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A TWO-COMPONENT PHENOMENOLOGY FOR THE EVOLUTION OF MHD TURBULENCE

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ABSTRACT

Incompressible MHD turbulence with a mean magnetic field $\boldsymbol{B}_0$ develops anisotropic spectral structure and can be simply described only by including at least two distinct fluctuation components. These are conveniently referred to as "*waves*," for which propagation effects are important, and "*quasi-2D*" turbulence, for which nonlinear effects dominate over propagation ones. The quasi-2D component has wavevectors approximately perpendicular to $\boldsymbol{B}_0$. These two idealized ingredients capture the essential physics of propagation (high frequency fluctuations) and strong turbulence (low frequency fluctuations.) Here we present a two-component energy-containing range phenomenology for the evolution of homogeneous MHD turbulence.

1. INTRODUCTION

Magnetohydrodynamic (MHD) turbulence is expected to be active in many space physics and astrophysics systems, including the solar corona and solar wind. Given the apparent intractability of the governing nonlinear equations it is desirable to develop models for the evolution of MHD turbulence. Here we report on the initial phase of such a development, concentrating on the simpler case of zero cross helicity.

The key idea is to decompose the velocity ($\boldsymbol{v}$) and magnetic ($\boldsymbol{b}$) fluctuations into "wave-like" and non-wavelike components. The decomposition is accomplished in Fourier space using an orthogonal projection. Thus we write,

$$\boldsymbol{v} = \boldsymbol{v}^{\rm 2D} + \boldsymbol{v}^{\rm waves}, \qquad (1)$$
$$\boldsymbol{b} = \boldsymbol{b}^{\rm 2D} + \mathbf{b}^{\rm waves}. \qquad (2)$$

Throughout we measure magnetic fields in Alfvén speed units $\mathbf{b}/\sqrt{4\pi\rho} \to \boldsymbol{b}$.

Account is taken of both the self interactions of each component and also the couplings between the two components. As some of the couplings are resonant they have the potential to dominate the remaining couplings.

In incompressible MHD there are two timescales associated with excitation at each wavevector $\boldsymbol{k}$ (ignoring dissipative timescales). These are the Alfvén (or wave) timescale and the nonlinear timescale, respectively defined as

$$\tau_{\rm A}(\boldsymbol{k}) = \frac{1}{k_\parallel B_0}, \qquad (3)$$
$$\tau_{\rm NL}(\boldsymbol{k}) = \frac{1}{k \bar{u}_k}, \qquad (4)$$

where $\bar{u}_k^2 \approx$ the energy in the scales with $|\boldsymbol{k}| \approx k$. At each $\boldsymbol{k}$, the faster timescale has the leading-order influence on the dynamics. However, because of the anisotropy of $\tau_{\rm A}(\boldsymbol{k})$ with respect to the mean field direction, the boundary in Fourier space where $\tau_{\rm NL}(\boldsymbol{k}) = \tau_{\rm A}(\boldsymbol{k})$ is also, in general, anisotropic (Montgomery and Turner 1981; Montgomery 1982; Higdon 1984; Goldreich and Sridhar 1995; Kinney and McWilliams 1998; Oughton et al. 2004). Note that this would be true even if the energy spectrum itself was (somehow) maintained to be isotropic. Figure 1a is a sketch in Fourier space showing the nature of this boundary for a strong B_0 and an assumed Kolmogorov-type spectrum in the (roughly) perpendicular directions. This boundary is referred to by various names including the *critical balance* boundary and the *reduced MHD* boundary.

Also shown (Fig. 1b) are typical directions of spectral transfer inside and outside the $\tau_{\rm NL} = \tau_{\rm A}$ boundary. Note that in each case, the dominant sense of transfer is in the perpendicular directions. This is an aspect of the dynamics which modeling of the turbulence should reflect.

2. DEVELOPMENT

The starting point for the development of the model is the (dissipative) incompressible MHD equations, with a mean magnetic field $\boldsymbol{B}_0$. Substituting Eqs. 1 and 2 into the MHD equations, and switching to Elsässer variables for each component, defined as

$$\boldsymbol{z}_\pm = \boldsymbol{v}^{\rm 2D} \pm \boldsymbol{b}^{\rm 2D}, \qquad (5)$$
$$\boldsymbol{w}_\pm = \boldsymbol{v}^{\rm waves} \pm \boldsymbol{b}^{\rm waves}, \qquad (6)$$

one finds for the "plus" fields,

$$\begin{aligned}\frac{\partial}{\partial t}(\boldsymbol{z}_+ + \boldsymbol{w}_+) &\sim -(\boldsymbol{z}_- + \boldsymbol{w}_-)\cdot\boldsymbol{\nabla}(\boldsymbol{z}_+ + \boldsymbol{w}_+) \\ &\quad + \mathbf{B}_0\cdot\boldsymbol{\nabla}(\boldsymbol{z}_+ + \boldsymbol{w}_+) \\ &= -\boldsymbol{z}_-\cdot\boldsymbol{\nabla}\boldsymbol{z}_+ - \boldsymbol{w}_-\cdot\boldsymbol{\nabla}\boldsymbol{z}_+ \\ &\quad -\boldsymbol{z}_-\cdot\boldsymbol{\nabla}\boldsymbol{w}_+ - \boldsymbol{w}_-\cdot\boldsymbol{\nabla}\boldsymbol{w}_+ \\ &\quad +\mathbf{B}_0\cdot\boldsymbol{\nabla}\boldsymbol{z}_+ + \mathbf{B}_0\cdot\boldsymbol{\nabla}\boldsymbol{w}_+ \qquad (7)\end{aligned}$$

A similar equation holds for the "minus" fields. An advantage of the two-component decomposition is that the

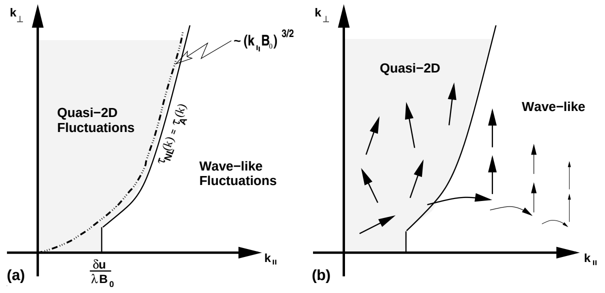

Figure 1. (a) Sketch of the quasi-2D/wave-like regions, and boundary, in Fourier space. The boundary is given by the curve $\tau_{\rm NL}(\boldsymbol{k}) = \tau_{\rm A}(\boldsymbol{k})$. A moderate to strong B_0 is assumed. Note the dependence of the boundary on B_0. In particular as B_0 gets weaker the boundary and shaded (quasi-2D) region expand to fill more and more of Fourier space. (b) Indication of typical spectral transfer directions and strengths. In both regions the dominant behaviour is (quasi-)perpendicular transfer. After Oughton et al. (2004).

generic nature of each term is evident in Eq. 7. For example, reading from left to right there are terms involving 2D-2D interactions, wave-2D, 2D-wave, and wave-wave interactions. The final two terms are associated with propagation effects.

Equations for the evolution of the Elsässer energies are obtained by forming dot products of Eq. 7 and its "minus" sibling with $\boldsymbol{z}_\pm$, $\boldsymbol{w}_\pm$, and spatially averaging (homogeneous turbulence is assumed).

The next step is to model the nonlinear terms in the Elsässer energy equations, using ideas based on the well-verified "u^3/ℓ" decay phenomenology of hydrodynamics. That is, on the pair of equations $\mathrm{d}u^2/\mathrm{d}t \sim -u^3/\ell$, $\mathrm{d}\ell/\mathrm{d}t \sim u$ (von Kármán and Howarth 1938; Dryden 1943; Hossain et al. 1995). Note that these have analytic solutions. Some MHD extensions of this phenomenology have been considered previously (*e.g.,* Hossain et al. 1995; Matthaeus et al. 1996).

For this first presentation of the model we specialise to the case of zero cross helicity, *i.e.,* $\langle \boldsymbol{v}\cdot\boldsymbol{b}\rangle = 0$. It is then convenient to employ the notation

$$Z^2 = \langle z_+^2\rangle = \langle z_-^2\rangle, \qquad (8)$$

$$W^2 = \langle w_+^2\rangle = \langle w_-^2\rangle. \qquad (9)$$

After taking account of the relevant nonlinear and Alfvén timescales in each term in the Elsässer energy equations we obtain the following modeled equations for the quasi-2D and wave-like energies,

$$\begin{aligned}\frac{\mathrm{d}Z^2}{\mathrm{d}t} &= -\frac{Z^3}{\ell}\Gamma_z^{zz} - \frac{WZ^2}{\ell}\Gamma_z^{wz} \\ &\quad +WZ^2\left[\frac{\Gamma_w^{zw}}{\lambda} - \frac{\Gamma_w^{zz}}{\ell}\right] \\ &\quad +W^2Z\left[\frac{\Gamma_w^{ww}}{\lambda} - \frac{\Gamma_w^{wz}}{\ell}\right], \qquad (10)\end{aligned}$$

$$\begin{aligned}\frac{\mathrm{d}W^2}{\mathrm{d}t} &= -\frac{ZW^2}{\lambda}\Gamma_z^{zw} - \frac{W^3}{\lambda}\Gamma_w^{ww} \\ &\quad -WZ^2\left[\frac{\Gamma_w^{zw}}{\lambda} - \frac{\Gamma_w^{zz}}{\ell}\right] \\ &\quad -W^2Z\left[\frac{\Gamma_w^{ww}}{\lambda} - \frac{\Gamma_w^{wz}}{\ell}\right], \qquad (11)\end{aligned}$$

where the B_0-dependent attenuation factors are

$$\Gamma_c^{ab} = \frac{1}{1+\tau_{\rm NL}^{ab}/\tau_{\rm A}^{c}} = \left[1 + \frac{\ell_b}{a}/\frac{\ell_c^{\|}}{B_0}\right]^{-1} \qquad (12)$$

with ℓ_a a characteristic lengthscale of a, $\ell^{\|}$ a characteristic parallel lengthscale, and a, b, c represent the appropriate $\boldsymbol{z}_\pm$, $\boldsymbol{w}_\pm$ fluctuation. Lengthscales associated with the quasi-2D component are denoted using ℓ, while those for the wave-like component are denoted with λ (with its connotations of wavelength). The Γ factors arise from constructing the spectral transfer time associated with each term in equations like 7 (*e.g.,* Matthaeus and Zhou 1989).

In each equation, the first two terms model the turbulent cascades while the last two model the (conservative) exchange of energy between the two components. Indeed, consideration of the various types of interactions between quasi-2D and wave-like modes leads to the following conceptual classification of the terms,

$$\begin{aligned}\frac{\mathrm{d}Z^2}{\mathrm{d}t} &= \text{quasi-2D cascade (Kolmogorov-like)} \\ &\quad + \text{non-res. replenishment from waves} \\ &\quad + \text{non-res. loss to waves} \qquad (13)\\ \frac{\mathrm{d}W^2}{\mathrm{d}t} &= \text{resonant } z\text{-}w\ (\perp) \text{ cascade (Shebalin-like)} \\ &\quad + \text{non-res. } w_+\text{-}w_- \text{ cascade (Kraichnan-like)} \\ &\quad + \text{non-res. gain from quasi-2D interactions} \\ &\quad + \text{non-res. loss to quasi-2D component.} \qquad (14)\end{aligned}$$

Note, in particular, that the distinction between cascade type effects and exchange type effects comes out quite cleanly.

2.1 Evolution of the lengthscales

In order to close the model, equations for the behaviour of the lengthscales are also required.

For the perpendicular ones, ℓ and λ, we employ separate conservation laws for the quasi-2D and wave-like components:

$$\ell Z^n = \text{const}, \quad (15)$$
$$\lambda W^m = \text{const}, \quad (16)$$

with n and m independent constants. Note that n or $m = 1$ implies evolution at constant Reynolds number, while n or $m = 2$ implies constant area under the associated correlation function (Hossain et al. 1995; Zank et al. 1996; Matthaeus et al. 1996)

The evolution of the parallel lengthscales, $\ell_\parallel$ and $\lambda_\parallel$, is somewhat more difficult to model, since not all types of interactions alter them. We are again using the u^3/ℓ type phenomenology as a guide [cf. Hossain et al. (1995); Matthaeus et al. (1996); Zank et al. (1996)].

2.2 Limits

Here we briefly consider two limits of particular interest, namely $B_0 \to 0$ and $B_0 \to \infty$.

In the $B_0 \to 0$ case, all the Γ factors approach unity. For consistency of interpretation it is also necessary that $W^2/Z^2 \to 0$. This can be understood heuristically with reference to Fig. 1a. As B_0 decreases, the equal timescale (aka critical balance) boundary expands away from the $k_\perp$ axis and the wave-like region occupies a smaller and smaller region of $\boldsymbol{k}$-space. Indeed, in the $B_0 \to 0$ limit there is no wave-like region.

Thus, to leading-order only quasi-2D (note that the name quasi-2D is quite misleading in this limit) fluctuations remain, and one recovers essentially the standard hydrodynamic u^3/ℓ phenomenology,

$$\frac{\mathrm{d}Z^2}{\mathrm{d}t} = -\frac{Z^3}{\ell}, \qquad \frac{\mathrm{d}\ell}{\mathrm{d}t} = Z. \quad (17)$$

This is the sense in which the free decay of zero cross helicity MHD turbulence, without a mean field, is hydrodynamic-like. Note also that isotropy has been recovered.

Turning now to the $B_0 \to \infty$ limit, we find that the leading-order terms for both Z and W are associated with perpendicular cascades. They have the important property that they are independent of the mean field strength B_0.

For the wave-like component, the leading-order term is due to the dominance of the resonant interaction of a quasi-2D fluctuation with a wave-like fluctuation. The importance of these resonant terms was first identified by Shebalin et al. (1983) and subsequently expanded upon (*e.g.,* Bondeson 1985; Oughton et al. 1994, 1998; Matthaeus et al. 1998).

$$\frac{\mathrm{d}Z^2}{\mathrm{d}t} = -\frac{1}{2}\frac{Z^3}{\ell} - \frac{WZ^2}{\ell}\frac{1}{1+Z/W}, \quad (18)$$
$$\frac{\mathrm{d}W^2}{\mathrm{d}t} = -\frac{ZW^2}{\lambda}\frac{1}{1+\lambda/\ell}. \quad (19)$$

Note the factor of a half in the first term on the RHS of Eq. 18, which is a natural consequence of the quasi-2D/wave-like decomposition employed in this phenomenology. Recall that Hossain et al. (1995) compared an earlier MHD u^3/ℓ-type phenomenology to direct simulation results and found, empirically, that the best fits for energy decay were obtained with just such a factor of a half.

3. CONCLUSIONS AND FURTHER WORK

The development of a two-component model of MHD turbulence with a mean field has been outlined. The components are a *wave-like* piece for which propagation effects are important, and a *quasi-2D* piece for which nonlinear interactions generate the leading-order dynamics. The components are coupled and there are typically both resonant and non-resonant terms in the evolution equations for the energy and lengthscales characterizing each component.

We anticipate that such phenomenologies will be useful in modeling the evolution of systems such as the solar corona and the solar wind.

As for future work, we are currently working on completing the development of equations for the evolution of the parallel lengthscales, and also on extending the model to take account of non-zero cross helicity effects and forcing terms.

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