On D-inverse Constellations: an Alternative View of Ordered Groupoids

A thesis

submitted in partial fulfilment

of the requirements for the degree

of

Master of Science in Mathematics

at

The University of Waikato

by

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THE UNIVERSITY OF
WAIKATO
Te Whare Wānanga o Waikato

2024

Abstract

A groupoid is a category in which every arrow has an inverse. The ESN (Ehresmann-Schein-Nambooripad) theorem states that the category of inverse semigroups is isomorphic to the category of inductive groupoids, which are groupoids with additional order-theoretic structure. Inductive groupoids are special types of ordered groupoids; the latter shares most of the properties of inductive groupoids but do not correspond to any type of semigroup. Despite this, many of the main facts about inverse semigroups carry over to ordered groupoids, and ordered groupoids have been shown to be an important tool in the study of inverse semigroups. Since then, it has been found that a particular type of partial algebra we call D-inverse constellations are equivalent to ordered groupoids, but have an arguably simpler, purely algebraic definition. This thesis will explore and illustrate the value of working with D-inverse constellations rather than ordered groupoids, presenting an alternative formulation and proof of the ESN Theorem using D-inverse constellations.

Acknowledgements

I would like to acknowledge Dr. Tim Stokes for his unwavering patience and support over the course of writing this work.
Not only did he teach me how to pioneer my way through a new area of mathematics but rough times led to a frustratingly long delay, yet I was never forsaken.

My family also dedicated a lot of their time to my aid during said delay, they should also be greatly acknowledged.

Last of all, this would not have been possible without those who came before me in this field, thank you for laying out the fundamentals for me to develop upon.

Contents

1 Introduction

As with many papers of this nature, necessity dictates we state the very important ESN (Ehresmann-Schein-Nambooripad) theorem. This was originally published in [6] by Lawson, 1998.

Theorem 1.1. *The category of inductive groupoids (with inductive functors) is isomorphic to the category of inverse semigroups (with semigroup homomorphisms).*

The implications of this theorem are far-reaching. It gives us an alternative perspective on *inverse semigroups* and the knowledge to construct new ones from *inductive groupoids*. Picked up by *semigroup* theorists such as Lawson, much of *inverse semigroup* structure can be investigated through a generalization of *inductive groupoids*. These are called *ordered groupoids* but to work with them is still quite a task. The problem is *ordered groupoids* have a surplus of axioms, some not so easy to grasp in practice. It seems this may not be the most efficient solution.

Subsequently, Gould and Hollings published [1] in 2009 defining a brand new generalization of categories. These so-called *constellations* were developed to prove a more general ESN-type theorem with *left-restriction semigroups* and *inductive constellations*. Gould and Hollings also proved another ESN-type theorem with certain kinds of *inductive constellations* and *inverse semigroups* but these are no less complex than *inductive groupoids*.

Finally, Gould and Stokes wrote [4] in 2024, defining and developing *D-inverse constellations*. They proved an ESN-type theorem bridging the connection between *D-inverse constellations* and *Ordered groupoids*. The importance of this fact is highlighted by the simplicity of how *D-inverse constellations* are defined. They are merely *constellations* with one added axiom. No less, it is an axiom that is easy to understand and refer back to.

This thesis aims to build upon *D-inverse constellation* theory. It will alternatively prove the ESN-type theorem in [4] as well as the ESN-theorem itself. Examples of *ordered groupoids* developed by Lawson will also be explored by taking the *D-inverse constellation* approach. The final goal is to show *D-inverse constellations* are easier to work with than their *ordered groupoid* counterparts. This will be illustrated by proving analogous statements in *ordered groupoid* theory using *D-inverse constellations*.

I will now briefly outline each chapter. Chapter 2 will introduce each important algebraic structure and partial algebra in this work. It will give many key results required in the following chapters and few proofs that are fundamental to *D-inverse constellation* theory. Chapter 3 defines and develops *inductive D-inverse constellations*. The focus is to prove these are *inductive constellations* as in [1] and they correspond to *inverse semigroups* in an ESN-type theorem. It also provides a detailed explanation on how to translate between an *inductive D-inverse constellation* and its equivalent *inverse semigroup*. Chapter 4 is a short investigation on *identity-separating congruences* on *D-inverse constellations*. These are shown to be precisely the same as *identity-separating ordered congruences* on *ordered groupoids*. Chapters 5-8 follow along Lawson's texts [6] and [7] by defining equivalent constructions of *D-inverse constellations* to his *ordered groupoids*. Taking the *D-inverse constellation* approach will, in most cases, emphasize the efficiency of working with them instead.

2 Preliminaries

As is common in modern algebra, functions on arguments will be given by the notation xf instead of $f(x)$ to distinguish them from any given unary operation D on a set X. These instead will be denoted by $D(x)$ for $x \in X$. Composition of functions is hence carried out such that $x(gf)$ is given by $(xg)f$.

We begin with definitions of all required algebraic structures and the constructions defined on each. Key results from other works will be given if required later on.

§2.1 Inverse Semigroups

The majority of content within this section of the preliminaries have been acquired from [6] and [10]. The rest will cited as required.

We introduce some algebraic structures with everywhere-defined binary operations, the most general in this work being **semigroups**. It is common notation for products in a semigroup $(\mathbf{S}, \cdot)$ to be represented by concatenation rather than of the form $x \cdot y$ for binary operation $\cdot$. Therefore, this notation will be adopted.

Definition 2.1. *Let* $\mathbf{S}$ *be a set equipped with binary operation* $\cdot$. *Then* $(\mathbf{S}, \cdot)$ *is a semigroup if* $\forall x, y, z \in \mathbf{S}$*:*

(S1) $x(yz) = (xy)z$.

Semigroups are one of the most basic algebraic structures on which many others are built. The reader may notice groups are one such structure and the above property is the axiom of associativity.

The next algebraic structure is perhaps the most important in this work. These can be viewed as a generalization of a group in which does not have an identity element.

Definition 2.2. *Let* $(\mathbf{S}, \cdot)$ *be a semigroup. Then* $(\mathbf{S}, \cdot, ')$ *with unary operation* $'$ *is an* ***inverse semigroup*** *if* $\forall x, y \in \mathbf{S}$*:*

(IS1) $x'' = x$;

(IS2) $x = xx'x$;

(IS3) $xx'yy' = yy'xx'$.

Equivalently, $\forall x \in \mathbf{S}$*:*

*(IS*1)* $\exists x' \in \mathbf{S}$ *that is unique such that* $x = xx'x$ *and* $x' = x'xx'$.

While the second definition seems more efficient than the first, the former gives more information about how elements interact with respect to the $'$ operation. In most cases this is preferable.

In any inverse semigroup we have what is called the **natural partial order** (sometimes referred to as the NPO in this work) as in [6]. That is, for an inverse semigroup $\mathbf{S}$, there is a partial order $\leq$ satisfying $\forall x, y \in \mathbf{S}$:

$$x \leq y \iff x = xx'y.$$

Interestingly enough, this partial order has many equivalent forms such as $x \leq y \iff x = yx'x$ or even as general as $x \leq y \iff x = ey$ for some idempotent e. The natural partial order will be a recurring theme in this work.

The final algebraic structures of this section are even further generalizations of an inverse semigroup. The first of which are **left-restriction semigroups**, fundamental to the constructions in [1].

Definition 2.3. *Let* $(\mathbf{S}, \cdot)$ *be a semigroup. Then* $(\mathbf{S}, \cdot, D)$ *is a* ***left-restriction semigroup*** *if* $\forall x, y \in \mathbf{S}$*:*

(LR1) $D(x)x = x$*;*

(LR2) $D(x)D(y) = D(y)D(x)$*;*

(LR3) $D(x)D(x) = D(D(x)) = D(x)$*;*

(LR4) $D(xy)D(x) = D(xy)$*;*

(LR5) $xD(y) = D(xy)x$*.*

The below results explicitly justify my claim that they are generalizations of inverse semigroups.

Result 2.4. *(Proposistion 2.1 of [10]): Every inverse semigroup,* $\mathbf{S}$*, is a left-restriction semigroup if one defines* $D(x) = xx'$ *for all* $x \in \mathbf{S}$*.*

Result 2.5. *(Corollary 2.13 of [10]): If* $\mathbf{S}$ *is a left-restriction semigroup in which every element* $x \in \mathbf{S}$ *has a unique D-inverse* x' *satisfying* $D(x) = xx'$ *and* $D(x') = x'x$*, then* $\mathbf{S}$ *is an inverse semigroup.*

Corollary 2.6. $\mathbf{S}$ *is a left-restriction semigroup where every* $x \in \mathbf{S}$ *has a unique inverse* $x' \in \mathbf{S}$ *satisfying* $D(x) = xx'$ *and* $D(x') = x'x$ *if and only if it is an inverse semigroup in which we define* $D(x) = xx'$*.*

The second generalization of inverse semigroups are **$\star$-semigroups** or sometimes termed **semigroups with involution**.

Definition 2.7. *Let* $(\mathbf{S}, \cdot)$ *be a semigroup. Then* $(\mathbf{S}, \cdot, ^*)$ *is a* ***$\star$-semigroup*** *if* $\forall x, y \in \mathbf{S}$*:*

($\star$1) $x^{**} = x$*;*

($\star$2) $(xy)^* = y^*x^*$*.*

As the reader might notice, it is far easier to see $\star-$semigroups are generalizations of inverse semigroups than left-restriction semigroups. These algebraic structures will be explored in more detail in Chapter 4.

§2.2 Groupoids to Constellations

This section of the preliminaries does away with the concept of algebras with everywhere defined products between elements and instead explores partial algebras.

From [5], a **partial algebra** is a non-empty set, $\mathcal{A}$, equipped with a collection of partial operations F. This means it has at least one operator, say $f \in F$ of arity $\sigma(f)$ (henceforth σ will only be used to refer to the arity of an operation) and $x_1, x_2, ..., x_{\sigma(f)} \in \mathcal{A}$ such that $f(x_1, x_2, ..., x_{\sigma(f)})$ may or may not be defined. It is easier to think of what it means to be defined by making an example. Take a subset, $\mathcal{A}$, of an algebra, $\mathcal{B}$, and equip it with the same operations as before. If such an operation on the appropriate number of elements in $\mathcal{A}$ gives an element in $\mathcal{A}$, then it is defined. Otherwise, the element belongs to only $\mathcal{B}$ and we would say it is not defined. In this work we need only be concerned with partial operators on partial algebras of arity two or less.

Another concept in the theory of partial algebras are identities. Consider the partial algebra, $\mathcal{A}$, with partial binary operation, $\star$. We say $e \in \mathcal{A}$ is a **left identity** if $\forall x \in \mathcal{A}$, $\exists e \star x$ implies $e \star x = x$, a **right identity** if $\exists x \star e$ implies $x \star e = x$ and simply an **identity** if it is either or.

Our first example of a partial algebra is certainly the most commonly known and actually has two equivalent definitions. The first is called the category theoretic definition and the other is called the algebraic definition. The one used in this work will be the latter. All content relating to categories were obtained from [6].

Definition 2.8. *Let* $\mathbf{C}$ *be a class (or a set) equipped with partial binary operation* $\circ$ *unary operations* D *and* R*. Then* $(\mathbf{C}, \circ, D, R)$ *is a* ***category*** *if the following holds* $\forall x, y, z \in \mathbf{C}$*.*

(Cat1) $\exists x \circ (y \circ z) \iff \exists (x \circ y) \circ z$ *and they are equal;*

(Cat2) $\exists x \circ y$ *and* $\exists y \circ z \Rightarrow \exists x \circ (y \circ z)$*;*

(Cat3) For each $x \in \mathbf{C}$*,* $\exists D(x), R(x) \in \mathbf{C}$*, identities such that* $D(x) \circ x = x$ *and* $x \circ R(x) = x$*.*

For the sake of brevity, $\exists$ is the symbol we use to say a product is defined. Earlier we gave an example of when a partial operation is defined for partial algebras in general. However, in categories specifically, it is easiest to understand what the $\exists x \circ y$ notation means by referring back to x and y's usual names, **arrows**, which are mappings. We then say $\exists x \circ y$ if and only if $x \circ y$ is defined in terms of composition of mappings (i.e. the range of x is equal to the domain of y).

Albeit non-trivially, it follows $R(x)$ and $D(x)$ in $(CAT3)$ are unique being left and right identities of x respectively. Furthermore, we denote $\mathbf{C}_o = \{D(s), R(s) : s \in \mathbf{C}\}$ as the set of identities in $\mathbf{C}$.

Structure preserving maps (morphisms) between categories are called **functors**. These have the following properties.

Definition 2.9. *Let* $\mathbf{C}$ *and* $\mathbf{D}$ *be categories and let* $\theta : \mathbf{C} \to \mathbf{D}$*. Then* θ *is called a* ***Functor*** *if* $\forall x, y \in \mathbf{C}$*:*

(F1) $\exists x \circ y \Rightarrow \exists x\theta \circ y\theta$ *and* $(x \circ y)\theta = x\theta \circ y\theta$*;*

(F2) $D(x\theta) = D(x)\theta$*;*

(F3) $R(x\theta) = R(x)\theta$*.*

As referenced in the ESN Theorem, morphisms such as functors need to be considered when showing categories of algebraic structures are isomorphic. Though what a morphism's properties will entail are dependent on the algebraic structure itself (for example, they are semigroup homomorphisms for inverse semigroups).

The definition of categories can be extended to have a notion of inverses.

Definition 2.10. *Let* $\mathbf{G}$ *be a category. Then* $\mathbf{G}$ *is a* ***groupoid*** *if:*

(G1) $\forall x \in \mathbf{G}$, $\exists x' \in \mathbf{G} \Rightarrow x \circ x' = D(x)$ *and* $x' \circ x = R(x)$*.*

The study of these structures is important in of itself because of their relation to groups. It is not hard to see every group is a groupoid while monoids give categories.

A particular type of groupoid has been studied extensively by Lawson called **ordered groupoids**, defined by the following.

Definition 2.11. *Let* $\mathbf{G}$ *be a groupoid equipped with a partial order* $\leq$*. Then* $\mathbf{G}$ *is an* ***ordered groupoid*** *if* $\forall w, x, y, z \in \mathbf{G}$*:*

(OG1) $x \leq y \Rightarrow x' \leq y'$*;*

(OG2) $x \leq z$, $y \leq w$, $\exists x \circ y$ *and* $\exists z \circ w \Rightarrow x \circ y \leq z \circ w$*;*

(OG3) For e *an identity element, if* $e \leq D(x)$*, then* $\exists e|x \in \mathbf{G}$ *called the* ***restriction of*** $\boldsymbol{x}$ ***to*** $\boldsymbol{e}$ *that is unique satisfying* $e|x \leq x$ *and* $D(e|x) = e$*;*

(OG4) For e *an identity element, if* $e \leq R(x)$*, then* $\exists x|e \in \mathbf{G}$ *called the* ***corestriction of*** $\boldsymbol{x}$ ***to*** $\boldsymbol{e}$ *that is unique satisfying* $x|e \leq x$ *and* $R(x|e) = e$*.*

Furthermore, $\mathbf{G}$ *is an* ***inductive groupoid*** *if the* $\wedge$ *(greatest lower bound with respect to* $\leq$*) operation on identity elements of* $\mathbf{G}$ *satisfy the property:*

(I) For e *and* f *are identity elements,* $\exists e \wedge f \in \mathbf{G}$ *and it is an identity element.*

In [6], Lawson also provides the groundwork for an alternative definition for ordered groupoids without reference to restrictions nor corestrictions. This is as follows.

Result 2.12. *(Proposition 4.1.4 of [6]): Let* $\mathbf{G}$ *be an groupoid with partial order* $\leq$*. Then* $\mathbf{G}$ *is an ordered groupoid if and only if* $\mathbf{G}$ *satisfies* $(OG1)$*,* $(OG2)$ *and the following:*

(OI) The set of identities $\mathbf{G}_o$ *is an order ideal of* $\mathbf{G}$*;*

(OG3)* $\forall x \in \mathbf{G}$ *and* $e \in \mathbf{G}_o$*, if* $e \leq R(x)$*, then* $\exists y \in \mathbf{G}$ *such that* $y \leq x$ *and* $R(y) = e$*.*

A final note on inductive groupoids and a direct consequence of the ESN-theorem. From any inductive groupoid, one can build an inverse semigroup as we see now.

Result 2.13. *(Proposition 4.1.7 of [6]): Let* $\mathbf{G}$ *be an inductive groupoid and define the operation* $\otimes$ *on* $\mathbf{G}$ *where* $\forall x, y \in \mathbf{G}$*:*

$$x \otimes y = (x|e) \circ (e|y) \text{ where } e = R(x) \wedge D(y).$$

Then $(\mathbf{G}, \otimes)$ *is an inverse semigroup.*

Here, the inverse of an element x under $\otimes$ remains the same as it did under $\circ$. Conversely, one may start with an inverse semigroup $(\mathbf{S}, \cdot)$ and obtain the inductive groupoid, $(\mathbf{S}, \circ, \leq, R, D)$ by defining the partial binary operation $\circ$ on $\mathbf{S}$ to have the property $\exists x \circ y \iff x'x = yy'$. The partial order on $\mathbf{S}$, $\leq$, then becomes the natural partial order on $\mathbf{S}$ under $\cdot$. One must also define $D(x) = xx'$ and $R(x) = x'x$.

We introduce a more recently constructed algebraic structure in modern algebra which originally debuted in [1].

Definition 2.14. *Let* $\mathcal{P}$ *be a class (or a set) equipped with partial binary operation* $\cdot$ *and unary operation* D *such that* $D(\mathcal{P}) = \{D(x) : x \in \mathcal{P}\} \subseteq \mathcal{P}$*. Then* $(\mathcal{P}, \cdot, D)$ *is a* ***constellation*** *if the following axioms hold* $\forall x, y, z \in \mathcal{P}$*:*

(CONST1) $\exists x \cdot (y \cdot z) \Rightarrow \exists (x \cdot y) \cdot z$ *and they are equal;*

(CONST2) $\exists x \cdot y$ *and* $\exists y \cdot z \Rightarrow \exists x \cdot (y \cdot z)$*;*

(CONST3) $\exists D(x) \in D(\mathcal{P})$ *that is unique such that* $\exists D(x) \cdot x$ *and is equal to* x*;*

(CONST4) If $e \in D(\mathcal{P})$ *and* $\exists x \cdot e$*, then* $x \cdot e = x$*.*

In $(CONST3)$, we call $D(x)$ the **unique left identity of x**. Furthermore, $D(\mathcal{P})$ is the set of elements we call **projections**, $e \in \mathcal{P}$, such that $D(e) = e$. Consequently, we simply call $D(\mathcal{P})$ the set of projections of $\mathcal{P}$. In fact, there lies an interesting result in how constellations can also be defined.

Lemma 2.15. *Let* $\mathcal{P}$ *be a constellation, definitions of* $(CONST3)$ *and* $(CONST4)$ *are equivalent to:*

*(Const*3) For each* $x \in \mathcal{P}$*,* $\exists e$*, a unique right identity such that* $\exists e \cdot x$ *and is equal to* x*.*

Proof. *Since we know every projection is a right identity, the aim is to show every right identity is a projection. So let* $e \in \mathcal{P}$ *be a right identity. Then:*

$$\begin{aligned} e &= D(e) \cdot e, \text{ by } (CONST3); \\ &= D(e), \text{ since } e \text{ is a right identity.} \end{aligned}$$

So $e \in D(\mathcal{P})$*, as required.* □

Originally in [1], elements of $D(\mathcal{P})$ were defined to be instead idempotent elements of the partial algebra. Though it follows that the set of projections and the set of idempotent elements of a constellation are the same. This was proven in [2].

The reader will notice the definition for constellations is not so dissimilar to categories. Rather, constellations are said to be more or less "one sided categories". A result of this 'one-sidedness' is the way products in a constellation are defined. That is, considering the constellation where its elements are partial mappings gives $\exists x \cdot y \iff R(x) \subseteq D(y)$ as opposed to $R(x) = D(y)$ if R and D are defined as usual.

Continuing on, much like how functors are important to categories, there is a corresponding definition for morphisms on constellations.

Definition 2.16. *Let* $\mathcal{P}$ *and* $\mathcal{Q}$ *be constellations. Then* $\theta : \mathcal{P} \to \mathcal{Q}$ *is a* ***radiant*** *if* $\forall x, y \in \mathcal{P}$*:*

(R1) If $\exists x \cdot y$*, then* $\exists x\theta \cdot y\theta$ *and* $(x \cdot y)\theta = x\theta \cdot y\theta$*;*

(R2) $D(x)\theta = D(x\theta)$*.*

Key results that arise from constellations and are useful are the following.

Result 2.17. *(Lemma 2.3 of [1]): Let* $\mathcal{P}$ *be a constellation with* $x, y \in \mathcal{P}$*. Then:*

$$\exists x \cdot y \iff \exists x \cdot D(y).$$

Result 2.18. *(Lemma 2.2 of [1]): Let* $\mathcal{P}$ *be a constellation with* $x, y \in \mathcal{P}$*. Then:*

$$\exists x \cdot y \Rightarrow D(x \cdot y) = D(x).$$

The main constellation under consideration in [1] is as follows.

Definition 2.19. *Let* $\mathcal{P}$ *be a constellation equipped with a partial order* $\leq$ *and* $\wedge$ *operation (glb with respect to* $\leq$*) on* $D(\mathcal{P})$*. Then* $\mathcal{P}$ *is an* ***inductive constellation*** *if* $\forall w, x, y, z \in \mathcal{P}$*:*

(I1) $x \leq y \Rightarrow D(x) \leq D(y)$*;*

(I2) $x \leq z$*,* $y \leq w$*,* $\exists x \cdot y$ *and* $\exists z \cdot w \Rightarrow x \cdot y \leq z \cdot w$*;*

(I3) For $e \in D(\mathcal{P})$*, if* $e \leq D(x)$*, then* $\exists e|x \in \mathcal{P}$ *called the* ***restriction of*** $\boldsymbol{x}$ ***to*** $\boldsymbol{e}$ *that is unique satisfying* $e|x \leq x$ *and* $D(e|x) = e$*;*

(I4) $\forall e \in D(\mathcal{P})$, $\exists x|e \in \mathcal{P}$ *called the* ***corestriction of x to e*** *that is the maximum element such that* $x|e \leq x$ *and* $\exists(x|e) \cdot e$;

(I5) $e \in D(\mathcal{P})$, $\exists x \cdot y \Rightarrow D((x \cdot y)|e) = D(x|D(y|e))$;

(I6) If $e, f \in D(\mathcal{P})$ *then if the restriction of* e *to* f *exists then it coincides with the corestriction of* e *to* f;

(I7) $\forall e, f \in D(\mathcal{P})$, $\exists e \wedge f \in D(\mathcal{P})$ *and is equal to* $e|f$.

This definition is reminiscent of the axiomatization of inductive groupoids. Thanks to work from Stokes in [8] and [9], we have alternative ways to prove $(I3)$-$(I5)$ in practice. We begin with a result from [8] of inductive constellations that simplifies $(I3)$, which requires the introduction of the **natural quasiorder**.

Definition 2.20. *Let* $\mathcal{P}$ *be a constellation and* $x, y \in \mathcal{P}$. *Then the natural quasiorder on* $\mathcal{P}$ *is the quasiorder satisfying:*

$$x \leq y \iff x = D(x) \cdot y.$$

Consequently, it is not hard to see $\leq$ on $D(\mathcal{P})$ satisfies $D(x) \leq D(y) \iff \exists D(x) \cdot D(y)$. If the natural quasiorder on a constellation is a partial order, then it will be referred to as the natural partial order. This is exactly as was introduced for inverse semigroups where $D(x)$ is the idempotent.

Result 2.21. *(Proposition 3.3 of [8]): Any quasiorder on an constellation* $\mathcal{P}$ *satisfying* $(I1)$, $(I2)$, $(I3)$ *and agreeing with the natural quasiorder when restricted to* $D(P)$ *is the natural quasiorder with:*

$$e|x = e \cdot x \text{ whenever } e \leq D(x).$$

So $(I3)$ can be simplified if the partial order on an inductive constellation is the natural partial order.

Similarly for $(I4)$ and $(I5)$, notice since there is no concept of range in a constellation, an alternative definition of corestrictions is required. This aspect can be rather difficult to work with if one is trying to prove a particular partial algebra is an inductive constellation. Hence the need for the following definition and result from [9].

Definition 2.22. *Let* $\mathcal{P}$ *be a constellation. Then* $\mathcal{P}$ *is said to be* ***normal*** *if* $\forall e, f \in D(\mathcal{P})$:

$$\exists e \cdot f \text{ and } \exists f \cdot e \Rightarrow e = f.$$

Result 2.23. *(Lemma 2.3 of [9]): Let $\mathcal{P}$ be a constellation with $x, y \in \mathcal{P}$ and $e \in D(\mathcal{P})$. Then $\mathcal{P}$ satisfies $(I4)$ and $(I5)$ if and only if $\mathcal{P}$ is normal and for some operation, $\star : \mathcal{P} \times D(\mathcal{P}) \to D(\mathcal{P})$:*

$$\exists (y \cdot x) \cdot e \iff \exists y \cdot (x \star e).$$

It then follows the corestriction: $x|e = (x \star e) \cdot x$.

We now incorporate the partial order to its morphisms, but to do so one must extend the definition of radiants.

Definition 2.24. *Let $\mathcal{P}, \mathcal{Q}$ be inductive constellations and $\rho : \mathcal{P} \to \mathcal{Q}$ be a radiant. Then ρ is an* ***ordered radiant*** *if $\forall x, y \in \mathcal{P}$:*

(IR1) $x \leq y \Rightarrow x\rho \leq y\rho$ in $\mathcal{Q}$;

(IR2) $e \in D(\mathcal{P}) \Rightarrow (x|e)\rho = x\rho|e\rho$.

Finally, this was enough to establish the following important result in [1].

Result 2.25. *(Theorem 4.13 of [1]): The category of inductive constellations (with ordered radiants) is isomorphic to the category of left-restriction semigroups (with (2,1)-morphisms).*

We use the method outlined below from [1] to translate between inductive constellations and left-restriction semigroups. To build an inductive constellation from a left-restriction semigroup $(\mathbf{S}, \star, D)$ we define $\forall x, y \in \mathbf{S}$:

$$\exists x \cdot y \iff x = x \star D(y). \text{ Then } x \cdot y = x \star y.$$

Then $(\mathbf{S}, \cdot, D)$ is an inductive constellation if we keep $D(x)$ to be the same as in the left-restriction semigroup. However, to build a left-restriction semigroup from an inductive constellation $(\mathcal{P}, \cdot, D)$, we define $D(x)$ to be the same as it was in $\mathcal{P}$ and the binary operation, $\otimes$, to be $\forall x, y \in \mathcal{P}$:

$$x \otimes y = x|D(y) \cdot y.$$

This is always defined by $(I4)$ and **Result 2.17**. Then $(\mathcal{P}, \otimes, D)$ is a left-restriction semigroup by similarly defining $D(x)$ to be the same as in the inductive constellation.

The last kind of constellation in this section was introduced in [4] and is the main subject of this work.

Definition 2.26. *Let $\mathcal{P}$ be a constellation. Then $\mathcal{P}$ is an* ***D-inverse constellation*** *if:*

(IC) For each $x \in \mathcal{P}$, $\exists x' \in \mathcal{P}$ that is unique such that $x \cdot x' = D(x)$ and $x' \cdot x = D(x')$.

In fact, (IC) stems from another definition given in [4] in which we call x' the **D-inverse** of x.

It is sometimes convenient to look at D-inverse constellations as **constellations with range** where one would define $R(x) = D(x')$, similar to what you would expect in a groupoid. There are of course more technicalities for a constellation with range however, they are beyond the scope of this work (see [3] for more on these).

In practice, proving (IC) can be quite tedious, our saving grace is the following result and remains the go to method for doing so.

Result 2.27. *(Proposition 3.9 of [4]): Let $\mathcal{P}$ be a constellation. Then every $e \in D(\mathcal{P})$ is a D-inverse of itself and each element $x \in \mathcal{P}$ has at most one D-inverse if and only if $\mathcal{P}$ is normal.*

D-inverse constellations are very intuitive to work with. In fact, every D-inverse constellation is equipped with the natural partial order as a result of (IC). A consequence of this are the following lemmas.

Lemma 2.28. *Let $\mathcal{P}$ be a D-inverse constellation with $x, y \in \mathcal{P}$. Then if $\leq$ is the natural partial order.*

$$x \leq y \iff x' \leq y'.$$

Proof. *It suffices to prove only one direction as the other follows by symmetry. Suppose $x \leq y$, then:*

$x = D(x) \cdot y;$

$x' \cdot x = (x' \cdot D(x)) \cdot y,$ *by* $(CONST1)$ *and* $(CONST2)$;

$D(x') = x' \cdot y,$ *by* $(CONST4)$;

$D(x') \cdot y' = x' \cdot (y \cdot y'),$ *by* $(CONST1)$ *and* $(CONST2)$;

$D(x') \cdot y' = x' \cdot D(y) = x',$ *by* $(CONST4)$.

So indeed $x' \leq y'$, as required. □

Lemma 2.29. *Let $\mathcal{P}$ be a D-inverse constellation with $x, y \in \mathcal{P}$. Then if $\leq$ is the natural partial order.*

$$\exists x \cdot y \iff D(x') \leq D(y).$$

Proof. *First suppose $\exists x \cdot y$, then:*

$D(x') = x' \cdot x$ *implies* $\exists (x' \cdot x) \cdot y$ *(by (CONST1) and (CONST2)), so:*

$\exists (x' \cdot x) \cdot D(y) = D(x') \cdot D(y)$, *so* $D(x') \leq D(y)$, *as required.*

Conversely, suppose $D(x') \leq D(y)$. Then:

$\exists D(x') \cdot D(y) = (x' \cdot x) \cdot D(y)$ *implies* $\exists (x' \cdot x) \cdot y$ *(by* ***Result 2.17****).*

Note $\exists x \cdot (x' \cdot x)$ *by* $(CONST2)$*;*

$\exists (x \cdot (x' \cdot x)) \cdot y$ *by* $(CONST1)$ *and* $(CONST2)$*;*

$\exists ((x \cdot x') \cdot x) \cdot y$ *by* $(CONST1)$*;*

$= (D(x) \cdot x) \cdot y = x \cdot y$*, as required.* □

Lemma 2.28 is merely a simple tool in working with D-inverse constellations. However, **Lemma 2.29** tells us if we know where every projection is on the natural partial order, then we know every product that exists in the D-inverse constellation and vice versa.

A result reminiscent of commonly named "Socks and Shoes Lemma" also applies to these partial algebras.

Result 2.30. *(Proposition 3.9 of [4]): Let* $\mathcal{P}$ *be a D-inverse constellation with* $x, y \in \mathcal{P}$ *such that* $\exists x \cdot y$ *and* $\exists y' \cdot x'$*. Then:*

$$(x \cdot y)' = y' \cdot x'.$$

Yet the most important aspect of D-inverse constellations is the following result. Its proof heavily on D-inverse constellations being certain kinds of constellations with range and ordered groupoids being certain kinds of ordered categories. These (and their morphisms) were shown to be equivalent under another ESN-type theorem.

Result 2.31. *(Theorem 4.13 of [4]): The category of ordered groupoids (with ordered functors) is isomorphic to the category of D-inverse constellations (with radiants).*

The significance of this result is similar to that of the ESN-theorem in that one can build an equivalent D-inverse constellation from an ordered groupoid and vice versa. The method in [4] to "translate" from one to the other is simple. Let $(\mathcal{P}, \cdot, D)$ be a D-inverse constellation. Then $\forall x, y \in \mathcal{P}$, define:

$$\exists x \circ y = x \cdot y \iff D(x') = D(y).$$

By redefining $D(x') = R(x)$ and taking the partial order to be $x \leq y \iff x = D(x) \cdot y$, one obtains the ordered groupoid $(\mathcal{P}, \circ, \leq, D, R)$. Conversely suppose $(\mathbf{G}, \circ, \leq, D, R)$ is an ordered groupoid, define:

$$x \cdot y = x \circ R(x)|y.$$

Then $(\mathbf{G}, \cdot, D)$ is a D-inverse constellation. It is not difficult to see should $R(x) = D(y)$, then $x \cdot y = x \circ y$. The following lemma directly follows from this definition.

Lemma 2.32. *Let* $(\mathcal{P}, \cdot, D)$ *be a D-inverse constellation and* $(\mathcal{P}, \circ, D, R, \leq)$ *be its corresponding ordered groupoid. Let* $\exists x \cdot y \in \mathcal{P}$*. Then* $D((x \cdot y)') = D(y'|R(x))$ *where* $y'|R(x)$ *is as defined in* $(\mathcal{P}, \circ, D, R, \leq)$*. Furthermore,* $\exists D((x \cdot y)') \cdot D(y')$*.*

Proof. *Recall* $x \cdot y = x \circ R(x)|y$ *so* $(x \cdot y)' = (R(x)|y)' \circ x'$ *and:*

$$D((x \cdot y)') = D((R(x)|y)' \circ x') = D((R(x)|y)') = D(y'|R(x)).$$

By $(OG4)$ *we have* $y'|R(x) \leq y'$. *So* $D(y'|R(x)) \leq D(y')$ *but* $\leq$ *is the natural partial order with respect to* $\cdot$ *by* ***Result 2.31***.

Therefore, $\exists D(y'|R(x)) \cdot D(y')$, *i.e.* $\exists D((x \cdot y)') \cdot D(y')$, *as required.* □

The conditions for products under $\cdot$ to exist then rely solely on the partial order on the ordered groupoid. Particularly, it is determined by when the restriction itself exists. Furthermore, since $\leq$ satisfies $(I1)$-$(I3)$ by definition, **Result 2.21** implies $\leq$ is always the natural partial order on the D-inverse constellation $(\mathcal{P}, \cdot, D)$. While never mentioned in Chapters 5-8, it is precisely this method that was employed to find the equivalent D-inverse constellations to formerly known ordered groupoids. The reader will see the perhaps unnecessary proofs where the resultant D-inverse constellation is shown to actually be a D-inverse constellation. These proofs are carried out to illustrate the ease of the procedure assuming we hadn't started with an ordered groupoid.

A final important aspect of partial algebras commonly considered are **congruences** and **quotients**. To save repetition for ordered groupoids then D-inverse constellations, it makes sense to define congruences and quotients on partial algebras as a whole. As such, the notation will resemble that given in [5].

Definition 2.33. *Let* $(\mathcal{A}, F)$ *be a partial algebra with* F *a set of all operations on* $\mathcal{A}$ *and* ρ *an equivalence relation on* $\mathcal{A}$. *Then* ρ *is a* ***congruence*** *on* $\mathbf{G}$ *if the following holds* $\forall f \in F$ *and* $a_j, b_j \in \mathcal{A}, j \in \mathbb{N}$*:*

(CON) if $a_j \rho b_j, \forall j \leq \sigma(f)$, $\exists f(a_1, a_2, ..., a_{\sigma(f)})$ *and* $\exists f(b_1, b_2, ..., b_{\sigma(f)})$, *then:*

$$f(a_1, a_2, ..., a_{\sigma(f)}) \rho f(b_1, b_2, ..., b_{\sigma(f)}).$$

The ***Quotient*** *of* $\mathcal{A}$ *with respect to* ρ, $(\mathcal{A}/\boldsymbol{\rho}, F)$ *is then given by the set:*

$$\mathcal{A}/\boldsymbol{\rho} = \{[x] \in \mathcal{A} : x \rho y \Rightarrow [x] = [y]\}.$$

Any operation $f \in F$ *is defined to satisfy* $\forall [a]_j \in \mathcal{A}/\boldsymbol{\rho}$ *where* $\forall j \leq \sigma(f)$*:*

(Q) $\exists f([a]_1, [a]_2, ..., [a]_{\sigma(f)}) \iff \exists a_j \in [a]_j$ *such that* $\exists f(a_1, a_2, ..., a_{\sigma(f)}) \in \mathcal{A}$.

For partial algebras, it is not true all quotients will form another of that same type of partial algebra. However, Lawson found certain kinds of congruences on groupoids that will always form a quotient groupoid. Note my use of x^* does not refer to x under an involution operator. The element x^* is merely another element which in most cases is defined to be congruent to x.

Result 2.34. *(Lemma 4.3.3 of [6]): Let ρ be a congruence on a groupoid, $\mathbf{G}$, satisfying $\forall x, y \in \mathbf{G}$:*

(OCON) $R(x)\rho D(y) \to \exists x^*, y^* \in \mathbf{G}$ *such that* $x\rho x^*$, $y\rho y^*$ *and* $R(x^*) = D(y^*)$;

(AC) $\exists [x] \circ ([y] \circ [z]) \iff \exists ([x] \circ [y]) \circ [z]$ *and they are equal.*

Then $\mathbf{G}/\rho$ is a groupoid.

Below is another property Lawson investigated in [6]. Let $\mathbf{G}$ be a groupoid and ρ be a congruence on $\mathbf{G}$. Then $\forall x, y \in \mathbf{G}$:

(QCON) $R(x)\rho D(y) \implies \exists x^* \in \mathbf{G}$ such that $x\rho x^*$ and $R(x^*) = D(y)$.

While it is not explicitly written as such, this is equivalent to the **right-strong** property of congruences for groupoids, alternatively expressed as:

(QCON*) $\exists [x] \circ [y] \implies \exists x^* \in [x]$ such that $\exists x^* \circ y$.

On the other hand we would say the congruence is **left-strong** if $\exists [x] \circ [y] \implies \exists y^* \in [y]$ such that $\exists x \circ y^*$. If a congruence is both left-strong and right-strong, we simply say it is a **strong** congruence. In this work, we shall use these definitions for congruences on constellations with respect to the $\cdot$ operation.

Leading us to the following result by Lawson.

Result 2.35. *(Lemma 4.3.7 of [6]): Let $\mathbf{G}$ be a groupoid and ρ be a congruence on $\mathbf{G}$. If ρ satisfies $(QCON)$, then ρ satisfies $(OCON)$ and (AC).*

Corollary 2.36. *Let $\mathbf{G}$ be a groupoid and ρ be a congruence on $\mathbf{G}$ satisfying $(QCON)$. Then $\mathbf{G}/\rho$ is a groupoid.*

So congruences on groupoids that are right strong form quotient groupoids. Similar to this we have.

Result 2.37. *(Proposition 2.24 of [2]): Let $\mathcal{P}$ be a constellation and ρ a right strong congruence on $\mathcal{P}$. Then $\mathcal{P}/\rho$ is a constellation.*

Lemma 2.38. *Let $\mathbf{G}$ be an ordered groupoid and ρ be a congruence on $\mathbf{G}$ such that $\mathbf{G}/\rho$ is an ordered groupoid. Let the partial order $\leq$ on $\mathbf{G}$ be described by: $[x] \leq [y]$ if and only if for each $y^* \in [y]$ then $\exists x^* \in [x]$ such that $x^* \leq y^*$. Let $\mathcal{P}$ be the D-inverse constellation corresponding to $\mathbf{G}$. Then $\mathcal{P}/\rho$ is a D-inverse constellation and is the corresponding D-inverse constellation to $\mathbf{G}/\rho$.*

Proof. *By applying the method for building an equivalent D-inverse constellation from the ordered groupoid $\mathbf{G}/\rho$, we obtain $(\mathbf{G}/\boldsymbol{\rho}, \star)$ is a D-inverse constellation.*

We must confirm this D-inverse constellation is indeed the quotient D-inverse constellation of $\mathcal{P}$ with respect to ρ. In other words, $\forall [x], [y] \in \mathbf{G}/\rho$:

$$[x] \cdot [y] = [x] \star [y] = [x] \circ [R(x)] | [y].$$

So let $\exists [R(x)] | [y]$ for some $[x], [y] \in \mathbf{G}/\rho$. Then $[R(x)] \leq [D(y)]$ meaning for each $D(z) \in [D(y)], \exists w \in [R(x)]$ such that $w \leq D(z)$.

Therefore w is an identity element such that $R(w) \leq D(z)$.

With this fact, we know $\exists R(w)|z$.

Observe there is at least one $z^* \in [y]$ such that $D(z^*) = D(z) \in [D(y)]$ and at least one $w^* \in [x]$ such that $R(w^*) = w \in [R(x)]$.

Finally we have $\exists [x] \cdot [y]$ because $\exists w^* \in [x]$ and $\exists z^* \in [y]$ such that $\exists w^* \circ (R(w^*)|z^*)$.

In other words, $\exists [x] \cdot [y]$ because $\exists w^* \in [x]$ and $\exists z^* \in [y]$ such that $\exists w^* \cdot z^*$. □

3 Inductive D-inverse constellations

We recall the inductive property introduced in [6] and use it to define the key partial algebra of this chapter.

Definition 3.1. *Let $\mathcal{P}$ be a D-inverse constellation. Then $\mathcal{P}$ is an **inductive D-inverse constellation** if its projections form a meet-semilattice under the natural partial order.*

The purpose of this chapter is to establish a correspondence between the category of inductive D-inverse constellations with the category of inverse semigroups. It will also highlight some interesting properties that may prove these partial algebras worthy of further investigation. Explicitly, the initial plan is to show every inductive D-inverse constellation is an inductive constellation (as defined in preliminaries) and utilize **Result 2.25** in a clever way. As a consequence, the reader will see how this provides an alternate proof of the ESN-theorem. To begin with, a rather simple lemma.

Lemma 3.2. *Let $\mathcal{P}$ be a D-inverse constellation with natural partial order $\leq$. Then if $x \in \mathcal{P}$ and $e, f \in D(\mathcal{P})$, we have:*

$$\exists x \cdot e \text{ and } \exists x \cdot f \iff \exists x \cdot (e \wedge f).$$

Proof. *Let $x \in \mathcal{P}$ and $e, f \in D(\mathcal{P})$ and suppose $\exists x \cdot e$ and $\exists x \cdot f$. Then:*

*$D(x') \leq e$ and $D(x') \leq f$, by **Lemma 2.29**;*

so $D(x') \leq e \wedge f$;

so $\exists D(x') \cdot (e \wedge f)$ by NPO and it is equal to $(x' \cdot x) \cdot (e \wedge f)$. Furthermore:

$\exists (x \cdot (x' \cdot x)) \cdot (e \wedge f)$, by $(CONST1)$ and $(CONST2)$;

$= ((x \cdot x') \cdot x)) \cdot (e \wedge f)$ by $(CONST1)$;

$= (D(x) \cdot x) \cdot (e \wedge f) = x \cdot (e \wedge f)$, as required.

Conversely, suppose $\exists x \cdot (e \wedge f)$. Then:

*$D(x') \leq e \wedge f$ by **Lemma 2.29**, implying:*

$D(x') \leq e$ and $D(x') \leq f$. It follows $\exists x \cdot e$ and $\exists x \cdot f$, as required. □

This lemma is indispensable when relating inductive D-inverse constellations to the properties exhibited in inductive constellations, for example.

Proposition 3.3. *Let $\mathcal{P}$ be an inductive D-inverse constellation with $x, y \in \mathcal{P}$ and $e \in D(\mathcal{P})$. Define $x \star e = D(((e \wedge D(x') \cdot x'))')$. Then:*

$$\exists (y \cdot x) \cdot e \iff \exists y \cdot (x \star e).$$

Proof. *First suppose* $\exists(y \cdot x) \cdot e$, *notice also* $\exists(y \cdot x) \cdot D(x')$ *as well. Then we have:*

$\exists(y \cdot x) \cdot (e \wedge D(x'))$ *(By **Lemma 3.2**);*

By $e \wedge D(x') \leq D(x')$, $(CONST2)$ *and **Result 2.17**, we have* $\exists(y \cdot x) \cdot (e \wedge D(x'))$ *and* $\exists e \wedge D(x')) \cdot x'$. *This implies:*

$\exists(y \cdot x) \cdot ((e \wedge D(x')) \cdot x')$. *Then since* $\exists((e \wedge D(x')) \cdot x') \cdot (e \wedge D(x') \cdot x')'$, *we have:*

$\exists((y \cdot x) \cdot ((e \wedge D(x')) \cdot x')) \cdot (e \wedge D(x') \cdot x')'$ *by* $(CONST1)$ *and* $(CONST2)$;

$\exists((y \cdot x) \cdot ((e \wedge D(x')) \cdot x')) \cdot (x \star e)$ *by **Result 2.17**;*

$\exists(((y \cdot x) \cdot (e \wedge D(x'))) \cdot x') \cdot (x \star e)$ *by* $(CONST1)$;

$\exists((y \cdot x) \cdot x') \cdot (x \star e)$ *by* $(CONST4)$;

$\exists(y \cdot (x \cdot x')) \cdot (x \star e)$ *by* $(CONST2)$ *since* $\exists y \cdot x$ *and* $\exists x \cdot x'$;

$= (y \cdot D(x)) \cdot (x \star e)$;

so $\exists y \cdot (x \star e)$ *by* $(CONST4)$, *as required.*

Now instead suppose $\exists y \cdot (x \star e)$. *Observe:*

$x \star e = ((e \wedge D(x')) \cdot x')' \cdot ((e \wedge D(x')) \cdot x')$;

$= (((e \wedge D(x)) \cdot x')' \cdot (e \wedge D(x'))) \cdot x'$, *by* $(CONST1)$.

$(NOTE1)$: $\exists((e \wedge D(x)) \cdot x')' \cdot (e \wedge D(x'))$;

$= ((e \wedge D(x')) \cdot x')' \cdot x'$, *by* $(CONST4)$.

$(NOTE2)$: $x \star e = ((e \wedge D(x')) \cdot x')' \cdot x'$;

$\exists((e \wedge D(x')) \cdot x')' \cdot (x' \cdot x)$ *by* $(CONST2)$.

$(NOTE3)$: *by* $D(x') = x' \cdot x$ *and* $(CONST4)$ *this is also equal to* $((e \wedge D(x')) \cdot x')'$;

$\exists(((e \wedge D(x')) \cdot x')' \cdot x') \cdot x$ *by* $(CONST1)$;

$= (x \star e) \cdot x$ *by* $(NOTE2)$.

Recall $(NOTE1)$: $\exists((e \wedge D(x)) \cdot x')' \cdot (e \wedge D(x'))$. *Then:*

$\exists(e \wedge D(x)) \cdot x')' \cdot e$ *by **Lemma 3.2**;*

$= ((x \star e) \cdot x) \cdot e$ *by* $(NOTE3)$.

Furthermore, $\exists y \cdot (x \star e)$;

$\exists y \cdot (((x \star e) \cdot x) \cdot e)$ *by* $(CONST2)$;

$\exists(y \cdot (((x \star e) \cdot x))) \cdot e$ *by* $(CONST1)$;

$\exists((y \cdot (x \star e)) \cdot x) \cdot e$ *by* $(CONST1)$;

so $\exists(y \cdot x) \cdot e$ *by* $(CONST4)$, *as required.* □

It follows from **Result 2.23** and **Proposition 3.3** that $(I4)$ and $(I5)$ hold in any inductive D-inverse constellation. The general formula for corestrictions as defined rather indirectly in $(I4)$ can be obtained.

Corollary 3.4. *Let* $\mathcal{P}$ *be an inductive D-inverse constellation with* $x \in \mathcal{P}$ *and* $e \in D(\mathcal{P})$. *Then the **corestriction of e on x** is given by* $x|e = (e \wedge D(x') \cdot x')'$ *That is,* $x|e$ *is the greatest element with respect to the natural partial order on* $\mathcal{P}$ *such that* $\exists(x|e) \cdot e$ *and* $x|e \leq x$.

With every piece set in place, it is time to prove the main theorem of this chapter.

Theorem 3.5. *Let $\mathcal{P}$ be an inductive D-inverse constellation. Then $\mathcal{P}$ is an inductive constellation. Furthermore, the category of inductive D-inverse constellations (with radiants) is isomorphic to the category of inverse semigroups (with semigroup homomorphisms).*

Proof. *We begin by showing that $\mathcal{P}$ is an inductive constellation. Take the natural partial order on $\mathcal{P}$ to be the partial order $\leq$. Then $\leq$ immediately satisfies $(I1)$-$(I3)$. $(I4)$ and $(I5)$ follow from **Proposition 3.3**, so it remains to show $(I6)$ and $(I7)$ hold.*

(I6): *Let $e, f \in D(\mathcal{P})$ with $e \leq f$. Then $\exists e|f$ and $e|f = e \cdot f = e$ by **Result 2.21** and $(CONST4)$. By **Corollary 3.4**:*

$$\begin{aligned}
f|e &= ((e \wedge D(f')) \cdot f')'; \\
&= ((e \wedge f') \cdot f')'; \\
&= ((e \wedge f) \cdot f)' \text{ by } f' = f; \\
&= (e \wedge f)', \text{ by } (CONST4); \\
&= e \wedge f; \\
&= e, \text{ since } e \leq f; \\
&= e|f, \text{ as required.}
\end{aligned}$$

(I7): *Follows from the proof of $(I6)$ $(f|e = e \wedge f)$ and commutativity of $\wedge$.*

Therefore, $\mathcal{P}$ is an inductive constellation.

*Recall **Result 2.25**. Then (with their respective morphisms) the category of inductive D-inverse constellations is isomorphic to the category of certain left-restriction semigroups. These are in fact the left-restriction semigroups with unique D-inverses, so by **Corollary 2.6**, we have shown what was required.* □

The significance of the above theorem stems from the implications of the ESN-Theorem. That is, we can construct an inverse semigroup from any inductive D-inverse constellation and vise versa. We key importance is that we can translate between the two without any loss of information.

The question remains, how does one translate between the two? Fortunately, it is exactly as described in [1] when translating between inductive constellations and left-restriction semigroups. Given an inverse semigroup **S**, define the D-inverse of an element to be the same as its inverse in **S** and define $\cdot$ on **S** such that $\forall x, y \in \mathbf{S}$.

$$\exists x \cdot y \iff x = xyy' \text{ and then } x \cdot y = xy.$$

The reverse is less simple but likewise, D-inverses remain as inverses in the constructed inverse semigroup. This construction makes use of **Corollary 3.4** to replace corestrictions of elements in a usual inductive constellation with its explicit formula obtained in inductive D-inverse constellations. See below.

Corollary 3.6. *Let $\mathcal{P}$ be an inductive D-inverse constellation and define $\otimes$ on $\mathcal{P}$ such that $\forall x, y \in \mathcal{P}$:*

$$x \otimes y = ((D(y) \wedge D(x')) \cdot x')' \cdot y.$$

Then $(\mathcal{P}, \otimes)$ is an inverse semigroup.

Most properties of inverse semigroup theory can in fact be studied without the inductive property in the corresponding ordered groupoid so D-inverse constellations may be employed in the same way. We then see as a consequence of the inductive property, it follows that.

Corollary 3.7. *The category of inductive D-inverse constellations (with radiants) is isomorphic to the category of inductive groupoids (with inductive functors).*

Therefore, (ESN-Theorem): the category of inductive groupoids (with inductive functors) is isomorphic to the category of inverse semigroups (with semigroup homomorphisms).

A final note on inductive D-inverse constellations that may be of interest though is not investigated further. It is possible to find D-inverses of products. For brevity, the following makes use of the definition of corestrictions as expressed in **Corollary 3.4**, though of course there is no need to since we have a formula for such elements.

Lemma 3.8. *Let $\mathcal{P}$ be an inductive D-inverse constellation with $x, y \in \mathcal{P}$. Then if $\exists x \cdot y$:*

$$(x \cdot y)' = (y'|D(x')) \cdot x'.$$

Proof. *If $\exists x \cdot y$, then by $(CONST1)$ and $(CONST2)$, $\exists (x' \cdot x) \cdot y = D(x') \cdot y$. We also have that:*

$x = D(x) \cdot x = (x \cdot x') \cdot x = x \cdot (x' \cdot x) = x \cdot D(x')$ By $(CONST1)$, $(CONST2)$ and $(CONST3)$;

$x \cdot y = (x \cdot D(x')) \cdot y = x \cdot (D(x') \cdot y)$ By $(CONST2)$.

Therefore, $(x \cdot y)' = (x \cdot (D(x') \cdot y))'$.

*It follows from **Result 2.30** if $\exists((D(x') \cdot y))' \cdot x'$, then it is equal to $(x \cdot y)'$.*

*Observe $(D(x') \cdot y)' = ((D(x') \wedge D(y)) \cdot y)'$ by **Lemma 2.29** and is equal to $y'|D(x')$ by **Corollary 3.4**.*

*Similarly, by **Corollary 3.4**, $\exists (y'|D(x')) \cdot D(x')$, and by **Result 2.17**, it follows that: $\exists (y'|D(x')) \cdot x'$, as required.* □

4 Identity Separating Congruences

This chapter will be brief but explores a more general relationship to ordered groupoids and inverse semigroups. The motivation behind this chapter lies in [6]. However, so far the notions of so-called **ordered congruences** and **special functors** do not seem to have elegant descriptions in terms of D-inverse constellations.

That said, there have been some discoveries about **identity-separating (or IS) ordered congruences** on ordered groupoids. Therefore, **(IS)-congruences** on D-inverse constellations should also be investigated. Before that, a definition.

Definition 4.1. *Let* $\mathbf{G}$ *be an ordered groupoid and* ρ *a congruence on* $\mathbf{G}$. *Then* ρ *is an* ***(IS)-ordered congruence*** *if it satisfies:*

(IS) if e *and* f *are identities and* $e\rho f$*, then* $e = f$*;*

(OC) if $x, y, y^* \in \mathbf{G}$*,* $x \leq y$ *and* $y\rho y^*$*, then* $\exists x^* \in \mathbf{G}$ *such that* $x\rho x^*$ *and* $x^* \leq y^*$*.*

Note this is an alternate definition to that given in [6], but was shown to be equivalent by Lawson in the same text. An (IS)-congruence on a D-inverse constellation is then a congruence satisfying (IS) and $x\rho y \Rightarrow D(x') = D(y')$.

Lemma 4.2. *Let* $\mathcal{P}$ *be a D-inverse constellation with (IS)-congruence* ρ*. Then* $\forall x, y \in \mathcal{P}$*:*

$$x\rho y \iff x'\rho y'$$

Proof. *It suffices to show only one direction since the other follows by symmetry. Let* $x, y \in \mathcal{P}$ *and* $x\rho y$*. Then* $D(x') = D(y')$ *and* $D(x) = D(y)$ *so* $\exists x' \cdot y, \exists y' \cdot x$ *by* ***Lemma 2.29****. Therefore:*

$y'\rho y'$, $x\rho y$, $\exists y' \cdot x$ and $y' \cdot y$ imply $y' \cdot x\rho y' \cdot y = D(y')$.

By symmetry $x' \cdot y\rho D(x')$.

But $D(x') = D(y')$ so $D(x')\rho y' \cdot x$. Also with $x'\rho x'$, $\exists D(x') \cdot x$ and $\exists (y' \cdot x) \cdot x'$, we have: $D(x') \cdot x'\rho(y' \cdot x) \cdot x'$. In other words:

$x'\rho y' \cdot (x \cdot x') = y' \cdot D(x) = y'$ by $(CONST2)$, $(CONST3)$ and $(CONST4)$.

We have $x'\rho y'$, as required. □

Proposition 4.3. *Let* $(\mathbf{G}, \circ, \leq, D, R)$ *be an ordered groupoid. Then* ρ *is an (IS)-ordered congruence on* $(\mathbf{G}, \circ, \leq, D, R)$ *precisely when* ρ *is an (IS)-congruence on* $(\mathbf{G}, \cdot, D)$*.*

Proof. *First suppose* ρ *is an (IS)-ordered congruence. Then it suffices to show it is a D-inverse constellation congruence. We first show* ρ *is a left-congruence on* $(\mathbf{G}, \cdot, D)$*.*

Let $x, y, z \in \mathbf{G}$ *with* $y\rho z$*,* $\exists x \cdot y$ *and* $\exists x \cdot z$*. Then we want to show:*

$$x \cdot y\rho x \cdot z, \text{ i.e. } x \circ R(x)|y\rho x \circ R(x)|z.$$

This holds if $R(x)|y \ \rho \ R(x)|z$. Observe by (OC), $y\rho z$ and $R(x)|y \le y$ implies $\exists z^* \in \mathbf{G}$ such that $z^* \le z$ and:

$$z^* \rho R(x)|y \text{ implies } D(z^*)\rho D(R(x)|y) = R(x) \text{ so } D(z^*) = R(x).$$

But these are precisely the properties of the restriction $R(x)|z$ so $z^* = R(x)|z$ by uniqueness and $R(x)|y\rho R(x)|z$, as required.

Now we show ρ is a right-congruence of $(\mathbf{G}, \cdot, D)$, let $x, y, z \in \mathbf{G}$ such that $x\rho y$, $\exists x \cdot z$ and $\exists y \cdot z$. Then $R(x)\rho R(y)$ implies $R(x) = R(y)$ so $R(x)|z = R(y)|z$ and:

$$x \cdot z = x \circ R(x)|z\rho y \circ R(y)|z = y \cdot z, \text{ as required.}$$

$x\rho y \Rightarrow x'\rho y'$ since ρ is an ordered congruence hence $D(x') = D(y')$. So ρ is an (IS)-congruence on $(\mathbf{G}, \cdot, D)$.

Now instead suppose ρ is an (IS)-congruence on the D-inverse constellation $(\mathbf{G}, \cdot, D)$. Then clearly ρ is an (IS)-congruence on $(\mathbf{G}, \circ, \le, D, R)$ since $x \circ y = x \cdot y$ when $D(x') = D(y)$. It remains to show (OC) holds.

Let $x, y, y^* \in \mathbf{G}$ such that $y\rho y^*$ and $x = D(x) \cdot y$ (i.e. $x \le y$). Then $y\rho y^*$ implies $D(y)\rho D(y^*)$ so $D(y) = D(y^*)$ by (IS). Therefore:

$x = D(x) \cdot y$;
so $x \cdot y' = D(x) \cdot (y \cdot y'))$ by $(CONST2)$;
$= D(x) \cdot D(y)$;
$= D(x) \cdot D(y^*)$;
$= (D(x) \cdot y^*) \cdot (y^*)'$ by $(CONST1)$;
so $\exists x^*$ such that $x^* = D(x) \cdot y^*$.
So $x^* = D(x^*) \cdot y^*$ by definition of NPO.

Therefore since $\exists D(x) \cdot y$, $\exists D(x) \cdot y^*$ and $y\rho y^*$. It follows $D(x) \cdot y\rho D(x) \cdot y^*$, in other words $x\rho x^*$, as required.

So ρ is an (IS)-ordered congruence. □

We proceed to show an analogous proof of a proposition by Lawson, but taking the inductive D-inverse constellation approach. Note that the operation $\otimes$ is as defined in the preliminaries translating from inductive constellation to left-restriction semigroups, but by **Theorem 3.5** we get an inverse semigroup instead.

Proposition 4.4. *Let $\mathcal{P}$ be an inductive D-inverse constellation. Then ρ an (IS)-congruence precisely when ρ is an idempotent-separating congruence on $(\mathcal{P}, \otimes)$.*

Proof. *First suppose ρ is an idempotent-separating congruence on $(\mathcal{P}, \otimes)$. Then the idempotent elements of $(\mathcal{P}, \cdot, D)$ are simply projections so (IS) is satisfied.*

All products and inverses of $(\mathcal{P}, \cdot, D)$ can be written in terms of certain products and inverses respectfully in $(\mathcal{P}, \otimes)$. So it follows ρ is an (IS)-congruence on $(\mathcal{P}, \cdot, D)$.

Now suppose ρ is an (IS)-congruence on $(\mathcal{P}, \cdot, D)$. Then we show products in $(\mathcal{P}, \otimes)$ of congruent elements are congruent. Let $w, x, y, z \in \mathcal{P}$ with $x\rho w$ and $y\rho z$. We aim to show:

$$x \otimes y \rho w \otimes z, \quad i.e\ x|D(y) \cdot y \rho w|D(z) \cdot z.$$

It suffices to show $x|D(y) = ((D(y) \wedge D(x')) \cdot x')' \rho w|D(z) = ((D(z) \wedge D(w')) \cdot w')'$. *Since* $x \rho w$ *we have* $D(x') = D(w')$ *and* $y \rho z$ *implies* $D(y) = D(z)$*. Therefore:*

$$\begin{aligned} D(y) \wedge D(x') &= D(z) \wedge D(w') \text{ so by \textbf{Lemma 3.2}, we have:} \\ & (D(y) \wedge D(x')) \cdot x' \ \rho \ (D(z) \wedge D(w')) \cdot w'. \end{aligned}$$

So $((D(y) \wedge D(x')) \cdot x')' \ \rho \ ((D(z) \wedge D(w')) \cdot w')'$ *by* ***Lemma 4.2****, as required.* □

With the simple definition of (IS)-congruences on D-inverse constellations along with the equivalence of their somewhat less simple (IS)-ordered congruence counterparts, we gain another reason to adopt the D-inverse constellations approach.

On the other hand, an elegant D-inverse constellation equivalent of ordered congruences and special functors of ordered groupoids has yet to be discovered. Until such a description is found, further investigation may have to be carried out with the ordered groupoid perspective. So despite Lawson's further observations of these concepts, these will not be outlined in this work.

5 D-inverse constellations and Right-cancellative Categories

In [7], Lawson was able to construct ordered groupoids from left-cancellative categories and vice versa. It is important for the reader to note however, the conventions carried out by Lawson and those used in this work tend to be reversed. For example, the product $a \circ b$ is read "first b, then a" in [6], while in this work it is read "first a, then b". Therefore, the left-cancellative categories Lawson utilizes and the right-cancellative categories utilized in this work are one and the same.

§5.1 D-inverse constellations from Right-Cancellative Categories

This section contains the bulk of this chapter and aims to construct D-inverse constellations from right-cancellative categories as Lawson did with ordered groupoids from left-cancellative categories.

Let $(\mathbf{C}, \circ)$ be a right-cancellative category in the sense that $\forall a, b, c \in \mathbf{C}, a \circ c = b \circ c \Rightarrow a = b$. Then from $\mathbf{C} \times \mathbf{C}$, define:

$$\mathcal{U} = \{(a, b) \in \mathbf{C} \times \mathbf{C} : D(a) = D(b)\}.$$

On $\mathcal{U}$ define $D((a, b)) = (a, a)$ and a partial binary operation $\cdot$ such that $\forall (a, b), (c, d) \in \mathcal{U}$:

$$(a, b) \cdot (c, d) = (a, p \circ d) \iff \exists p \in \mathbf{C} \text{ such that } b = p \circ c.$$

Here $\exists p \circ d$ since:

$$\exists p \circ c \iff \exists p \circ D(c) = p \circ D(d) \text{ and } \exists p \circ D(d) \iff \exists p \circ d.$$

Proposition 5.1. *Let $\mathcal{U}$ be the set of ordered pairs as described above and let $\theta : \mathbf{C} \to \mathbf{D}$ be a functor where defining $\Theta : \mathcal{U} \to \mathcal{V}$ such that $(a, b) \in \mathcal{U}$ gives $(a, b)\Theta = (a\theta, b\theta)$. Then $(\mathcal{U}, \cdot, D)$ is a constellation and Θ is a radiant on $\mathcal{U}$.*

Proof. *First we must show axioms $(CONST1)$-$(CONST4)$ are satisfied.*

(CONST1): *Let $(a, b), (c, d), (s, t) \in \mathcal{U}$ with $\exists (a, b) \cdot [(c, d) \cdot (s, t)]$. Then:*

$$\begin{aligned}
(a, b) \cdot [(c, d) \cdot (s, t)] &= (a, b) \cdot (c, p \circ t), \text{ where } d = p \circ s; \\
&= (a, q \circ (p \circ t)), \text{ where } b = q \circ c; \\
&= (a, (q \circ p) \circ t) \text{ by } (CAT1); \\
&= (a, q \circ d) \cdot (s, t), \text{ since } q \circ d = q \circ (p \circ s) = (q \circ p) \circ s \text{ by } (CAT1); \\
&= [(a, b) \cdot (c, d)] \cdot (s, t), \text{ since } b = q \circ c, \text{ as required.}
\end{aligned}$$

(CONST2): *Let $(a, b), (c, d), (s, t) \in \mathcal{U}$ with $\exists (a, b) \cdot (c, d)$ and $\exists (c, d) \cdot (s, t)$. Then $(a, b) \cdot (c, d) = (a, p \circ d)$ where $b = p \circ c$ and also $(c, d) \cdot (s, t) = (c, q \circ t)$ where $d = q \circ t$.*

So $\exists (a, b) \cdot (c, q \circ t)$, since p satisfies $b = p \circ c$;

$= (a,b)\cdot[(c,d)\cdot(s,t)]$, *as required.*

(CONST3): *$D(a)$ is such an element of $\mathbf{C}$ where $a = D(a)\circ a$. Noting for $(a,b)\in\mathcal{U}$ we also have $D((a,b)) = (a,a)$ hence:*

$$\begin{aligned}\exists(a,a)\cdot(a,b) &= (a, D(a)\circ b);\\ &= (a, D(b)\circ b);\\ &= (a,b),\ \textit{as required.}\end{aligned}$$

To show uniqueness, suppose $\exists D((c,d)) = (c,c)$ such that $\exists(c,c)\cdot(a,b) = (a,b)$. Then $\exists p\in\mathbf{C}$ such that $c = p\circ a$ and $(c,c)\cdot(a,b) = (c, p\circ b)$ implying $a = c$ so $D((a,b)) = D((c,d))$.

Therefore, $(CONST3)$ is satisfied.

(CONST4): *Suppose for $(a,b), D((c,d))\in\mathcal{U}$, $\exists(a,b)\cdot D((c,d)) = (a,b)\cdot(c,c)$. Then $\exists p\in\mathbf{C}$ such that $b = p\circ c$ and:*

$$\begin{aligned}(a,b)\cdot(c,c) &= (a, p\circ c);\\ &= (a,b),\ \textit{as required.}\end{aligned}$$

So $(\mathcal{U},\cdot,D)$ is a constellation.

It remains to show Θ is a radiant on $\mathcal{U}$.

(R1): *Let $(a,b),(c.d)\in\mathcal{U}$ and $\exists(a,b)\cdot(c,d)$. Then $b = p\circ c$ for some $p\in\mathbf{C}$ and $(a,b)\cdot(c,d) = (a, p\circ d)$. Observe:*

$$b = p\circ c \textit{ implies } b\theta = (p\circ c)\theta = p\theta\circ c\theta \textit{ (by F1), so:}$$

$$\begin{aligned}((a,b)\cdot(c,d))\Theta &= (a, p\circ d)\Theta;\\ &= (a\theta, (p\circ d)\theta);\\ &= (a\theta, p\theta\circ d\theta)\ \textit{by } (F1);\\ &= (a\theta, b\theta)\cdot(c\theta, d\theta)\ \textit{by } b\theta = p\theta\circ c\theta;\\ &= (a,b)\Theta\cdot(c,d)\Theta,\ \textit{as required.}\end{aligned}$$

(R2): *Let $(a,b)\in\mathcal{U}$ then:*

$D((a,b))\Theta = (a,a)\Theta = (a\theta, a\theta)$ and:

$D((a,b)\Theta) = D((a\theta, b\theta)) = (a\theta, a\theta)$, as required.

Thus Θ is a radiant on $\mathcal{U}$. □

Define $u(a^*,b^*) = (u\circ a^*, u\circ b^*)$. Let us introduce the binary relation δ on $\mathcal{U}$ where $\forall(a,b),(a^*,b^*)\in\mathcal{U}$:

$$(a,b)\delta(a^*,b^*) \iff (a,b) = u(a^*,b^*) \text{ where } u \text{ is an isomorphism in } \mathbf{C}.$$

Proposition 5.2. *The relation δ defined as above is a congruence relation defined on $\mathcal{U}$.*

Proof. *The first task is to show δ is an equivalence relation.*

(Reflexive): *$\forall(a,b) \in \mathcal{U}$, $D(a)$ is such an element where:*

$D(a)(a,b) = (D(a) \circ a, D(a) \circ b) = (a, D(b) \circ b) = (a,b)$ *implying* $(a,b)\delta(a,b)$*, as required.*

(Symmetric): *Suppose for $(a,b),(c,d) \in \mathcal{U}$ that:*

$$(a,b)\delta(c,d). \text{ Then } (a,b) = u(c,d) \text{ for some isomorphism } u \in \mathbf{C}.$$

Then $(c,d) = u^{-1}(a,b)$ implies $(c,d)\delta(a,b)$, as required.

(Transitive): *Let $(a,b),(c,d),(s,t) \in \mathcal{U}$ with $(a,b)\delta(c,d)$ and $(c,d)\delta(s,t)$. Then $(c,d) = u(s,t)$ for some isomorphism $u \in \mathbf{C}$ and similarly:*

$$\begin{aligned}
(a,b) &= v(c,d) \text{ for some isomorphism } v \in \mathbf{C};\\
&= v(u(s,t));\\
&= v(u \circ s, u \circ t);\\
&= (v \circ (u \circ s), v \circ (u \circ t));\\
&= ((v \circ u) \circ s, (v \circ u) \circ t);\\
&= (v \circ u)(s,t).
\end{aligned}$$

Therefore since $v \circ u$ is an isomorphism, we have$(a,b)\delta(s,t)$, as required. So δ is an equivalence relation.

(Con $\cdot$): *Let $(a,b),(a^*,b^*),(c,d),(c^*,d^*) \in \mathcal{U}$ with $(a,b)\delta(a^*,b^*)$ and $(c,d)\delta(c^*,d^*)$. Furthermore let $\exists(a,b)\cdot(c,d)$ and $\exists(a^*,b^*)\cdot(c^*,d^*)$ with $p,q \in \mathbf{C}$. Then:*

$$\begin{aligned}
(a,b)\cdot(c,d) &= (a, p \circ d) \text{ for } b = p \circ c;\\
(a^*,b^*)\cdot(c^*,d^*) &= (a^*, q \circ d^*) \text{ for } b^* = q \circ c^*.
\end{aligned}$$

Suppose $u,v \in \mathbf{C}$ are isomorphisms relating (a,b) to (a^*,b^*) and (c,d) to (c^*,d^*) respectively. Then $a = u \circ a^*$, so it suffices to show $p \circ d = u \circ (q \circ d^*)$ and it follows $(a,b)\cdot(c,d)\delta(a^*,b^*)\cdot(c^*,d^*)$. Observe:

$b = p \circ c = p \circ (v \circ c^*) = (p \circ v) \circ d^*$. Also:

$b = u \circ b^* = u \circ (q \circ d^*) = (u \circ q) \circ d^*$;

$p \circ v = u \circ q$ (by right cancellative property). Therefore:

$$\begin{aligned}
p \circ d &= p \circ (v \circ d^*);\\
&= (p \circ v) \circ d^*;\\
&= (u \circ q) \circ d^*;\\
&= u \circ (q \circ d^*) \text{ by } (CAT1), \text{ as required.}
\end{aligned}$$

(Con D): *Let $(a,b),(c,d) \in \mathcal{U}$ with $(a,b)\delta(c,d)$. Then $a = u \circ c$ for some isomorphism $u \in \mathbf{C}$ and:*

$$D((a,b)) = (a,a) = u(c,c) = uD((c,d)) \text{ implies } D((a,b))\delta D((c,d)), \text{ as required.}$$

So δ is a congruence relation. □

Proposition 5.3. *Let $(\mathcal{U}, \cdot, D)$ be the constellation constructed as in* ***Proposition 5.1*** *and δ is the congruence on $\mathcal{U}$ as in* ***Proposition 5.2****. Then δ is a strong congruence.*

Proof. *Let $(a,b),(c,d) \in \mathcal{U}$ such that $\exists (a,b)\cdot(c,d)$. Then we must show $\forall (a^*,b^*),(c^*,d^*) \in \mathcal{U}$ congruent to (a,b) and (c,d) respectively that $\exists (a^*,b^*)\cdot(c^*,d^*)$. We do so by letting $u,v \in \mathbf{C}$ be isomorphisms such that $\exists u \circ a$ and $\exists v \circ c$, then $\exists (u\circ a, u \circ b)\cdot(v \circ c, v \circ d)$. Observe that:*

$\exists v \circ c$ implies $\exists (v^{-1}\circ v)\circ c = R(v)\circ c = c$, and similar for d. Therefore:

$$\exists (a,b)\cdot(c,d) \text{ implies } \exists (a,b)\cdot((v^{-1}\circ v)\circ c, (v^{-1}\circ v)\circ d), \text{ so:}$$

$$\begin{aligned}
b &= r\circ((v^{-1}\circ v)\circ c) \text{ for some } r \in \mathbf{C};\\
&= (r\circ(v^{-1}\circ v))\circ c \text{ by } (CAT1);\\
&= ((r\circ v^{-1})\circ v)\circ c \text{ by } (CAT1);\\
&= (r\circ v^{-1})\circ(v\circ c), \text{ by } (CAT1). \text{ So } r\circ v^{-1} \text{ is such an isomorphism where:}\\
&\quad \exists (a,b)\cdot(v\circ c, v\circ d). \text{ Furthermore, } b = (r\circ v^{-1})\circ(v\circ c) \text{ and } \exists u\circ b \text{ implies:}\\
&\quad u\circ b = u\circ((r\circ v^{-1})\circ(v\circ c));\\
&= (u\circ(r\circ v^{-1}))\circ(v\circ c) \text{ by } (CAT1). \text{ So } u\circ(r\circ v^{-1}) \text{ is such an isomorphism where:}\\
&\quad \exists (u\circ a, u\circ b)\cdot(v\circ c, v\circ d), \text{ as required.}
\end{aligned}$$

We have δ is a strong congruence. □

Theorem 5.4. *Let $(\mathcal{U}, \cdot, D)$ be the constellation constructed as in* ***Proposition 5.1*** *and δ is the congruence on $\mathcal{U}$ as in* ***Proposition 5.2****. Then $(\mathcal{U}/\delta, \cdot, D)$ is a D-inverse constellation by defining the congruence class of any element $(a,b) \in \mathcal{U}$ as $[a,b]$ and its D-inverse as $[a,b]' = [b,a]$.*

Proof. *By* ***Result 2.37*** *and* ***Proposition 5.3****, $(\mathcal{U}/\delta, \cdot, D)$ is a constellation. Note that since δ is a strong congruence, products exist under the same conditions as in $\mathcal{U}$.*

Let $[a,b] \in \mathcal{U}/\delta$ then first we show $'$ is well-defined, suppose $(a,b) = u(a^*,b^*)$ for some isomorphism $u \in \mathbf{C}$. Then $[a,b] = [a^*,b^*]$ and $(b,a) = u(b^*,a^*)$. Therefore:

$$[a,b]' = [b,a] = [b^*,a^*] = [a^*,b^*]', \text{ as required.}$$

Next is to show $[b,a]$ is a D-inverse of $[a,b]$. That is, $\exists [a,b]\cdot[b,a]$ and it equals $D([a,b]) = [a,a]$. The other direction: $[b,a]\cdot[a,b] = D([b,a])$ will follow by symmetry.

Observe $D(a)$ is such an isomorphism such that $b = D(a)\circ b$ since $D(a) = D(b)$ and:

$$\exists [a,b]\cdot[b,a] = [a, D(a)\circ a] = [a,a] = D([a,b]), \text{ as required.}$$

*It remains to show the D-inverses are unique, which is a consequence of $\mathcal{P}$ being normal by **Result 2.27**. So for $[a,a],[c,c] \in D(\mathcal{U}/\delta)$ let $\exists[a,a]\cdot[c,c]$ and $\exists[c,c]\cdot[a,a]$. Then:*

$[a,a]\cdot[c,c] = [a, p\circ c]$ *with* $a = p\circ c$ *and:*

$[c,c]\cdot[a,a] = [c, q\circ c]$ *with* $c = q\circ a$ *so:*

$D(a)\circ a = a = p\circ c = p\circ(q\circ a) = (p\circ q)\circ a$ *implies* $p\circ q = D(a)$ *by right cancellative property.*

By symmetry, $q\circ p = D(c)$ *so* p *and* q *are isomorphisms.*

Therefore, $(a,a) = p(c,c)$ *where* p *is an isomorphism hence we have* $[a,a] = [c,c]$*, as required.*

So $(\mathcal{U}/\delta,\cdot,D)$ *is a D-inverse constellation.* □

From now on we will denote $\mathcal{U}/\delta = \mathcal{P}(\mathbf{C})$. Simply put, $\mathcal{P}(\mathbf{C})$ is the "D-inverse constellation constructed from $\mathbf{C}$".

The final step is to show $\mathcal{P}(\mathbf{C})$ gives Lawson's corresponding ordered groupoid $\mathbf{G}(\mathbf{C})$ in [6]. To remind the reader of the definitions used to construct $\mathbf{G}(\mathbf{C})$, the equivalence classes on $\mathcal{U}$ are identical. The partial order $\leq$ is given by:

$$[a,b] \leq [c,d] \iff (a,b) = p(c,d) \text{ for some } p \in \mathbf{C}.$$

The operations on $\mathbf{G}(\mathbf{C})$ were defined to be $\forall(a,b),(c,d) \in \mathcal{U}$:

$$D([a,b]) = [a,a], R([a,b]) = [b,b], [a,b]^{-1} = [b,a] \text{ and:}$$

$$[a,b] \star [c,d] = [a, u\circ d] \text{ if } R([a,b]) = D([c,d]).$$

Theorem 5.5. *Let $(\mathcal{P}(\mathbf{C}),\cdot,D,\lesssim)$ be the constructed D-inverse constellation with $\lesssim$ is the natural partial order on $\mathcal{P}(\mathbf{C})$. Then if $[a,b],[c,d] \in \mathcal{P}(\mathbf{C})$ defining $R([a,b]) = [b,b]$ and restricting products so that $[a,b]\cdot[c,d] \iff R([a,b]) = D([c,d])$, we obtain $(\boldsymbol{G}(\mathbf{C}),\star,D,R,\leq)$.*

Proof. *By **Result 2.31** we have $(\mathcal{P}(\mathbf{C})),\otimes,D,R,\lesssim)$ is an ordered groupoid where $\otimes$ is the product operation of the derived category. Defining the range operation on $\mathcal{P}(\mathbf{C})$ gives precisely that of $\boldsymbol{G}(\mathbf{C})$ as does every other operation in $(\mathcal{P}(\mathbf{C}),\otimes,D,R,\lesssim)$. Therefore, it remains to show the partial orders $\leq$ and $\lesssim$ are equivalent.*

$(\leq\Rightarrow\lesssim)$*: Suppose $[a,b] \leq [c,d]$. Then $(a,b) = p(c,d)$ for some $p \in \mathbf{C}$, we also have $a = p\circ c$. Note $D([a,b]) = [a,a]$ and observe:*

$$\exists[a,a]\cdot[c,d] = [a,p\circ d] = [a,b]. \text{ So } [a,b] \lesssim [c,d], \text{ as required.}$$

$(\lesssim\Rightarrow\leq)$*: Suppose $[a,b] \lesssim [c,d]$. Then $D([a,b]) = [a,a]$, $\exists[a,a]\cdot[c,d]$, $a = p\circ c$ for some isomorphism $p \in \mathbf{C}$ tells us:*

$[a,b] = [a,a]\cdot[c,d] = [a,p\circ d]$*, so* $\exists u$*, an isomorphism such that:*

$a = u\circ a = u\circ(p\circ c) = (u\circ p)\circ c$ *and* $b = u\circ(p\circ d) = (u\circ p)\circ d$.

Therefore, $(a,b) = (u\circ p)(c,d)$ *so* $(a,b) \leq (c,d)$*, as required.*

Thus $(\mathcal{P}(\mathbf{C}),\otimes,D,R,\lesssim)$ *is the ordered groupoid* $(\boldsymbol{G}(\mathbf{C}),\star,D,R,\leq)$. □

§5.2 Right-Cancellative Categories from D-inverse constellations

This section will be brief since work has already been done on constructing categories from constellations in [2] with so-called **canonical extensions**. It turns out applying the same construction on D-inverse constellations will give right-cancellative categories, as will be seen. We begin by defining $\mathcal{P}$ is a D-inverse constellation, then define the set:

$$\mathbf{C}(\mathcal{P}) = \{(x, e) \in \mathcal{P} \times D(\mathcal{P}) : \exists x \cdot e\}.$$

I remark this is precisely the set Lawson uses in [7] on ordered groupoids albeit in disguise. Keep in mind the partial order on D-inverse constellations is the natural partial order. Observe:

$$\exists x \cdot e \iff \exists D(x') \cdot e \iff D(x') \leq e = R(x) \leq e \text{ (as } R \text{ is defined upon translation).}$$

For elements $(x, e), (y, f) \in \mathbf{C}(\mathcal{P})$ we can define the operations D and R such that $D((x, e)) = (D(x), D(x))$ and $R((x, e)) = (e, e)$. We also get $\exists (x, e) \circ (y, f)$ if $R((x, e)) = D((y, f))$, (i.e. $e = D(y)$). It then follows $\exists x \cdot y$ so we define:

$$(x, e) \circ (y, f) = (x \cdot y, f).$$

Proposition 5.6. *Let $\mathcal{P}$ be a D-inverse constellation and $\mathbf{C}(\mathcal{P})$ its canonical extension. Then $\mathbf{C}(\mathcal{P})$ is a right-cancellative category with isomorphisms of the form $(x, D(x'))$.*

Proof. *Let $(x, e), (y, f), (z, g) \in \mathbf{C}(\mathcal{P})$ with $\exists (x, e) \circ (y, f)$ and $\exists (z, g) \circ (y, f)$ such that:*

$(x, e) \circ (y, f) = (z, g) \circ (y, f)$, then $e = D(y) = g$, and:
$(x \cdot y, f) = (z \cdot y, f)$;
so $x \cdot y = z \cdot y$;
hence $x = z$, by right-cancellative property;
therefore $(x, e) = (z, g)$, as required.

Let $(x, e) \in \boldsymbol{C}(\mathcal{P})$, then let $(y, f) \in \boldsymbol{C}(\mathcal{P})$ be an element such that:

$$\exists (x, e) \circ (y, f) = (x \cdot y, f) = D((x, e));$$
$$\exists (y, f) \circ (x, e) = (y \cdot x, e) = D((y, f)).$$

Then $e = D(y), f = D(x), y \cdot x = D(y)$ and $x \cdot y = D(x)$. So by uniqueness of D-inverses in $\mathcal{P}$, $x = y'$. So $(x, e) = (x, D(y)) = (x, D(x'))$. Therefore, isomorphisms are of the form $(x, D(x'))$ whose inverse is $(x', D(x))$, as required. □

§5.3 $\mathbf{C}\mathcal{P}(\mathbf{C})$ and $\mathcal{P}\mathbf{C}(\mathcal{P})$

Analogous to Lawson's "Forward and back" section in [7], this section aims to highlight the alternative use of D-inverse constellations in the construction of $\mathbf{C}\mathcal{P}(\mathbf{C})$ and $\mathcal{P}\mathbf{C}(\mathcal{P})$. First we describe the elements of each.

Let $\mathbf{C}$ be a right-cancellative category. Then $\mathbf{C}\mathcal{P}(\mathbf{C})$ is the canonical extension of $\mathcal{P}(\mathbf{C})$ whose elements are of the form:

$$u = ([x, y], [z, z]) \text{ such that } D(u) = ([x, x], [x, x]) \text{ and } R(u) = ([z, z], [z, z]).$$

Taking $u = ([x, y], [z, z])$ and $v = ([a, b], [c, c])$ then $\exists u \circ v$ if $R(u) = ([z, z], [z, z]) = ([a, a], [a, a]) = D(v)$. That is, the product exists if $z = w \circ a$ for some isomorphism $w \in \mathbf{C}$. It follows that $\exists p \in \mathbf{C}$ such that $y = p \circ a$ and the final product will be:

$$([x, y], [z, z]) \circ ([a, b], [c, c]) = ([x, y] \cdot [a, b], [c, c]) = ([x, p \circ b], [c, c]).$$

We need to define the **dense** and **full** properties. The importance of these properties will not be highlighted in this work but merely stated for the following theorem. Note these are as defined in [7].

Definition 5.7. *Let $\mathbf{C}$ be a category and $\mathbf{B}$ be a subcategory of $\mathbf{C}$. Then $\mathcal{B}$ is said to be **dense** in $\mathbf{C}$ if for each identity $e \in \mathbf{C}_o$, there exists an identity $f \in \mathbf{C}_o$ and an isomorphism $x \in \mathbf{C}$ such that $R(x) = e$ and $D(x) = f$.*

Definition 5.8. *Let $\mathbf{C}$ be a category and $\mathbf{B}$ be a subcategory of $\mathbf{C}$. Then $\mathcal{B}$ is said to be **full** in $\mathbf{C}$ if every $x \in \mathbf{C}$ such that $D(x), R(x) \in \mathbf{B}$ implies $x \in \mathbf{B}$.*

Proposition 5.9. *Let $\mathbf{C}$ be a right-cancellative category. Define $\alpha : \mathbf{C} \to \mathbf{C}\mathcal{P}(\mathbf{C})$ such that for $x \in \mathbf{C}$:*

$$x\alpha = ([D(x), x], [R(x), R(x)]).$$

Then α is an embedding as a full, dense subcategory of $\mathbf{C}\mathcal{P}(\mathbf{C})$.

Proof. *First we show α is injective, let $x, y \in \mathbf{C}$ such that $x\alpha = y\alpha$ then $[D(x), x] = [D(y), y]$ so $\exists u \in \mathbf{C}$, an isomorphism such that:*

$$D(x) = u \circ D(y) \text{ and } x = u \circ y.$$

We see from above $u = D(x)$ since $D(y)$ is a right identity and $x = y$ because $D(x)$ is a left identity. So α is injective.

We now show α is a functor.

(F1): *Let $x, y \in \mathbf{C}$ such that $\exists x \circ y$. Then $R(x) = D(y)$ and:*

$$(x \circ y)\alpha = ([D(x \circ y), x \circ y], [R(x \circ y), R(x \circ y)]) = ([D(x), x \circ y], [R(y), R(y)]).$$

$\exists (x\alpha) \circ (y\alpha)$ if $\exists u \in \mathbf{C}$, an isomorphism such that $R(x) = u \circ D(y)$, observe $u = D(y)$ satisfies this. Therefore:

$(x\alpha) \circ (y\alpha) = ([D(x), x] \cdot [D(y), y], [R(y), R(y)])$.

So $\exists p \in \mathbf{C}$ such that $x = p \circ D(y) = p$ and:

$([D(x), x] \cdot [D(y), y], [R(y), R(y)]) = ([D(x), p \circ y], [R(y), R(y)])$;

$= ([D(x), x \circ y], [R(y), R(y)])$;

$= (x \circ y)\alpha$, *as required.*

(F2): *Observe:*

$$D(x\alpha) = ([D(x), D(x)], [D(x), D(x)]) = ([D(D(x)), D(x)], [R(D(x)), R(D(x))]) = D(x)\alpha;$$

$$R(x\alpha) = ([R(x), R(x)], [R(x), R(x)]) = ([D(R(x)), R(x)], [R(R(x)), R(R(x))]) = R(x)\alpha.$$

So α is a functor.

α is a full embedding if $u = ([x, y], [e, e]) \in \mathbf{C}\mathcal{P}(\mathbf{C})$ and $D(u), R(u) \in img(\alpha)$ imply $u \in img(\alpha)$. $D(u) = ([x, x], [x, x]) \in img(\alpha)$ implies x is an identity since $D(x) = R(x) = x$. Furthermore, $[x, y] \in \mathcal{P}(\mathbf{C})$ implies $D(x) = x = D(y)$.

Lastly, $[x, y] \cdot [e, e]$ implies $\exists p \in \mathbf{C}$ such that $y = p \circ e = p$ since e is an identity. So $\exists y \circ e$ implies $e = R(y)$ by uniqueness of $R(y)$. Therefore:

$$([x, y], [e, e]) = ([D(y), y], [R(y), R(y)]) \in img(\alpha), \text{ as required.}$$

α is a dense embedding if for each identity $e \in \mathbf{C}\mathcal{P}(\mathbf{C})$ there is an identity $f \in img(\alpha)$ and an isomorphism $u \in \mathbf{C}\mathcal{P}(\mathbf{C})$ such that $D(u) = f$ and $R(u) = e$. An identity of $\mathbf{C}\mathcal{P}(\mathbf{C})$ is given by:

$$([x, x], [x, x]).$$

Recall isomorphisms are of the form $(a, D(a'))$ so consider the isomorphism $u = ([D(x), x], [x, x])$. Then:

$D(u) = (D([D(x), x]), D([D(x), x])) = ([D(x), D(x)], [D(x), D(x)]) = D(x)\alpha$ *and:*

$R(u) = ([x, x], [x, x])$, *as required.*

So α is a dense embedding. □

This theorem equivalently states, every right-cancellative category can be constructed from a single D-inverse constellation. This was an expected consequence of Lawson's work in [7] and **Result 2.31**.

Now that we have looked at $\mathbf{C}\mathcal{P}(\mathbf{C})$, it is time to investigate $\mathcal{P}\mathbf{C}(\mathcal{P})$ whose elements are of the form:

$$u = [(x, e), (y, f)] \text{ such that } D(u) = [(x, e), (x, e)] \text{ and } [(x, e), (y, f)]' = [(y, f), (x, e)].$$

Taking $u = [(x, e), (y, f)]$ and $v = [(s, g), (t, h)]$ then $\exists u \cdot v$ if and only if $\exists (p, i)$ such that $(y, f) = (p, i) \circ (s, g) = (p \cdot s, g)$. In other words, the product exists if and only if $f = g$ and $\exists p$ such that $y = p \cdot s$. Then:

$$u \cdot v = [(x, e), (p, i) \circ (t, h)] = [(x, e), (p \cdot t, h)].$$

We continue to follow Lawson and now construct another D-inverse constellation and prove it is isomorphic to $\mathcal{P}\mathbf{C}(\mathcal{P})$. This way, we will not have to deal with heinous equivalence classes. Define:

$$\bar{P} = \{< e, x, f > \in D(\mathcal{P}) \times \mathcal{P} \times D(\mathcal{P}) : \exists D(x) \cdot e \text{ and } \exists D(x') \cdot f\}.$$

Define $D(< e, x, f >) = < e, D(x), e >$ and $< e, x, f >' = < f, x', e >$ with partial product:

$$< e, x, f > \cdot < g, y, h > = < e, x \cdot y, h > \iff \exists x \cdot y \text{ and } f = g.$$

For brevity, whenever we define products such as the above to exist, we will do away with needing to state the $f = g$ condition. We will do so by continuing to define $< e, x, f >$ as is but instead defining $< g, y, h > = < f, y, h >$. The product then exists if and only if $\exists x \cdot y$, a condition we will see makes $\bar{P}$ easy to work with. The condition $f = g$ will only be considered if a product in $\bar{P}$ exists as a result of something else.

Proposition 5.10. *Let $\bar{P}$ be as described above. Then $(\bar{P}, \cdot, D, ')$ is a D-inverse constellation.*

Proof. *We aim to show $(CONST1)$-$(CONST4)$ and (IC) are satisfied.*

(CONST1): *Follows directly from $(CONST1)$ in $\mathcal{P}$ since for $< e, x, f >, < f, y, g >, < g, z, h > \in \bar{P}$:*

$\exists < e, x, f > \cdot (< f, y, g > \cdot < g, z, h >)$ *implies* $\exists x \cdot (y \cdot z)$ *so:*

$\exists (x \cdot y) \cdot z$ *by $(CONST1)$ implying* $\exists (< e, x, f > \cdot < f, y, g >) \cdot < g, z, h >$.

It is not hard to see that the products are equal because $x \cdot (y \cdot z) = (x \cdot y) \cdot z$.

(CONST2): *Follows directly from $(CONST2)$ in $\mathcal{P}$. I.e. for $< e, x, f >, < f, y, g >, < g, z, h > \in \bar{P}$:*

$\exists x \cdot y, \exists y \cdot z$ *if and only if* $\exists x \cdot (y \cdot z)$, *so:*

$\exists < e, x, f > \cdot < f, y, g >, \exists < f, y, g > \cdot < g, z, h >$ *if and only if:*

$\exists < e, x, f > \cdot (< f, y, g > \cdot < g, z, h >)$.

(CONST3): *Let $< e, x, f > \in \bar{P}$. Then $D(< e, x, f >) = < e, D(x), e >$.*

Observe $\exists < e, D(x), e > \cdot < e, x, f >$ and is equal to $< e, D(x) \cdot x, f > = < e, x, f >$.

To show uniqueness, let $< g, h, g > \in D(\bar{P})$. Then:

$< g, h, g > \cdot < e, x, f > = < g, h \cdot x, f > = < e, x, f >$, *so:*

$g = e$ *and* $h \cdot x = x$ *so* $h = D(x)$ *by $(CONST3)$ in $\mathcal{P}$.*

(CONST4): *Suppose for* $< e, x, f > \in \bar{P}$ *and* $< f, g, f > \in D(\bar{P})$*, that* $\exists < e, x, f > \cdot < f, g, f >$*. Then* $\exists x \cdot g$ *and is equal to* x *by* $(CONST4)$ *in* $\mathcal{P}$ *so:*

$$< e, x, f > \cdot < f, g, f > = < e, x \cdot g, f > = < e, x, f >, \text{ as required.}$$

So $\bar{P}$ *is a constellation.*

(IC): *Let* $< e, x, f > \in \bar{P}$*. Then* $< e, x, f >' = < f, x', e >$*, so since* $\exists x \cdot x' = D(x)$ *and* $\exists x' \cdot x = D(x')$*, we have:*

$$\begin{aligned} \exists < e, x, f > \cdot < f, x', e > &= < e, D(x), e > = D(< e, x, f >); \\ \exists < f, x', e > \cdot < e, x, f > &= < f, D(x'), f > = D(< f, x', e >). \end{aligned}$$

So $< f, x', e >$ *is a D-inverse of* $< e, x, f >$*. It remains to show D-inverses are unique. By* ***Result 2.27*** *it is enough to show* $\bar{P}$ *is normal.*
Let $< e, f, e >, < e, g, e > \in D(\bar{P})$ *such that* $\exists < e, f, e > \cdot < e, g, e >$ *and* $\exists < e, g, e > \cdot < e, f, e >$*. Then* $\exists f \cdot g$ *and* $g \cdot f$ *so* $h = g$ *by* $\mathcal{P}$ *is normal.*
Therefore:

$$< e, f, e > = < e, g, e >, \text{ as required.}$$

So $\bar{P}$ *is a D-inverse constellation.* □

Proving $\bar{P} \cong \mathcal{P}\mathbf{C}(\mathcal{P})$ requires us to find an isomorphism $\alpha : \bar{P} \to \mathcal{P}\mathbf{C}(\mathcal{P})$. Lawson's method is very similar to ours as one would expect, with one key difference. Define the function:

$$< e, x, f > \alpha = [(x', e), (D(x'), f)].$$

Proposition 5.11. *Let* $\alpha : \bar{P} \to \mathcal{P}\mathbf{C}(\mathcal{P})$ *be as described above. Then* α *is an isomorphism of D-inverse constellations.*

Proof. *First we show every element in* $\mathcal{P}\mathbf{C}(\mathcal{P})$ *can be expressed uniquely with respect to a given element of* $\mathcal{P}$*. Observe when* $[(x, e), (y, h)] \in \mathcal{P}\mathbf{C}(\mathcal{P})$*, then* $D(x) = D(y)$*. Therefore,* $y' \cdot D(y) = y' \cdot D(x)$ *so* $(y', D(x))$ *is an isomorphism in* $\mathbf{C}(\mathcal{P})$ *such that* $\exists (y', D(x))((x, e), (y, h))$ *and:*

$$\begin{aligned} (y', D(x))((x, e), (y, h)) &= ((y', D(x)) \circ ((x, e)), (y', D(x)) \circ (y, h)); \\ &= ((y' \cdot x, e), (y' \cdot y), h); \\ &= ((y' \cdot x, e), (D(y'), h)); \\ &= ((y' \cdot x, e), (D(y' \cdot x), h)). \end{aligned}$$

Therefore, taking $w = y' \cdot x$ *gives:*

$$[(x, e), (y, h)] = [(w, e), (D(w), h)], \text{ as required.}$$

As this is the case, it is easy to see α is surjective.

We now check α is injective. Suppose $< e, x, f >, < g, y, h > \in \bar{P}$ such that $< e, x, f > \alpha = < g, y, h > \alpha$. Then:

$$[(x', e), (D(x'), f)] = [(y', g), (D(y'), h)].$$

So $\exists (w, D(w'))$ an isomorphism in $\mathbf{C}(\mathcal{P})$ such that $\exists (w, D(w'))((x', e), (D(x'), f))$ and it equals $((y', g), (D(y'), h))$. But:

$(w, D(w'))((x', e), (D(x'), f)) = ((w, D(w')) \circ (x', e), (w, D(w')) \circ (D(x'), f))$;
$= ((w \cdot x', e), (w \cdot D(x'), f))$;
$= ((w \cdot x', e), (w, f))$;
so $e = g, f = h$, and $w = D(y')$;
so $y' = w \cdot x' = D(y') \cdot x'$, (i.e. $y' \lesssim x'$ w.r.t. the NPO);
$x' \lesssim y'$ by symmetry (using $(w, D(w'))'$ in the other direction);
so $x' = y'$, hence $x = y$ by uniqueness of D-inverses. Therefore:

$$< e, x, f > = < g, y, h >, \text{ as required.}$$

So α is a bijection, it remains to show it is a radiant.

(R1): *Let $< e, x, f >, < f, y, h > \in \bar{P}$ be such that $\exists < e, x, f > \cdot < f, y, h >$. Then it is equal to $< e, x \cdot y, g >$. Therefore:*

$$< e, x \cdot y, g > \alpha = [((x \cdot y)', e), (D((x \cdot y)'), g)].$$

We need to check the conditions for $\exists (< e, x, f > \alpha) \cdot (< f, y, h > \alpha)$, i.e. when:

$$\exists [(x', e), (D(x'), f)] \cdot [(y', f), (D(y'), g)].$$

First of all $f = f$ so the product is defined when $\exists p \in \mathcal{P}$ such that $D(x') = p \cdot y'$ which occurs if and only if $p = D(x') \cdot y$. But $\exists D(x') \cdot y$ if and only if $\exists x \cdot y$. (Note: if α instead mapped elements as Lawson did we would get this product existed if and only if $\exists x' \cdot y'$).

And so the product gives:

$$\begin{aligned} [(x', e), (D(x'), f)] \cdot [(y', f), (D(y'), g)] &= [(x', e), ((D(x') \cdot y) \cdot D(y'), g)]; \\ &= [(x', e), (D(x') \cdot y, g)]. \end{aligned}$$

Although this is not quite the correct form, I claim the isomorphism of $\mathbf{C}(\mathcal{P})$, $((x \cdot y)' \cdot x, D(x')))$, will prove the congruence classes of $< e, x \cdot y, g > \alpha$ and $(< e, x, f > \alpha) \cdot (< f, y, h > \alpha)$ are equal. It does not appear obvious $\exists (x \cdot y)' \cdot x$ but observe:

$$\exists (x \cdot y)' \cdot D(x \cdot y) = (x \cdot y)' \cdot D(x) \text{ by } \textbf{Result 2.18} \text{ so:}$$
$$\exists (x \cdot y)' \cdot x \text{ by } \textbf{Result 2.17}, \text{ as required.}$$

Certainly $\exists x' \cdot (x \cdot y)$ *and is equal to* $((x \cdot y)' \cdot x)'$ *by* ***Result 2.30*** *so* $D(x') = D(x' \cdot (x \cdot y)) = D(((x \cdot y)' \cdot x)')$.

Thus indeed $((x \cdot y)' \cdot x, D(x'))$ *is an isomorphism of* $\mathbf{C}(\mathcal{P})$.

Since $D(x') = D(x')$, *we have* $\exists((x \cdot y)' \cdot x, D(x'))((x', e), (D(x') \cdot y, g))$ *and it is equal to:*

$((((x \cdot y)' \cdot x, D(x')) \circ (x', e)), (((x \cdot y)' \cdot x, D(x')) \circ (D(x') \cdot y, g)))$;

$= ((((x \cdot y)' \cdot x) \cdot x', e), (((x \cdot y)' \cdot x) \cdot (D(x') \cdot y), g))$;

$= (((x \cdot y)' \cdot (x \cdot x'), e), ((((x \cdot y)' \cdot x) \cdot D(x')) \cdot y, g))$ *by* $(CONST1)$ *and* $(CONST2)$;

$= (((x \cdot y)' \cdot D(x), e), (((x \cdot y)' \cdot x) \cdot y, g))$ *by* $(CONST4)$;

$= (((x \cdot y)', e), ((x \cdot y)' \cdot (x \cdot y), g))$ *by* $(CONST2)$ *and* $(CONST4)$;

$= (((x \cdot y)', e), (D((x \cdot y)'), g))$. *Therefore we have:*

$[(x', e), (D(x') \cdot y, g)] = [((x \cdot y)', e), (D((x \cdot y)'), g)]$, *as required.*

(R2): *Let* $< e, x, f > \in \bar{P}$. *Then:*

$D(< e, x, f > \alpha) = D([(x', e), (D(x'), f)]) = [(x', e), (x', e)]$ *and:*

$D(< e, x, f >)\alpha = < e, D(x), e > \alpha = [(D(x), e), (D(x), e)]$. *But notice* $(x', D(x))$ *is an isomorphism such that:*

$\exists(x', D(x))((D(x), e), (D(x), e))$ *since* $D(x) = D(x)$ *so:*

$(x', D(x)) \circ (D(x), e) = (x' \cdot D(x), e) = (x, e)$ *by* $(CONST4)$ *so:*

$(x', D(x))((D(x), e), (D(x), e)) = ((x', e), (x', e))$. *Finally we have:*

$$D(< e, x, f >)\alpha = [(D(x), e), (D(x), e)] = [(x', e), (x', e)] = D(< e, x, f > \alpha).$$

Hence α *is an isomorphism.* □

It happens that α instead maps to the inverse of elements Lawson's isomorphism maps to in [6]. In the end, this does not matter since D-inverses are unique in D-inverse constellations. This means by symmetry we still obtain the corresponding D-inverse constellation to Lawson's ordered groupoid in [6].

6 From $\star$-Semigroups to D-inverse constellations

Lawson states in [6], underlying every **C*-algebra** is a $\star$-semigroup. While C*-algebras themselves will not be visited in this work, it is enough to know they are algebras equipped with an involution operator that is also a fairly constrained metric space.

The purpose of this chapter is to show it is possible to take arbitrary $\star$-semigroups hence C*-algebras and use them to construct D-inverse constellations.

Let $\mathbf{S}$ be a $\star$-semigroup. Then we denote the set of **partial isometries**, $PI(\mathbf{S}) \subseteq \mathbf{S}$ by:

$$PI(\mathbf{S}) = \{x \in \mathbf{S} : x = xx^*x\}.$$

Proposition 6.1. *Let $\mathbf{S}$ be a $\star-$semigroup. $\forall x, y \in \mathbf{S}$, define the partial product $\exists x \cdot y \iff x = xyy^*$ such that $x \cdot y = xy$. Define $D(x) = x \cdot x^*$. Then $(PI(\mathbf{S}), \cdot, D)$ is a D-inverse constellation.*

Proof. *Let $x, y \in PI(\mathbf{S})$. We first show closure under $\cdot$.*

Let $\exists x \cdot y$. Then $x = xyy^*$ and by associativity and $(\star 2)$ we have:

$$xy(xy)^*xy = xy(y^*x^*)xy = (xyy^*)x^*xy = (xx^*x)y = xy \text{ implies } x \cdot y \in PI(\mathbf{S}).$$

Observe by $(\star 1)$ we have:

$$x = xx^*x = xx^*(x^*)^* \text{ implies } \exists x \cdot x^*.$$

Closure of $PI(\mathbf{S})$ under D follows by closure under $\cdot$.

Pressing on with axioms $(CONST1)$-$(CONST4)$ and (IC).

(CONST1): *Let $x, y, z \in PI(\mathbf{S})$ and let $\exists x \cdot (y \cdot z)$. Then in $\mathbf{S}$ we have $y = yzz^*$ and $x = x(yz)(yz)^*$ such that:*

$$x = x(yz)(yz)^* = x(yz)(z^*y^*) = x(yzz^*)y^* = xyy^* \text{ implies } \exists x \cdot y, \text{ and:}$$

$$xy = x(yzz^*) = (xy)(zz^*) \text{ implies } \exists (xy) \cdot z.$$

But $xy = x \cdot y$ so $\exists (x \cdot y) \cdot z$ and it is equal to $x \cdot (y \cdot z)$ by associativity in $\mathbf{S}$. So $(CONST1)$ is satisfied.

(CONST2): *Let $x, y, z \in PI(\mathbf{S})$. We show if $\exists x \cdot y$ and $\exists y \cdot z$ then $\exists x \cdot (y \cdot z)$. That is, it suffices to prove $\exists x \cdot (yz)$.*
Note $x = xyy^*$ and $y = yzz^*$, so:

$$x = xyy^* = x(yzz^*)y^* = x(yz)(z^*y^*) = x(yz)(yz)^* \text{ implies } \exists x \cdot (yz).$$

(CONST3): *Let $x \in PI(\mathbf{S})$. We first show $\exists D(x) \cdot x$ and it equals x.*

$D(x) = x \cdot x^* = xx^*$ and $D(x^*) = x^* \cdot x^{**} = x^* \cdot x$ imply $\exists (x \cdot x^*) \cdot x$ by $(CONST2)$ and is equal to $D(x) \cdot x$ so:

$$D(x) \cdot x = (xx^*) \cdot x = (xx^*)x = x \text{ since } x \in PI(\mathbf{S}).$$

It remains to show $D(x)$ is unique.
Suppose $\exists e \in D(PI(\mathbf{S}))$ such that $\exists e \cdot x$ and equals x. Then:

$$e = exx^* = (ex)x^* = (e \cdot x)x^* = xx^* = D(x).$$

(CONST4): *Let $x \in PI(\mathbf{S})$ and $e \in D(PI(\mathbf{S}))$ with $\exists x \cdot e$. Then $\exists y \in PI(\mathbf{S})$ such that $e = yy^* = y \cdot y^*$, so we have:*

$$x \cdot e = x \cdot (y \cdot y^*) \text{ implying } \exists x \cdot y \text{ by } (CONST2) \text{ but:}$$
$$x = xyy^* = x \cdot (yy^*) = x \cdot e.$$

So $(CONST4)$ is satisfied and we have $(PI(\mathbf{S}), \cdot, D)$ is a constellation.

(IC): *By definition, we already have $\forall x \in PI(\mathbf{S}), \exists x^*$ such that $D(x) = x \cdot x^*$ and $D(x^*) = x^* \cdot x$. Therefore, it must be shown these D-inverses are unique.*
*By **Result 2.27** it suffices to show $(PI(\mathbf{S}), \cdot, D)$ is normal. Let $e, f \in D(PI(\mathbf{S}))$ with $\exists e \cdot f$ and $\exists f \cdot e$.*
Then $fe = f \cdot e = f$ and $ef = e \cdot f = e$ by $(CONST4)$ so in $\mathbf{S}$ we have:

$$e = e^* = (ef)^* = f^* e^* = fe = f.$$

Finally, $(PI(\mathbf{S}), \cdot, D)$ is normal hence a D-inverse constellation. □

Since $(PI(\mathbf{S}), \cdot, D)$ is a D-inverse constellation, it follows $PI(\mathbf{S})$ has the natural partial order. This aspect helps with forming inverse semigroups by the D-inverse constellation constructed in particular cases.

It is not hard to see this corresponds to Lawson's ordered groupoid in [6] since the derived category product $x \circ y$ exists when $D(x^*) = D(y)$ and $x = xyy^*$ and the order he uses is precisely the natural partial order.

One can take a step further provided $\mathbf{S}$ is a $\star$-monoid. $\star$-monoids are $\star$-semigroups that also have a **global identity** element. We denote this global identity as 1 and is defined to have the property:

$$\forall x \in \mathbf{S}, \, x = 1x = x1.$$

We also require the following definition before introducing the notion of **special partial isometries**.

Definition 6.2. *Let $\mathbf{S}$ be a $\star$-semigroup. Then $\mathbf{S}$ is said to be **ordered** if there is a partial order $\subseteq$ on $\mathbf{S}$ such that $\forall x, y \in \mathbf{S}$:*

(O⋆1) $x \subseteq y \Rightarrow x^* \subseteq y^*$;

(O⋆2) $x \subseteq y, s \subseteq t \Rightarrow xs \subseteq yt$.

So let $\mathbf{S}$ be an ordered-$\star$-monoid with partial order $\subseteq$. Then the set of special partial isometries is a subset of $PI(\mathbf{S})$ we denote as $SPI(\mathbf{S})$ defined by:

$$SPI(\mathbf{S}) = \{x \in PI(\mathbf{S}) : 1 \subseteq xx^*, x^*x\}.$$

Proposition 6.3. *Let $(\mathbf{S}, \subseteq)$ be an ordered-$\star$-monoid. Then $(SPI(\mathbf{S}), \cdot, D)$ is a sub-D-inverse constellation of $(PI(\mathbf{S}), \cdot, D)$.*

Proof. *First of all, $SPI(\mathbf{S})$ is closed under the D operation by definition.*

Let $x, y \in SPI(\mathbf{S})$ and let $\exists x \cdot y$ in $PI(\mathbf{S})$. Then $x = xyy^*$ and ($\star$2) give:

$$1 \subseteq xx^* = (xyy^*)x^* = xy(y^*x^*) = xy(xy)^*, \text{ as required.}$$

So $SPI(\mathbf{S})$ is closed under $\cdot$, hence a sub-D-inverse constellation. □

The addition of the $\subseteq$ order turns out to be the reverse of the natural partial order on $D(SPI(\mathbf{S}))$ as proven in [6]. This sort of makes sense since $\forall e \in D(SPI(\mathbf{S})), \exists e \cdot 1$ (i.e. $e \leq 1$).

With the D-inverse constellation built, one can form an inverse semigroup if the natural partial order forms a meet semilattice. One would then obtain the same results Lawson has for the $C^\star$-algebras of $n \times n$ complex matrices, the group coset $\star$-monoid, and the dual-symmetric $\star$-monoid.

7 Groups Actions on a Poset

The so-called **star-injective** and **star-surjective** properties of a functor were used quite often in [6]. As we follow along Lawson's work, it is easier to define them now and reference back later. When both conditions are satisfied by the functor it is called a covering functor. For brevity we will define them collectively in the radiant analogue.

Definition 7.1. *Let $\mathcal{P}$ and $\mathcal{Q}$ be D-inverse constellations and $\alpha : \mathcal{P} \to \mathcal{Q}$ be a radiant. Then α is a* ***covering radiant*** *or equivalently,* ***star-bijective*** *if it satisfies:*

(CR1) $\forall x, y \in \mathcal{P}$, $D(x) = D(y)$ *and* $x\alpha = y\alpha \Rightarrow x = y$ ***(star-injectivity)****;*

(CR2) if $f \in D(\mathcal{P})$ *and* $y \in \mathcal{Q}$ *such that* $D(y) = f\alpha$*, then* $\exists x \in \mathcal{P}$ *such that* $D(x) = f$ *and* $x\alpha = y$ ***(star-surjectivity)****.*

Another construction by Lawson in [6] starts with both a group, **G**, and a poset (not necessarily on **G**), **X**. Aiming to construct a D-inverse constellation from these structures, we require a little more framework. Hence another definition.

Definition 7.2. *Let* $\mathbf{G}$ *be a group and* $\mathbf{X}$ *be a poset. Then* $\mathbf{G}$ *is said to act on the right of* $\mathbf{X}$ *by order automorphisms if there exists a function from* $\mathbf{G} \times \mathbf{X} \to \mathbf{G}$ *denoted by* $(e, g) \to e \star g$ *satisfying* $\forall e, f \in \mathbf{X}$ *and* $\forall g, h \in \mathbf{G}$*:*

(GA1) $e \star 1 = e$;

(GA2) $(e \star g) \star h = e \star (gh)$;

(GA3) $\exists e \leq f \iff e \star g \leq f \star g$.

Proposition 7.3. *Let* $\mathbf{X}$ *be a poset and* $\mathbf{G}$ *be a group that acts on the right of* $\mathbf{X}$ *by order automorphisms. Then define* $\forall (g, x), (h, y) \in \mathbf{G} \times \mathbf{X}$*:*

$$D((g,x)) = (1, x \star g'), \quad (g,x)' = (g', x \star g') \text{ and:}$$
$$(g,x) \cdot (h,y) = (gh, x \star h) \iff x \star h \leq y.$$

Then $(\mathbf{G} \times \mathbf{X}, \cdot)$ *is a D-inverse constellation.*

Proof. ***(CONST1):*** *Let* $(g, x), (h, y), (i, z) \in \mathbf{G} \times \mathbf{X}$ *such that* $\exists (g, x) \cdot ((h, y) \cdot (i, z))$*. Then* $y \star i \leq z$ *and:*

$$\begin{aligned}
(g,x)\cdot((h,y)\cdot(i,z)) &= (g,x)\cdot(hi, y\star i) \text{ so } x\star(hi) \leq y\star i \text{ and:}\\
&= (ghi, x\star(hi)) \text{ but by } (GA2),\ x\star(hi) = (x\star h)\star i \leq y\star i \leq z \text{ so:}\\
&= (gh, x\star h)\cdot(i,z) \text{ and by } (GA3),\ (x\star h)\star i \leq y\star i \text{ implies } x\star h \leq y;\\
&= (gh, x\star h)\cdot(i,z) = ((g,x)\cdot(h,y))\cdot(i,z), \text{ as required.}
\end{aligned}$$

(CONST2): *Let* $(g, x), (h, y), (i, z) \in \mathbf{G} \times \mathbf{X}$ *such that* $\exists (g, x) \cdot (h, y)$ *and* $\exists (h, y) \cdot (i, z)$*.*

Then $y \star i \leq z$ *and* $(h, y) \cdot (i, z) = (hi, y \star i)$*.*

Observe $x \star h \leq y$ implies $(x \star h) \star i \leq y \star i$ by $(GA3)$.

By $(GA2)$, $(x \star h) \star i = x \star (hi)$.

So $\exists (g, x) \cdot (hi, y \star i)$ and it is equal to $(g, x) \cdot ((h, y) \cdot (i, z))$, as required.

(CONST3): *Let $(g, x) \in \mathbf{G} \times \mathbf{X}$. Then $D((g, x)) = (1, x \star g')$. By $(GA1)$ and $(GA2)$:*

$$x = x \star 1 = x \star (g'g) = (x \star g') \star g.$$

So $\exists (1, x \star g') \cdot (g, x)$ and it is equal to $(1g, (x \star g') \star g) = (g, x)$.

We now show $(1, x \star g')$ is unique with this property. Suppose $\exists (1, w) \in D(\mathbf{G} \times \mathbf{X})$ such that $\exists (1, w) \cdot (g, x)$ and it is equal to (g, x). Then by $(GA1)$ and $(GA2)$:

$$x = w \star g \text{ so } x \star g' = (w \star g) \star g' = w \star (gg') = w \star 1 = w, \text{ as required.}$$

(CONST4): *Let $(g, x) \in \mathbf{G} \times \mathbf{X}$ and $(1, w) \in D(\mathbf{G} \times \mathbf{X})$ such that $\exists (g, x) \cdot (1, w)$. Then $(g, x) \cdot (1, w) = (g1, x \star 1) = (g, x)$ by $(GA1)$.*

So $\mathbf{G} \times \mathbf{X}$ is a constellation.

(IC): *Let $(g, x) \in \mathbf{G} \times \mathbf{X}$. Then $(g, x)' = (g', x \star g')$ and by $(GA1)$ and $(GA2)$:*

$$D((g', x \star g')) = (1, (x \star g') \star g) = (1, x \star (g'g)) = (1, x \star 1) = (1, x).$$

Recall $(x \star g') \star g = x$ so $\exists (g', x \star g') \cdot (g, x)$ and it is equal to:

$$(g'g, (x \star g') \star g) = (1, x) = D((g', x \star g')) = D((g, x)').$$

Trivially, $x \star g' = x \star g'$, so $\exists (g, x) \cdot (g', x \star g')$ and it is equal to

$$(gg', x \star g') = (1, x \star g') = D((g, x)).$$

*We prove uniqueness of inverses by utilizing **Result 2.27**. Let $(1, x), (1, y) \in D(\mathbf{G} \times \mathbf{X})$ such that $\exists (1, x) \cdot (1, y)$ and $\exists (1, y) \cdot (1, x)$. Then:*

$$x = x \star 1 \leq y \text{ and } y = y \star 1 \leq x \text{ by (GA1) so } x = y.$$

*Hence $\mathbf{G} \times \mathbf{X}$ is normal satisfying (IC) by **Result 2.27**. Therefore, $\mathbf{G} \times \mathbf{X}$ a D-inverse constellation.* □

The above D-inverse constellation is the constellation equivalent of the ordered groupoid Lawson calls $P(G, X)$ or the "semidirect product of a poset by a group". One will find if you keep only the products $(g, x) \cdot (h, y)$ such that $D((g, x)) = D((h, y)')$ then $x \star h = y$ and $(g, x) \cdot (h, y) = (gh, y)$ gives precisely Lawson's ordered groupoid.

Proposition 7.4. *Let $\mathbf{G} \times \mathbf{X}$ be as described above. Then $\pi : \mathbf{G} \times \mathbf{X} \to \mathbf{G}$ defined by $(g, x)\pi = g$ is a surjective covering radiant. Furthermore, when X is a meet semilattice, $\mathbf{G} \times \mathbf{X}$ is the inductive D-inverse constellation corresponding to the inverse semigroup called the semidirect product of a semilattice by a group.*

Proof. *It is clear π is surjective since $\forall g \in \mathbf{G}, \exists (g,x) \in \mathbf{G} \times \mathbf{X}$. Let $(g,x),(h,y) \in \mathbf{G} \times \mathbf{X}$ such that $\exists (g,x)\cdot(h,y)$. Then it is equal to $(gh, x \star h)$.*

Since $\mathbf{G}$ is a group, we can regard it as a D-inverse constellation where every product exists and $\forall g \in \mathbf{G}$, $D(g) = gg' = 1$ so:

$$(g,x)\pi(h,y)\pi = gh = (gh, x \star h)\pi.$$

Furthermore, $\forall (g,x) \in \mathbf{G} \times \mathbf{X}$ $D((g,x))\pi = 1 = D(g) = D((g,x)\pi)$ so π is a radiant.

We now show it is a covering radiant.

(CR1): *Let $(g,x),(h,y) \in \mathbf{G} \times \mathbf{X}$ such that $D((g,x)) = D((h,y))$ and $(g,x)\pi = (h,y)\pi$. Then $g = h$ so $(1, x \star g) = (1, y \star h) = (1, y \star g)$.*

$x \star g = y \star g$ implies $x = y$ by $(GA3)$.

Therefore, $(g,x) = (h,y)$, as required.

(CR2): *Let $(1,x) \in D(\mathbf{G} \times \mathbf{X})$ and $g \in \mathbf{G}$. Then trivially $(1,x)\pi = D(g)$.*

Take $\exists (g, x \star g) \in \mathbf{G} \times \mathbf{X}$. It is an element such that $D((g, x \star g)) = (1,x)$ and $(g, x \star g)\pi = g$, precisely as required.

So π is a covering radiant.

(Inductive property): *Let X be a meet semilattice and define for $(1,x),(1,y) \in D(\mathbf{G}\times\mathbf{X})$:*

$$(1,x) \wedge (1,y) = (1, x \wedge y).$$

By definition $\exists (1,x)\cdot(1.y)$ if and only if $x \star 1 = x \leq y$ by $(GA1)$. It follows the natural partial order on $D(\mathbf{G} \times \mathbf{X})$ also forms a meet semilattice.

So $\mathbf{G} \times \mathbf{X}$ is an inductive inverse constellation.

It remains to show the products in $(\mathbf{G} \times \mathbf{X}, \otimes)$ are of the form $\forall (g,x),(h,y) \in \mathbf{G} \times \mathbf{X}$:

$$(g,x) \otimes (h,y) = (gh, y \wedge (x \star h)).$$

So first we must evaluate $(g,x)|D((h,y))$. Since $(g,x)\otimes(h,y) = ((g,x)|D((h,y)))\cdot(h,y)$.

$$\begin{aligned}
(g,x)|D((h,y)) &= (g,x)|(1, y \star h'); \\
&= (((1, y \star h') \wedge (1,x)) \cdot (g', x \star g'))'; \\
&= ((1, (y \star h') \wedge x) \cdot (g', x \star g'))'; \\
&= (g', ((y \star h') \wedge x) \star g')'; \\
&= (g, (((y \star h') \wedge x) \star g') \star g); \\
&= (g, ((y \star h') \wedge x) \star (g'g)) \text{ by } (GA2); \\
&= (g, ((y \star h') \wedge x) \star 1); \\
&= (g, (y \star h') \wedge x) \text{ by } (GA1).
\end{aligned}$$

Therefore we have:

$$(g,x)\otimes(h,y)=(g,(y\star h')\wedge x)\cdot(h,y)=(gh,((y\star h')\wedge x)\star h)=(gh,y\wedge(x\star h)).$$

Precisely as we claimed. □

Now that we have a D-inverse constellation to work with, the goal now is to follow along with Lawson's work utilizing $(\mathbf{G}\times\mathbf{X},\cdot)$.

Below is a prime example of what makes D-inverse constellations easier to work with over ordered groupoids. Although only a lemma, the proof relies only on simple definitions and results. However, tackling this problem from the ordered groupoid perspective requires thorough understanding of the definition for restrictions as well as their properties with relation to ordered functors.

Lemma 7.5. *Let $\theta:\mathcal{P}\to\mathcal{Q}$ be a star-injective radiant between D-inverse constellations. Then $\forall x,y\in\mathcal{P}$:*

$$x=D(x)\cdot y \iff \exists D(x)\cdot D(y) \text{ and } x\theta=D(x\theta)\cdot y\theta.$$

Proof. *Let $x,y\in\mathcal{P}$ such that $x=D(x)\cdot y$. $\exists D(x)\cdot D(y)$ is immediate by **Result 2.17** and $x\theta=D(x\theta)\cdot y\theta$ by $(R1)$ and $(R2)$.*

*So suppose instead $\exists D(x)\cdot D(y)$ and $x\theta=D(x\theta)\cdot y\theta$. Then $\exists D(x)\cdot D(y)$ implies $\exists D(x)\cdot y$ by **Result 2.17**. By $(R1)$ and $(R2)$ we have:*

$$x\theta=D(x\theta)\cdot y\theta=D(x)\theta\cdot y\theta=(D(x)\cdot y)\theta.$$

Finally by star-injectivity $x=D(x)\cdot y$. □

Another way to describe this lemma is as Lawson does. $x\le y \iff D(x)\le D(y)$ and $x\theta\le y\theta$ but $\le$ is the natural partial order.

The final proposition of this chapter pertains to a special case of $P(G,X)$ not pursued in this work nor [6]. However, it may be interesting to revisit in its own right at some later stage.

Proposition 7.6. *Let $\phi:\Pi\to\mathbf{G}$ be a covering radiant from the D-inverse constellation Π onto the group $\mathbf{G}$. Then $\mathbf{G}$ acts on the poset $(D(\Pi),\le)$ by order automorphisms where $\le$ is the natural partial order and there is an isomorphism of D-inverse constellations $\theta:\Pi\to\mathbf{G}\times D(\Pi)$ such that $\theta\pi=\phi$.*

Proof. *Define for $g\in\mathbf{G}$ and $e\in D(\Pi)$:*

$$e\star g=D(x') \text{ for some } x\in\Pi \text{ such that } e=D(x) \text{ and } x\phi=g.$$

We show $\star$ is a group action by order automorphisms.

(GA1): *Let $e\in D(\Pi)$ then $e\star 1=D(x')$ for some $x\in\Pi$ such that $e=D(x)$ and $x\phi=1$.*

But then $e\phi=1$ by $(R2)$.

Therefore, $x = e$ by star-injectivity and $D(x') = D(e') = e$ so $e \star 1 = e$.

(GA2): *Let $e \in D(\Pi)$ and $g, h \in \mathbf{G}$ such that $e \star g = D(x')$ and $(e \star g) \star h = D(y')$ for some $x, y \in \Pi$.*

*Then $D(x') = D(y)$ so $\exists x \cdot y$ and $\exists y' \cdot x'$. We also have $D(x \cdot y) = D(x) = e$ by **Result 2.18**.*

Furthermore, $x\phi = g$ and $y\phi = h$ meaning $(x \cdot y)\phi = (x\phi) \cdot (y\phi) = gh$ by $(R1)$.

*Finally by **Result 2.18** and **Result 2.30**, we have:*

$$e \star (gh) = D((x \cdot y)') = D(y' \cdot x') = D(y') = (e \star g) \star h.$$

(GA3): *Let $e, f \in D(\Pi)$ and $g \in \mathbf{G}$ such that $\exists e \cdot f$ (i.e. $e \leq f$). Then let $e \star g = D(x')$ and $f \star g = D(y')$ for some $x, y \in \Pi$. We aim to show $\exists (e \star g) \cdot (f \star g)$.*

Observe $x\phi = y\phi = g$ so $D(x\theta) = D(y\theta)$ implying $\exists D(x\phi) \cdot y\theta$ and:

$$D(x\phi) \cdot y\phi = 1g = g = x\phi.$$

*Furthermore, $e \cdot f = D(x) \cdot D(y)$ so $x = D(x) \cdot y$ by **Lemma 7.5**.*

*Finally by **Lemma 2.28** and **Result 2.17**:*

$x = D(x) \cdot y$ implies $x' = D(x') \cdot y'$ so $\exists D(x') \cdot D(y')$ and $D(x') \cdot D(y') = (e \star g) \cdot (f \star g)$.

Thus $\mathbf{G}$ acts on $(D(\Pi), \leq)$ by order automorphisms.

Now define $\theta : \Pi \to \mathbf{G} \times D(\Pi)$ by:

$$x\theta = (x\phi, D(x')).$$

Then we show θ is an isomorphism of D-inverse constellations.

(R1): *Note I make heavy use of the notation $e \cdot f$ instead of $e \leq f$ for $e, f \in D(\Pi)$ in this section of the proof.*

*Suppose $x, y \in \Pi$ such that $\exists x \cdot y$. Then $\exists D(x') \cdot D(y)$ by **Lemma 2.29**. So:*

$\exists x\theta \cdot y\theta$ if and only if $\exists (x\phi, D(x')) \cdot (y\phi, D(y'))$;

but $\exists (x\phi, D(x')) \cdot (y\phi, D(y'))$ if and only if $\exists (D(x') \star y\phi) \cdot D(y')$, i.e. $(D(x') \star y\phi) \leq D(y')$;

but $\exists (D(x') \star y\phi) \cdot D(y')$ if and only if $\exists ((D(x') \star y\phi) \star y'\phi) \cdot (D(y') \star y'\phi)$ by $(GA3)$;

$= (D(x') \star ((y\phi)(y'\phi))) \cdot (D(y') \star y'\phi)$ by $(GA2)$;

$= (D(x') \star 1) \cdot (D(y') \star y'\phi)$;

$= D(x') \cdot (D(y') \star y'\phi)$ by $(GA1)$.

But y' is such an element where $D(y') = D(y')$ and $y'\phi = y'\phi$ So $D(y') \star y'\phi = D(y)$.

Indeed this means $\exists D(x') \cdot (D(y') \star y'\phi)$.

Therefore, $\exists (x\phi, D(x')) \cdot (y\phi, D(y'))$ and by $(R1)$ of ϕ, it is equal to:

$$((x\phi)(y\phi), D(x') \star y\phi) = ((x \cdot y)\phi, D(x') \star y\phi).$$

But:

$$(x \cdot y)\phi = ((x \cdot y)\phi, D((x \cdot y)').$$

Thus it remains to show $D(x') \star y\phi = D((x \cdot y)')$.

Consider the ordered groupoid $(\Pi, \circ, \leq, D, R)$ *corresponding to* $(\Pi, \cdot, D)$ *by* ***Result 2.31***.

Then by ***Lemma 2.32***, $D((x \cdot y)') = D((D(x')|y)')$.

It is precisely $D(x')|y$ *that satisfies* $D(x') = D(D(x')|y)$ *and:*

$$(D(x')|y)\phi = D(x')\phi|y\phi = 1|(y\phi) = y\phi.$$

Therefore, $D(x') \star y\phi = D((D(x')|y)') = D((x \cdot y)')$ *so:*

$$(x \cdot y)\theta = ((x \cdot y)\phi, D((x \cdot y)') = ((x\phi)(y\phi), D(x') \star y\phi) = (x\theta) \cdot (y\theta).$$

(R2): *Let* $x \in \Pi$. *Then:*

$$D(x)\theta = (D(x)\phi, D(D(x)')) = (1, D(x))) \text{ by } (R2) \text{ of } \phi \text{ and:}$$
$$D(x\theta) = D((x\phi, D(x'))) = (1, D(x') \star x'\phi).$$

By definition, $D(x') \star x'\phi = D(z')$ *for some* $z \in \Pi$ *such that* $D(x') = D(z)$ *and* $z\phi = x'\phi$.

Observe that $z = x'$ *satisfies this so* $D(x') \star x'\phi = D((x')') = D(x)$. *Therefore:*

$$D(x)\theta = (1, D(x)) = (1, D(x') \star x'\phi) = D(x\theta).$$

So θ *is a radiant. It remains to show* θ *is a bijection. Suppose* $x, y \in \Pi$ *such that* $x\theta = y\theta$. *Then:*

$$(x\phi, D(x')) = (y\phi, D(y')).$$

So $x\phi = y\phi$ *and* $D(x') = D(y')$.

But $x\phi = y\phi$ *implies* $x'\phi = y'\phi$ *so by* $(CR1)$ *of* ϕ *we have* $x' = y'$ *which implies* $x = y$.

So θ *is injective. We now show* θ *is surjective.*

Let $(g, e) \in \mathbf{G} \times D(\Pi)$.

T viewing $\mathbf{G}$ *as a D-inverse constellation.*

Therefore, with respect to ϕ, *we take* $f = e$ *and* $y = g'$ *as in the definition of* $(CR2)$; *notice trivially* $D(g') = 1 = D(z)\phi, \forall z \in \Pi$.

So $\exists x \in \Pi$ *such that* $D(x) = e$ *and* $x\phi = g'$.

Doing so we get $x'\theta = (x'\phi, D(x)) = (g, e)$. *So* θ *is surjective hence an isomorphism.*

Finally, let $x \in \Pi$. *Then:*

$$x(\theta\pi) = (x\theta)\pi = (x\phi, D(x'))\pi = x\phi.$$

So $\theta\pi = \phi$, *as required.* □

8 Enlargements

In [6], Lawson's chapter of the same name introduces **enlargements** of ordered subgroupoids. Therefore, we shall do the same for enlargements of D-inverse subconstellations.

Definition 8.1. *Let $\mathcal{P}$ be a D-inverse subconstellation of $\mathcal{Q}$. Then $\mathcal{Q}$ is an* ***enlargement*** *of $\mathcal{P}$ if it satisfies the following:*

(EN1) if $e \in D(\mathcal{P})$ and $f \in D(\mathcal{Q})$ such that $\exists f \cdot e$, then $f \in D(\mathcal{P})$;

(EN2) if $x \in \mathcal{Q}$ and $D(x), D(x') \in \mathcal{P}$, then $x \in \mathcal{P}$;

(EN3) if $e \in D(\mathcal{Q})$, then $\exists x \in \mathcal{Q}$ such that $D(x') = e$ and $D(x) \in \mathcal{P}$.

The above definition can be easily seen to correspond to the definitions of ordered groupoids since the domain elements in $(\mathbf{G}, \circ, \leq, D, R)$ are precisely the domain elements in $(\mathbf{G}, \cdot, D)$ and range elements can be rewritten as $R(x) = D(x'), \forall x \in \mathbf{G}$.

The main focus of this chapter is to provide another example of D-inverse constellations analogous to Lawson's constructed ordered groupoid in [6] where one can construct an enlargement of some other D-inverse subconstellation. Unlike previous examples, the D-inverse constellation approach may not be the easiest. Lawson observes a nifty property that by-passes having to check for groupoid axioms. I at least, was not able to follow his strategy, hence have opted for proving the below proposition the old fashioned way.

Proposition 8.2. *Let $\mathcal{P}$ and $\mathcal{Q}$ be D-inverse constellations and $p : \mathcal{P} \to \mathcal{Q}$ be a star-injective radiant. Then define the set:*

$$\mathcal{N} = \{(e, x, y) \in D(\mathcal{P}) \times \mathcal{Q} \times \mathcal{Q} : ep = D(x), D(x') = D(y)\}.$$

Define D and $'$ on $\mathcal{N}$ by:

$$D((e, x, y)) = (e, x, D(x')),\ (e, x, y)' = (e, x \cdot y, y').$$

Define the partial product $\cdot$ by:

$$(e, x, y) \cdot (f, w, z) = (e, x, y \cdot z) \iff \exists e \cdot f,\ \exists D(y') \cdot D(w') \text{ and } x \cdot y = D(x) \cdot w.$$

Then $(\mathcal{N}, D, ', \cdot)$ is a D-inverse constellation.

Proof. ***(CONST1):*** *Let $(e, x, y), (f, w, z), (g, s, t) \in \mathcal{N}$ such that $\exists (e, x, y) \cdot ((f, w, z) \cdot (g, s, t))$. Then $\exists f \cdot g$, $\exists D(z') \cdot D(s')$, $w \cdot z = D(w) \cdot s$ and it equals:*

$(e, x, y) \cdot ((f, w, z) \cdot (g, s, t)) = (e, x, y) \cdot (f, w, z \cdot t)$.

So $\exists e \cdot f$, $\exists D(y') \cdot D(w')$ and $x \cdot y = D(x) \cdot w$ and:

$$(e, x, y) \cdot (f, w, z \cdot t) = (e, x, y \cdot (z \cdot t)) = (e, x, (y \cdot z) \cdot t) \text{ by } (CONST1).$$

Note $\exists e \cdot f, \exists f \cdot g$ *implies* $\exists e \cdot g$ *by* $(CONST2)$ *and* $(CONST4)$.

By ***Lemma 2.32*** $\exists D((y \cdot z)') \cdot D(z')$ *so with* $\exists D(z') \cdot D(s')$, *we have* $\exists D((y \cdot z)') \cdot D(s')$ *by* $(CONST2)$ *and* $(CONST4)$.

By $(CONST2)$, $\exists D(x) \cdot w$ *and* $\exists w \cdot z$ *imply* $\exists D(x) \cdot (w \cdot z)$.

Substituting $x \cdot y = D(x) \cdot w$, $w \cdot z = D(w) \cdot s$ *alongside axioms* $(CONST1)$ *and* $(CONST4)$ *give:*

$$(x \cdot y) \cdot z = (D(x) \cdot w) \cdot z = D(x) \cdot (w \cdot z) = D(x) \cdot (D(w) \cdot s) = D(x) \cdot s.$$

Altogether, $\exists e \cdot g$, $\exists D((y \cdot z)') \cdot D(s')$ *and* $(x \cdot y) \cdot z = D(x) \cdot s$ *so* $\exists (e, x, (y \cdot z)) \cdot (g, s, t)$.

Furthermore, $\exists e \cdot f$, $\exists D(y') \cdot D(w')$ *and* $x \cdot y = D(x) \cdot w = D(x \cdot y) \cdot w$ *(By* ***Result 2.18****)* *imply* $\exists (e, x, y) \cdot (f, w, z)$ *so:*

$$(e, x, (y \cdot z) \cdot t) = (e, x, (y \cdot z)) \cdot (g, s, t) = ((e, x, y) \cdot (f, w, z)) \cdot (g, s, t).$$

(CONST2): *Let* $(e, x, y), (f, w, z), (g, s, t) \in \mathcal{N}$ *such that* $\exists (e, x, y) \cdot (f, w, z)$ *and* $\exists (f, w, z) \cdot (g, s, t)$. *Then* $(f, w, z) \cdot (g, s, t) = (f, w, z \cdot t)$.

By $\exists (e, x, y) \cdot (f, w, z)$, *we have* $\exists e \cdot f$, $\exists D(y') \cdot D(w')$ *and* $x \cdot y = D(x) \cdot w$.

So we get $\exists (e, x, y) \cdot (f, w, z \cdot t)$. *Indeed this is equal to* $(e, x, y) \cdot ((f, w, z) \cdot (g, s, t))$.

(CONST3): *Let* $(e, x, y) \in \mathcal{N}$. *Then* $D((e, x, y)) = (e, x, D(x'))$.

It is not hard to see $\exists e \cdot e$ $\exists D(x') \cdot D(x')$ *and* $x \cdot D(x') = x$ *so* $\exists D((e, x, y)) \cdot (e, x, y)$ *and it is equal to* $(e, x, D(x') \cdot y) = (e, x, D(y) \cdot y) = (e, x, y)$ *by* $(CONST3)$ *in* $\mathcal{Q}$.

We must also show uniqueness. Suppose $\exists (f, w, z) \in \mathcal{N}$ *such that* $(f, w, z) \cdot (e, x, y) = (e, x, y)$.

Then by definition $(f, w, z) \cdot (e, x, y) = (f, w, z \cdot y)$ *so* $f = e$, $w = x$ *and* $z \cdot y = y$ *implying* $z = D(y) = D(x')$ *by* $(CONST3)$ *in* $\mathcal{Q}$.

Indeed $(f, w, z) = (e, x, D(x'))$.

(CONST4): *Let* $(e, x, y) \in \mathcal{N}$ *and* $(f, z, D(z')) \in D(\mathcal{N})$ *such that* $\exists (e, x, y) \cdot (f, z, D(z'))$. *Then it is equal to* $(e, x, y \cdot D(z')) = (e, x, y)$ *by* $(CONST4)$ *in* $\mathcal{Q}$, *as required.*

(IC): *Let* $(e, x, y) \in \mathcal{N}$. *Then* $(e, x, y)' = (e, x \cdot y, y')$ *and* $D((e, x, y)) = (e, x, D(x')) = (e, x, D(y))$.

Trivially $\exists e \cdot e$ *and* $x \cdot y = x \cdot y$. *We also have* $\exists D(y') \cdot D((x \cdot y)')$ *since* $\exists y' \cdot x'$ *by definition* $(D(x') = D(y))$ *and* ***Result 2.30*** *implies* $D((x \cdot y)') = D(y' \cdot x') = D(y')$.

So $\exists (e, x, y) \cdot (e, x, y)'$ *and:*

$$(e, x, y) \cdot (e, x, y)' = (e, x, y) \cdot (e, x \cdot y, y') = (e, x, y \cdot y') = (e, x, D(y)) = D((e, x, y)).$$

Note $D((e, x, y)') = D((e, x \cdot y, y')) = (e, x \cdot y, D((x \cdot y)')) = (e, x \cdot y, D(y'))$ *by* $\exists y' \cdot x'$, ***Result 2.30*** *and* ***Result 2.18***.

Once again we know $\exists e \cdot e$, $\exists D(y') \cdot D(x)$ *and* $(x \cdot y) \cdot y' = x \cdot (y \cdot y') = x \cdot D(y) = x = D(x) \cdot x$ *by* $(CONST2)$, $(CONST3)$ *and* $(CONST4)$ *in* $\mathcal{Q}$. *So* $\exists (e, x, y)' \cdot (e, x, y)$ *and:*

$(e,x,y)' \cdot (e,x,y) = (e, x\cdot y, y') \cdot (e,x,y) = (e, x\cdot y, y' \cdot y) = (e, x\cdot y, D(y')) = D((e,x,y)')$.

It remains to show uniqueness of inverses.

By ***Result 2.27****, we need only show* $\mathcal{Q}$ *is normal. Let* $(e,x,D(x')), (f,y,D(y')) \in D(\mathcal{N})$ *such that* $\exists(e,x,D(x'))\cdot(f,y,D(y'))$ *and* $\exists(f,y,D(y'))\cdot(e,x,D(x'))$*. Then* $\exists e\cdot f$ *and* $\exists f \cdot e$ *so* $e = f$ *since* $\mathcal{P}$ *is normal.*

Observe $\exists(e,x,D(x'))\cdot(f,y,D(y'))$ *implies* $x = x\cdot D(x') = D(x)\cdot y$ *so by symmetry* $\exists(f,y,D(y'))\cdot(e,x,D(x'))$ *implies* $y = D(y)\cdot x$.

By the natural partial order on $\mathcal{Q}$ *we have* $x = y$.

Leaving us with $(e,x,D(x')) = (f,y,D(y'))$*, as required.* □

It is unclear whether Lawson decided axioms ($OG1$), ($OG2$) and (OI) were too trivial or omitted for brevity in his analogous proof in [6]. The truth of the matter may tip the scales towards or even further away for the D-inverse constellation approach.

We continue to follow along with Lawson's constructions and properties for D-inverse constellations.

Proposition 8.3. *Let* $\mathcal{N}$ *be the D-inverse constellation above. Define the functions* $\pi : \mathcal{N} \to \mathcal{Q}$ *and* $\zeta : \mathcal{P} \to \mathcal{N}$ *by* $(e,x,y)\pi = y$ *and* $x\zeta = (D(x), D(x)p, xp)$ *respectively. Then:*

(1) π *is a covering radiant;*

(2) $\forall x,y \in \mathcal{P}$, $x = D(x)\cdot y \iff x\zeta = D(x\zeta)\cdot y\zeta$;

(3) $\zeta\pi = p$.

Proof. ***(1):*** *We first prove* π *is a radiant.*

(R1): *Let* $(e,x,y),(f,w,z) \in \mathcal{N}$ *such that* $\exists(e,x,y)\cdot(f,w,z)$*. Then this product is equal to* $(e,x,y\cdot z)$ *and* $(e,x,y)\pi \cdot (f,w,z)\pi = y\cdot z = (e,x,y\cdot z)\pi$.

(R2): *Let* $(e,x,y),(f,w,z) \in \mathcal{N}$*. Then* $D((e,x,y)) = (e,x,D(x'))$ *so* $(e,x,D(x'))\pi = D(x') = D(y) = D((e,x,y)\pi)$.

Thus π *is a radiant. It remains to show it is star-bijective.*

(CR1): *Let* $D((e,x,y)) = D((g,s,t))$ *and* $(e,x,y)\pi = (g,s,t)\pi$*. Then* $y = t$ *and* $(e,x,D(x')) = (g,s,D(s'))$ *imply* $e = g$ *and* $x = s$.

So $(e,x,y) = (g,s,t)$ *hence* π *is star-injective.*

(CR2): *Consider* $(e,x,D(x')) \in D(\mathcal{N})$ *and* $v \in \mathcal{Q}$ *such that* $D(v) = (e,x,D(x'))\pi = D(x')$.

Then $(e,x,v) \in \mathcal{N}$ *is an element with properties* $D((e,x,v)) = (e,x,D(x'))$ *and* $(e,x,v)\pi = v$ *hence* π *star-surjective.*

Therefore, π *is a covering radiant.*

(2): *First suppose* $x,y \in \mathcal{P}$ *such that* $x = D(x)\cdot y$.

Then $D(x\zeta) = D((D(x), D(x)p, xp)) = (D(x), D(x)p, D(D(x)')p) = (D(x), D(x)p, D(x)p)$ *and* $y\zeta = (D(y), D(y)p, yp)$.

Observe $\exists D(x) \cdot y$ *implies* $D(x) \cdot D(y)$ *by* ***Result 2.17*** *so* $\exists D(x)p \cdot D(y)p$ *by* $(R1)$ *of* p *and note it is equal to* $D(D(x)'p) \cdot D(D(y)'p)$.

Furthermore, $D(x)p \cdot D(y)p = D(x)p$ *by* $(R2)$ *and* $(CONST4)$ *implies* $D(x)p \cdot D(x)p = D(D(x)p) \cdot D(y)p$.

Altogether, $\exists D(x) \cdot D(y)$, $\exists D(D(x)'p) \cdot D(D(y)'p)$ *and* $D(x)p \cdot D(x)p = D(D(x)p) \cdot D(y)p$ *imply* $\exists (D(x), D(x)p, D(x)p) \cdot (D(y), D(y)p, yp)$. *But this is* $D(x\zeta) \cdot y\zeta$ *so we have:*

$D(x\zeta) \cdot y\zeta = (D(x), D(x)p, D(x)p) \cdot (D(y), D(y)p, yp)$;

$= (D(x), D(x)p, D(x)p \cdot yp)$;

$= (D(x), D(x)p, (D(x) \cdot y)p)$ *by* $(R1)$ *of* p;

$= (D(x), D(x)p, xp)$;

$= x\zeta$, *as required.*

Suppose instead $x\zeta = D(x\zeta) \cdot y$. *Then* $\exists D(x) \cdot D(y)$ *and* $xp = D(x)p \cdot yp$ *imply* $x = D(x) \cdot y$ *by* ***Lemma 7.5***.

(3): *Let* $x \in \mathcal{P}$. *Then* $x\zeta\pi = (x\zeta)\pi = (D(x), D(x)p, xp)\pi = xp$. □

Define the relation ρ on $\mathcal{N}$ by:

$(e, x, y)\rho(f, w, z) \iff y = z$, and $\exists s \in \mathcal{P}$ such that $D(s) = f$, $D(s') = e$ and $w = (sp) \cdot x$.

At this point, my work slightly diverges from Lawson's. The reason is he shows ρ is an ordered congruence on his ordered groupoid. A straight-forward equivalent notion of these for D-inverse constellations has yet to be determined.

Proposition 8.4. *Let* $\mathcal{N}$ *be as defined above and* $\rho^\natural : \mathcal{N} \to \mathcal{N}/\boldsymbol{\rho}$ *such that* $(e, x, y)\rho^\natural = [e, x, y]$. *Then:*

(1) If $(e, x, y)\rho(f, w, z)$ *and* $(e, x, y) = D((e, x, y)) \cdot (f, w, z)$, *then* $(e, x, y) = (f, w, z)$;

(2) ρ *is a congruence on* $\mathcal{N}$;

(3) $\mathcal{N}/\boldsymbol{\rho}$ *is a D-inverse constellation;*

(4) $i = \zeta\rho^\natural$ *is an embedding.*

Proof. ***(1):*** *Let* $(e, x, y), (f, w, z) \in \mathcal{N}$ *such that* $(e, x, y)\rho(f, w, z)$ *and* $(e, x, y) = D((e, x, y)) \cdot (f, w, z)$. *Then* $y = z$ *and* $\exists s \in \mathcal{P}$ *such that* $D(s') = e, D(s) = f$ *and* $w = sp \cdot x$ *by definition of* ρ. *Therefore:*

$$\begin{aligned}(e, x, y) &= D((e, x, y)) \cdot (f, w, z);\\ &= (e, x, D(x')) \cdot (f, w, y) \text{ which gives } x = x \cdot D(x') = D(x) \cdot w.\end{aligned}$$

But $w = sp \cdot x = sp \cdot (D(x) \cdot w) = (sp \cdot D(x)) \cdot w = sp \cdot w$ *by* $(CONST1)$ *and* $(CONST4)$ *so* $sp = D(w)$.

Note also $w = sp \cdot x = D(w) \cdot x$*, so we have* $w = x$ *by the natural partial order on* $\mathcal{Q}$*.*

Therefore, $sp = D(w) = fp$ *and* $D(s) = f$ *imply* $s = f$ *by* $(CR1)$ *of* p*.*

Consequently $e = D(s') = D(s) = f$*. Finally we have* $(e, x, y) = (f, w, z)$*.*

(2): *By the same argument as Lawson in [6],* ρ *is an equivalence relation and respects the* D *operation. It is then sufficient to show* ρ *respects* $\cdot$ *in* $\mathcal{N}$*.*

We first show ρ *is a right-congruence with respect to* $\cdot$*. Let* $(e, x, y), (f, w, z), (g, s, t) \in \mathcal{N}$ *and suppose* $\exists(e, x, y) \cdot (g, s, t), \exists(f, w, z) \cdot (g, s, t)$ *and* $(e, x, y)\rho(f, w, z)$*. Then:*

$$\begin{aligned}(e, x, y) \cdot (g, s, t) &= (e, x, y \cdot t) \text{ with } \exists e \cdot g,\ \exists D(y') \cdot D(s') \text{ and } x \cdot y = D(x) \cdot s;\\ (f, w, z) \cdot (g, s, t) &= (f, w, z \cdot t) \text{ with } \exists f \cdot g,\ \exists D(z') \cdot D(s') \text{ and } w \cdot z = D(w) \cdot s.\end{aligned}$$

But $y = z$ *and* $\exists s \in \mathcal{P}$ *such that* $D(s) = f$*,* $D(s') = e$ *and* $w = (xp) \cdot s$ *by* $(e, x, y)\rho(f, w, z)$*.*

These are precisely the conditions for $(e, x, y \cdot t)\rho(f, w, z \cdot t)$*.*

So ρ *is a right-congruence with respect to* $\cdot$*.*

Alternatively suppose $(e, x, y), (f, w, z), (g, s, t) \in \mathcal{N}$ *such that* $\exists(e, x, y) \cdot (f, w, z)$*,* $\exists(e, x, y) \cdot (g, s, t)$ *and* $(f, w, z)\rho(g, s, t)$*. Then:*

$$\begin{aligned}(e, x, y) \cdot (f, w, z) &= (e, x, y \cdot z) \text{ with } \exists e \cdot f,\ \exists D(y') \cdot D(w') \text{ and } x \cdot y = D(x) \cdot w;\\ (e, x, y) \cdot (g, s, t) &= (e, x, y \cdot t) \text{ with } \exists e \cdot g,\ \exists D(y') \cdot D(s') \text{ and } x \cdot y = D(x) \cdot s.\end{aligned}$$

By $(f, w, z)\rho(g, s, t)$*, we have* $z = t$*, so actually* $(e, x, y \cdot z) = (e, x, y \cdot t)$ *and* $(e, x, y \cdot z)\rho(e, x, y \cdot t)$ *trivially.*

So ρ *is a left-congruence with respect to* $\cdot$ *hence* ρ *is a congruence on* $\mathcal{N}$*.*

(3): *This lacks an elegant proof so we opt to utilize **Lemma 2.38**. Therefore, through work in [6, Proposistion 8.3.2] and **Lemma 2.38** we have* $\mathcal{N}/\boldsymbol{\rho}$ *is a D-inverse constellation.*

(4): *By **Proposition 8.3**,* ζ *is injective. It then suffices to show* $\rho^\natural$ *is injective with respect to elements in the image of* ζ *for* $i = \zeta\rho^\natural$ *to be as well.*

Let $[D(x), D(x)p, xp], [D(y), D(y)p, yp] \in img(\zeta)$ *such that* $[D(x), D(x)p, xp] = [D(y), D(y)p, yp]$*.*

Then $xp = yp$ *and* $\exists s \in \mathcal{P}$ *such that* $D(s) = D(y)$*,* $D(s') = D(x)$ *and* $D(y)p = (sp) \cdot D(x)p = sp$ *by* $(CONST4)$ *and* $(R2)$ *of* p*.*

So $D(y) = s$ *by* $(CR1)$ *of* p*. It follows* $D(y) = s = D(s') = D(x)$*.*

Therefore $(D(x), D(x)p, xp) = (D(y), D(y)p, yp)$*, as required.*

It remains to show i *is a radiant.*

(R1): *Let* $x, y \in \mathcal{P}$ *such that* $\exists x \cdot y$*. Then* $(x \cdot y)i = [D(x), D(x)p, (x \cdot y)p]$*.*

$\exists(xi) \cdot (yi)$ *if and only if* $\exists[D(x), D(x)p, xp] \cdot [D(y), D(y)p, yp]$*.*

Observe $(D(x'), x'p, xp) \in [D(x), D(x)p, xp]$ *since* $s = x$ *is such an element where* $D(s) = D(x), D(s') = D(x')$ *and* $D(x)p = sp \cdot x'p$ *by* $(R1)$ *of* p*.*

$\exists D(x') \cdot D(y)$ *by **Lemma 2.29** so* $\exists D(D(x')'p) \cdot D(D(y)'p)$*.*

Altogether, $\exists D(x') \cdot D(y)$, $\exists D(D(x')'p) \cdot D(D(y)'p)$ *and* $x'p \cdot xp = D(x')p \cdot D(y)p$ *gives* $\exists (D(x'), x'p, xp) \cdot (D(y), D(y)p, yp)$ *and it is equal to* $(D(x'), x'p, (x \cdot y)p)$ *by* $(R1)$ *of* p.

In other words, $\exists (xi) \cdot (yi)$ *and it is equal to* $[D(x'), x'p, (x \cdot y)p]$.

By the exact same argument as $(D(x'), x'p, xp) \in [D(x), D(x)p, xp]$ *we have* $(D(x'), x'p, (x \cdot y)p) \in [D(x), D(x)p, (x \cdot y)p]$. *It follows:*

$$(xi) \cdot (yi) = [D(x'), x'p, (x \cdot y)p] = [D(x), D(x)p, (x \cdot y)p] = (x \cdot y)i.$$

(R2): *Let* $x \in \mathcal{P}$. *We show* $D(x)i = D(xi)$.

$$D(xi) = D([D(x), D(x)p, xp]) = [D(x), D(x)p, D(x)p] = D(x)p.$$

So i *is an embedding, as required.* □

This final theorem covers an example of an enlargement. Though the only one in this work, Lawson was able to extend this theory with ordered groupoids to prove the Maximum Enlargement theorem. Initial work in this area of my thesis aimed to cover exactly that for D-inverse constellations. Unfortunately this was held back by the elusive ordered congruences and special functors.

Theorem 8.5. *Let* $p : \mathcal{P} \to \mathcal{Q}$ *be a star-injective radiant between D-inverse constellations. Then there is a D-inverse constellation* $\mathcal{R}$*, an embedding* $i : \mathcal{P} \to \mathcal{R}$ *and a covering radiant* $\theta : \mathcal{R} \to \mathcal{Q}$ *such that* $\mathcal{R}$ *is an enlargement of* $(img(i), \cdot, D)$ *and* $i\theta = p$.

Proof. *Let* $\mathcal{N}, \rho, i, \zeta$ *and* $\rho^\natural$ *be as described above and let* $\mathcal{N}/\boldsymbol{\rho} = \mathcal{R}$. *Then* $i = \zeta\rho^\natural$ *is an embedding by* ***Proposition 8.4***.

Define $\theta : \mathcal{R} \to \mathcal{Q}$ *by:*

$$[e, x, y]\theta = y.$$

By the definition of ρ, θ *is well defined* ($\forall (f, w, z) \in [e, x, y]$*, we have* $y = z$) *and trivially a radiant. It remains to show* θ *is star-bijective.*

(CR1): *Let* $[e, x, y], [f, w, z] \in \mathcal{R}$ *such that* $[e, x, y]\theta = [f, w, z]\theta$ *and* $D([e, x, y]) = D([f, w, z])$. *Then* $y = z$ *and* $[e, x, D(x')] = [f, w, D(w')]$.

It directly follows $[e, x, y] = [f, w, z]$.

(CR2): *Let* $[e, x, D(x')] \in \mathcal{R}$ *and* $y \in \mathcal{Q}$ *such that* $[e, x, D(x')]\theta = D(y)$.

Then $D(x') = D(y)$ *so* $[e, x, y] \in \mathcal{R}$ *and observe* $[e, x, y]$ *is such an element where* $D([e, x, y]) = [e, x, D(x')]$ *and* $[e, x, y]\theta = y$*, as required.*

So θ *is a covering radiant.*

We know $(img(i), \cdot, D)$ *is a D-inverse subconstellation of* $\mathcal{R}$ *since* $(img(i), \circ, \leq, D, R)$ *is an ordered subgroupoid of* $(\mathcal{R}, \circ, \leq, D, R)$ *by* ***Result 2.31***.

We proceed to show $\mathcal{R}$ *is an enlargement of* $(img(i), \cdot, D)$.

(EN1): *Let* $[D(x), D(x)p, D(x)p] \in D(img(i))$ *and* $[e, y, D(y')] \in D(\mathcal{R})$ *where* $\exists [e, y, D(y')] \cdot [D(x), D(x)p, D(x)p]$.

Observe if $(e, ep, D(x)p)\rho(D(x), D(x)p, D(x)p)$ *then* $\exists s$ *such that* $D(s') = e$, $D(s) = D(x)$ *and* $D(x)p = (sp) \cdot ep = sp$ *by* $(CONST4)$ *and* $(R2)$ *of* p.

So $D(D(x)) = D(s)$ *and* $D(x)p = sp$ *implies* $s = D(x)$ *by* $(CR1)$ *of* p. *Therefore:*

$$D(x) = s = D(s') = e \text{ so } (e, ep, D(x)p) = (D(x), D(x)p, D(x)p).$$

Then $\exists (f, z, D(z')) \in [e, y, D(y')]$ *such that* $\exists (f, z, D(y')) \cdot (D(x), D(x)p, D(x)p)$ *and* $z = z \cdot D(y') = D(z) \cdot D(x)p = D(z)$ *by* $(CONST4)$ *and* $(R2)$ *of* p.

So $[f, z, D(z')] = [f, D(z), D(z)] = [f, fp, fp] \in D(img(i))$, *as required.*

(EN2): *Let* $[e, x, y] \in \mathcal{R}$ *such that* $[e, x, D(x')], [e, x \cdot y, D(y')] \in D(img(i))$. *Then* $\exists [f, fp, fp]$ *which is equal to* $[e, x, D(x')]$ *and* $\exists [g, gp, gp]$ *which is equal to* $[e, x \cdot y, D(y')]$.

Observe $[e, x, D(x')] = [f, fp, fp]$ *so* $\exists s \in \mathcal{P}$ *such that* $D(s) = f, D(s') = e$ *and:*

$$D(x') = fp = sp \cdot x \text{ so } sp = x', s'p = x.$$

Similarly we have:

$[g, gp, gp] = [e, x \cdot y, D(y')]$ *so* $\exists t \in \mathcal{P}$ *such that* $D(t) = e, D(t') = g$ *and* $x \cdot y = tp \cdot fp = tp$.

Now $e = D(s') = D(t)$ *so* $\exists s \cdot t$ *hence by* $D(x') = D(y)$, $(CONST1)$, $(CONST3)$ *and* $(R1)$ *of* p *we have:*

$$(s \cdot t)p = sp \cdot tp = x' \cdot (x \cdot y) = (x' \cdot x) \cdot y = D(x') \cdot y = D(y) \cdot y = y.$$

So $[e, x, y] = [e, s'p, (s \cdot t)p] = [D(s \cdot t), D(s \cdot t)p, (s \cdot t)p]$ *since* s *is such an element where* $D(s) = D(s \cdot t)$, $D(s') = e$ *and* $D(s)p = D(s \cdot t)p = sp \cdot s'p$.

Indeed $[e, x, y] = [D(s \cdot t), D(s \cdot t)p, (s \cdot t)p] \in img(i)$, *as required.*

(EN3): *Let* $[e, x, D(x')] \in D(\mathcal{R})$. *Then* $[e, D(x), x] \in \mathcal{R}$ *is such an element where:*

$$D([e, D(x), x]) = [e, D(x), D(x)] = [e, ep, ep] \in img(i);$$
$$D([e, D(x), x]') = D([e, D(x) \cdot x, x']) = [e, x, D(x')].$$

So $\mathcal{R}$ *is an enlargement of* $(img(i), \cdot, D)$.

The final task is to show $i\theta = p$. *Let* $x \in \mathcal{P}$. *Then:*

$$x(i\theta) = (xi)\theta = [D(x), D(x)p, xp]\theta = xp.$$

Precisely as was claimed. □

It is quite obvious in this particular case, the D-inverse constellation approach was not the best choice. This chapter has highlighted an example where utilizing the corresponding ordered groupoid is more efficient with more key concepts at their disposal.

9 Conclusion

This work defined and developed *inductive D-inverse constellations*. It was found that *inductive D-inverse constellations* (with *radiants*) correspond to *inverse semigroups* (with *semigroup homomorphisms*). This and work from Gould and Stokes in [4] proved the ESN-theorem itself as a by-product.

Certain kinds of congruences on *D-inverse constellations* were considered. Work by Lawson in [6] showed *(IS)-ordered congruences* on *ordered groupoids* are precisely *idempotent-separating congruences* on *inverse semigroups*. It so happens that *(IS)-ordered congruences* on *ordered groupoids* are precisely *(IS)-congruences* on *D-inverse constellations* upon translation. Therefore, this is the case for *(IS)-congruences* on *inductive D-inverse constellations* and *idempotent-separating congruences* on *inverse semigroups*. This was alternatively shown using the translation method from [1] to translate between *inductive D-inverse constellations* and *inverse semigroups*.

In general, constructions of *D-inverse constellations* equivalent to those of Lawson's are easier and more efficient to work with. However, it may not be that all theory on *ordered groupoids* can be elegantly carried over to *D-inverse constellations*. Prime examples of such concepts are *ordered congruences* and *special functors*. At the very least, *D-inverse constellations* may be used as a tool for *ordered groupoid* theory. At the very most, *D-inverse constellations* may become the first choice for research in *inverse semigroup* theory.

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