnetic field in an optical crystal with a non-linear second order susceptibility $\chi^{(2)}$. The frequencies of the three modes namely ω_p (pump), ω_i (idler) and ω_s (signal) obey the relation $\omega_p = \omega_s + \omega_i$. The corresponding wave vectors of the three waves satisfy the phase matching condition $\underset{\sim}{k}_s + \underset{\sim}{k}_i = \underset{\sim}{k}_p$. In the degenerate case the frequencies of the signal and idler modes coincide ($\omega_s = \omega_i = 1/2\,\omega_p$). The system may be described by the following Hamiltonian [Drummond et.al., 1981b].

$$H = H_{REV} + H_{IRREV} \tag{4.1}$$

with

$$\begin{aligned} H_{REV} = {} & \hbar\omega a_1^\dagger a_1 + 2\hbar\omega a_2^\dagger a_2 \\ & + i\hbar\kappa/2(a_1^{\dagger 2} a_2 - a_1^2 a_2^\dagger) \\ & + i\hbar(\varepsilon_1 a_1^\dagger e^{-i\omega t} - \varepsilon_1^* a_1 e^{i\omega t}) \\ & + i\hbar(\varepsilon_2 a_2^\dagger e^{-2i\omega t} - \varepsilon_2 a_2 e^{2i\omega t}) \end{aligned}$$

$$H_{IRREV} = (a_1\Gamma_1^\dagger + a_1^\dagger\Gamma_1) + (a_2\Gamma_2^\dagger + a_2^\dagger\Gamma_2)$$

Here $a_i^\dagger, a_i$ are creation and annihilation operators for the signal (i = 1) and pump (i = 2) modes. The coupling strength κ is real. Each mode is damped by coupling to a reservoir where Γ_i are reservoir operators. The frequency of the pump is 2ω. We have included the possibility that the signal/idler mode is also coherently driven. This model may also be considered as describing sub/second harmonic generation and has been analysed by Drummond, McNeil and Walls [1980, 1981b]. We shall follow their treatment.

Using techniques outlined in chapter one, to eliminate the heat bath operators we obtain the following master equation for the reduced density operator of the system.

$$\frac{\partial \rho}{\partial t} = \frac{1}{i\hbar}[H_{REV}, \rho] + \gamma_1(2a_1\rho a_1^\dagger - a_1^\dagger a_1\rho - \rho a_1^\dagger a_1) + \gamma_2(2a_2\rho a_2^\dagger - a_2^\dagger a_2\rho - \rho a_2^\dagger a_2) \tag{4.2}$$

where γ_i are the mode damping constants and we have assumed the reservoir to be at zero temperature.

Using the generalised P-representation of chapter one, equation (4.2) may be converted to a c-number Fokker-Planck equation. Expanding ρ by

$$\rho = \int P(\underset{\sim}{\alpha}, \underset{\sim}{\beta}) \frac{|\alpha\rangle\langle\beta^*|}{\langle\beta^*|\alpha\rangle} d\underset{\sim}{\alpha}\, d\underset{\sim}{\beta} \tag{4.3}$$

where $|\underset{\sim}{\alpha}\rangle = |\alpha_1, \alpha_2\rangle$

$|\underset{\sim}{\beta}\rangle = |\beta_1, \beta_2\rangle$

and $d\underset{\sim}{\alpha}\, d\underset{\sim}{\beta} = d\alpha_1\, d\alpha_2 \cdot d\beta_1 \cdot d\beta_2$

(the correspondence between c-numbers and operators is

$$(\alpha_1, \beta_1) \leftrightarrow (a_1, a_1^\dagger) \qquad (\alpha_2, \beta_2) \leftrightarrow (a_2, a_2^\dagger)$$

We obtain the following Fokker-Planck equation in the interaction picture

$$\begin{aligned}\frac{\partial P(\underset{\sim}{\alpha}, \underset{\sim}{\beta})}{\partial t} = \Big\{ &\frac{\partial}{\partial \alpha_1}\left[\gamma_1\alpha_1 - \varepsilon_1 - \kappa\beta_1\alpha_2\right] + \frac{\partial}{\partial \beta_1}\left[\gamma_1\beta_1 - \varepsilon_1^* - \kappa\alpha_1\beta_2\right] \\ &+ \frac{\partial}{\partial \alpha_2}\left[\gamma_2\alpha_2 - \varepsilon_2 + \frac{\kappa}{2}\alpha_1^2\right] \\ &+ \frac{\partial}{\partial \beta_2}\left[\gamma_2\beta_2 - \varepsilon_2^* + \frac{\kappa}{2}\beta_1^2\right] \\ &+ \frac{1}{2}\left[\frac{\partial^2}{\partial \alpha_1^2}(\kappa\alpha_2) + \frac{\partial^2}{\partial \beta_1^2}(\kappa\beta_2)\right]\Big\} P(\underset{\sim}{\alpha}, \underset{\sim}{\beta})\end{aligned} \tag{4.4}$$

Using the positive P-representation and the Ito rules, the above Fokker-Planck equation leads to the stochastic differential equations

$$\frac{\partial}{\partial t}\begin{pmatrix}\alpha_1\\ \\ \beta_1\end{pmatrix} = \begin{pmatrix}\varepsilon_1 + \kappa\beta_1\alpha_2 - \gamma_1\alpha_1\\ \\ \varepsilon_1^* + \kappa\alpha_1\beta_2 - \gamma_1\beta_1\end{pmatrix} + \begin{pmatrix}\kappa\alpha_2 & 0\\ \\ 0 & \kappa\beta_2\end{pmatrix}^{1/2}\cdot\begin{pmatrix}\eta_1(t)\\ \\ \eta_2(t)\end{pmatrix} \tag{4.4a}$$

and

$$\frac{\partial}{\partial t}\begin{pmatrix}\alpha_2\\ \\ \beta_2\end{pmatrix} = \begin{pmatrix}\varepsilon_2 - \kappa/2.\,\alpha_1^2 - \gamma_2\alpha_2\\ \\ \varepsilon_2^* - \kappa/2.\,\beta_1^2 - \gamma_2\beta_2\end{pmatrix} \tag{4.4b}$$

where $\eta_i(t)$ are delta correlated stochastic forces with zero mean.

We note that the stochastic equations for the pump mode contain no fluctuating forces at zero temperature. This enables the adiabatic elimination of the pump mode to be carried out in a straightforward fashion. We thus assume that $\gamma_2 \gg \gamma_1$ and eliminate the pump mode to obtain the following Fokker-Planck equation for the signal mode alone,

$$\begin{aligned}\frac{\partial P}{\partial t} = \Bigg\{ & \frac{\partial}{\partial\alpha_1}\left[\gamma_1\alpha_1 - \varepsilon_1 - \frac{\kappa}{\gamma_2}\left(\varepsilon_2 - \frac{\kappa\alpha_1^2}{2}\right).\beta_1\right]\\ & + \frac{\partial}{\partial\beta_1}\left[\gamma_1\beta_1 - \varepsilon_1^* - \frac{\kappa}{\gamma_2}\left(\varepsilon_2^* - \frac{\kappa\beta_1^2}{2}\right).\alpha_1\right]\\ & + 1/2\cdot\left[\frac{\partial^2}{\partial\alpha_1^2}\cdot\frac{\kappa}{\gamma_2}\left(\varepsilon_2 - \frac{\kappa}{2}\alpha_1^2\right) + \frac{\partial^2}{\partial\beta_1^2}\cdot\frac{\kappa}{\gamma_2}\left(\varepsilon_2^* - \frac{\kappa}{2}\beta_1^2\right)\right]\Bigg\} P(\alpha_1\beta_1)\end{aligned} \tag{4.5}$$

Thus the drift vector

$$A = \begin{pmatrix}\gamma_1\alpha_1 - \varepsilon_1 - \dfrac{\kappa}{\gamma_2}\left(\varepsilon_2 - \dfrac{\kappa\alpha_1^2}{2}\right)\beta_1\\ \\ \gamma_1\beta_1 - \varepsilon_1^* - \dfrac{\kappa}{\gamma_2}\left(\varepsilon_2^* - \dfrac{\kappa\beta_1^2}{2}\right).\,\alpha_1\end{pmatrix} \tag{4.6}$$

and the diffusion matrix

$$D = \begin{pmatrix} \frac{\kappa}{\gamma_2}\left(\varepsilon_2 - \frac{\kappa}{2}\alpha_1^2\right) & 0 \\ 0 & \frac{\kappa}{\gamma_2}\left(\varepsilon_2^* - \frac{\kappa}{2}\cdot\beta_1^2\right) \end{pmatrix} \tag{4.7}$$

Before obtaining the exact steady state solutions to equation (4.5) it is useful to consider the approximate solution obtained by linearising equation (4.5) about the deterministic steady state solutions, α_0, α_0^*.

It is shown by Drummond et.al. [1980] that in the deterministic limit $\gamma_2 \gg \gamma_1$ the solution for α_2 is

$$\alpha_2^0 = \frac{(\varepsilon_2 - \kappa/2\,.\,\alpha_1^2)}{\gamma_2} \tag{4.8}$$

while the equation for the fundamental mode is

$$\dot{\alpha}_1 = \varepsilon_1 - \gamma_1\alpha_1 + \frac{\kappa\varepsilon_2}{\gamma_2}\cdot\alpha_1^* - \frac{\kappa^2}{2\gamma_2}\cdot|\alpha_1|^2\cdot\alpha_1 \tag{4.9}$$

(where we have used the result that in the deterministic limit $\beta_1 \doteq \alpha_1^*$). We now summarise the results of Drummond et.al. [1980].

When only the pump mode is driven coherently (i.e. $\varepsilon_1 = 0$) we obtain the process of sub-harmonic generation where input photons of frequency 2ω are converted to photons of frequency ω. The steady state solutions are

$$\alpha_1^0 = \begin{cases} 0 & |\varepsilon_2| < \varepsilon_2^c \\ \pm\left[\frac{2}{\kappa}\left(\varepsilon_2 - \varepsilon_2^c\right)\right]^{1/2} & \varepsilon_2 \geqslant \varepsilon_2^c \\ \pm i\left[\frac{2}{\kappa}\left(|\varepsilon_2| - \varepsilon_2^c\right)\right]^{1/2} & \varepsilon_2 \leqslant -\varepsilon_2^c \end{cases} \tag{4.10}$$

where $\mathcal{E}_2^c = \gamma_1 . \gamma_2 / \kappa$ is the threshold for parametric oscillation. In what follows we shall assume that $\mathcal{E}_2$ is real and positive and thus we will only consider "in phase" solutions, that is α_1^o real.

If both modes are coherently driven so that $\mathcal{E}_1$ and $\mathcal{E}_2$ are both non-zero and positive the in-phase solution obeys the equation

$$\alpha \quad + \left(2 \frac{\gamma_1 \gamma_2}{\kappa^2} - \frac{2 \mathcal{E}_2}{\kappa} \right) \alpha - \frac{2 \gamma_2 \mathcal{E}_1}{\kappa^2} = 0 \tag{4.11}$$

The discriminant of this cubic is

$$\Delta = (2/3\kappa)^3 . (\mathcal{E}_2^c - \mathcal{E}_2)^3 + (\gamma_2 \mathcal{E}_1 / \kappa)^2 \tag{4.12}$$

The solutions are given by z_i where

$$\begin{aligned} z_1 &= s_1 + s_2 \\ z_2 &= -1/2 . (s_1 + s_2) + \frac{i\sqrt{3}}{2} (s_1 - s_2) \\ z_3 &= -1/2 (s_1 + s_2) - \frac{i\sqrt{3}}{2} (s_1 - s_2) \end{aligned} \tag{4.13}$$

where $$s_{1,2} = \left\{ \frac{\gamma_2 \mathcal{E}_1}{\kappa^2} \pm \Delta^{1/2} \right\}^{1/3} \tag{4.14}$$

When $$\mathcal{E}_2 = \mathcal{E}_2^c , \quad \Delta = \left(\frac{\gamma_2 \mathcal{E}_1}{\kappa^2} \right)^2$$

and $$s_1 = \left(\frac{2 \gamma_2 \mathcal{E}_1}{\kappa^2} \right)^{1/3} , \quad s_2 = 0$$

Since $\Delta \geqslant 0$ there is one real solution, given by

$$\alpha_1^o = \left(\frac{2 \gamma_2 \mathcal{E}_1}{\kappa^2} \right)^{1/3} \tag{4.15}$$

We now return to the linearised solution of equation (4.5). Expanding about the steady state solutions we find that

$$A \simeq \begin{bmatrix} -\left(\varepsilon_1 + \frac{K^2}{\gamma_2}\alpha_0 |\alpha_0|^2\right) + \alpha_1\left(\gamma_1 + \frac{K^2}{\gamma_2}|\alpha_0|^2\right) - \beta\frac{K}{\gamma_2}\left(\varepsilon_2 - \frac{K}{2}\alpha_0^2\right) \\ -\left(\varepsilon_1^* + \frac{K^2}{\gamma_2}\alpha_0^* |\alpha_0|^2\right) + \beta\left(\gamma_1 + \frac{K^2}{\gamma_2}|\alpha_0|^2\right) - \alpha\frac{K}{\gamma_2}\left(\varepsilon_2^* - \frac{K}{2}\alpha_0^{*2}\right) \end{bmatrix} \tag{4.16}$$

while

$$D \simeq \begin{bmatrix} \frac{K}{\gamma_2}\left(\varepsilon_2 - \frac{K}{2}\alpha_0^2\right) & 0 \\ 0 & \frac{K}{\gamma_2}\left(\varepsilon_2^* - \frac{K}{2}\alpha_0^{*2}\right) \end{bmatrix} \tag{4.17}$$

where for convenience we have dropped the subscript.

The Fokker-Planck equation satisfied by equations (4.16) and (4.17) satisfies the potential conditions

$$\frac{\partial V_1}{\partial \beta} = \frac{\partial V_2}{\partial \alpha} \tag{4.18}$$

where

$$V_i = \sum_k (D^{-1})_{ik} \{ 2A_k - \sum_j \frac{\partial}{\partial x_j} D_{kj}(x) \}$$

Thus the steady state solution is

$$P(\alpha,\beta) = \mathcal{N} \exp\left\{2\beta\alpha - (a\alpha^2 + a^*\beta^2 - b\alpha - b^*\beta)\right\} \tag{4.19}$$

where

$$a = \frac{(\gamma_1\gamma_2 + K^2|\alpha_0|)}{K(\varepsilon_2 - K/2\ \alpha_0^2)} \tag{4.20}$$

$$b = \frac{2(\gamma_2\varepsilon_1 + K^2\alpha_0|\alpha_0|^2)}{K.(\varepsilon_2 - K/2\ \alpha_0^2)} \tag{4.21}$$

As we are only considering in phase solutions we may use equation (4.11) to write equation (4.19) in Gaussian form. Using equation (4.11) it is easily verified that

$$2\, \alpha_0\, (a - 1) \quad = \quad b$$

and thus equation (4.19) may be written as

$$P(\alpha, \beta) \quad = \mathcal{N}' \exp \left\{ 2(\alpha-\alpha_0)(\beta-\beta_0) - a\left[(\alpha-\alpha_0)^2 + (\beta-\beta_0)^2\right] \right\} \qquad (4.22)$$

Comparison of equation (4.22) with the complex P-representation for a squeezed state (equation (3.45)) we see that this model produces squeezed states in the linearised approximation with the reduction of fluctuations determined by the parameter a . The variances in $\hat{X}_1$ and $\hat{X}_2$ follow by inspection of equation (4.22) together with (equation (1.54))

$$V(\hat{X}_1) \quad = \quad 1/4 . \left(1 + \frac{1}{a-1} \right) \qquad (4.23)$$

$$V(\hat{X}_2) \quad = \quad 1/4 . \left(1 - \frac{1}{a+1} \right) \qquad (4.24)$$

We now discuss two special cases.

Firstly we consider the situation of pure sub-harmonic generation ($\varepsilon_1 = 0$). In this case

$$a \quad = \begin{cases} \dfrac{\varepsilon_2^c}{\varepsilon_2} & \varepsilon_2 < \varepsilon_2^c \\[2em] 2 . \dfrac{\varepsilon_2}{\varepsilon_2^c} - 1 & \varepsilon_2 > \varepsilon_2^c \end{cases} \qquad (4.25)$$

Thus below threshold we find in the linearised analysis that

$$V(\hat{X}_1) \quad = \quad 1/4 . \left(1 + \varepsilon_2 / (\varepsilon_2^c - \varepsilon_2) \right) \qquad (4.26a)$$

$$V(\hat{X}_2) \quad = \quad 1/4 \left(1 - \varepsilon_2 / (\varepsilon_2^c + \varepsilon_2) \right) \qquad (4.26b)$$

while above threshold

$$V(\hat{X}_1) = 1/4.\left(1 + \frac{\mathcal{E}_2/2}{\mathcal{E}_2 - \mathcal{E}_2^c}\right) \qquad (4.27a)$$

$$V(\hat{X}_2) = 1/4.\left(1 + \frac{\mathcal{E}_2^c}{2\mathcal{E}_2}\right) \qquad (4.27b)$$

We see that the squeezing is obtained in the $\hat{X}_2$ quadrature. The quadrature with increased fluctuations $(\hat{X}_1)$ is amplified. Furthermore the maximum reduction in fluctuations occurs at $\mathcal{E}_2 = \mathcal{E}_2^c$ where $V(\hat{X}_2) = 1/8$. At this point the fluctuations in $\hat{X}_1$ diverge indicating the breakdown of the linearised analysis. Above threshold both $V(\hat{X}_1)$ and $V(\hat{X}_2)$ return to the coherent value of $1/4$ as $\mathcal{E}_2$ is increased.

We now consider the case of sub/second harmonic generation in which both $\mathcal{E}_1$ and $\mathcal{E}_2$ are non-zero. We consider the pump field amplitude held fixed at $\mathcal{E}_2^c$, the threshold for parametric oscillation and vary the idler amplitude $\mathcal{E}_1$. By comparing the complex P-representation for a pure squeezed state with equation (4.22) we expect that the quadrature carrying the reduced fluctuations will change as the sign of a is changed. (We assume that the $\hat{X}_1$ axis is always aligned in phase with the amplitude of the output field). Since α_0 is now an increasing function of $\mathcal{E}_1$ the sign of a will change at $\mathcal{E}_1 = \mathcal{E}_1^c = (2\gamma_1^3\gamma_2/\kappa^2)^{1/3}$. At this point $a \to \infty$ and thus $V(\hat{X}_1) = V(\hat{X}_2) = 1/4$, indicating the squeeze parameter has become zero. For $\mathcal{E}_1 > \mathcal{E}_1^c$ the $\hat{X}_1$ quadrature is squeezed. Furthermore since the output is real and proportional to $\mathcal{E}_1^{1/3}$ we see that we now have reduced fluctuations in the amplified quadrature. As $\mathcal{E}_1 \to \infty$ we find that $V(\hat{X}_1) \to 1/6$ and $V(\hat{X}_2) \to 1/2$.

This change in the direction of squeezing while keeping the phase of the output unchanged should result in $g^{2}(0)$ decreasing from a value greater than one to something less than one, indicating photon antibunching (see section 3.1). We thus have the remarkable effect that by increasing the intensity of one driving field we are able to decrease the photon number fluctuations. This result is indeed confirmed by the exact analysis we shall shortly present.

It is necessary to make a few comments concerning the validity of the linearised analysis. It has been shown by Lugiato and Strini [1982] that the linearised analysis is expected to be correct when $\gamma_1 \gamma_2 / K^2 \gg 1$, in which case the fluctuations around the steady state are small. This prediction may also be seen by writing equation (4.20) evaluated at $\mathcal{E}_2 = \mathcal{E}_2^c$ (with $\mathcal{E}_1 = 0$) as

$$a = \frac{(1 + (K^2/\gamma_1 \gamma_2)|\alpha_0|^2)}{(1 - (K^2/2\gamma_1 \gamma_2)|\alpha_0|^2)} \tag{4.28}$$

The maximum reduction in fluctuations will occur whenever $a = 1$. The deviations from this depend on how fluctuations increase $|\alpha_0|^2$ near threshold. However if $K^2/\gamma_1\gamma_2 \ll 1$ these fluctuations are negligible in equation (4.28) and $a \simeq 1$. Thus, $V(\hat{X}_2) = 1/8$ when $\gamma_1 \gamma_2 / K^2 \gg 1$. Of course in this limit the threshold for parametric oscillation is very large.

Interestingly the exact Fokker-Planck equation (equation 4.5) itself satisfies potential conditions and an exact steady state solution may be obtained, thus enabling an exact determination of the steady state statistics. The solution as given by Drummond et.al. [1981] is

$$P(\alpha,\beta) = \mathcal{N} \exp\left\{ 2\alpha\beta + \frac{2\bar{\gamma}_1\gamma_2}{K^2}\ln(C^2 - K^2\alpha^2) + \left(\frac{2\bar{\gamma}_1\gamma_2}{K^2}\right)^* \ln(C^{*2} - K^2\beta^2) + \frac{2\gamma_2\varepsilon_1}{CK}.\ln\left(\frac{C+K\alpha}{C-K\alpha}\right) + 2\left(\frac{\gamma_2\varepsilon_1}{CK}\right)^* \ln\left(\frac{C^*+K\beta}{C^*-K\beta}\right) \right\} \tag{4.31}$$

where

$$C = \sqrt{2K\varepsilon_2}$$

$$\bar{\gamma}_1 = \gamma_1 - K^2/2\gamma_2$$

The moments defined by

$$I_{nn'} \equiv \langle a^{n} a^{+n'} \rangle = \int_{C_1}\int_{C_2} \alpha^{n}.\beta^{n'}.P(\alpha,\beta)\,d\alpha\,d\beta \tag{4.30}$$

are evaluated by choosing the contours to correspond to those which define the Gauss hypergeometric functions. These integration paths encircle each pole and transverse the Riemann sheets so that the initial and final values of the integrand are equal, allowing partial integration to be defined. The result of Drummond et.al. [1981]

$$I_{nn'} = \mathcal{N}' \sum_{m=0}^{\infty} \frac{2^m}{m!} \left(-\frac{C}{K}\right)^{m+n} \cdot \left(-\frac{C^*}{K}\right)^{m+n'} \cdot {}_2F_1(-(m+n), j_1, j_2; 2)$$

$$\times\ {}_2F_1(-(m+n'), j_1^*, j_2^*; 2)$$

where

$$j_1 = \frac{2\bar{\gamma}_1\gamma_2}{K^2} + \frac{2\gamma_2\varepsilon}{CK} \quad ; \quad j_2 = \frac{4\bar{\gamma}_1\cdot\gamma_2}{K^2}$$

In order to investigate the quadrature phase variances it is necessary to consider

$$V(\hat{X}_1) = 1/4.\left(\langle(\alpha+\beta)^2\rangle - \langle\alpha+\beta\rangle^2\right) + 1/4$$

$$V(\hat{X}_2) = -1/4\left(\langle(\alpha-\beta)^2\rangle - \langle\alpha-\beta\rangle^2\right) + 1/4$$

We firstly consider the case of pure subharmonic generation $(\varepsilon_1 = 0)$ In figure 4.1 we have plotted the semiclassical intensity and the mean intensity $\langle\alpha\beta\rangle$. We see that there is a second order phase transition at ε_2^c . It is interesting to note that the exact intensity lies below the semiclassical intensity above threshold. This is due to the presence of high order quantum correlations.

In figure 4.2 $V(\hat{X}_2)$ is plotted as a function of the scaled driving field ε_2 . We see that as the driving field is increased from zero $V(\hat{X}_2)$ decreases from its coherent value of 1/4 to a minimum at $\varepsilon_2 = \varepsilon_2^c$, the threshold for parametric oscillation.

While the precise minimum value obtained depends on the size of $\gamma_1\gamma_2/\kappa^2$ the minimum possible value of 1/8 is closely approached as ε_2 increases. This corresponds to a reduction in fluctuations by a factor of two. As ε_2 is increased above threshold $V(\hat{X}_2)$ increases again approaching a value of 1/4 for large driving fields. These results are in accord with the linearised analysis.

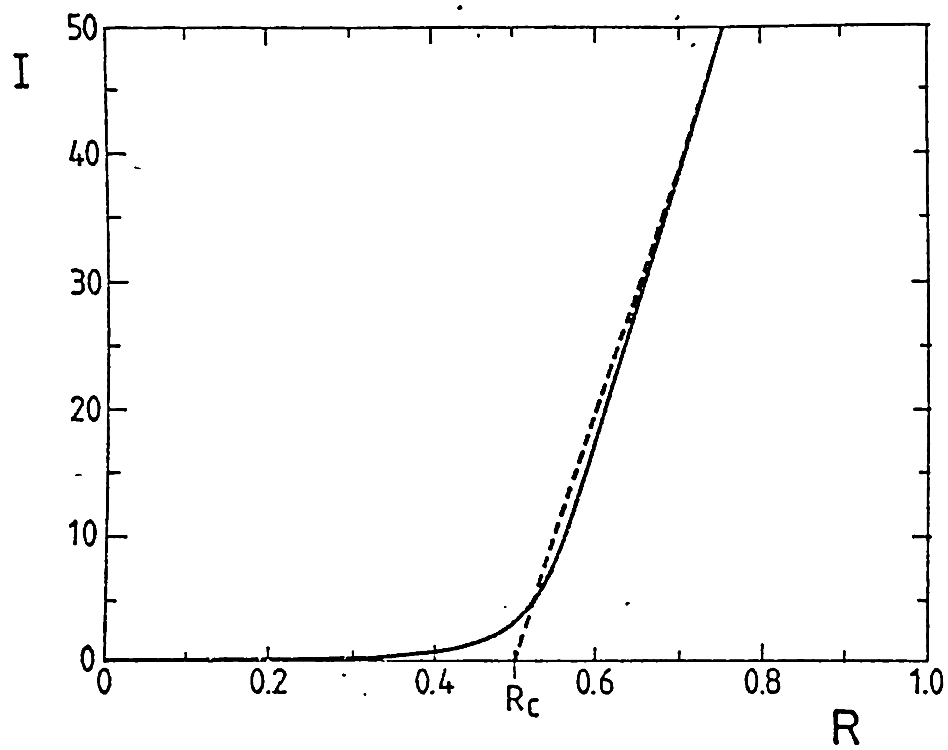

Figure 4.1 Intensity output for degenerate parametric oscillation. The semiclassical intensity (---) and the exact intensity (——) are plotted against driving field. ($\chi = 1.0$; $\gamma_1 = 0.5$ $\gamma_2 = 100$)

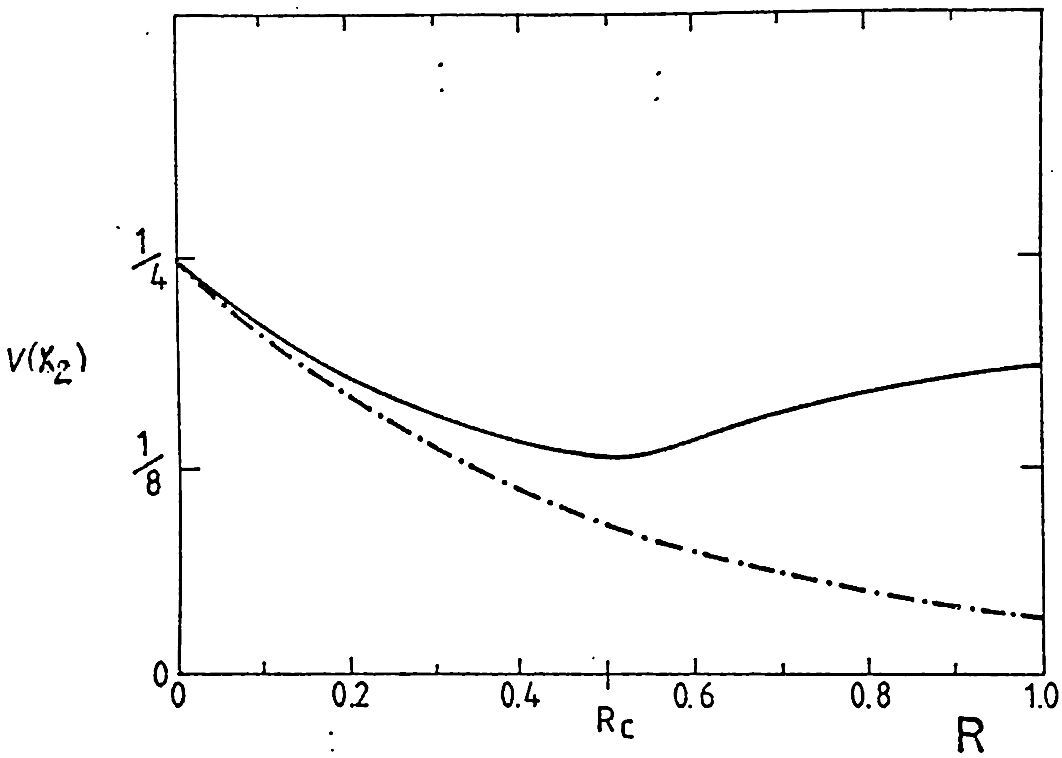

Figure 4.2 A plot of $V(\hat{X}_2)$ versus driving field, $R = \mathcal{E}_2 \cdot K/\gamma_2$, with $K = 1$, $\gamma_1 = 0.5$, $\gamma_2 = 100$. $\mathcal{E}_2^c = 50$
For comparison $V(\hat{X}_2)$ for a pure squeezed state is also shown (-.-.-)

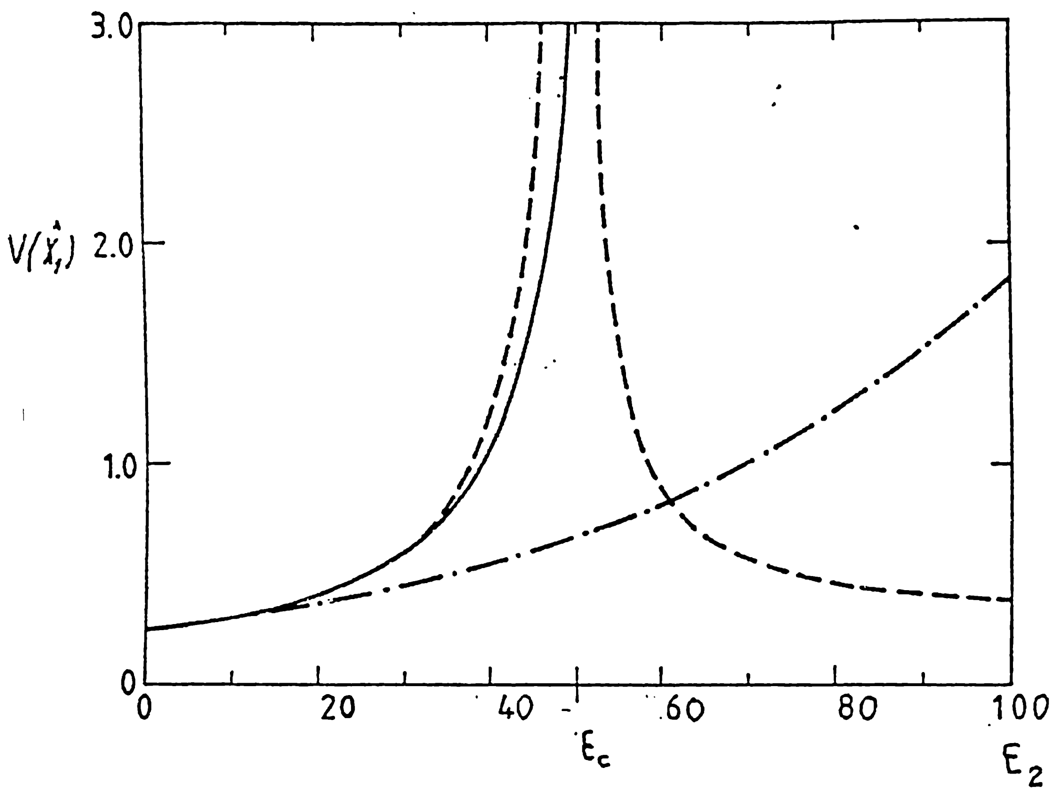

Figure 4.3 A plot of $V(\hat{Y}_1)$ versus driving field for exact moments (——) and linearised approximation (----). The result for an ideal squeezed state is also shown (-·-·-).

In figure 4.3 [Milburn and Walls, 1981] we have plotted $V(\hat{X}_1)$ versus driving field. We see that $V(\hat{X}_1)$ shows a corresponding increase at threshold. However as the driving field is increased above threshold we find that $V(\hat{X}_1)$ continues to increase. This is due to the splitting of the distribution function (equation (4.22)) into two peaks above threshold (each centred on the positive and negative solutions in equation (4.10)).

If we use one of these peaks evaluated in the linearised approximation to determine $V(\hat{X}_1)$ we see that it also returns to the coherent value of 1/4 for large driving fields.

The above exact analysis is in accord with the predictions of the linearised theory. It is clear that the sub-harmonic generator will produce nearly ideal squeezed states below threshold, however there is a limit to the reduction in fluctuations which may be obtained. The maximum reduction in fluctuations that may be obtained in an ideal (zero thermal fluctuations) degenerate parametric oscillator is a factor of two for a device operated in the region of threshold. Thermal fluctuations would tend to reduce this factor still further.

This minimum reduction in fluctuations confirms the simplistic analysis of chapter three which suggested that for steady state operation the degree of squeezing is necessarily limited.

We now turn to the case where both sub and second harmonics are coherently driven. In figure 4.4 we have plotted $V(\hat{X}_1)$, $V(\hat{X}_2)$ and $g^2(0)$ versus $\mathcal{E}_1$ with $\mathcal{E}_2$ held fixed at threshold $(\mathcal{E}_2 = \mathcal{E}_2^c)$ for convenience we have scaled the quadrature phase variances by a factor

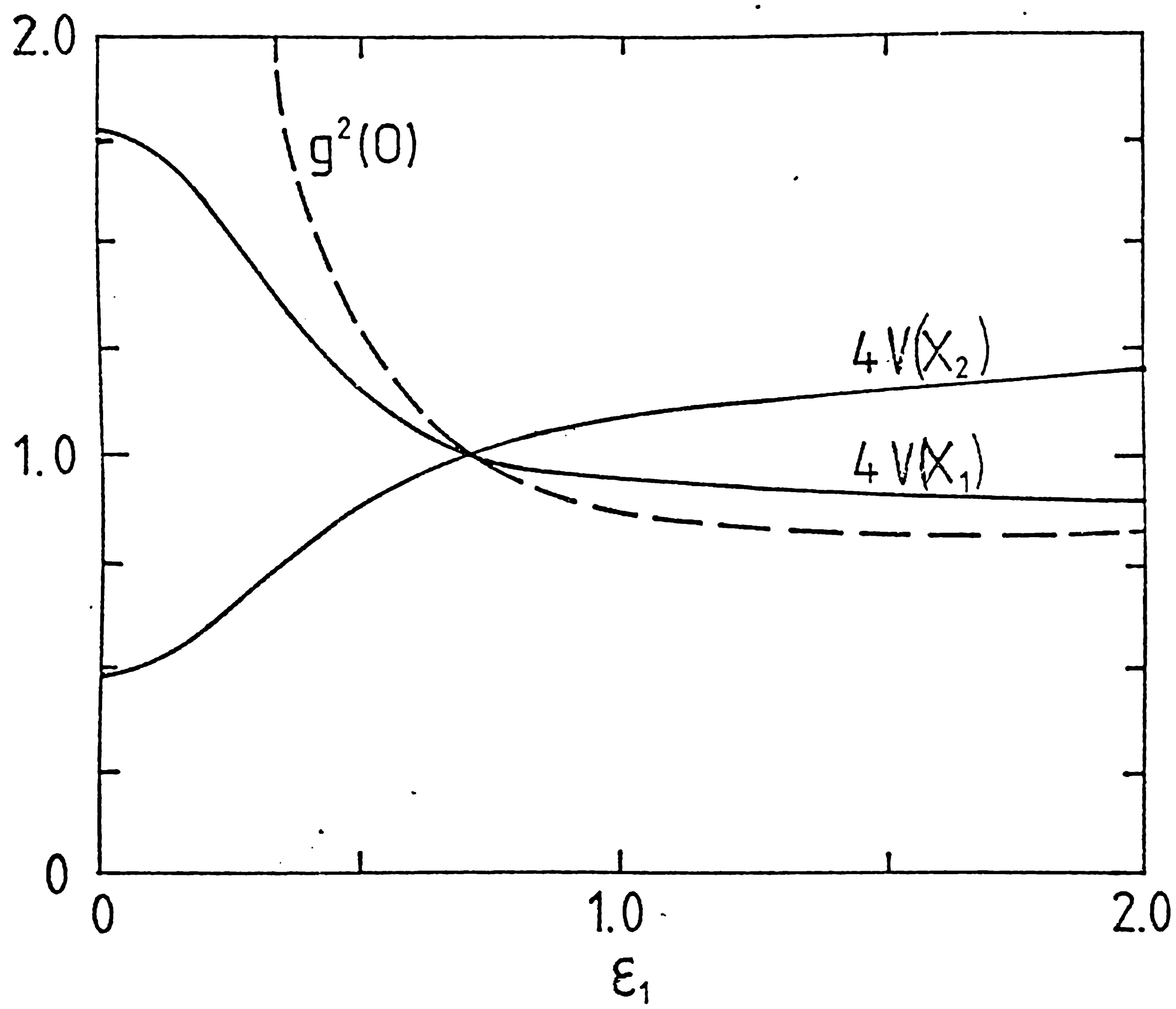

Figure 4.4 $V(X_1)$ (——), $V(X_2)$ (——) and $g^2(0)$ (– – –) versus ε_1 for sub/second harmonic generation $\varepsilon_2 = \varepsilon_2^c = 5.0$, $\gamma_1 = 1$, $\gamma_2 = 100$, $K = 20.0$.

of four [see Milburn and Walls, 1982]. At $\mathcal{E}_1 = 0$, the component carrying the excitation $\hat{X}_1$ is not squeezed, the $\hat{X}_2$ quadrature however is squeezed. The photon correlation function $g^2(0)$ is greater than unity reflecting the enhanced amplitude fluctuations, as discussed in section 3.1.

As $\mathcal{E}_1$ is increased beyond $\mathcal{E}_1^c$ (where the sign of the squeeze parameter changes) the quadrature carrying the reduced fluctuations changes from the imaginary $\hat{X}_2$ component to the real $\hat{X}_1$ component. $g^2(0)$ becomes less than unity as the quadrature carrying the coherent excitation is now squeezed. This confirms the results of the linearised analysis, which for the parameters chosen in figure 4.4 predict $\mathcal{E}_1^c = 1/\sqrt{2}$. Note that the linearised analysis suggests that at $\mathcal{E}_1 = \mathcal{E}_1^c$, $V(\hat{X}_1) = V(\hat{X}_2) = 1/4$ and thus $g^2(0)$ should be unity. This is also confirmed by the exact analysis. The addition of the extra field has permitted the quadrature carrying the reduced fluctuations to be amplified, at the expense of a smaller reduction.

It seems clear that for a degenerate parametric oscillator operated in the steady state the greatest reduction in fluctuations possible is a factor of two. This conclusion is also reached by Lugiato and Strini [1982] who relax the condition that $\gamma_2 \gg \gamma_1$ and proceed via a linearised analysis.

4.2 Squeezing in Nonlinear Absorption and Dispersive Optical Bistability

We now consider two further single mode systems of nonlinear optics. Once again the medium is assumed to be contained in a cavity and the cavity mode is both damped and coherently driven by an external driving field. We shall treat the two cases where the medium

(a) is a two photon absorber

(b) has a cubic nonlinear polarizability leading to dispersive optical bistability

The model we shall present has been treated in depth by Drummond et.al., [1981b] and Drummond and Walls [1980].

A Hamiltonian which describes both systems is

$$H = \sum_{i=1}^{5} H_i$$

$$\begin{aligned} H_1 &= \hbar\omega_c a^\dagger a \\ H_2 &= i\hbar(\mathcal{E} e^{-i\omega_L t} a^\dagger - \mathcal{E}^* e^{i\omega_L t} a) \\ H_3 &= a\Gamma_c^\dagger + a^\dagger \Gamma_c \\ H_4 &= \hbar\chi'' a^{\dagger 2} a^2 \\ H_5 &= a^{\dagger 2}\Gamma_2 + a^2 \Gamma_2^\dagger \end{aligned} \qquad (4.32)$$

where H_1 describes the cavity mode a with frequency ω_c. H_2 describes the coupling with the coherent driving field with amplitude $\mathcal{E}$ and frequency ω_L. H_3 describes the coupling to the cavity reservoirs $\Gamma_c, \Gamma_c^\dagger$. H_4 describes a nonlinear dispersive medium with a cubic nonlinear susceptibility. H_5 describes a intracavity two photon absorber with reservoir operators $\Gamma_2, \Gamma_2^\dagger$. In the interaction picture of frequency ω_L the master equation is

$$\frac{\partial \rho}{\partial t} = \sum_{j=1}^{5} L_j(\rho)$$

$$\begin{aligned}
L_1(\rho) &= -i\Delta\,[a^{\dagger}a, \rho] \\
L_2(\rho) &= [\varepsilon a^{\dagger} - \varepsilon^{*}a, \rho] \\
L_3(\rho) &= \gamma\,(2a\rho a^{\dagger} - \rho a^{\dagger}a - a^{\dagger}a\rho) \\
L_4(\rho) &= -i\chi''\,[a^{\dagger 2}a^{2}, \rho] \\
L_5(\rho) &= \chi'\,(2a^{2}\rho a^{\dagger 2} - \rho a^{\dagger 2}a^{2} - a^{\dagger 2}a^{2}\rho)
\end{aligned} \tag{4.33}$$

where χ' is the two photon absorption rate, γ is the cavity mode damping rate, $\Delta = \omega_c - \omega_L$ and the temperature of the cavity is taken to be zero, so that only quantum fluctuations are present. Using the complex P-representation Drummond et.al. [1981] have obtained the following Fokker-Planck equation

$$\frac{\partial P(\alpha,\beta)}{\partial t} = \Big[\frac{\partial}{\partial\alpha}\left(\kappa\alpha + 2\chi\alpha^{2}\beta - \varepsilon\right) + \frac{\partial}{\partial\beta}\left(\kappa\beta + 2\chi^{*}\beta^{2}\alpha - \varepsilon^{*}\right)$$

$$- \chi\frac{\partial^{2}}{\partial\alpha^{2}}\alpha^{2} - \chi^{*}\frac{\partial^{2}}{\partial\beta^{2}}\beta^{2}\Big]P(\alpha,\beta) \tag{4.34}$$

where $\chi = \chi' + i\chi''$

and $\kappa = \gamma + i\Delta$

Before proceeding to the exact steady state solution to equation (4.27) it is instructive to consider the result of a linearised analysis.

It is shown by Drummond [1979] that the covariance matrix for α, β (equation (1.54)) in a linearised analysis is given by

$$C = \frac{1}{2a'\lambda} \begin{pmatrix} -d\,a^* a' & |d|^2 a' \\ |d|^2 a' & -d^* a\, a' \end{pmatrix} \qquad (4.35)$$

where

$$a = \kappa + 4\chi n \quad ; \quad a' = \gamma + 4\chi' n$$

$$d = 2\chi\alpha_0^2 \quad ; \quad \lambda = |a|^2 - |d|^2$$

and $n = |\alpha_0|^2$, with α_0 being the deterministic steady state solution. This linearised analysis is expected to be a good approximation for n large. At $|a|^2 = |d|^2$ points of instability arise at which the linearised analysis breaks down.

Using equation (1.54) we find

$$C(\hat{X}_1, \hat{X}_2) = \frac{1}{4a'\lambda} \begin{pmatrix} \mathrm{Re}(z) + w & \mathrm{Im}(z) \\ \mathrm{Im}(z) & -\mathrm{Re}(z) + w \end{pmatrix} + \tfrac{1}{4} \cdot I \qquad (4.36)$$

where $z = -da^*a'$ and $w = |d|^2 a'$. This matrix is off diagonal which implies that in general, the squeezing direction may be different from the direction of coherent excitation.

We now turn to the simpler case of two photon absorption, $\chi'' = 0$ on resonance $(\Delta = 0)$. If we choose the driving field real, it is easily verified that one steady state solution α_0 is real, as are a and d . The covariance matrix $C(\hat{X}_1 \hat{X}_2)$ is then found to be diagonal with

$$V(\hat{X}_1) = 1/4.\left(1 - d/(a+d)\right)$$

$$V(\hat{X}_2) = 1/4.\left(1 + d/(a-d)\right)$$

If the linear damping γ is negligible in relation to χn we find

$$V(\hat{X}_1) = 1/6 \qquad ; \qquad V(\hat{X}_2) = 1/2$$

We see that the reduction of fluctuations appears in the quadrature in phase with the driving field and is asymptotically independent of the driving field. Any appreciable linear damping will further diminish this reduction in fluctuations. We note also that in this case it is not possible for $|a|^2$ to equal $|d|^2$ and thus there is no critical point divergence.

To consider the situation of dispersive optical bistability $(\chi' = 0)$ we choose the driving field to be purely imaginary. Then $\mathrm{Im}(\alpha_0) = \gamma\,\mathrm{Re}(\alpha_0)/(\Delta + 2\chi n)$ and α_0 is approximately real when $\gamma \ll (\Delta + 2\chi n)$. The relevant parameters take the form

$$a = \gamma + i(4\chi'' n + \Delta)$$
$$a' = \gamma$$
$$d = 2i\chi'' n$$

For certain values of the parameters it is possible for $|a|^2 = |d|^2$ and we expect a critical point to occur at which the linearised analysis breaks down. This occurs when

$$\lambda = \gamma^2 + \Delta^2 + 8\Delta\chi'' n + 12\chi''^2 n = 0$$

At the critical point the fluctuations in both $\hat{X}_1$ and $\hat{X}_2$ diverge. This divergence of the linearised analysis occurs at the instability points responsible for the bistable nature of the output intensity [Drummond and Walls, 1980].

The condition for bistable operation is, $\Delta\chi'' < 0$ [Drummond and Walls, 1980]. Hence for a positive χ'' we require Δ to be negative.

The covariance matrix in $\hat{X}_1$ and $\hat{X}_2$ is no longer diagonal and is given by,

$$C(\hat{X}_1,\hat{X}_2) = \frac{1}{4\lambda}\begin{pmatrix} -2\chi''\Delta n - 4\chi''^2 n & -2\chi'' n^2 \gamma \\ -2\chi'' n^2 \gamma & 2\chi''\Delta n + 12\chi''^2 n \end{pmatrix} + 1/4\, I \tag{4.37}$$

If the detuning is sufficiently large and we are in a region where $\chi n/\Delta$ and $\gamma n/\Delta$ are small,

$$C(\hat{X}_1\hat{X}_2) \approx \frac{1}{4\lambda}\begin{pmatrix} 2n\,|\chi''\Delta| & 0 \\ 0 & -2n|\chi''\Delta| \end{pmatrix} + 1/4.I$$

where we have assumed we are operating in a bistable regime where $\chi''\Delta < 0$. Thus

$$V(\hat{X}_1) \approx 1/4\ (1 + 2n|\chi''\Delta|/\lambda)$$

$$V(\hat{X}_2) = 1/4.\ (1 - 2n|\chi''\Delta|/\lambda)$$

We thus expect to get a small reduction of fluctuations in the $\hat{X}_2$ quadrature on the low transmission branch, for parameters which validate the approximations stated above.

On the high transmission branch the terms in n^2 are expected to dominate. If we consider the case where $\gamma/\chi'' << 1$, the covariance matrix is approximately diagonal. As n becomes large we then have,

$$V(\hat{X}_1) = 1/4.(1 - 4\chi''^2 n^2/\lambda)$$

$$V(\hat{X}_2) = 1/4.(1 + 12\chi''^2 n^2/\lambda)$$

As $n^2 \to \infty$, $V(\hat{X}_1) \to 1/6$ and $V(\hat{X}_2) \to 1/2$ We expect that on the high transmission branch the fluctuations in the $\hat{X}_1$ quadrature are reduced. Since this quadrature carries the coherent excitation (when $\gamma\,\mathrm{Re}(\alpha)/(\Delta + 2\chi'' n) \ll 1$) we expect to see a $g^2(0)$ less than unity on the high transmission branch.

The many approximations in the previous linearised analysis means that such an analysis can only give a general indication of how the quadrature phase fluctuations behave. Fortunately the Fokker-Planck equation (equation (4.27)) satisfies potential conditions and thus admits a steady state solution. We turn to this solution to obtain the exact statistics of nonlinear absorption and dispersive optical bistability.

The steady state solution as obtained by Drummond et.al. [1981] is,

$$P(\alpha, \beta) = \alpha^{c-2} \cdot \beta^{c^*-2} \exp\left\{ \varepsilon/\chi\,(1/\alpha + 1/\beta) + 2\alpha\beta \right\} \qquad (4.38)$$

where $c = (\gamma + i\Delta)/\chi$ and the phase of the driving field has been chosen so that ε/χ is real. The evaluation of the moments $I_{nn'}$ (equation (4.30)) using this solution proceeds by defining each path of integration to be a Hankel path from $-\infty$ on the real axis around the origin in an anticlockwise direction and back to $-\infty$. The result of Drummond et.al. is

$$I_{nn'} = \left(\frac{\varepsilon}{\chi}\right)^{n+n'} \cdot \frac{\Gamma(c).\,\Gamma(c^*).\,{}_0F_2(n'+c^*, n+c, 2|\varepsilon/\chi|^2)}{\Gamma(n'+c^*).\,\Gamma(n+c)\,{}_0F_2(c^*, c, 2|\varepsilon/\chi|^2)} \qquad (4.39)$$

where ${}_0F_2$ is a generalised hypergeometric function.

In figure 4.5 we plot $g^2(0)$ $V(\hat{X}_1)$ and $V(\hat{X}_2)$ for the case of two photon absorption. As expected a small reduction in fluctuations is obtained in the $\hat{X}_1$ quadrature. Since the output is real we expect this system to exhibit a $g^2(0)$ less than unity. That this is indeed the case can be seen in figure 4.5. The minimum value for $V(\hat{X}_1)$ is approximately 0.2, in accord with the result of $1/6$ expected from the linearised analysis.

In figure 4.6 we plot $g^2(0)$ $V(\hat{X}_1)$ and $V(\hat{X}_2)$ for dispersive optical bistability. As expected a slight reduction in fluctuations is obtained in the $\hat{X}_2$ quadrature on the low transmission branch. The fluctuations in $\hat{X}_1$ and $\hat{X}_2$ increase at threshold. Above threshold the fluctuations in both quadratures are increased above the coherent value of 1/4. Since the parameters have been chosen so that $\langle\alpha\rangle$ is real, we expect to see $g^2(0)$ greater than unity below threshold. This is seen to be the case. For the parameters chosen a slight phase shift between the direction of squeezing and the direction of coherent excitation may be expected. This would account for the fact that on the high transmission branch the fluctuations in $\hat{X}_1$ and $\hat{X}_2$ appear to be greater than those expected from the linearised analysis. This hypothesis is further suggested by the fact that $g^2(0)$ is less than unity indicating that the quadrature in phase with the output field does carry reduced fluctuations.

We conclude that neither two photon absorption or dispersive optical bistability are good candidates for producing significant squeezing in the steady state.

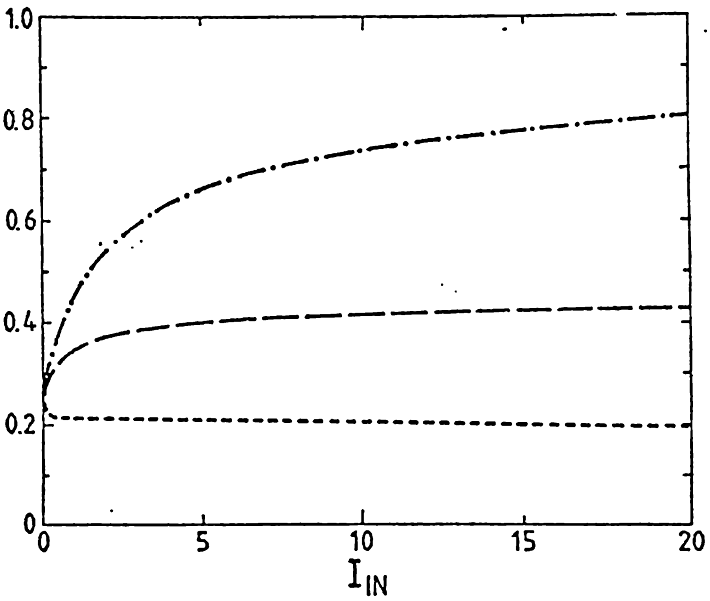

Figure 4.5 Two Photon Absorber. Plots of $g^2(0)$ (-·-·-), $V(\hat{X}_1)$ (----) and $V(\hat{X}_2)$ (——) versus driving field intensity.

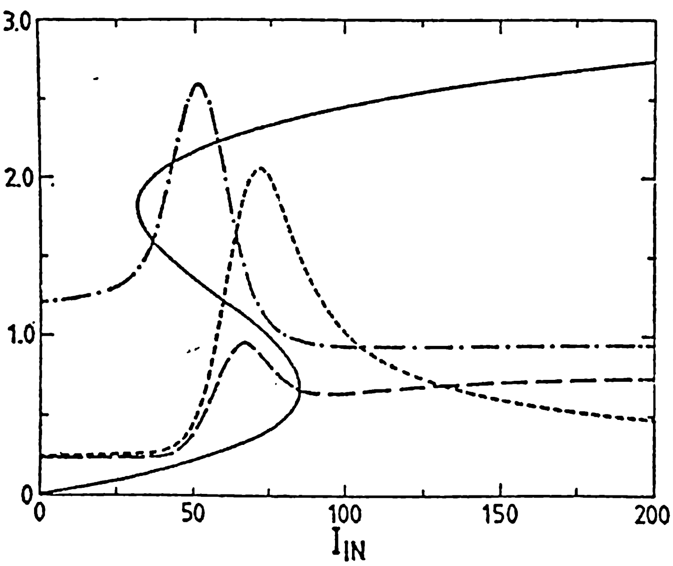

Figure 4.6 Dispersive Bistability. Plots of the semiclassical intensity (——), $g^2(0)$ (-·-·-), $V(\hat{X}_1)$ (---) $V(\hat{X}_2)$ (——) versus driving field intensity $\gamma = 2.5$, $\Delta = -10$, $\chi'' = 1.0$.

CHAPTER 5

QUANTUM NON-DEMOLITION MEASUREMENTS

5.1 Introduction

In this chapter our consideration of quantum fluctuations takes up a new theme, viz., the influence of quantum fluctuations when making highly accurate measurements.

In the definitive formulation of quantum mechanics given by von Neumann, it is explicitly postulated that a measurement made on a pure quantum system (i.e. described entirely within a quantum framework) will alter it in an indeterminate way, unless the system happens to be in an eigenstate of the self- adjoint operator corresponding to the measured observable.

Generally, for measurements on a macroscopic body, a classical description is adequate, in which case the value of an observable may be determined precisely and repeatedly by sufficiently accurate measurements. Each measurement in itself does not alter the state of the system in any way, and the results of the measurements may be entirely predictable.

Recently however a class of measurement schemes intended to determine the effect of a gravitational wave on a large mass, have required such accuracy that the detector must be described entirely by Quantum Mechanics (see C.M. Caves, 1981 for a review and a comprehensive list of references).

When a gravitational wave interacts with a large mass detector, it produces mechanical vibrations or phonon quanta in the detector. The gravitational wave itself may be treated as entirely classical since the quantum occupation number is very large, e.g. $n \simeq 10^{75}$ for the gravitational wave burst emitted by a supernova [Thorne, 1980]. One may thus model the detection process as a classical force interacting with the normal modes of a mechanical oscillator. In the simplest case one may imagine the force to be coupled to only one resonant mode and model the interaction as a classical force driving the position coordinate of a simple harmonic oscillator.

Although gravitational radiation is very classical it interacts only very weakly with terrestrial detectors. For example the radiation resulting from the collision of two black holes at the centre of the galaxy, continually driving a resonant bar with a Q factor of 10^6, at 3 khz, will produce displacements $\simeq 10^{-15}$m [Davies, 1980]. If the bar were in its ground state the quantum mechanical uncertainty in its position is given as $(\hbar/2m\omega)^{1/2}$ which is $\simeq 10^{-14}$m, for a one tonne bar at 3 khz. We see that the force is so weak as to produce displacements smaller than the quantum mechanical uncertainties in the ground state of the oscillator, and the detector must necessarily be described as a quantum mechanical object.

In order to monitor or perhaps merely detect a gravitational wave one must make a sequence of measurements on the oscillator the results of which must be entirely predictable in the absence of the gravitational wave. As one must necessarily work at a level where the quantum nature of the detector is manifest such a sequence of measurements would not in general lead to a determinate sequence of results.

For example, if one determined the initial position of the oscillator to an accuracy $\Delta \hat{q}_i$ this necessarily produces an uncertainty in momentum given by $\Delta \hat{p}_i \geq \hbar / 2 \Delta q_i$. Under free evolution we have that

$$\hat{q}(t) = \hat{q}(0) \cos \omega t + (\hat{p}(0)/m\omega) \sin \omega t \tag{5.1}$$

where ω is the frequency of the oscillator and m is the mass. Thus

$$V(\hat{q}(t)) = V(\hat{q}(0)) \cos^2 \omega t + \frac{V(\hat{p}(0))}{(m\omega)^2} \sin^2 \omega t \tag{5.2}$$

We see that uncertainties in $\hat{p}$ "feed back" into the variable of interest. If one made a measurement of $\hat{q}$ a time τ after $t = 0$ (but not at $\sin \omega\tau = 0$, a feature we will consider later) one would obtain a result entirely unpredictable from the result of the initial measurement. One would then not be able to attribute any change in $\hat{q}$ to a gravitational wave.

Quantum mechanics however does not preclude a determinate sequence of measurements in general. For example if one chose to measure the number operator $\hat{N} = a^{\dagger}a$, then in the absence of a gravitational wave, the results of a sequence of measurements is completely predictable (for perfectly accurate measurements). In fact the results will be identical as $\hat{N}$ is a constant of the motion.

This simple example suggests there may be other detector observables and appropriate measurement schemes for which a determinate sequence of results is possible. This is indeed the case and such measurement schemes have come to be known as Quantum Nondemolition (QND) measurements.

If one were to insist on a definition then the following sentence from C.M. Caves [1981b] would seem appropriate. "Quantum Nondemolition (measurement) refers to techniques for monitoring a weak force acting on a harmonic oscillator, the force being so weak that it changes the oscillators amplitude by an amount less than the amplitude of zero-point fluctuations".

As many QND schemes seek merely to detect a classical force (e.g. Quantum Counting techniques) one might like to add "... monitoring OR DETECTING a weak force ..." to the above definition.

The essential feature of any QND scheme is the ability to find a variable which may be measured again and again giving completely predictable results in the absence of the gravitational wave.

To make the discussion a little more precise let us now consider the interaction of a classical force with an oscillator.

The Hamiltonian representing the interaction of a classical force $F(t)$ with an oscillator may be written [Caves et.al., 1980]

$$H = H_0 + F(t)\hat{q} \tag{5.3}$$

where H_o is the free oscillator Hamiltonian and $\hat{q}$ is the oscillators position co-ordinate. In terms of Bose operators $\hat{q} = (\hbar/2m\omega)^{1/2}(a + a^{\dagger})$ where m is the mass of the oscillator and ω its fundamental frequency. In the interaction picture the Hamiltonian becomes

$$H_I = (\hbar/2m\omega)^{1/2}F(t)(a e^{-i\omega t} + a^{\dagger}e^{i\omega t}) \tag{5.4}$$

The equation of motion for the oscillators amplitude is then

$$\frac{da}{dt} = -i\,(1/2\hbar m\omega)^{1/2} \,.\, F(t)\, e^{i\omega t}$$

with the solution

$$a(\tau) = a(0) + \alpha(\tau) \qquad (5.5)$$

where

$$\alpha(\tau) = -i\,(1/2\hbar m\omega)^{1/2} \,.\, \int_0^{\tau} F(t)\, e^{i\omega t}\, dt \qquad (5.6)$$

Clearly the force simply displaces the oscillators complex amplitude by $\alpha(\tau)$. If the force were sinusoidal and resonant, $F(t) = F_o \sin\omega t$ the resulting change in the oscillators position co-ordinate $\delta\langle\hat{q}\rangle$ would be $\delta\langle\hat{q}\rangle = F_o\tau/2m\omega$. If the oscillator were in its ground state or indeed any coherent state, this force could only be detected with certainty if this displacement was larger than the variance in $\hat{q}$ for a coherent state. This places a limit on the minimum size of the force,

$$F_o \geqslant (2/\tau)\,.\,(2\hbar m\omega)^{1/2} \qquad (5.7)$$

Equation (5.7) is known as the STANDARD QUANTUM LIMIT (SQL). It is often convenient to refer to the uncertainty in $\hat{q}$ and $\hat{p}$ for a coherent state as the SQL as this enforces equation (5.7).

Before proceeding a number of useful quantities for an oscillator need to be defined. The quadrature phase amplitudes for a harmonic oscillator $\hat{X}_1$ and $\hat{X}_2$ are defined by

$$\hat{X}_1 \equiv (\hbar/2m\omega)^{1/2} \,.\, (a\, e^{i\omega t} + a^{\dagger} e^{-i\omega t}) \qquad (5.8)$$

$$\hat{X}_2 \equiv -i(\hbar/2m\omega)^{1/2} \,.\, (a\, e^{i\omega t} - a^{\dagger} e^{-i\omega t}) \qquad (5.9)$$

Thus

$$a = (m\omega/2\hbar)^{1/2} \; (\hat{X}_1 + i\,\hat{X}_2)\, e^{-i\omega t}$$

and $\hat{X}_1$ and $\hat{X}_2$ are the real and imaginary parts of the oscillators complex amplitude.

It needs to be emphasized that $\hat{X}_1$ and $\hat{X}_2$ as defined in equations (5.8) and (5.9) are Schrodinger operators with explicit time dependence. This is also seen when $\hat{X}_1$ and $\hat{X}_2$ are written in terms of $\hat{q}$ and $\hat{p}$.

$$\hat{X}_1 = \hat{q}\,\cos\omega t - (\hat{P}/m\omega)\,.\,\sin\omega t \tag{5.8}$$

$$\hat{X}_2 = \hat{q}\,\sin\omega t + (\hat{P}/m\omega)\,\cos\omega t \tag{5.9}$$

It is easily seen that the complex amplitude $\hat{X}_1 + i\,\hat{X}_2$ is a constant of the motion for a free oscillator, as both $\hat{X}_1$ and $\hat{X}_2$ are explicitly time independent Heisenberg operators for a free oscillator.

The operators $\hat{X}_1$ and $\hat{X}_2$ defined here are to be considered distinct from similar quantities defined in chapter two (equations (2.13a,b), there they were defined as dimensionless position and momentum operators. The two definitions are equivalent in the interaction picture however.

The commutation relation for $\hat{X}_1$ and $\hat{X}_2$ is

$$[\,\hat{X}_1,\, \hat{X}_2\,] = i\hbar/m\omega \tag{5.10}$$

with the corresponding uncertainty principal

$$V(\hat{X}_1)^{1/2}\, V(\hat{X}_2)^{1/2} \geq \hbar/2m\omega \tag{5.11}$$

If the oscillator is in a coherent state

$$V(\hat{X}_1)^{1/2} = V(\hat{X}_2)^{1/2} = (\hbar/2m\omega)^{1/2} \qquad (5.12)$$

A complex amplitude diagram is a useful device for visualising the nature of the quadrature phase operators (figure 5.1). This diagram has a similar meaning to the complex amplitude diagrams defined in chapter three. The major difference is that the $\hat{X}_1$ axis and the $\hat{X}_2$ axis are now rotating at frequency ω with respect to the $\hat{p}, \hat{q}$ axes for a freely evolving oscillator. As the complex amplitude $\langle a \rangle$ is also rotating at this frequency it remains fixed with respect to the $\hat{X}_1, \hat{X}_2$ axes, as does the error ellipse for the variances in $\hat{X}_1$ and $\hat{X}_2$. We thus need only consider the complex amplitude in the $\hat{X}_1, \hat{X}_2$ reference frame. The fact that the error circle (or ellipse) is rotating with respect to the $\hat{p}$ and $\hat{q}$ axis provides a clear illustration of how uncertainties in $\hat{p}$ feed back into $\hat{q}$.

Following Caves et.al. [1980] we now distinguish three types of measurement which may be made on an oscillator.

The usual measurements made on an oscillator seek to determine its complex amplitude a [Yuen, 1981, Caves, et.al., 1980, Caves, 1981b]. These measurements may be made using heterodyning techniques, and making direct measurements on a secondary system coupled to the oscillator. Since such measurements are a simultaneous determination of $\hat{X}_1$ and $\hat{X}_2$, their accuracy is limited by relation (5.11). Thus the uncertainty in $\hat{X}_1$ and $\hat{X}_2$ is $\Delta \hat{X}_1 = \Delta \hat{X}_2 \geq (\hbar/2m\omega)^{1/2}$, (where $\Delta \hat{X}_i \equiv V(\hat{X}_i)^{1/2}$). Such measurements have come to be known as amplitude and phase measurements as a simultaneous knowledge of $\hat{X}_1$

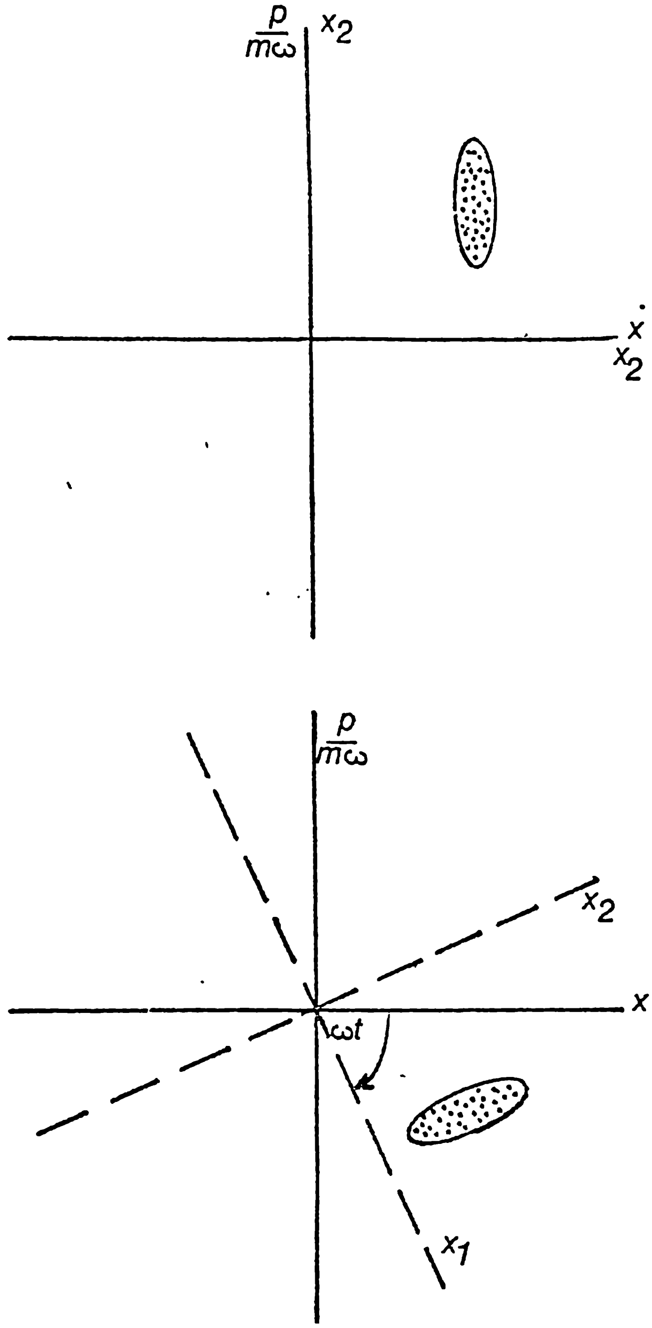

Figure 5.1 (a) Error ellipse for an oscillator with reduced fluctuations in the $\hat{X}_1$ quadrature, perhaps resulting from a measurement of $\hat{X}_1$.

(b) The same error ellipse a short time later under free evolution. Relative to the $\hat{q}$, $\hat{p}/m\omega$ axis it rotates, relative to the $\hat{X}_1$, $\hat{X}_2$ axes it is stationary [after C.M. Caves, 1981].

and $\hat{X}_2$ corresponds to simultaneous knowledge of the oscillators amplitude $|\hat{X}_1 + i\hat{X}_2|$ and phase $\phi = \tan^{-1}(\hat{X}_2/\hat{X}_1)$. The minimum errors in equation (5.12) are also known as the SQL for amplitude and phase measurements as they enforce the requirement in equation (5.7).

The change in $\langle \hat{X}_1 + i\hat{X}_2 \rangle$ produced by the classical force is given by

$$\delta \langle \hat{X}_1 + i\hat{X}_2 \rangle = (2\hbar/m\omega)^{1/2}\, \alpha(\tau)$$

where $\alpha(\tau)$ is given by equation (5.6). To detect the force this change must be greater than the possible error in the values of $\hat{X}_1$ and $\hat{X}_2$ for amplitude and phase measurements given by equation (5.12). That is we require $|\delta\langle \hat{X}_1 + i\hat{X}_2 \rangle| \gtrsim 2\,(\hbar/2m\omega)^{1/2}$ or $|\alpha(\tau)| \gtrsim 1$. This is the general statement of the SQL as given by Caves et.al. [1980].

Amplitude and phase measurements which can never beat the SQL, are NOT QND measurements. We now consider measurement schemes which circumvent the SQL.

There are two classes of QND measurements which may be made on an oscillator;

1) QUANTUM COUNTING

2) BACK ACTION EVADING MEASUREMENTS.

The first of these was proposed by V. Braginsky [1968] and was among the earliest QND measurement schemes.

To see how quantum counting can beat the SQL, imagine that at time $t = 0$ a precise measurement of the number operator was made with result N_0. The oscillator is then in the pure number state $|N_0\rangle$.

We now assume a classical force acts for a time τ. As this simply produces a displacement in the oscillators complex amplitude by $\alpha(\tau)$, the unitary time displacement operator $U(\tau)$ is given by $U(\tau) = D(\alpha(\tau))$ where $D(\alpha)$ is the displacement operator (equation (2.20)). In the Schrodinger picture the state of the oscillator at time τ is then given by $|\Psi(\tau)\rangle = D(\alpha(\tau))\,|N_o\rangle$. The variance of $a^{\dagger}a$ in the state $|\Psi(\tau)\rangle$ is easily obtained as,

$$V(\hat{N}(\tau)) = (N_o + 1)\,|\alpha(\tau)|^2 \tag{5.13}$$

If we were to measure the number at time τ and $V.(N(\tau))^{1/2} < 1$ we would obtain the same result as at $t = 0$ since we cannot measure a change of less than one quanta, and conclude that the force had not acted. Thus in order that the force produce any change at all we require $V(N(\tau))^{1/2} \gg 1$ or

$$|\alpha(\tau)| \gtrsim 1/\sqrt{N_o + 1} \tag{5.14}$$

Thus when N_o is large we may detect the presence of an arbitrarily weak force. Note that the condition (5.14) is $1/\sqrt{N_o}$ times the SQL ($|\alpha(\tau)| \gtrsim 1$).

To understand how quantum counting is used to detect a classical force consider the following scenario. Imagine that we make a sequence of very accurate measurements of $\hat{N}$. If there is no classical force acting on the oscillator one MUST obtain a sequence of identical results $\{N_o, N_o, N_o \ldots\}$ where N_o was the result of the first measurement. In fact even if a classical force does act but is so weak that equation (5.14) does not hold, one MUST also obtain the sequence of identical results. However if the force is strong enough to be detected (i.e. equation (5.14) holds) one MUST eventually obtain a result different from N_o and conclude that the force had acted.

It should be emphasised however that one cannot with certainty determine the strength of the force that acted. The force simply broadens the initial kroneckar delta form of the number distribution into a double peaked distribution [see Caves et.al., 1980]. For example if $N_0 = 30$ and a result $N' = 29$ was obtained one could only conclude that $0.05 \lesssim |\alpha(\tau)| \lesssim 0.3$ [Caves et.al., 1980]. Furthermore one could not construct the time dependence of the force (see section 5.2).

In order for the quantum counting QND scheme to work it is essential that the number remain a constant of the motion even in the presence of any transducers or amplifiers coupled to the detector oscillator. Thus one must find ways to count the number of quanta in the detector without changing the number of quanta. In chapter six we give two examples of models that achieve this. Ordinary photoelectric detection of quanta certainly changes the number of quanta and thus is a "demolition" counting technique. We must therefore look to non-demolition quantum counting measurements. This explains the origin of the name "Quantum nondemolition" measurements, which now, of course has come to mean for more than simply quantum counting measurement schemes.

Unruh [1978] pointed out that true non-demolition quantum counting requires a transducer coupling interaction that is quadratic in the detectors complex amplitude, and has suggested models by which this may be achieved. In chapter six we propose and analyse in detail two non-demolition quantum counting schemes.

As an alternative to the essentially nonlinear coupling required for QND quantum counting schemes there are the BACK ACTION EVADING (BAE) schemes proposed by Thorne et.al. [1978], which require only a linear coupling to the detector oscillators amplitude. The idea of BAE schemes is to evade the SQL by measuring only ONE quadrature $(\hat{X}_1 \text{ or } \hat{X}_2)$ of the oscillators complex amplitude rather than both.

We can now make contact with the discussion of squeezed states in the earlier part of this thesis. If one makes a precise measurement of $\hat{X}_1$ a self adjoint operator, at time $t = 0$, then the oscillator will be placed in a near eigenstate of $\hat{X}_1$. This is a consequence of "measurements of the first kind" discussed in the introduction. Thus after measurement the state of the oscillator is such that

$$V(\hat{X}_1) \quad << \quad \hbar/2m\omega \quad << \quad V(\hat{X}_2)$$

that is, the oscillator finds itself in an arbitrarily "squeezed state". Since $\hat{X}_1$ is a constant of the motion the oscillator will remain in this near eigenstate. Furthermore since the classical force only produces a displacement of the complex amplitude, it cannot change the variance in $\hat{X}_1$. After a time τ one measures $\hat{X}_1(\tau)$ and obtains the (almost) completely predictable result $\xi(\tau)$ for $\hat{X}_1(\tau)$ given by

$$\xi(\tau) \quad = \quad \xi(0) \; + \; (2\hbar/m\omega)^{1/2} \; \mathrm{Re}\,(\alpha(\tau)) \tag{5.15}$$

where $\mathrm{Re}\,(\alpha(\tau)) = 1/2\,(\alpha(\tau) + \alpha^*(\tau))$. After the measurement the oscillator will be in an even closer eigenstate of $\hat{X}_1(t)$. The measurement will realiably detect the force whenever $|\xi(\tau) - \xi(0)| \gtrsim 2\;V(X_1)^{1/2}$ that is when

$$\mathrm{Re}(\alpha(\tau)) \quad \gtrsim \quad V(\hat{X}_1)^{1/2}/(\hbar/2m\omega)^{1/2} \tag{5.16}$$

which is less than the SQL whenever $V(\hat{X}_1) < \hbar/2m\omega$.

In terms of the complex amplitude diagram we may say that the classical force merely "displaces" the error ellipse without changing its shape. Providing the ellipse is narrow enough in the $\hat{X}_1$ quadrature the presence of any displacement is easily detected. The idea is to "squeeze" the fluctuations in the quadrature we wish to measure.

As $\hat{X}_1$ can be measured repeatedly over smaller and smaller intervals $(\tau \to 0)$ (although this requires a large coupling energy to any transducer) one may determine the precise time dependence of $F(t)$ by [Caves, 1981]

$$\lim_{\tau \to 0} \left\{ \frac{Re(\alpha(\tau))}{\tau} \right\} = \frac{F(t)}{\sqrt{2\hbar m\omega}} \cdot \sin \omega t$$

The essential feature of BAE measurements is that the variance in $\hat{X}_1$ and $\hat{X}_2$ does not increase under free evolution. Thus any transducer coupled to the oscillator must preserve this quality. Clearly if the transducer-detector coupling Hamiltonian is only proportional to $\hat{X}_1$ this property is guaranteed. In the presence of noise either thermal or quantum this property will also be diminished as we saw in section 3.5.

Later in this chapter we consider in more detail examples of particular BAE measurement schemes.

An interesting type of QND measurement is the stroboscopic measurement. From equation (5.2) we see that for times τ such that $\sin \omega \tau = 0$ i.e. for $\tau = n\pi/\omega$ the variance in the position returns to its original value. If at time $t = 0$ we made a precise measurement of $\hat{q}$ the oscillator would be placed in a near eigenstate of $\hat{q}$ and it would return to this near eigenstate every half period.

Thus providing we make an instantaneous measurement of $\hat{q}_V$ only every half period we can beat the SQL and reveal with arbitrary accuracy the strength of the force. This is a stroboscopic measurement and $\hat{q}_V$ is referred to as a stroboscopic QND variable [Braginsky et.al., 1978; Caves et.al., 1980]. In chapter six we will consider a strobscopic QND variable which arises in a non-demolition quantum counting model.

Given a detector system how does one decide what the appropriate QND variables are? Caves [Caves et.al., 1980; Caves, 1981] [see also Unruh, 1979] has given a precise prescription for obtaining the QND variables. Let $\hat{A}(t)$ be the Schrodinger picture operator corresponding to an observable $\mathcal{A}$, which may have explicit time dependence, however any explicit time dependence does not change the eigenvalue spectrum of the operator in question (in chapter six we will see the consequences of removing this assumption). It is further assumed that ideal (arbitrarily precise) instantaneous measurements of $\hat{A}(t)$ are possible, which regardless of the state of the system before measurement leave it in an eigenstate of $\hat{A}(t)$ with eigenvalue corresponding to the result of the measurement.

Caves [1981b] has given the following precise definition of a QND measurement: "... if in any sequence of measurements of $\mathcal{A}$ the results of each measurement after the first can be predicted with no uncertainty from the results of the preceeding measurement. Such a sequence of measurements of $\mathcal{A}$ is called a sequence of QUANTUM NON-DEMOLITION MEASUREMENTS".

The criteria for determining whether $\hat{A}(t)$ is a QND variable (i.e. one for which a QND measurement is possible) is

$$[\hat{A}^{I}(t), \hat{A}^{I}(t)] = 0 \tag{5.17}$$

where $A^{I}(t)$ is the interaction picture representation of $\hat{A}(t)$ (i.e. the Heisenberg picture operator in the absence of external interactions).

There are a number of equivalent interpretations of equation (5.17) perhaps the most precise is; if the system is in an eigenstate of $\hat{A}(t)$ at time t it remains in such an eigenstate at all times, although the eigenvalues may change in time. Alternatively if at any time $V(\hat{A}(t)) = 0$, it remains zero for all time.

The simplest example of a QND observable is any quantity which is a constant of the motion.

The QND criteria equation (5.17) enables one to determine the QND variables of an isolated quantum system. As our object is to monitor a weak force interacting with the oscillator the question arises as to whether a given QND variable continues to satisfy equation (5.17) in the presence of the force. The answer is no in general, and the simplest example is the number operator $\hat{N}$ for a harmonic oscillator.

Under free evolution $\hat{N}$ is a constant of the motion, and thus is a QND variable, however it is not a constant of the motion in the presence of the force. Worse still in the presence of the force $\hat{N}$ is coupled to variables it does not commute with. To see this it is only necessary to write the equation of motion for $\hat{N}$ using the Hamiltonian in equation (5.3)

$$\begin{aligned} \frac{d\hat{N}(t)}{dt} &= -i\frac{F(t)}{\hbar}.[\hat{N}, \hat{q}] \\ &= \frac{F(t)}{\hbar m\omega}\cdot\hat{p}(t) \end{aligned} \tag{5.18}$$

Clearly any uncertainties in $\hat{p}^{(0)}$ will feed back into $\hat{N}(t)$. Thus although $\hat{N}$ can be used to DETECT the presence of a weak force it can never reveal its strength or time dependence precisely. QND variables such as $\hat{N}$ which do not remain QND variables in the presence of the force have been given the name QNDR (QND Readout) [Unruh, 1979]. Observables which remain QND in the presence of the force are known either as QNDF variables [Caves, et.al., 1980] or QNDD (QND Detection) variables [Unruh, 1979].

The above example suggests how a QND variable $\hat{A}(t)$ may remain QNDF. It is only necessary that the commutator of $\hat{A}(t)$ with the Hamiltonian representing the interaction with the force be some non-zero c-number or any operator which commutes with $\hat{A}(t)$ at all times. For an oscillator this means

$$[\hat{A}(t), \hat{q}] = \begin{cases} \lambda \\ \hat{B}(t) \end{cases} \tag{5.19}$$

where $\lambda \neq 0$ and $[\hat{A}(t), \hat{B}(t)] = 0$

It is now clear why $\hat{X}_1(t)$ in the BAE scheme allows one to monitor the force so precisely. We see that

$$[\hat{X}_1(t), \hat{q}] = -i\hbar/m\omega . \sin \omega t$$

thus the commutator is a c-number, and the detector will remain in an eigenstate of $\hat{X}_1(t)$ even in the presence of the force, although the eigenvalue will change.

Clearly an oscillator responding to a force is of little use as a detector if we cannot monitor its state. Thus we necessarily require that the detector be coupled to some subsequent readout/meter amplification stage. The question then arises as to whether this interaction destroys the QND property of the chosen detector variable. The answer is no, providing the readout coupling interaction is chosen appropriately.

The criteria for deciding on optimal readout coupling schemes suggests itself from the preceeding analysis of the interaction between the detector and force. The simplest statement of this criteria is; if the only observable of the detector that appears in the detector-readout interaction Hamiltonian is the detector QND variable then the unitary evolution of the QND variable is unaffected by the interaction with the measuring apparatus. Or more succinctly

$$[\hat{A}(t), H_I] = 0 \tag{5.20}$$

where H_I is the detector-readout interaction Hamiltonian. This ensures that the QND observable $\hat{A}(t)$ is isolated from variables it does not commute with. We say that $\hat{A}(t)$ has "evaded" the back action of the readout system or meter. We may refer to (5.20) as the back action evading criteria. The back action evading criteria must hold for both BAE and quantum counting QND measurements.

The introduction of the readout system or meter as we shall refer to it, enables the more realistic description of the QND measurement scheme, when the meter interacts with the system for a time τ, which is not vanishingly small, and the detector is not put into a pure eigenstate of the QND variable. An analysis of the QND measurement sequence may then be divided into two stages. The first stage involves solving for the time dependent unitary evolution of the coupled detector meter system. During this stage correlations between the state of the detector and meter arise. After a time τ (i.e. the "measurement" duration) the unitary evolution is suspended and a readout of the meter variable is made, whereupon the meter is projected into an eigenstate corresponding to the value read, i.e. the state of the meter is reduced. The second stage of the analysis then, involves a determination of the non-unitary effect of meter state reduction upon the detector.

Before proceeding to an analysis of a particular BAE QND measurement scheme an important assumption must be emphasised. For all the QND schemes discussed in this thesis we assume that there is no prior special preparation of the state of the detector, "the state of the system is determined by the measurement process itself" [Caves, 1981]. Correlation between the states of the detector and meter arise only under their coupled unitary evolution.

5.2 Back Action Evasion Measurements Via Parametric Amplification

5.2.1 Deterministic Analysis

We now present an analysis of a BAE measurement of $\hat{X}_1$ for a detector by coupling the detector to a meter modelled as another oscillator. The detector-meter interaction Hamiltonian is modelled after an optical parametric amplifier coupling, suggested by Hillery and Scully [1981] [see also Milburn, Lane and Walls, 1982].

A parametric frequency conversion interaction, has been discussed in detail by Lane [Milburn et.al., 1982].

The model also takes into account the unavoidable presence of quantum fluctuations due to spontaneous damping of both oscillators.

The unitary evolution of the coupled system including damping is treated using a master equation and associated Fokker-Planck equation, based on the generalised P-representation (see chapter one for a summary of these techniques). The second stage of the analysis, meter state reduction is carried out by performing appropriate integration over the P-representation for the total system.

The Hamiltonian for this model is

$$H = \hbar\omega_a a^\dagger a + \hbar\omega_b b^\dagger b + H_I + a\Gamma_a^\dagger + a^\dagger\Gamma_a + b\Gamma_b^\dagger + b^\dagger\Gamma_b \qquad (5.21)$$

where $a(b)$ is the Bose operator for the detector (meter) mode. Γ_a, Γ_b are heat bath operators for the detector and meter modes respectively. ω_a and ω_b are the oscillator frequencies. We further assume that the heat baths are at absolute zero. The interaction Hamiltonian H_I takes the form,

$$H_I = -\hbar K\left(a b\, e^{i(\omega_a+\omega_b)t} + h.c.\right) \qquad (5.22)$$

If we now define $\hat{X}_1, \hat{X}_2$ and $\hat{Y}_1, \hat{Y}_2$ to be the quadrature phase operators for the detector and meter respectively the interaction Hamiltonian may be written (we assume unit mass)

$$H_I = -K\sqrt{\omega_a\omega_b}\,(\hat{X}_1\hat{Y}_1 - \hat{X}_2\hat{Y}_2) \qquad (5.23)$$

It is now clear that this interaction does not satisfy the back action evading criteria and we expect to see a corresponding feed back of fluctuations into $\hat{X}_1$.

We firstly consider the straight forward deterministic equations of motion to understand how QND measurements may be made on this system.

The solutions to the Heisenberg equations of motion in the interaction picture are

$$\hat{X}_1(t) = \hat{X}_1(0)\cosh Kt + (\omega_b/\omega_a)^{1/2}\,\hat{Y}_2(0)\sinh Kt \qquad (5.24a)$$

$$\hat{X}_2(t) = \hat{X}_2(0)\cosh Kt + (\omega_b/\omega_a)^{1/2}\,\hat{Y}_1(0)\,\sinh Kt \qquad (5.24b)$$

$$\hat{Y}_1(t) = \hat{Y}_1(0)\,\mathrm{Cosh}\,Kt + (\omega_a/\omega_b)^{1/2}.\hat{X}_2(0)\,\mathrm{Sinh}\,Kt \tag{5.24c}$$

$$\hat{Y}_2(t) = \hat{Y}_2(0)\,\mathrm{Cosh}\,Kt + (\omega_a/\omega_b)^{1/2}\hat{X}_1(0)\,\mathrm{Sinh}\,Kt \tag{5.24d}$$

$\hat{X}_1(t)$ is now clearly seen to be a QND variable for this system, i.e. $[\hat{X}_1(t), \hat{X}_1(t')] = 0$. From equations (5.24a,d) we have

$$\hat{X}_1(t) = (\omega_b/\omega_a)\,(\hat{Y}_2(t)\,\mathrm{Coth}\,Kt - \hat{Y}_2(0)/\mathrm{Sinh}\,Kt) \tag{5.25}$$

Using equation (5.25) we can infer values for $\hat{X}_1(t)$ by making measurements of $\hat{Y}_2(t)$.

The failure of back action evasion is now easily demonstrated. Using equation (5.24a) we have

$$V(\hat{X}_1(t)) = V(\hat{X}_1(0))\,\mathrm{Cosh}^2 Kt + \omega_b/\omega_a . V(\hat{Y}_2(0))\,\mathrm{Sinh}^2 Kt \tag{5.26}$$

(where we have assumed the detector and meter were initially uncorrelated). We see that even if $V(\hat{Y}_2(0))$ is zero $V(\hat{X}_1(t))$ will grow with time unless $V(\hat{X}_1(0)) = 0$. Thus only if the detector and meter are in a simultaneous eigenstate of $\hat{X}_1$ and $\hat{Y}_2$ will the system remain in an eigenstate. As such perfect eigenstates are impossible to achieve in practice $V(\hat{X}_1(t))$ will always be an increasing function of time. The problem arises entirely from the presence of the $\hat{X}_2\hat{Y}_2$ term in the interaction which does not permit $\hat{X}_1(t)$ to be a constant of the motion.

One measurement strategy would proceed as follows. We prepare the state of the meter at $t = 0$ so that $\langle Y_2(0)\rangle = 0$. The interaction is then allowed to proceed for a time t at which point $\hat{Y}_2(t)$ of the meter is measured with result $y_2(t)$. Using equation (5.25) the mean values at time t are related by,

$$\langle \hat{X}_1(t) \rangle = (\omega_b/\omega_a)^{1/2} . \langle \hat{Y}_2(t) \rangle \, \mathrm{Coth}\, Kt \tag{5.26}$$

Thus a measurement of $\hat{Y}_2(t)$ with result $y_2(t)$ may be used to infer a value $x_1(t)$ for $\hat{X}_1(t)$ given by

$$x_1(t) = (\omega_b/\omega_a)^{1/2} . \, y_2(t) \, \mathrm{Coth}\, Kt \tag{5.27}$$

The possible error in the inferred value $\Delta x_1(t)$ is determined by the standard deviation in $\hat{Y}_2(t)$ at the time of measurement, i.e.

$$\Delta x_1(t) = (\omega_b/\omega_a)^{1/2} . \, V(\hat{Y}_2(t))^{1/2} \, \mathrm{Coth}\, Kt \tag{5.28}$$

Using equation (5.24d) this becomes

$$\Delta x_1(t) = \mathrm{Cosh}\, Kt \left\{ \frac{\omega_b}{\omega_a} V(\hat{Y}_2(0)) \, \mathrm{Coth}^2 Kt + V(\hat{X}_1(0)) \right\}^{1/2} . \tag{5.29}$$

Thus even if the state of the meter had been chosen so that $V(\hat{Y}_2(0)) \simeq 0$ the determination of $x_1(t)$ must be more uncertain at time t, than it was at time zero. This is a consequence of the failure of back action evasion.

It is interesting to note however that the signal to noise ratio, $\Delta x_1(t)/x_1(t)$ given by (see equation (5.24a))

$$\frac{\Delta x_1(t)}{x_1(t)} = \frac{[\omega_b/\omega_a . V(\hat{Y}_2(0)) \, \mathrm{Coth}\, Kt + V(\hat{X}_1(0))]^{1/2}}{\langle X_1(0) \rangle} \tag{5.30}$$

is bounded as t becomes large. In fact for $\omega_b/\omega_a \, V(\hat{Y}_2(0)) << V(\hat{X}_1(0))$ it is given as $V(\hat{X}_1(0))^{1/2}/\langle X_1(0) \rangle$ which is what it was prior to the interaction.

If the system happens to be in a simultaneous eigenstate of both $\hat{X}_1$ and $\hat{Y}_2$ at the start of the measurement (i.e. $V(\hat{X}_1(0)) = V(\hat{Y}_2(0)) = 0$) the determination of $x_1(t)$ is certain. However if the system were in such a simultaneous eigenstate it would have divergent energy $V(\hat{X}_2) \to \infty$ and $V(\hat{Y}_1) \to \infty$ and the entire system would disintegrate, before any measurement was possible. This essential limitation to all QND schemes has been stated by Braginsky [Braginsky, 1981]. For a mechanical oscillator the breakdown is mechanical and determined by the Youngs modulus of the material. In an electromagnetic resonator it is determined by the breakdown electric field for the dielectric in the resonator. For an empty cavity this is proportional to its volume.

An alternative measurement scheme for the parametric amplifier has been suggested by Hillery [1982]. In this scheme one attempts to infer $\hat{X}_1(0)$ by measurements of $\hat{Y}_2(t)$, that is we attempt to measure $\hat{X}_1$ at the beginning of a measurement time t , rather than at the end.

If we once again prepare the state of the meter so that $\langle \hat{Y}_2(0) \rangle = 0$ equation (5.24d) gives the following relation between the mean values.

$$\langle \hat{Y}_2(t) \rangle = (\omega_a/\omega_b)^{1/2} . \langle \hat{X}_1(0) \rangle \sinh \kappa t \tag{5.31}$$

Thus from a measurement of $\hat{Y}_2(t)$ with result $y_2(t)$ one infers a value $x_1(0)$ for $\hat{X}_1(0)$ given by

$$x_1(0) = (\omega_b/\omega_a)^{1/2} . y_2(t) / \sinh \kappa t \tag{5.32}$$

The uncertainty $\Delta \mathcal{X}_1(0)$ in this inferred value is

$$\Delta \mathcal{X}_1(0) = \left\{ \omega_D/\omega_a \; V(\hat{Y}_2(0)) \coth kt + V(\hat{X}_1(0)) \right\}^{1/2} \tag{5.33}$$

which unlike $\Delta \mathcal{X}_1(t)$ remains bounded. In fact for t large and $(\omega_D/\omega_a).V(\hat{Y}_2(0)) \ll V(\hat{X}_1(0))$ we have that $\Delta \mathcal{X}_1(0) \simeq V(\hat{X}_1(0))^{1/2}$. Not surprisingly the uncertainty in the determination of $\hat{X}_1(0)$ is determined by its uncertainty at the start of the measurement. As in determinations of $\mathcal{X}_1(t)$ if the system happens to be in a simultaneous eigenstate of $\hat{X}_1$ and $\hat{Y}_2$ at the start of the measurement $\mathcal{X}_1(0)$ may be determined with no uncertainty. In view of the energy considerations given previously we exclude such unphysical states.

Either measurement scheme, at least on a superficial level, would appear to offer a way of monitoring a classical force. If one was to make a determinate sequence of measurements of $\hat{X}_1(t)$ then a perfect BAE monitoring of a classical force would be possible. Alternatively if one was able to know $\hat{X}_1(0)$ with certainty for a sequence of measurements then one could detect a classical force acting between each measurement, by the changes it would produce in $\hat{X}_1(0)$ at the start of each measurement. In both cases little more can be said until we know what happens to the fluctuations in $\hat{X}_1$ as a result of each measurement and this entails a detailed analysis of the effects of meter state reduction.

We now turn to a more detailed analysis of the unitary evolution including damping. We have demonstrated above how the fundamental dynamics of the interaction limit the certainty in a determination of $\hat{X}_1(t)$ or $\hat{X}_1(0)$. Our object now is to determine what additional limitations are imposed by spontaneous damping of the system at absolute zero.

5.2.2 Inclusion of Damping

Using standard techniques (see chapter 1) the following master equation may be derived from the Hamiltonian (equation (5.21)), in the interaction picture

$$\frac{\partial \rho}{\partial t} = (1/i\hbar)\,[H_I, \rho] + \gamma_1/2\,(2a\rho a^{\dagger} - a^{\dagger}a\rho - \rho a^{\dagger}a)$$

$$+ \gamma_2/2\,(2b\rho b^{\dagger} - b^{\dagger}b\rho - \rho b^{\dagger}b) \tag{5.34}$$

where ρ is the density operator for the coupled system and where γ_1 and γ_2 are the damping constants for the detector and meter respectively.

It should be noted however that in deriving equation (5.34) we have assumed that each system is damped independently regardless of the strength of the coupling of the two modes. It has been shown by Walls [1970] that this is reasonable only if κ is not too large. Since a large κ is desirable for fast measurements the master equation (5.34) must be considered as only an approximation when κ is large. Deviations resulting when κ is large, however, are not expected to significantly modify the dynamics.

Using the complex P-representation we expand ρ in terms of coherent states

$$\rho = \int_{C_i} d\alpha_1 d\alpha_2 d\beta_1 d\beta_2\, P(\underset{\sim}{\gamma}, t)\, \frac{|\alpha_1, \alpha_2\rangle\langle \beta_1^*, \beta_2^*|}{\langle \beta_1^*, \beta_2^* | \alpha_1, \alpha_2\rangle} \tag{5.35}$$

where $\underset{\sim}{\gamma}$ is a column vector given as

$$\underset{\sim}{z} = \begin{pmatrix} \alpha_1 \\ \beta_1 \\ \alpha_2 \\ \beta_2 \end{pmatrix}$$

and we have the following correspondences

$$a \leftrightarrow \alpha_1 \quad ; \quad b \leftrightarrow \alpha_2$$
$$a^{+} \leftrightarrow \beta_1 \quad ; \quad b^{+} \leftrightarrow \beta_2$$

There are actually four independent contour integrals $(i = 1-4)$ involved in equation (5.35) in the complex space of each variable. Using equation (5.35) the master equation yields the following Fokker-Planck equation

$$\frac{\partial P(\underset{\sim}{z},t)}{\partial t} = \left\{ \nabla_{\underset{\sim}{z}}^{T} A \underset{\sim}{z} + 1/2\, \nabla_{\underset{\sim}{z}}^{T} D\, \nabla_{\underset{\sim}{z}} \right\} P(\underset{\sim}{z},t) \tag{5.36}$$

where $\nabla_{\underset{\sim}{z}}^{T} = \left(\frac{\partial}{\partial\alpha_1}, \frac{\partial}{\partial\beta_1}, \frac{\partial}{\partial\alpha_2}, \frac{\partial}{\partial\beta_2} \right)$

and

$$A = \begin{pmatrix} \gamma_1/2 & 0 & 0 & -iK \\ 0 & \gamma_1/2 & iK & 0 \\ 0 & -iK & \gamma_2/2 & 0 \\ iK & 0 & 0 & \gamma_2/2 \end{pmatrix} \tag{5.37}$$

$$D = \begin{pmatrix} 0 & 0 & iK & 0 \\ 0 & 0 & 0 & -iK \\ iK & 0 & 0 & 0 \\ 0 & -iK & 0 & 0 \end{pmatrix} \tag{5.38}$$

The general solution to equation (5.36) (which is a multivariable Ornstein-Uhlenback equation [Gardiner, 1982]) is the multivariate Gaussian,

$$P(\underset{\sim}{z}, t) = \exp\left\{-1/2\,(\underset{\sim}{z} - \langle \underset{\sim}{z}(t)\rangle)^T C^{-1}(\underset{\sim}{z})(\underset{\sim}{z} - \langle \underset{\sim}{z}(t)\rangle)\right\} \qquad (5.39)$$

where $\langle \underset{\sim}{z}(t)\rangle^T = (\langle a(t)\rangle, \langle a^\dagger(t)\rangle, \langle b(t)\rangle, \langle b^\dagger(t)\rangle)$

and $C(\underset{\sim}{z})$ is the covariance matrix.

The initial states of the detector and meter are taken to be general squeezed state (see equation (2.75))

detector; $|v_1 ; r_1\rangle$

meter; $|v_2 ; r_2\rangle$.

where v_1 (v_2) is the amplitude of the detector (meter) and r_1 (r_2) is the squeeze parameter for the detector (meter).

The explicit form of $\langle \underset{\sim}{z}(t)\rangle$ is

$$\langle \underset{\sim}{z}(t)\rangle = \begin{pmatrix} 1/2\,(q^2 e^{-\Delta t/4} + p^2 e^{\Delta t/4})\,v_1 + 4iK/\Delta \,.\, \sinh(\Delta t/4)\, v_2^* \\ 1/2\,(q^2 e^{-\Delta t/4} + p^2 e^{\Delta t/4})\,v_1^* - 4iK/\Delta .\, \sinh(\Delta t/4)\, v_2 \\ 1/2\,(p^2 e^{\Delta t/4} + q^2 e^{\Delta t/4})\,v_2 + 4iK/\Delta .\, \sinh(\Delta t/4)\, v_1^* \\ 1/2\,(p^2 e^{-\Delta t/4} + q^2 e^{\Delta t/4})\,v_2^* + 4iK/\Delta\, \sinh(\Delta t/4)\, v_1 \end{pmatrix} \qquad (5.40)$$

where

$$\Delta = \sqrt{(\gamma_1 - \gamma_2)^2 + 16K^2}$$

$$p = \left(1 + \frac{(\gamma_2 - \gamma_1)}{\Delta}\right)^{1/2}$$

$$q = \left(1 - \frac{(\gamma_2 - \gamma_1)}{\Delta}\right)^{1/2}$$

The full expression for the covariance matrix is somewhat involved and is given in appendix two. From the covariance matrix we obtain

$$V(\hat{X}_1(t)) = (\hbar/2\omega_a).\left\{1 + [(e^{-2r_1} - 1)A^2 + (e^{2r_2} - 1)B^2]e^{-(\gamma_1+\gamma_2)t/2} + \frac{16K^2}{\Delta^2}\left(\frac{x}{p^2} - \frac{2y}{pq} + \frac{z}{q^2}\right)\right\} \tag{5.41}$$

$$V(\hat{X}_2(t)) = (\hbar/2\omega_b).\left\{1 + [(e^{2r_2} - 1)A^2 + (e^{-2r_1} - 1)B^2]e^{-(\gamma_1+\gamma_2)t/2} + \frac{16K}{\Delta^2}\left(\frac{x}{p^2} - \frac{2y}{pq} + \frac{z}{q^2}\right)\right\} \tag{5.42}$$

where

$$A = \frac{8K^2}{\Delta^2}\left(\frac{e^{\Delta t/4}}{p^2} + \frac{e^{-\Delta t/4}}{q^2}\right)$$

$$B = \frac{4K}{\Delta}.\ \mathrm{Sinh}\left(\frac{\Delta t}{4}\right)$$

$$x = \frac{-8K^2}{\Delta(\gamma_1+\gamma_2+\Delta)}.\left(1 - e^{-(\gamma_1+\gamma_2+\Delta)t/2}\right)$$

$$y = \frac{2K(\gamma_2-\gamma_1)}{(\gamma_2+\gamma_1)}.\left(1 - e^{-(\gamma_1+\gamma_2)t/2}\right)$$

$$z = \frac{8K^2}{\Delta(\gamma_1+\gamma_2-\Delta)}.\left(1 - e^{-(\gamma_1+\gamma_2-\Delta)t/2}\right)$$

We now discuss some special cases with $\gamma_1 = \gamma_2 = \gamma$.

Consider the situation where $K \gg \gamma$. If the detector is initially in an eigenstate of $\hat{X}_1$ (*i.e.* $r_1 \to \infty$) and the meter is in a coherent state we have

$$V(\hat{X}_1(t)) = \frac{\hbar}{4\omega_a}.\left(\mathrm{Cosh}\,2Kt - e^{-\gamma t}\right)$$

$$V(\hat{Y}_2(t)) = \frac{\hbar}{4\omega_a}\left(\mathrm{Cosh}\,2Kt + e^{-\gamma t}\right)$$

If κ is large, t may be chosen so that $\gamma t \ll 1$ and we find $V(\hat{X}_1(t)) \simeq (\hbar/2\omega_a)\sinh^2 \kappa t$ Thus even if damping is negligible the failure of back action increases the fluctuations in $\hat{X}_1$. If the time of measurement is large $\gamma t \gg 1$ we find $V(X_1(t)) \simeq \frac{\hbar}{2\omega_a}(\sinh^2 \kappa t + 1/2)$ Thus spontaneous damping makes a bad situation even worse.

We now consider what happens if the damping is significant; $\gamma_1 = \gamma_2 = \gamma$ and $\gamma \gg \kappa$ If the system begins in the ideal state of a simultaneous eigenstate of $\hat{X}_1(0)$ and $\hat{Y}_2(0)$ we find,

$$V(\hat{X}_1(t)) = \hbar/2\omega_a . (1 - e^{-\gamma t})$$
$$V(\hat{Y}_2(t)) = \hbar/2\omega_b (1 - e^{-\gamma t})$$

In accordance with section (3.5) the interaction with vacuum modes causes the fluctuations in $\hat{X}_1$ and $\hat{Y}_2$ to return to the SQL. Even in the best possible case for QND measurements, the presence of significant damping will degrade the certainty in the determination of $\hat{X}_1$. For a good QND measurement we require the measurement time to be much shorter than the time over which the system is damped. As short measurement times require κ large for a significant signal we thus require $\kappa \gg \gamma$. In the following discussion we will assume this to be the case.

5.2.3 Meter State Reduction

Having obtained the complex P-function for the coupled system it is now possible to determine the effect on the detector of a perfect measurement made on the meter.

Upon readout of a perfect measurement of $\hat{Y}_2(t)$ the meter collapses into an eigenstate of $\hat{Y}_2(t)$ with eigenvalue $y_2(t)$ the result obtained. This produces a non-unitary change in the state of the detector which then becomes the initial state for the next measurement (We could however reprepare the state of the meter before another measurement). This enables us to consider a sequence of measurements of $\hat{X}_1(t)$. Of course such perfect $\hat{Y}_2(t)$ measurements are not possible, however as long as the variance in $\hat{Y}_2(t)$ is reduced much below $V(X_1(t))$ at readout, a perfect $\hat{Y}_2(t)$ measurement is a good assumption.

To determine the effect of a meter measurement we use the projection postulate outlined in section 1.5

If $\rho(t)$ is the (Schrodinger picture) density operator of the coupled detector-meter system at the time of the readout, the density operator for the total system after readout, $\bar{\rho}(t)$, is given by (see equation (1.55))

$$\bar{\rho}(t) = \mathcal{N}\, \hat{P}_{\hat{Y}_2(t)}(y_2(t))\, \rho(t)\, \hat{P}_{\hat{Y}_2(t)}(y_2(t)) \qquad (5.43)$$

where $\hat{P}_{\hat{Y}_2}(y_2(t))$ is a projector onto the subspace spanned by $|y_2(t), t\rangle$ eigenstate and $\mathcal{N}^{-1} = \mathrm{Tr}\left(\rho(t)\, \hat{P}_{\hat{Y}_2}(y_2(t))\right)$. The state of the detector after readout is then obtained by tracing out over meter variables

$$\bar{\rho}_D(t) = \mathrm{Tr}_m\left(\bar{\rho}(t)\right) \qquad (5.44)$$

If we introduce the interaction picture density operator $\rho^I(t)$ by

$$\rho(t) = \exp\left\{-\frac{it}{\hbar}(H_m + H_D)\right\} \rho^I(t) \exp\left\{\frac{it}{\hbar}(H_m + H_D)\right\} \qquad (5.45)$$

where H_m (H_0) is the free Hamiltonian for the meter (detector), the state of the detector after readout in the interaction picture is

$$\bar{\rho}_D^I(t) = \mathrm{Tr}_m \left\{ \rho^I(t) \, | y_2(t), 0 \rangle \langle y_2(t), 0 | \right\} \tag{5.46}$$

where we have used the property [Caves et.al., 1980, equation C7]

$$| y_2(t), t \rangle = \exp\left\{ -\frac{it}{\hbar} H_m \right\} | y_2(t), 0 \rangle \tag{5.47}$$

As we have been working entirely in the interaction picture (that is $\rho(t)$ and $P(\underset{\sim}{z}, t)$ in equation (5.35) are in the interaction picture) we will drop the superscript on $\bar{\rho}_D^I(t)$ and it is to be understood that all operators are in the interaction picture. Using equations (5.35) and (5.46) the complex P-representation for the detector after readout is

$$\bar{P}_D(\alpha, \beta; t) = \mathcal{N} \oint_{C_2} \oint_{C_2^\dagger} d\alpha_2 \, d\beta_2 \; P(\underset{\sim}{z}, t) \frac{\langle y_2(t), 0 | \alpha_2 \rangle \langle \beta_2^* | y_2(t), 0 \rangle}{\langle \beta_2^* | \alpha_2 \rangle} \tag{5.48}$$

where $P(\underset{\sim}{z}, t)$ is the full complex P-representation for the coupled detector-meter system and C_2, $C_2^\dagger$ are the same contours used in calculating moments from $P(\underset{\sim}{z}, t)$ in α_2 and β_2 space.

To evaluate this integral we note that $\hat{Y}_2(0) = \hat{P}/\omega$ (equation (5.9) and thus $| y_2(t), 0 \rangle$ is in fact a momentum eigenstate. The momentum representation for coherent states may then be used to show [Louisell, 1973].

$$\langle y_2(t), 0 | \alpha_2 \rangle = \left(\frac{1}{\hbar \omega_b \pi} \right)^{1/4} . \exp\left\{ -\omega b/2\hbar \, . \, y_2^2(t) - i\sqrt{\frac{2\omega b}{\hbar}} \, \alpha_2 \, . \, y_2(t) - 1/2 \, |\alpha_2|^2 + 1/2 \, \alpha_2^2 \right\} \tag{5.49}$$

Substituting equations (5.39) and (5.49) into (5.48) enables us to write

$$\bar{P}_D(t) = \int d\alpha_2 \int d\beta_2 \exp\left\{ -1/2 \left[(\underset{\sim}{z} - \langle \underset{\sim}{z} \rangle)^T C^{-1} (\underset{\sim}{z} - \langle \underset{\sim}{z} \rangle) + (\underset{\sim}{z} - \underset{\sim}{x})^T B (\underset{\sim}{z} - \underset{\sim}{x}) \right\} \tag{5.50}$$

where

$$\underline{x} = (0, 0, i(\omega_b/2\hbar)^{1/2} y_2(t), -i(\omega_b/2\hbar)^{1/2} y_2(t)) \qquad (5.51)$$

and

$$B = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & 1 & -1 \end{pmatrix} \qquad (5.52)$$

The contours used to evaluate this integral are either the real or imaginary axis in α_2, β_2 space depending on the sign of the quadratic term in the argument of the exponential in equation (5.50). Upon integration $\bar{P}_D(t)$ becomes a two dimensional Gaussian in α_1 and β_1 with mean $\langle \bar{\underline{\xi}}_1 \rangle$ and covariance matrix $\bar{C}(\underline{\xi}_1)$ where $\underline{\xi}_1^T = (\alpha_1, \beta_1)$. The general form of $\langle \bar{\underline{\xi}}_1 \rangle$ and $\bar{C}(\underline{\xi}_1)$ including damping are given in appendix two. It is sufficient for our purposes to give $\overline{\langle \hat{X}_1(t) \rangle}$ and $V(\overline{\hat{X}_1(t)})$ (i.e. the mean and variance for $\hat{X}_1(t)$ after a readout at time t), in the absence of damping.

If we assume both detector and meter were initially in coherent states with $\langle \hat{Y}_2(0) \rangle = 0$ after one readout of $\hat{Y}_2(t)$ at $t = \tau$ with result $y_2(\tau)$, the state of the detector immediately after readout is such that

$$\overline{V(\hat{X}_1(\tau))}^{(1)} = \frac{\hbar}{2\omega_a} \cdot \frac{1}{\operatorname{Cosh} 2k\tau} \qquad (5.53)$$

$$\overline{V(\hat{X}_2(\tau))}^{(1)} = \hbar/2\omega_a \,.\, \operatorname{Cosh} 2k\tau \qquad (5.54)$$

$$\overline{\langle X_1(\tau) \rangle}^{(1)} = \langle X_1(\tau) \rangle \cdot \frac{1}{\operatorname{Cosh} 2k\tau} + \frac{2}{(1 + \operatorname{Coth}^2 k\tau)} \cdot x_1(\tau) \qquad (5.55)$$

where $x_1(\tau)$ is the inferred value of $\hat{X}_1(\tau)$ and the superscript (1) specifies that we have made only one readout of the meter.

The limit of arbitrarily fast (τ small) and accurate measurements is $\kappa\tau \to \infty$. In this limit we find

$$\begin{aligned} &\overline{V(\hat{X}_1(\tau))}^{(1)} \longrightarrow 0 \\ &\overline{V(\hat{X}_2(\tau))}^{(1)} \longrightarrow \infty \\ &\overline{\langle \hat{X}_1(t) \rangle}^{(1)} \longrightarrow x_1(\tau) \end{aligned} \tag{5.56}$$

Thus after such a perfect measurement of $\hat{X}_1(\tau)$ the detector is in an eigenstate of $\hat{X}_1(\tau)$ with eigenvalue equal to the inferred result.

At the end of such a perfectly accurate measurement the coupled detector-meter system is in a simultaneous eigenstate of $\hat{X}_1(\tau)$, $\hat{Y}_2(\tau)$ in which, in the absence of damping, it will remain. This purely mathematical result however is impossible to achieve in reality. It requires that κ be infinitely large and thus the energy of the total coupled system is infinitely large. As discussed previously such a situation would result in the breakdown of the entire system. (A further demonstration of the unphysical nature of $\hat{X}_1$, $\hat{Y}_2$ eigenstates is obtained when we consider them as squeezed states in the limit $r_1 \to \infty$, $r_2 \to -\infty$. Using equation (3.5) we see that the average energy $\langle a^\dagger a \rangle$ of such a state diverges). In practice then, the variance $\overline{V(\hat{X}_1(t))}$ must remain finite. Of course the above deviation assumes that we have made a perfect measurement of $\hat{Y}_2(t)$. This is an approximation which remains valid provided $\overline{V(Y_2(\tau))} \ll \overline{V(X_1(t))}$. In fact Caves et.al. [1980] have shown that in the large κ limit the effect of imprecise meter readout is entirely negligible.

If however $\overline{V(X_1(t))}^{(1)}$ is not zero the fluctuations in $\hat{X}_1$ after the readout must immediately begin to increase under free evolution. The question arises as to whether the reduction of fluctuations on readout is greater than the increase in fluctuation during a subsequent period of free evolution.

The variance in $\hat{X}_1$ at a time t after a readout at time τ is given by (equation (5.26))

$$V(\hat{X}_1(t+\tau)) = \frac{\hbar}{2\omega_a} \cdot \frac{\cosh^2 \kappa t}{\cosh 2\kappa\tau} \tag{5.57}$$

At $t = \tau$ in the limit $\kappa \to \infty$

$$V(\hat{X}_1(2\tau)) \longrightarrow \hbar/4\omega_a \; . \; = 1/2 \, . \; V(\hat{X}_1(0))$$

The fluctuations in $\hat{X}_1$ at a time 2τ are much less than they would have been had not a meter readout been made at time τ . Even for κ large, the increase in fluctuations is bounded.

This result suggests that in a sequence of measurements of duration τ , the fluctuations in $\hat{X}_1$ could be made arbitrarily small and thus $\hat{X}_1$ may be determined to an arbitrary degree of certainty.

To test this hypothesis we now consider making a second meter readout at time τ after the first meter readout. This is done by noting that the first measurement leaves the detector in a squeezed state with squeeze parameter r , such that $e^{2r} = \cosh 2\kappa\tau$ (equation (5.53)). The meter is assumed to be reprepared in a coherent state.

The post readout variances are then found by substituting $C(\underset{\sim}{\gamma})$ for these initial states (appendix two) into equation (5.50) and integrating. The variances immediately upon readout after a measurement of duration τ, for a detector in a squeezed state at the start of the measurement, are given by,

$$\overline{V(\hat{X}_1(\tau))} = \frac{\hbar}{2\omega_a} \cdot \left\{ 1 - \frac{2}{g+1} \right\} \tag{5.58}$$

$$\overline{V(\hat{X}_2(\tau))} = \frac{\hbar}{2\omega_a} \left\{ 1 + \frac{2}{g+1} \right\} \tag{5.59}$$

where

$$g = \frac{\mathrm{Coth}^2 \kappa\tau}{\left(\frac{\mathrm{Sinh}\, 2r}{(\mathrm{Cosh}\, 2r + 1) . \mathrm{Sinh}^2 \kappa\tau} + 1 \right)} \tag{5.60}$$

and r is the squeeze parameter for the state of the detector.

Thus the variance in $\hat{X}_1$ immediately after a second readout at a time τ after the first readout (itself a time τ after $t = 0$) is given by

$$\overline{V(\hat{X}_1(2\tau))}^{(2)} = \frac{\hbar}{2\omega_a} \cdot \frac{1}{(2\mathrm{Cosh}^4 \kappa\tau - 1)} \tag{5.61}$$

To compare this with $\overline{V(x_1(\tau))}^{(1)}$ the variance after the first readout, we write equation (5.53) as

$$\overline{V(\hat{X}_1(\tau))}^{(1)} = \frac{\hbar}{2\omega_a} \frac{1}{(2\mathrm{Cosh}^2 \kappa\tau - 1)}$$

After the second readout we see that the fluctuations in $\hat{X}_1$ are reduced still further, despite the increase in fluctuations under free evolution.

If we now left the meter in the near $\hat{Y}_2$ eigenstate and the near $\hat{X}_1$ eigenstate (represented in equation (5.61)) resulting from the second readout, and let the system evolve freely for a further time τ, the possible error in the inferred value of $\hat{X}_1$ at a third measurement is given by equation (5.29) as

$$\Delta x_1^2(\tau) \approx \frac{\hbar}{2\omega_a} \cdot \frac{\mathrm{Cosh}^2 \kappa\tau}{(2\,\mathrm{Cosh}^4 \kappa\tau - 1)} \tag{5.62}$$

(We have assumed the variance in $\hat{Y}_2$ as a result of the second measurement is much less than the variance in $\hat{X}_1$ after the second measurement). We see that this error may be made arbitrarily small, no matter how short the measurement times, by making κ sufficiently large. Thus after two initial measurements the degenerate parametric amplifier is working like a perfect QND measurement scheme, and all future determinations of $\hat{X}_1(t)$ may be made with any degree of certainty.

It is as well now, to summarise the picture for the back action evasion measurements that has emerged from the previous discussion. At time $t = 0$, we prepare the meter in a coherent state and couple it to the detector. The total system is allowed to evolve freely for a time τ, whereupon a readout of the meter is made and a value $x_1^{(1)}(\tau)$ inferred for $\hat{X}_1(\tau)$. This value is, of course, completely unpredictable as we had no knowledge of the initial state of the detector. The meter is now reprepared in a coherent state and the system allowed to evolve for a further time τ. As we know that the variance in $\hat{X}_1$ prior to a second readout is less than the variance prior to the first readout (equation (5.57)) the inferred value for $\hat{X}_1$ must be close to $x_1^{(1)}(\tau)$ obtained from the first measurement, however it cannot be predicted with arbitrary certainty. Let the value inferred for $\hat{X}_1$ as a result of the second measurement be $x_1^{(2)}(\tau)$. If we now let the system evolve for a further time τ and make a third measurement of $\hat{X}_1$, the result obtained must be arbitrarily close to $x_1^{(2)}(\tau)$ as the variance in $\hat{X}_1$ prior to the third readout can be made arbitrarily small (equation (5.62)). Thus the value obtained from the third measurement can be predicted with arbitrary certainty from the result of the second measurement, no matter how short the measurement time. This is precisely what is required of a QND measurement.

An approximate back action evasion measurement has been discussed in detail by Hillery and Scully [1982].

CHAPTER 6

QUANTUM COUNTING QND MEASUREMENTS

6.1 Quadratic Coupling of the First Kind

As discussed in chapter five one of the earliest QND schemes to be proposed sought to monitor the number of quanta contained in a detector oscillator without altering the number distribution. The essential basis of such schemes was the realization that the number operator was a constant of the motion for a free oscillator and thus a QND variable [Braginsky, 1968].

We also saw in the previous chapter that while quantum counting techniques could reveal the presence of an arbitrarily weak force, it would not be possible to determine its precise strength.

For quantum counting QND to work we require that the number operator $(\hat{N} \equiv a^\dagger a)$ remain a constant of the motion in the presence of the meter interaction that is, $\hat{N}$ must satisfy the back action evasion criteria of equation (5.20). As Unruh [1979] was the first to point out this requires a coupling quadratic in the oscillators complex amplitude. Such a coupling would give rise to an interaction Hamiltonian that commutes with the detector number operator.

In this chapter we present an analysis of two such quadratic coupling schemes based on a quantum optical four wave mixing interaction.

Let the coupled detector meter system be described by the following Hamiltonian [Milburn and Walls, 1982c]

$$H = \hbar\omega_a a^\dagger a + \hbar\omega_b b^\dagger b + \hbar\chi a^\dagger a b^\dagger b \qquad (6.1)$$

where $a(b)$ is an annihilation operator for the detector (meter) mode, ω_a and ω_b are the oscillator frequencies and χ is the coupling constant. Clearly the interaction Hamiltonian is quadratic in the oscillators quadrature phase amplitude $(a^\dagger a = \omega/2\hbar \cdot (\hat{X}_1^2 + \hat{X}_2^2) - 1/2)$ and thus satisfies the back action evasion criteria. For ease of reference we shall refer to this coupling scheme as "quadratic coupling of the first kind".

Such an interaction Hamiltonian could also represent the quantised fields of two optical modes coupled by a third order susceptibility as in four wave mixing.

Clearly $a^\dagger a$ is a detector QND variable. Furthermore any analytic function of $a^\dagger a$ will also be a QND variable. Thus the number distribution $P(n_a)$ of the detector cannot change as a result of unitary evolution.

The solution to the Heisenberg equations of motion in the interaction picture are,

$$a(t) = \exp\{-i\chi b^\dagger b t\} \cdot a(0) \tag{6.2a}$$

$$b(t) = \exp\{-i\chi a^\dagger a t\} \cdot b(0) \tag{6.2b}$$

Let $\hat{Y}_1(t)$ and $\hat{Y}_2(t)$ denote the quadrature phase operators of the meter. Then

$$\hat{Y}_1(t) = \hat{D}_1(t)\, \hat{Y}_1(0) + \hat{D}_2(t) \cdot \hat{Y}_2(0) \tag{6.3}$$

where the operators $\hat{D}_1$ and $\hat{D}_2$ are defined by

$$\hat{D}_1(t) \equiv \cos(\chi a^\dagger a t) \tag{6.4a}$$

$$\hat{D}_2(t) \equiv \sin(\chi a^\dagger a t) \tag{6.4b}$$

We note that $\hat{D}_1$ and $\hat{D}_2$ are also QND operators for the detector.

From equation (6.3) we have

$$\langle \hat{Y}_1(t) \rangle = \langle \hat{D}_1(t) \rangle \langle \hat{Y}_1(0) \rangle + \langle \hat{D}_2(t) \rangle \langle \hat{Y}_2(0) \rangle \qquad (6.5)$$

Equation (6.5) allows us to infer values for $\hat{D}_1(t)$ from measurements of $\hat{Y}_1(t)$.

If we prepare the detector in an initial coherent state such that $\langle \hat{Y}_2(0) \rangle = 0$ and $\langle Y(0) \rangle = (2\hbar/\omega_b)^{1/2} \cdot y_1(0)$, a measurement of $\hat{Y}_1$ at time τ with result $y_1(\tau)$ can be used to infer a value $d_1(\tau)$ for $\hat{D}_1(\tau)$ given by

$$d_1(\tau) = (\omega_b/2\hbar)^{1/2} \cdot y_1(\tau)/y_1(0) \qquad (6.6)$$

The error in this inferred value $\Delta d_1(\tau)$ is given by,

$$\Delta d(\tau) = \left(\frac{\omega_b}{2\hbar}\right)^{1/2} \cdot \frac{V(\hat{Y}_1(\tau))^{1/2}}{y_1(0)} \qquad (6.7)$$

where $V(\hat{Y}_1(\tau))$ is the variance in $\hat{Y}_1$ at the time of readout.

If the meter is initially in the coherent state $|y_1(0)\rangle$ we find that

$$V(Y_1(t)) = \frac{\hbar}{2\omega_b} + \frac{2\hbar}{\omega_b} \cdot y_1^2(0) \cdot V(\hat{D}_1(t)) \qquad (6.8)$$

and thus

$$\Delta d_1(\tau) = \left\{\frac{1}{4y_1^2(0)} + V(\hat{D}_1(\tau))\right\}^{1/2} \qquad (6.9)$$

We see that it is impossible to predict with certainty the outcome of a measurement of $\hat{D}_1$ if it is very uncertain at the time of readout. However if $V(\hat{D}_1(\tau))$ is small then by preparing the meter in a highly excited coherent state ($y_1(0)$ large) we can make a determination of $d_1(\tau)$ arbitrarily certain.

Before proceeding to an analysis of meter state reduction we first discuss some properties of the operator $\hat{D}_1(t)$ (similar statements can be made concerning $\hat{D}_2(t)$).

The eigenstates of $\hat{D}_1(\tau)$ are the number states $|n\rangle$, with eigenvalue $d_1(\tau)$ given by

$$d_1(\tau) = \cos(\chi n \tau) \tag{6.10}$$

However, due to the periodic nature of the cosine function, the number states $|n\rangle$, where

$$n = \left| \frac{2\pi \ell \pm \cos^{-1}(d_1(\tau))}{\chi \tau} \right| \tag{6.11}$$

with $\ell = 0, 1, 2 \ldots$

(and n is restricted to be an integer) are degenerate.

It should be noted however that while the number states with n given by equation (6.11), are also eigenstates of $\hat{D}_1(\tau')$ $(\tau \neq \tau')$ these states are no longer degenerate. We have

$$\hat{D}_1(\tau') \cdot \left| \frac{2\pi\ell \pm \Theta}{\chi \tau} \right\rangle = d(\tau') \cdot \left| \frac{2\pi\ell \pm \Theta}{\chi\tau} \right\rangle$$

where

$$\Theta = \cos^{-1}(d_1(\tau))$$

and

$$d_1(\tau') = \cos[(2\pi\ell \pm \Theta)\,\tau'/\tau] \tag{6.12}$$

These eigenvalues are dependent on the value of ℓ . Thus, while a linear superposition of the states specified by equation (6.11) is also an eigenstate of $\hat{D}_1(\tau)$, it is not an eigenstate of $\hat{D}_1(\tau')$.

The periodicity of the cosine function also implies that if the system were in a pure number state $|n\rangle$ a single measurement of $\hat{D}_1$ at time t_1 could not be used to infer a unique value for n .

However a second measurement of $\hat{D}_1$ at a time t_2 such that $\chi(t_1 - t_2)$ is not a rational number, allows n to be inferred uniquely from the results of the two measurements. This may be seen as follows. Let y be the result obtained for $\hat{D}_1(t_1)$ and z be the result obtained for $\hat{D}_1(t_2)$, then with $\lambda = \chi t_1$ and $\mu = \chi t_2$ we need to solve the two equations.

$$\cos(\lambda n) = y$$
$$\cos(\mu n) = z$$

or

$$n\lambda = \cos^{-1} y + 2\pi l$$
$$n\mu = \cos^{-1} z + 2\pi m$$

where l, m are arbitrary integers. Thus

$$n = \frac{(\cos^{-1} y - \cos^{-1} z)}{\lambda - \mu} + \frac{2\pi(l-m)}{\lambda - \mu}$$

$$= \frac{\cos^{-1} y - \cos^{-1} z}{\lambda - \mu} + \frac{2\pi L}{\lambda - \mu}$$

where L is also an integer.

The separation between two possible values of n is given by $2\pi(L_1 - L_2)/(\lambda - \mu)$ which must itself be an integer or zero. If it is not an integer then a unique value for n may be inferred, (i.e. $L_1 - L_2 = 0$). That is a unique value for n cannot be inferred if

$$\lambda - \mu = 2\pi\, p/q$$

or

$$\chi(t_1 - t_2) = p/q$$

where p, q are integers. Only if $\chi(t_1 - t_2)$ is irrational can n be inferred uniquely.

In any practical scheme of course, it is only possible to choose $\chi(t_1 - t_2)$ to be a fixed number of decimal places and it may be impossible to ensure $n\chi(t_1 - t_2)$ is not an integer multiple of over the range of n values appropriate to the detector description.

An instantaneous perfectly accurate measurement of $\hat{D}_1(t)$ with result $d_1(\tau)$ will leave the detector in a linear combination of eigenstates corresponding to this measured value. Which linear combinations occur depend on which eigenstates were present in the pre-measurement number distribution of the detector. Once the measurement is made and the detector state has collapsed to this linear combination of number states, the corresponding number distribution cannot change as a result of free evolution of the coupled system. A subsequent perfectly accurate measurement of $\hat{D}_1$ at the same time τ after the first, must then yield the same result $d_1(\tau)$. Of course the value of $\hat{D}_1$ at any time other than $t = \tau$ is still unpredictable.

This behaviour is very much the behaviour of a stroboscopic QND variable. Indeed if the variance of $\hat{D}_1$ is zero at any time it periodically returns to zero. For example at $t = 0$. $V(\hat{D}_1(0)) = 0$ for any detector state, and it returns to zero every period. In figure 6.1 we have plotted the variance in $\hat{D}_1$ as a function of time for an initial coherent state. The stroboscopic nature of is clearly evident.

A measurement at time τ after the initial time places the system in a $\hat{D}_1$ eigenstate, i.e. $V(\hat{D}_1(\tau)) = 0$. The variance

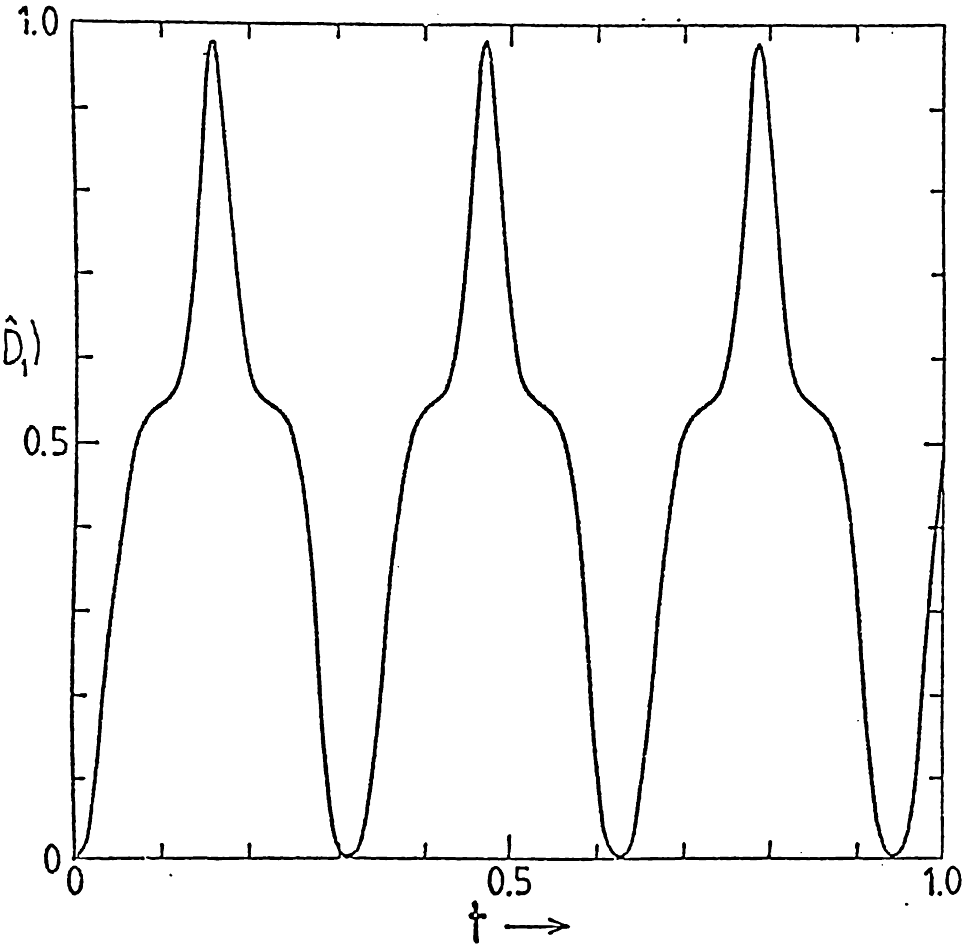

Figure 6.1 The variance in $\hat{D}_1$ versus time for an oscillator in a coherent state with $\bar{n} = 1.0$.

of $\hat{D}_1$ then returns to zero every τ seconds. In this case the period of the strobscopic QND variable (i.e. the time between successive values of zero variance) is determined by the time of the first measurement.

We now proceed to the second step in a QND measurement analysis; a determination of the non-unitary effect of meter state reduction.

The state of the detector after a readout of $\hat{Y}_1$ at time τ is, in the Schrodinger picture, given by

$$\rho(\tau) = \mathcal{N} \langle y_1(\tau), \tau | \exp[-iH\tau/\hbar] \, \rho(0) \exp[iH\tau/\hbar] \, | y_1(\tau), \tau \rangle \tag{6.13}$$

where $y_1(\tau)$ is the result obtained for $\hat{Y}_1$ and $|y_1(\tau), \tau\rangle$ is the corresponding $\hat{Y}_1(\tau)$ eigenstate, $\rho(0)$ is the initial density operator and H is the system Hamiltonian. The normalization constant is given by $\mathcal{N}^{-1} = Tr(\rho(\tau) |y_1(\tau), \tau\rangle\langle y_1(\tau), \tau|)$ (see section 1.5).

The distribution $\bar{P}(n)$ of the detector after readout is given by

$$\bar{P}(n) = \langle n | \bar{\rho}_D(\tau) | n \rangle$$
$$= \mathcal{N} \left| \langle y_1(\tau), \tau | \exp\left\{ -i(\omega_b + \chi n) b^\dagger b \tau \right\} | \psi \rangle \right|^2 . P(n) \tag{6.14}$$

where $|\psi\rangle$ is the initial state of the meter and $P(n)$ is the detector initial number distribution and we have used the Hamiltonian given in equation (6.1).

We now assume the meter is initially in a coherent state $|y_1(0)\rangle$ with $y_1(0)$ real. Then $\langle \hat{Y}_2(0) \rangle = 0$ and

$$\langle \hat{Y}_1(0) \rangle = \left(\frac{2\hbar}{\omega_b} \right)^{1/2} . y_1(0) \tag{6.15}$$

The eigenstates of $\hat{Y}_1(t)$ are obtained from the eigenstates of $\hat{Y}_1(0)$ by [Caves et.al., 1980]

$$| y_1(t), t \rangle = \exp\left\{ -i\omega_b b^\dagger b t \right\} | y_1(t), 0 \rangle$$

(see equation (5.47)). We next note that in fact $\hat{Y}_1(0) = \hat{q}$ (see equation (5.8)) and thus $|y_1(t), 0\rangle$ is a position eigenstate. We are now in a position to evaluate equation (6.14).

We firstly expand $|y_1(t), 0\rangle$ in terms of number states by [Louisell, 1973]

$$|y_1(t), 0\rangle = \left(\frac{\omega_b}{\pi\hbar} \right)^{1/4} \exp\left\{ \frac{-\omega_b y_1^2(t)}{2\hbar} \right\} . \sum_{\ell=0}^{\infty} \frac{H_\ell\left(\sqrt{\omega_b/\hbar} . y_1(t) \right)}{2^{\ell/2} . \sqrt{\ell!}} \tag{6.16}$$

where $H_\ell(x)$ is a Hermite polynomial of order ℓ. We also expand the coherent state $|y_1(0)\rangle$ in terms of number states,

$$| y_1(0) \rangle = \exp\left(-1/2 \, y_1^2(0) \right) \sum_{\ell=0}^{\infty} \frac{y_1(0)^\ell}{\ell!} | \ell \rangle \tag{6.17}$$

Equation (6.16) and (6.17) together with the following identity [Gradsteyn and Ryzhik, 1965]

$$\sum_{\ell=0}^{\infty} \frac{a^{\ell}}{2^{\ell}\ell!} H_{\ell}(x) = \exp\left(ax - a^2/4\right) \tag{6.18}$$

equation (6.14) becomes

$$\bar{P}(n) = \mathcal{N} \exp\left\{-2\left[y_1(0)\cos\chi n\tau - \sqrt{\frac{\omega_b}{2\hbar}} \cdot y_1(\tau)\right]^2\right\} P(n) \tag{6.19}$$

using equations (6.6) and (6.15), equation (6.19) may be written

$$\bar{P}(n) = \mathcal{N}' \exp\left\{-2 y_1^2(0)\left[\cos\chi n\tau - d_1(\tau)\right]^2\right\} P(n) \tag{6.20}$$

where $d_1(\tau)$ is the inferred value of $\hat{D}_1$.

If we choose $y_1(0)$ large, i.e. the meter is prepared in a highly excited coherent state, then $\bar{P}(n) = 0$ except when $d_1(\tau) = \cos\chi n\tau$ We conclude that $\bar{P}(n)$ will be multipeaked and concentrated at values of n given by

$$n = \left|\frac{2\pi\ell \pm \cos^{-1}(d_1(\tau))}{\chi\tau}\right| \tag{6.21}$$

where $\ell = 0, 1, 2 \ldots$ and n is an integer. As these values of n correspond to the degenerate eigenstates of $\hat{D}_1(\tau)$ with eigenvalue $d_1(\tau)$ we see that after a readout the detector is in a near eigenstate of $\hat{D}_1$ with eigenvalue equal to the inferred result.

In figure 6.2(a),(b) we have plotted equation (6.20) for $P(n)$ poissonian (i.e. the detector initially in a coherent state). As expected the post readout distribution corresponds to a near eigenstate of $\hat{D}_1$ for $y_1(0)$ large (6.2(b)) and the determination of $d_1(\tau)$ is most precise. After such a measurement the detector is in a linear superposition of the degenerate eigenstates of $\hat{D}_1$.

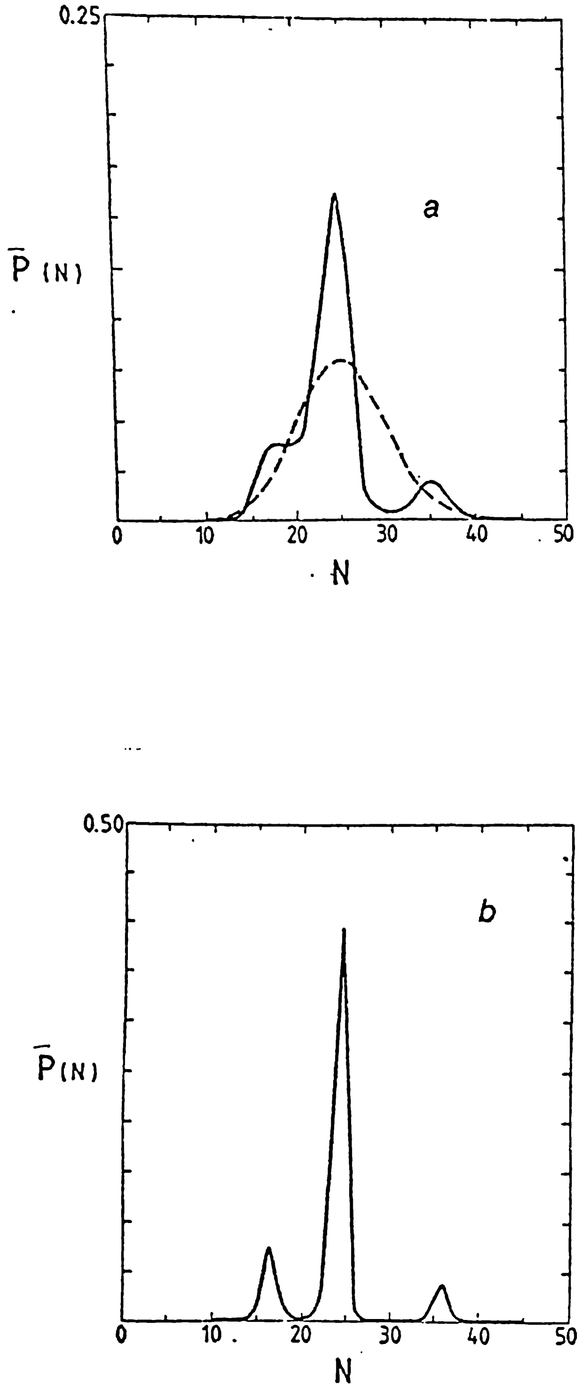

Figure 6.2 Post readout number distribution for quadratic coupling of the first kind (solid line) for an initial coherent state distribution (dashed line). Two different meter states are shown (a) $\gamma_1^2(0) = 2.0$ (b) $\gamma_1^2(0) = 15.0$
In both (a) and (b) $\alpha_1(\tau) = 0.5$, $\tau = 0.1\pi$.

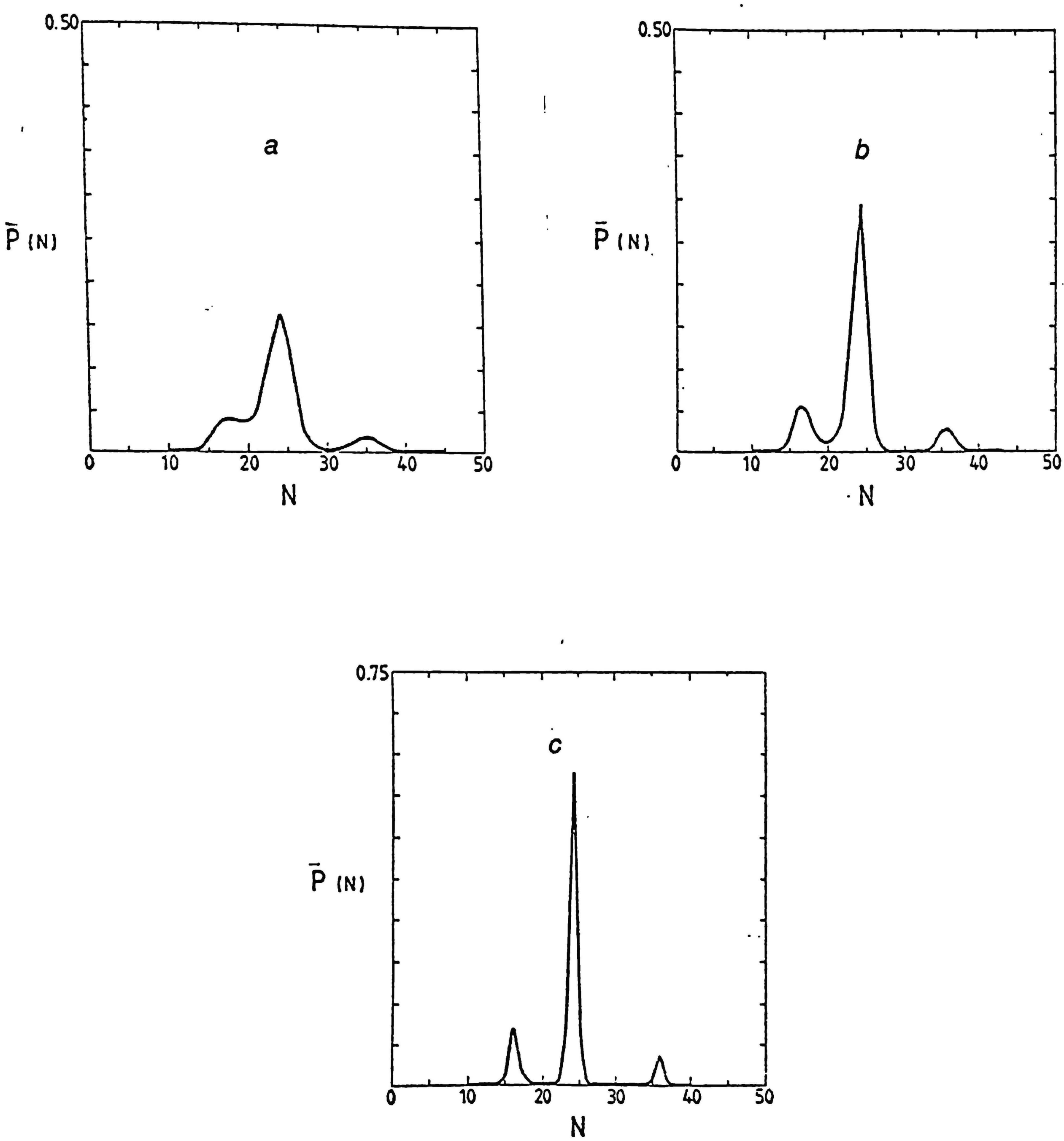

Figure 6.3 Number distribution after three succesive readouts a, b, c, for an initial coherent state. $\mathcal{G}_1^2(0) = 1.0$
$\alpha_1(\tau) = 0.3$, $\chi\tau = 0.1\pi$

If $y_1(0)$ is not sufficiently large a couple of measurements made after commensurate evolution times will place the detector in an arbitrarily near eigenstate of $\hat{D}_1$. This may be seen in figures 6.3(a),(b),(c) where we have plotted $\bar{P}(n)$ after three consecutive measurements. One must of course reprepare the meter in the same state prior to each measurement.

As each measurement of $\hat{D}_1$ after a time τ places the detector in a closer eigenstate the values $d_1(\tau)$ for each measurement approach a limiting constant sequence. This is what is required of a QND measurement.

It should be noted that an initially precise determination of $\hat{D}_1$ at time t_1 cannot be used to infer a precise value for $\hat{D}_1$ at a later time t_2 . The reason is that the degenerate eigenstates of $\hat{D}_1(t_1)$ are not degenerate with respect to $\hat{D}_1(t_2)$. Thus while a linear superposition of eigenstates of $\hat{D}_1(t_1)$ is also an eigenstate of $\hat{D}_1(t_1)$ it is not an eigenstate of $\hat{D}_1(t_2)$. In the problem just considered a measurement of $\hat{D}_1$ at time t_1 places the detector in just such a linear superposition of eigenstates. Since $a^\dagger a$ is a constant of the motion the detector remains in this state under free evolution and thus is not an eigenstate of $\hat{D}_1$ at time t_2 . This may also be seen as a consequence of the fact that $\hat{D}_1$ is a stroboscopic QND variable.

As $a^\dagger a$ is a constant of the motion the detector once placed in an eigenstate of $\hat{D}_1(\tau)$ will remain there. This enables a determinate sequence of measurements of $\hat{D}_1(\tau)$ to be obtained, given perhaps one or two preparatory measurements.

However, if a classical force acts on the oscillator in an initial number state the oscillator is driven into a double peaked distribution (see Caves et.al, 1980, equation (2.26)). Such a state will not be an eigenstate of $\hat{Q}_1(t)$ (we assume the force is sufficiently strong for equation (5.14) to occur). Thus if the force acts at any time during a determinate sequence of $a_1(\tau)$ one would obtain a result not commensurate with that obtained from measurements prior to the action of the force, and conclude that a classical force had been detected.

6.2 Quadratic Coupling of the Second Kind

We now consider the two harmonic oscillators to be coupled via the interaction Hamiltonian [Milburn and Walls, 1982d].

$$H_I = \hbar\chi' a^\dagger a \left(b\, \varepsilon^*(t) + b^\dagger \varepsilon(t) \right) \tag{6.22}$$

where $\varepsilon(t) = \varepsilon\, e^{i\omega t}$. Such a Hamiltonian could represent two electromagnetic field modes coupled by a third order susceptibility as in a four wave mixing process with one mode in a highly populated coherent excitation and treated clasically.

The obvious QND variable is, once again $\hat{N}_a \equiv a^\dagger a$. Furthermore the interaction satisfies the back action evasion criteria. The equations of motion in the interaction picture are

$$\frac{d}{dt} b(t) = -i\chi \hat{N}_a e^{-i\nu t} \tag{6.23a}$$

$$\frac{d\hat{N}_a}{dt} = 0 \tag{6.23b}$$

where $\chi = \chi'\varepsilon$ and $\nu = \omega - \omega_b$

The solution to (6.23a) is

$$b(t) = \xi(t)\,\hat{N}_a + b(0) \tag{6.24}$$

where $$\xi(t) = \frac{\chi}{\nu}.(e^{-i\nu t} - 1)$$

Equation (6.24) implies that in the interaction picture the unitary time development operator is given by

$$U(t) = \exp.\left\{ \hat{N}_a \left(\xi(t) b^{+} - \xi^{*}(t)\, b \right) \right\} \tag{6.25}$$

Defining $\hat{N}_b(t) \equiv b^{+}(t).b(t)$ as the meter number operator we have upon substitution of (6.24) that

$$\hat{N}_b(t) = |\xi(t)|^2 \hat{G}_a + \hat{N}_a \left(\xi^{*}(t)\, b(0) + \xi(t)\, b^{+}(0) \right) + \hat{N}_b(0) \tag{6.26}$$

where

$$\hat{G}_a \equiv (\hat{N}_a)^2 \tag{6.27}$$

We now assume the meter has been prepared in the initial number state $|n_b(0)\rangle$, perhaps resulting from a previous measurement of $\hat{N}_b(t)$. Then

$$\langle \hat{N}_b(t) \rangle = |\xi(t)|^2 \langle \hat{G}_a \rangle + n_b(0) \tag{6.28}$$

From a measurement of $\hat{N}_b$ at time τ with result $n_b(\tau)$ we may infer a value g_a for $\hat{G}_a$ given by

$$g_a = (n_b(\tau) - n_b(0)) / |\xi(\tau)|^2 \tag{6.29}$$

The possible error in this inferred value Δg_a is given by

$$\Delta g_a = \frac{V(N_b(\tau))^{1/2}}{|\xi(\tau)|^2} \tag{6.30}$$

For the assumed initial meter state $|n_b(0)\rangle$ we find

$$\Delta g_a = \left\{ V(\hat{G}_a) + 2\,\frac{\langle \hat{G}_a \rangle (n_b + 1/2)}{|\xi(\tau)|^2} \right\}^{1/2} \tag{6.31}$$

when the measurement time τ , is very small we find

$$\Delta g_a \approx \left\{ V(G_a) + 2 \cdot \frac{\langle G_a \rangle}{(\chi \tau)^2} \cdot (n_b + 1/2) \right\}^{1/2} \tag{6.32}$$

(equation (6.32) is exact on resonance, i.e. $\nu = 0$). Once again we see that we cannot predict with certainty a value for $\hat{G}_a$ if it is initially very uncertain prior to the measurement. However if $V(\hat{G}_a)$ is small we may determine a value for $\hat{G}_a$ with certainty, no matter how small the measurement time τ , by ensuring that χ is sufficiently large. This is the usual limit for arbitrarily accurate instantaneous quantum measurements.

We now turn to a consideration of the non-unitary change of the detector upon readout of the meter variable. If $\rho(\tau)$ is the Schrodinger picture density operator of the coupled detector meter system at the time of readout, the density operator of the total system after readout is

$$\rho(\tau) = \mathcal{N} \, |n_b(\tau)\rangle\langle n_b(\tau)| \, \rho(\tau) \, |n_b(\tau)\rangle\langle n_b(\tau)| \tag{6.33}$$

where $\mathcal{N}^{-1} = Tr\left(\rho(t) \, |n_b(t)\rangle\langle n_b(t)|\right)$. We next note that if $\rho^I(t)$ is the total density operator in the interaction picture the photon number distribution $P(n_a)$, for the detector is given by

$$P(n_a) = \langle n_a | \rho_D(t) | n_a \rangle = \langle n_a | \rho_D^I(t) | n_a \rangle$$

(i.e. it is the same in either picture).

It is easily verified using equation (5.45) that the post readout density operator is given in the interaction picture by

$$\bar{\rho}^I(t) = \mathcal{N} \, |n_b(\tau)\rangle\langle n_b(\tau)| \, \rho^I(\tau) \, |n_b(\tau)\rangle\langle n_b(\tau)|$$

If the initial density operator is given by

$$\rho(0) = |\psi\rangle \, |n_b(0)\rangle\langle n_b(0)| \, \langle\psi| \tag{6.34}$$

where ψ refers to the state of the detector and $n_b(0)$ refers to the state of the meter. Using equations (6.25), (6.34), (6.33), (1.56) and (1.58), the post readout number distribution is given by,

$$\bar{P}(n_a) = \left| \langle n_b(\tau) | \exp\left\{ n_a(\xi(\tau) b^+ - \xi^*(\tau) b \right\} . | n_b(0) \rangle \right|^2 P(n_a) \quad (6.35)$$

where $P(n_a) = |\langle n_a | \psi \rangle|^2$

Using a result (equation (2.26) of Caves et.al., 1980) equation (6.35) becomes,

$$\bar{P}(n_a) = \frac{N!}{M!} \left\{ L_N^K(x) \right\}^2 . x^k . e^{-x} \quad P(n_a) \qquad (6.36)$$

where

$$M = n_b(\tau) \quad ; \quad N = n_b(0)$$

$$K = M - N = g_a |\xi(\tau)|^2$$

$$x = (n_a |\xi(\tau)|)^2$$

and $L_N^K(x)$ is the generalised Laguerre polynomial.

To obtain some insight into the meaning of equation (6.36) we consider the limiting form of $\bar{P}(n_a)$ for small τ, with $\chi\tau$ large. (The limit of arbitrarily accurate instantaneous measurements).

Consider the function $f(x)$,

$$f(x) = \left\{ L_N(x) \right\}^2 . x^k . e^{-x} \qquad (6.37)$$

It is easily verified that $f'(x) = 0$ when

$$2x \frac{d L_N^K(x)}{dx} + (K - x) L_N^K(x) = 0 \qquad (6.38)$$

In general this equation has many roots, indicating that $\bar{P}(n_a)$ is multipeaked. However if x and K are scaled by some parameter A, as they are in equation (6.36) where $A = (\chi\tau)^2$, equation (6.38) may be written

$$2y \frac{d}{dy} \left(\frac{L_N^K(Ay)}{A} \right) + (g_a - y) L_N^K(Ay) = 0$$

where $y = n_a^2$. The first term in this expression is of the order of A^{N-1} whereas the second term is of the order of A^N . Thus for large A equation (6.38) has one root at $x = K$ i.e. $n_a = \sqrt{g_a}$ The probability distribution after readout is then seen to be concentrated around $n_a = \sqrt{g_a}$. The number state $|\sqrt{g_a}\rangle$ is of course an eigenstate of $\hat{G}_a$ with eigenvalue g_a , the inferred result of the measurement. Thus no matter how small the measurement time τ , by choosing the coupling sufficiently large so that $\chi\tau$ is large, we find that after readout the detector will be in an eigenstate of the measured observable with eigenvalue corresponding to the result of the measurement.

In the simplest case we may prepare the state of the detector so that $n_b(0) = 0$. Since $L_0^K(x) \equiv 1$, the function $f(x)$ has only one peak at $x = K$, for all values of $\chi\tau$. In this case $f''(x)_K$ diverges to $-\infty$ as K becomes large, indicating that in this limit $\bar{P}(n_a)$ becomes sharply peaked.

This behaviour is evident in figure 6.4 where we have plotted $\bar{P}(n_a)$ against n_a for two different values of $\chi\tau$. For the values of the chosen parameters $\sqrt{g_a} = 20$. We see that as $\chi\tau$ increases the post-readout distribution becomes more norrowly concentrated on $n_a = \sqrt{g_a}$.

Since $a^\dagger a$ is a constant of the motion the detector once placed in a near eigenstate of $\hat{G}_a$ will remain there. This is precisely the situation which occurred for first quadratic coupling considered in section 6.1 Thus subsequent measurements must yield the constant sequence $\{..g_a..\}$ of results. Any departure from this result may be taken as evidence of the presence of an external force.

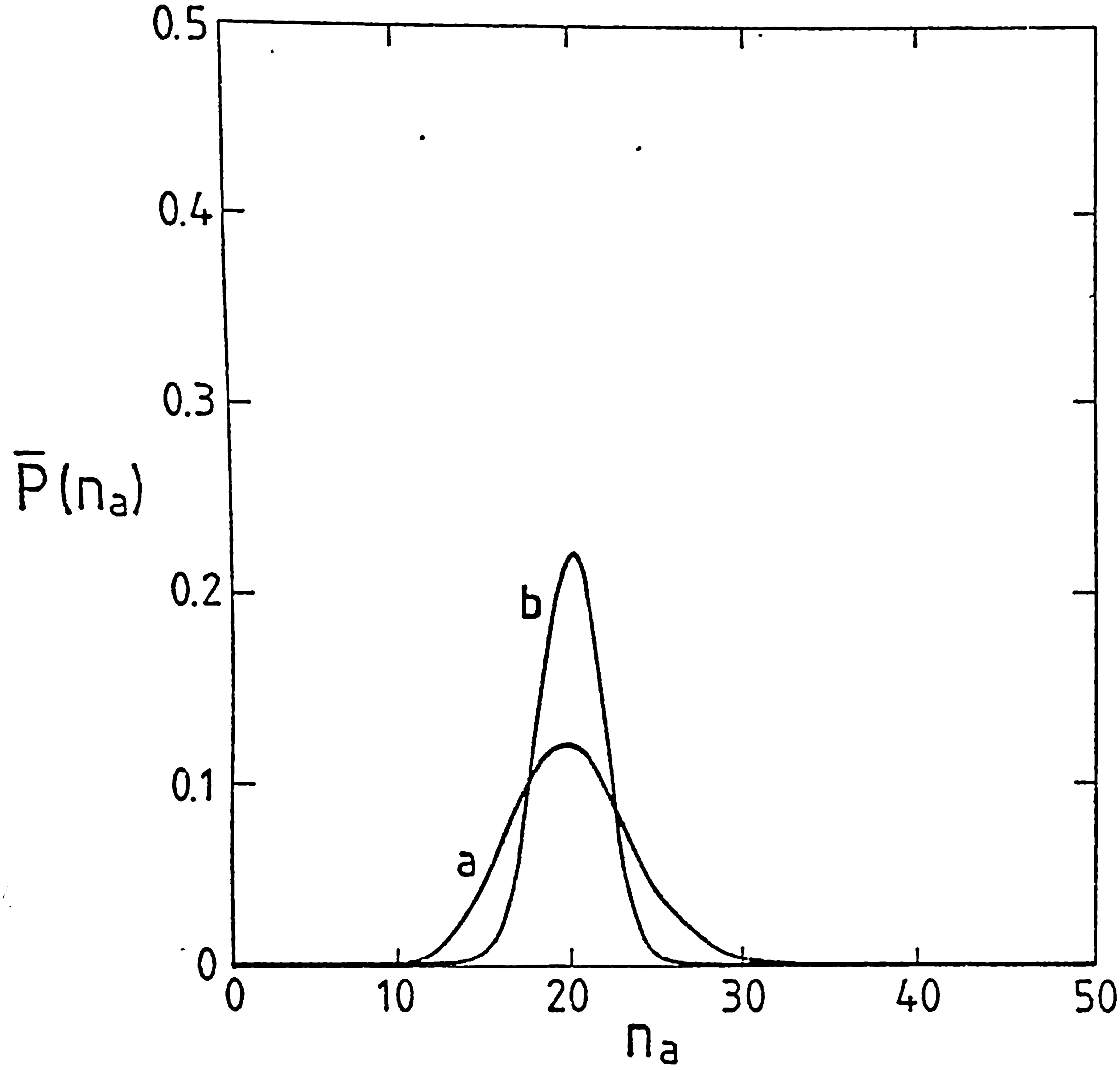

Figure 6.4 Post-readout detector number distribution.

(a) $n_b(t) = 4$, $n_b(0) = 0$, $\lambda\tau = 0.1$

(b) $n_b(t) = 25$, $n_b(0) = 0$, $\lambda\tau = 0.25$

In both cases the initial state of the detector was such that $\bar{n}_a = 20$

APPENDIX ONE

WANG AND UHLENBECKS SOLUTION OF A LINEAR FOKKER-PLANCK EQUATION

We consider a system described by n complex variables given as the vector

$$\underset{\sim}{z} = \begin{pmatrix} z_1 \\ z_2 \\ \vdots \\ z_n \end{pmatrix} \tag{A1.1}$$

The system is then taken to satisfy the general linear Fokker-Planck equation

$$\frac{\partial P(\underset{\sim}{z}, t)}{\partial t} = \left\{ -\nabla_{\underset{\sim}{z}}^{T} M \underset{\sim}{z} + 1/2 \ \nabla_{\underset{\sim}{z}}^{T} . N . \nabla_{\underset{\sim}{z}} \right\} \tag{A1.2}$$

where

$$\nabla_{\underset{\sim}{z}} = \begin{pmatrix} \frac{\partial}{\partial z_1} \\ \vdots \\ \frac{\partial}{\partial z_n} \end{pmatrix} \tag{A1.3}$$

and M and N are both $n \times n$ matrices and N is positive definite and symmetric.

The first step in solving (A1.2) requires the diagonalization of the drift matrix M

$$\Lambda = S^{-1} M S = Diag(\lambda_1, \ldots \lambda_n) \tag{A1.4}$$

We then make a change of variable by

$$\underset{\sim}{u} = S^{-1} \underset{\sim}{z} \tag{A1.5}$$

The initial condition is then taken to be

$$P(\underset{\sim}{z}, 0) = \delta^{n}(\underset{\sim}{z} - \underset{\sim}{z}_0) = |\det S^{-1}| . \delta^{n}(\underset{\sim}{u} - \underset{\sim}{u}_0)$$

Solving equation (A1.2) in the new variables we obtain

$$P(\underset{\sim}{z},t) = \frac{|\det S^{-1}|}{(2\pi)^{n/2}|C_1|^{1/2}} \exp.\left\{-1/2\ \underset{\sim}{u}(t)^T\ C_1^{-1}(\underset{\sim}{u})\ \underset{\sim}{u}(t)\right\} \tag{A1.6}$$

where

$$\underset{\sim}{u}(t) = \underset{\sim}{u} - e^{\Lambda t}\underset{\sim}{u}_o \tag{A1.7}$$

and

$$C_1(u) = \frac{-\pi_{ij}}{\lambda_i+\lambda_j}\left(1 - e^{(\lambda_i+\lambda_j)t}\right) \tag{A1.8}$$

with

$$\pi = S^{-1} N (S^{-1})^T \tag{A1.9}$$

The covariance matrix in the original variables $C_1(\underset{\sim}{z})$ is then given by

$$C_1(\underset{\sim}{z}) = S\ C_1(\underset{\sim}{u})\ S^T \tag{A1.10}$$

For non-delta function initial conditions we need to take into account the decay from the initial covariance matrix. We then need to add to $C_1(\underset{\sim}{z})$ given in equation (A1.10), the term $e^{-\Lambda t} C_o(\underset{\sim}{z})\, e^{-A^T t}$ where C_o is the initial covariance matrix [see Gardiner, 1982]. Thus

$$C(\underset{\sim}{z}) = C_1(\underset{\sim}{z}) + e^{-At}.C_o(\underset{\sim}{z}).e^{-A^T t} \tag{A1.11}$$

APPENDIX TWO

GENERAL COVARIANCE MATRIX FOR THE PARAMETRIC AMPLIFIER QND SCHEME

In this appendix we give the expression for the covariance matrix of the coupled oscillator QND model discussed in chapter 5.

The covariance matrix for the coupled detector-meter system in the presence of damping and for initial squeezed states, may be written as the sum of two terms:

$$\underset{\sim}{C}(z) = \underset{\sim}{C}_1(z) + \underset{\sim}{C}_0(z) \tag{A2.1}$$

where

$$C_1(z) = \begin{pmatrix} 0 & a & -ib & 0 \\ a & 0 & 0 & ib \\ -ib & 0 & 0 & a \\ 0 & ib & a & 0 \end{pmatrix} \tag{A2.2}$$

and

$$a = \frac{8K^2}{\Delta^2}\cdot\left(\frac{x}{p^2} - \frac{2y}{pq} + \frac{z}{q^2}\right) \tag{A2.3}$$

$$b = \frac{2K}{\Delta}\left(x - z - \frac{(\gamma_2-\gamma_1)}{2K}\cdot y\right) \tag{A2.4}$$

(see chapter 5 for further definitions

$$C_0(z) =$$

$$\begin{pmatrix} s_1A^2 - s_2B^2 & c_1A^2 + c_2B^2 & iAB(c_1+c_2) & -iAB(s_1-s_2) \\ c_1A^2 + c_2B^2 & s_1A^2 - s_2B^2 & iAB(s_1-s_2) & -iAB(c_1+c_2) \\ iAB(c_1+c_2) & iAB(s_1-s_2) & -s_1B^2 + s_2A^2 & c_1B^2 + c_2A^2 \\ -iAB(s_1-s_2) & -iAB(c_1+c_2) & c_1B^2 + c_2A^2 & -s_1B^2 + s_2A^2 \end{pmatrix} \frac{e^{-(\gamma_1+\gamma_2)t/2}}{2} \tag{A2.5}$$

where

$$s_i = -\sinh(2r_i)$$

$$c_i = \cosh 2r_i - 1$$

and r_1 and r_2 are the squeeze parameters for the detector and meter respectively.

The expressions for the covariance matrix and mean values for the state of the detector after readout of the meter variable $y_2(t)$, (with $r_1 = r_2 = 0$) are;

$$\overline{C(\alpha,\beta)} = (M^{-1} + N^{-1})^{-1} \qquad \text{(A2.6)}$$

where

$$M^{-1} = \frac{1}{a^2 - b^2} \begin{pmatrix} 0 & a \\ a & 0 \end{pmatrix}$$

$$N^{-1} = \frac{b^2}{a[2(a^2-b^2)+a]} \begin{pmatrix} 1 & -\left(1 + \frac{a}{a^2-b^2}\right) \\ -\left(1 + \frac{a}{a^2-b^2}\right) & 1 \end{pmatrix}$$

$$\overline{\langle \underset{\sim}{z}_{\delta 1} \rangle} = \langle \underset{\sim}{z}_{\delta 1}(t) \rangle + (M^{-1} + N^{-1})^{-1} \underset{\sim}{\omega}^{-1} \qquad \text{(A2.7)}$$

where

$$\underset{\sim}{\omega} = \frac{a^2 - b^2}{b} \cdot \sqrt{\frac{2\omega_b}{\hbar}} \cdot \left(y_2(t) - \langle \hat{y}_2(t) \rangle\right) \cdot \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

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