Nondegenerate three-dimensional complex Euclidean superintegrable systems and algebraic varieties

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A classical (or quantum) second order superintegrable system is an integrable n-dimensional Hamiltonian system with potential that admits $2n-1$ functionally independent second order constants of the motion polynomial in the momenta, the maximum possible. Such systems have remarkable properties: multi-integrability and multiseparability, an algebra of higher order symmetries whose representation theory yields spectral information about the Schrödinger operator, deep connections with special functions, and with quasiexactly solvable systems. Here, we announce a complete classification of nondegenerate (i.e., four-parameter) potentials for complex Euclidean 3-space. We characterize the possible superintegrable systems as points on an algebraic variety in ten variables subject to six quadratic polynomial constraints. The Euclidean group acts on the variety such that two points determine the same superintegrable system if and only if they lie on the same leaf of the foliation. There are exactly ten nondegenerate potentials. © *2007 American Institute of Physics.* [DOI: 10.1063/1.2817821]

I. INTRODUCTION

For any complex three-dimensional (3D) conformally flat manifold, we can always find local coordinates x,y,z such that the classical Hamiltonian takes the form

$$H = \frac{1}{\lambda(x,y,z)}(p_1^2+p_2^2+p_3^2) + V(x,y,z), \quad (x,y,z) = (x_1,x_2,x_3), \tag{1}$$

i.e., the complex metric is $ds^2=\lambda(x,y,z)(dx^2+dy^2+dz^2)$. This system is *superintegrable* for some potential V if it admits five functionally independent constants of the motion (the maximum number possible) that are polynomials in the momenta p_j. (Some authors require that the constants of the motion be "globally defined." We restrict to polynomials, but allow singularities in the potential and metric, in order to make direct contact with quantum mechanics. Also we do not assume, but prove, that our systems are integrable.) It is *second order superintegrable* if the constants of the motion are quadratic, i.e., of the form

$$S = \sum a^{ji}(x,y)p_jp_i + W(x,y,z). \tag{2}$$

That is, $\{H,S\}=0$, where

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$$\{f,g\}=\sum_{j=1}^{n}(\partial_{x_j}f\partial_{p_j}g-\partial_{p_j}f\partial_{x_j}g)$$

is the Poisson bracket for functions $f(\mathbf{x},\mathbf{p})$, $g(\mathbf{x},\mathbf{p})$ on phase space.[1–8] There is a similar definition of second order superintegrability for quantum systems with formally self-adjoint Schrödinger and symmetry operators whose classical analogs are those given above, and these systems correspond one to one.[9] (In particular, the terms in the Hamiltonian that are quadratic in the momenta are replaced by the Laplace-Beltrami operator on the manifold, and Poisson brackets are replaced by operator commutators in the quantum case.) Historically, the most important superintegrable system is the Euclidean space Kepler-Coulomb problem where $V=\alpha/\sqrt{x^2+y^2+z^2}$. (Recall that this system not only has angular momentum and energy as constants of the motion but a Laplace vector that is conserved.) Superintegrable systems have remarkable properties. In particular, every trajectory of a solution of the Hamilton equations for such a system in six-dimensional phase space lies on the intersection of five independent constant of the motion hypersurfaces in that space, so that the trajectory can be obtained by algebraic methods alone, with no need to solve Hamilton's equations directly. Other common properties for second order superintegrable systems include multiseparability (which implies multi-integrability, i.e., integrability in distinct ways)[1–8,10–12] and the existence of a quadratic algebra of symmetries that closes at order 6. The quadratic algebra in the quantum case gives information relating the spectra of the constants of the motion, including the Schrödinger operator.

Many examples of 3D and n-dimensional superintegrable systems are known, although, in distinction to the two-dimensional (2D) case, they have not been classified.[13–19] Here, we employ theoretical methods based on integrability conditions to obtain a complete classification of Euclidean systems, with nondegenerate potentials. To make it clear how these systems relate to general second order superintegrable systems, we introduce some terminology. A set of second order symmetries for a classical superintegrable system is either linearly independent (LI) or linearly dependent (LD). LI sets can be functionally independent (FI) in the six-dimensional phase space in two ways: they are strongly FI if they are functionally independent even when the potential is set equal to zero and they are weakly FI if the functional independence holds only when the potential is turned on (example: the isotropic oscillator). Otherwise, they are functionally dependent (FD). A LI set can be functionally linearly dependent (FLD) if it is linearly dependent at each regular point, but the linear dependence varies with the point. An LI set can be FLD in two ways: it is weakly FLD if the functional linear dependence holds only with the potential turned off and strongly FLD if the functional linear dependence holds even with the potential turned on. Otherwise, the set is functionally linearly independent (FLI). The Calogero and Generalized Calogero potentials are FD and FLD-S.[9] One property of FLD systems is that their potentials satisfy a first order linear partial differential equation. Thus, they can be expressed in terms of a function of only two variables. In that sense, they are degenerate. This paper is concerned with a classification of functionally linearly independent potentials. As shown in Ref. 20, if a 3D second order superintegrable system is FLI, then the potential V is must satisfy a system of coupled partial differential equations (PDEs) of the form

$$V_{22}=V_{11}+A^{22}V_1+B^{22}V_2+C^{22}V_3,\quad V_{33}=V_{11}+A^{33}V_1+B^{33}V_2+C^{33}V_3, \tag{3}$$

$$V_{12}=A^{12}V_1+B^{12}V_2+C^{12}V_3,\quad V_{13}=A^{13}V_1+B^{13}V_2+C^{13}V_3,$$
$$V_{23}=A^{23}V_1+B^{23}V_2+C^{23}V_3. \tag{4}$$

The analytic functions A^{ij},B^{ij},C^{ij} are determined uniquely from the Bertrand-Darboux (BD) equations for the five constants of the motion and are analytic except for a finite number of poles. If the integrability conditions for these equations are satisfied identically, then the potential is said to be *nondegenerate*. A nondegenerate potential (which is actually a vector space of potential functions) is characterized by the following property. At any regular point $\mathbf{x}_0=(x_0,y_0,z_0)$, i.e., a point where the A^{ij},B^{ij},C^{ij} are defined and analytic and the constants of the motion are functionally independent, we can prescribe the values of $V(\mathbf{x}_0)$, $V_1(\mathbf{x}_0)$, $V_2(\mathbf{x}_0)$, $V_3(\mathbf{x}_0)$, $V_{11}(\mathbf{x}_0)$ arbitrarily and obtain a unique solution of (4). Here, $V_1=\partial V/\partial x$, $V_2=\partial V/\partial y$, etc. The four parameters for a nondegenerate potential (in addition to the usual additive constant) are the maximum number of parameters that can appear in a superintegrable system. A FLI superintegrable system is *degenerate* if the potential function satisfies additional restrictions in addition to Eq. (4). These restrictions can arise in two ways, either as additional equations arising directly from the BD equations or as restrictions that occur because the integrability conditions for Eq. (4) are not satisfied identically. In any case, the number of free parameters for a degenerate potential is strictly fewer than 4. In this sense, the nondegenerate potentials are those of maximal symmetry, though the symmetry is not meant in the traditional Lie group or Lie algebra sense. Nondegenerate potentials admit no nontrivial Killing vectors. Our concern in this paper is the classification of all 3D FLI nondegenerate potentials in complex Euclidean space. In Ref. 21, we have begun the study of fine structure for second order 3D superintegrable systems, i.e., the structure and classification theory of systems with various types of degenerate potentials.

Our plan of attack is as follows. First, we give a brief review of the fundamental equations that characterize second order FLI systems with nondegenerate potential in a 3D conformally flat space. Then, we review the structure theory that has been worked out for these systems, including multiseparability and the existence of a quadratic algebra. We will recall the fact that all such systems are equivalent via a Stäckel transform to a superintegrable system on complex Euclidean 3-space or on the complex 3-sphere. Thus, a classification theory must focus on these two spaces. Due to the multiseparability of these systems, we can use the separation of variable theory to help attack the classification problem. In Ref. 22 we showed that associated with each of the seven Jacobi elliptic coordinate generically separable systems for complex Euclidean space, there was a unique superintegrable system with a separable eigenbasis in these coordinates. Thus, the only remaining systems were those that separated in nongeneric orthogonal coordinates alone, e.g., Cartesian coordinates, spherical coordinates, etc. The possible nongeneric separable coordinates are known[23] so, in principle, the classification problem could be solved. Unfortunately, that still left so many specific coordinate systems to check that classification was a practical impossibility. Here, we present a new attack on the problem based on characterizing the possible superintegrable systems with nondegenerate potentials as points on an algebraic variety. Specifically, we determine a variety in ten variables subject to six quadratic polynomial constraints. Each point on the variety corresponds to a superintegrable system. The Euclidean group $E(3,\mathbf{C})$ acts on the variety such that two points determine the same superintegrable system if and only if they lie on the same leaf of the foliation. The differential equations describing the spatial evolution of the system are just those induced by the Lie algebra of the subgroup of Euclidean translations. A further simplification is achieved by writing the algebraic and differential equations in an explicit form so that they transform irreducibly according to representations of the rotation subgroup $SO(3,\mathbf{C})$. At this point, the equations are simple enough to check directly which superintegrable systems arise that permit separation in a given coordinate system. We show that in addition to the seven superintegrable systems corresponding to separation in one of the generic separable coordinates, there are exactly three superintegrable systems that separate only in nongeneric coordinates. Furthermore, for every system of orthogonal separable coordinates in complex Euclidean space, there corresponds at least one nondegenerate superintegrable system that separates in these coordinates. The method of proof of these results should generalize to higher dimensions.

II. CONFORMALLY FLAT SPACES IN THREE DIMENSIONS

Here, we review some basic results about 3D second order superintegrable systems in conformally flat spaces. For each such space, there always exists a local coordinate system x, y, z and a nonzero function $\lambda(x,y,z)=\exp G(x,y,z)$ such that the Hamiltonian is (1). A quadratic constant of the motion (or generalized symmetry) (2) must satisfy $\{H,S\}=0$, i.e.,

$$a_i^{ii} = -G_1 a^{1i} - G_2 a^{2i} - G_3 a^{3i}$$

$$2a_i^{ij} + a_j^{ii} = -G_1 a^{1j} - G_2 a^{2j} - G_3 a^{3j}, \quad i \neq j \tag{5}$$

$$a_k^{ij} + a_j^{ki} + a_i^{jk} = 0, \quad i,j,k \text{ distinct}$$

and

$$W_k = \lambda \sum_{s=1}^{3} a^{sk} V_s, \quad k = 1,2,3. \tag{6}$$

(Here, a subscript j denotes differentiation with respect to x_j.) The requirement that $\partial_{x_\ell} W_j = \partial_{x_j} W_\ell, \ell \neq j$, leads from (6) to the second order BD partial differential equations for the potential,

$$\sum_{s=1}^{3} [V_{sj} \lambda a^{s\ell} - V_{s\ell} \lambda a^{sj} + V_s((\lambda a^{s\ell})_j - (\lambda a^{sj})_\ell)] = 0. \tag{7}$$

For second order superintegrabilty in 3D, there must be five functionally independent constants of the motion (including the Hamiltonian itself). Thus, the Hamilton-Jacobi equation admits four additional constants of the motion,

$$S_h = \sum_{j,k=1}^{3} a_{(h)}^{jk} p_k p_j + W_{(h)} = L_h + W_{(h)}, \quad h = 1, \ldots, 4.$$

We assume that the four functions S_h together with H are functionally linearly independent in the six-dimensional phase space. In Ref. 20, it is shown that the matrix of the 15 BD equations for the potential has rank at least 5; hence, we can solve for the second derivatives of the potential in the form (3). If the matrix has rank >5, then there will be additional conditions on the potential and it will depend on fewer parameters: $D_{(s)}^1 V_1 + D_{(s)}^2 V_2 + D_{(s)}^3 V_3 = 0$. Here, the A^{ij}, B^{ij}, C^{ij}, $D_{(s)}^i$ are functions of x, symmetric in the superscripts, that can be calculated explicitly. Suppose now that the superintegrable system is such that the rank is exactly 5 so that the relations are only (3). Further, suppose that the integrability conditions for system (3) are satisfied identically. In this case, the potential is nondegenerate. Thus, at any point $\mathbf{x}_0$, where the A^{ij}, B^{ij}, C^{ij} are defined and analytic, there is a unique solution $V(\mathbf{x})$ with arbitrarily prescribed values of $V_1(\mathbf{x}_0)$, $V_2(\mathbf{x}_0)$, $V_3(\mathbf{x}_0)$, $V_{11}(\mathbf{x}_0)$ [as well as the value of $V(\mathbf{x}_0)$ itself]. The points $\mathbf{x}_0$ are called *regular*.

Assuming that V is nondegenerate, we substitute the requirement (3) into the BD equations (7) and obtain three equations for the derivatives a_i^{jk}. Then, we can equate coefficients of V_1, V_2, V_3, V_{11} on each side of the conditions $\partial_1 V_{23} = \partial_2 V_{13} = \partial_3 V_{12}$, $\partial_3 V_{23} = \partial_2 V_{33}$, etc., to obtain integrability conditions, the simplest of which are

$$A^{23} = B^{13} = C^{12}, \quad B^{12} - A^{22} = C^{13} - A^{33}, \quad B^{23} = A^{31} + C^{22}, \quad C^{23} = A^{12} + B^{33}. \tag{8}$$

It follows that the 15 unknowns can be expressed linearly in terms of the ten functions

$$A^{i2}, A^{13}, A^{22}, A^{23}, A^{33}, B^{12}, B^{22}, B^{23}, B^{33}, C^{33}. \tag{9}$$

In general, the integrability conditions satisfied by the potential equations take the following form. We introduce the vector $\mathbf{w} = (V_1, V_2, V_3, V_{11})^T$ and the matrices $\mathbf{A}^{(j)}$, $j = 1,2,3$, such that

$$\partial_{x_j} \mathbf{w} = \mathbf{A}^{(j)} \mathbf{w}, \quad j = 1,2,3. \tag{10}$$

The integrability conditions for this system are

$$A_i^{(j)} - A_j^{(i)} = A^{(i)}A^{(j)} - A^{(j)}A^{(i)} \equiv [A^{(i)},A^{(j)}]. \tag{11}$$

The integrability conditions (8) and (11) are analytic expressions in x_1, x_2, x_3 and must hold identically. Then, the system has a solution V depending on four parameters (plus an arbitrary additive parameter).

Using the nondegenerate potential condition and the BD equations, we can solve for all of the first partial derivatives a_i^{jk} of a quadratic symmetry to obtain the 18 basic symmetry equations, (27) in Ref. 20, plus the linear relations (8). Using the linear relations, we can express C^{12}, C^{13}, C^{22}, C^{23}, and B^{13} in terms of the remaining ten functions. Each a_i^{jk} is a linear combination of the $a^{\ell m}$ with coefficients that are linear in the ten variables and in the G_s.

Since this system of first order partial differential equations is involutive, the general solution for the six functions a^{jk} can depend on at most six parameters, the values $a^{jk}(\mathbf{x}_0)$ at a fixed regular point $\mathbf{x}_0$. For the integrability conditions, we define the vector-valued function

$$\mathbf{h}(x,y,z) = (a^{11},a^{12},a^{13},a^{22},a^{23},a^{33})^T$$

and directly compute the 6×6 matrix functions $\mathcal{A}^{(j)}$ to get the first-order system $\partial_{x_j}\mathbf{h}=\mathcal{A}^{(j)}\mathbf{h}$, $j=1,2,3$. The integrability conditions for this system are

$$\mathcal{A}_i^{(j)}\mathbf{h} - \mathcal{A}_j^{(i)}\mathbf{h} = \mathcal{A}^{(i)}\mathcal{A}^{(j)}\mathbf{h} - \mathcal{A}^{(j)}\mathcal{A}^{(i)}\mathbf{h} \equiv [\mathcal{A}^{(i)},\mathcal{A}^{(j)}]\mathbf{h}. \tag{12}$$

By assumption, we have five functionally linearly independent symmetries, so at each regular point the solutions sweep out a five-dimensional subspace of the six-dimensional space of symmetric matrices. However, from the conditions derived above, there seems to be no obstruction to construction of a six-dimensional space of solutions. Indeed, in Ref. 20, we show that this construction can always be carried out.

Theorem 1: $[(5)\Rightarrow(6)]$ *Let V be a nondegenerate potential corresponding to a conformally flat space in three dimensions that is superintegrable, i.e., suppose that V satisfies the equations (3) whose integrability conditions hold identically, and there are five functionally independent constants of the motion. Then, the space of second order symmetries for the Hamiltonian $H=(p_x^2+p_y^2+p_z^2)/\lambda(x,y,z)+V(x,y,z)$ (excluding multiplication by a constant) is of dimension $D=6$.*

Thus, at any regular point (x_0,y_0,z_0) and given constants $\alpha^{kj}=\alpha^{jk}$, there is exactly one symmetry S (up to an additive constant) such that $a^{kj}(x_0,y_0,z_0)=\alpha^{kj}$. Given a set of five functionally independent second order symmetries $\mathcal{L}=\{S_\ell:\ell=1,\ldots,5\}$ associated with the potential, there is always a sixth second order symmetry S_6 that is functionally dependent on $\mathcal{L}$ but linearly independent.

Since the solution space of the symmetry equations is of dimension $D=6$, it follows that the integrability conditions for these equations must be satisfied identically in the a^{ij}. As part of the analysis in Ref. 20, we used the integrability conditions for these equations and for the potential to derive the following:

(1) An expression for each of the first partial derivatives $\partial_\ell A^{ij}$, $\partial_\ell B^{ij}$, $\partial_\ell C^{ij}$, for the ten independent functions as homogeneous polynomials of order at most 2 in the $A^{i'j'}$, $B^{i'j'}$, $C^{i'j'}$. There are $30=3\times 10$ such expressions in all. [In the case $G\equiv 0$, the full set of conditions can be written in the convenient form (59) and (61)].

(2) Exactly five quadratic identities for the ten independent functions, see (31) in Ref. 20 In Euclidean space, these identities take the form $I^{(a)}-I^{(e)}$ in (24) of the present paper.

In Ref. 20, we studied the structure of the spaces of third, fourth, and sixth order symmetries (or constants of the motion) of H. Here, the order refers to the highest *order* terms in the momenta. We established the following results.

Theorem 2: *Let V be a superintegrable nondegenerate potential on a conformally flat space. Then, the space of third order constants of the motion is four dimensional and is spanned by Poisson brackets $R_{jk}=\{S_j,S_k\}$ of the second order constants of the motion. The dimension of the space of fourth order symmetries is* 21 *and is spanned by second order polynomials in the six basis* *symmetries S_h. (In particular, the Poisson brackets $\{R_{jk},S_\ell\}$ can be expressed as second order polynomials in the basis symmetries.) The dimension of the space of sixth order symmetries is* 56 *and is spanned by third order polynomials in the six basis symmetries S_h. (In particular, the products $R_{jk}R_{\ell h}$ can be expressed by third order polynomials in the six basis symmetries.)*

There is a similar result for fifth order constants of the motion, but it follows directly from the Jacobi identity for the Poisson bracket. This establishes the quadratic algebra structure of the space of constants of the motion: it is closed under the Poisson bracket action.

From the general theory of variable separation for Hamilton-Jacobi equations[23,24] and the structure theory for Poisson brackets of second order constants of the motion, we established the following result.[22]

Theorem 3: *A superintegrable system with nondegenerate potential in a 3D conformally flat space is multiseparable. That is, the Hamilton-Jacobi equation for the system can be solved by additive separation of variables in more than one orthogonal coordinate system.*

The corresponding Schrödinger eigenvalue equation for the quantum systems can be solved by multiplicative separation of variables in the same coordinate systems.

Finally, in Ref. 22, we studied the Stäckel transform for 3D systems, an invertible transform that maps a nondegenerate superintegrable system on one conformally flat manifold to a nondegenerate superintegrable system on another manifold. Our principal result was the following.

Theorem 4: *Every superintegrable system with nondegenerate potential on a 3D conformally flat space is equivalent under the Stäckel transform to a superintegrable system on either 3D flat space or the 3-sphere.*

III. GENERIC SEPARABLE COORDINATES FOR EUCLIDEAN SPACES

Now, we turn to the classification of second order nondegenerate superintegrable systems in 3D complex Euclidean space. A subclass of these systems can be obtained rather easily from the separation of variable theory. To make this clear, we recall some facts about generic elliptical coordinates in complex Euclidean n-space and their relationship to superintegrable systems with nondegenerate potentials (see Ref. 25 for more details).

Consider a second order superintegrable system of the form $H=\sum_{k=1}^{n}p_k^2+V(\mathbf{x})$ in Euclidean n-space expressed in Cartesian coordinates x_k. In analogy with the 3D theory, the potential is nondegenerate if it satisfies a system of equations of the form

$$\begin{aligned} V_{jj}-V_{11}&=\sum_{\ell=1}^{n}A^{jj,\ell}(\mathbf{x})V_\ell, \quad j=2,\ldots,n,\\ V_{kj}&=\sum_{\ell=1}^{n}A^{kj,\ell}(\mathbf{x})V_\ell, \quad 1\leqslant k<j\leqslant n, \end{aligned} \tag{13}$$

where all of the integrability conditions for this system of partial differential equations are identically satisfied.[20,26] There is an important subclass of such nondegenerate superintegrable systems that can be constructed for all $n\geqslant 2$ based on their relationship to variable separation in generic Jacobi elliptic coordinates. The prototype superintegrable system which is nondegenerate in n-dimensional flat space has the Hamiltonian

$$H=\sum_{i=1}^{n}\left(p_i^2+\alpha x_i^2+\frac{\beta_i}{x_i^2}\right)+\delta. \tag{14}$$

This system is superintegrable with nondegenerate potential and a basis of $n(n+1)/2$ second order symmetry operators given by

$$P_i = p_i^2 + \alpha x_i^2 + \frac{\beta_i}{x_i^2}, \quad J_{ij} = (x_i p_j - x_j p_i)^2 + \beta_i \frac{x_j^2}{x_i^2} + \beta_j \frac{x_i^2}{x_j^2}, \quad i \neq j.$$

Although there appear to be "too many" symmetries, all are functionally dependent on a subset of $2n-1$ functionally independent symmetries. A crucial observation is that the corresponding Hamilton-Jacobi equation $H=E$ admits additive separation in n generic elliptical coordinates.

$$x_i^2 = c^2 \prod_{j=1}^{n} (u_j - e_i) \Bigg/ \prod_{k \neq i} (e_k - e_i)$$

simultaneously *for all* values of the parameters with $e_i \neq e_j$ if $i \neq j$ and $i, j = 1, \ldots, n$. (Similarly, the quantum problem $H\Psi = E\Psi$ is superintegrable and admits multiplicative separation.) Thus, the equation is multiseparable and separates in a continuum of elliptic coordinate systems (and in many others besides). The n involutive symmetries characterizing a fixed elliptic separable system are polynomial functions of the e_i, and requiring separation for all e_i simultaneously sweeps out the full $n(n+1)/2$ space of symmetries and uniquely determines the nondegenerate potential. The infinitesimal distance in Jacobi elliptical coordinates u_j has the form

$$ds^2 = -\frac{c^2}{4} \sum_{i=1}^{n} \frac{\Pi_{j \neq i}(u_i - u_j)}{\Pi_{k=1}^{n}(u_i - e_k)} du_i^2 = -\frac{c^2}{4} \sum_{i=1}^{n} \frac{\Pi_{j \neq i}(u_i - u_j)}{P(u_i)} du_i^2, \tag{15}$$

where $P(\lambda) = \Pi_{k=1}^{n}(\lambda - e_k)$. However, it is well known that (15) is a flat space metric for any polynomial $P(\lambda)$ of order $\leq n$ and that each choice of such a $P(\lambda)$ defines an elliptic-type multiplicative separable solution of the Laplace-Beltrami eigenvalue problem (with constant potential) in complex Euclidean n-space.[23] The distinct cases are labeled by the degree of the polynomial and the multiplicities of its distinct roots. If for each distinct case we determine the most general potential that admits separation for all e_i compatible with the multiplicity structure of the roots, we obtain a unique superintegrable system with nondegenerate potential and $n(n+1)/2$ second order symmetries.[22,25] These are the generic superintegrable systems. (Thus, for $n=3$, there are seven distinct cases for $-\frac{1}{4}P(\lambda)$,

$$(\lambda - e_1)(\lambda - e_2)(\lambda - e_3), \quad (\lambda - e_1)(\lambda - e_2)^2, \quad (\lambda - e_1)^3,$$

$$(\lambda - e_1)(\lambda - e_2), \quad (\lambda - e_1)^2, \quad (\lambda - e_1), \quad 1,$$

where $e_i \neq e_j$ for $i \neq j$. The first case corresponds to Jacobi elliptic coordinates.) The number of distinct generic superintegrable systems for each integer $n \geq 2$ is $\Sigma_{j=0}^{n} p(j)$, where $p(j)$ is the number of integer partitions of j.

All of the generic separable systems, their potentials, and their defining symmetries can be obtained from the basic Jacobi elliptic system in n dimensions by a complicated but well defined set of limit processes.[22,25,27] In addition to these generic superintegrable systems, there is an undetermined number of nongeneric systems. For $n=2$, all the systems have been found, and now we give the results for $n=3$.

We review some of the details from Ref. 22 to show how each of the generic separable systems in three dimensions uniquely determines a nondegenerate superintegrable system that contains it. We begin by summarizing the full list of orthogonal separable systems in complex Euclidean space and the associated symmetries. (All of these systems have been classified[23] and all can be obtained from the ultimate generic Jacobi elliptic coordinates by limiting processes.[27,28]) Here, a "natural" basis for first order symmetries (Killing vectors) is given by $p_1 \equiv p_x$, $p_2 \equiv p_y$, $p_3 \equiv p_z$, $J_1 = yp_z - zp_y$, $J_2 = zp_x - xp_z$, $J_3 = xp_y - yp_x$ in the classical case and $p_1 = \partial_x$, $p_2 = \partial_y$, $p_3 = \partial_z$, $J_1 = y\partial_z - z\partial_y$, $J_2 = z\partial_x - x\partial_z$, $J_3 = x\partial_y - y\partial_x$ in the quantum case. (In the operator characterizations for the quantum case, the classical product of two constants of the motion is replaced by the symmetrized product of the corresponding operator symmetries.) The free Hamiltonian is $H_0 = p_1^2 + p_2^2 + p_3^2$. In each case below we list the coordinates. The constants of the motion that characterize

these coordinates can be found in Ref. 22 We use the bracket notation of Bôcher[27] to characterize each separable system.
[2111]

$$x^2=c^2\frac{(u-e_1)(v-e_1)(w-e_1)}{(e_1-e_2)(e_1-e_3)},\quad y^2=c^2\frac{(u-e_2)(v-e_2)(w-e_2)}{(e_2-e_1)(e_2-e_3)},$$

$$z^2=c^2\frac{(u-e_3)(v-e_3)(w-e_3)}{(e_3-e_1)(e_3-e_2)},$$

[221]

$$x^2+y^2=-c^2\left[\frac{(u-e_1)(v-e_1)(w-e_1)}{(e_1-e_2)^2}\right]-\frac{c^2}{e_1-e_2}[(u-e_1)(v-e_1)+(u-e_1)(w-e_1)$$
$$+(v-e_1)(w-e_1)],$$

$$(x-iy)^2=c^2\frac{(u-e_1)(v-e_1)(w-e_1)}{e_1-e_2},\quad z^2=c^2\frac{(u-e_2)(v-e_2)(w-e_2)}{(e_2-e_1)^2}.$$

[23]

$$x-iy=\frac{1}{2}c\left(\frac{u^2+v^2+w^2}{uvw}-\frac{1}{2}\frac{u^2v^2+u^2w^2+v^2w^2}{u^3v^3w^3}\right),$$

$$z=\frac{1}{2}c\left(\frac{uv}{w}+\frac{uw}{v}+\frac{vw}{u}\right),\quad x+iy=cuvw.$$

[311]

$$x=\frac{c}{4}\left(u^2+v^2+w^2+\frac{1}{u^2}+\frac{1}{v^2}+\frac{1}{w^2}\right)+\frac{3}{2}c,$$

$$y=-\frac{c}{4}\frac{(u^2-1)(v^2-1)(w^2-1)}{uvw},\quad z=i\frac{c}{4}\frac{(u^2+1)(v^2+1)(w^2+1)}{uvw}.$$

[32]

$$x+iy=uvw,\quad x-iy=-\left(\frac{uv}{w}+\frac{uw}{v}+\frac{vw}{u}\right),\quad z=\tfrac{1}{2}(u^2+v^2+w^2).$$

[41]

$$x+iy=u^2v^2+u^2w^2+v^2w^2-\tfrac{1}{2}(u^4+v^4+w^4),\quad x-iy=c^2(u^2+v^2+w^2),\quad z=2icuvw.$$

[5]

$$x+iy=c(u+v+w),\quad x-iy=\frac{c}{4}(u-v-w)(u+v-w)(u+w-v),$$

$$z=-\frac{c}{4}(u^2+v^2+w^2-2(uv+uw+vw)).$$

We summarize the remaining degenerate separable coordinates.

Cylindrical-type coordinates. All of these have one symmetry in common: $L_1=p_3^2$. The seven systems are, polar, Cartesian, light cone, elliptic, parabolic, hyperbolic, and semihyperbolic.

Complex sphere coordinates. These all have the symmetry $L_1=J_1^2+J_2^2+J_3^2$ in common. The five systems are spherical, horospherical, elliptical, hyperbolic, and semicircular parabolic.

Rotational types of coordinates. There are three of these systems, each of which is characterized by the fact that the momentum terms in one defining symmetry form a perfect square, whereas the other two are not squares.

In addition to these orthogonal coordinates, there is a class of nonorthogonal heat-type separable coordinates that are related to the embedding of the heat equation in two dimensions into 3D complex Euclidean space.[23] These coordinates are not present in real Euclidean space, only in real Minkowski spaces. The coordinates do not have any bearing on our further analysis as they do not occur in nondegenerate systems in three dimensions. This is because they are characterized by an element of the Lie algebra p_1+ip_2 (not squared, i.e., a Killing vector) so they cannot occur for a nondegenerate system.

Note that the first seven separable systems are "generic," i.e., they occur in one-, two-, or three-parameter families, whereas the remaining systems are special limiting cases of the generic ones. Each of the seven generic Euclidean separable systems depends on a scaling parameter c and up to three parameters e_1,e_2,e_3. For each such set of coordinates, there is exactly one nondegenerate superintegrable system that admits separation in these coordinates *simultaneously for all values of the parameters* c,e_j. Consider the system,[23] for example. If a nondegenerate superintegrable system separates in these coordinates for all values of the parameter c, then the space of second order symmetries must contain the five symmetries

$$H=p_x^2+p_y^2+p_z^2+V, \quad S_1=J_1^2+J_2^2+J_3^2+f_1, \quad S_2=J_3(J_1+iJ_2)+f_2,$$

$$S_3=(p_x+ip_y)^2+f_3, \quad S_4=p_z(p_x+ip_y)+f_4.$$

It is straightforward to check that the 12×5 matrix of coefficients of the second derivative terms in the 12 BD equations associated with symmetries $S_1,\ldots,S_4$ has rank 5 in general. Thus, there is at most one nondegenerate superintegrable system admitting these symmetries. Solving the BD equations for the potential, we find the unique solution

$$V(\mathbf{x}) \coloneqq \alpha(x^2+y^2+z^2)+\frac{\beta}{(x+iy)^2}+\frac{\gamma z}{(x+iy)^3}+\frac{\delta(x^2+y^2-3z^2)}{(x+iy)^4}.$$

Finally, we can use the symmetry conditions for this potential to obtain the full six-dimensional space of second order symmetries. This is the superintegrable system III on the following table. The other six cases yield corresponding results.

Theorem 5: *Each of the seven "generic" Euclidean separable systems determines a unique nondegenerate superintegrable system that permits separation simultaneously for all values of the scaling parameter c and any other defining parameters e_j. For each of these systems, there is a basis of five (strongly) functionally independent and six linearly independent second order symmetries. The corresponding nondegenerate potentials and basis of symmetries are*

I [2111]

$$V=\frac{\alpha_1}{x^2}+\frac{\alpha_2}{y^2}+\frac{\alpha_3}{z^2}+\delta(x^2+y^2+z^2),$$
$$\mathcal{P}_i=p_{x_i}^2+\delta x_i^2+\frac{\alpha_i}{x_i^2}, \quad \mathcal{J}_{ij}=(x_ip_{x_j}-x_jp_{x_i})^2+\alpha_i^2\frac{x_j^2}{x_i^2}+\alpha_j^2\frac{x_i^2}{x_j^2}, \quad i\geqslant j. \tag{16}$$

II [221]

$$V=\alpha(x^2+y^2+z^2)+\beta\frac{x-iy}{(x+iy)^3}+\frac{\gamma}{(x+iy)^2}+\frac{\delta}{z^2},$$

$$\mathcal{S}_1=JJ+f_1,\quad \mathcal{S}_2=p_z^2+f_2,\quad \mathcal{S}_3=J_3^2+f_3, \tag{17}$$

$$\mathcal{S}_4=(p_x+ip_y)^2+f_4,\quad L_5=(J_2-iJ_1)^2+f_5.$$

III [23]

$$V=\alpha(x^2+y^2+z^2)+\frac{\beta}{(x+iy)^2}+\frac{\gamma z}{(x+iy)^3}+\frac{\delta(x^2+y^2-3z^2)}{(x+iy)^4},$$

$$\mathcal{S}_1=JJ+f_1,\quad \mathcal{S}_2=(J_2-iJ_1)^2+f_2,\quad \mathcal{S}_3=J_3(J_2-iJ_1)+f_3, \tag{18}$$

$$\mathcal{S}_4=(p_x+ip_y)^2+f_4,\quad \mathcal{S}_5=p_z(p_x+ip_y)+f_5.$$

IV [311]

$$V=\alpha(4x^2+y^2+z^2)+\beta x+\frac{\gamma}{y^2}+\frac{\delta}{z^2},$$

$$\mathcal{S}_1=p_x^2+f_1,\quad \mathcal{S}_2=p_y^2+f_2,\quad \mathcal{S}_3=p_zJ_2+f_3, \tag{19}$$

$$\mathcal{S}_4=p_yJ_3+f_4,\quad \mathcal{S}_5=J_1^2+f_5.$$

V [32]

$$V=\alpha(4x^2+y^2+z^2)+\beta x+\frac{\gamma}{(y+iz)^2}+\frac{\delta(y-iz)}{(y+iz)^3},$$

$$\mathcal{S}_1=p_x^2+f_1,\quad \mathcal{S}_2=J_1^2+f_2,\quad \mathcal{S}_3=(p_z-ip_y)(J_2+iJ_3)+f_3, \tag{20}$$

$$\mathcal{S}_4=p_zJ_2-p_yJ_3+f_4,\quad \mathcal{S}_5=(p_z-ip_y)^2+f_5.$$

VI [41]

$$V=\alpha(z^2-2(x-iy)^3+4(x^2+y^2))+\beta(2(x+iy)-3(x-iy)^2)+\gamma(x-iy)+\frac{\delta}{z^2},$$

$$\mathcal{S}_1=(p_x-ip_y)^2+f_1,\quad \mathcal{S}_2=p_z^2+f_2,\quad \mathcal{S}_3=p_z(J_2+iJ_1)+f_3, \tag{21}$$

$$\mathcal{S}_4=J_3(p_x-ip_y)-\frac{i}{4}(p_x+ip_y)^2+f_4,\quad \mathcal{S}_5=(J_2+iJ_1)^2+4ip_zJ_1+f_5.$$

VII [5]

$$V=\alpha(x+iy)+\beta\big(\tfrac{3}{4}(x+iy)^2+\tfrac{1}{4}z\big)+\gamma\big((x+iy)^3+\tfrac{1}{16}(x-iy)+\tfrac{3}{4}(x+iy)z\big)$$
$$+\delta\big(\tfrac{5}{16}(x+iy)^4+\tfrac{1}{16}(x^2+y^2+z^2)+\tfrac{3}{8}(x+iy)^2z\big),$$

$$\mathcal{S}_1=(J_1+iJ_2)^2+2iJ_1(p_x+ip_y)-J_2(p_x+ip_y)+\tfrac{1}{4}(p_y^2-p_z^2)-iJ_3p_z+f_1,$$
$$\mathcal{S}_2=J_2p_z-J_3p_y+i(J_3p_x-J_1p_z)-\frac{i}{2}p_yp_z+f_2,\quad \mathcal{S}_3=(p_x+ip_y)^2+f_4, \tag{22}$$
$$\mathcal{S}_4=J_3p_z+iJ_1p_y+iJ_2p_x+2J_1p_x+\frac{i}{4}p_z^2+f_3,\quad \mathcal{S}_5=p_z(p_x+ip_y)+f_5.$$

In Ref. 22, we proved what was far from obvious, the fact that *no other* nondegenerate superintegrable system separates for *any* special case of ellipsoidal coordinates, i.e., fixed parameter.

Theorem 6: *A 3D Euclidean nondegenerate superintegrable system admits separation in a special case of the generic coordinates [2111], [221], [23], [311], [32], [41], or [5], respectively, if and only if it is equivalent via a Euclidean transformation to system [I], [II], [III], [IV], [V], [VI], or [VII], respectively*.

This does not settle the problem of classifying all 3D nondegenerate superintegrable systems in complex Euclidean space, for we have not excluded the possibility of such systems that separate only in degenerate separable coordinates. In fact, we have already studied two such systems in Ref. 20,

$[O]$

$$V(x,y,z)=\alpha x+\beta y+\gamma z+\delta(x^2+y^2+z^2),$$

$[OO]$

$$V(x,y,z)=\frac{\alpha}{2}\left(x^2+y^2+\frac{1}{4}z^2\right)+\beta x+\gamma y+\frac{\delta}{z^2}. \tag{23}$$

IV. POLYNOMIAL IDEALS

In this section, we introduce a very different way of studying and classifying superintegrable systems, through polynomial ideals. Here, we confine our analysis to 3D Euclidean superintegrable systems with nondegenerate potentials. Thus, we can set $G\equiv 0$ in the 18 fundamental equations for the derivatives $\partial_i a^{jk}$. Due to the linear conditions (8), all of the functions A^{ij}, B^{ij}, C^{ij} can be expressed in terms of the ten basic terms (9). Since the fundamental equations admit six linearly independent solutions a^{hk}, the integrability conditions $\partial_i a_\ell^{hk}=\partial_\ell a_i^{hk}$ for these equations must be satisfied identically. As follows from Ref. 20, these conditions plus the integrability conditions (11) for the potential allow us to compute the 30 derivatives $\partial_\ell D^{ij}$ of the ten basic terms [Eq. (60) in what follows]. Each is a quadratic polynomial in the ten terms. In addition, there are five quadratic conditions remaining, Eq. (31) in Ref. 20 with $G\equiv 0$.

These five polynomials determine an ideal Σ'. Already, we see that the values of the ten terms at a fixed regular point must uniquely determine a superintegrable system. However, choosing those values such that the five conditions $I^{(a)}-I^{(e)}$, listed below, are satisfied will not guarantee the existence of a solution because the conditions may be violated for values of (x,y,z) away from the chosen regular point. To test this, we compute the derivatives $\partial_i\Sigma'$ and obtain a single new condition, the square of the quadratic expression $I^{(f)}$, listed below. The polynomial $I^{(f)}$ extends the ideal. Let $\Sigma\supset\Sigma'$ be the ideal generated by the six quadratic polynomials, $I^{(a)},\cdots,I^{(f)}$,

$$I^{(a)}=-A^{22}B^{23}+B^{23}A^{33}+B^{12}A^{13}+A^{23}B^{22}-A^{12}A^{23}-A^{23}B^{33},$$

$$I^{(b)} = (A^{33})^2 + B^{12}A^{33} - A^{33}A^{22} - A^{12}B^{33} - A^{13}C^{33} + A^{12}B^{22} - B^{12}A^{22} + A^{13}B^{23} - (A^{12})^2,$$

$$I^{(c)} = B^{23}C^{33} + B^{12}A^{33} + (B^{12})^2 + B^{22}B^{33} - (B^{33})^2 - A^{12}B^{33} - (B^{23})^2,$$

$$I^{(d)} = - B^{12}A^{23} - A^{33}A^{23} + A^{13}B^{33} + A^{12}B^{23},$$

$$I^{(e)} = - B^{23}A^{23} + C^{33}A^{23} + A^{22}B^{33} - A^{33}B^{33} + B^{12}A^{12},$$

$$I^{(f)} = A^{13}C^{33} + 2A^{13}B^{23} + B^{22}B^{33} - (B^{33})^2 + A^{33}A^{22} - (A^{33})^2 + 2A^{12}B^{22} + (A^{12})^2 - 2B^{12}A^{22} + (B^{12})^2 + B^{23}C^{33} - (B^{23})^2 - 3(A^{23})^2. \tag{24}$$

It can be verified with the Gröbner basis package of MAPLE that $\partial_i\Sigma \subseteq \Sigma$, so that the system is closed under differentiation! This leads us to a fundamental result.

Theorem 7: *Choose the 10-tuple (9) at a regular point, such that the six polynomial identities (24) are satisfied. Then, there exists one and only one Euclidean superintegrable system with nondegenerate potential that takes on these values at a point.*

We see that all possible nondegenerate 3D Euclidean superintegrable systems are encoded into the six quadratic polynomial identities. These identities define an algebraic variety that generically has dimension 6, though there are singular points, such as the origin $(0,\ldots,0)$, where the dimension of the tangent space is greater. This result gives us the means to classify all superintegrable systems.

An issue is that many different 10-tuples correspond to the same superintegrable system. How do we sort this out? The key is that the Euclidean group $E(3,\mathbf{C})$ acts as a transformation group on the variety and gives rise to a foliation. The action of the translation subgroup is determined by the derivatives $\partial_k D^{ij}$ that we have already determined (and will list below). The action of the rotation subgroup on the D^{ij} can be determined from the behavior of the canonical equations (3) under rotations. The local action on a 10-tuple is then given by six Lie derivatives that are a basis for the Euclidean Lie algebra $e(3,\mathbf{C})$. For "most" 10-tuples $\mathbf{D}_0$ on the six-dimensional variety, the action of the Euclidean group is locally transitive with isotropy subgroup only the identity element. Thus, the group action on such points sweeps out a solution surface homeomorphic to the six parameter $E(3,\mathbf{C})$ itself. This occurs for the generic Jacobi elliptic system with potential

$$V = \alpha(x^2 + y^2 + z^2) + \frac{\beta}{x^2} + \frac{\gamma}{y^2} + \frac{\delta}{z^2}.$$

At the other extreme, the isotropy subgroup of the origin $(0,\ldots,0)$ is $E(3,C)$ itself, i.e., the point is fixed under the group action. This corresponds to the isotropic oscillator with the potential

$$V = \alpha(x^2 + y^2 + z^2) + \beta x + \gamma y + \delta z.$$

More generally, the isotropy subgroup at $\mathbf{D}_0$ will be H and the Euclidean group action will sweep out a solution surface homeomorphic to the homogeneous space $E(3,C)/H$ and define a unique superintegrable system. For example, the isotropy subalgebra formed by the translation and rotation generators $\{P_1, P_2, P_3, J_1 + iJ_2\}$ determines a new superintegrable system $[A]$ with the potential

$$V = \alpha((x - iy)^3 + 6(x^2 + y^2 + z^2)) + \beta((x - iy)^2 + 2(x + iy)) + \gamma(x - iy) + \delta z.$$

Each class of Stäckel equivalent Euclidean superintegrable systems is associated with a unique isotropy subalgebra of $e(3,\mathbf{C})$, although not all subalgebras occur. (Indeed, there is no isotropy subalgebra conjugate to $\{P_1, P_2, P_3\}$.) One way to find all superintegrable systems would be to determine a list of all subalgebras of $e(3,\mathbf{C})$, defined up to conjugacy, and then for each subalgebra to determine if it occurs as an isotropy subalgebra. Then, we would have to resolve the degeneracy problem in which more than one superintegrable system may correspond to a single isotropy subalgebra.

To begin our analysis of the ideal Σ, we first determine how the rotation subalgebra $so(3,\mathbf{C})$ acts on the ten variables (9) and their derivatives and decompose the representation spaces into $so(3,\mathbf{C})$-irreducible pieces. The A^{ij}, B^{ij}, and C^{ij} are ten variables that, under the action of rotations, split into two irreducible blocks of dimensions 3 and 7,

$$X_{+1} = A^{33} + 3B^{12} - 2A^{22} + i(3A^{12} + B^{33} + B^{22}), \tag{25}$$

$$X_0 = -\sqrt{2}(C^{33} + 2A^{13} + B^{23}), \tag{26}$$

$$X_{-1} = -A^{33} - 3B^{12} + 2A^{22} + i(3A^{12} + B^{33} + B^{22}), \tag{27}$$

$$Y_{+3} = A^{22} + 2B^{12} + i(B^{22} - 2A^{12}), \tag{28}$$

$$Y_{+2} = \sqrt{6}(A^{13} - B^{23} + 2iA^{23}), \tag{29}$$

$$Y_{+1} = \frac{\sqrt{3}}{\sqrt{5}}(3A^{22} - 2B^{12} - 4A^{33} + i(B^{22} - 2A^{12} - 4B^{33})), \tag{30}$$

$$Y_0 = \frac{2}{\sqrt{5}}(2C^{33} - A^{13} - 3B^{23}), \tag{31}$$

$$Y_{-1} = \frac{\sqrt{3}}{\sqrt{5}}(2B^{12} + 4A^{33} - 3A^{22} + i(B^{22} - 2A^{12} - 4B^{33})), \tag{32}$$

$$Y_{-2} = \sqrt{6}(A^{13} - B^{23} - 2iA^{23}), \tag{33}$$

$$Y_{-3} = -A^{22} - 2B^{12} + i(B^{22} - 2A^{12}). \tag{34}$$

Quadratics in the variables can also be decomposed into irreducible blocks. There are two one-dimensional representations, three of dimension 5, one of dimension 7, two of dimension 9, and one of dimension 13,

$$Z_0^{(1a)} = X_0^2 - 2X_{-1}X_{+1}, \tag{35}$$

$$Z_0^{(1b)} = Y_0^2 - 2Y_{-1}Y_{+1} + 2Y_{-2}Y_{+2} - 2Y_{-3}Y_{+3}, \tag{36}$$

$$Z_{\pm 2}^{(5a)} = X_{\pm 1}^2, \tag{37}$$

$$Z_{\pm 1}^{(5a)} = \sqrt{2}X_0X_{\pm 1}, \tag{38}$$

$$Z_0^{(5a)} = \frac{\sqrt{2}}{\sqrt{3}}(X_0^2 + X_{-1}X_{+1}), \tag{39}$$

$$Z_{\pm 2}^{(5b)} = Y_{\pm 1}^2 - \frac{\sqrt{10}}{\sqrt{3}}Y_0Y_{\pm 2} + \frac{\sqrt{5}}{\sqrt{3}}Y_{\mp 1}Y_{\pm 3}, \tag{40}$$

$$Z_{\pm 1}^{(5b)} = \frac{1}{\sqrt{3}}Y_0Y_{\pm 1} - \frac{\sqrt{5}}{\sqrt{2}}Y_{\mp 1}Y_{\pm 2} + \frac{5}{\sqrt{6}}Y_{\mp 2}Y_{\pm 3}, \tag{41}$$

$$Z_0^{(5b)} = \frac{\sqrt{2}}{\sqrt{3}}Y_0^2 - \frac{\sqrt{3}}{\sqrt{2}}Y_{-1}Y_{+1} + \frac{5}{\sqrt{6}}Y_{-3}Y_{+3}, \tag{42}$$

$$Z_{\pm 2}^{(5c)} = X_{\mp 1}Y_{\pm 3} + \frac{1}{\sqrt{15}}X_{\pm 1}Y_{\pm 1} - \frac{1}{\sqrt{3}}X_0 Y_{\pm 2}, \tag{43}$$

$$Z_{\pm 1}^{(5c)} = \frac{1}{\sqrt{5}}X_{\pm 1}Y_0 - \frac{2\sqrt{2}}{\sqrt{15}}X_0 Y_{\pm 1} + \frac{\sqrt{2}}{\sqrt{3}}X_{\mp 1}Y_{\pm 2}, \tag{44}$$

$$Z_0^{(5c)} = -\frac{\sqrt{3}}{\sqrt{5}}X_0 Y_0 + \frac{\sqrt{2}}{\sqrt{5}}X_{-1}Y_{+1} + \frac{\sqrt{2}}{\sqrt{5}}X_{+1}Y_{-1}. \tag{45}$$

There is one 7-dimensional representation with the highest weight vector

$$Z_{+3}^{(7)} = X_0 Y_{+3} - \frac{1}{\sqrt{3}}X_{+1}Y_{+2}, \tag{46}$$

two 9-dimensional representations with highest weight vectors

$$Z_{+4}^{(9a)} = Y_{+2}^2 - \frac{2\sqrt{3}}{\sqrt{5}}Y_{+1}Y_{+3}, \tag{47}$$

$$Z_{+4}^{(9b)} = X_{+1}Y_{+3}, \tag{48}$$

and one 13-dimensional representation

$$Z_{+3}^{(13)} = Y_{+3}^2. \tag{49}$$

A linear combination of representations of the same dimension is another representation and if we define

$$Z_m = 2Z_m^{(5a)} - 5Z_m^{(5b)} + 5Z_m^{(5c)} \quad \text{for } m = -2, -1, 0, +1, +2, \tag{50}$$

$$W_0 = 8Z_0^{(1a)} - 5Z_0^{(1b)}, \tag{51}$$

the algebraic variety defining the nondegenerate superintegrable systems is given by

$$Z_m = W_0 = 0 \quad \text{for } m = -2, -1, 0, +1, +2. \tag{52}$$

If J_x, J_y, and J_z are Lie derivatives corresponding to rotation about the x, y, and z axes, we define

$$J_+ = iJ_x + J_y, \quad J_- = iJ_x - J_y, \quad J_3 = iJ_z,$$

then

$$J_+ f_m = \sqrt{(l-m)(l+m+1)}f_{m+1},$$

$$J_- f_m = \sqrt{(l+m)(l-m+1)}f_{m-1}, \tag{53}$$

$$J_3 f_m = m f_m,$$

where f_m is taken as one of X_m, Y_m, Z_m, or W_0.

Derivatives of X_m and Y_m are quadratics in these variables. The derivatives of X_m are linear combinations of the quadratics from the representations of dimensions 1 and 5. In particular,

$$\partial_i X_j \in \{2Z_m^{(5a)} + 5Z_m^{(5b)} : m = 0, \pm 1, \pm 2\} \cup \{Z_0^{(1A)}\}. \tag{54}$$

Hence, the quadratic identities (52) can be used to write these derivatives as a sum of terms each of degree at least 1 in X_m. This means that whenever all of X_m vanish at a point, their derivatives also vanish and hence the set $\{X_{-1}, X_0, X_{+1}\}$ is a relative invariant.

The derivatives of Y_m are linear combinations of the quadratics from the representations of dimensions 5 and 9,

$$\partial_i Y_j \in \{2Z_m^{(5a)} + 5Z_m^{(5b)}: -2 \leqslant m \leqslant +2\} \cup \{5Z_m^{(9a)} - 24Z_m^{(9b)}: -4 \leqslant m \leqslant +4\}. \tag{55}$$

Hence, they can be written as a sum of terms each of degree at least 1 in Y_m, so

$$\{Y_{-3}, Y_{-2}, Y_{-1}, Y_0, Y_{+1}, Y_{+2}, Y_{+3}\}$$

is a relative invariant set. Note that from the dimension of the spaces containing the derivatives of X_m and Y_m, there must be at least three linear relations among the derivatives of X_m and seven among the derivatives of Y_m.

In a similar way, we can find relative invariant sets of quadratics carrying a representation of the Lie algebra $so(3,\mathbf{C})$. For example, the following are relative invariant sets:

$$\begin{aligned}
R_1 &= \{X_{-1}, X_0, X_{+1}\},\\
R_2 &= \{Y_{-3}, Y_{-2}, Y_{-1}, Y_0, Y_{+1}, Y_{+2}, Y_{+3}\},\\
R_3 &= \{4Z_m^{(5a)} - 15Z_m^{(5b)}: m = 0, \pm 1, \pm 2\} \cup \{Z_0^{(1A)}\},\\
R_4 &= \{3Z_m^{(5a)} - 5Z_m^{(5b)}: m = 0, \pm 1, \pm 2\} \cup \{Z_0^{(1A)}\},\\
R_5 &= \{8Z_m^{(5a)} - 5Z_m^{(5b)}: m = 0, \pm 1, \pm 2\},\\
R_6 &= R_5 \cup \{5Z_m^{(9a)} + 6Z_m^{(9b)}: m = 0, \pm 1, \pm 2, \pm 3, \pm 4\}.
\end{aligned} \tag{56}$$

Recall that the known superintegrable nondegenerate potentials are

$$V_{\mathrm{I}} = \alpha(x^2 + y^2 + z^2) + \frac{\beta}{x^2} + \frac{\gamma}{y^2} + \frac{\delta}{z^2},$$

$$V_{\mathrm{II}} = \alpha(x^2 + y^2 + z^2) + \frac{\beta(x - iy)}{(x + iy)^3} + \frac{\gamma}{(x + iy)^2} + \frac{\delta}{z^2},$$

$$V_{\mathrm{III}} = \alpha(x^2 + y^2 + z^2) + \beta(x + iy)^2 + \frac{\gamma z}{(x + iy)^3} + \frac{\delta(x^2 + y^2 - 3z^2)}{(x + iy)^4},$$

$$V_{\mathrm{IV}} = \alpha(4x^2 + y^2 + z^2) + \beta x + \frac{\gamma}{y^2} + \frac{\delta}{z^2},$$

$$V_{\mathrm{V}} = \alpha(4z^2 + x^2 + y^2) + \beta z + \frac{\gamma}{(x + iy)^2} + \frac{\delta(x - iy)}{(x + iy)^3},$$

$$V_{\mathrm{VI}} = \alpha(4x^2 + 4y^2 + z^2 - 2(x - iy)^3) + \beta(2x + 2iy - 3(x - iy)^2) + \gamma(x - iy) + \frac{\delta}{z^2},$$

$$\begin{aligned}V_{\mathrm{VII}} = {}&\alpha(x + iy) + \beta(3(x + iy)^2 + z) + \gamma(16(x + iy)^3 + x - iy + 12z(x + iy)) + \delta(5(x + iy)^4 + x^2 + y^2\\ &+ z^2 + 6(x + iy)^2 z),\end{aligned}$$

$$V_O = \alpha(x^2 + y^2 + z^2) + \beta x + \gamma y + \delta z,$$

$$V_{OO} = \alpha(4x^2 + 4y^2 + z^2) + \beta x + \gamma y + \frac{\delta}{z^2},$$

$$V_A = \alpha((x - iy)^3 + 6(x^2 + y^2 + z^2)) + \beta((x - iy)^2 + 2x + 2iy) + \gamma(x - iy) + \delta z. \tag{57}$$

The correspondence between relative invariant sets and potentials is in the accompanying table.

V	R_1	R_2	R_3	R_4	R_5	R_6
I						
II						
III			0			
IV						
V				0		
VI					0	
VII	0		0	0	0	
O	0	0	0	0	0	0
OO					0	0
A	0		0	0	0	0

The action of the Euclidean translation generators on the ten basis monomials can also be written in terms of the irreducible representations of $so(3,\mathbf{C})$. (Indeed, these equations are much simpler than when written directly in terms of A^{ij}, B^{ij}, C^{ij}.) Using the notation

$$\partial_{\pm} = i\partial_y \pm \partial_x, \tag{58}$$

$$Z_m^{(5X)} = 5Z_m^{(5b)} + 2Z_m^{(5a)}, \quad Z_m^{(9Y)} = 24Z_m^{(9b)} - 5Z^{(9a)}. \tag{59}$$

we obtain the fundamental differential relations

$$\partial_- X_{+1} = \tfrac{1}{30\sqrt{6}} Z_0^{(5X)} - \tfrac{1}{9} Z_0^{(1A)}, \quad \partial_+ X_{+1} = \tfrac{1}{30} Z_{+2}^{(5X)},$$

$$\partial_z X_{+1} = -\tfrac{1}{60} Z_{+1}^{(5X)}, \quad \partial_- X_0 = \tfrac{1}{30\sqrt{2}} Z_{-1}^{(5X)},$$

$$\partial_+ X_0 = \tfrac{1}{30\sqrt{2}} Z_{+1}^{(5X)}, \quad \partial_z X_0 = -\tfrac{1}{30\sqrt{3}} Z_0^{(5X)} - \tfrac{1}{9\sqrt{2}} Z_0^{(1A)}, \tag{60}$$

$$\partial_- X_{-1} = \tfrac{1}{30} Z_{-2}^{(5X)}, \quad \partial_+ X_{-1} = \tfrac{1}{30\sqrt{6}} Z_0^{(5X)} - \tfrac{1}{9} Z_0^{(1A)},$$

$$\partial_z X_{-1} = -\tfrac{1}{60} Z_{-1}^{(5X)},$$

$$\partial_- Y_{+3} = \tfrac{1}{180\sqrt{7}} Z_{+2}^{(9Y)} + \tfrac{1}{35} Z_{+2}^{(5X)}, \quad \partial_+ Y_{+3} = \tfrac{1}{90} Z_{+4}^{(9Y)},$$

$$\partial_z Y_{+3} = -\tfrac{1}{180\sqrt{2}} Z_{+3}^{(9Y)}, \quad \partial_- Y_{+2} = \tfrac{1}{60\sqrt{21}} Z_{+1}^{(9Y)} + \tfrac{\sqrt{2}}{35\sqrt{3}} Z_{+1}^{(5X)},$$

$$\partial_+ Y_{+2} = \tfrac{1}{60\sqrt{3}} Z_{+3}^{(9Y)}, \quad \partial_z Y_{+2} = -\tfrac{1}{30\sqrt{42}} Z_{+2}^{(9Y)} + \tfrac{1}{35\sqrt{6}} Z_{+2}^{(5X)},$$

$$\partial_- Y_{+1} = \tfrac{1}{30\sqrt{42}} Z_0^{(9Y)} + \tfrac{\sqrt{2}}{35\sqrt{5}} Z_0^{(5X)}, \quad \partial_+ Y_{+1} = \tfrac{1}{12\sqrt{105}} Z_{+2}^{(9Y)} + \tfrac{1}{35\sqrt{15}} Z_{+2}^{(5X)},$$

$$\partial_z Y_{+1} = -\tfrac{1}{12\sqrt{210}} Z_{+1}^{(9Y)} + \tfrac{2}{35\sqrt{15}} Z_{+1}^{(5X)}, \quad \partial_- Y_0 = \tfrac{1}{18\sqrt{70}} Z_{-1}^{(9Y)} + \tfrac{1}{35\sqrt{5}} X_{-1}^{(5X)},$$

$$\partial_+ Y_0 = \tfrac{1}{18\sqrt{70}} Z_{+1}^{(9Y)} + \tfrac{1}{35\sqrt{5}} X_{+1}^{(5X)}, \quad \partial_z Y_0 = -\tfrac{1}{45\sqrt{14}} Z_0^{(9Y)} + \tfrac{\sqrt{3}}{35\sqrt{10}} X_0^{(5X)},$$

$$\partial_- Y_{-1} = \tfrac{1}{12\sqrt{105}} Z_{-2}^{(9Y)} + \tfrac{1}{35\sqrt{15}} Z_{-2}^{(5X)}, \quad \partial_+ Y_{-1} = \tfrac{1}{30\sqrt{42}} Z_0^{(9Y)} + \tfrac{\sqrt{2}}{35\sqrt{5}} Z_0^{(5X)}, \tag{61}$$

$$\partial_z Y_{-1} = -\tfrac{1}{12\sqrt{210}} Z_{-1}^{(9Y)} + \tfrac{2}{35\sqrt{15}} Z_{-1}^{(5X)}, \quad \partial_- Y_{-2} = \tfrac{1}{60\sqrt{3}} Z_{-3}^{(9Y)},$$

$$\partial_+ Y_{-2} = \tfrac{1}{60\sqrt{21}} Z_{-1}^{(9Y)} + \tfrac{\sqrt{2}}{35\sqrt{3}} Z_{-1}^{(5X)}, \quad \partial_z Y_{-2} = -\tfrac{1}{30\sqrt{42}} Z_{-2}^{(9Y)} + \tfrac{1}{35\sqrt{6}} Z_{-2}^{(5X)},$$

$$\partial_- Y_{-3} = \tfrac{1}{90} Z_{-4}^{(9Y)}, \quad \partial_+ Y_{-3} = \tfrac{1}{180\sqrt{7}} Z_{-2}^{(9Y)} + \tfrac{1}{35} Z_{-2}^{(5X)}, \tag{62}$$

$$\partial_z Y_{-3} = -\tfrac{1}{180\sqrt{2}} Z_{-3}^{(9Y)}.$$

In the following table, we describe each of the known superintegrable systems in terms of variables adapted to the rotation group action. For this, it is convenient to choose the ten constrained variables in the form X_i, $i=1,\ldots,3$, and $Y_j, j=1,\ldots,7$, with d_X and d_Y, respectively, as the number of independent variables on which these variables depend. These are defined by

$$X_1 = 2A^{13} + B^{23} + C^{33} = -\frac{X_0}{\sqrt{2}}, \quad X_2 = 2A^{22} - A^{33} - 3B^{12} = \frac{X_{-1} - X_{+1}}{2},$$

$$X_3 = 3A^{12} + B^{33} + B^{22} = \frac{X_{-1} + X_{+1}}{2}, \quad Y_1 = \tfrac{1}{2}(Y_{+3} - Y_{-3}),$$

$$Y_2 = \frac{1}{2i}(Y_{+3} + Y_{-3}), \quad Y_3 = \frac{1}{2i\sqrt{6}}(Y_{+2} - Y_{-2}), \quad Y_4 = \frac{1}{2\sqrt{6}}(Y_{+2} + Y_{-2}), \tag{63}$$

$$Y_5 = \frac{\sqrt{5}}{2\sqrt{3}}(Y_{+1} - Y_{-1}), \quad Y_6 = \frac{\sqrt{5}}{2i\sqrt{3}}(Y_{+1} + Y_{-1}), \quad Y_7 = \frac{\sqrt{5}}{2} Y_0.$$

	$\Sigma_{j=1}^3 X_j^2$	$[X_1, X_2, X_3]$	d_X d_Y	$[Y_1, Y_2, Y_3, Y_4, Y_5, Y_6, Y_7]$
V_{I}	$\frac{9}{x^2}+\frac{9}{y^2}+\frac{9}{z^2}$	$\left[-\frac{3}{x},-\frac{3}{y},\frac{3}{z}\right]$	3 3	$\left[\frac{3}{x},-\frac{3}{y},0,0,-\frac{3}{x},-\frac{3}{y},-\frac{6}{z}\right]$
V_{II}	$\frac{9}{z^2}$	$\left[-\frac{6}{x+iy},-\frac{6i}{x+iy},\frac{3}{z}\right]$	2 3	$\left[-\frac{6(x-iy)}{(x+iy)^2},-\frac{6i(x-iy)}{(x+iy)^2},0,0,-\frac{6}{x+iy},-\frac{6i}{x+iy},-\frac{6}{z}\right]$
V_{III}	0	$\left[-\frac{9}{x+iy},-\frac{9i}{x+iy},0\right]$	1 3	$\left[-\frac{6(x^2+y^2-2z^2)}{(x+iy)^3},-\frac{6i(x^2+y^2-2z^2)}{(x+iy)^3},\frac{6iz}{(x+iy)^2},\frac{6z}{(x+iy)^2},\frac{6}{x+iy},\frac{6i}{x+iy},0\right]$
V_{IV}	$\frac{9}{y^2}+\frac{9}{z^2}$	$\left[0,-\frac{3}{y},\frac{3}{z}\right]$	2 2	$\left[0,-\frac{3}{y},0,0,0,-\frac{3}{y},-\frac{6}{z}\right]$
V_{V}	0	$\left[-\frac{6}{x+iy},-\frac{6i}{x+iy},0\right]$	1 2	$\left[-\frac{6(x-iy)}{(x+iy)^2},-\frac{6i(x-iy)}{(x+iy)^2},0,0-\frac{6}{x+iy},-\frac{6i}{x+iy},0\right]$
V_{VI}	$\frac{9}{z^2}$	$\left[0,0,\frac{3}{z}\right]$	1 1	$\left[6,-6i,0,0,0,0,-\frac{6}{z}\right]$
V_{VII}	0	[0, 0, 0]	0 1	$[-48(x+iy),-48i(x+iy),12i,12,0,0,0]$

	$\Sigma_{j=1}^{3}X_j^2$	$[X_1,X_2,X_3]$	d_X d_Y	$[Y_1,Y_2,Y_3,Y_4,Y_5,Y_6,Y_7]$
V_O	0	[0, 0, 0]	0 0	[0, 0, 0, 0, 0, 0, 0]
V_{OO}	$\frac{9}{z^2}$	$\left[0,0,\frac{3}{z}\right]$ 33	1 1	$\left[0,0,0,0,0,0,-\frac{6}{z}\right]$
V_A	0	[0, 0, 0]	0	$[-2,2i,0,0,0,0,0]$

In principle, one could classify all possibilities by referring to distinct cases exhibited in the accompanying table. Here, however, we use the preceding algebraic and differential conditions, together with the coordinates in which the corresponding nondegenerate system could separate, to demonstrate that our ten known superintegrable systems are the only ones possible.

V. COMPLETION OF THE PROOF

We know that in addition to the generic superintegrable systems, the only possible superintegrable systems are those that are multiseparable in nongeneric coordinates. Our strategy is to consider each nongeneric separable system in a given standard form and use the integrability conditions associated with the corresponding separable potential. If a superintegrable system permits separation in these coordinates, then by a suitable Euclidean transformation, we can assume that the system permits separation in this standard form. This information is then used together with the six algebraic conditions $I^{(a)},\ldots,I^{(f)}$, (24), to deduce all the information available from algebraic conditions. At that point, the differential equations (60) for D^{ij} can be solved in a straightforward manner to obtain the final possible superintegrable systems. In some cases, the algebraic conditions alone suffice and the differential equations are unnecessary. We proceed on a case by case basis.

A. Cylindrical systems

For cylindrical-type systems, the potential splits off the z variable, i.e., the potential satisfies $V_{13}=0, V_{23}=0$ in Eq. (3). This implies that $A^{13}=B^{13}=C^{13}=0$ and $A^{23}=B^{23}=C^{23}=0$. From the equations for X_i $(i=1,2,3)$ and Y_j $(j=1,\ldots,7)$, we can deduce that $Y_7=-2X_3$. In addition, it is also easy to conclude that $Y_3=Y_4=0$ and $X_1=Y_5, X_2=Y_6$.

If we add the requirement of Cartesian coordinate separation, then $A^{12}=B^{12}=C^{12}=0$. If $X_3=0$, we obtain potential V_0. If $X_3\neq 0$, then $X_3=3/z$. If $X_1=X_2=0$, then we have potential V_{00}. If one of X_1,X_2 is not zero, this leads directly to potential V_{I}.

For separation in cylindrical coordinates $x=r\cos\theta$, $y=r\sin\theta$, z, the following conditions must apply:

$$V_{xz}=0, \quad V_{yz}=0,$$

$$(x^2-y^2)V_{xy}+xy(V_{yy}-V_{xx})+3xV_y-3yV_x=0.$$

The last condition is equivalent to $\partial_\theta(r\partial_r(r^2V))=0$, where $r^2=x^2+y^2$. Solving the algebraic conditions that result, we determine that

$$X_1=Y_5=-G\left(1+\frac{y^2}{x^2}\right)-\frac{3}{x}, \quad X_2=Y_6=G\left(\frac{x}{y}+\frac{y}{x}\right)-\frac{3}{y},$$

$$Y_1 = G\left(-3 + \frac{y^2}{x^2}\right) + \frac{3}{x}, \quad Y_2 = G\left(\frac{x}{y} - 3\frac{y}{x}\right) - \frac{3}{y}, \quad Y_3 = Y_4 = 0,$$

where G is an unknown function. In addition, we deduce that $Y_7 = -2X_3$. It is then easy to show from the differential equations that $X_3 = 3/z$ or 0 and that $G = 0$. We conclude that separation of this type occurs in cases $V_{\rm I}$ and $V_{\rm IV}$.

For parabolic cylinder coordinates $x = \frac{1}{2}(\xi^2 - \eta^2)$, $y = \xi\eta$, z, the conditions on the potential have the form

$$V_{xz} = 0, \quad V_{yz} = 0 \;\; 2xV_{xy} + y(V_{yy} - V_{xx}) + 3V_y = 0.$$

This implies that

$$X_1 = -2F, \quad X_2 = 2\frac{x}{y}F - \frac{3}{y}, \quad X_3 = -C,$$

$$Y_1 = -2F, \quad Y_2 = 2\frac{x}{y}F - \frac{3}{y}, \quad Y_3 = Y_4 = 0,$$

$$Y_5 = -2F, \quad Y_6 = 2\frac{x}{y}F - \frac{3}{y}, \quad Y_7 = 2C.$$

The remaining differential equations require that $F = 0$ and $C = 3/z$. This type occurs in case $V_{\rm IV}$.

For elliptic cylinder coordinates $x = \cosh A \cos B$, $y = \sinh A \sin B$, z, the integrability conditions for the potential have the form

$$V_{zx} = 0, \quad V_{yz} = 0, \quad (x^2 - y^2 - 1)V_{xy} + xy(V_{yy} - V_{xx}) + 3(xV_y - yV_x) = 0.$$

This and the algebraic conditions imply

$$X_1 = \left(\frac{x}{y} + \frac{y}{x} + \frac{1}{xy}\right)G - \frac{3}{x}, \quad X_2 = \left(-1 - \frac{x^2}{y^2} + \frac{1}{y^2}\right)G - \frac{3}{y}, \quad X_3 = -C,$$

$$Y_1 = \left(3\frac{x}{y} - \frac{y}{x} - \frac{1}{xy}\right)G + \frac{3}{x}, \quad Y_2 = \left(-\frac{x^2}{y^2} + 3 + \frac{1}{y^2}\right)G - \frac{3}{y}, \quad Y_3 = Y_4 = 0,$$

$$Y_5 = \left(\frac{x}{y} + \frac{y}{x} + \frac{1}{xy}\right)G - \frac{3}{x}, \quad Y_6 = \left(-1 - \frac{x^2}{y^2} + \frac{1}{y^2}\right)G - \frac{3}{y}, \quad Y_7 = 2C.$$

The remaining differential equations require $G = 0$ and $C = -3/z$ or 0 corresponding to systems $V_{\rm I}$ and $V_{\rm IV}$.

In semihyperbolic coordinates $x + iy = 4i(u+v)$, $x - iy = 2i(u-v)^2$, the extra integrability condition is

$$(1 + ix + y)(V_{xx} - V_{yy}) + 2(-2i - x + iy)V_{xy} + 3iV_x - 3V_y = 0.$$

The algebraic conditions yield the requirements

$$X_1 = Y_5 = G, \quad X_2 = -G, \quad X_3 = -C, \quad Y_3 = Y_4 = 0,$$

$$Y_1 = \tfrac{3}{2}i + \tfrac{i}{2}(x - iy)G, \quad Y_2 = -\tfrac{3}{2} + \tfrac{1}{2}(-x + iy)G, \quad Y_6 = iG, \quad Y_7 = 2C.$$

This leads to potentials V_A and $V_{\rm VI}$.

For hyperbolic coordinates $x + iy = rs$, $x - iy = (r^2 + s^2)/rs, z$, the integrability condition is

$$(1+ixy)(V_{yy}-V_{xx})+i(x^2-y^2-2)V_{xy}+3i(xV_y-yV_x)=0.$$

The algebraic conditions imply $Y_7=2X_3=2C$ and

$$X_1=Y_5=(xy-iy^2-2i)G-\frac{6}{x+iy},\quad X_2=Y_6=-(x^2-ixy-2)G-\frac{6i}{x+iy},$$

$$Y_1=\frac{3yx^2-2ix-y^3-2y}{x+iy}G-\frac{6(x-iy)}{(x+iy)^2},\quad Y_2=-\frac{x^3-3xy^2-2x+2iy}{x+iy}G-i\frac{6(x-iy)}{(x+iy)^2}.$$

This yields potential $V_{\rm II}$.

B. Radial-type coordinates

We consider systems that have a radial coordinate r as one of the separable coordinates. The two other coordinates are separable on the complex 2D sphere. We first consider spherical coordinates $x=r\sin\theta\cos\varphi$, $y=r\sin\theta\sin\varphi$, $z=r\cos\theta$. The integrability conditions on the potential have the form

$$(x^2-y^2)V_{xy}+xzV_{yz}-yzV_{xz}+xy(V_{yy}-V_{xx})+3xV_y-3yV_x=0,$$

$$(x^2-z^2)V_{xz}+xz(V_{zz}-V_{xx})+xyV_{yz}-zyV_{xy}+3xV_z-3zV_x=0,$$

$$(y^2-z^2)V_{yz}+yz(V_{zz}-V_{yy})+xyV_{xz}-zxV_{xy}+3yV_z-3zV_y=0,$$

$$xV_{yz}-yV_{xz}=0.$$

Note that the first three conditions are not independent and only two are required. For any potential that separates in spherical coordinates, one additional condition is required. Indeed, if r, u, and v are any form of separable spherical-type coordinates, then the potential must have the functional form

$$V=f(r)+g(u,v)/r^2, \tag{64}$$

it being understood that u and v are coordinates on the complex 2D sphere and r is the radius. It is then clear that $r^2V=r^2f(r)+g(u,v)$. As a consequence, there are the conditions $\partial_r\partial_\lambda(r^2V)=0$, where $\lambda=u,v$. Noting that

$$x\partial_xF+y\partial_yF+z\partial_zF=DF=r\partial_rF$$

and that

$$J_1F=y\partial_zF-z\partial_yF=a(u,v)\partial_uF+b(u,v)\partial_vF,$$

with similar expressions for J_2F and J_3F, we conclude that the conditions (64) are equivalent to any two of the three conditions $(1/r^2)J_iD(r^2V)=0$. These are indeed the three conditions we have given. If we now solve all the algebraic conditions, we determine that

$$X_1=Y_5=-\frac{(x^2+y^2)}{xy}G-\frac{3}{x},\quad X_2=\frac{(x^2+y^2)}{y^2}G-\frac{3}{y},\quad X_3=\frac{3}{z},\quad Y_7=-\frac{6}{z},$$

$$Y_1=-\frac{3x^2-y^2}{xy}G+\frac{3}{x},\quad Y_2=\frac{x^2-3y^2}{y^2}G-\frac{3}{y},\quad Y_3=Y_4=0.$$

From this, we see that the remaining differential equations give $G=0$ and we obtain solution $V_{\rm I}$.

We now consider horospherical coordinates on a complex 2-sphere, viz,

$$x+iy=-i\frac{r}{v}(u^2+v^2), \quad x-iy=i\frac{r}{v}, \quad z=-ir\frac{u}{v}.$$

The extra integrability condition in this case is

$$z(V_{xx}-V_{yy})+2izV_{xy}-(x+iy)(V_{xz}+iV_{yz})=0.$$

Solving the algebraic conditions, we conclude that

$$X_1=iX_2=\frac{(x+iy)}{z}G-\frac{6}{x+iy}, \quad X_3=\frac{(x+iy)^2}{z^2}G+\frac{3}{z},$$

$$Y_1=iY_2=-4\frac{z}{(x+iy)}G-6\frac{(x-iy)}{(x+iy)^2}, \quad Y_3=iY_4=-2iG,$$

$$Y_5=iY_6=-4\frac{(x+iy)}{z}G-\frac{6}{(x+iy)}, \quad Y_7=-2\frac{(x+iy)^2}{z^2}G-\frac{6}{z}.$$

The derivative conditions give $G=0$, so this corresponds to solution V_{II}.

Conical coordinates are also radial type,

$$x^2=r^2\frac{(u-e_1)(v-e_1)}{(e_1-e_2)(e_1-e_3)}, \quad y^2=r^2\frac{(u-e_2)(v-e_2)}{(e_2-e_1)(e_2-e_3)},$$

$$z^2=r^2\frac{(u-e_3)(v-e_3)}{(e_3-e_2)(e_3-e_1)}.$$

The extra integrability condition is

$$\begin{aligned}3(e_2-e_3)yzV_x+3(e_3-e_1)xzV_y+3(e_1-e_2)xyV_z+xyz[(e_2-e_3)V_{xx}+(e_3-e_1)V_{yy}+(e_1-e_2)V_{zz}]\\+z[(e_3-e_1)y^2+(e_2-e_3)x^2+(e_2-e_1)z^2]V_{xy}+y[(e_1-e_2)z^2+(e_2-e_3)x^2+(e_1-e_3)y^2]V_{xz}\\+x[(e_1-e_2)z^2+(e_3-e_2)x^2+(e_1-e_3)y^2]V_{yz}=0.\end{aligned}$$

The algebraic conditions yield immediately solution V_{I} with

$$X_1=-\frac{3}{x}, \quad X_2=-\frac{3}{y}, \quad X_3=\frac{3}{z}, \quad Y_1=\frac{3}{x}, \quad Y_2=-\frac{3}{y},$$

$$Y_3=Y_4=0, \quad Y_5=-\frac{3}{x}, \quad Y_6=-\frac{3}{y}, \quad Y_7?=-\frac{6}{x}.$$

For degenerate-type elliptic polar coordinates (type 1), we can write

$$x+iy=\frac{r}{\cosh A\cosh B}, \quad 2x=r\left[\frac{\cosh A}{\cosh B}+\frac{\sinh B}{\sinh A}\right], \quad z=r\tanh A\tanh B.$$

The extra integrability condition is

$$\begin{aligned}3(x+iy)^2V_z-3xzV_x-3i(2x+iy)zV_y-2i(x+iy)(z^2+ixy)V_{yz}-2(y^2+z^2)(x+iy)V_{xz}\\+2iz(z^2+y^2)V_{xy}+z(x+iy)^2V_{zz}+z(z^2+y^2)V_{xx}-z(x^2+z^2+2ixy)V_{yy}=0.\end{aligned}$$

Solving the algebraic conditions, we deduce that

$$X_1 = -\frac{2}{x}(y-ix)G - \frac{6}{x+iy}, \quad X_2 = -i\frac{2}{x}(y^2-x^2+z^2-ixy)(y-ix)G - \frac{6i}{x+iy},$$

$$X_3 = -\frac{2i}{xz}(z^2+y^2-ixy)(y-ix)^2G + \frac{3}{z}, \quad Y_1 = -\frac{1}{x}(-y^3+3x^2y+2z^2y-6iz^2x)G - 6\frac{(x-iy)}{(x+iy)^2},$$

$$Y_2 = -\frac{i}{x}(-3ixy^2+ix^3+2z^2y-6iz^2x)G - 6i\frac{x-iy}{(x+iy)^2}, \quad Y_3 = iY_4 = 2z\frac{(y-ix)^2}{x}G,$$

$$Y_5 = -\frac{3}{x}(-3y^2+5ixy+2z^2)(y-ix)G - \frac{6}{x+iy},$$

$$Y_6 = -\frac{i}{6}(-8y^2+13ixy+3x^2+2z^2)(y-ix)G - \frac{6i}{x+iy},$$

$$Y_7 = -\frac{2i}{xz}(-2y^2+2ixy+3z^2)(y-ix)^2G - \frac{6}{z}.$$

The differential conditions require $G=0$, leading to a type V_{II} potential.

For degenerate elliptic coordinates (type 2) on the complex 2-sphere, we have

$$x+iy = ruv, \quad x-iy = \tfrac{1}{4}r(u^2+v^2)^2u^3v^3, \quad z = -\tfrac{i}{2}r\frac{u^2-v^2}{uv}.$$

The corresponding integrability condition is

$$3(z^2+ixy-y^2)V_x + 3i(z^2-x^2-ixy)V_y - 3iz(y-ix)V_z - i(-ixy^2+y^3+iz^2x+yz^2)V_{xx} + i(ix^3-x^2y+iz^2x+tz^2)V_{xx} + i(-ix+y)(x^2+y^2)V_{zz} + 2(x^2y+yz^2+ixy^2+ixz^2)V_{xy} - 2iz(x^2+y^2)V_{yz} - 2z(x^2+y^2)V_{xz} = 0.$$

The solutions to the algebraic conditions are

$$X_1 = -2iz(ix+2y)(y-ix)G - \frac{9}{x+iy}, \quad X_2 = 2z(y+ix)(y-ix)G - \frac{9i}{x+iy},$$

$$X_3 = 2(-ix+y)(x^2+y^2)G, \quad Y_3 = 2i(yz^2+iz^2x-ixy^2-x^2y)G + \frac{6iz}{(x+iy)^2},$$

$$Y_1 = -iY_2 = i\frac{(-3y^2-3x^2+4z^2)z(ix+y)}{ix-y}G + 6\frac{(-x^2-y^2+2z^2)}{(x+iy)^3},$$

$$Y_4 = (2yz^2+2iz^2x-y^3+ixy^2+x^2y-ix^3)G + \frac{6}{(x+iy)^2}, \quad Y_5 = iz(y+3ix)(y-ix)G + \frac{6}{x+iy},$$

$$Y_6 = -z(ix+3y)(y-ix)G + \frac{6i}{x+iy}, \quad Y_7 = (-ix+y)(x^2+y^2)G.$$

The differential conditions hold only if $G=0$. This is system V_{III}.

C. Spheroidal coordinates

We take these as

$$x = \sinh A \cos B \cos \varphi, \quad y = \sinh A \cos B \sin \varphi, \quad z = \cosh A \sin B.$$

The integrability conditions for the potential are

$$-3zV_x + 3xV_z + zx(V_{zz} - V_{xx}) - zyV_{xy} + (1 + x^2 + y^2 - z^2)V_{zx} = 0,$$

$$-3zV_y + 3yV_z + zy(V_{zz} - V_{yy}) - zxV_{xy} + (1 + x^2 + y^2 - z^2)V_{zy} = 0,$$

$$yV_{zx} - xV_{zy} = 0.$$

The solutions of the algebraic conditions are

$$X_1 = Y_5 = -\frac{y}{x}(x^2 + y^2)G - \frac{3}{x}, \quad X_2 = Y_6 = (x^2 + y^2)G - \frac{3}{y}, \quad X_3 = \frac{3}{z},$$

$$Y_1 = -\frac{y}{x}(-y^2 + 3x^2)G + \frac{3}{rx}, \quad Y_2 = (-3y^2 + x^2)G - \frac{3}{y}, \quad Y_7 = -\frac{6}{z}.$$

From the differential conditions, we see that $G=0$ and obtain potential V_{I}.

D. Horospherical coordinates

These are

$$x + iy = \sqrt{\rho\nu}, \quad x - iy = 4\frac{\rho + \nu - \rho\nu\mu}{\sqrt{\rho\nu}}, \quad z = 2\sqrt{\rho\nu\mu}.$$

The corresponding integrability conditions for the potential are

$$(x^2 - ixy - z^2)V_{zx} + (yx - iy^2 + iz^2)V_{zy} + i(x + iy)zV_{xy} + zx(V_{zz} - V_{xx}) + izy(V_{yy} - V_{zz}) = 0,$$

$$(x^2 - y^2)V_{xy} + xy(V_{yy} - V_{xx}) + zxV_{zy} - yzV_{zx} - 3yV_x + 3xV_y = 0,$$

$$z(V_{xx} - V_{yy}) - 2izV_{xy} + (ix + y)V_{zy} + (-x + iy)V_{zx} = 0.$$

The solutions to all the algebraic conditions are

$$X_1 = -iX_2 = -\frac{i(x + iy)}{z}G - \frac{6}{x + iy}, \quad X_3 = \frac{i(x + iy)^2}{z^2}G + \frac{3}{z},$$

$$Y_1 = -iY_2 = -\frac{4iz}{x + iy}G - 6\frac{x - iy}{(x + iy)^2}, \quad Y_3 = iY_4 = 2G,$$

$$Y_5 = -iY_6 = 4\frac{i(x + iy)}{z}G - \frac{6}{x + iy}, \quad Y_7 = -2i\frac{(x + iy)^2}{z^2}G - \frac{6}{z}.$$

The differential conditions require $G=0$ and this gives potential V_{II}.

E. Rotational parabolic coordinates

For these coordinates, $x=\xi\eta\cos\varphi$, $y=\xi\eta\sin\varphi$, $z=\frac{1}{2}(\xi^2-\eta^2)$. The required conditions on the potential are

$$xy(V_{yy}-V_{xx})+(x^2-y^2)V_{xy}-yzV_{yz}+xzV_{xz}-3yV_x+3xV_y=0,$$

$$x^2(V_{xx}-V_{zz})+y^2(V_{yy}-V_{zz})+2xyV_{xy}+2zxV_{yz}+2xzV_{zx}+3xV_x+3yV_y=0,$$

$$xV_{zy}-yV_{zx}=0.$$

These integrability conditions directly produce the solution

$$X_1=-\tfrac{3}{x},\quad X_2=-\tfrac{3}{y},\quad X_3=0,\quad Y_1=\tfrac{3}{x},\quad Y_2=-\tfrac{3}{y},$$

$$Y_3=Y_4=0,\quad Y_5=-\tfrac{3}{x},\quad Y_6=-\tfrac{3}{y},\quad Y_7=0.$$

This is a permuted version of potential V_{IV}.

We have covered all possibilities for separable coordinates and found exactly which superintegrable system separates in each coordinate system It follows that our list of ten superintegrable systems is complete. Another interesting consequence of this analysis is the following.

Theorem 8: *For every orthogonal separable coordinate system, there is at least one nondegenerate superintegrable system that separates in these coordinates.*

On the other hand, no nondegenerate superintegrable system permits separation in nonorthogonal heat-type coordinates. Potential V_{VII} is the only generic system that separates in generic coordinates alone.

VI. DISCUSSION AND OUTLOOK

A referee has asked us to comment on the relation of our results to the list of maximal superintegrable systems in real 3D Euclidean space that are contained in Table I of Evans' ground breaking 1990 paper.[2] Our results are for nondegenerate potentials in complex flat space. Of our ten systems, four are real in real Euclidean space and six are real in Minkowski space. Evans' results are based on the assumption of multiseparability, whereas we have *proved* multiseparability. Evans' Table I listed five systems of which two are nondegenerate (four-parameter) potentials and three are degenerate (three-parameter) potentials. He also found the isotropic oscillator nondegenerate potential but listed it separately. Thus, Evans listed three of the four nondegenerate potentials on real Euclidean space, omitting only V_{OO}. He did not mention that, in fact, these nondegenerate potentials admit six linearly independent second order symmetries nor did he call attention to the quadratic algebra generated by the symmetries. Evans' remaining three (three-parameter) potentials are of the type studied in our paper[29] on fine structure, where we show that such systems admit exactly five second order symmetries, due to an obstruction, and there is no finite quadratic algebra.

The basic structure and classification problems for 2D second order superintegrable systems have been solved.[14,30–33] For 3D systems, the corresponding problems are much more complicated, but we have now achieved a verifiably complete classification of the possible nondegenerate potentials in 3D Euclidean space. There are 10 such potentials, as compared to 11 in two dimensions. To finish the classification of nondegenerate potentials for all 3D conformally flat spaces, the main task remaining is the classification on the 3-sphere. This is because all conformally flat systems can be obtained from flat space and the 3-sphere by Stäckel transforms. The new idea used here that made the complete verifiable classification practical was the association of nondegenerate superintegrable systems with points on an algebraic variety on which the Euclidean group acts to produce foliations. In the future, we hope to refine this approach to give a direct classification using only the algebraic variety and group action. Here, we also had to rely on basic results from the separation of variable theory to simplify the calculations. In distinction to the 2D case, which is special, the 3D classification problem seems to have all of the ingredients that go into the corresponding nondegenerate potential classification problem in n dimensions, though the number of nondegenerate potentials grows rapidly with dimension. The algebraic variety approach should be generalizable to this case.

In addition to nondegenerate potentials for 3D superintegrable systems, there is also a "fine structure," i.e., a hierarchy of various classes of degenerate potentials with fewer than four parameters. The structure and classification theory for these systems has just begun, with initial results for three-parameter FLI systems.[21,29] Sometimes, a quadratic algebra structure exists and sometimes it does not. Extension of these methods to complete the fine structure analysis for 3D systems appears relatively straightforward. The analysis can be extended to two-parameter and one-parameter potentials with five functionally linearly independent second order symmetries. Here, first order PDEs for the potential appear as well as second order, and Killing vectors may occur. Another class of 3D superintegrable systems is that for which the five functionally independent symmetries are functionally linearly dependent. This class is related to the Calogero potential[34–36] and necessarily leads to first order PDEs for the potential, as well as second order.[9] However, the integrability methods discussed here should be able to handle this class with no special difficulties. On a deeper level, we hope that the algebraic geometry approach alone can be extended to determine the possible superintegrable systems in all these cases.

Finally, the algebraic geometry related results that we have described in this paper suggest strongly that there is an underlying geometric structure to superintegrable systems that is not apparent from the usual presentations of these systems.

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