

Agreeable semigroups

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Abstract

This paper concerns the theory of partial maps under composition and more generally, the RC-semigroups introduced in [7] (semigroups with a unary operation called (right) closure). Many of the motivating examples have a natural meet-semilattice structure; the inverse semigroup of all injective partial transformations of a set and the semigroup of all binary operations under composition are two examples. We here view the semilattice meet as an additional operation, thereby obtaining a variety of algebras with one unary and two binary operations. The two non-semigroup operations are then shown to be captured by a single binary operation, via the notion of an agreeable semigroup. We look at a number of properties of these structures including their congruences (which are uniquely determined by their restriction to certain idempotents), a relationship with so-called interior semigroups, and a natural category associated with a large variety of RC-semigroups (which includes all inverse semigroups). For example, we show that the existence of equalisers in this category is intimately connected with the existence of the natural meet-semilattice structure.

1 Introduction

In an inverse semigroup of injective partial maps on a set, one may effectively associate with each such partial map f , a map that fixes the domain of f and is undefined everywhere else (the map $f^{-1} \circ f$). Likewise one may choose to associate with each partial map f on a set, the identity map on the domain of f . If these partial maps are closed under composition then we obtain a unary semigroup. Up to isomorphism, such structures form a variety that is axiomatised by a quite simple set of identities, and numerous aspects of the theory of inverse semigroups have a natural extension in this setting (see [7] and below). The study of these *twisted RC-semigroups* was begun in [7].

We take as the starting point of this paper a slightly broader class of structures called *right closure semigroups* or *RC-semigroups*, although particular attention is given to twisted RC-semigroups. All results have natural left sided duals. A subsemigroup of an RC-semigroup that is closed under the C operation will be called an *RC-subsemigroup*. Formal definitions are given in Section 2.

We begin (in Section 3) by considering the consequences of assuming that the natural partial order on an RC-semigroup is a meet-semilattice ordering; for the twisted case, we obtain a representation in terms of partial maps extending the representation of twisted RC-semigroups obtained in [7]. The meet operation is then used to give an interior semigroup structure in the sense of [8]. Section 8 concerns the congruences of these structures, which turn out to be rigidly determined by the closed elements (elements fixed by C). In Sections 4, 5 and 6 we look at a term equivalence between the semilattice ordered structures and semigroups with a binary operation $*$ which again has a natural interpretation via the representation as partial maps. Finally in Section 9 we examine a category related to RC-semigroups satisfying a fairly natural restriction (weaker than twistedness) and we describe when this category has equalisers in terms of the existence of the operation $*$ (equivalently, semilattice meet).

2 Preliminaries

In [7], the variety of *RC-semigroups* is defined to be the variety of unary semigroups with unary C satisfying the identities

1. $xC(x) = x$;
2. $C(x)C(y) = C(y)C(x)$;
3. $C(C(x)) = C(x)$; and
4. $C(xy)C(y) = C(xy)$.

It is shown that in any RC-semigroup S there is a subsemilattice L such that for any $x \in S$, $C(x) = \min\{e \in L \mid xe = x\}$. Indeed RC-semigroups can be alternatively defined as semigroups in which such a subsemilattice exists. LC-semigroups (left closure semigroups) can be defined by reversing all of the above laws.

Every inverse semigroup admits at least one RC-semigroup structure by defining $C(x)$ to be $x^{-1}x$. This will be called the *usual (right) closure* on an inverse semigroup. As well as the five laws above, it is easily verified that an inverse semigroup with the usual closure satisfies the additional law $C(x)y = yC(xy)$. In general an RC-semigroup satisfying $C(x)y = yC(xy)$ is called a *twisted* RC-semigroup. (Likewise the variety of twisted LC-semigroups is given within the variety of all LC-semigroups by the law $xC(y) = C(xy)x$.) The RC-semigroup law $C(C(x)y) = C(xy)$ is called the *right congruence condition*; it is implied by the twisted law.

We now recall from [7] some basic facts for RC-semigroups that are useful in what follows. By a *closed element* of an RC-semigroup S we mean an element $x \in S$ for which $x = C(x)$. By $C(S)$, we mean the set of all closed elements of S .

Lemma 2.1 (i) *The relation $\bar{\mathcal{L}}$ on a RC-semigroup S defined by $(a, b) \in \bar{\mathcal{L}} \Leftrightarrow C(a) = C(b)$ is a right congruence if and only if S satisfies the right congruence condition.*

- (ii) The relation \leq_C on an RC-semigroup S given by $a \leq_C b \Leftrightarrow a = bC(a)$ (equivalently, $a \leq_C b \Leftrightarrow (\exists e \in C(S)) a = be$) is a partial order that is stable under left and right multiplication, and closure when S is twisted (and coincides with the usual order when S is an inverse semigroup with its usual closure). This order will be called the C-order.

Some important examples of RC-semigroups that we use frequently below are the following

$\mathcal{B}(X)$: the RC-semigroup of all binary operations on X under composition and with the closure $C(f)$ of a relation f being the relation Δ_X restricted in its domain to the domain of f .

$\mathcal{P}(X)$: the sub-RC-semigroup of $\mathcal{B}(X)$ consisting of all partial maps of the set X under composition and with the closure $C(x)$ of a map f given by the identity map on the domain of f .

$\mathcal{I}(X)$: the inverse sub-semigroup of $\mathcal{P}(X)$ consisting of all injective elements of $\mathcal{P}(X)$.

Both $\mathcal{P}(X)$ and $\mathcal{I}(X)$ are twisted RC-semigroups while $\mathcal{B}(X)$ satisfies the right congruence condition [7].

Any twisted RC-semigroup S can be faithfully represented as an RC-subsemigroup of $\mathcal{P}(S)$ by setting $\psi_a(x) = ax$ for all $x \in C(a)S = \{C(a)x \mid x \in S\}$, generalising the Vagner-Preston representation of an inverse semigroup: see [7]. This representation will be used on numerous occasions below, and we refer to it as the *natural representation of a twisted RC-semigroup as partial maps*. In general if x is a partial transformation of a set then $\text{dom}(x)$, $\text{ran}(x)$ and $\text{fix}(x)$ will be used to denote the domain, range and elements fixed by x .

We note that the class of RC-semigroups is also closely related in various degrees to numerous structures in the literature that have stemmed from the *abundant semigroups* of Fountain [4]. In particular the class of weakly left ample semigroups (studied in [5, 6] for example) is exactly the class of twisted LC-semigroups in which every idempotent is closed. (Weakly right ample semigroups are the right handed dual and correspond to twisted RC-semigroups). These structures may be seen as a half way point between inverse semigroups with the standard closure and twisted LC-semigroups in their full generality. Nevertheless many of the results in the RC-semigroup setting imply results in the weakly right ample setting, and similarly some results from weakly right ample semigroups may be extended to RC-semigroups. One advantage with the current approach is that the classes we deal with form a variety and so standard equational reasoning is applicable; also we are here primarily concerned with semigroups of partial maps with domain information and in this setting it is twisted RC-semigroups that are the structures of interest (because of the natural representation). The class of LC-semigroups are also a subclass of the γ -semigroups introduced in [1] (see also [2]); in fact they coincide with the class of type SL γ -semigroups.

Some other similar ideas were introduced in [9], [11], [12], [13] and [14]. The primary goal in these papers is an algebraic theory of functions. Many of the structures introduced in these papers are semigroups with two unary operations; a left closure and a right closure (in our terminology).

3 Introducing global semilattice structure

Recall that in the RC-semigroup S , $a \leq_C b$ means that there exists $\alpha \in C(S)$ for which $a = b\alpha$; equivalently, $a = bC(a)$ (see Lemma 2.1). In many natural cases, such as most of the motivating examples appearing in [7] (including $\mathcal{P}(X)$), there is a meet-semilattice structure compatible with this C-order, in the sense that $a \wedge b = \max\{c \in S \mid c \leq_C a, b\}$ exists for all $a, b \in S$.

It is possible to characterise equationally when an RC-semigroup S has such a meet-semilattice order. Evidently this is the case if and only if the RC-semigroup is also a semilattice, with the C-order coinciding with the natural semilattice order (in which the semilattice operation is interpreted as meet). Thus we require $a \leq_C b$ if and only if $a = a \wedge b$. Note that $a \leq_C b$ if and only if $a = bC(a)$, so this last equivalence can be captured by the two laws $bC(a) = bC(a) \wedge b$ and $a \wedge b = bC(a \wedge b)$.

However, the following equational form is somewhat more convenient to work with.

Proposition 3.1 *Let S be an RC-semigroup with additional binary operation \wedge . Then S is a meet-semilattice under the C-order, with \wedge the meet operation, if and only if the following hold for all $a, b \in S$:*

1. $a \wedge a = a$ and $a \wedge b = b \wedge a$;
2. $a \wedge b = aC(a \wedge b)$; and
3. $(a \wedge b)\alpha = (a\alpha) \wedge (b\alpha)$ for all $\alpha \in C(S)$.

Proof. If S is a meet-semilattice under the C-order and \wedge is the meet operation, then certainly the first two conditions hold. As for the third, certainly $(a \wedge b)\alpha \leq_C a\alpha, b\alpha$, so $(a \wedge b)\alpha \leq_C (a\alpha) \wedge (b\alpha)$. Conversely, if $d \leq_C a\alpha, b\alpha$, then $d = a\alpha\beta$ for some $\beta \in C(S)$, so $d = d\alpha$. Moreover, $d \leq_C a, b$, so $d \leq_C a \wedge b$, and so $d = d\alpha \leq_C (a \wedge b)\alpha$. Letting $d = (a\alpha) \wedge (b\alpha)$ shows that the third condition holds.

Now suppose the three conditions hold. Then certainly $a \wedge b \leq_C a, b$ by the first and second conditions. If $d \leq_C a, b$, then $d = aC(d) = bC(d)$, so $(a \wedge b)C(d) = aC(d) \wedge bC(d) = d \wedge d = d$, and so $d \leq_C a \wedge b$. Hence $a \wedge b = \max\{c \in S \mid c \leq_C a, b\}$. \square

Let us call an RC-semigroup equipped with a meet-semilattice structure as above a *semilattice-ordered RC-semigroup*, or a SLORC. (If we consider twisted RC-semigroups, the natural order is stable under left and right multiplication [7], and so this terminology is consistent with existing definitions of semilattice ordered semigroup; see [3] for

example.) Obviously a given RC-semigroup can be a SLORC in at most one way. Notationally, we shall assume the semigroup operation takes priority over the semilattice meet.

One might just as well require that the C-order of an RC-semigroup be a join-semilattice. However, letting \vee be this join in any such RC-semigroup S , the law $x = (x \vee y)C(x)$ will surely be satisfied since $a \leq_C a \vee b$ for all $a, b \in S$; then letting $x = a$ and $y = C(a)$, we obtain $a = (a \vee C(a))C(a)$, while letting $x = C(a)$ and $y = a$ yields $C(a) = (C(a) \vee a)C(C(a)) = a$. Hence $C(S) = S$ and so S is simply a lattice for which $C(a) = a$ for all a . However, as we shall see, the case of SLORC's is rather richer.

Proposition 3.2 *If S is a SLORC then $(a \wedge b)\alpha = (a\alpha) \wedge b$ and $(a\alpha) \wedge (a\beta) = a\alpha\beta$ for all $a, b \in S$ and $\alpha, \beta \in C(S)$.*

Proof. Certainly $(a\alpha) \wedge (b\alpha) \leq_C (a\alpha) \wedge b$, so to prove the equality it remains to show that $(a\alpha) \wedge b \leq_C b\alpha$. But $(a\alpha) \wedge b = b\delta$ for some $\delta \in C(S)$, so $(a\alpha) \wedge b = b\alpha\delta \leq_C b\alpha$ as required.

Secondly, $a\alpha\beta \leq_C a\alpha$ and $a\beta$, so $a\alpha\beta \leq_C a\alpha \wedge a\beta$. Conversely, $(a\alpha) \wedge (a\beta) = a\alpha\gamma = a\beta\delta$ for some $\gamma, \delta \in C(S)$, so $(a\alpha) \wedge (a\beta) = a\alpha\gamma\beta\delta \leq_C a\alpha\beta$. \square

We now obtain an extension of the notion of twistedness for SLORC's. First note that the twisted RC-semigroup $S = \mathcal{P}(X)$ (X a set), with $C(f)$ the identity restricted to the domain of f for all $f \in S$, is a SLORC, since $f \wedge g$, the restriction of f (or g) to where f, g agree, is the greatest lower bound of f, g under the C-order.

If S is a SLORC that is twisted as an RC-semigroup and satisfies the additional property that $(a \wedge b)c = (ac) \wedge (bc)$ for all $a, b, c \in S$, then we say S is a *twisted SLORC*. The semigroup $S = \mathcal{P}(X)$ just discussed is one such, as is routine (though not totally trivial) to verify. Therefore every twisted RC-semigroup is an RC-subsemigroup of a twisted SLORC. On the other hand the following example shows that SLORC's which are twisted as RC-semigroups need not be twisted as SLORC's.

Example 3.3 *The three element chain (viewed as a meet-semilattice) $T = \{0, a, 1\}$ with $1 > a > 0$ with closed elements 1 and 0 is a twisted RC-semigroup whose natural order makes it a SLORC (with bottom element 0 and with 1 and a incomparable). However $(a \wedge 1)a = 0a = 0$ while $aa \wedge 1a = a$.*

Importantly, the natural representation theorem for twisted RC-semigroups as subalgebras of $\mathcal{P}(X)$ extends to twisted SLORC's.

Theorem 3.4 *Let S be a twisted SLORC. Then S is embeddable as a sub-SLORC of $\mathcal{P}(S)$.*

Proof. We must show that the RC-semigroup embedding $\theta : S \rightarrow \mathcal{P}(S)$, given by $\theta(a) = \psi_a$ as before, also respects \wedge , defined on $\mathcal{P}(S)$ as above.

Now if $C(f)x = x$, $C(g)x = x$ and $fx = gx$ then certainly $C(f \wedge g)x = xC((f \wedge g)x) = xC(fx \wedge gx) = xC(fx \wedge gx) = xC(fx) = C(f)x = x$, and then $(f \wedge g)x = fx \wedge gx =$

$fx \wedge fx = fx$. Thus $\psi_f(x) = \psi_g(x)$ implies both equal $(f \wedge g)(x)$. Conversely, if $C(f \wedge g)x = x$ then surely $C(f)x = C(g)x = x$ and moreover $fx = fC(f \wedge g)x = (f \wedge g)x$, from the first condition for SLORC's. Hence wherever $\psi_{f \wedge g}$ is defined, so are ψ_f and ψ_g and moreover all three agree. \square

It is interesting to compare Theorem 3.4 with one of the main results of [3]: this says that, up to isomorphism, the class of semilattice ordered semigroups coincides (up to isomorphism) with the class whose members are sets of binary relations closed under composition and intersection. Theorem 3.4 above demands additional structure but the representation is more refined and generalises existing representations such as the Vagner-Preston representation for inverse semigroups. The representation in [3] is quite distinct from that in Theorem 3.4 and sends every element to an assymmetric binary relation.

The situation when a twisted SLORC S is an inverse semigroup with the usual closure is of particular interest. These examples can be seen to coincide with the class of inverse semigroups whose natural order is a semilattice ordering. Note that for any set X , the sub-RC-semigroup $\mathcal{I}(X)$ of $\mathcal{P}(X)$ under its standard closure is also a sub-SLORC. The representation in Theorem 3.4 actually represents such a structure as a SLORC of partial injective maps and therefore actually gives an embedding (respecting \cdot , \wedge , C and $^{-1}$) of S into $\mathcal{I}(S)$ considered as a SLORC. The C operation here plays no essential role since it is term definable in terms of \cdot and $^{-1}$. These structures can then be equationally defined using the usual inverse semigroup axioms along with the laws $(x \wedge y)z^{-1}z = xz^{-1}z \wedge yz^{-1}z$, $x \wedge y = x(x \wedge y)^{-1}(x \wedge y)$.

As is the case with semigroups, every RC-semigroup S can be made into an RC-monoid by adjoining an identity element: just set $C(1) = 1$ (which must hold in any RC-monoid). We denote this monoid by S^1 . Moreover if S satisfies the right congruence condition or is twisted, then the same is true of S^1 . Similarly, every SLORC can be made into a SLORC monoid by adjoining an identity element, although we need to be more careful in defining the \wedge operation.

Proposition 3.5 *If S is a SLORC then the RC-semigroup S^1 formed by adjoining a new identity element also respects the existing \wedge operation extended according to $1 \wedge 1 = 1$ and $1 \wedge x = x \wedge 1 = x \wedge C(x)$ (for all x).*

Proof. First note that if we have a SLORC that is a monoid then $x \wedge C(x) = x \wedge 1C(x) = (x \wedge 1)C(x) = xC(x) \wedge 1 = x \wedge 1$. On the other hand if 1 is adjoined as described then certainly we have an RC-semigroup and each of the SLORC axioms given in Proposition 3.1 are easily seen to be satisfied since if we assign 1 to a variable in one of these axioms, then expressions of the form $x \wedge 1$ or $1 \wedge x$ can be replaced by expressions $x \wedge C(x)$, which no longer involve 1 (or if we assign 1 to all the variables, then both sides of the identities take the value 1). \square

In fact this construction is easily seen to make a twisted SLORC into a twisted SLORC monoid.

4 Agreeable semigroups

Now if S is a SLORC and we define $a * b = C(a \wedge b)$, then $a * a = C(a \wedge a) = C(a)$ and $a(a * b) = aC(a \wedge b) = a \wedge b$, so we can express the operations C and \wedge purely in terms of $*$.

Definition 4.1 *The variety of agreeable semigroups is specified by the following equations (and we give the semigroup multiplication precedence over $*$):*

1. $a(a * a) = a$;
2. $(a * a)(b * b) = (b * b)(a * a)$;
3. $(a * a) * (a * a) = a * a$;
4. $(ab * ab)(b * b) = ab * ab$;
5. $a(a * b) = b(b * a)$;
6. $a(a * b)(c * c) = a(c * c)(a(c * c) * b(c * c))$; and
7. $a(a * b) * a(a * b) = a * b$.

We use the term “agreeable” here because, at least in the twisted case, $a * b$ represents the restriction of the identity to those domain elements where the partial maps a, b are both defined and are equal (this is proved in Corollary 6.2 below).

Theorem 4.2 *The variety of agreeable semigroups is term equivalent to the variety of SLORC’s.*

Proof. Let S be an agreeable semigroup. Defining $C(a) = a * a$ for all $a \in S$, the first four rules above immediately show that C is a right closure operation on S . Also letting $a \wedge b = a(a * b)$ for all $a, b \in S$, the first rule implies $a \wedge a = a$ and the fifth implies $a \wedge b = b \wedge a$, while the sixth and seventh imply that $(a \wedge b)C(d) = aC(d) \wedge bC(d)$ and $a \wedge b = aC(a \wedge b)$ for all $a, b, c \in S$, so S is a SLORC. Conversely, if S is a SLORC, letting $a * b = C(a \wedge b)$ for all $a, b \in S$, all of the above rules except the last arise on direct translation of the SLORC rules, while the last follows from the fact that $C(aC(a \wedge b)) = C(a \wedge b)$ for all $a, b \in S$.

It remains to show that these translations are mutually inverse. Starting with an SLORC S and letting $a * b = C(a \wedge b)$, we obtain an agreeable semigroup, in which $a * a = C(a)$ and $a(a * b) = a \wedge b$ as above. Conversely, starting with an agreeable semigroup S and letting $C(a) = a * a$ and $a \wedge b = a(a * b)$, we obtain a SLORC in which $C(a \wedge b) = a(a * b) * a(a * b) = a * b$ by the final rule. \square

Hence agreeable semigroups and SLORC’s are essentially the same things. Because they involve only one ‘extra’ operation, we focus on agreeable semigroups from this point on, although we are free to introduce the other operations: when we view an

agreeable semigroup as an RC-semigroup (resp. SLORC), we mean with $C(a) = a * a$ (resp. $a \wedge b = a(a * b)$) for all a (resp. a, b). Conversely, given an RC-semigroup structure on S , this may or may not give rise to a (unique) SLORC structure on S and hence agreeable semigroup structure; if it does, we say S is agreeable *relative to* the given closure operation C on S : the operation $*$ is completely determined by $C(S)$.

5 Normal agreeable semigroups

It is possible to characterise agreeable semigroups amongst RC-semigroups satisfying the identity $C(a)C(b) = C(aC(b))$. We call such an RC-semigroup *normal*, and any agreeable semigroup which is normal as an RC-semigroup we call normal also. In fact the next theorem achieves a more general result.

Theorem 5.1 *If S is a normal agreeable semigroup then $a * b = \max\{\alpha \in C(S) \mid \alpha \leq_C C(a)C(b) \text{ and } a\alpha = b\alpha\}$ for all $a, b \in S$.*

Conversely if S is an RC-semigroup in which

$$a \star b = \max\{\alpha \in C(S) \mid \alpha \leq_C C(a)C(b) \text{ and } a\alpha = b\alpha\}$$

*exists for all $a, b \in S$, then S is an agreeable semigroup with respect to the operation defined by $a * b = C(a(a \star b))$. If S is normal then \star and $*$ coincide.*

Proof. Suppose S is a normal agreeable semigroup. Then for all $a, b \in S$, certainly $a(a * b) = b(a * b)$, and $a * b = C(a \wedge b) = C(aC(a) \wedge b) = C((a \wedge b)C(a)) = C(a \wedge b)C(a) = (a * b)(a * a)$, so $a * b \leq_C a * a = C(a)$, and similarly $a * b \leq_C C(b)$, so $a * b \leq_C C(a)C(b)$. If also $a\alpha = b\alpha$ and $\alpha \leq_C C(a)C(b)$, then $(a * b)\alpha = C(a \wedge b)\alpha = C((a \wedge b)\alpha) = C(a\alpha \wedge b\alpha) = C(a\alpha \wedge a\alpha) = C(a \wedge a)\alpha = C(a)\alpha = \alpha$, so indeed $a * b = \max\{\alpha \in C(S) \mid \alpha \leq_C C(a)C(b) \text{ and } a\alpha = b\alpha\}$.

Conversely, if S is an RC-semigroup and the element $a \star b = \max\{\alpha \in C(S) \mid \alpha \leq_C C(a)C(b) \text{ and } a\alpha = b\alpha\}$ exists for all $a, b \in S$, then let $a \wedge b$ be defined as $a(a \star b)$ (or, equivalently, $b(a \star b)$). If $a\gamma = b\gamma$ for some $\gamma \in C(S)$ then $b\gamma = bC(b)\gamma = b\gamma C(b) = a\gamma C(b) = aC(a)C(b)\gamma = bC(a)C(b)\gamma$ and likewise $b\gamma = aC(a)C(b)\gamma$ which since $a\gamma = b\gamma$ shows that $C(a)C(b)\gamma \leq_C a \star b$. But then $a\gamma = aC(a)C(b)\gamma \leq_C a(a \star b)$. Hence $a(a \star b)$ is the greatest lower bound of a, b under the C-order on S . This shows that S is an agreeable semigroup with respect to the operation $*$ given by $a * b = C(a(a \star b))$.

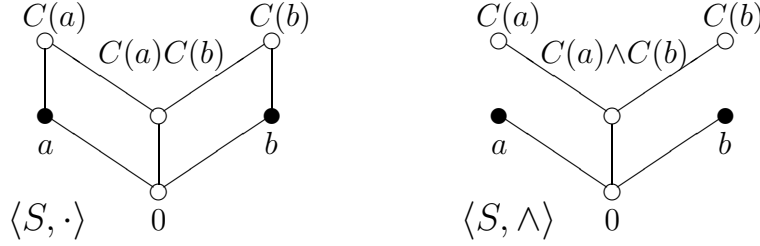
To complete the proof it remains to show that if S is normal then $a * b = a \star b$. This follows since $a * b = C(a(a \star b)) = C(a)C(a \star b) = C(a \star b) = a \star b$. \square

The variety of normal RC-semigroup contains the class of RC-semigroups satisfying the right congruence condition (since using the identity $C(xy) = C(C(x)y)$ we have $C(x)C(y) = C(C(x)C(y)) = C(xC(y))$). The following example shows that this containment is proper, even for agreeable semigroups.

Example 5.2 Let B be the RC-semilattice of subsets of $\{0, 1\}$ with closed elements $\{0, 1\}$ and \emptyset . The C -order on B is a semilattice that makes B an agreeable semigroup with \wedge operation given by (for all $x, y \subseteq \{0, 1\}$ with $x \neq y$) $x \wedge x = x$ and $x \wedge y = \emptyset$. However as an RC-semigroup B satisfies the law $C(x)C(y) = C(xC(y))$ but not the right congruence condition.

The next example demonstrates that without the restriction of normality in the first half of Theorem 5.1, we may have $a * b \neq \max\{\alpha \in C(S) \mid \alpha \leq_C C(a)C(b) \text{ and } a\alpha = b\alpha\}$ and $a\alpha = b\alpha$ and also that without the restriction of normality in the second half, we need not have $a * b = a * b$.

Example 5.3 Let S be the RC-semilattice on the set $\{a, b, C(a), C(b), C(a)C(b), 0 = C(0)\}$ whose multiplication is given by the left Hasse diagram below (here closed elements are denoted by hollow circles). Then S is a (non-normal) agreeable semigroup for which $a * b = \max\{\alpha \in C(S) \mid \alpha \leq_C C(a)C(b) \text{ and } a\alpha = b\alpha\}$ exists for all $a, b \in S$ but such that $C(a(a * b)) \neq a * b$.



Proof. It is routinely verified that the C -order on this RC-semigroup gives the semilattice whose Hasse diagram is on the right. Thus S is a SLORC and therefore also an agreeable semigroup. However $a * b = C(a \wedge b) = C(0) = 0$ while $aC(a)C(b) = 0 = bC(a)C(b)$, showing that $\max\{\alpha \in C(S) \mid \alpha \leq_C C(a)C(b) \text{ and } a\alpha = b\alpha\} = C(a)C(b) \neq a * b$. The identity $C(x)C(y) = C(xC(y))$ fails with $x = a$, $y = C(a)C(b)$ for example. \square

We can deduce from this example that there are non-normal agreeable semigroups for which $a * b$ is not equal to $\max\{\alpha \in C(S) \mid \alpha \leq_C C(a)C(b) \text{ and } a\alpha = b\alpha\}$ for all a, b . However normality is not equivalent to the condition $(\forall a, b) a * b = \max\{\alpha \in C(S) \mid \alpha \leq_C C(a)C(b) \text{ and } a\alpha = b\alpha\}$ since restricting all the operations in S of Example 5.3 to the set $\{a, 0, C(a), C(a)C(b)\}$ gives a second non-normal agreeable semigroup for which $x * y = \max\{\alpha \in \{0, C(a), C(a)C(b)\} \mid \alpha \leq_C C(x)C(y) \text{ and } x\alpha = y\alpha\}$ for all x, y .

The normality condition is expressible in terms of $*$ in an agreeable semigroup, and in this case the agreeable semigroup axioms can be simplified considerably.

Proposition 5.4 For the agreeable semigroup S , the following are equivalent;

1. S is normal;

2. S satisfies $(a * b)\alpha = a\alpha * b\alpha$ for all $a, b \in S$, $\alpha \in C(S)$;

3. S satisfies $(a * b)\alpha = a\alpha * b$ for all $a, b \in S$, $\alpha \in C(S)$.

Proof. If S is normal then for all $a, b \in S$ and $\alpha \in C(S)$, $(a * b)\alpha = C(a \wedge b)\alpha = C((a \wedge b)\alpha) = C(a\alpha \wedge b\alpha) = a\alpha * b\alpha$. Conversely, if $(a * b)\alpha = a\alpha * b\alpha$ holds for all $a, b \in S$ and $\alpha \in C(S)$, then $C(a)\alpha = (a * a)\alpha = a\alpha * a\alpha = C(a\alpha)$, so S is normal. The equivalence of the final law follows from the following argument, valid in any agreeable semigroup: for all $a, b \in S$ and $\alpha \in C(S)$, $a\alpha * b = C(a\alpha \wedge b) = C(a\alpha C(a\alpha \wedge b)) = C(a\alpha C(a\alpha \wedge b))C(\alpha) = C(a\alpha \wedge b)\alpha = (a\alpha * b)\alpha$. Hence if S satisfies the second condition, then $a\alpha * b = (a\alpha * b)\alpha = a\alpha^2 * b\alpha = a\alpha * b\alpha = (a * b)\alpha$ as required. That the third condition implies the second condition is immediate. \square

Theorem 5.5 *The variety of normal agreeable semigroups can be defined (amongst semigroups with additional binary operation $*$) by the following identities.*

1. $a(a * a) = a$;

2. $a * b = b * a$;

3. $a(a * b) = b(a * b)$;

4. $(a * b) * (c * d) = (a * b)(c * d)$; and

5. $(a * b)(c * d) = a(c * d) * b$.

Proof. Suppose S is a normal agreeable semigroup; then it is a normal SLORC if we define $C(a) = a * a$ and $a \wedge b = a(a * b)$. The first rule is immediate from the agreeable semigroup rules, while $a * b = C(a \wedge b) = C(b \wedge a) = b * a$. This and the fifth axiom of Definition 4.1 imply the third rule. Now

$$\begin{aligned} (a * b) * (c * d) &= C(C(a \wedge b) \wedge C(c \wedge d)) \\ &= C(C(a \wedge b)C(c \wedge d)) \text{ by Proposition 3.2} \\ &= C(a \wedge b)C(c \wedge d) \\ &= (a * b)(c * d), \end{aligned}$$

and so the fourth rule holds. The fifth rule is immediate. Hence S satisfies the above identities.

Conversely, assume S is a semigroup with additional binary operation $*$ satisfying the above identities. Let C be given by $C(a) = a * a$. Now for all $a, b \in S$, $aC(a) = a(a * a) = a$, $C(a)C(b) = (a * a)(b * b) = (a * a) * (b * b) = (b * b) * (a * a) = (b * b)(a * a) = C(b)C(a)$, and

$$\begin{aligned} C(C(a)) &= (a * a) * (a * a) \\ &= (a * a)(a * a) \\ &= (a(a * a) * a) \\ &= a * a \\ &= C(a), \end{aligned}$$

$C(ab)C(b) = (ab * ab)(b * b) = ab(b * b) * ab = ab * ab = C(ab)$. Hence S is a normal RC-semigroup. Now let $a \wedge b = a(a * b)$ for all $a, b \in S$. Now for all $a, b, c \in S$, $a \wedge a = a(a * a) = a$, $a \wedge b = a(a * b) = b(b * a) = b \wedge a$,

$$aC(a \wedge b) = a(a(a * b) * a(a * b)) = a(a * a)(a * b) = a(a * b) = a \wedge b, \text{ and}$$

$C(a)C(b) = C(aC(b))$ from the previous lemma. Hence S is a SLORC which is normal as an RC-semigroup. Since $C(a \wedge b) = a(a * b) * a(a * b) = (a * a)(a * b) = (a(a * a) * b) = a * b$, it follows that S is a normal agreeable semigroup. \square

The following is immediate.

Corollary 5.6 *If S is a normal agreeable semigroup, then $(a * b)(b * c) = (a * b)(a * c)$.*

If an agreeable semigroup is a monoid, then $1 = 1C(1) = C(1)$, and in terms of the $*$ operation, $1 * 1 = 1$. The agreeable semigroup axioms for normal agreeable monoids can be streamlined still further.

Corollary 5.7 *The monoid S equipped with an additional binary operation $*$ is a normal agreeable monoid if and only if, for all $a, b, c, d \in S$,*

1. $a(a * a) = a$;
2. $a * b = b * a$;
3. $a(a * b) = b(a * b)$; and
4. $(a * b)(c * d) = a(c * d) * b$.

Proof. Obviously any normal agreeable monoid satisfies these four laws. Conversely, suppose S is a monoid equipped with an operation $*$ satisfying the four laws. Now for all $a, b, c, d \in S$,

$$\begin{aligned} (a * b) * (c * d) &= 1(a * b) * (c * d) \\ &= (1 * (c * d))(a * b) \\ &= (1(c * d) * 1)(a * b) \\ &= (1 * 1)(c * d)(a * b) \\ &= (c * d)(a * b), \end{aligned}$$

so by commutativity of $*$, the remaining non-immediate normal agreeable semigroup law follows. \square

6 Twisted agreeable semigroups

SLORC twistedness is easily captured using a single agreeable semigroup identity.

Proposition 6.1 *The agreeable semigroup S is twisted as a SLORC if and only if it satisfies the identity $(a * b)c = c(ac * bc)$ for all $a, b, c \in S$.*

Proof. If S is a twisted SLORC, then $(a * b)c = C(a \wedge b)c = cC((a \wedge b)c) = cC(ac \wedge bc) = c(ac * bc)$ for all $a, b, c \in S$. Conversely, if S is an agreeable semigroup for which the identity holds, then for all $a, b, c \in S$, $C(a)b = (a * a)b = b(ab * ab) = bC(ab)$ and $(a \wedge b)c = a(a * b)c = ac(ac * bc) = ac \wedge bc$ as required. \square

We call an agreeable semigroup which is twisted as a SLORC *twisted*. Because twistedness implies normality, we therefore can apply Theorem 5.5 and obtain six identities that define twisted agreeable semigroups.

The next result follows from Theorem 3.4.

Corollary 6.2 *Every twisted agreeable semigroup S is embeddable in the agreeable semigroup $\mathcal{P}(S)$, in which $f * g$ is the restriction of the identity to where f, g are both defined and agree, under the embedding $a \mapsto \psi_a$, where $\psi_a(x) = ax$ for all $x \in \text{dom}(\psi_a) = \{x \in S \mid (a * a)x = x\}$.*

We remark that in the monoid case, the twisted agreeable semigroup axioms can be simplified still further, with the law $a(a * a) = a$ replaced by $1 * 1 = 1$: for if the latter holds (as well as the other laws), then $a(a * a) = (1 * 1)a = a$. Hence from Corollary 5.7, we have

Corollary 6.3 *The monoid S equipped with an additional binary operation $*$ is a twisted agreeable monoid if and only if, for all $a, b, c, d \in S$,*

1. $1 * 1 = 1$;
2. $a * b = b * a$;
3. $a(a * b) = b(a * b)$;
4. $(a * b)(c * d) = a(c * d) * b$; and
5. $(a * b)c = c(ac * bc)$.

Corollary 6.4 *Every twisted agreeable monoid S is embeddable in the agreeable monoid $\mathcal{P}(S)$ in which $f * g$ is the restriction of the identity to where f, g are both defined and agree, under the embedding $a \mapsto \psi_a$, where $\psi_a(x) = ax$ for all $x \in \text{dom}(\psi_a) = \{x \in S \mid (a * a)x = x\}$.*

Proof. Now $\text{dom}(\psi_1) = C(1)S = S$, and so $\psi_1(x) = 1x = x$ for all $x \in S$. Hence ψ_1 is the identity on $\mathcal{P}(S)$. \square

7 Connections with interior semigroups

Let S be a semigroup with a subsemigroup L that is a semilattice, with $I(a) = \max\{e \in L \mid ae = e\}$ existing for all $a \in S$. Then I satisfies

1. $aI(a) = I(a)$;
2. $I(a)I(b) = I(b)I(a)$;
3. $I^2(a) = I(a)$; and
4. $I(a)I(b) = I(ab)I(b)$

for all $a, b \in S$. Moreover $L = \{I(a) \mid a \in S\}$. Conversely, if a semigroup S is equipped with unary I satisfying these four rules, then $I(S) = \{I(a) \mid a \in S\}$ is a subsemigroup of S which is a semilattice, and $I(a) = \max\{e \in I(S) \mid ae = e\}$. Note that the above axioms reduce to the more familiar ones for an interior operator if S is a semilattice.

We call such a semigroup with unary I as above an *I-semigroup*, as in [8]. An important family of examples arises by considering the semigroup $\mathcal{P}(X)$ of all partial maps on a topological space X , with $I(f)$ defined not to be the (identity map restricted to the) interior of the domain of f but rather the (identity map restricted to the) interior of the subset of X fixed by f .

There is a natural connection between $*$ operations and interior operations in an RC-semigroup. For instance, the existence of the former implies the existence of the latter.

Proposition 7.1 *Let S be an agreeable semigroup. Then I defined by $I(a) = a * C(a)$ is an interior operation on S , with $I(S) = C(S)$.*

Proof. First note that $a \wedge C(a) \leq_C C(a)$, and so $a \wedge C(a) \in C(S)$, so $I(a) = a * C(a) = C(a \wedge C(a)) = a \wedge C(a)$. Hence $aI(a) = a(a * C(a)) = a \wedge C(a) = I(a)$. Now suppose $\alpha \in C(S)$ is such that $a\alpha = \alpha$. Then $\alpha C(a) = a\alpha C(a) = aC(a)\alpha = a\alpha = \alpha$, so $\alpha \leq_C C(a)$. But then (using Proposition 3.2) $I(a)\alpha = (a \wedge C(a))\alpha = a\alpha \wedge C(a) = \alpha \wedge C(a) = \alpha C(a) = \alpha$, so $I(a) = \max\{\alpha \in C(S) \mid a\alpha = \alpha\}$. Hence I is an interior operation on S . \square

Let $S = \mathcal{I}(X)$ be the inverse semigroup of all one-to-one partial maps on X , f' the inverse of $f \in S$. Then the standard closure on this (or any other) inverse semigroup is defined by $C(f) = f'f$, $f \in S$. This makes S twisted. Moreover $C(S) = E(S)$, the set of all idempotents of S and also the set of all restrictions of the identity. Giving X the discrete topology and letting $I(f)$ be the same as on $\mathcal{P}(X)$, we obtain a twisted RC-semigroup which is also interior, and satisfies $I(S) = C(S)$. Also satisfied are $I(e) = e$ for all $e \in E(S)$ and $I(f)g = gI(g'fg)$.

Generalising, suppose S is an inverse semigroup with a' the inverse of a . Suppose S is endowed with its standard twisted closure, that is, $C(a) = a'a$ for all $a \in S$. Suppose S is also an interior semigroup with $I(e) = e$ for all $e \in E(S)$; hence $I(S) = C(S) = E(S)$.

Finally suppose $I(a)b = bI(b'ab)$ holds for all $a, b \in S$. Now if $x \in \text{dom}(\psi_{I(a)})$, then $C(I(a))x = x$, that is, $I(a)x = x$, so $\psi_{I(a)}(x) = x$. Also, $ax = aI(a)x = I(a)x = x$ and $C(a)x = xC(ax) = xC(x) = x$, so $\psi_a(x) = x$. Conversely, if $\psi_a(x) = x$, then $ax = x$ and $C(a)x = x$, so $I(a)x = xI(x'ax) = xI(x'x) = xx'x = x$, and of course $C(I(a))x = I(a)x = x$. This all shows that $\psi_{I(a)}$ is the identity restricted to $\text{fix}(\psi_a)$. Thus we have abstractly captured algebras of one-to-one partial maps under the operations of composition, inversion, and the naturally defined unaries C and I . Note that C can be viewed as derived (via $C(a) = a'a$), and we can define any algebra S in the class of interest as follows:

1. $(S, \cdot, ')$ is an inverse semigroup;
2. (S, \cdot, I) is an interior semigroup;
3. $I(a'a) = a'a$ for all $a \in S$; and
4. $I(a)b = bI(b'ab)$ for all $a, b \in S$.

We call such an S a *DII-semigroup* (discrete interior inverse semigroup). Clearly the class of DII-semigroups is a variety of semigroups with two additional unary operations. Summarising, we have

Proposition 7.2 *If S is a DII-semigroup then S is embeddable as a DII-semigroup in the inverse semigroup $\mathcal{I}(S)$ with the usual closure and with $I(x) = \text{fix}(x)$.*

A converse of Proposition 7.1 holds for DII-semigroups; indeed there is a term equivalence between the variety of DII-semigroups and a suitable variety of twisted agreeable inverse semigroups.

Theorem 7.3 *Let S be a DII-semigroup. Then S is twisted agreeable relative to its standard inverse semigroup closure with $a * b = I(a'b)$ for all $a, b \in S$, and satisfies $a * bc = b'a * c$ for all $a, b, c \in S$.*

*Conversely, let S be an inverse semigroup which is twisted agreeable relative to its standard closure, and satisfies $a * bc = b'a * c$ for all $a, b, c \in S$. Then if one defines $I(a) = a * C(a)$ for all $a \in S$, S is a DII-semigroup.*

This correspondence is a term equivalence of varieties.

Proof. Note that the class of inverse semigroups which are twisted agreeable relative to their standard closures and satisfying the two given laws is indeed a variety of agreeable inverse semigroups, since the link between the standard closure and $*$ is given by the law $a * a = a'a$.

Start with a DII-semigroup S , and define $a * b = I(a'b)$ for all $a, b \in S$. Using Proposition 7.2, $a * b = I(a'b)$ is represented as the largest restriction of the identity agreeing with $a'b$, that is, the largest such restriction for which a, b are both defined and are equal, that is $a * b$ has its usual meaning in $\mathcal{I}(S)$ and so is a twisted agreeable semigroup operation on this and hence on S itself. Evidently $*$ is the agreeable operation

related to $C(a) = a'a$. Finally, for all $a, b, c \in S$, $a*bc = bc*a = I((bc)'a) = I(c'(b'a)) = b'a*c$.

Conversely, suppose S is an inverse semigroup which is twisted agreeable relative to its standard closure and satisfies $a*bc = b'a*c$ for all $a, b, c \in S$. Then for all $a \in S$, $a*C(a) = aa'a*C(a) = C(a')a*C(a) = a*C(a')'C(a) = (a*C(a'))C(a) = (aC(a)*C(a')) = a*C(a')$, and for all $a, b \in S$, $a*b = b*a = b*aC(a) = a'b*a'a = aa'b*a = aa'bC(a'b)*a = (aa'b*a)C(a'b)$, so $(a*b)C(a'b) = a*b$; hence by symmetry also $(a*b)C(b'a) = a*b$, and so $a*b = (aa'b*a)C(a'b)C(b'a)$. Therefore

$$\begin{aligned}
a'b*C(a'b) &= a'b*C(b'a) \\
&= a'bC(a'b)*C(b'a) \\
&= (a'b*a'bb'a)C(a'b) \\
&= (aa'b*bb'a)C(a'b) \\
&= (aa'b*C(b'a)a)C(a'b) \\
&= aa'b*aC(b'a)C(a'b) \\
&= (aa'b*a)C(b'a)C(a'b) \\
&= a*b.
\end{aligned}$$

Now define $I(a) = a*C(a)$ for all $a \in S$. Then I is an interior operation with $I(S) = C(S)$ by Proposition 7.1, and for all $a, b \in S$ and $e \in E(S)$, $I(e) = e*C(e) = e*e'e = e*e = C(e) = e$, and furthermore $bI(b'ab) = b(b'ab*C(b'ab)) = b((a'b)'b*C((a'b)'b)) = b(a'b*b) = b(a'C(a')b*b) = b(ab*C(a')b) = (a*C(a'))b = (a*C(a))b = I(a)b$, so S is a DII-semigroup.

Finally, starting with a DII-semigroup, and letting $a*b = I(a'b)$, $*$ satisfies $a*C(a') = a*C(a)$ from the above, and so $a*C(a) = a*C(a') = I(C(a')'a) = I(C(a')a) = I(aa'a) = I(a)$, whereas starting with a twisted agreeable inverse semigroup for which $a*bc = b'a*c$ holds, and letting $I(a) = a*C(a)$, we have $I(a'b) = a'b*C(a'b) = a*b$ from the above, so we have a term equivalence. \square

In the monoid case, the identity $a*bc = b'a*c$ in the above can be replaced by $a*b = b'a*1$; the former obviously implies the latter, whereas if the latter holds, then $a*bc = (bc)'a*1 = c'b'a*1 = b'a*c$ for all a, b, c .

8 Congruences on agreeable semigroups

Throughout this section, we continue to view agreeable semigroups as SLORC's in the usual way whenever convenient.

Every congruence on an agreeable semigroup S corresponds to an RC-semigroup congruence on S however the reverse is not true. For example let G^0 be a group G with adjoined zero element. By defining $C(0) = 0$ and $C(g) = e$ (the group identity element) for all $g \in G = G^0 \setminus \{0\}$, one obtains a (twisted) RC-semigroup in which the C-order is a semilattice order. Note that $x \leq_C y$ implies that $x = yC(x)$ which gives $x = y$ if

$x \neq 0$ and $x = 0$ otherwise. Thus $x * y = c(x \wedge y) = C(0) = 0$ unless $x = y$ in which case $x \wedge y = x(x * y) = x(x * x) = x = y$. Also if $z \neq 0$ then $xz = yz$ if and only if $x = y$ and so $(x \wedge y)z = xz \wedge yz$, while if $z = 0$ then $(x \wedge y)z = 0 = xz \wedge yz$. Hence G^0 is a twisted agreeable semigroup. Considered as an RC-semigroup, it is easily verified that every semigroup congruence on G is the restriction to G of a unique RC-semigroup congruence on G^0 . However G^0 is in fact congruence free as an agreeable semigroup since if $x \neq y$ and ρ is a congruence for which $x\rho y$ then without loss of generality we may assume that $C(x) = e$ (since $x \neq y$ implies at least one of $C(x)$ and $C(y)$ must equal e) while $C(x) = (x * x) \rho (x * y) = 0$. That is, $\rho = \nabla_{G^0}$.

The following elementary fact indicates that agreeable semigroups (regardless of whether or not the semigroup is twisted as an RC-semigroup or as an agreeable semigroup) should have a well-behaved congruence theory.

Lemma 8.1 *If S is an agreeable semigroup and θ is a congruence (with respect to \cdot and $*$) then θ is determined by its restriction to $C(S)$. Specifically, $x\theta y$ if and only if $C(x) \theta C(y) \theta (x * y)$.*

Proof. The only if direction is trivial. Now assume that θ is a congruence for which $C(x) \theta C(y) \theta (x * y)$. Then $x = xC(x) \theta x(x * y) = x \wedge y$ and so $x\theta y$ (by symmetry) as required. \square

Of course to obtain a full description of the congruences on an agreeable semigroup one needs to do more than describe semigroup congruences on the subsemilattice $C(S)$. Note that if ρ is a congruence on $C(S)$ and $C(a) \rho a * b \rho C(b)$ for some a and b in S , then if ρ can be extended to a congruence ρ' on all of S we must have $a\rho'b$. From this it follows that $ax\rho'bx$ (for all x) and so $C(ax) = (ax * ax) \rho (ax * bx) \rho (bx * bx) = C(bx)$. In general if S is an agreeable semigroup and ρ is a congruence on $C(S)$ satisfying (for all $a, b, x \in S$)

$$C(a) \rho (a * b) \rho C(b) \Rightarrow C(ax) \rho (ax * bx) \rho C(bx)$$

then ρ will be called a *normal congruence* on $C(S)$. We now show that for normal agreeable semigroups, normal congruences on the closed elements correspond exactly to congruences on the entire agreeable semigroup.

Theorem 8.2 *Let S be a normal agreeable semigroup. The restriction of any congruence on S to $C(S)$ is a normal congruence and conversely any normal congruence ρ on $C(S)$ extends (uniquely) to a congruence on S given by $\rho' = \{(a, b) \in A \times A \mid C(a) \rho (a * b) \rho C(b)\}$.*

Proof. The first statement is established in comments preceding the theorem. Now suppose ρ is a normal congruence on $C(S)$. By Lemma 8.1 it suffices to show that ρ' as defined in the theorem is a congruence (that its restriction to $C(S)$ coincides with ρ follows from $C(C(x)) = C(x)$). By normality, ρ' is stable under right multiplication. We now show that it is stable under left multiplication. Let a and b be ρ' related and x be any

element of S . Then $C(xa) = C(xa)C(a) \rho C(xa)(a * b) = C(xa(a * b)) = C(xb(a * b)) = C(xb)(a * b) \rho C(xb)C(b) = C(xb)$. Also, $(xa * xb) \leq_C C(xa) \leq_C C(a) \rho (a * b)$ and so $(xa * xb) \rho (xa * xb)(a * b) = xa(a * b) * xb = xb(a * b) * xb = (xb * xb)(a * b) = C(xb)(a * b) \rho C(xb)$. Hence $xa \rho' xb$ as required.

Finally we verify that ρ' respects $*$. Suppose $a \rho' b$ and $c \rho' d$. Since $a * c$ and $b * d$ are closed elements it suffices to show that $a * c \rho (a * c) * (b * d) \rho b * d$. Firstly note that $C(a) \rho a * b \rho C(b)$ and $C(c) \rho c * d \rho C(d)$. Now because ρ is a congruence on $C(S)$, from repeated use of Corollary 5.6 we obtain $a * c = (a * c)C(a)C(c) \rho (a * c)(a * b)(c * d) = (a * c)(b * d)(c * d)$, so $(a * c)(b * d) \rho ((a * c)(b * d)(c * d))(b * d) = a * c$. By symmetry we also have $(a * c)(b * d) \rho b * d$, giving $a * c \rho (a * c) * (b * d) \rho b * d$ as required. \square

An immediate consequence of Lemma 8.1 is that every agreeable semigroup for which $|C(S)| = 1$ is congruence free. (We avoid the use of the word ‘‘simple’’ because of its alternative use in the theory of semigroups). However if $C(x) = C(y)$ for all x and y we also obtain $C(x \wedge y) = C(x) = C(y)$ and then have that $x = y$ for all x, y . But if the case $|C(S)| = 1$ is equivalent to the law $x = y$, then any agreeable semigroup for which $|C(S)| = 2$ must be congruence free. One such example was constructed above: any group with adjoined zero admits an (twisted) agreeable semigroup structure.

The following lemma follows immediately from the fact that $C(xy) \leq_C C(y)$ and $C(x \wedge y) \leq_C C(y)$.

Lemma 8.3 *Let S be an agreeable semigroup and $e \in C(S)$. Then $S_e = \{x \in S \mid C(x) \leq_C e\}$ is a sub-agreeable semigroup of S .*

We now have that

Lemma 8.4 *If S is an agreeable semigroup and 0 is a zero for $C(S)$, then $0 = x0$ for all $x \in S$.*

Proof. First note that by Lemma 8.3, the sub-agreeable semigroup S_0 has only one closed element and therefore is trivial. Now notice that $x0 \in S_0$ since $C(x0) \leq_C C(0)$. \square

Proposition 8.5 *A semigroup S can be given the structure of an agreeable semigroup with $|C(S)| = 2$ if and only if S has a right identity element 1 and right zero element 0 .*

Proof. First if S has a right identity 1 and right zero element 0 then by letting $C(x) = 1$ for all $x \in S \setminus \{0\}$ and $C(0) = 0$ we obtain $x \wedge y = 0$ if $x \neq y$ and $x \wedge x = x$, making S an agreeable semigroup with $|C(S)| = 2$.

Now suppose S admits an agreeable semigroup structure with $|C(S)| = 2$. Then by Lemma 8.4, S has a closed right zero, 0 say. Also if $x, y \neq 0$ then $C(x) = C(y) = 1$, say, and $x1 = xC(x) = x$, making 1 a right identity element. \square

For twisted agreeable semigroups we can say a little more.

Proposition 8.6 *A semigroup S can be given the structure of a twisted agreeable semigroup with $|C(S)| = 2$ if and only if it is the disjoint union of three sets X, Y, Z such that X is a right cancellative monoid, Y is a right zero semigroup and Z is a null semigroup with zero element 0 such that the following hold: for all $x \in X$ and $s_1, s_2 \in S$, $s_1x = s_2x \Rightarrow s_1 = s_2$, $s_11 = s_1$ and $0x = 0$; elements of Y are right zeros of S ; 0 is a right zero for all of S , and other products conform to the ‘gross’ structure outlined by the following table:*

\cdot	X	Y	Z
X	X	Y	0
Y	Y	Y	0
Z	Z	Y	0

Remark: The example G^0 at the start of this section can be constructed by taking $X = G$, $Z = \{0\}$ and $Y = \emptyset$.

Proof. The if part follows as in the proof of Proposition 8.5 except that it must be verified that the resulting structure is twisted as an RC-semigroup and as an agreeable semigroup. This is routine and will be omitted. As before, we assume $C(0) = 0$ and for all other $s \in S$, $C(s) = 1$.

Now assume that S can be given the structure of a twisted agreeable semigroup with $|C(S)| = 2$. As before, S must have a right identity 1 and a left zero 0 such that $C(x) = 1$ for all x except for $0 = C(0)$. Also, $x \neq y$ implies that $x \wedge y = 0$. It is possible that 1 is not a left identity for all elements of S . However $1x = C(1)x = xC(1x)$ and since $C(1x)$ is either 0 or 1 , we have $1x \neq x$ implies $1x = 0$. A similar argument shows that $0x \neq 0$ implies that $0x = x$. Now let X be the set $\{x \in S \setminus \{0\} \mid 0x = 0, 1x = x\}$, Y be the set $\{y \in S \setminus \{0\} \mid 0y = y\}$ and Z be the set $\{z \in S \mid 1z = 0\}$. Routine arguments show that $X \cup \{0\}$, $Y \cup \{0\}$ and Z are subsemigroups of S (whose intersection is the set $\{0\}$); for example if $x_1, x_2 \in X \cup \{0\}$ then $0x_1x_2 = 0x_2 = 0$ and $1x_1x_2 = x_1x_2$. Now note that if $z \in Z$ and $s \in S$, then $sz = (s1)z = s0 = 0$, so that Z is a null subsemigroup of S ; moreover, $SZ = \{0\}$. If $s \in S$ and $y \in Y$, then $sy = s(0y) = 0y = y$ and so y is a right identity for all S and therefore Y (without 0) is a right zero subsemigroup of S which does not include 1 ; moreover, for all $y \in Y$, $Sy = \{y\}$. Similarly, if $s \in S \setminus \{0\}$ and $x \in X$ and $sx = 0$ then $x = 1x = C(s)x = xC(sx) = x0 = 0$, contradicting the fact that $0 \notin X$ but $x \in X$. Hence X (without 0) is a subsemigroup of S . Furthermore products on the right by elements of X are right cancellative since if $s_1, s_2 \in S$ and $x \in X$, $s_1 \neq s_2$ implies $0 = (s_1 \wedge s_2)x = s_1x \wedge s_2x$ which implies that $s_1x \neq s_2x$ (s_1x and s_2x cannot both be zero since we have shown that $sx = 0$ implies that $s = 0$).

These facts show that X , Y and Z partition S into subsemigroups as described in the proposition. To complete the proof, we need to compare any remaining products between different members of X , Y and Z . If $x \in X$ and $z \in Z$ then we have $0zx = zx$ and $zx \in Z$. If $x \in X$ and $y \in Y$, then $yx = 0yx$ and so $yx \in Y$. This gives the gross structure outlined in the table above. \square

In contrast, note that it follows easily from a result in [7] that a congruence free twisted RC-semigroup with $|C(S)| = 2$ has at most 8 elements.

A further way of constructing large congruence free agreeable semigroups is via $\mathcal{P}(X)$.

Proposition 8.7 *If X is a set and $\mathcal{P}(X)$ is endowed with the usual (twisted) agreeable semigroup structure, then $\mathcal{P}(X)$ is congruence free.*

Proof. For $A \subseteq X$, denote by 1_A the map with domain A given by $1_A(x) = x$. By Lemma 8.1, a congruence on $\mathcal{P}(X)$ can be described by its restriction to elements of the form 1_A . Let A and B be two distinct subsets of X and θ be a congruence on $\mathcal{P}(X)$ such that $1_A\theta 1_B$. We may assume that $A \setminus B$ is non-empty and take $a \in A \setminus B$. Let f_a be the map whose domain is X and is given by $f_a(x) = a$. Since θ is a congruence, by Theorem 8.2, $C(1_A)\theta C(1_B)$ implies $C(1_A f_a)\theta C(1_B f_a)$. However $C(1_A f_a) = 1_X$ while $C(1_B f_a) = 1_\emptyset$. Hence $1_X\theta 1_\emptyset$, from which it easily follows that $\theta = \nabla_{\mathcal{P}(X)}$. \square

As a consequence, every (finite) twisted agreeable semigroup can be embedded in a (finite) congruence free twisted agreeable semigroup. This property is common to several other well known classes such as the class of groups (every group G is embeddable in the simple group $A_{|G|+2}$; see [10] for example).

Once again, the case for inverse agreeable semigroups is of particular interest. The situation for congruences on $\mathcal{I}(X)$ however is quite different to that for $\mathcal{P}(X)$.

Proposition 8.8 *If X is a finite set and $\mathcal{I}(X)$ is the inverse semigroup of all partial injective maps from X into X endowed with the usual (twisted) agreeable semigroup structure (and the usual inverse), then the lattice of agreeable semigroup congruences on $\mathcal{I}(X)$ is isomorphic to the $|X|$ element chain. In fact every congruence on $\mathcal{I}(X)$ is of the form $\theta_n = \{(f, g) \in \mathcal{I}(X) \times \mathcal{I}(X) \mid f = g \text{ or } |\text{dom}(f)|, |\text{dom}(g)| \leq n\}$ for some $n \leq |X|$.*

Proof. We assume the same notation as in Proposition 8.7. First let α be a cardinal number less than $|X|$. We construct an equivalence θ_α on $C(\mathcal{I}(X))$ by defining for $f, g \in \mathcal{I}(X)$, $f\theta_\alpha g$ if and only if the domain of both f and g has cardinality at most α . We now prove this is a congruence. Since $\mathcal{I}(X)$ is twisted, by Lemma 8.1 it suffices to show that θ is a normal congruence on $C(\mathcal{I}(X))$. Indeed it suffices to show that for any set A with $|A| \leq \alpha$ and for x any element of $\mathcal{I}(X)$, the domain of $1_A x$ is of cardinality at most α . This is immediate since both x and 1_A are injective. Hence θ_α is a congruence. (Note that this argument does not use the finiteness of X .)

Now assume there is a congruence θ on $\mathcal{I}(X)$ and two distinct subsets A and B of X for which $1_A\theta 1_B$ and such that $A \setminus B$ is non empty. We may also assume without loss of generality that $A \supset B$ (and therefore $|A| > |B|$) since $1_A\theta 1_B$ implies $1_A\theta 1_{A \cap B}\theta 1_B$ and one of $A \cap B \subset A$ or $A \cap B \subset B$ holds if A and B are distinct. Now if B is nonempty, then let f be a permutation of A for which $f(B) \neq B$. Since θ is a congruence, $1_A = 1_A f\theta 1_B f$ while $1_B f$ has domain a proper subset of B . Since X is finite, we may repeat this argument to eventually obtain $1_A\theta 1_\emptyset$. If $A = X$ then $\theta = \theta_{|X|} = \nabla$ and

is of the required form. Otherwise, let D be any other set of cardinality A and let b be a bijection of D onto A . Then $1_D = 1_A b \theta 1_\emptyset$ $b = 1_\emptyset$. That is, we have $1_D \theta 1_\emptyset$ for all sets D of cardinality at most $|A|$. Taking the largest set A with this property we have $\theta = \theta_{|A|}$ as required. \square

This description of agreeable semigroup congruences on $\mathcal{I}(X)$ is similar to the familiar situation for semigroup congruences on $\mathcal{I}(X)$; in particular the semigroup congruences on $\mathcal{I}(X)$ (for X finite) also form a chain [15].

The following example demonstrates that if the set X is infinite then the situation is more complicated.

Example 8.9 *Let X be countably infinite and θ be the semilattice congruence on $C(\mathcal{I}(X))$ given by*

$$\{(1_A, 1_B) \in C(\mathcal{I}(X)) \times C(\mathcal{I}(X)) \mid 1_A = 1_B \text{ or } A \text{ and } B \text{ have finite symmetric difference}\}.$$

Then θ is a normal congruence and so defines a congruence on $\mathcal{I}(X)$.

Proof. It suffices to show that if A and B have finite symmetric difference and x is any element of $\mathcal{I}(X)$ then $C(1_A x) \theta C(1_B x) \theta 1_A x * 1_B x$. Now $C(1_A x)$ is the identity map on the domain of x restricted in its range to the set A . Let D be the range of x . Then $A \cap D$ and $B \cap D$ have finite symmetric difference and are the ranges of $1_A x$ and $1_B x$ respectively. Since x is injective, the domain of $1_B x$ has finite symmetric difference with the domain of $1_A x$; alternatively, the map $1_B x$ differs from $1_A x$ in only finitely many places and where they are both defined they agree. This also shows that the domain of $1_A x \wedge 1_B x$ has finite symmetric difference with the domain of $1_A x$ and $1_B x$. Hence $C(1_A x) \theta C(1_B x) \theta 1_A x * 1_B x$, and so θ is a normal congruence as required. \square

We conclude this section by showing that RC-semigroup congruences extend to other operations if they satisfy certain natural conditions satisfied by pointwise operations on $\mathcal{P}(X)$ inherited from algebraic structure on X . Thus, we say S is an *enriched RC-semigroup* if S is an RC-semigroup possessing additional operations of various arities such that, for each such operation ρ , of arity n say,

1. $\rho(a_1, a_2, \dots, a_n)\alpha = \rho(a_1\alpha, a_2\alpha, \dots, a_n\alpha)\alpha$, and
2. $C(\rho(a_1, a_2, \dots, a_n)) \leq_C C(a_1)C(a_2) \cdots C(a_n)$,

for all $a_1, a_2, \dots, a_n \in S$ and $\alpha \in C(S)$.

Evidently these two identities are satisfied for the RC-semigroup $S = \mathcal{P}(X)$ of all partial maps $X \rightarrow X$, where X is a universal algebra endowed with a closure operator and ρ is the pointwise n -ary operation on S inherited from the n -ary operation ρ' on X , with the domain of $\rho(f_1, f_2, \dots, f_n)$, $f_i \in S$ the intersection of the domains of the f_i . (Indeed the α term is not needed on the right of the first of the defining identities in such examples, but the arguments to follow work at the apparently greater level of generality used in our definition.)

Theorem 8.10 *Let S be an enriched agreeable semigroup. If θ is a congruence of (S, \cdot, C, \wedge) , then θ is a congruence respecting all other operations on S .*

Proof. Suppose ρ is n -ary on S , $a_i \theta b_i$, $i = 1, 2, \dots, n$. Then for each i , $C(a_i) \theta a_i * b_i \theta C(b_i)$, and so

$$\begin{aligned}
\rho(a_1, a_2, \dots, a_n) &= \rho(a_1, a_2, \dots, a_n)C(\rho(a_1, a_2, \dots, a_n)) \\
&= \rho(a_1, a_2, \dots, a_n)C(\rho(a_1, a_2, \dots, a_n))C(a_1)C(a_2) \cdots C(a_n) \\
&= \rho(a_1, a_2, \dots, a_n)C(a_1)C(a_2) \cdots C(a_n) \\
&\theta \rho(a_1, a_2, \dots, a_n)(a_1 * b_1)(a_2 * b_2) \cdots (a_n * b_n) \\
&= \rho(a_1 \alpha, a_2 \alpha, \dots, a_n \alpha) \alpha, \text{ where } \alpha = (a_1 * b_1)(a_2 * b_2) \cdots (a_n * b_n) \\
&= \rho(b_1 \alpha, b_2 \alpha, \dots, b_n \alpha) \alpha \\
&= \rho(b_1, b_2, \dots, b_n) \alpha \\
&\theta \rho(b_1, b_2, \dots, b_n)C(b_1)C(b_2) \cdots C(b_n) \\
&= \rho(b_1, b_2, \dots, b_n)C(\rho(b_1, b_2, \dots, b_n)) \\
&= \rho(b_1, b_2, \dots, b_n),
\end{aligned}$$

and so $\rho(a_1, a_2, \dots, a_n) \theta \rho(b_1, b_2, \dots, b_n)$. □

9 RC-semigroups, categories and equalizers

There is a category associated with every RC-semigroup satisfying the right congruence condition $C(xy) = C(C(x)y)$. We are able to characterise when this derived category has equalizers in $C(S)$ in terms of the existence of an agreeable semigroup structure on S of sufficient strength (no stronger than twisted).

Motivated by the partial map case, it is natural to try to define a category $\mathcal{C}(S)$ based on the RC-semigroup S as follows. First, if S has no identity, enlarge it by introducing an element 1, which can be viewed as closed by setting $C(1) = 1$; call the result S' . (Indeed this can be done even if S does have an identity; the ideas to follow apply in either case.) Let the “objects” of $\mathcal{C}(S)$ be the elements of $C(S')$, and let the “arrows” be all $f_{a,\beta} : C(a) \rightarrow \beta$ where $a \in S$ and $\beta \in C(S)$ are such that $\beta a = a$. The motivating idea comes from the twisted case, where we view $f_{a,\beta}$ as a map determined by $a \in S$ from the domain of a (namely $C(a)$, under the natural representation), into the set β containing its range.

Composition of “arrows” $f_{a,\beta} : \alpha \rightarrow \beta$ and $f_{b,\gamma} : \beta \rightarrow \gamma$ should be defined if and only if $C(b) = \beta$, via $f_{b,\gamma} \circ f_{a,\beta} = f_{ba,\gamma}$, and the identity on the object α is $f_{\alpha,\alpha}$. (In the case of $\mathcal{P}(X)$ or indeed $\mathcal{B}(X)$, this definition is exactly what it should be in order to capture composition of functions or relations.)

We begin with a purely RC-semigroup-theoretic result, describing when this $\mathcal{C}(S)$ structure defines a category.

Proposition 9.1 *Given an RC-semigroup S , the above construction makes $\mathcal{C}(S)$ a category if and only if S satisfies the right congruence condition.*

Proof. Suppose S satisfies the right congruence condition. Then so does S' as is easily checked. If $f_{b,\gamma} \circ f_{a,\beta}$ is defined then $\gamma b = b$, $\beta a = a$, and $C(b) = \beta$. Hence $\gamma b a = b a$, and so $f_{b a, \gamma}$ is an arrow. Then $\text{dom}(f_{b a, \gamma}) = C(b a) = C(C(b) a) = C(\beta a) = C(a)$, while $\text{ran}(f_{b a, \gamma}) = \gamma$ by definition, so $f_{b a, \gamma}$ has the correct domain and range.

Next we show associativity of arrow composition. Suppose $(f_{a,\alpha} \circ f_{b,\beta}) \circ f_{c,\gamma}$ exists. Then it must equal the arrow $f_{ab,\alpha} \circ f_{c,\gamma} = f_{abc,\alpha}$. But also, we must have $\alpha a = a$, $\beta b = b$ and $\gamma c = c$, as well as $C(a) = \beta$ and $C(ab) = \gamma$. Hence $\gamma = C(ab) = C(C(a)b) = C(\beta b) = C(b)$, so $f_{b,\beta} \circ f_{c,\gamma}$ exists and equals $f_{bc,\beta}$. But $\beta = C(a)$, so $f_{a,\alpha} \circ (f_{b,\beta} \circ f_{c,\gamma}) = f_{a,\alpha} \circ f_{bc,\beta}$ exists and equals $f_{abc,\alpha}$. Hence if $(f_{a,\alpha} \circ f_{b,\beta}) \circ f_{c,\gamma}$ exists then $(f_{a,\alpha} \circ f_{b,\beta}) \circ f_{c,\gamma} \circ (f_{a,\alpha} \circ f_{b,\beta}) \circ f_{c,\gamma}$. The argument showing that if $f_{a,\alpha} \circ (f_{b,\beta} \circ f_{c,\gamma})$ exists then so does $(f_{a,\alpha} \circ f_{b,\beta}) \circ f_{c,\gamma}$ and that the two are equal, is very similar.

Finally, a typical arrow with domain α is $f_{a,\beta}$, where $C(a) = \alpha$, and then $f_{a,\beta} \circ f_{\alpha,\alpha} = f_{a\alpha,\beta} = f_{a,\beta}$, while a typical arrow with range β is $f_{a,\beta}$, where $\beta a = a$, so $f_{\beta,\beta} \circ f_{a,\beta} = f_{\beta a,\beta} = f_{a,\beta}$.

Conversely, suppose $\mathcal{C}(S)$ is a category. Then for $a, b \in S$, there exists $\alpha \in C(S')$ for which $\alpha a = a$, so $f_{a,\alpha} \circ f_{C(a)b,C(a)} = f_{aC(a)b,\alpha}$ will exist, meaning that the domain of $f_{ab,\alpha}$ must be the same as that of $f_{C(a)b,C(a)}$, and so $C(C(a)b) = C(aC(a)b) = C(ab)$. Hence S satisfies the right congruence condition. \square

We will call this the *derived category* of the RC-semigroup S satisfying the right congruence condition. Recall that a typical example of such an RC-semigroup is $\mathcal{B}(X)$ with the standard closure or any RC-subsemigroup of this such as $\mathcal{P}(X)$ or $\mathcal{I}(X)$; note that this includes all inverse semigroups with their usual closure.

An *equalizer* for two arrows $f : A \rightarrow B$ and $g : A \rightarrow B$ in a category is an arrow $h : X \rightarrow A$ such that $fh = gh$, and whenever $fk = gk$ for any arrow $k : Y \rightarrow A$, there is a unique arrow $d : Y \rightarrow X$ for which $k = hd$. The object X is determined uniquely up to isomorphism and correspondingly for h , which must be monic.

Note that the category SET, with objects all sets and arrows all maps between sets, has equalizers: the equalizer of $f, g : A \rightarrow B$ is $h : X \rightarrow A$, where $X = \{x \in A \mid f(x) = g(x)\}$ and h is the inclusion map into A . A special case of this is obtained by considering the category \mathcal{C} , whose objects are all subsets of some fixed universal set X and whose arrows are the maps between subsets of X ; equalizers are as for SET. However, the category \mathcal{C} is isomorphic to the derived category of the twisted RC-semigroup $S = \mathcal{P}(X)$. Equalizers in \mathcal{C} category are all in $C(S)$. (Strictly, we here mean “can be chosen in $C(S)$ ”, since given one equalizer of the two elements, another can be obtained from any object having domain isomorphic to that of the first equalizer.)

It is possible to characterise those RC-semigroups S whose derived category has its equalizers in $C(S)$ in terms of certain kinds of agreeable semigroups, at least in the monoid case.

Theorem 9.2 *For the RC-semigroup S satisfying the right congruence condition, the derived category $\mathcal{C}(S)$ has equalizers in $C(S)$ if and only if S is an agreeable semigroup (with $C(a) = a * a$ for all $a \in S$) satisfying the implication*

$$aC(b)c = bC(a)c \Rightarrow (a * b)c = C(a)C(b)c,$$

in which case the equalizer of $f_{a,\alpha}$ and $f_{b,\alpha}$ (with $C(a) = C(b)$) is $f_{a*b,C(a)}$.

Proof. Let S be an RC-semigroup satisfying the right congruence condition. To say that $\mathcal{C}(S)$ has equalizers in $C(S)$ is to say that for any two arrows $f_{a,\beta}$ and $f_{b,\beta}$ with the same domains and ranges, so that $C(a) = C(b)$, there exists an arrow $f_{\alpha,C(a)}$ (with $\alpha \in C(S)$) for which $a\alpha = b\alpha$, and with the property that for any arrow $f_{c,C(a)}$ such that $ac = bc$, there exists a unique $f_{d,\alpha}$ such that $f_{c,C(a)} = f_{\alpha,C(a)} \circ f_{d,\alpha} = f_{\alpha d,C(a)}$.

Algebraically, this says that for all $a, b \in S$ for which $C(a) = C(b)$, if there is $\beta \in C(S')$ for which $\beta a = a$ and $\beta b = b$, then there exists $\alpha \in C(S)$ for which $\alpha \leq_C C(a)$ and $a\alpha = b\alpha$, and if $ac = bc$ for any c for which $C(a)c = c$, then there is a unique $d \in S$ satisfying $\alpha d = d$ for which $c = \alpha(\alpha d) = d$. Obviously d is unique if it exists, so the problem is reduced to establishing the existence of such a d , which in turn hinges on whether $\alpha c = c$.

Now suppose S is an agreeable semigroup which satisfies the stated implication. Then for all $a, b \in S$ and $\beta \in C(S')$ for which $C(a) = C(b)$ and $\beta a = a$, $\beta b = b$, there exists $\alpha = a * b \leq_C C(a)$ with $a\alpha = b\alpha$. Also, if $ac = bc$ for any $c \in S$ for which $C(a)c = c$, then $aC(b)c = bC(a)c$, so from the implication, $\alpha c = C(b)c = c$. Hence $f_{\alpha,C(a)}$ is an equalizer of $f_{a,\beta}$ and $f_{b,\beta}$.

Conversely, suppose S is an RC-monoid and $\mathcal{C}(S)$ has equalizers as described, so that the algebraic condition of the paragraph before last is satisfied. Suppose that $a, b \in S$ are such that $C(a) = C(b)$. Now of course, $1a = a$, $1b = b$, so taking c to be in $C(S)$, with $C(a)c = c$, that is $c \leq_C C(a) = C(b)$, and $ac = bc$, we know that there exists $\alpha \in C(S)$ for which $\alpha \leq_C C(a)$ and $a\alpha = b\alpha$, and $\alpha c = c$, that is, $c \leq_C \alpha$. Hence $\alpha = \max\{\delta \in C(S) \mid \delta \leq_C C(a), a\delta = b\delta\}$. Call this element $a * b$; so far it only exists if $C(a) = C(b)$. However, for general $a, b \in S$, note that $C(aC(a)C(b)) = C(aC(b)) = C(C(a)C(b)) = C(a)C(b)$ by the right congruence condition, and similarly $C(bC(a)C(b)) = C(a)C(b)$, so $a * b = (aC(a)C(b)) * (bC(a)C(b))$ is well-defined, and is the largest $\alpha \in C(S)$ below $C(a)C(b)$ for which $aC(a)C(b)\alpha = bC(a)C(b)\alpha$, that is, $a\alpha = b\alpha$. Hence by Theorem 5.1, this implies that S is an agreeable semigroup. Note that in the derived category, the equalizer of $f_{a,\beta}$ and $f_{b,\beta}$ (with $C(a) = C(b)$) is $f_{a*b,C(a)}$.

If now any $h \in S$ satisfies $aC(b)h = bC(a)h$, then

$$(aC(a)C(b))(C(a)C(b)h) = (bC(a)C(b))(C(a)C(b)h),$$

so $a'h' = b'h'$ where $a' = aC(a)C(b)$, $b' = bC(a)C(b)$ and $h' = C(a)C(b)h$, and so $C(a') = C(a)C(b) = C(b')$ as above, and $C(a')h' = h'$, so $(a * b)h' = h'$. That is, $(a * b)h = C(a)C(b)h$ as required. \square

The class of agreeable semigroups satisfying the implication

$$aC(b)c = bC(a)c \Rightarrow (a * b)c = C(a)C(b)c$$

is in fact a variety, containing the variety of twisted agreeable semigroups.

Theorem 9.3 *Let S be an agreeable semigroup which satisfies the right congruence condition as an RC-semigroup. Then S satisfies the implication*

$$aC(b)c = bC(a)c \Rightarrow (a * b)c = C(a)C(b)c$$

if and only if it satisfies the identity

$$(a * b)c(aC(b)c * bC(a)c) = C(a)C(b)c(aC(b)c * bC(a)c).$$

Proof. If the identity is satisfied, and if $aC(b)c = bC(a)c$, then we have

$$\begin{aligned} C(a)C(b)c &= C(a)C(b)cC(C(a)C(b)c) \\ &= C(a)C(b)cC(aC(b)c) \\ &= C(a)C(b)c(aC(b)c * aC(b)c) \\ &= C(a)C(b)c(aC(b)c * bC(a)c) \\ &= (a * b)C(aC(b)c * bC(a)c) \\ &= (a * b)cC(aC(b)c) \\ &= (a * b)cC(C(a)C(b)c) \\ &= (a * b)C(a)C(b)cC(C(a)C(b)c) \\ &= (a * b)C(a)C(b)c \\ &= (a * b)c, \end{aligned}$$

for all a, b, c . Conversely, if the implication is satisfied, then because

$$aC(b)c(aC(b)c * bC(a)c) = bC(a)c(aC(b)c * bC(a)c)$$

for any a, b, c , it must be that

$$(a * b)c(aC(b)c * bC(a)c) = C(a)C(b)c(aC(b)c * bC(a)c)$$

as required. □

It is of interest to try to simplify these conditions somewhat. Note that the implication in the statement of Theorem 9.3 may be equivalently formulated as

$$ac = bc \ \& \ C(a) = C(b) \Rightarrow (a * b)c = C(a)C(b)c$$

since $aC(b)c = aC(a)C(b)c$ and $a*b \leq_C C(a)C(b)$ hold always. Perhaps this is equivalent to the simpler implication $ac = bc \Rightarrow (a * b)c = C(a)C(b)c$. The latter certainly implies the former, but it is hard to see if the converse holds. Certainly the obvious argument will not show it, since it is easy to produce relations a, b, c on a set for which $aC(b)c = bC(a)c$ yet $ac \neq bc$. In any case, it is not hard to modify the above proof to show that the agreeable semigroup S satisfying the right congruence condition as an RC-semigroup, satisfies the implication

$$ac = bc \Rightarrow (a * b)c = C(a)C(b)c$$

if and only if it satisfies the identity

$$(a * b)c(ac * bc) = C(a)C(b)c(ac * bc).$$

On the other hand, it is possible that the identities simply define all twisted agreeable semigroups. (Certainly each is implied by twistedness, as follows immediately from the fact that $(a * b) \leq_C C(a)$.) We do not know the answer to this in general, but if the RC-semigroup not only satisfies the right congruence condition but is twisted (as an RC-semigroup), this is indeed the case.

Corollary 9.4 *For the twisted RC-semigroup S , the derived category $\mathcal{C}(S)$ has equalizers in $C(S)$ if and only if S is a twisted agreeable semigroup with $C(a) = a * a$ for all $a \in S$.*

Proof. Let S be a twisted agreeable semigroup. Then certainly the identity in Theorem 9.3 is satisfied, and so by Theorem 9.2, the derived category $\mathcal{C}(S)$ has equalizers in $C(S)$.

Conversely, if S a twisted RC-semigroup (and hence satisfies the right congruence condition), and its derived category has equalizers in $C(S)$, then it is an agreeable semigroup (with $C(a) = a * a$ for all $a \in S$) and the identity $(a * b)c(aC(b)c * bC(a)c) = C(a)C(b)c(aC(b)c * bC(a)c)$ holds by Theorem 9.2 and Theorem 9.3. Consider again the proof of Theorem 3.4: we give a modified version of it here with the stronger assumptions of that theorem replaced by the above assumptions. Thus we show that the twisted RC-semigroup natural embedding also respects $*$.

Now if $C(f)x = x$, $C(g)x = x$ and $fx = gx$ then $(f * g)x(fx * gx) = (f * g)x(fC(g)x * gC(f)x) = C(f)C(g)x(fx * gx)$, and so

$$\begin{aligned} (f * g)x &= (f * g)C(f)x \\ &= (f * g)xC(fx) \\ &= (f * g)x(fx * fx) \\ &= (f * g)x(fx * gx) \\ &= C(f)C(g)x(fx * gx) \\ &= C(f)C(g)x(fx * fx) \\ &= C(f)C(g)xC(fx) \\ &= C(f)C(g)C(f)x \\ &= x. \end{aligned}$$

Thus $\psi_f(x) = \psi_g(x)$ implies $\psi_{f * g}(x) = x$. Conversely, if $\psi_{f * g}(x) = x$ then $(f * g)x = x$, and so $C(f)x = C(g)x = x$. Thus $fx = f(f * g)x = g(f * g)x = gx$. Hence wherever $\psi_{f * g}$ is defined, so are ψ_f and ψ_g and moreover all three agree.

This shows that S is embeddable as an agreeable semigroup in $\mathcal{P}(S)$, and so S is a twisted agreeable semigroup since $\mathcal{P}(S)$ is. \square

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