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$\mathcal{L}$-invariants and congruences for Galois representations of dimension 3, 4, and 8

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Abstract

The arithmetic of Galois representations plays a central role in modern number theory. In this thesis we consider representations arising from tensor products of the two-dimensional representations attached to modular forms by Deligne. In particular, we shall study the Iwasawa theory of the adjoint representation, as well as certain double and triple products of Deligne's representations.

In the first half we will undertake a computational study of $\mathcal{L}$-invariants attached to symmetric squares of modular forms. Let f be a primitive modular form of weight k and level N, and $p \nmid N$ a prime greater than two for which the attached representation is ordinary. The p-adic L-function for $\mathrm{Sym}^2 f$ always vanishes at $s = 1$, even though the complex L-function does not have a zero there. The $\mathcal{L}$-invariant itself appears on the right-hand side of the formula

$$\frac{\mathrm{d}}{\mathrm{d}s}\mathbf{L}_p(\mathrm{Sym}^2 f, s)\bigg|_{s=k-1} = \mathcal{L}_p(\mathrm{Sym}^2 f) \times (1-\alpha_p^{-2}p^{k-2})(1-\alpha_p^{-2}p^{k-1}) \times \frac{L_\infty(\mathrm{Sym}^2 f, k-1)}{\pi^{k-1}\langle f, f\rangle_N}$$

where $X^2 - a_p(f)X + p = (X-\alpha_p)(X-\beta_p)$ with $\alpha_p \in \mathbb{Z}_p^\times$.

Now let E be an elliptic curve over $\mathbb{Q}$ with associated modular form f_E, and $p \neq 2$ a prime of good ordinary reduction. We devise a method to calculate $\mathcal{L}_p(\mathrm{Sym}^2 f_E)$ effectively, then show it is non-trivial for almost all pairs of elliptic curves E of conductor $N_E \leq 300$ with $4|N_E$, and ordinary primes $p < 17$. Hence, in these cases at least, the order of the zero in $\mathbf{L}_p(\mathrm{Sym}^2 f_E, s)$ at $s = 1$ is exactly one. We also generalise this method to compute symmetric square $\mathcal{L}$-invariants for modular forms of weight $k > 2$.

In the second half we will establish congruences between p-adic L-functions. In the late 1990s, Vatsal showed that a congruence modulo p^ν between two newforms implied a congruence between their respective p-adic L-functions. We shall prove an analogous statement for both the double product and triple product p-adic L-functions, $\mathbf{L}_p(\mathbf{f}\otimes\mathbf{g})$ and $\mathbf{L}_p(\mathbf{f}\otimes\mathbf{g}\otimes\mathbf{h})$: the former is cyclotomic in its nature, while the latter is over the weight-space.

As a corollary, we derive transition formulae relating analytic λ-invariants for pairs of congruent Galois representations for $V_{\mathbf{f}} \otimes V_{\mathbf{g}}$, and for $V_{\mathbf{f}} \otimes V_{\mathbf{g}} \otimes V_{\mathbf{h}}$.

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Chapter 1

Introduction

The main purpose of Iwasawa theory is to link together the arithmetic world with the analytic world. The two principal objects are the analytic p-adic L-function which interpolates the normalised critical values, and the algebraic p-adic L-function which is traditionally the characteristic power series of some large Selmer group. The so-called 'Main Conjecture' predicts that they are equal, up to a unit of course. In this thesis we will be concerned solely with the analytic p-adic L-function.

As an illustration, in 1964 Kubota and Leopoldt interpreted the famous Kummer congruences between Bernoulli numbers by demonstrating that they imply the existence of a p-adically continuous function $\zeta_{p,s_0}(s)$ with the interpolation property

$$\zeta_{p,s_0}(k) = (1 - p^{k-1})\zeta(1-k), \quad k \in \mathbb{N}.$$

Here $\zeta(s)$ denotes the complex Riemann zeta function, and each choice of $s_0 \in \{0, 1, \ldots, p-2\}$ gives one of the $p-1$ 'branches' of the p-adic zeta function. The precise p-adic L-functions that we will be of interest to us are defined in Chapter 3. In this chapter we will only state our goals in general terms.

Let p be an odd prime, and M be a pure motive over $\mathbb{Q}$ that is ordinary at p. Coates and Perrin-Riou [10] give a conjectural recipe for attaching to M a p-adic L-function $\mathbf{L}_p(M, -, 0)$, and describe its expected behaviour at a critical point $s = 0$. At each Dirichlet character χ of conductor p^{n_χ}, the p-adic L-function should be related to its complex counterpart $L_\infty(M, \chi, s)$ via the equation

$$\mathbf{L}_p(M, \chi, 0) = \mathcal{E}_p(M, \chi^{-1}, 0) \cdot \frac{L_\infty(M, \chi, 0)}{\Omega_\infty^{\mathrm{sign}(\chi)}(M)}$$

for a suitable choice of Archimedean periods $\Omega_\infty^{\pm}(M) \in \mathbb{C}^\times$. The p-adic multiplier term $\mathcal{E}_p(M, \chi^{-1}, s)$ is described fully in Equation (4.14) of *op. cit.* and consists of a Gauss sum, an Euler factor at p, and a power of the unit root of Frobenius.

Curiously, the p-adic multiplier may vanish at $s = 0$ even when $L_\infty(M, \chi, s)$ does not, in which case we say that M has an exceptional p-adic zero. Let us factorise out the trivial zero contribution into $\mathcal{E}_p(M, \chi, s) = \mathcal{E}_p^\dagger(M, \chi, s) \times \mathcal{E}_p^{\mathrm{triv}}(M, \chi, s)$, where $\mathcal{E}_p^\dagger(M, \chi, 0) \neq 0$ and $\mathrm{order}_{s=0}\left(\mathcal{E}_p^{\mathrm{triv}}(M, \chi, s)\right) = \mathbf{e}_p$. Greenberg [31] has associated an explicit invariant $\mathcal{L}_p^{\mathrm{Gr}}(M) \in \mathbb{Q}_p$, and he predicts that

$$\left.\frac{\mathrm{d}^{\mathbf{e}_p}\mathbf{L}_p(M, \chi, s)}{\mathrm{d}s^{\mathbf{e}_p}}\right|_{s=0} = \mathcal{L}_p^{\mathrm{Gr}}(M) \times \mathcal{E}_p^\dagger(M, \chi, 0) \times \frac{L_\infty(M, \chi, 0)}{\Omega_\infty(M)} .$$

One is naturally left to address the following problem.

Question. *For a given motive M as described above, and for an ordinary prime p satisfying the exceptional zero condition, is Greenberg's $\mathcal{L}$-invariant term $\mathcal{L}_p^{\mathrm{Gr}}(M)$ non-zero?*

For example, let f be a primitive eigenform of weight $k \geq 2$, level N and trivial nebentypus. The symmetric square motives $M = \mathrm{Sym}^2(f)(k-1)$ and $M = \mathrm{Sym}^2(f)(k)$ both exhibit exceptional p-adic zero phenomena at ordinary primes $p \nmid N$. Recently there has been interest in this topic [3, 34, 56], in particular with the construction of global cohomology classes (an Euler system) for the motive $M(f \otimes f)$ in [49].

Under some standard assumptions, Hida has shown [40] that in a Λ-adic family of modular forms $\{\mathcal{F}_k\}_{k \in \mathcal{W}}$, the quantity $\mathcal{L}_p^{\mathrm{Gr}}\big(\mathrm{Sym}^2(\mathcal{F}_k)(k-1)\big)$ can vanish at only finitely many points in the weight-space $\mathcal{W}$. It therefore seems appropriate to address the problem of non-vanishing for the symmetric square

$\mathcal{L}$-invariant, albeit from a computational perspective. This is our goal in the first half of this thesis, with our main result being the following.

Theorem 1.1. *For every elliptic curve E with associated modular form f_E and conductor $N_E \leq 300$ such that $4 \mid N_E$, and every prime $p \leq 13$ for which E has good ordinary reduction, the invariant $\mathcal{L}_p^{\mathrm{Gr}}\big(\mathrm{Sym}^2(f_E)(1)\big)$ is non-zero, with at most ten exceptions* [1].

Two important quantities in Iwasawa theory are the μ-invariant (the minimum p-adic valuation of the coefficients of a power series) and the λ-invariant (the number of zeroes of the power series on the open unit disc). The second half of the thesis is concerned with addressing the following interesting problem:

Question. *How does the analytic λ-invariant appearing in the Main Conjecture vary as we switch between two p^ν-congruent $G_{\mathbb{Q}}$-representations?*

We give an answer for the analytic p-adic L-functions attached to double and triple product Galois representations, at least in certain common situations, by proving:

Theorem 1.2. *Suppose that $\mathbf{f}$, $\mathbf{g}^{(\mathrm{I})}$, and $\mathbf{g}^{(\mathrm{II})}$ are p-ordinary modular forms such that $a_l(\mathbf{g}^{(\mathrm{I})}) \equiv a_l(\mathbf{g}^{(\mathrm{II})}) \bmod p^\nu$ for all primes l not in the set $S_{\mathbf{g}}$ consisting of those primes that divide the level of $\mathbf{g}^{(\mathrm{I})}$ or $\mathbf{g}^{(\mathrm{II})}$. At each branch $j \in \{0, \ldots, p-2\}$:*

(i) $\mathbf{L}_{p,S_{\mathbf{g}}}\big(\mathbf{f} \otimes \mathbf{g}^{(\mathrm{I})}, \omega^j\big) \equiv \mathbf{L}_{p,S_{\mathbf{g}}}\big(\mathbf{f} \otimes \mathbf{g}^{(\mathrm{II})}, \omega^j\big) \mod p^{\mu_{\mathrm{cyc}}^{(j)}+\nu}$ *, and*

(ii) $\lambda\Big(\mathbf{L}_p\big(\mathbf{f} \otimes \mathbf{g}^{(\mathrm{I})}, \omega^j\big)\Big) = \lambda\Big(\mathbf{L}_p\big(\mathbf{f} \otimes \mathbf{g}^{(\mathrm{II})}, \omega^j\big)\Big) + \sum_{l \in S_{\mathbf{g}}} \mathbf{e}_l^{(\mathrm{II})}(\omega^j) - \mathbf{e}_l^{(\mathrm{I})}(\omega^j)$.

Here $\mu_{\mathrm{cyc}}^{(j)}$ denotes the minimum of the μ-invariants for $\mathbf{L}_p\big(\mathbf{f} \otimes \mathbf{g}^{(\mathrm{I})}, \omega^j\big)$ and $\mathbf{L}_p\big(\mathbf{f} \otimes \mathbf{g}^{(\mathrm{II})}, \omega^j\big)$, and $\mathbf{e}_l^{(\star)}(\omega^j)$ is the λ-invariant of the power series interpolating the Euler factor $L_l\big(\mathbf{f} \otimes \mathbf{g}^{(\star)} \otimes \omega^j, s\big)$ at a prime l.

[1] These possible exceptions are due to being unable to compute the $\mathcal{L}$-invariants to an accuracy sufficient to establish their non-vanishing, we do not believe that they are genuine exceptions.

We also obtain an analogous result for triple product L-functions, which we refrain from elaborating on here due to the complicated nature of the L-function's construction. We will simply remark that the congruences in the triple product case are subject to some restrictions on the levels of the newforms that are necessary for the p-adic L-functions to exist (see Theorems 6.3 and 6.6 for precise statements), and that they are in the weight variable as opposed to the cyclotomic variable.

Let us outline the structure of this thesis. Chapters 2 and 3 are dedicated to background material to aid in the digestion of the remaining chapters. Specifically, Chapter 2 focusses on modular forms: the critical values that are interpolated by p-adic L-functions may be expressed in terms of inner products of modular forms, so the material in this chapter equips us with useful tools for manipulating these expressions. In Chapter 3 we recall the definitions of complex L-functions attached to the symmetric square of the Galois representation associated with a modular form, as well as to tensor products of pairs and triples of such representations. We also review the interpolation properties of their p-adic analogues.

In Chapter 4 we devise a method for computing the $\mathcal{L}$-invariant of the symmetric square of an elliptic curve having good ordinary reduction at an odd prime p. We then use this method to perform the computations necessary to prove Theorem 1.1. Chapter 5 generalises this method to allow for modular forms of higher weight, and we compare our approach with the modular symbols technique of Dummit et al. [23].

In Chapter 6 we wade through a mire of technical calculations. Essentially we show that the various operators appearing in the expressions relating critical values of double and triple product L-functions to Petersson inner products are congruence preserving. Having done this, Chapter 7 is but a short stroll along the boardwalk to prove Theorem 1.2 and its triple product counterpart.

Chapter 2

Modular forms

In this chapter we lay out the foundations of classical modular forms. We then encounter families of p-adic modular forms, in particular Hida families passing through p-ordinary forms. The chapter concludes with a short discussion of nearly holomorphic modular forms.

2.1 Classical modular forms

We begin with an overview of the classical theory of modular forms. Authoritative sources for the material in this section are the books of Miyake [51], and Diamond and Shurman [22].

2.1.1 Basic definitions

We begin by describing the action of the group $\mathrm{GL}_2^+(\mathbb{R})$ on the upper half-plane $\mathfrak{h} = \{z \in \mathbb{C} \mid \mathrm{Im}(z) > 0\}$, and on the complex-valued functions defined on $\mathfrak{h}$. For any $z \in \mathfrak{h}$ and $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{GL}_2^+(\mathbb{R})$ we set

$$\gamma z = \frac{az+b}{cz+d} \in \mathfrak{h}.$$

For any function $f : \mathfrak{h} \to \mathbb{C}$ and positive integer k, we define the weight k action of γ on f as

$$\left(f\big|_k \gamma\right) = (\det \gamma)^{k/2}(cz+d)^{-k} f(\gamma z).$$

Definition 2.1 (Congruence subgroup)**.** *A congruence subgroup* Γ *is any subgroup of the modular group* $\mathrm{SL}_2(\mathbb{Z})$ *that contains*

$$\Gamma(N) = \mathrm{Ker}\left(\mathrm{SL}_2(\mathbb{Z}) \to \mathrm{SL}_2(\mathbb{Z}/N\mathbb{Z})\right)$$

for some positive integer N*, and the least such* N *is called the level of* Γ*.*

Of particular importance are the congruence subgroups

$$\Gamma_0(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}) \;\middle|\; c \equiv 0 \pmod{N} \right\}$$

and

$$\Gamma_1(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}) \;\middle|\; c \equiv 0 \pmod{N},\ a \equiv d \equiv 1 \pmod{N} \right\}.$$

Definition 2.2 (Modular forms)**.** *For any congruence subgroup* Γ *of level* N *and positive integer* k*, we call a function* $f : \mathfrak{h} \to \mathbb{C}$ *a modular form of weight* k *for* Γ *if the following properties hold:*

1. f *is holomorphic on the extended upper half-plane* $\mathfrak{h}^* = \mathfrak{h} \cup \mathbb{P}^1(\mathbb{Q})$*,*

2. f *is invariant under the weight* k *action of* Γ*.*

The set of all modular forms of weight k *for some fixed congruence subgroup* Γ *has the structure of a complex vector space, which we will denote* $\mathcal{M}_k(\Gamma)$*.*

The study of modular forms may be reduced to that of modular forms on the congruence subgroups $\Gamma_1(N)$, therefore we will frequently write $\mathcal{M}_k(N)$ for $\mathcal{M}_k(\Gamma_1(N))$. Moreover, $\mathcal{M}_k(N)$ itself decomposes into

$$\mathcal{M}_k(N) = \bigoplus_{\chi} \mathcal{M}_k(N, \chi) \tag{2.1}$$

where the sum is over all Dirichlet characters modulo N. Here $\mathcal{M}_k(N,\chi)$ denotes the χ-eigenspace of $\mathcal{M}_k(N)$, that is the space of elements of $\mathcal{M}_k(N)$ with the property that

$$f\big|_k \gamma = \chi(d) f \qquad \text{for all } \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(N).$$

A typical element $f \in \mathcal{M}_k(N)$ admits a Fourier expansion at the cusp ∞ of the form

$$f(z) = \sum_{n=0}^{\infty} a_n(f) q^n$$

where $q = \exp(2\pi i z)$.

2.1.2 Subspaces of modular forms

Next we define various subspaces of modular forms, which arise naturally as stable subspaces of $\mathcal{M}_k(N)$ under the action of the Hecke algebra of level N.

Definition 2.3 (Cusp forms)**.** *A modular form $f \in \mathcal{M}_k(N)$ exhibiting the additional property that f vanishes at all of its cusps, that is $f(z) = 0$ for all $z \in \mathbb{P}^1(\mathbb{Q})$, is called a cusp form. The subspace of cusp forms in $\mathcal{M}_k(N)$ is denoted by $\mathcal{S}_k(N)$.*

Definition 2.4 (Petersson inner product)**.** *Given any two modular forms f and g in $\mathcal{M}_k(N)$, such that at least one of them is a cusp form, we set*

$$\langle f, g \rangle_N = \int_{\Gamma_0(N)\backslash\mathfrak{h}} \overline{f(z)} g(z) y^k \frac{\mathrm{d}x\,\mathrm{d}y}{y^2}.$$

It follows from our normalisation of the Petersson inner product that

$$\langle f, g \rangle_M = [\Gamma_0(N) : \Gamma_0(M)] \times \langle f, g \rangle_N$$

whenever $N \mid M$, and f and g belong to $\mathcal{M}_k(N)$.

Definition 2.5 (Eisenstein series)**.** *A modular form $f \in \mathcal{M}_k(N)$ is called an Eisenstein series if $\langle f, g \rangle_N = 0$ for every $g \in \mathcal{S}_k(N)$. The subspace of all such forms is called the Eisenstein subspace, and is denoted throughout by $\mathcal{E}_k(N)$.*

We remark that $\mathcal{E}_k(N)$ is the orthogonal complement of $\mathcal{S}_k(N)$ inside $\mathcal{M}_k(N)$ under the Petersson inner product. It follows that we have the decomposition

$$\mathcal{M}_k(N) = \mathcal{S}_k(N) \oplus \mathcal{E}_k(N). \tag{2.2}$$

This decomposition induces analogous decompositions which respect the χ-eigenspaces $\mathcal{M}_k(N,\chi)$, as χ ranges over Dirichlet characters modulo N.

Suppose that N and M are positive integers with $M \mid N$, and that d is a positive divisor of $\frac{N}{M}$. Then there is a natural degeneracy map $V_d : \mathcal{S}_k(M) \to \mathcal{S}_k(N)$ given by setting $V_d\left(f(q)\right) = f(q^d)$. We define the old subspace of $\mathcal{S}_k(N)$ to be the images of these maps for proper divisors M of N, that is

$$\mathcal{S}_k^{\text{old}}(N) = \bigoplus_{\substack{M|N \\ M\neq N}} \bigoplus_{d|\frac{N}{M}} V_d(\mathcal{S}_k(M)).$$

Now if we define the new subspace $\mathcal{S}_k^{\text{new}}(N)$ to be the orthogonal complement of $\mathcal{S}_k^{\text{old}}(N)$ with respect to the Petersson inner product, we have a tautological decomposition

$$\mathcal{S}_k(N) = \mathcal{S}_k^{\text{old}}(N) \oplus \mathcal{S}_k^{\text{new}}(N).$$

2.1.3 Hecke operators on modular forms

Throughout this section we will take f to be a generic element of $\mathcal{M}_k(N,\chi)$. The degeneracy map V_m acts on f by the rule

$$\begin{aligned}\left(f\middle|V_m\right)(z) &= f(mz)\\ &= m^{-k/2} f\Big|_k \begin{pmatrix} m & 0 \\ 0 & 1 \end{pmatrix}(z).\end{aligned}$$

The map V_m sends $\mathcal{M}_k(N,\chi)$ to $\mathcal{M}_k(mN,\widetilde{\chi})$, and also maps $\mathcal{S}_k(N,\chi)$ to $\mathcal{S}_k(mN,\widetilde{\chi})$ [51, Theorem 4.6.1]; here $\widetilde{\chi}$ denotes the character modulo mN induced by χ. It is clear that $f\big|V_m$ will have the Fourier expansion

$$\left(f\middle|V_m\right)(z) = \sum_{n=0}^{\infty} a_n(f) q^{mn}.$$

For an integer $d \nmid N$, the diamond operator $\langle d\rangle : \mathcal{M}_k(N) \to \mathcal{M}_k(N)$ is defined by setting

$$\left(f\middle|\langle d\rangle\right)(z) = \left(f\big|_k\alpha\right)(z),$$

for any $\alpha = \left(\begin{smallmatrix} a & b \\ c & \delta \end{smallmatrix}\right) \in \Gamma_0(N)$ with $\delta \equiv d \pmod{N}$. The diamond operator respects the decomposition in Equation (2.1), and for any $f \in \mathcal{M}_k(N,\chi)$ we have $f\big|\langle d\rangle = \chi(d)f$.

Given a prime number q, the Hecke operator U_q is defined by setting

$$f\big|U_q = q^{k/2-1} \sum_{j=0}^{q-1} f\,\Big|_k \left(\begin{smallmatrix} 1 & j \\ 0 & q \end{smallmatrix}\right).$$

The modular form $f\big|U_q$ will have the Fourier expansion

$$\left(f\big|U_q\right)(z) = \sum_{n=0}^{\infty} a_{nq}(f) q^n.$$

We may then define the Hecke operator T_q by the relation

$$f\big|T_q = f\big|U_q + q^{k-1} f\big|\langle q\rangle \circ V_q.$$

Note that T_q coincides with U_q whenever $q \mid N$.

Definition 2.6. *We call f an eigenform if it is an eigenfunction of each Hecke operator T_q for every prime number q. An eigenform in $\mathcal{S}_k^{\text{new}}(N, \chi)$, normalised so that $a_1(f) = 1$, is then called a primitive eigenform.*

By Hecke's inverse theory, the T_q-eigenvalues of a primitive form are its Fourier coefficients, and we therefore have the identities

$$\begin{aligned} f\big|T_q &= a_q(f) f, \qquad \text{if } q \nmid N \\ f\big|U_q &= a_q(f) f, \qquad \text{if } q \mid N \end{aligned}$$

where f is a primitive form of exact level N. We can extend the operators T_n to all of $n \in \mathbb{N}$ by inductively defining

$$T_{q^r} = T_q T_{q^{r-1}} - q^{k-1} \langle q\rangle T_{q^{r-2}}, \qquad r \geq 2$$

(where T_1 is the identity operator) and setting

$$T_{mn} = T_m T_n, \qquad \gcd(m, n) = 1.$$

It is sometimes useful to know the relation between the Fourier coefficients of f and those of $f|T_n$. If $f \in \mathcal{M}_k(N, \chi)$ then by [51, Theorem 4.5.11] we have

$$a_m(f|T_n) = \sum_{d \,|\, \gcd(m,n)} \chi(d) d^{k-1} a_{mn/d^2}(f). \tag{2.3}$$

Definition 2.7 (Hecke algebra)**.** *The Hecke algebras* $\mathcal{H}_k(N,\chi)$ *and* $h_k(N,\chi)$ *are defined to be the subalgebras of* $\mathrm{End}(\mathcal{M}_k(N,\chi))$ *and* $\mathrm{End}(\mathcal{S}_k(N,\chi))$ *respectively, that are generated by* $\{T_n : n \in \mathbb{N}\} \cup \{\langle d\rangle : d \mid N\}$.

The Fricke involution W_N is defined by

$$\left(f\middle|W_N\right)(z) = (\sqrt{N}z)^{-k} f\left(-\frac{1}{Nz}\right)$$
$$= f\Big|_k \begin{pmatrix} 0 & -1 \\ N & 0 \end{pmatrix}(z),$$

and W_N induces the following isomorphisms [51, Theorem 4.3.2]:

$$\mathcal{M}_k(N,\chi) \simeq \mathcal{M}_k(N,\overline{\chi}),$$
$$\mathcal{S}_k(N,\chi) \simeq \mathcal{S}_k(N,\overline{\chi}),$$
$$\mathcal{E}_k(N,\chi) \simeq \mathcal{E}_k(N,\overline{\chi}).$$

Note that the operator W_N satisfies the relation

$$(f|W_N)|W_N = (-1)^k f.$$

Moreover, if f is a primitive form of level N then there exists a complex number γ_f of modulus 1, such that $f^\rho|W_N = \gamma_f f$ [51, Theorem 4.6.15]. Here $f^\rho(z) = \overline{f(-\overline{z})} = \sum_{n=0}^{\infty} \overline{a_n(f)} q^n$ denotes the conjugate primitive form.

Finally we must define the trace operator. Given a complete set of representatives $\mathcal{R}$ for $\Gamma_0(M)\backslash\Gamma_0(N)$ where $N \mid M$, we define

$$f\Big|\,\mathrm{Tr}_N^M = \sum_{\gamma\in\mathcal{R}} \overline{\chi}(\gamma) f\Big|_k \gamma \qquad \text{for all } f \in \mathcal{M}_k(M,\chi)$$

where we have written $\chi(\gamma) = \chi(d)$ for each $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$. The trace operator sends $\mathcal{M}_k(M,\chi)$ to $\mathcal{M}_k(N,\chi)$, and has the neat property that

$$\langle f, g\rangle_M = \langle f\Big|\,\mathrm{Tr}_N^M, g\rangle_N$$

for every $f \in \mathcal{M}_k(M)$ and $g \in \mathcal{M}_k(N)$ with one of f, g a cusp form. We can express 'Tr' in terms of Hecke operators and the involution W_N:

$$f\Big|\,\mathrm{Tr}_N^{Nq} = \begin{cases} q^{1-k/2} \times f\Big|W_{Nq} U_q W_N, & \text{if } q \mid N \\ q^{1-k/2} \times f\Big|W_{Nq} T_q W_N, & \text{if } q \nmid N. \end{cases}$$

Let us conclude this section with a list of relations involving these operators, that will be very useful in future calculations:

$$f\Big|W_N \circ V_m = m^{-k/2} \times f\Big|W_{Nm}, \tag{2.4}$$

$$f\Big|V_m \circ W_{Nm} = m^{-k/2} \times f\Big|W_N, \tag{2.5}$$

$$f\Big|W_N \circ T_q = f\Big|T_q^* \circ W_N, \tag{2.6}$$

$$f\Big|W_N \circ U_q = f\Big|U_q^* \circ W_N, \tag{2.7}$$

$$f\Big|W_N \circ \langle d\rangle = f\Big|\langle d\rangle \circ W_N^*, \tag{2.8}$$

Here T_q^*, U_q^*, and W_N^* denote the adjoint Hecke operators with respect to the Petersson inner product.

2.2 Hida families

In this section we introduce the notion of p-adic and Λ-adic modular forms; a standard reference is [38], especially Chapter 7. Throughout $p \geq 3$ will denote a fixed prime number. We begin with some notation.

Let us denote by μ_{p-1} the group of $(p-1)$-st roots of unity in $\mathbb{Z}_p$. The Teichmüller character $\omega : \mathbb{Z}_p^\times \to \mu_{p-1}$ is defined by setting $\omega(x)$ to be the unique $(p-1)$-st root of unity such that $\omega(x) \equiv x \pmod{p}$, or equivalently $\omega(x) = \lim_{n\to\infty} x^{p^n}$. We may also take the projection to principal local units $\langle\rangle : \mathbb{Z}_p^\times \to 1 + p\mathbb{Z}_p$ by choosing $\langle x\rangle$ to satisfy $x = \omega(x)\langle x\rangle$; in so doing, we have a unique direct product decomposition

$$\mathbb{Z}_p^\times = \mu_{p-1} \times \left(1 + p\mathbb{Z}_p\right).$$

We will also put $u := 1 + p$, which is a topological generator of $1 + p\mathbb{Z}_p$.

Let χ be a Dirichlet character modulo N, and A a subring of $\mathbb{C}$ that contains $\mathbb{Z}[\chi]$. It is well known that

$$\mathcal{M}_k(N, \chi) \cap A[\![q]\!] = \left(\mathcal{M}_k(N, \chi) \cap \mathbb{Z}[\chi][\![q]\!]\right) \otimes_{\mathbb{Z}[\chi]} A. \tag{2.9}$$

This simple fact motivates our definitions of spaces of p-adic modular forms.

Definition 2.8. *Let χ be a Dirichlet character modulo N, and A be a subring of $\mathbb{C}_p$ that contains $\mathbb{Z}[\chi]$. The space of p-adic modular forms over A with respect to χ is*

$$\mathcal{M}_k(N,\chi;A) = \big(\mathcal{M}_k(N,\chi) \cap \mathbb{Z}[\chi][\![q]\!]\big) \otimes_{\mathbb{Z}[\chi]} A,$$

and the subspace of p-adic cusp forms over A with respect to χ is

$$\mathcal{S}_k(N,\chi;A) = \big(\mathcal{S}_k(N,\chi) \cap \mathbb{Z}[\chi][\![q]\!]\big) \otimes_{\mathbb{Z}[\chi]} A.$$

That is, we regard p-adic modular forms as A-linear combinations of q-expansions of those elements of $\mathcal{M}_k(N,\chi)$ that have coefficients in $\mathbb{Z}[\chi]$. The action of the Hecke operators T_n naturally carries over to these spaces via Equation (2.3).

Definition 2.9 (Ordinary projector)**.** *The ordinary projector e is given by*

$$e = \lim_{n\to\infty} U_p^{n!}.$$

It follows from [38, Lemma 7.2.1] that e exists as an idempotent element of both $\mathcal{H}_k(N,\chi;A)$ and $h_k(N,\chi;A)$, and so we are able to define ordinary subspaces of modular forms as those cut out by e.

Definition 2.10 (Ordinary subspaces)**.** *The space of ordinary modular forms is given by*

$$\mathcal{M}_k^{\mathrm{ord}}(N,\chi;A) = \{f|e : f \in \mathcal{M}_k(N,\chi;A)\},$$

similarly, the space of ordinary cusp forms is

$$\mathcal{S}_k^{\mathrm{ord}}(N,\chi;A) = \{f|e : f \in \mathcal{S}_k(N,\chi;A)\}.$$

If f is an eigenform of U_p then it is clear that

$$f|e = \begin{cases} f & \text{if } |a_p(f)|_p = 1 \\ 0 & \text{if } |a_p(f)|_p < 1, \end{cases}$$

so the effect of the ordinary projector is to kill off the primitive forms that do not have p-adic unit eigenvalues $a_p(f)$. Remarkably, the rank of $\mathcal{S}_k^{\mathrm{ord}}(N,\chi;\mathbb{Z}_p)$ is constant as the weight k varies.

Theorem 2.11 (Hida)**.** *For any $k \geq 2$ and Dirichlet character χ we have*

$$\mathrm{rank}_{\mathbb{Z}_p} \mathcal{S}_k^{\mathrm{ord}}(N, \chi\omega^{-k}; \mathbb{Z}_p) = \mathrm{rank}_{\mathbb{Z}_p} \mathcal{S}_2^{\mathrm{ord}}(N, \chi\omega^{-2}; \mathbb{Z}_p).$$

In order to define Λ-adic modular forms, we fix a base ring $\mathcal{O}$, the ring of integers of some finite extension of $\mathbb{Q}_p$, and let $\Lambda = \mathcal{O}[\![X]\!]$. We may think of a Λ-adic modular form as a formal q-expansion, with coefficients that simultaneously interpolate the Fourier coefficients of a family of classical modular forms; the following makes this idea precise.

Definition 2.12 (Λ-adic modular form)**.** *Let χ be a character modulo N_0p^r, where $\gcd(N_0, p) = 1$, with values in $\mathcal{O}^\times$.*

- *A Λ-adic modular form of tame level N_0 and character χ is a formal q-expansion*
$$F(q) = \sum_{n=0}^{\infty} A_n(F; X)q^n \in \Lambda[\![q]\!]$$
with the property that for all but finitely many weights k, the specialisation $F(u^k - 1)$ lies in $\mathcal{M}_k(N_0p^r, \chi; \mathcal{O})$. The space of all Λ-adic forms of fixed tame level N_0 and character χ is denoted $\mathcal{M}(N_0p^\infty, \chi)$.

- *If $F(u^k - 1)$ is in $\mathcal{S}_k(N_0p^r, \chi; \mathcal{O})$ for all but finitely many k, then we call F a Λ-adic cusp form. The space of all Λ-adic cusp forms of fixed tame level N_0 and character χ is denoted by $\mathcal{S}(N_0p^\infty, \chi)$.*

- *Any weight k for which $F(u^k - 1)$ yields a classical modular form will be called an admissible weight for F.*

The Hecke operators defined in Section 2.1.3 may be extended to Λ-adic modular forms. For any Λ-adic modular form F, we define $F|T_n$ to be the series in $\Lambda[\![q]\!]$ with coefficients

$$A_m(F|T_n; X) = \sum_{b \,|\, \gcd(m,n)} \kappa(\langle b \rangle)(X)\chi(b)b^{-1}A_{mn/b^2}(F; X)$$

where the sum is over divisors of $\gcd(m, n)$ coprime to p and the character $\kappa : 1 + p\mathbb{Z}_p \to \Lambda^\times$ is given by $\kappa(u^s) = (1 + X)^s$ (c.f. Equation (2.3)). A straightforward calculation shows that if $F(u^k - 1) \in \mathcal{M}_k(N, \chi\omega^{-k}; \mathcal{O})$ then we also have $(F|T_n)(u^k - 1) = F(u^k - 1)|T_n \in \mathcal{M}_k(N, \chi\omega^{-k}; \mathcal{O})$.

Likewise, the notions of newforms and eigenforms carry over to Λ-adic modular forms in a natural way. As in the classical case, we define the Hecke algebras $\mathcal{H}(N_0p^\infty, \chi)$ and $h(N_0p^\infty, \chi)$ to be the subrings of $\mathrm{End}_\Lambda(\mathcal{M}(N_0p^\infty, \chi))$ and $\mathrm{End}_\Lambda(\mathcal{S}(N_0p^\infty, \chi))$ respectively that are generated by the Hecke operators T_n.

Definition 2.13. *Let F be a Λ-adic modular form of tame level N_0.*

- *We say that F is a newform if the specialisation $F(u^k - 1)$ is new at level N_0 for all but finitely many k.*
- *If the specialisation $F(u^k - 1)$ is an eigenform for all but finitely many k, then we call F an eigenform.*
- *If F is both a new at level N_0 and an eigenform, and F is normalised so that $A_1(F; X) = 1$, then we say that F is a primitive Λ-adic form.*

We also have a notion of an ordinary Λ-adic modular form.

Definition 2.14. *A Λ-adic modular form F is ordinary if*

$$F(u^k - 1) \in \mathcal{M}_k^{\mathrm{ord}}(N_0p^r, \chi)$$

for all but finitely many k. We denote by $\mathcal{M}^{\mathrm{ord}}(N_0p^\infty, \chi)$ and $\mathcal{S}^{\mathrm{ord}}(N_0p^\infty, \chi)$ the spaces of ordinary modular forms and ordinary cusp forms respectively.

The Hecke algebra $\mathcal{H}(N_0p^\infty, \chi)$ contains a unique element e that projects $\mathcal{M}_k(N_0p^\infty, \chi)$ onto $\mathcal{M}_k^{\mathrm{ord}}(N_0p^\infty, \chi)$, and satisfies the equation $(F|e)(u^k - 1) = F(u^k - 1)|e$ ([38, Proposition 7.3.1]).

The following result shows that we may think of the space $\mathcal{S}^{\mathrm{ord}}(N_0p^\infty, \chi)$ as p-adically interpolating the spaces $\mathcal{S}_k^{\mathrm{ord}}(N_0p, \chi\omega^{-k}; \mathbb{Z}_p)$ as the weight varies.

Theorem 2.15. *Let χ be a Dirichlet character with conductor N_0p.*

1. *The space $\mathcal{S}^{\mathrm{ord}}(N_0p^\infty, \chi)$ is free of finite rank over Λ, in particular*

$$\mathrm{rank}_\Lambda\, \mathcal{S}^{\mathrm{ord}}(N_0p^\infty, \chi) = \mathrm{rank}_{\mathbb{Z}_p}\, \mathcal{S}_2^{\mathrm{ord}}(N_0p, \chi\omega^{-2}; \mathbb{Z}_p).$$

2. *After a suitable extension of coefficients to a finite extension K of* $\mathrm{Frac}(\Lambda)$, *the space $\mathcal{S}^{\mathrm{ord}}(N_0p^\infty, \chi) \otimes_\Lambda K$ has a basis of Hecke eigenforms. Moreover, after specialising this basis at a weight $k \geq 2$ we obtain a basis of Hecke eigenforms for the space $\mathcal{S}_k^{\mathrm{ord}}(N_0p, \chi\omega^{-k}; \mathcal{O})$ where $\mathcal{O}$ is the ring of integers of some finite extension of $\mathbb{Q}_p$.*

Definition 2.16. *Let $F \in \mathcal{S}^{\mathrm{ord}}(N_0p^\infty, \chi) \otimes_\Lambda \Lambda_K$ where Λ_K is the integral closure of Λ in some finite extension K of* $\mathrm{Frac}(\Lambda)$. *We call the set*

$$\{F(u^k - 1) \mid k \text{ is admissible for } F\}$$

a Hida family. If f is an ordinary cusp form contained in some Hida family, then we say that the family passes through f.

Proposition 2.17. *For every cusp form f, there exists a unique (up to Galois conjugacy) Hida family passing through f.*

2.3 Nearly holomorphic functions

We recall properties of the Maass-Shimura differential operator '$\delta_w^{(r)}$' from [60], and then we give some background on the projection mapping 'Hol_∞'.

Let $w, r \geq 0$ be integers, and consider the operator $\delta_w := \frac{1}{2\pi i}\left(\frac{w}{2iy} + \frac{\partial}{\partial z}\right)$ where as usual $\frac{\partial}{\partial z} = \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right)$ for all $z = x + iy$. One can take an r-fold composition

$$\delta_w^{(r)} := \delta_{w+2r-2} \circ \cdots \circ \delta_{w+2} \circ \delta_w$$

with the understanding that if $r = 0$, then $\delta_w^{(0)}$ just refers to the identity operator.

If G is a holomorphic modular form of weight w, level N and character ψ, then $\delta_w^{(r)}(G)$ has weight $w + 2r$, level N and character ψ although it may no longer be holomorphic; in fact $\delta_w^{(r)}(G)$ is an element of the set

$$\left\{\sum_{j=0}^{r} y^{-j} \cdot h_j \;\middle|\; h_j \text{ is holomorphic}\right\}.$$

It follows that $\delta_w^{(r)}(G) \in \mathcal{C}^\infty(\mathfrak{h})$ belongs to the larger space of $\mathcal{C}^\infty$-modular forms, denoted by $\mathcal{M}^\infty_{w+2r}\big(\Gamma(N)\big)$, and exhibits 'moderate growth' in the sense of [33, 53]. Specifically, a form $H \in \mathcal{M}^\infty_w\big(\Gamma(N)\big)$ is said to have *moderate growth at* $\gamma \in \mathrm{SL}_2(\mathbb{Z})$ if for all $z \in \mathfrak{h}$ and $s \in \mathbb{C}$ with $\mathrm{Re}(s) \gg 0$, the complex integrals

$$\int_{\tau=u+iv\in\mathfrak{h}} \big(H\big|_w\gamma\big)(\tau)\cdot\big(\overline{\tau}-z\big)^{-w-2r}\big|\overline{\tau}-z\big|_\infty^{-2s}\,\big(\mathrm{Im}(\tau)\big)^{w+2r+s}\cdot\frac{\mathrm{d}u\mathrm{d}v}{v^2} \qquad (2.10)$$

are absolutely convergent, and admit an analytic continuation to the point $s = 0$.

Definition 2.18. *If* $H(z) = \sum_{m\in\mathbb{Z}} A_H(y,m)\cdot e^{2\pi imx} \in \mathcal{M}^\infty_w(N,\psi)$ *denotes an arbitrary* $\mathcal{C}^\infty$*-modular form with* $w \geq 2$ *and* $A_H(y,m) \in \mathcal{C}^\infty(\mathbb{R}^+)$*, then we define*

$$\mathrm{Hol}_\infty(H) \;:=\; \sum_{n=0}^{\infty} a(n,H)\cdot q^n \;\in\; \mathbb{C}\big[\big[q\big]\big]$$

where at each integer $n > 0$*, the* n*-th Fourier coefficient is given by*

$$a(n,H) \;=\; \lim_{s\to 0^+}\left(\frac{(4\pi n)^{w-1}}{\Gamma(w-1)}\cdot\int_0^\infty A_H(y,n)\,e^{-2\pi ny}y^{w+s-2}\cdot\mathrm{d}y\right).$$

Theorem 2.19. *(Gross-Zagier and Panchishkin [33, 53]) Let us suppose that* $H(z) \in \mathcal{M}^\infty_w(N,\psi)$ *is a* $\mathcal{C}^\infty$*-modular form which exhibits the two extra properties:*

(i) the coefficients $A_H(y,m) = 0$ *for all* $m \leq 0$*, and*

(ii) $H\big|_w\gamma \in \mathcal{M}^\infty_w\big(\Gamma(N)\big)$, $\gamma \in \mathrm{SL}_2(\mathbb{Z})$ *has moderate growth, cf. Equation (2.10).*

Then $a(0,H) = 0$*, moreover* $\mathrm{Hol}_\infty(H)$ *belongs to* $\mathcal{M}_w(N,\psi)$ *i.e. it is a classical holomorphic modular form, and lastly it satisfies the inner product identity*

$$\big\langle F, \mathrm{Hol}_\infty(H)\big\rangle_N = \big\langle F, H\big\rangle_N \quad \textit{at every } F \in \mathcal{S}_w(N,\psi).$$

We will return to these $\mathcal{C}^\infty$-modular forms in Chapter 6, where they will be used to analyse both the double product and triple product L-functions.

Chapter 3

L-functions attached to Galois representations

In this chapter we survey various complex and p-adic L-functions that are going to be used later in this thesis. The archetypal complex L-function is the Riemann zeta function

$$\zeta(s) = \sum_{n=1}^{\infty} n^{-s}$$

which converges for all complex s with $\mathrm{Re}(s) > 1$. Some properties of $\zeta(s)$ which will be shared in some form by the other complex L-functions in this chapter are:

- the Riemann zeta function has an Euler product expansion

$$\zeta(s) = \prod_{p \text{ prime}} \frac{1}{1 - p^{-s}}$$

  which also converges when $\mathrm{Re}(s) > 1$;

- analytic continuation to a meromorphic function on the entire complex plane via the functional equation

$$\zeta(s) = 2^s \pi^{s-1} \sin\left(\frac{\pi s}{2}\right) \Gamma(1-s)\zeta(1-s)$$

  relating the value at s to the value at $1 - s$;

- the values of the zeta function at negative integers are rational, specifically we have

$$\zeta(1-k) = -\frac{B_k}{k}$$

for each integer $k \geq 1$, where B_k is the k-th Bernoulli number defined by the Taylor series expansion

$$\frac{t}{e^t - 1} = \sum_{k=0}^{\infty} B_k \frac{t^k}{k!}.$$

The final property allows us to construct a p-adic analogue of the Riemann zeta function; it is these special values which will be interpolated by the Kubota-Leopoldt L-function constructed in the latter half of the next section.

3.1 Constructing p-adic L-functions

Before continuing our discussion of L-functions, we will develop the tools necessary for constructing p-adic L-functions, and illustrate their use by constructing the Kubota-Leopoldt L-function from scratch.

3.1.1 Distributions and measures

We begin by defining distributions and measures, and using them to develop a theory of p-adic integration. The exposition mainly follows that of Chapter II of [47] and Chapter 12 of [69].

Definition 3.1 (Distribution). *Let X be a compact-open subset of $\mathbb{Q}_p$. A p-adic distribution on X is a $\mathbb{Q}_p$-linear homomorphism*

$$\mu : \mathrm{Step}(X) \to \mathbb{C}_p$$

where $\mathrm{Step}(X)$ *is the $\mathbb{Q}_p$-vector space of locally constant functions from X to $\mathbb{Q}_p$.*

The set X will typically be either $\mathbb{Z}_p$ or $\mathbb{Z}_p^\times$. Given a locally constant function $f : X \to \mathbb{Q}_p$ we will generally write $\int_X f \mathrm{d}\mu$ instead of $\mu(f)$ for value of μ at f.

Equivalently, we may think of p-adic distributions as finite additive functions on the compact-open subsets of X.

Any distribution restricted to a function on compact-open subsets by setting $\mu(U) = \int_X \delta_U \mathrm{d}\mu$ is finitely additive. Conversely, if we are given a finitely additive function μ, we can set $\int_X \delta_U \mathrm{d}\mu = \mu(U)$ where δ_U is the characteristic function on U. This naturally extends to a $\mathbb{Q}_p$-linear function on $\mathrm{Step}(X)$, since any locally constant function can be written as a linear combination of characteristic functions. For example, the function μ_α defined by setting

$$\mu_\alpha(U) = \begin{cases} 1, & \text{if } \alpha \in U \\ 0, & \text{otherwise} \end{cases}$$

for some fixed $\alpha \in \mathbb{Z}_p$ can easily seen to be additive. Therefore it extends to a distribution on $\mathbb{Z}_p$, which we call the Dirac distribution concentrated at α.

Proposition 3.2. *Let μ be a $\mathbb{C}_p$-valued function defined on the intervals in $X \subseteq \mathbb{Q}_p$ with the property that for every interval $a + p^N\mathbb{Z}_p \subseteq X$ we have,*

$$\mu(a + p^N\mathbb{Z}_p) = \sum_{b=0}^{N-1} \mu(a + bp^N + p^{N+1}\mathbb{Z}_p).$$

The function μ extends uniquely to a p-adic distribution on X.

Any compact-open subset of X can be written as a finite union of intervals, say $U = \bigcup I_i$. We may therefore extend μ to a function on the compact-open subsets of X by defining $\mu(U) = \sum \mu(I_i)$ (if μ is to be additive, then it must have this property). That μ does not depend on the partition of U into intervals follows by applying the equality given in the statement of the proposition to a refinement of two different partitions of U (see the proposition on page 32 of [47] for details).

This proposition may be used to construct p-adic distributions, for example the Haar distribution μ_{Haar} which is defined by setting

$$\mu_{\mathrm{Haar}}\left(a + p^N\mathbb{Z}_p\right) = \frac{1}{p^N}.$$

This distribution is an example of a Bernoulli distribution (which will be defined in Section 3.1.2, and will play a role in the construction of the ζ-function). Observe that μ_{Haar} grows p-adically as $N \to \infty$, as is the case for all Bernoulli distributions. This prevents us from extending the definition of $\int f \mathrm{d}\mu_{\text{Haar}}$ beyond locally constant functions.

In order to make sense of $\int f \mathrm{d}\mu$ for all continuous functions f, we will need to look at a subset of distributions called measures.

Definition 3.3 (Measure)**.** *A p-adic distribution μ on $X \subseteq \mathbb{Q}_p$ is called a measure on X if it is p-adically bounded, that is if there is some number $K \in \mathbb{R}$ such that*

$$|\mu(U)|_p \leq K$$

for every compact-open set $U \subseteq X$.

Now if we have some function f in $C(X)$, the space of continuous $\mathbb{Q}_p$-valued functions on X, and if $\{f_n\}$ is a Cauchy sequence of functions in $\mathrm{Step}(X)$ that converges to f, then it follows from the boundedness of μ and the strong triangle inequality that

$$\left| \int_X f_n \mathrm{d}\mu - \int_X f_m \mathrm{d}\mu \right|_p \to 0$$

as $m, n \to \infty$ (see Proposition 12.1 of [69]). Therefore we may define

$$\int_X f \mathrm{d}\mu = \lim_{n \to \infty} \int_X f_n \mathrm{d}\mu$$

which leads us the following result.

Proposition 3.4. *If μ is a bounded measure on a compact-open set $X \subseteq \mathbb{Q}_p$ then the map*

$$\int_X f \mathrm{d}\mu : \mathrm{Step}(X) \to \mathbb{C}_p$$

extends uniquely to a continuous map

$$\int_X f \mathrm{d}\mu : C(X) \to \mathbb{C}_p$$

where $C(X)$ denotes the space of continuous functions on X.

Later we will make use of the following important technical result [47, p. 84].

Proposition 3.5. *Let $f, g : X \to \mathbb{Q}_p$ be two continuous functions such that $|f(x) - g(x)|_p \leq \varepsilon$ for all $x \in X$, and μ a measure on X that is bounded by $K \in \mathbb{R}$. Then we have the inequality*

$$\left| \int_X f \mathrm{d}\mu - \int_X g \mathrm{d}\mu \right| \leq \varepsilon K.$$

As an example, consider the Dirac distribution μ_α which is clearly a measure. We have $\int f \mathrm{d}\mu_\alpha = f(\alpha)$ for every $f \in \mathrm{Step}(X)$ and hence for every $f \in C(X)$ also. However the Haar distribution is not a measure, and while $\int_X f_n \mathrm{d}\mu_{\mathrm{Haar}}$ might converge as $n \to \infty$ for some Cauchy sequence $\{f_n\} \to f \in C(X)$, the value of the limit is not independent of the choice of Cauchy sequence $\{f_n\}$ (see the discussion in [69, p. 238]).

In fact, the requirement that a distribution be bounded is stronger than necessary. We are also able to develop a theory of p-adic integration using the more delicate notion of an h-admissible measure, as introduced by Amice and Vélu [1] and Višik [68].

Definition 3.6 (h-admissible measure)**.** *Given a positive integer h, an h-admissible measure on $\mathbb{Z}_p$ is a linear homomorphism $\mu : C^h(\mathbb{Z}_p^\times) \to \mathbb{C}_p$ that satisfies the growth condition*

$$\left| \sup_{a \in \mathbb{Z}_p} \int_{a + p^N \mathbb{Z}_p} (x_p - a_p)^i \mathrm{d}\mu \right|_p = o\left(\left| p^N \mathbb{Z}_p \right|_p^{i-h} \right)$$

for all i such that $0 \leq i < h$. Here x_p denotes the inclusion maps $\mathbb{Z}_p^\times \hookrightarrow \mathbb{C}_p^\times$, and $C^h(\mathbb{Z}_p^\times)$ is the space of $\mathbb{C}_p$-valued functions that can be locally represented by polynomials of degree less than h.

Let $f : \mathbb{Z}_p^\times \to \mathbb{C}_p$ be a function for which $\mathrm{d}^{h-1} f / \mathrm{d}x_p^{h-1}$ exists and is Lipschitz continuous. For each positive integer N we define the Riemann sum

$$S_N(f) = \sum_{e \in (\mathbb{Z}/p^N\mathbb{Z})^\times} \int_{e + p^N \mathbb{Z}_p} \sum_{i=0}^{h-1} \frac{f^{(i)}(e)}{i!} (x_p - e_p)^i \mathrm{d}\mu.$$

Now $S_N(f)$ converges as $N \to \infty$ ([68, Definition-Lemma 1.6]), and so we may define

$$\int_{\mathbb{Z}_p^\times} f \mathrm{d}\mu = \lim_{N \to \infty} S_N(f).$$

3.1.2 Example: The Kubota-Leopoldt L-function

Now we will use the theory of p-adic integration from the previous section in order to construct a p-adic analytic function that interpolates the special values of the Riemann zeta function.

To begin with we need to p-adically interpolate the function $f(s) = n^s$. To do this, we must choose a positive integer n that is not divisible by p, and fix a number $s_0 \in \{0, 1, 2, \ldots, p-2\}$. Now if s and s' are two non-negative integers, both congruent to s_0 modulo $p-1$, such that $|s - s'|_p \leq 1/p^N$, then it can be shown that $|n^s - n^{s'}|_p \leq 1/p^{N+1}$. Furthermore, the set

$$S_{s_0} = \{s \in \mathbb{N} : s \equiv s_0 \pmod{p-1}\}$$

is dense in $\mathbb{Z}_p$ so each function $f : S_{s_0} \to \mathbb{Z}_p$ defined by $f(s) = n^s$ extends uniquely to a continuous function $f : \mathbb{Z}_p \to \mathbb{Z}_p$.

Recall that the Riemann ζ-function has special values $\zeta(1-k) = -B_k/k$ where the k-th Bernoulli number B_k is defined by the Taylor series expansion

$$\frac{t}{e^t - 1} = \sum_{k=0}^{\infty} B_k \frac{t^k}{k!}.$$

Our goal is to p-adically interpolate the numbers $-B_k/k$. To this end we define Bernoulli distributions using Bernoulli polynomials, which for each non-negative integer k is defined to be the polynomial $B_k(x)$ so that we have the Taylor series expansion

$$\frac{te^{xt}}{e^t - 1} = \sum_{k=0}^{\infty} B_k(x) \frac{t^k}{k!}.$$

Definition 3.7 (Bernoulli distribution)**.** *For each positive integer k define a map $\mu_{B,k}$ on intervals $a + p^N\mathbb{Z}_p$ by setting*

$$\mu_{B,k}\left(a + p^N\mathbb{Z}_p\right) = p^{N(k-1)} B_k\left(\frac{a}{p^N}\right).$$

This map satisfies the property required by Proposition 3.2, and therefore extends to a distribution on $\mathbb{Z}_p$.

A simple calculation shows that

$$\int_{\mathbb{Z}_p} \mathrm{d}\mu_{B,k} = B_k, \tag{3.1}$$

so we may refine our main goal to interpolating $(-1/k)\int_{\mathbb{Z}_p} \mathrm{d}\mu_{B,k}$. As mentioned in the previous section, Bernoulli distributions are not bounded and so before we proceed, we must modify them slightly in order to obtain a true measure.

Definition 3.8. *Let $\alpha \neq 1$ be a rational integer not divisible by p. The α-regularised Bernoulli distribution $\mu_{k,\alpha}$ is defined by*

$$\mu_{k,\alpha}(U) = \mu_{B,k}(U) - \alpha^{-k}\mu_{B,k}(\alpha U)$$

for all compact-open subsets $U \subseteq \mathbb{Z}_p^\times$.

These distributions, for different k, are all closely related to $\mu_{1,\alpha}$ as we see in the following result.

Proposition 3.9. *If we define d_k to be the least common denominator of the coefficients of $B_k(X)$, then*

$$d_k\mu_{k,\alpha}\left(a + p^N\mathbb{Z}_p\right) \equiv d_k k a^{k-1}\mu_{1,\alpha}\left(a + p^N\mathbb{Z}_p\right) \quad \left(\operatorname{mod} p^N\right).$$

This is proved in [47, Theorem II.5.5]. It follows immediately that these distributions are indeed bounded measures, and furthermore

$$\int_U \mathrm{d}\mu_{k,\alpha} = k\int_U x^{k-1}\mathrm{d}\mu_{1,\alpha} \tag{3.2}$$

for all compact-open U.

We claim that the integral on the right hand side of this equation can be p-adically interpolated. We saw earlier that if $k' \equiv k \pmod{(p-1)p^N}$ then

$$\left|x^{k'-1} - x^{k-1}\right|_p \leq \frac{1}{p^{N+1}}$$

for $x \in \mathbb{Z}_p^\times$, and so by Proposition 3.5

$$\left|\int_{\mathbb{Z}_p^\times} x^{k'-1}\mathrm{d}\mu_{1,\alpha} - \int_{\mathbb{Z}_p^\times} x^{k-1}\mathrm{d}\mu_{1,\alpha}\right|_p \leq \frac{1}{p^{N+1}}. \tag{3.3}$$

Therefore if we fix an $s_0 \in \{0, 1, 2, \ldots, p-2\}$, then the function $\int_{\mathbb{Z}_p^\times} x^{k-1}\mathrm{d}\mu_{1,\alpha}$, defined for all $k \in S_{s_0}$, extends to a continuous function

$$\int_{\mathbb{Z}_p^\times} x^{s_0+s(p-1)-1}\mathrm{d}\mu_{1,\alpha}$$

which is defined for all p-adic integers s. We then calculate for each non-negative integer k,

$$\int_{\mathbb{Z}_p^\times} \mathrm{d}\mu_{k,\alpha} = (1-\alpha^{-k})(1-p^{k-1})B_k, \tag{3.4}$$

and combining this with Equations (3.1) and (3.2) yields

$$(1-p^{k-1})\left(-\frac{B_k}{k}\right) = \frac{1}{\alpha^{k-1}-1}\int_{\mathbb{Z}_p^\times} x^{k-1}\mathrm{d}\mu_{1,\alpha}. \tag{3.5}$$

At this point we make the observation that the factor $1-p^{k-1}$ in Equation (3.4) arises because we are forced to integrate over $\mathbb{Z}_p^\times$ instead of all of $\mathbb{Z}_p$, since x^{k-1} can only be interpolated when $p \nmid x$. The factor $1-\alpha^{-k}$ is an artefact of the α-regularisation of the Bernoulli distribution in Definition 3.8.

From the preceding discussion we are able to p-adically interpolate the right hand side of Equation (3.5), so we are now in a strong position to define the Kubota-Leopoldt p-adic zeta function.

Definition 3.10. *For any fixed prime number p and $s_0 \in \{0,1,\ldots,p-2\}$, we define*

$$\zeta_{p,s_0}(s) = \frac{1}{\alpha^{-(s_0+(p-1)s)}-1}\int_{\mathbb{Z}_p^\times} x^{s_0+(p-1)s-1}\mathrm{d}\mu_{1,\alpha} \tag{3.6}$$

for $s \in \mathbb{Z}_p$ (and with $s \neq 0$ if $s_0 = 0$).

It can be shown that $\zeta_{p,s_0}(s)$ is independent of the choice of α, see for example [47, Theorem II.6.8]. Because of Equation (3.3), it follows that the function $\zeta_{p,s_0}(s)$ is p-adically continuous. Furthermore, by Equation (3.5) it p-adically interpolates the special values of the Riemann zeta function.

If $k \in \mathbb{Z}$ and $k \equiv s_0 \pmod{p-1}$, then

$$\zeta_{p,s_0}(k) = (1-p^{k-1})\left(-\frac{B_k}{k}\right) = (1-p^{k-1})\zeta(1-k).$$

Note that the case $s=0$ and $s_0=0$ is excluded because here $\alpha^{-(s_0+(p-1)s)} = 1$, and as the denominator on the right hand side of Equation (3.6) vanishes, the p-adic zeta function has a "p-adic pole" corresponding to the pole at $s=1$ of the complex Riemann zeta function $\zeta(s)$.

3.2 L-functions attached to modular forms

Deligne [20] has constructed Galois representations associated with modular forms of weight $k \geq 2$. We begin our short survey by defining L-functions attached to modular forms. Throughout, f will denote a primitive form of level N_f and character χ that is p-ordinary, i.e. $\mathrm{ord}_p\big(a_p(f)\big) = 0$, for some fixed prime $p \geq 3$.

3.2.1 Hecke L-functions

Given a newform $f \in \mathcal{S}_k(N, \chi)$ with Fourier expansion $f(z) = \sum_{n=1}^{\infty} a_n q^n$, we define the L-function attached to f by

$$L(f, s) = \sum_{n=1}^{\infty} a_n(f) n^{-s}.$$

The function $L(f, s)$ converges for $\mathrm{Re}(s) > 1 + k/2$, and satisfies the functional equation

$$\Lambda(f, s) = \pm\Lambda(f, k - s)$$

where $\Lambda(f, s) = (2\pi)^{-s} N^{s/2} \Gamma(s) L(f, s)$. If f is a normalised eigenform, then its L-function has an Euler product expansion

$$L(f, s) = \prod_{l} \left[(1 - \alpha_l l^{-s})(1 - \beta_l l^{-s})\right]^{-1}$$

where $X^2 - a_l(f)X + \chi(l)l^{k-1} = (X - \alpha_l)(X - \beta_l)$ is the Hecke polynomial of f at l.

3.2.2 Galois representations associated with f

Let K denote a number field containing the Fourier coefficients of f, let $\mathfrak{p}$ be a prime of K lying above p, and write $K_{\mathfrak{p}}$ for the completion of K at $\mathfrak{p}$. By the work of Deligne, if f has weight $k \geq 2$ then we may associate with it a 2-dimensional Galois representation V_f, characterised by the condition

$$\det\Big(1 - X\mathrm{Frob}_l^{-1}\big|V_f\Big) = (1 - \alpha_l X)(1 - \beta_l X)$$

for all primes $l \nmid pN_f$, where Frob_l is an arithmetic Frobenius element. Thus we see that the L-function

$$L(V_f, s) = \prod_l \det\left(1 - X\mathrm{Frob}_l^{-1} | V_f\right)^{-1}\Big|_{X=l^{-s}}$$

attached to V_f is in fact the same as $L(f, s)$. It is by the deep work of Carayol [6] that we know these L-functions agree even at bad primes.

There is a canonical $G_{\mathbb{Q}}$-stable $\mathcal{O}_{K_{\mathfrak{p}}}$ lattice T_f^* inside V_f^* which is generated by the étale cohomology of $Y_1(N)_{\overline{\mathbb{Q}}}$ with $\mathbb{Z}_p$-coefficients. Since the representation V_f is ordinary at p, it has a 2-step filtration

$$0 = \mathrm{Fil}^2(V_f) \subseteq \mathrm{Fil}^1(V_f) \subseteq \mathrm{Fil}^0(V_f) = V_f$$

when viewed as a $G_{\mathbb{Q}_p}$-module. The quotients $\mathrm{Fil}^i(V_f)/\mathrm{Fil}^{i+1}(V_f)$ have local Galois actions as follows:

$$\frac{\mathrm{Fil}^0(V_f)}{\mathrm{Fil}^1(V_f)} = \theta(\alpha),$$
$$\frac{\mathrm{Fil}^1(V_f)}{\mathrm{Fil}^2(V_f)} = \kappa^{k-1}\theta\left(\frac{\beta}{p^{k-1}}\right).$$

Here κ denotes the cyclotomic character and $\theta(x)$ is the unramified character that maps the arithmetic Frobenius element at p to x.

3.3 Symmetric square L-functions

One may also attach L-functions to the symmetric square Galois representations associated with modular forms, i.e. the symmetric squares of the representations described in Section 3.2.2. We define complex L-functions attached to these representations and also describe their p-adic analogues. We maintain the notation of Section 3.2.

3.3.1 Galois representations associated with $\mathrm{Sym}^2 f$

Let ψ be a Dirichlet character of conductor N_ψ coprime to p. We will follow the convention of identifying ψ with a character of $G_{\mathbb{Q}}$ by composing it with the mod N_ψ cyclotomic character so that $\psi(\mathrm{Frob}_l) = \psi(l)$ for $l \nmid N_\psi$. We also denote by $\nu_{k,\psi}$ the point in weight-space $\mathcal{W} = \mathrm{Hom}_{\mathrm{cont}}(\mathbb{Z}_p^\times, \mathbb{C}_p^\times)$ defined by $\nu_{k,\psi}(z) = \psi(z)z^k$.

Definition 3.11. *The symmetric square representation for f twisted by ψ is given by*

$$W_f = \mathrm{Sym}^2\, V_f^*(\nu_{1,\psi}).$$

The space W_f is a 3-dimensional irreducible $K_\mathfrak{p}$-linear representation of $G_\mathbb{Q}$, unramified outside pN_fN_ψ, that is crystalline at p if $p \nmid N_f$. Moreover, because we picked an ordinary prime for f, W_f has as 3-step filtration

$$0 = \mathrm{Fil}^3(W_f) \subseteq \mathrm{Fil}^2(W_f) \subseteq \mathrm{Fil}^1(W_f) \subseteq \mathrm{Fil}^0(W_f) = W_f$$

when viewed as a $G_{\mathbb{Q}_p}$-module. The local Galois actions on the graded pieces are as follows:

$$\begin{aligned}
\frac{\mathrm{Fil}^0(W_f)}{\mathrm{Fil}^1(W_f)} &= \kappa\theta(\alpha^2\psi(p)), \\
\frac{\mathrm{Fil}^1(W_f)}{\mathrm{Fil}^2(W_f)} &= \kappa^k\theta\left(\chi(p)\psi(p)\right), \\
\frac{\mathrm{Fil}^2(W_f)}{\mathrm{Fil}^3(W_f)} &= \kappa^{2k-1}\theta\left(\frac{\beta^2}{p^{2k-2}}\psi(p)\right).
\end{aligned}$$

3.3.2 Complex L-functions attached to $\mathrm{Sym}^2 f$

In this section we introduce the primitive and imprimitive L-functions attached to $\mathrm{Sym}^2 f$. Let ψ be a Dirichlet character of conductor N_ψ coprime to N_f. For each prime l, we define

$$P_l(\mathrm{Sym}^2 f \otimes \psi, X) = \det\left(1 - \mathrm{Frob}_l X \middle| W_f^{I_l}\right) \in K(\chi)[X]$$

where $I_l \subset G_\mathbb{Q}$ is an inertia group at l, and W_f is the representation described in the previous section.

Definition 3.12. *The primitive L-function attached to* $\mathrm{Sym}^2 f \otimes \psi$ *is given by the Euler product*

$$L_\infty(\mathrm{Sym}^2 f \otimes \psi, s) = \prod_l P_l(\mathrm{Sym}^2 f \otimes \psi, l^{-s})^{-1}$$

which converges when $\mathrm{Re}(s) > k$.

This definition makes sense because each polynomial $P_l(\mathrm{Sym}^2 f \otimes \psi, X)$ is independent of any choices made. For example, if $l \nmid N_f$ then the Euler factor at l is of the form

$$P_l(\mathrm{Sym}^2 f \otimes \psi, X) = (1 - \alpha_l^2 \psi(l) X)(1 - \alpha_l \beta_l \psi(l) X)(1 - \beta_l^2 \psi(l) X),$$

and the Euler factors for primes $l \mid N_f$ are calculated in [57].

The following algebraicity result, due to Sturm [62, 63], is of paramount importance.

Proposition 3.13. *Let s be an integer such that either $s \in \{1, \ldots, k-1\}$ and $\psi(-1) = (-1)^{s+1}$, or $s \in \{k, \ldots, 2k-2\}$ and $\psi(-1) = (-1)^s$. If we set $\delta = 0$ whenever $1 \leq s \leq k-1$, and $\delta = 1$ whenever $k \leq s \leq 2k-2$, then*

$$\frac{L_\infty(\mathrm{Sym}^2 f \otimes \psi, s)}{\pi^{k-1} \langle f, f \rangle_{N_f}} \left(\frac{\tau(\chi^{-1}\psi^{-1})}{(2\pi)^{s-k+1}} \right)^{1+\delta} \in K \subset \overline{\mathbb{Q}}.$$

We call these the critical values of $L_\infty(\mathrm{Sym}^2 f \otimes \psi, s)$. Here the complex number $\tau(\omega) = \sum_{j=1}^{N_\omega} \omega(j) e^{2\pi i j / N_\omega}$ denotes the Gauss sum of a Dirichlet character ω modulo N_ω.

It is these critical values that will be interpolated by the p-adic analogue of $L_\infty(\mathrm{Sym}^2 f \otimes \psi, s)$.

We will also need an imprimitive version of this object.

Definition 3.14. *The imprimitive L-function attached to $\mathrm{Sym}^2 f$ twisted by ψ is*

$$D(f, \psi, s) = L_{N_f N_\psi}(\chi^2 \psi^2, 2s - 2k + 2) \sum_{n=1}^{\infty} \psi(n) a_{n^2}(f) n^{-s}$$

where $L_N(\omega, s)$ denotes the Dirichlet L-function twisted by ω with Euler factors at primes dividing N removed.

If we adopt the convention that $(\alpha_l, \beta_l) = (a_l(f), 0)$ whenever $l \mid N_f$, then we have the following Euler product expansion for the imprimitive L-function:

$$D(f, \psi, s) = \prod_l \left[(1 - \psi(l) \alpha_l^2 l^{-s})(1 - \psi(l) \alpha_l \beta_l l^{-s})(1 - \psi(l) \beta_l^2 l^{-s}) \right]^{-1}.$$

3.3.3 p-adic L-functions attached to $\mathrm{Sym}^2 f$

We now describe the p-adic L-functions that interpolate the special values of the primitive symmetric square L-functions. We will assume that f is a modular form such that $h = \lfloor 2\mathrm{ord}_p\big(a_p(f)\big)\rfloor + 1 \leq k-1$.

We let $\Gamma = \mathbb{Z}_p^\times$, $\Lambda(\Gamma) = \mathbb{Z}_p[\![\Gamma]\!]$, and identify $\Lambda(\Gamma)$ with the ring of $\mathbb{Z}_p$-valued analytic functions on the weight-space $\mathcal{W}$, i.e. the rigid analytic space parametrising continuous characters of Γ. The existence of the following p-adic L-function was proven in [13] and [57].

Theorem 3.15. *There exists a p-adic L-function $\mathbf{L}_p(\mathrm{Sym}^2 f\otimes\psi,\sigma) : \mathcal{W} \to \mathbb{C}_p$ of type $o(\log^h)$, where $h = \lfloor 2\mathrm{ord}_p\big(a_p(f)\big)\rfloor + 1$, that is uniquely determined by the following interpolation property:*

1. *If $1 \leq s \leq k-1$ and ε is a finite character of Γ satisfying $\varepsilon(-1) = (-1)^{s+1}\psi(-1)$ then*

$$\mathbf{L}_p(\mathrm{Sym}^2 f\otimes\psi,\nu_{s,\varepsilon}) = \frac{(-1)^{s-k+1}\varepsilon(-1)\Gamma(s)}{2^{2k}i^a}\frac{\tau(\varepsilon)}{(2\pi i)^{s-k+1}} \times \mathcal{E}_p(s,\varepsilon)\frac{L_\infty(\mathrm{Sym}^2 f\otimes\psi\varepsilon^{-1},s)}{\pi^{k-1}\langle f,f\rangle_{N_f}}$$

where $a \in \{0,1\}$ such that $\psi(-1) = (-1)^{k+a}$, and

$$\mathcal{E}_p(s,\varepsilon) = \begin{cases} \Big(p^{s-1}\psi(p)^{-1}\alpha_p^{-2}\Big)^r & \text{if } N_\varepsilon = p^r > 1 \\ (1-p^{s-1}\psi(p)^{-1}\alpha_p^{-2}) \\ \quad\times(1-\psi(p)\alpha_p\beta_p p^{-s}) & \text{if } N_\varepsilon = 1. \\ \quad\times(1-\psi(p)\beta_p^2 p^{-s}) \end{cases}$$

2. *If $k \leq s \leq 2k-2$ and ε is a finite character of Γ with $\varepsilon(-1) = (-1)^s\psi(-1)$ then*

$$\mathbf{L}_p(\mathrm{Sym}^2 f\otimes\psi,\nu_{s,\varepsilon}) = \frac{\Gamma(s-k+1)\Gamma(s)}{2^{2s+1}}\varepsilon(N_{\psi\chi})\tau(\varepsilon)^2 \times \mathcal{E}'_p(s,\varepsilon)\frac{L_\infty(\mathrm{Sym}^2 f\otimes\psi\varepsilon^{-1},s)}{\pi^{2s-k+1}\langle f,f\rangle_{N_f}}$$

where

$$\mathcal{E}'_p(s,\varepsilon) = \begin{cases} \left(p^{s-1}\psi(p)^{-2}\chi(p)^{-1}\alpha_p^{-2}\right)^r & \text{if } N_\varepsilon = p^r > 1 \\ \\ (1 - p^{s-1}\psi(p)^{-1}\alpha_p^{-2}) & \\ \quad \times (1-\psi(p)\alpha_p\beta_p p^{-s}) & \text{if } N_\varepsilon = 1. \\ \quad \times (1 - p^{s-1}\psi(p)^{-1}\alpha_p^{-1}\beta_p^{-1}) & \end{cases}$$

In each case N_ε denotes the conductor of ε.

3.4 Symmetric square $\mathcal{L}$-invariants

In this section we describe in detail Greenberg's arithmetic $\mathcal{L}$-invariant for the symmetric square of an elliptic curve, and also the analytic $\mathcal{L}$-invariant. We then discuss the relationship between the arithmetic and analytic $\mathcal{L}$-invariants, and the significance of their non-triviality in the context of deformation theory. While we only discuss in detail the symmetric square $\mathcal{L}$-invariants for elliptic curves, these notions generalise naturally to include symmetric square representations attached to modular forms of weight $k > 2$.

Throughout this section, E will denote an elliptic curve over $\mathbb{Q}$ (so that E is necessarily modular by the work in [5, 70]), and f will be its associated modular form. We also fix an ordinary prime p.

3.4.1 Greenberg's $\mathcal{L}$-invariant

Consider the Galois representation

$$W = \mathrm{Sym}^2\left(H_{\text{ét}}\big(\overline{E}, \mathbb{Q}_p(1)\big)^*\right) \cong \mathbb{Q}_p \otimes_{\mathbb{Z}_p} \mathrm{Sym}^2\big(\mathrm{Ta}_p(E)\big)$$

where $\mathrm{Ta}_p(E) = \varprojlim_n E_{p^n}$ is the p-adic Tate module of E.

Let Σ denote a finite set of primes containing p and the primes of bad reduction for E. Associated to the $\mathrm{Gal}\big(\mathbb{Q}_\Sigma/\mathbb{Q}\big)$-representation W in [4] are the Bloch-Kato Selmer groups

$$H^1_{f,\{p\}}(\mathbb{Q},W) \;:=\; \mathrm{Ker}\left(H^1\big(\mathrm{Gal}(\mathbb{Q}_\Sigma/\mathbb{Q}),W\big) \overset{\oplus\mathrm{res}_l}{\longrightarrow} \bigoplus_{l\in\Sigma,\, l\neq p} H^1\big(I_l,W\big)\right)$$

and

$$H^1_f(\mathbb{Q},W) \;:=\; \mathrm{Ker}\left(H^1_{f,\{p\}}(\mathbb{Q},W) \overset{\mathrm{res}_p}{\longrightarrow} \frac{H^1\big(G_{\mathbb{Q}_p},W\big)}{H^1_f\big(G_{\mathbb{Q}_p},W\big)}\right)$$

where $H^1_f(G_{\mathbb{Q}_p},W)$ denotes the kernel of the mapping from $H^1(G_{\mathbb{Q}_p},W)$ to $H^1(G_{\mathbb{Q}_p},W\otimes B_{\mathrm{cris}})$. Flach et al. [26, 70, 21] have shown $H^1_f(\mathbb{Q},W)=\{0\}$, which implies that $\dim_{\mathbb{Q}_p} H^1_{f,\{p\}}(\mathbb{Q},W)=1$. Let us fix a generator η of this line, so that $H^1_{f,\{p\}}(\mathbb{Q},W)=\mathbb{Q}_p\cdot\eta$.

We now explain how to choose coordinates. Observe that $H^1\big(G_{\mathbb{Q}_p},W\big) = H^1\big(G_{\mathbb{Q}_p},\mathrm{Fil}^1W\big)$, an assertion that can be checked from the local formula

$$\begin{aligned}\dim_{\mathbb{Q}_p} H^1\big(G_{\mathbb{Q}_p},U\big) \;&=\; \dim_{\mathbb{Q}_p}\big(U\otimes B_{\mathrm{cris}}\big)^{G_{\mathbb{Q}_p}}\\ &\quad + \dim_{\mathbb{Q}_p} H^0\big(G_{\mathbb{Q}_p},U\big)\\ &\quad + \dim_{\mathbb{Q}_p} H^0\big(G_{\mathbb{Q}_p},U^*(1)\big)\end{aligned}$$

which yields the value $3+0+0$ if $U=W$, and the value $2+0+1$ if $U=\mathrm{Fil}^1W$. By applying Kummer theory, we make the natural identification $H^1\big(G_{\mathbb{Q}_p},\mathbb{Q}_p(1)\big)\cong\mathbb{Q}_p\otimes_{\mathbb{Z}_p}\Big(\varprojlim_n \mathbb{Q}_p^\times/\mathbb{Q}_p^{\times\, p^n}\Big)$, from which one obtains the homomorphism

$$\begin{aligned}\mathfrak{q}: H^1\big(G_{\mathbb{Q}_p},W\big) = H^1\big(G_{\mathbb{Q}_p},\mathrm{Fil}^1W\big) \;&\overset{\mathrm{mod}\ \mathrm{Fil}^2}{\longrightarrow}\; H^1\big(G_{\mathbb{Q}_p},\mathrm{Fil}^1W/\mathrm{Fil}^2W\big)\\ &\overset{\sim}{\longrightarrow}\; \mathbb{Q}_p\otimes_{\mathbb{Z}_p}\Big(\varprojlim_n \mathbb{Q}_p^\times/\mathbb{Q}_p^{\times\, p^n}\Big).\end{aligned}$$

Furthermore, on the right-hand target space there is an isomorphism

$$\mathbb{Q}_p\otimes_{\mathbb{Z}_p}\Big(\varprojlim_n \mathbb{Q}_p^\times/\mathbb{Q}_p^{\times\, p^n}\Big) \overset{\sim}{\longrightarrow} \mathbb{Q}_p\times\mathbb{Q}_p$$

sending $q\mapsto\big(\log_p(q),\mathrm{ord}_p(q)\big)$.

Definition 3.16. *The arithmetic $\mathcal{L}$-invariant is defined to be the slope of $\mathfrak{q}\circ\mathrm{res}_p(\eta)$ inside the vector space $H^1\big(G_{\mathbb{Q}_p},\mathrm{Fil}^1W/\mathrm{Fil}^2W\big)\cong\mathbb{Q}_p\times\mathbb{Q}_p$, i.e.*

$$\mathcal{L}_p^{\mathrm{Gr}}(\mathrm{Sym}^2E) \;:=\; \frac{\log_p\Big(\mathfrak{q}\big(\mathrm{res}_p(\eta)\big)\Big)}{\mathrm{ord}_p\Big(\mathfrak{q}\big(\mathrm{res}_p(\eta)\big)\Big)}$$

which is independent of the choice of generator η for the $\mathbb{Q}_p$-line $H^1_{f,\{p\}}(\mathbb{Q},W)$.

3.4.2 The analytic $\mathcal{L}$-invariant

In fact, there is a more analytic way to introduce the $\mathcal{L}$-invariant if we work with the p-adic L-function directly. Recall from Section 3.3.2 that, provided $\mathrm{Re}(s) > 2$, the complex symmetric square L-function for E is given by an Euler product

$$L_\infty(\mathrm{Sym}^2 E, s) = \prod_{\text{primes } l} \det\left(1 - \mathrm{Frob}_l^{-1} X \;\middle|\; \mathrm{Sym}^2 H_{\text{ét}}\big(\overline{E}, \mathbb{Q}_q(1)\big)^{I_l}\right)^{-1} \Big|_{X=l^{-s}}$$

and if the prime number l does not divide the $\mathbb{Q}$-conductor N_E of the elliptic curve, then

$$\det\left(1 - \mathrm{Frob}_l^{-1} X \;\middle|\; \mathrm{Sym}^2 H_{\text{ét}}\big(\overline{E}, \mathbb{Q}_q(1)\big)^{I_l}\right) = \big(1 - \alpha_l^2 X\big)\big(1 - \beta_l^2 X\big)\big(1 - lX\big)$$

where $1 - a_l(E)X + lX^2 = \big(1 - \alpha_l X\big)\big(1 - \beta_l X\big)$ is the factorisation of the Hecke polynomial at l. Gelbart and Jacquet [29] showed that the function $L_\infty(\mathrm{Sym}^2 E, s)$ has an analytic continuation to all $s \in \mathbb{C}$, and satisfies a functional equation linking the value at s with the value at $3 - s$.

Since E has good ordinary reduction at p, by Theorem 3.15, there exists an analytic function $\mathcal{F}(X) \in X \cdot \mathbb{Z}_p[\![X]\!] \otimes \mathbb{Q}$ such that

$$\mathcal{F}\big(\chi(1+p) - 1\big) \;=\; \frac{\tau(\overline{\chi})}{\alpha_p^{2m_\chi}} \times \frac{L_\infty(\mathrm{Sym}^2 E \otimes \chi, 1)}{(2\pi i)^{-1}\Omega_E^+\Omega_E^-}$$

at all non-trivial characters χ of conductor $\mathfrak{f}_\chi = p^{m_\chi} > 1$ satisfying $\chi\big|_{\mathbb{F}_p^\times} = \mathbf{1}$, while $\mathcal{F}(0) = 0$. Here α_p is the p-adic unit root of $X^2 - a_p(E)X + p$, secondly $\tau(\overline{\chi})$ denotes a Gauss sum for χ^{-1}, and lastly $\Omega_E^\pm$ are real/imaginary periods associated to a minimal Weierstrass equation for $E_{/\mathbb{Z}}$.

Definition 3.17. *We write* $\mathbf{L}_p(\mathrm{Sym}^2 E, -) : \mathbb{Z}_p \to \mathbb{Q}_p$ *for the Mazur-Mellin transform*

$$\mathbf{L}_p(\mathrm{Sym}^2 E, s) \;:=\; \mathcal{F}\big((1+p)^{s-1} - 1\big),$$

so that $\mathbf{L}_p(\mathrm{Sym}^2 E, s)$ *has an exceptional zero at* $s = 1$.

In the late 1980s, Coates and Greenberg made the following prediction about its first derivative.

Conjecture 3.18. *If E has good ordinary reduction at p, the $\mathcal{L}$-invariant given by the ratio*

$$\mathcal{L}_p^{\mathrm{an}}(\mathrm{Sym}^2 E) \;:=\; \frac{\mathrm{d}}{\mathrm{d}s}\mathbf{L}_p(\mathrm{Sym}^2 E, s)\Big|_{s=1} \times \left((1-\alpha_p^{-2})(1-p\alpha_p^{-2})\cdot\frac{L_\infty(\mathrm{Sym}^2 E, 1)}{(2\pi i)^{-1}\Omega_E^+\Omega_E^-}\right)^{-1}$$

is a **non-zero** *p-adic number, so in particular*

$$\mathrm{order}_{s=1}\left(\mathbf{L}_p(\mathrm{Sym}^2 E, s)\right) \;=\; 1.$$

As will be discussed at length in Section 3.4.3, in most situations the work of Citro, Dasgupta and Hida [9, 15, 40] implies that $\mathcal{L}_p^{\mathrm{an}}(\mathrm{Sym}^2 E) = \mathcal{L}_p^{\mathrm{Gr}}(\mathrm{Sym}^2 E)$, so we may shift between these two definitions as appropriate. In particular, the non-vanishing of the analytic $\mathcal{L}$-invariant means that the line $H^1_{f,\{p\}}(\mathbb{Q}, V)$ has a non-trivial slope inside $\mathbb{Q}_p \otimes_{\mathbb{Z}_p} \left(\varprojlim_n \mathbb{Q}_p^\times/\mathbb{Q}_p^{\times\, p^n}\right) \cong \mathbb{Q}_p \times \mathbb{Q}_p$.

Remarks. (a) If E has complex multiplication, then a result of Ferrero and Greenberg [25] implies that $\mathcal{L}_p^{\mathrm{an}}(\mathrm{Sym}^2 E) = \log_p(\alpha_p^{-2})$; therefore in the CM case, Conjecture 3.18 is at least known to be true.

(b) If E has split multiplicative reduction at p, under certain restrictions Rosso [56] recently proved $\mathcal{L}_p^{\mathrm{an}}(\mathrm{Sym}^2 E) = \log_p(q_E)/\mathrm{ord}_p(q_E)$ where q_E is the Tate period of the rigid analytic curve; moreover $\log_p(q_E) \neq 0$ by [2, Theorem 3], so Conjecture 3.18 holds in this situation too.

(c) We should also point out that in the case where E has split multiplicative reduction at p, the Tate period q_E is a universal norm for the $\mathbb{Z}_p$-extension $F_\infty/\mathbb{Q}_p$ cut out by

$$\mathrm{Im}\Big(H^1\big(G_{\mathbb{Q}_p}, \mathrm{Sym}^2\mathrm{Ta}_p(E)\big) \overset{\mathrm{mod\ Fil}^1}{\longrightarrow} H^1\big(G_{\mathbb{Q}_p}, \mathbb{Z}_p\big)\Big)$$

inside $H^1\big(G_{\mathbb{Q}_p}, \mathbb{Z}_p\big) = \mathrm{Hom}\big(G_{\mathbb{Q}_p}, \mathbb{Z}_p\big) \cong \mathbb{Z}_p^2$. Under the Tate local pairing

$$H^1\big(G_{\mathbb{Q}_p}, \mathbb{Q}_p(1)\big) \times H^1\big(G_{\mathbb{Q}_p}, \mathbb{Q}_p\big) \to \mathbb{Q}_p,$$

the line $\mathfrak{q} \circ \mathrm{res}_p\left(H^1_{f,\{p\}}(\mathbb{Q}, V)\right)$ will then be orthogonal to the subspace $\mathrm{Hom}\big(\mathrm{Gal}(F_\infty/\mathbb{Q}_p), \mathbb{Q}_p\big)$. Applying exactly the same reasoning as [31, p154], it follows that the slopes $\log_p(q_E)/\mathrm{ord}_p(q_E)$ and $\log_p\Big(\mathfrak{q}\big(\mathrm{res}_p(\eta)\big)\Big)/\mathrm{ord}_p\Big(\mathfrak{q}\big(\mathrm{res}_p(\eta)\big)\Big)$ are actually equal.[1]

3.4.3 The connection with deformation theory

We now discuss these $\mathcal{L}$-invariants in the context of Λ-adic cusp forms. For a given elliptic curve $E_{/\mathbb{Q}}$ and a good ordinary prime $p \geq 3$, one can lift the p-stabilisation $f_0 \in \mathcal{S}_2\big(\Gamma_0(pN_E)\big)$ to an $\mathbb{I}$-adic eigenform, $\mathcal{F}$, where $\mathbb{I}$ denotes a suitable finite, flat extension of $\mathbb{Z}_p[\![X]\!]$, isomorphic to the irreducible component of the universal ordinary Hecke algebra carrying the form f_0.

For a sufficiently small choice of p-adic disk $\mathcal{W} \subset \mathbb{Z}_p$ centred on $k = 2$, each specialisation

$$\mathcal{F}_k \;:=\; \mathcal{F}\big|_{X=(1+p)^{k-2}-1} \;\in\; \mathcal{S}_k\big(\Gamma_0(N_E p^\infty), \omega_p^{2-k}\big) \quad \text{for } k \in \mathcal{W} \cap \mathbb{Z}_{\geq 2}$$

yields a classical cuspidal Hecke eigenform, with the q-expansion $\mathcal{F}_k(q) = \sum_{n=1}^{\infty} a\big(\mathcal{F}_k, n\big) q^n$. One can then interpolate each q^n-coefficient to yield a function, $a\big(\mathcal{F}(X), n\big)$, on the disk $\mathcal{W}$.

If $n = p$, then the derivative of $a\big(\mathcal{F}(X), p\big)$ with respect to X is rigid meromorphic on $\mathcal{W}$. Hida established in [39, Prop 7.1] under suitable hypotheses (which are true, for instance, if the versal deformation ring $\mathcal{R}_E$ is Gorenstein) that $\frac{\mathrm{d}a(\mathcal{F},p)}{\mathrm{d}X}$ is non-zero, and can thus vanish at only finitely many unspecified bad weights. Furthermore, the main formula in [40, Thm 1.1] yields

$$\mathcal{L}_p^{\mathrm{Gr}}\big(\mathrm{Sym}^2(\mathcal{F}_k)(k)\big) = -2\log_p(1+p)\cdot a\big(\mathcal{F}_k, p\big)^{-1} \cdot \left.\frac{\mathrm{d}a(\mathcal{F},p)}{\mathrm{d}X}\right|_{X=(1+p)^{k-2}-1} \tag{3.7}$$

for every weight $k \in \mathcal{W} \cap \mathbb{Z}_{\geq 2}$, where $\mathcal{L}_p^{\mathrm{Gr}}\big(-\big)$ again denotes Greenberg's algebraic $\mathcal{L}$-invariant.

Note that the Gorenstein property of the versal deformation ring $\mathcal{R}_E$ above has been verified for numerous elliptic curves E, and ordinary primes $p \geq 3$

[1] In the case of split multiplicative reduction the $\mathcal{L}$-invariant for $\mathrm{Sym}^2 E$ is the same as the $\mathcal{L}$-invariant for E, and it is further conjectured (by Greenberg) that the $\mathcal{L}$-invariants for $\mathrm{Sym}^m E$ should be independent of $m > 0$.

(see [5, 36, 70]). For example, it is known to hold if the conductor N_E of the elliptic curve is a square-free integer.

Remarks. (a) Let $\mathbf{L}_p\big(\mathcal{F}_k \otimes \mathcal{F}_k, s\big)$ denote the analytic p-adic L-function constructed in [35], which interpolates the special values $L\big(\mathcal{F}_k \otimes \mathcal{F}_k \otimes \chi, k\big)$. From Dasgupta's result in [15, Thm 1], one has a factorisation

$$\mathbf{L}_p\big(\mathcal{F}_k \otimes \mathcal{F}_k, s\big) = \star \cdot \zeta_p\big(s-k+1, \omega_p^0\big) \cdot \mathbf{L}_p\big(\mathrm{Sym}^2(\mathcal{F}_k), s\big).$$

Here $\star$ consists of some Euler factors which are non-zero at classical weights, and so $\star\big|_{s=k} \neq 0$.

(b) Allowing $s \to k$ and observing that $\mathrm{Res}_{s=k}\Big(\zeta_p\big(s-k+1, \omega_p^0\big)\Big) = 1-p^{-1}$, the above implies

$$\mathbf{L}_p\big(\mathcal{F}_k \otimes \mathcal{F}_k, k\big) = \mathcal{L}_p^{\mathrm{an}}\big(\mathrm{Sym}^2(\mathcal{F}_k)(k)\big) \cdot \mathcal{E}_p(\mathcal{F}_k) \cdot \frac{L_\infty(\mathrm{Sym}^2\mathcal{F}_k, k)}{\Omega_{\infty,\mathrm{Sym}^2(\mathcal{F}_k)}} \quad (3.8)$$

with $\mathcal{E}_p(\mathcal{F}_k) = \star\big|_{s=k} \cdot \big(1-p^{-1}\big)\big(1-\alpha_p(\mathcal{F}_k)^{-2}p^{k-1}\big)\big(1-\beta(\mathcal{F}_k)_p^{-2}p^{-k}\big) \neq 0$.

(c) Under the same assumptions as [40, Thm 1.1], Citro proves in [9, Thm 1] that

$$\mathbf{L}_p\big(\mathcal{F}_k \otimes \mathcal{F}_k, k\big) = \mathcal{L}_p^{\mathrm{Gr}}\big(\mathrm{Sym}^2(\mathcal{F}_k)(k)\big) \cdot \mathcal{E}_p(\mathcal{F}_k) \cdot \frac{L_\infty(\mathrm{Sym}^2\mathcal{F}_k, k)}{\Omega_{\infty,\mathrm{Sym}^2(\mathcal{F}_k)}}. \quad (3.9)$$

Using Equations (3.8) and (3.9), Dasgupta [15, Thm 4] then reads off Greenberg's prediction that

$$\mathcal{L}_p^{\mathrm{Gr}}\big(\mathrm{Sym}^2(\mathcal{F}_k)(k)\big) = \mathcal{L}_p^{\mathrm{an}}\big(\mathrm{Sym}^2(\mathcal{F}_k)(k)\big) = \mathcal{L}_p^{\mathrm{an}}\big(\mathrm{Sym}^2(\mathcal{F}_k)(k-1)\big)$$

(note the second equality is a consequence of the p-adic functional equation for $\mathrm{Sym}^2(\mathcal{F}_k)$).

A corollary of these remarks is that we can replace the algebraic $\mathcal{L}$-invariant in Equation (3.7) with either analytic version. In particular, at weight two Hida's formula now becomes

$$\frac{\mathrm{d}a(\mathcal{F}, p)}{\mathrm{d}X}\bigg|_{X=0} = -\frac{\alpha_p}{2\log_p(1+p)} \cdot \mathcal{L}_p^{\mathrm{an}}\big(\mathrm{Sym}^2 E\big), \quad (3.10)$$

hence the derivative of $a(\mathcal{F},p)$ at zero coincides with $\frac{\mathcal{L}_p^{\rm an}(\mathrm{Sym}^2E)}{p}$, up to an explicit p-adic unit. Of course, this could just end up being the equation "$0=0$" in disguise!

Dummit, Hablicsek, Harron, Jain, Pollack and Ross [23] have a direct method to calculate $a(\mathcal{F},p)'(0)$ through the use of overconvergent modular symbols, and they have computed four examples in *op. cit.*, thereby establishing the non-triviality of $\mathcal{L}_p^{\rm an}(\mathrm{Sym}^2E)$ in these cases. Their results further determine power series expansions for $a(\mathcal{F}_k,p)$, as a function of k, over the weight-space $\mathcal{W}$.

The non-triviality of this $\mathcal{L}$-invariant has a key consequence for the Iwasawa Main Conjecture for Sym^2E over the cyclotomic $\mathbb{Z}_p$-extension $\mathbb{Q}^{\rm cyc}$ of $\mathbb{Q}$. The property that $\mathcal{L}_p^{\rm an}(\mathrm{Sym}^2E)\neq 0$ allows one to deduce that the order of the *algebraic p-adic L-function* at $s=1$ is exactly one. Here the algebraic p-adic L-function denotes the Mazur-Mellin transform of a generator, for the characteristic ideal of $\mathrm{Hom}_{\rm cont}\Big(\mathrm{Sel}_{p^\infty}\big(\mathrm{Sym}^2E(1)/\mathbb{Q}^{\rm cyc}\big),\mathbb{Q}/\mathbb{Z}\Big)$ over the cyclotomic Iwasawa algebra $\mathbb{Z}_p\big[\big[\mathrm{Gal}(\mathbb{Q}^{\rm cyc}/\mathbb{Q})\big]\big]$ – we refer the reader to [56, Sect 10] for a fuller discussion.

3.5 Double product L-functions

We now describe the complex double product L-function attached to a pair of modular forms of different weights, as well as its p-adic counterpart.

Definition 3.19. *Let $f=\sum_{n=1}^\infty a_n(f)q^n\in\mathcal{S}_k(N,\chi)$ and $g=\sum_{n=1}^\infty a_n(g)q^n\in\mathcal{M}_l(N,\psi)$ where $k>l>1$. We define the convolution L-function*

$$D(s,f,g)=\sum_{n=1}^{\infty}a_n(f)a_n(g)n^{-s}.$$

The series $D(s,f,g)$ converges for $\mathrm{Re}(s)\gg 0$, and can be expressed in terms of the Petersson inner product at certain of its special values.

Theorem 3.20 (Shimura)**.** *If $f \in \mathcal{S}_k(N,\chi)$ and $g \in \mathcal{M}_l(N,\psi)$, with $l+2r<k$ for some non-negative integer r, then*

$$D(k-1-r,f,g) = c \times \langle f^\rho, g\cdot\delta_\lambda^{(r)} E^*_{\lambda,N}(z,\chi\psi)\rangle_N$$

where $\lambda = k-l-2r$ and

$$c = \frac{(-1)^r(4\pi)^{k-1}\Gamma(k-l-2r)}{\Gamma(k-1-r)\Gamma(k-l-r)}.$$

Here $E^*_{w,N}(z,\eta)$ is the Eisenstein series in [60, Eqn (2.3)] of weight $w \geq 0$, character η^{-1} and level N, given by the infinite series

$$E^*_{w,N}(z,s,\eta) \;=\; \sum_{\Gamma_\infty\backslash\Gamma_0(N)} \eta(\gamma)\cdot(cz+d)^{-w}\big|cz+d\big|_\infty^{-2s}, \quad \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}. \tag{3.11}$$

Definition 3.21. *We define the complex double product L-function by setting*

$$\Psi(s,f,g) = \gamma(s)L_{N_fN_g}(2s+2-k-l,\psi_f\psi_g)D(s,f,g)$$

where $\gamma(s) = (2\pi)^{-2s}\Gamma(s)\Gamma(s+1-l)$.

The function $\Psi(s,f,g)$ has analytic continuation to all of $\mathbb{C}$, and also satisfies a functional equation, see [45, 61]. The normalised critical values

$$\frac{\Psi(l+r,f,g)}{(2\pi)^{1-l}\langle f,f\rangle_N}$$

are algebraic for all integers r satisfying $0 \leq r \leq k-l-1$ [61, Theorem 4.2]. For the purposes of the next theorem, we will assume that f is p-ordinary for some odd prime p.

Theorem 3.22 (Hida and Panchishkin [37, 53])**.** *Let $f \in \mathcal{S}_k(N_f,\psi)$ and $g \in \mathcal{S}_l(N_g,\omega)$ with $l<k$. There is a bounded $\mathbb{C}_p$-analytic function that is uniquely determined by the interpolation property*

$$L_p(f\otimes g,\chi x_p^s) = i_p\left[(-1)^{-r}\omega(p^{n_\chi})\frac{\tau(\chi)^2p^{n_\chi(l+2r-1)}}{\alpha_p^{2n_\chi}}\frac{\Psi(l+r,f,g^\rho\otimes\chi)}{(-2\pi i)^{1-l}\langle f,f\rangle_{N_f}}\right]$$

for each finite character χ of conductor p^{n_χ}, and integer $r \in \{0,1,\ldots,k-l-1\}$.

The domain of this function is $\mathrm{Hom}_{\mathrm{cont}}(G,\mathbb{C}_p^\times)$, the p-adic analytic Lie group of continuous p-adic characters of the Galois group $G = \mathrm{Gal}(\mathbb{Q}(\mu_{p^\infty})/\mathbb{Q})$. Panchiskin's original construction required that $a_{N_f}(f)\cdot a_{N_g}(g)$ be non-zero, this restriction was mostly removed in [16].

3.6 Triple product L-functions

We shall closely follow the notation employed by Fukunaga and Hsieh in [28, 42]. In particular, $\mathbb{I}_i$ denotes a normal finite flat extension of the algebra $\Lambda_{\mathrm{wt}} = \mathcal{O}_K[\![\Gamma^{\mathrm{wt}}]\!]$ at each $i \in \{1,2,3\}$, with $\Gamma^{\mathrm{wt}} = 1 + p\mathbb{Z}_p$ and $[K:\mathbb{Q}_p] < \infty$. Let us fix a triple of $\mathbb{I}_i$-adic forms $\big(\mathbf{F}_1, \mathbf{G}^{(2)}, \mathbf{G}^{(3)}\big)$ such that $\mathbf{F}_1 := \mathbf{G}^{(1)} \in \mathcal{S}^{\mathrm{ord}}\big(C_1, \psi_1; \mathbb{I}_1\big)$ and also $\mathbf{G}^{(i)} \in \mathcal{S}^{\mathrm{ord}}\big(C_i, \psi_i; \mathbb{I}_i\big)$ for $i = 2, 3$ are each primitive families in the sense of Hida [37], and have expansions in $\mathbb{I}_i[\![q]\!]$.

For a choice of index $i \in \{1,2,3\}$, we consider the set of non-zero continuous $\mathcal{O}_K$-algebraic homomorphisms $\mathfrak{X}_i := \big\{\mathcal{Q}_m^{(i)} : \mathbb{I}_i \to \overline{\mathbb{Q}}_p\big\}_{m\in\mathbb{N}}$. Now given such a formal series $\mathbf{G}^{(i)} \in \mathbb{I}_i[\![q]\!]$ as described above, at every $m \geq 1$ one can take its specialisation

$$\mathbf{G}^{(i)}(m) \ := \ \sum_{n=0}^{\infty} \mathcal{Q}_m^{(i)}\big(a_n\big(\mathbf{G}^{(i)}\big)\big) \cdot q^n \ \in \ \overline{\mathbb{Q}}_p[\![q]\!]$$

which yields a normalised p-stabilised newform of weight $k^{(i)}(m)$, level $p^{e^{(i)}(m)}C_i$ and character $\psi_i \omega^{-k^{(i)}(m)} \epsilon_m^{(i)}$, where $\epsilon_m^{(i)}$ is the restriction of $\mathcal{Q}_m^{(i)}$ to $\Gamma^{\mathrm{wt}} \subset \Lambda_{\mathrm{wt}}$.

Definition 3.23. *If $\mathcal{R} = \mathbb{I}_1 \hat{\otimes}_{\mathcal{O}_K} \mathbb{I}_2 \hat{\otimes}_{\mathcal{O}_K} \mathbb{I}_3$ is the three-parameter weight algebra, then the unbalanced domain $\mathfrak{X}_{\mathcal{R}}^{\mathbf{F}_1}$ of interpolation points for $\mathcal{R}$ is given by*

$$\mathfrak{X}_{\mathcal{R}}^{\mathbf{F}_1} := \left\{ \underline{\mathcal{Q}} = \big(\mathcal{Q}_{m_1}^{(1)}, \mathcal{Q}_{m_2}^{(2)}, \mathcal{Q}_{m_3}^{(3)}\big) \in \mathfrak{X}_1 \times \mathfrak{X}_2 \times \mathfrak{X}_3 \ \middle| \ \begin{array}{c} k_1 + k_2 + k_3 \equiv 0 \ (\mathrm{mod}\, 2), \\ k_1 > k_2 + k_2 - 1, k_1 \geq 2 \end{array} \right\}$$

where we abbreviate $\big(k^{(1)}(m_1), k^{(2)}(m_2), k^{(3)}(m_3)\big)$ by instead using (k_1, k_2, k_3).

Let $\Pi'_{\underline{\mathcal{Q}}}$ be the product of the automorphic representations $\pi_{\mathbf{G}^{(i)}(m)}$ on $\mathrm{GL}_2(\mathbb{A})$ associated to the triple $\big(\mathbf{F}_1, \mathbf{G}^{(2)}, \mathbf{G}^{(3)}\big)(\underline{\mathcal{Q}})$, and define $\Pi_{\underline{\mathcal{Q}}} := \Pi'_{\underline{\mathcal{Q}}} \otimes \big(\chi_{\underline{\mathcal{Q}}}\big)_{\mathbb{A}}$ with

$$\chi_{\underline{\mathcal{Q}}} \ = \ \omega^{-\frac{k^{(1)}(m_1)+k^{(2)}(m_2)+k^{(3)}(m_3)}{2}} \cdot \big(\epsilon_m^{(1)}\epsilon_m^{(2)}\epsilon_m^{(3)}\big)^{\frac{1}{2}} \quad \text{at every point } \underline{\mathcal{Q}} \in \mathfrak{X}_{\mathcal{R}}^{\mathbf{F}_1}.$$

Passing from the automorphic viewpoint to the setting of Galois representations, one has an identification of complex L-series

$$\begin{aligned} L(\Pi_{\underline{\mathcal{Q}}}, s) = \ & \Gamma(\Pi_{\underline{\mathcal{Q}},\infty}, s) \\ & \times \prod_{l \in \mathrm{Spec}\mathbb{Z}} L_l\left(\mathbf{F}_1(m) \otimes \mathbf{G}^{(2)}(m) \otimes \mathbf{G}^{(3)}(m) \otimes \chi_{\underline{\mathcal{Q}}}\,, s + \frac{w-1}{2}\right) \end{aligned}$$

where $\Gamma(\Pi_{\underline{Q},\infty}, s) = \Gamma_{\mathbb{C}}(s + w/2) \cdot \prod_{i=1}^{3} \Gamma_{\mathbb{C}}\big(s + 1 - k_i^*\big)$ is the factor at infinity, $w = k^{(1)}(m_1) + k^{(2)}(m_2) + k^{(3)}(m_3) - 2$, and each $k_i^* = w/2 + 1 - k^{(i)}(m_i)$.

The following conditions (which are copied directly from those given in [28]) will guarantee the existence of a p-adic L-function attached to $\mathbf{F}_1 \otimes \mathbf{G}^{(2)} \otimes \mathbf{G}^{(3)}$ interpolating the special values

$$\iota_p \circ \iota_\infty^{-1}\left(\mathcal{E}_p\big(\mathbf{F}_{k_1} \otimes \mathbf{G}_{k_2}^{(1)} \otimes \mathbf{G}_{k_3}^{(2)} \otimes \chi_{\underline{k}}^{-1}\big)\frac{L\big(\mathbf{F}_{k_1} \otimes \mathbf{G}_{k_2}^{(1)} \otimes \mathbf{G}_{k_3}^{(2)} \otimes \chi_{\underline{k}}^{-1}, \frac{k_1+k_2+k_3-2}{2}\big)}{(-1)^{k_1} \cdot \Omega_\infty(\mathbf{F}_{k_1})^2}\right)$$

at $\underline{k} = (k_1, k_2, k_3)$ with $k_1 > k_2 + k_3 - 1$, where $\chi_{\underline{k}}$ is the unitarization of $\det\big(\Pi_{\underline{k}}^{(\star)}\big)^{1/2}$.

Hypothesis (T1) The primitive characters satisfy $\psi_1\psi_2\psi_3 = \mathbf{1}$.

Hypothesis (T2) The residual Galois representation $\overline{\rho}_{\mathbf{F}_1} : G_{\mathbb{Q}} \to \mathrm{GL}_2(\overline{\mathbb{F}}_p)$ is absolutely irreducible, and the semi-simplification of $\overline{\rho}_{\mathbf{F}_1}\big|_{G_{\mathbb{Q}_p}} \cong \theta_1 \oplus \theta_2$ with $\theta_1 \neq \theta_2$.

Hypothesis (T3) The value of $\gcd(C_1, C_2, C_3)$ is a square-free integer.

Hypothesis (T4) At each $\underline{Q} \in \mathfrak{X}_{\mathcal{R}}^{\mathbf{F}_1}$ and $l\big|C_1C_2C_3$, one has $\epsilon\big(1/2, \Pi_{\underline{Q},l}\big) = +1$ where $\epsilon\big(s, \Pi_{\underline{Q},l}\big)$ denotes the local ϵ-factor at a prime l, as defined by Ikeda in [44].

Theorem 3.24. *(Hsieh-Fukunaga [28, 42]) Under the Hypotheses* **(T1)**–**(T4)**, *there exists a unique element $\mathcal{L}^{\mathbf{F}_1}_{\mathbf{G}^{(2)},\mathbf{G}^{(3)}} \in \mathcal{R}$ satisfying the interpolation property*

$$\big(\mathcal{L}^{\mathbf{F}_1}_{\mathbf{G}^{(2)},\mathbf{G}^{(3)}}(\underline{Q})\big)^2 = \mathcal{E}_{\mathbf{F}_1(m)}(\Pi_{\underline{Q},p}) \cdot \frac{L(\Pi_{\underline{Q}}, 1/2)}{\sqrt{-1}^{2k^{(1)}(m_1)} \cdot \Omega^2_{\mathbf{F}_1(m)}}$$

at all unbalanced points $\underline{Q} \in \mathfrak{X}_{\mathcal{R}}^{\mathbf{F}_1}$, where the p-Euler factor $\mathcal{E}_{\mathbf{F}_1(m)}(\Pi_{\underline{Q},p})$ and the canonical period $\Omega_{\mathbf{F}_1(m)}$ are given in [28, (3.3.1) and Definition 3.3.4], respectively.

Definition 3.25. *The p-adic triple product L-function is given by*

$$\mathbf{L}_p\big(\mathbf{F}_1, \mathbf{G}^{(2)}, \mathbf{G}^{(3)}\big) = \big(\mathcal{L}^{\mathbf{F}_1}_{\mathbf{G}^{(2)},\mathbf{G}^{(3)}}\big)\big)^2.$$

The construction of $\mathcal{L}^{\mathbf{F}_1}_{\mathbf{G}^{(2)},\mathbf{G}^{(3)}}$ from *op. cit.* involves gluing '$\mathbf{G}^{(2)}\cdot\delta_\bullet^{(r)}\big(\mathbf{G}^{(3)}\big)(\underline{\mathcal{Q}})$' along the unbalanced points $\mathfrak{X}^{\mathbf{F}_1}_{\mathcal{R}}$ to produce an interpolating family

$$\mathbf{H}^{\mathrm{aux}} \in \mathcal{S}^{\mathrm{ord}}\big(N, \psi_{1,(p)}\overline{\psi}_1^{(p)}; \mathbb{I}_1\big) \otimes_{\mathbb{I}_1} \mathcal{R}.$$

One then sets

$$L^{\mathbf{F}_1}_{\mathbf{G}^{(2)},\mathbf{G}^{(3)}} := \text{ the first Fourier coefficient of } \eta_{\mathbf{F}_1} \cdot 1_{\mathbf{F}_1} \cdot \mathrm{Tr}_{N/C_1}\big(\mathbf{H}^{\mathrm{aux}}\big)$$

with $N := C_1C_2C_3$, and where the operators $\eta_{\mathbf{F}_1}, 1_{\mathbf{F}_1}$ will be introduced in Chapter 6 (in fact $\mathcal{L}^{\mathbf{F}_1}_{\mathbf{G}^{(2)},\mathbf{G}^{(3)}}$ and $L^{\mathbf{F}_1}_{\mathbf{G}^{(2)},\mathbf{G}^{(3)}}$ differ from each other by a very simple $\mathcal{R}$-unit).

In this chapter we have encountered a number of L-functions. The symmetric square L-functions in Section 3.3 will be used in Chapters 4 and 5, where we compute $\mathcal{L}$-invariants of the p-adic L-functions numerically. The double and triple product L-functions described in Sections 3.5 and 3.6 will be used in Chapters 6 and 7 where we show that a congruence between two pairs, or two triples, of modular forms implies a congruence between their respective p-adic L-functions.

Chapter 4

Computing $\mathcal{L}$-invariants for the symmetric square of an elliptic curve

By devising algorithms to compute $\mathbf{L}_p(\mathrm{Sym}^2 E, 1)'$ and $\mathcal{L}_p^{\mathrm{an}}(\mathrm{Sym}^2 E)$ numerically, and then implementing them in Sage [65], we have established the following result.

Theorem 4.1. *Let E be an elliptic curve over $\mathbb{Q}$ of conductor $N_E \leq 300$, with 4 dividing N_E.*

(i) If $p \in \{3, 5, 7\}$ is a prime of good ordinary reduction for E then Conjecture 3.18 is true, i.e. $\mathcal{L}_p^{\mathrm{an}}(\mathrm{Sym}^2 E) \neq 0$ and $\mathrm{order}_{s=1}\left(\mathbf{L}_p(\mathrm{Sym}^2 E, s)\right) = 1$.

(ii) If $p = 11$ is a prime of good ordinary reduction for E then Conjecture 3.18 is true, with the possible exceptions of the following elliptic curves

$$116a1, \quad 124b1, \quad 200a1, \quad 296a1.$$

(iii) If $p = 13$ is a prime of good ordinary reduction for E then Conjecture 3.18 is true, with the possible exceptions of the following elliptic curves:

$$140a1, \quad 200b1, \quad 232b1, \quad 244a1, \quad 272b1, \quad 280a1.$$

Here we employ Cremona's elliptic curve labelling from [12].

Remark. We have no reason to believe that $\mathcal{L}_p^{\mathrm{an}}(\mathrm{Sym}^2 E)$ actually vanishes for any of the possible exceptions listed in Theorem 4.1. We were simply unable to compute the $\mathcal{L}$-invariants to a high enough accuracy to prove their non-vanishing in these cases.

Recalling the discussion in Section 3.4.3, the next result immediately follows.

Corollary 4.2. *Suppose that E is an elliptic curve over $\mathbb{Q}$ of conductor $N_E \leq 300$ with $4|N_E$, and let $p \leq 13$ be a prime of good ordinary reduction for E. Provided that (E, p) is not one of the ten missing pairs listed in Theorem 4.1(ii)-(iii), we have that*

$$\left.\frac{\mathrm{d}a(\mathcal{F}, p)}{\mathrm{d}X}\right|_{X=0} \;=\; \delta_p(E) \cdot \frac{\mathcal{L}_p^{\mathrm{an}}(\mathrm{Sym}^2 E)}{p} \;\neq\; 0$$

where $\mathcal{F}$ denotes the Hida family lifting $f_E \in \mathcal{S}_2^{\mathrm{new}}\big(\Gamma_0(N_E)\big)$, and we define $\delta_p(E) := -\frac{p\alpha_p}{2\log_p(1+p)} \in \mathbb{Z}_p^\times$.

An outline of the method used to perform these calculations will presented in Section 4.2, but first we turn our attention to some technical results.

4.1 The analytic theory

Let $f_E \in \mathcal{S}_2^{\mathrm{new}}\big(\Gamma_0(N_E)\big)$ denote the primitive form associated to the modular elliptic curve E. Without loss of generality, we assume that the conductor N_E of the newform f_E is divisible by 4. Because $L_\infty(\mathrm{Sym}^2 E, s)$ is invariant under taking quadratic twists, one can always ensure that this holds by replacing E with its twist by the unique character of conductor 4 (if necessary). We also modify the quantities in Conjecture 3.18, as follows:

- we swap the motivic period $(2\pi i)^{-1}\Omega_E^+\Omega_E^-$ with the automorphic period $\pi\langle f_E, f_E\rangle_{N_E}$;
- we exchange the primitive L-function $L_\infty(\mathrm{Sym}^2 E \otimes \chi, s)$ with its imprimitive version

$$D(E, \chi, s) \;:=\; L_{\mathfrak{f}_\chi N_E}\big(\chi^2, 2s-2\big) \times \sum_{n=1}^{\infty} \frac{\chi(n) a_{n^2}(E)}{n^s} \;;$$

- we replace the p-adic L-function with

$$\mathbf{L}_p^{\mathrm{imp}}(\mathrm{Sym}^2 E, s) := \mathcal{F}^{\mathrm{imp}}\big((1+p)^{s-1}-1\big)$$

where

$$\mathcal{F}^{\mathrm{imp}}\big(\chi(1+p)-1\big) \;=\; \frac{\tau(\overline{\chi})}{\alpha_p^{2m_\chi}} \times \frac{D(E,\chi,1)}{\pi\langle f_E, f_E\rangle_{N_E}} \quad \text{if } \chi \neq \mathbf{1},$$

and $\mathcal{F}^{\mathrm{imp}}(0) = 0$.

Providing the imprimitive L-function is non-vanishing at $s=1$, the $\mathcal{L}$-invariant may be equivalently rewritten as

$$\mathcal{L}_p^{\mathrm{an}}(\mathrm{Sym}^2 E) \;:=\; \frac{\mathrm{d}}{\mathrm{d}s}\mathbf{L}_p^{\mathrm{imp}}(\mathrm{Sym}^2 E, s)\Big|_{s=1} \times \left((1-\alpha_p^{-2})(1-p\alpha_p^{-2}) \times \frac{D(E,1)}{\pi\langle f_E, f_E\rangle_{N_E}}\right)^{-1}. \quad (4.1)$$

The right-hand bracketed term in Equation (4.1) is reasonably straightforward to evaluate. (We will describe in Section 4.1.5 the modifications that need to be made to our method when $D(E,1)=0$.)

Lemma 4.3. *Assume E has minimal conductor amongst its quadratic twists. If the geometric conductor of $\mathrm{Sym}^2\big(h^1(E)\big)$ is denoted by $C_{\mathrm{Sym}^2 E} \in \mathbb{N}^2$, then one has the formula*

$$\frac{D(E,1)}{\pi\langle f_E, f_E\rangle_{N_E}} \;=\; \frac{4\cdot\sqrt{C_{\mathrm{Sym}^2 E}}}{N_E} \times \prod_{l\in S_1} \frac{H_l(l^{-1})}{H_l(l^{-2})}$$

where $L_\infty(\mathrm{Sym}^2 E \otimes \chi, s) \;=\; D(E,\chi,s) \times \prod_{l\in S_1} H_l\big(\chi(l)l^{-s}\big)^{-1}$ for a finite set of bad primes S_1.

Proof. If we define

$$\Lambda_\infty(\mathrm{Sym}^2 E, s) := \big(C_{\mathrm{Sym}^2 E}\big)^{s/2}\cdot \pi^{-s/2}\Gamma(s/2)(2\pi)^{-s}\Gamma(s) \times L_\infty(\mathrm{Sym}^2 E, s),$$

then the functional equation [11, Thm 2.2] for this completed L-function states that

$$\Lambda_\infty(\mathrm{Sym}^2 E, s) \;=\; \Lambda_\infty(\mathrm{Sym}^2 E, 3-s).$$

Combining this equation at $s=2$ with the formula $D(E,2) = \frac{8\pi^3}{N_E}\times\langle f_E, f_E\rangle_{N_E}$ for the imprimitive symmetric square L-function in [27, Equation (5)], the result follows easily. □

To calculate $\mathcal{L}_p^{\mathrm{an}}(\mathrm{Sym}^2 E)$ numerically, we must therefore evaluate the derivative $\frac{\mathrm{d}}{\mathrm{d}s}\mathbf{L}_p^{\mathrm{imp}}(\mathrm{Sym}^2 E, s)$ at $s = 1$ to a reasonable accuracy. If $\mu_E^{\mathrm{imp}} \in \mathrm{Meas}(\mathbb{Z}_p^\times, \mathbb{Q}_p)$ is the p-bounded measure corresponding to the power series $\mathcal{F}^{\mathrm{imp}}(X) \in X \cdot \mathbb{Z}_p[\![X]\!][1/p]$, then

$$\mathbf{L}_p^{\mathrm{imp}}(\mathrm{Sym}^2 E, s) \;=\; \int_{x \in \mathbb{Z}_p^\times} \langle x \rangle_p^{s-1} \cdot \, \mathrm{d}\mu_E^{\mathrm{imp}}(x) \quad \text{for every } s \in \mathbb{Z}_p,$$

where $\langle - \rangle_p : \mathbb{Z}_p^\times \twoheadrightarrow 1 + p\mathbb{Z}_p$ denotes the projection to the principal local units. Using a Riemann sum approximation for the covering $\mathbb{Z}_p^\times = \bigsqcup_e \big(e + p^m \mathbb{Z}_p\big)$, it follows that

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}s}\mathbf{L}_p^{\mathrm{imp}}(\mathrm{Sym}^2 E, s)\Big|_{s=1} \;&=\; \int_{x \in \mathbb{Z}_p^\times} \log_p \langle x \rangle_p \cdot \, \mathrm{d}\mu_E^{\mathrm{imp}}(x) \\ &\approx\; \sum_{e \in (\mathbb{Z}/p^m\mathbb{Z})^\times} \log_p \langle e \rangle_p \times \mu_E^{\mathrm{imp}}(e + p^m \mathbb{Z}_p). \end{aligned}$$

Question. *Given a class $e \in (\mathbb{Z}/p^m\mathbb{Z})^\times$, how do we calculate each moment $\mu_E^{\mathrm{imp}}(e + p^m\mathbb{Z}_p)$ efficiently?*

It is well known [11, 13, 57] that the moments $\mu_E^{\mathrm{imp}}(e + p^m\mathbb{Z}_p)$ can be written as an inner product of

$$f^0(z) \;:=\; \Big(f_E(z) - \beta_p f_E(pz)\Big)\Bigg| \begin{pmatrix} 0 & -1 \\ pN_E & 0 \end{pmatrix} \;\in\; \mathcal{S}_2\big(\Gamma_0(pN_E)\big)$$

with a certain modular form $R_{m,e} \in \mathcal{M}_2\big(\Gamma_0(pN_E)\big)$, whose Fourier coefficients are p-integral. The integrality of $\mu_E^{\mathrm{imp}}(-)$ is then controlled by that of $\frac{\langle f^0, R_{m,e} \rangle_{pN_E}}{\langle f_E, f_E \rangle}$ for varying m and e.

4.1.1 Petersson inner product identities for f^0

Recall that the functional equation for the completed Hasse-Weil L-function, $\Lambda_\infty(E, s)$, has the form $\Lambda_\infty(E, 2-s) = w_E \Lambda_\infty(E, s)$ where $w_E \in \{\pm 1\}$ denotes the root number for E over $\mathbb{Q}$. In terms of the associated newform,

$$f_E \big| W(N_E) \;=\; -w_E \cdot f_E \quad \text{under the action of } W(N_E) = \begin{pmatrix} 0 & -1 \\ N_E & 0 \end{pmatrix}.$$

Let $h(z)$ denote a weight 2 holomorphic modular form of level pN_E, and with trivial character. Our goal here is to derive the following technical result, which we repeatedly make use of later.

Lemma 4.4. *Letting $w_E \in \{\pm 1\}$ denote the root number for $E_{/\mathbb{Q}}$, one has the following identities:*

(i) if $\mathbb{C}\cdot h(z) \cap \big(\mathbb{C}\cdot f_E(z) \oplus \mathbb{C}\cdot f_E(pz)\big) = \{0\}$, then $\langle f^0, h\rangle_{pN_E} = 0$;

(ii) $\langle f^0, f_E(z)\rangle_{pN_E} = -w_E \cdot \frac{\alpha_p^2-1}{\alpha_p}\cdot \langle f_E, f_E\rangle_{N_E}$;

(iii) $\langle f^0, f_E(pz)\rangle_{pN_E} = -w_E \cdot \frac{\alpha_p^2-1}{\alpha_p^2}\cdot \langle f_E, f_E\rangle_{N_E}$.

Proof. Since $f_E^\rho = f_E$, the f_E-isotypic part of $\mathcal{M}_2\big(\Gamma_0(pN_E)\big)$ consists of the subspace $\mathbb{C}\cdot f_E(z) \oplus \mathbb{C}\cdot f_E(pz)$. Without loss of generality, assume $h(z)$ is an eigenform for the Hecke algebra at level pN_E. Then by multiplicity one, we can pick a prime $l \nmid pN_E$ such that $a_l(f_E) \neq a_l(h)$; consequently

$$a_l(f_E) \times \langle f^0, h\rangle_{pN_E} = \langle f^0\big|T_l^*, h\rangle_{pN_E} = \langle f^0, h\big|T_l\rangle_{pN_E} = a_l(h) \times \langle f^0, h\rangle_{pN_E}$$

in which case $\langle f^0, h\rangle_{pN_E} = 0$, so part (i) is true.

To establish statement (ii), let us first introduce the p-stabilisation

$$f_0(z) := f_E(z) - \beta_p f_E(pz) = \alpha_p^{-1}\cdot f_E\Big|\big(U_p - \beta_p I_2\big) \;\in\; \mathcal{S}_2\big(\Gamma_0(pN_E)\big). \quad (4.2)$$

This cusp form f_0 is related to f^0 through the formula

$$f^0(z) \;=\; f_E^\rho\big|W(pN_E) - \alpha_p p^{-1} f_E^\rho\big|W(N_E) \;=\; f_0^\rho\big|W(pN_E), \qquad (4.3)$$

where the involution $(-)^\rho$ above sends each $h(z) = \sum_{n\geq 1} h_n e^{2\pi i n z}$ to $h^\rho(z) = \sum_{n\geq 1} \overline{h_n} e^{2\pi i n z}$. Now using Equation (4.3) and observing that $f_E^\rho = f_E$, one obtains the equalities

$$\begin{aligned}
\langle f^0, f_E\rangle_{pN_E} &= \Big\langle f_E^\rho\big|W(pN_E), f_E\Big\rangle_{pN_E} - \overline{\alpha_p p^{-1}}\cdot\Big\langle f_E^\rho\big|W(N_E), f_E\Big\rangle_{pN_E}\\
&= \Big\langle f_E^\rho\Big|W(N_E)\Big|\left(\begin{smallmatrix} p & 0\\ 0 & 1\end{smallmatrix}\right), f_E\Big\rangle_{pN_E} - \beta_p p^{-1}\cdot\langle -w_E\cdot f_E, f_E\rangle_{pN_E}\\
&= \Big\langle -w_E\cdot f_E\Big|\left(\begin{smallmatrix} p & 0\\ 0 & 1\end{smallmatrix}\right)\Big|\mathrm{Tr}^{\Gamma_0(pN_E)}_{\Gamma_0(N_E)}, f_E\Big\rangle_{N_E}\\
&\qquad\qquad + w_E\beta_p p^{-1}\big[\Gamma_0(N_E):\Gamma_0(pN_E)\big]\cdot\langle f_E, f_E\rangle_{N_E}.
\end{aligned}$$

Note from the trace map identity $f_E\Big|\left(\begin{smallmatrix} p & 0\\ 0 & 1\end{smallmatrix}\right)\Big|\mathrm{Tr}^{\Gamma_0(pN_E)}_{\Gamma_0(N_E)} = f_E\big|T_p^* = \overline{a_p(E)}f_E$ together with the index formula $\big[\Gamma_0(N_E) : \Gamma_0(pN_E)\big] = p+1$, the above becomes

$$\begin{aligned}\big\langle f^0, f_E\big\rangle_{pN_E} &= w_E\cdot\left(-a_p(E)+\beta_p\cdot\frac{p+1}{p}\right)\cdot\big\langle f_E,f_E\big\rangle_{N_E}\\ &= -w_E\cdot\left(\frac{\alpha_p^2-1}{\alpha_p}\right)\cdot\big\langle f_E,f_E\big\rangle_{N_E}.\end{aligned}$$

Lastly to prove that (iii) is true, one knows from Equation (4.2) that

$$\begin{aligned}\big\langle f^0, f_E(pz)\big\rangle_{pN_E} &= \Big\langle f^0,\ \beta_p^{-1}\big(f_E(z)-f_0(z)\big)\Big\rangle_{pN_E}\\ &= \beta_p^{-1}\times\Big(\big\langle f^0,f_E\big\rangle_{pN_E}-\big\langle f^0,f_0\big\rangle_{pN_E}\Big).\end{aligned}$$

The first term $\langle f^0, f_E\rangle_{pN_E}$ is already determined from (ii) above. To compute the second term,

$$\big\langle f^0, f_0\big\rangle_{pN_E} \;=\; (p\alpha_p)^{-1}(\alpha_p-\beta_p)(p\alpha_p-\beta_p)\cdot\big\langle f_E^\rho\big|W(N_E),f_E\big\rangle_{N_E}$$

upon applying [30, Lemma 1], and clearly one has $\big\langle f_E^\rho\big|W(N_E),f_E\big\rangle_{N_E} = -w_E\big\langle f_E,f_E\big\rangle_{N_E}$. Combining these strands together:

$$\begin{aligned}&\big\langle f^0, f_E(pz)\big\rangle_{pN_E}\\ &\quad= \beta_p^{-1}\times\left(-w_E\cdot\frac{\alpha_p^2-1}{\alpha_p}+w_E\cdot\frac{(\alpha_p-\beta_p)(p\alpha_p-\beta_p)}{p\alpha_p}\right)\times\big\langle f_E,f_E\big\rangle_{N_E}\\ &\quad= -w_E\cdot\left(\frac{\alpha_p^2-1}{p}-\frac{(\alpha_p-\beta_p)(p\alpha_p-\beta_p)}{p^2}\right)\times\big\langle f_E,f_E\big\rangle_{N_E}\\ &\quad= -w_E\cdot\left(\frac{\alpha_p^2-1}{\alpha_p^2}\right)\times\big\langle f_E,f_E\big\rangle_{N_E},\end{aligned}$$

which completes the demonstration of (iii), and thereby the lemma. □

4.1.2 The q-expansion of the modular form $R_{m,e}$

The key ingredient in calculating the first derivative of $\mathbf{L}_p^{\mathrm{imp}}(\mathrm{Sym}^2E,s)$ at $s=1$, is that the moments of the measure $\mathrm{d}\mu_E^{\mathrm{imp}}(-)$ can be written in terms of the f^0-isotypic projection of a holomorphic modular form. More precisely, let us recall from [11, Eqs (3.22)-(3.23)] that

$$\mu_E^{\mathrm{imp}}(e+p^m\mathbb{Z}_p) \;=\; 2\alpha_p^{-2m}\times\frac{\big\langle f^0, R_{m,e}\big|U_p^{2m-1}\big\rangle_{pN_E}}{\big\langle f_E,f_E\big\rangle_{N_E}} \tag{4.4}$$

where $R_{m,e} \in \mathcal{M}_2\big(\Gamma_0(p^{2m}N_E)\big)$ is obtained by summing up products of certain theta-functions of weight $1/2$ with Eisenstein series of weight $3/2$ (the precise definitions will not be needed). Note also from [11, Lemma 3.10(ii)], the classical trace map identity

$$h\Big|U_p^{2m-1} = h\Big|W(p^{2m}N_E)\Big|\mathrm{Tr}_{\Gamma_0(pN_E)}^{\Gamma_0(p^{2m}N_E)}\Big|W(pN_E)$$

implies that $R_{m,e}\big|U_p^{2m-1}$ actually has level pN_E, so the inner product above is well-defined.

Remarks. (a) If $R_{m,e} = \sum_{n=0}^{\infty} r_n(m,e)q^n$ then it is clear that $R_{m,e}\big|U_p^{2m-1} = \sum_{n=0}^{\infty} r_{np^{2m-1}}(m,e)q^n$; furthermore, $r_0(m,e) = 0$ since the theta-functions of weight $1/2$ vanish at the cusp ∞.

(b) Applying [11, Theorem 3.11] each coefficient $r_n(m,e) \in \mathbb{Q}$, in fact if $p^{2m-1}\big|n$ then $r_n(m,e) \in \mathbb{Z}_p$; it follows that $R_{m,e}\big|U_p^{2m-1} \in q \cdot \mathbb{Z}_{(p)}\big[\big[q\big]\big]$.

(c) Assuming p^{2m-1} divides n, from [11, p133] the q^n-coefficient of $R_{m,e}$ is given by

$$r_n(m,e) = \frac{-2}{\phi(p^m)} \sum_{\chi \in \Delta_m} \sum_{(n_1,n_2) \in \mathcal{W}_n} \sum_{(a,b) \in \mathcal{V}_{n_2}} \mu(a)b \cdot \varepsilon_{n_2}(a) \times \chi(b^2 a)\chi^{-1}(n_1 e) \cdot L_{N_E}(\chi\varepsilon_{n_2}, 0). \qquad (4.5)$$

Here we have employed the notation:

- Δ_m denotes the set of *non-trivial* Dirichlet characters of conductor dividing p^m;
- $\mathcal{W}_n$ is the set of pairs $(n_1, n_2) \in \mathbb{N} \times \mathbb{N}$ coprime to p, and satisfying $n_1^2 \times \frac{N_E}{4} + n_2 = n$;
- $\mathcal{V}_{n_2}$ consists of pairs $(a,b) \in \mathbb{N} \times \mathbb{N}$ that are coprime to pN_E, such that $(ab)^2$ divides n_2;
- ε_{n_2} refers to the character of the imaginary quadratic field $\mathbb{Q}\big(\sqrt{-n_2 N_E}\big)$.

As usual, $L_{N_E}(\chi\varepsilon_{n_2}, s)$ indicates the $\chi\varepsilon_{n_2}$-twisted zeta-function with its Euler factors at the primes dividing N_E removed.

Definition 4.5. *(a) For an integer $t \geq 1$ and $y \in \mathbb{Z}$ with $p \nmid y$, one defines*

$$\vartheta_t(y) = \begin{cases} (p-1)^2/p^2 & \text{if } t \geq 2 \text{ and } y \equiv 1 \pmod{p^t} \\ -(p-1)/p^2 & \text{if } t \geq 2,\ y \not\equiv 1 \pmod{p^t} \text{ but } y \equiv 1 \pmod{p^{t-1}} \\ 0 & \text{if } t \geq 2 \text{ and } y \not\equiv 1 \pmod{p^{t-1}} \\ (p-2)/p & \text{if } t = 1 \text{ and } y \equiv 1 \pmod{p} \\ -1/p & \text{if } t = 1 \text{ and } y \not\equiv 1 \pmod{p}. \end{cases}$$

(b) For any $m \in \mathbb{N}$ and integers x, n_2 both coprime to p, we set

$$M_m^{(n_2)}(x) := \sum_{t=1}^{m} \sum_{\substack{j=1, \\ p \nmid j}}^{p^t} p^t \cdot \vartheta_t(xj) \times \frac{-1}{\mathfrak{f}_{\varepsilon_{n_2}}} \cdot \sum_{i=0}^{\mathfrak{f}_{\varepsilon_{n_2}}-1} \varepsilon_{n_2}(i) \cdot \left((i-j)p^{-t}\right)^{\sharp}$$

where $\left((i-j)p^{-t}\right)^{\sharp} \in \left\{0, \ldots, \mathfrak{f}_{\varepsilon_{n_2}} - 1\right\}$ is the unique integer congruent to $(i - j)p^{-t}$ mod $\mathfrak{f}_{\varepsilon_{n_2}}$.

The following yields an alternate expression for $r_n(m, e)$, designed for use in our programs.

Proposition 4.6. *If p^{2m-1} divides n, then the q^n-coefficient of $R_{m,e}$ is given by*

$$r_n(m, e) = \frac{-2}{\phi(p^m)} \sum_{(n_1,n_2) \in \mathcal{W}_n} \sum_{(a,b) \in \mathcal{V}_{n_2}} \sum_{d | N_E} \mu(ad) b \varepsilon_{n_2}(ad) M_m^{(n_2)}\left(ab^2 d(n_1 e)^*\right)$$

where $(n_1 e)^* \in \{1, \ldots, p^m - 1\}$ denotes the multiplicative inverse of $n_1 e$ modulo p^m.

Before we give the demonstration, we make a couple of observations.

Firstly, the main expense in computing $r_n(m, e)$ is in tabulating the values of ε_{n_2} necessary to compute $M_m^{(n_2)}(-)$. The length of time required to compute $r_n(m, e)$ is roughly proportional to the sum $\sum_{(n_1,n_2) \in \mathcal{W}_n} \mathfrak{f}_{\varepsilon_{n_2}}$, which has order $O(p^{3m})$ as a function of m.

Secondly, the quantity $\phi(p^m)^{-1}\cdot M_m^{(n_2)}\big(ab^2d(n_1e)^*\big)$ occurring above is actually p-integral. The reason is that $M_m^{(n_2)}(-)$ coincides with '$M_m(-)$' defined in [11, Eq (3.30)], and then by Lemma 3.12 of *op. cit.*, the latter is congruent to zero modulo p^{m-1}. However, once one has programmed in the function ϑ_t, our version $M_m^{(n_2)}(-)$ is the quicker to calculate numerically.

Proof. If one recalls the standard identity

$$L_{N_E}(\chi\varepsilon_{n_2},s)=\sum_{d|N_E}\mu(d)\chi(d)\varepsilon_{n_2}(d)d^{-s}\cdot L(\chi\varepsilon_{n_2},s),$$

then Equation (4.5) can be rewritten as

$$r_n(m,e) \;=\; \frac{-2}{\phi(p^m)}\sum_{(n_1,n_2)\in\mathcal{W}_n}\;\sum_{(a,b)\in\mathcal{V}_{n_2}}\;\sum_{d|N_E}\mu(ad)b\cdot\varepsilon_{n_2}(ad)$$
$$\cdot\sum_{\chi\in\Delta_m}\chi\left(\frac{ab^2d}{n_1e}\right)L(\chi\varepsilon_{n_2},0).$$

Therefore, it is enough to show that $\sum_{\chi\in\Delta_m}\chi(x)L(\chi\varepsilon_{n_2},0)$ is equal to the quantity $M_m^{(n_2)}(x)$. Now as each $L(\chi\varepsilon_{n_2},0)=-B_{1,\chi\varepsilon_{n_2}}$ with $B_{1,\chi\varepsilon_{n_2}}$ denoting a $\chi\varepsilon_{n_2}$-twisted Bernoulli number,

$$\begin{aligned}L(\chi\varepsilon_{n_2},0)&=\frac{-1}{\mathfrak{f}_\chi\mathfrak{f}_{\varepsilon_{n_2}}}\cdot\sum_{a=1}^{\mathfrak{f}_\chi\mathfrak{f}_{\varepsilon_{n_2}}}\chi\varepsilon_{n_2}(a)\cdot a\\&=\frac{-1}{\mathfrak{f}_{\varepsilon_{n_2}}}\cdot p^{-t}\cdot\sum_{i=1}^{\mathfrak{f}_{\varepsilon_{n_2}}}\sum_{j=1}^{p^t}\chi(a_{i,j})\varepsilon_{n_2}(a_{i,j})\cdot a_{i,j}\end{aligned}$$

where $\mathfrak{f}_\chi=p^t>1$ say, and the integers $a_{i,j}:=(i-1)p^t+j$. Moreover $\chi(a_{i,j})=\chi(j)$, so decomposing Δ_m into a disjoint union of $(\Delta_t-\Delta_{t-1})$'s yields

$$\begin{aligned}&\sum_{\chi\in\Delta_m}\chi(x)L(\chi\varepsilon_{n_2},0)\\&\quad=\sum_{t=1}^{m}\sum_{\chi\in\Delta_t-\Delta_{t-1}}\chi(x)L(\chi\varepsilon_{n_2},0)\\&\quad=\sum_{t=1}^{m}\sum_{\substack{j=1,\\ p\nmid j}}^{p^t}\left(p^{-t}\cdot\sum_{\chi\in\Delta_t-\Delta_{t-1}}\chi(xj)\right)\times\frac{-1}{\mathfrak{f}_{\varepsilon_{n_2}}}\cdot\sum_{i=1}^{\mathfrak{f}_{\varepsilon_{n_2}}}\varepsilon_{n_2}(a_{i,j})\cdot a_{i,j}.\end{aligned}$$

The lemma will now follow, provided one can verify that:

(i) $p^{-t}\cdot\sum_{\chi\in\Delta_t-\Delta_{t-1}}\chi(xj)$ equals $\vartheta_t(xj)$;

(ii) $\sum_{i=1}^{\mathfrak{f}_{\varepsilon_{n_2}}} \varepsilon_{n_2}(a_{i,j}) \cdot a_{i,j}$ coincides with $p^t \cdot \sum_{i=0}^{\mathfrak{f}_{\varepsilon_{n_2}}-1} \varepsilon_{n_2}(i) \cdot \big((i-j)p^{-t}\big)^{\sharp}$.

To establish statement (i), if $\overline{\Delta}_t = \Delta_t \cup \{\mathbf{1}\}$ for $t > 0$ with $\overline{\Delta}_0 = \{\mathbf{1}\}$ then

$$\begin{aligned} p^{-t} \cdot \sum_{\chi \in \Delta_t - \Delta_{t-1}} \chi(xj) & \\ &= p^{-t} \cdot \left(\sum_{\chi \in \overline{\Delta}_t} \chi(xj) \; - \; \sum_{\chi \in \overline{\Delta}_{t-1}} \chi(xj) \right) \\ &= p^{-t} \cdot \Big(\phi(p^t) \times \mathrm{char}_{1 \bmod p^t}(xj) \; - \; \phi(p^{t-1}) \times \mathrm{char}_{1 \bmod p^{t-1}}(xj) \Big) \end{aligned}$$

where $\mathrm{char}_{1 \bmod p^t}(y)$ returns 1 if p^t divides $y-1$, and returns 0 otherwise. It is then routine to check that the above formula agrees with $\vartheta_t(xj)$ from Definition 4.5.

To prove that (ii) is true, we first observe that

$$\sum_{i=1}^{\mathfrak{f}_{\varepsilon_{n_2}}} \varepsilon_{n_2}(a_{i,j}) \cdot a_{i,j} \;=\; p^t \sum_{i=0}^{\mathfrak{f}_{\varepsilon_{n_2}}-1} \varepsilon_{n_2}\big(ip^t + j\big) \cdot i \;+\; j \sum_{i=0}^{\mathfrak{f}_{\varepsilon_{n_2}}-1} \varepsilon_{n_2}\big(ip^t + j\big)$$

and the right-most summation is identically zero. Furthermore

$$p^t \cdot \sum_{i=0}^{\mathfrak{f}_{\varepsilon_{n_2}}-1} \varepsilon_{n_2}\big(ip^t + j\big) \cdot i \;=\; p^t \cdot \sum_{i=0}^{\mathfrak{f}_{\varepsilon_{n_2}}-1} \varepsilon_{n_2}(i) \cdot \big((i-j)p^{-t}\big)^{\sharp}$$

so statement (ii) is also verified. □

4.1.3 Expressing $R_{m,e}\big|U_p^{2m-1}$ in terms of a rational basis

The next stage is to write $R_{m,e}\big|U_p^{2m-1}$ in terms of an explicit rational basis of $\mathcal{M}_2\big(\Gamma_0(pN_E)\big)$. One first uses the decomposition

$$\mathcal{M}_2\big(\Gamma_0(pN_E)\big) \;=\; \mathcal{S}_2\big(\Gamma_0(pN_E)\big) \oplus \mathrm{Eis}_2\big(\Gamma_0(pN_E)\big)$$

where the second summand denotes the space of generalised Eisenstein series of weight two, level pN_E and trivial nebentypus. A basis of $\mathrm{Eis}_2(\Gamma_0(pN_E))$ can be computed in SAGE using the command

```
EisensteinForms(Gamma0(p*N_E), weight=2).
```

Turning our attention to the space of cusp forms,

$$\mathcal{S}_2\big(\Gamma_0(pN_E)\big) \;\cong\; \bigoplus_{M|pN_E}\;\bigoplus_{c|pN_E/M} \mathcal{S}_2^{\text{new}}\big(\Gamma_0(M)\big)\Big|V_c$$

and one can express an arbitrary cusp form as a linear combination of Hecke eigenforms. A basis of each subspace $\mathcal{S}_2^{\text{new}}\big(\Gamma_0(M)\big)$ may be computed via the command

```
NewForms(Gamma0(M), weight=2).
```

Write $\mathbf{d}_{\mathcal{S}}$ for the dimension of $\mathcal{S}_2\big(\Gamma_0(pN_E)\big)$, and let $\mathbf{d}_{\text{Eis}}$ be the dimension of $\text{Eis}_2\big(\Gamma_0(pN_E)\big)$. Then there exist coefficients $\delta_\bullet(m,e) \in \mathbb{Q}$ such that

$$\begin{aligned} R_{m,e}\Big|U_p^{2m-1} = \;& \delta_1(m,e)\cdot f_E(z) \;+\; \delta_2(m,e)\cdot f_E(pz) \\ & + \sum_{i=3}^{\mathbf{d}_{\mathcal{S}}} \delta_i(m,e)\cdot g_i \;+\; \sum_{j=1}^{\mathbf{d}_{\text{Eis}}} \delta_{j+\mathbf{d}_{\mathcal{S}}}(m,e)\cdot h_j \end{aligned} \tag{4.6}$$

where $\{f_E(z), f_E(pz), g_3(z), g_4(z), \;\ldots\;, g_{\mathbf{d}_{\mathcal{S}}}(z)\}$ is a basis of cuspidal eigenforms at level pN_E, and $\{h_1, \ldots, h_{\mathbf{d}_{\text{Eis}}}\}$ denotes an arbitrary $\mathbb{Q}$-basis for the Eisenstein component. Here we have adopted the labelling convention that $g_1(z) = f_E(z)$ and $g_2(z) = f_E(pz)$. We are then left with the task of determining the $\delta_\bullet(m,e)$'s, especially $\delta_1(m,e)$ and $\delta_2(m,e)$. To accomplish this we select an ordered tuple $\mathfrak{N} = \big[n_1, n_2, \;\ldots\;, n_{\mathbf{d}_{\mathcal{S}}+\mathbf{d}_{\text{Eis}}}\big] \in \mathbb{N}^{\mathbf{d}_{\mathcal{S}}+\mathbf{d}_{\text{Eis}}}$ of distinct positive integers, then consider the $(\mathbf{d}_{\mathcal{S}}+\mathbf{d}_{\text{Eis}}) \times (\mathbf{d}_{\mathcal{S}}+\mathbf{d}_{\text{Eis}})$-linear system of equations

$$r_{np^{2m-1}}(m,e) \;=\; \sum_{i=1}^{\mathbf{d}_{\mathcal{S}}} a_n(g_i)\cdot\delta_i(m,e) \;+\; \sum_{j=1}^{\mathbf{d}_{\text{Eis}}} a_n(h_j)\cdot\delta_{j+\mathbf{d}_{\mathcal{S}}}(m,e)$$

for each $n \in \mathfrak{N}$, arising from Equation (4.6). The corresponding q-coefficient matrix is given by

$$M = \begin{pmatrix} a_{n_1}(g_1) & \ldots & a_{n_1}(g_{\mathbf{d}_{\mathcal{S}}}) & a_{n_1}(h_1) & \ldots & a_{n_1}(h_{\mathbf{d}_{\text{Eis}}}) \\ a_{n_2}(g_1) & \ldots & a_{n_2}(g_{\mathbf{d}_{\mathcal{S}}}) & a_{n_2}(h_1) & \ldots & a_{n_2}(h_{\mathbf{d}_{\text{Eis}}}) \\ \vdots & & \vdots & \vdots & & \vdots \\ a_{n_{\mathbf{d}_{\mathcal{S}}+\mathbf{d}_{\text{Eis}}}}(g_1) & \ldots & a_{n_{\mathbf{d}_{\mathcal{S}}+\mathbf{d}_{\text{Eis}}}}(g_{\mathbf{d}_{\mathcal{S}}}) & a_{n_{\mathbf{d}_{\mathcal{S}}+\mathbf{d}_{\text{Eis}}}}(h_1) & \ldots & a_{n_{\mathbf{d}_{\mathcal{S}}+\mathbf{d}_{\text{Eis}}}}(h_{\mathbf{d}_{\text{Eis}}}) \end{pmatrix}$$

so we can write the system as $\underline{r(m,e)}^T = M \times \underline{\delta(m,e)}^T$, where $\underline{r(m,e)} = \big(r_{np^{2m-1}}(m,e)\big)_{n\in\mathfrak{N}}$ and $\underline{\delta(m,e)} = \big(\delta_i(m,e)\big)_{i=1,\ldots,\#\mathfrak{N}}$.

Hypothesis (det $M \neq 0$)**.** *The matrix $M = M(\mathfrak{N})$ is invertible for the choice of tuple $\mathfrak{N}$.*

Clearly one can always find an $\mathfrak{N}$ for which the above holds, otherwise $\{g_1,\ldots,g_{\mathbf{d}_{\mathcal{S}}},h_1,\ldots,h_{\mathbf{d}_{\mathrm{Eis}}}\}$ would not be a basis for $\mathcal{M}_2\big(\Gamma_0(pN_E)\big)$. In practice, we choose a tuple $\mathfrak{N}$ that will minimise $\sum_{n\in\mathfrak{N}}\sum_{(n_1,n_2)\in\mathcal{W}_n}\mathfrak{f}_{\varepsilon_{n_2}}$, and hence the time needed to compute the vector $\underline{r(m,e)}$.

Corollary 4.7. *If Hypothesis* (det $M \neq 0$) *is satisfied for a tuple $\mathfrak{N}$, and $W = \big(w_{i,j}\big)_{1\le i,j\le\#\mathfrak{N}}$ denotes the inverse matrix to $M = M(m,e,\mathfrak{N})$, then $\underline{\delta(m,e)}^T = W \times \underline{r(m,e)}^T$; in particular*

$$\delta_1(m,e) = \sum_{j=1}^{\#\mathfrak{N}} w_{1,j}\cdot \underline{r(m,e)}_j \quad \textit{and} \quad \delta_2(m,e) = \sum_{j=1}^{\#\mathfrak{N}} w_{2,j}\cdot \underline{r(m,e)}_j\ .$$

Therefore, to obtain these first two components of $\underline{\delta(m,e)}$, we must:

- calculate $g_1,\ldots g_{\mathbf{d}_{\mathcal{S}}}$ and $h_1,\ldots,h_{\mathbf{d}_{\mathrm{Eis}}}$ using SAGE;
- find an optimal choice of $\mathfrak{N}\in\mathbb{N}^{\mathbf{d}_{\mathcal{S}}+\mathbf{d}_{\mathrm{Eis}}}$ such that Hypothesis (det $M \neq 0$) holds;
- produce the vector of q-coefficients $\underline{r(m,e)} = \big(r_{np^{2m-1}}(m,e)\big)_{n\in\mathfrak{N}}$ from Proposition 4.6;
- evaluate the first two basis coefficients, i.e. $\delta_1(m,e)$ and $\delta_2(m,e)$, using Corollary 4.7.

The slowest part of the algorithm is the penultimate line, and as we need $\#\mathfrak{N} = \mathbf{d}_{\mathcal{S}} + \mathbf{d}_{\mathrm{Eis}}$ of these $r_{np^{2m-1}}(m,e)$'s, the time required for this step has order $O\big((\mathbf{d}_{\mathcal{S}}+\mathbf{d}_{\mathrm{Eis}})\times p^{3m}\big)$.

4.1.4 An explicit formula for $\mathcal{L}_p^{\text{an}}(\text{Sym}^2 E)$ modulo p^m, when $D(E,1) \neq 0$

We shall begin by expressing the moments of the measure $d\mu_E^{\text{imp}}$ in terms of the vector $\underline{\delta(m,e)}$. For each $m \geq 1$, define an integer $\nu_{m,p} = \nu_{m,p}(\mathcal{F}^{\text{imp}})$ by the rule

$$\nu_{m,p}(\mathcal{F}^{\text{imp}}) := \min\left\{\text{ord}_p\big(\delta_1(m,e)\big), \text{ord}_p\big(\delta_2(m,e)\big) \text{ where } e \in \big(\mathbb{Z}/p^m\mathbb{Z}\big)^\times\right\}.$$

Therefore to compute $\nu_{m,p}$ we must calculate the $2(p-1)p^{m-1}$ coefficients $\delta_i(m,e)$, for each $i \in \{1,2\}$.

The $\mathbb{Z}_p$-module $\mathfrak{L}_{m,p} \subset \mathbb{Q}_p$ generated by the $\delta_i(m,e)$'s evidently satisfies

$$\mathfrak{L}_{m,p} \;:=\; \mathbb{Z}_p \cdot \left\langle \delta_1(m,e), \delta_2(m,e) \;\middle|\; e \in (\mathbb{Z}/p^m\mathbb{Z})^\times \right\rangle \;=\; p^{\nu_{m,p}} \cdot \mathbb{Z}_p.$$

In particular, if all of the $\delta_i(m,e)$'s are p-integral then $\mathfrak{L}_{m,p} \subset \mathbb{Z}_p$, hence $\nu_{m,p}(\mathcal{F}^{\text{imp}}) \geq 0$.

Lemma 4.8. *For each integer $m \geq 1$ and congruence class $e \in \big(\mathbb{Z}/p^m\mathbb{Z}\big)^\times$,*

$$\mu_E^{\text{imp}}\big(e + p^m\mathbb{Z}_p\big) \;=\; \frac{-2\, w_E}{\alpha_p^{2m}} \cdot (1 - \alpha_p^{-2}) \times \Big(\alpha_p \cdot \delta_1(m,e) + \delta_2(m,e)\Big)$$

and these moments lie inside $p^{\nu_{m,p}}(1-\alpha_p^{-2}) \cdot \mathbb{Z}_p$.

Proof. Considering Equations (4.4) and (4.6) in turn, one deduces that

$$\begin{aligned}
\mu_E^{\text{imp}}(e + p^m\mathbb{Z}_p) &= 2\alpha_p^{-2m} \cdot \frac{\big\langle f^0 \;,\; R_{m,e}\big|U_p^{2m-1}\big\rangle_{pN_E}}{\big\langle f_E, f_E\big\rangle_{N_E}} \\
&= 2\alpha_p^{-2m} \cdot \frac{\big\langle f^0 \;,\; \delta_1(m,e) f_E(z) + \delta_2(m,e) f_E(pz) + \widetilde{R}(z)\big\rangle_{pN_E}}{\big\langle f_E, f_E\big\rangle_{N_E}}
\end{aligned}$$

where $\widetilde{R}(z) \in \mathcal{M}_2\big(\Gamma_0(pN_E)\big)$ intersects trivially with the isotypic subspace $\big(\mathbb{C} \cdot f_E \oplus \mathbb{C} \cdot f_E(pz)\big)$. If we make full use of Lemma 4.4, the three Petersson inner product identities imply

$$\begin{aligned}
\mu_E^{\text{imp}}(e + p^m\mathbb{Z}_p) \;=\; 2\alpha_p^{-2m} \cdot \Bigg(\delta_1(m,e) \cdot \left(-w_E \cdot \frac{\alpha_p^2 - 1}{\alpha_p}\right) \\
+\; \delta_2(m,e) \cdot \left(-w_E \cdot \frac{\alpha_p^2 - 1}{\alpha_p^2}\right) + \; 0\Bigg)
\end{aligned}$$

which is equivalent to the stated formula.

Note the integrality statement for $\mu_E^{\text{imp}}(-)$ follows as $\delta_i(m,e) \in p^{\nu_{m,p}} \cdot \mathbb{Z}_p$ and $\frac{-2w_E}{\alpha_p^{2m}} \in \mathbb{Z}_p^\times$. □

An important corollary of this result is that the power series $\mathcal{F}^{\text{imp}}(X)$ belongs to $p^{\nu_{m,p}} \cdot \mathbb{Z}_p[\![X]\!]$, hence the imprimitive p-adic L-function is p-integral if $\left|\delta_1(m,e)\right|_p, \left|\delta_2(m,e)\right|_p \leq 1$ for all e. Furthermore, if S_{ord} denotes the set of primes where E has good ordinary reduction over $\mathbb{Q}_p$, and S_{denom} consists of those primes dividing $6 \times \prod_{l \in S_1} H_l(l^{-1}) \times \frac{(2\pi i)^{-1}\Omega_E^+\Omega_E^-}{\pi\langle f_E, f_E\rangle_{N_E}}$ (cf. Lemma 4.3), an easy exercise verifies that

$$\mathcal{F}(X) \in p^{\nu_{m,p}} \cdot \mathbb{Z}_p[\![X]\!] \quad \text{at every prime } p \in S_{\text{ord}} - S_{\text{denom}}.$$

Consequently, the primitive p-adic L-function $\mathbf{L}_p(\text{Sym}^2 E, s)$ is a p-integral Iwasawa function at good ordinary primes $p \notin S_{\text{denom}}$ for which we have $\sup\left\{\nu_{m,p}(\mathcal{F}^{\text{imp}}) \;\middle|\; m \in \mathbb{N}\right\} \geq 0$.

For each m, the quantities $\nu_{m,p}$ give a lower bound on the μ-invariant of $\mathcal{F}(X)$ when $p \notin S_{\text{denom}}$. In all of our numerical calculations, we found that the exponent $\nu_{m,p}(\mathcal{F}^{\text{imp}})$ stabilised as a function of $m \geq 3$, and was only once smaller than -2 in value. In fact, this was the single instance where $\mathbf{L}'_p(\text{Sym}^2 E, 1) \notin \mathbb{Z}_p$, occurring at the prime $p = 3$ for the curve $E = 268\text{a}1$.

Theorem 4.9. *Provided that $D(E,1) \neq 0$, if one defines $\xi_{\text{Sym}^2 E} := \frac{D(E,1)}{\pi\langle f_E, f_E\rangle_{N_E}}$ and sets $\epsilon_p = \text{ord}_p\big((1-\alpha_p^{-2}) \cdot \xi_{\text{Sym}^2 E}\big)$, then the $\mathcal{L}$-invariant will satisfy the congruences*

$$\mathcal{L}_p^{\text{an}}(\text{Sym}^2 E) \equiv \frac{-2\, w_E \cdot \xi_{\text{Sym}^2 E}^{-1}}{\alpha_p^{2m}(1-p\alpha_p^{-2})} \times \sum_{e \in (\mathbb{Z}/p^m\mathbb{Z})^\times} \log_p\langle e\rangle_p \cdot \Big(\alpha_p \cdot \delta_1(m,e) + \delta_2(m,e)\Big) \mod p^{m+\nu_{m,p}-\epsilon_p}$$

for every integer $m \geq 1$.

Proof. From Lemma 4.8, our formulae for the moments of $\mathrm{d}\mu_E$ imply that

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}s}\mathbf{L}_p^{\mathrm{imp}}(\mathrm{Sym}^2 E, s)\Big|_{s=1} \\
&= \int_{x\in\mathbb{Z}_p^\times} \log_p\langle x\rangle_p \cdot \ \mathrm{d}\mu_E^{\mathrm{imp}}(x) \\
&\equiv \sum_{e\in(\mathbb{Z}/p^m\mathbb{Z})^\times} \log_p\langle e\rangle_p \cdot \mu_E^{\mathrm{imp}}(e+p^m\mathbb{Z}_p) \mod p^{m+\nu_{m,p}} \\
&\equiv \frac{-2\ w_E}{\alpha_p^{2m}} \cdot (1-\alpha_p^{-2}) \\
&\qquad \times \sum_{e\in(\mathbb{Z}/p^m\mathbb{Z})^\times} \log_p\langle e\rangle_p \cdot \Big(\alpha_p\cdot\delta_1(m,e)+\delta_2(m,e)\Big) \mod p^{m+\nu_{m,p}}.
\end{aligned}$$

Now using Equation (4.1) which is valid as $D(E,1)\neq 0$, the $\mathcal{L}$-invariant can be expressed as

$$\mathcal{L}_p^{\mathrm{an}}(\mathrm{Sym}^2 E) \ = \ \Big((1-\alpha_p^{-2})(1-p\alpha_p^{-2})\ \xi_{\mathrm{Sym}^2 E}\Big)^{-1} \times \frac{\mathrm{d}}{\mathrm{d}s}\mathbf{L}_p^{\mathrm{imp}}(\mathrm{Sym}^2 E, s)\Big|_{s=1}$$

and since $(1-\alpha_p^{-2})(1-p\alpha_p^{-2})\ \xi_{\mathrm{Sym}^2 E}\cdot\mathbb{Z}_p = p^{\epsilon_p}\cdot\mathbb{Z}_p$, the result follows directly. □

4.1.5 A general formula for $\mathbf{L}_p(\mathrm{Sym}^2 E, 1)'$ modulo p^m, even when $D(E,1)=0$

It is important to mention that for the six elliptic curves $176b1$, $196a1$, $200b1$, $240d1$, $272b1$, $300c1$, the value of $\mathbf{L}_p^{\mathrm{imp}}(\mathrm{Sym}^2 E, 1)'$ is zero at all primes p simply because $D(E,s)$ vanishes at $s=1$. One should note that the triviality of $\mathbf{L}_p^{\mathrm{imp}}(\mathrm{Sym}^2 E, 1)'$ does not imply either the triviality of $\mathbf{L}_p(\mathrm{Sym}^2 E, 1)'$, nor the triviality of $\mathcal{L}_p^{\mathrm{an}}(\mathrm{Sym}^2 E)$.

In order that our study of Conjecture 3.18 is not missing out any curves of conductor ≤ 300, for those six elliptic curves listed above with $D(E,1)=0$, we shall now describe a general method to approximate $\mathbf{L}_p(\mathrm{Sym}^2 E, 1)'$ that will work irrespective of whether $D(E,1)$ is zero. Let us begin by partitioning the set $S_1 = S_1(E)$, consisting of primes dividing N_E for which E had additive reduction, into a disjoint union of

$$S'_{1,-} := \{l \in S_1 - \{2\} \text{ such that } \#\Phi_l > 2 \text{ and } \mathrm{Gal}\big(\mathbb{Q}_l(E_4)/\mathbb{Q}_l\big) \text{ is abelian}\},$$
$$S''_{1,-} := \{l \in S_1 \text{ such that } \#\Phi_l = 2\}, \quad \text{and}$$
$$S_{1,+} := S_1 - S'_{1,-} - S''_{1,-}.$$

Here, if $l \neq 2$ then $\Phi_l \subset \mathrm{Gal}\big(\mathbb{Q}_l(E_4)/\mathbb{Q}_l\big)$ denotes the inertia subgroup, and if $l = 2$ then we write $\Phi_2 \subset \mathrm{Gal}\big(\mathbb{Q}_2(E_3)/\mathbb{Q}_2\big)$ for the inertia subgroup.

A careful reading of the argument in [11, pp119-121] indicates that for each prime $l \in S'_{1,-} \cup S''_{1,-}$, one has $H_l(X) = (1 - lX) \cdot \Upsilon_l(X)$ where

$$\Upsilon_l(X) := \begin{cases} 1 & \text{if } l \in S'_{1,-} \\ \big(1 - \hat{\alpha}_l^2 X\big)\big(1 - \hat{\beta}_l^2 X\big) & \text{if } l \in S''_{1,-}. \end{cases} \tag{4.7}$$

Alternatively, if a prime $l \in S_{1,+}$ then $H_l(X) = (1 + lX)$ unless either $l = 3$ and $\Phi_3 \cong C_4 \rtimes C_3$, or instead $l = 2$ and $\Phi_2 \in \{\mathrm{SL}_2(\mathbb{F}_3), Q_8\}$, in which case $H_l(X) = 1$.

It follows that for $s \in \mathbb{C}$, there is a natural separation of Euler factors given by

$$\prod_{l \in S_1} H_l(l^{-s}) = \left(\prod_{l \in S_{1,+}} H_l(l^{-s}) \times \prod_{l \in S''_{1,-}} \Upsilon_l(l^{-s})\right) \times \prod_{l \in S'_{1,-} \cup S''_{1,-}} \big(1 - l^{1-s}\big)$$

with the bracketed term non-zero at $s = 1$, while the other term has order $\#S'_{1,-} + \#S''_{1,-}$.

Definition 4.10. *For any $s \in \mathbb{Z}_p$, let us define the (period modified) p-adic L-function by*

$$\mathbf{L}_p^{\mathrm{aut}}(\mathrm{Sym}^2 E, s) := \frac{(2\pi i)^{-1}\Omega_E^+\Omega_E^-}{\pi\big\langle f_E, f_E\big\rangle_{N_E}} \times \mathbf{L}_p(\mathrm{Sym}^2 E, s).$$

Comparing the above with the imprimitive p-adic L-function, one can factorise the latter into

$$\mathbf{L}_p^{\mathrm{imp}}(\mathrm{Sym}^2 E, s) = \mathfrak{I}_p(s) \cdot \prod_{l \in S'_{1,-} \cup S''_{1,-}} \big(1 - \langle l\rangle_p^{s-1}\big) \times \mathbf{L}_p^{\mathrm{aut}}(\mathrm{Sym}^2 E, s) \tag{4.8}$$

where $\mathfrak{I}_p(s) \in \mathbb{Z}_p\langle\langle s \rangle\rangle$ is an Iwasawa function satisfying

$$\mathfrak{I}_p(1) \;=\; \prod_{l \in S_{1,+}} H_l(l^{-1}) \times \prod_{l \in S''_{1,-}} \Upsilon_l(l^{-1}).$$

It follows directly from this factorisation that

$$\mathrm{order}_{s=1}\big(\mathbf{L}_p^{\mathrm{imp}}(\mathrm{Sym}^2 E, s)\big) \;=\; \#S'_{1,-} + \#S''_{1,-} + \mathrm{order}_{s=1}\big(\mathbf{L}_p^{\mathrm{aut}}(\mathrm{Sym}^2 E, s)\big),$$

hence Conjecture 3.18 is equivalent to $\mathbf{L}_p^{\mathrm{imp}}(\mathrm{Sym}^2 E, s)$ vanishing with order $1 + \#S'_{1,-} + \#S''_{1,-}$ at the critical point $s = 1$.

To verify Coates and Greenberg's conjecture for a given elliptic curve E when $S'_{1,-} \cup S''_{1,-} \neq \emptyset$, we must therefore supply a method to calculate $\mathbf{L}_p^{\mathrm{aut}}(\mathrm{Sym}^2 E, 1)'$, then check it is non-zero.

Theorem 4.11. *For all integers $m \geq 1$, there are congruences*

$$\frac{\mathrm{d}}{\mathrm{d}s}\mathbf{L}_p^{\mathrm{aut}}(\mathrm{Sym}^2 E, s)\bigg|_{s=1} \equiv \frac{\sum_{e \in (\mathbb{Z}/p^m\mathbb{Z})^\times} \big(\log_p \langle e \rangle_p\big)^{1+\#S'_{1,-}+\#S''_{1,-}} \cdot \mu_E^{\mathrm{imp}}(e + p^m\mathbb{Z}_p)}{(1 + \#S'_{1,-} + \#S''_{1,-})! \times \mathfrak{I}_p(1) \times \prod_{l \in S'_{1,-} \cup S''_{1,-}} \log_p(1/l)}$$

modulo $p^{m+\nu_{m,p} - \mathrm{ord}_p\left(\mathfrak{I}_p(1) \times \prod_{l \in S'_{1,-} \cup S''_{1,-}} \log_p(1/l)\right)}$.

Proof. Let us first set $\kappa_p := \#S'_{1,-} + \#S''_{1,-} \geq 0$. We have the following Taylor series at $s = 1$:

- $\mathbf{L}_p^{\mathrm{imp}}(\mathrm{Sym}^2 E, s) \;=\; \dfrac{\mathrm{d}^{\kappa_p+1}\mathbf{L}_p^{\mathrm{imp}}(\mathrm{Sym}^2 E, s)}{\mathrm{d}s^{\kappa_p+1}}\bigg|_{s=1} \cdot \dfrac{(s-1)^{\kappa_p+1}}{(\kappa_p+1)!} + O\big((s-1)^{\kappa_p+2}\big)$
- $\mathbf{L}_p^{\mathrm{aut}}(\mathrm{Sym}^2 E, s) \;=\; \dfrac{\mathrm{d}\mathbf{L}_p^{\mathrm{aut}}(\mathrm{Sym}^2 E, s)}{\mathrm{d}s}\bigg|_{s=1} \cdot (s-1) \;+\; O\big((s-1)^2\big)$
- $\mathfrak{I}_p(s) \;=\; \mathfrak{I}_p(1) \cdot (s-1)^0 \;+\; O\big((s-1)^1\big)$
- $\big(1 - \langle l \rangle_p^{s-1}\big) \;=\; \log_p(1/l) \cdot (s-1)^1 \;+\; O\big((s-1)^2\big)$ for each prime $l \neq p$.

Plugging these directly into Equation (4.8), one reads off from the $(s-1)^{\kappa_p+1}$-term that

$$\frac{\mathrm{d}\mathbf{L}_p^{\mathrm{aut}}(\mathrm{Sym}^2 E, s)}{\mathrm{d}s}\bigg|_{s=1} \;=\; \frac{\frac{\mathrm{d}^{\kappa_p+1}\mathbf{L}_p^{\mathrm{imp}}(\mathrm{Sym}^2 E,s)}{\mathrm{d}s^{\kappa_p+1}}\Big|_{s=1}}{(\kappa_p+1)! \cdot \mathfrak{I}_p(1) \times \prod_{l \in S'_{1,-} \cup S''_{1,-}} \log_p(1/l)}\,.$$

Further, upon differentiating the Mazur-Mellin transform $(\kappa_p + 1)$-times, one easily deduces

$$\left.\frac{\mathrm{d}^{\kappa_p+1}\mathbf{L}_p^{\mathrm{imp}}(\mathrm{Sym}^2 E, s)}{\mathrm{d}s^{\kappa_p+1}}\right|_{s=1}$$
$$= \int_{x\in\mathbb{Z}_p^\times} \left(\log_p\langle x\rangle_p\right)^{\kappa_p+1} \cdot \; \mathrm{d}\mu_E^{\mathrm{imp}}(x)$$
$$\equiv \sum_{e\in(\mathbb{Z}/p^m\mathbb{Z})^\times} \left(\log_p\langle e\rangle_p\right)^{\kappa_p+1} \times \mu_E^{\mathrm{imp}}(e+p^m\mathbb{Z}_p) \bmod p^{m+\nu_p}.$$

Dividing by $(\kappa_p+1)! \cdot \mathfrak{I}_p(1) \times \prod_{l\in S'_{1,-}\cup S''_{1,-}} \log_p(1/l)$ yields the approximation.

□

Remarks. (a) The preceding theorem yields an effective method to calculate $\mathbf{L}_p^{\mathrm{aut}}(\mathrm{Sym}^2 E, 1)'$, as a formula for the moments of the measure $\mathrm{d}\mu_E^{\mathrm{imp}}$ has already been given in Lemma 4.8.

(b) The $\mathcal{L}$-invariant itself is then obtained simply by working out the ratio

$$\mathcal{L}_p^{\mathrm{an}}(\mathrm{Sym}^2 E) \;=\; \left.\frac{\mathrm{d}}{\mathrm{d}s}\mathbf{L}_p^{\mathrm{aut}}(\mathrm{Sym}^2 E, s)\right|_{s=1} \times \left((1-\alpha_p^{-2})(1-p\alpha_p^{-2}) \times \frac{L_\infty(\mathrm{Sym}^2 E, 1)}{\pi\langle f_E, f_E\rangle_{N_E}}\right)^{-1}. \tag{4.9}$$

(c) In the Section 4.1.6, we give a method to determine $S_{1,+}$, $S'_{1,-}$, $S''_{1,-}$ and also the $H_l(X)$'s.

(d) If $S'_{1,-} = S''_{1,-} = \emptyset$ so that $D(E,1) \neq 0$, then Theorem 4.11 and the $\mathcal{L}$-invariant equation specialise to the situation covered in Section 4.1.4 – here $\mathbf{L}_p^{\mathrm{aut}}$ and $\mathbf{L}_p^{\mathrm{imp}}$ have the same order at $s=1$.

4.1.6 Determining the set S_1, and the bad factors $H_l(X)$ with $l \in S_1$

The purpose of this section is to compute the decomposition $S_1 = S_{1,+} \cup S'_{1,-} \cup S''_{1,-}$, and the corresponding Euler factors $H_l(X)$. We retain the same notation and assumptions as Section 4.1.5. Let Δ_E denote the discriminant associated to a minimal Weierstrass equation for E over $\mathbb{Z}$.

Proposition 4.12. *(a) A prime $l \in S_1$ belongs to the subset $S''_{1,-}$ if and only if* $\mathrm{ord}_l\big(N_{E\otimes\theta}\big) = 0$ *at the character* $\theta = \varpi_l$ *if* $l > 2$*, or instead at* $\theta \in \Big\{\varpi_2, \big(\frac{-}{2}\big), \big(\frac{-}{-2}\big)\Big\}$ *if* $l = 2$*, in which case*

$$H_l(X) \ = \ (1-\hat{\alpha}_l^2 X)(1-\hat{\beta}_l^2 X)(1-lX)$$

where $1 - a_l(E\otimes\theta)X + lX^2 = (1-\hat{\alpha}_l X)(1-\hat{\beta}_l X)$.

(b) A prime $l \in S_1 - S''_{1,-} - \{2,3\}$ belongs to $S'_{1,-}$ if and only if either

- $\mathrm{ord}_l\big(\Delta_E\big) = 2, 4, 8, 10$ *and* $l \equiv 1 \pmod 3$, *or*
- $\mathrm{ord}_l\big(\Delta_E\big) = 3, 9$ *and* $l \equiv 1 \pmod 4$

in which case $H_l(X) = 1 - lX$.

(c) A prime $l \in S_1 - S''_{1,-} - \{2,3\}$ belongs to $S_{1,+}$ if and only if either

- $\mathrm{ord}_l\big(\Delta_E\big) = 2, 4, 8, 10$ *and* $l \equiv 2 \pmod 3$, *or*
- $\mathrm{ord}_l\big(\Delta_E\big) = 3, 9$ *and* $l \equiv 3 \pmod 4$

in which case $H_l(X) = 1 + lX$.

(d) For a prime $l \in \big(S_1 \cap \{2,3\}\big) - S''_{1,-}$, one determines whether it belongs to $S'_{1,-}$ or to $S_{1,+}$, and also its Euler factor $H_l(X)$, by using the tables in [11, p121] and Lemma 2.13 of op. cit.

Proof. Most of these statements follow from the description in [58] of the Galois representation $\rho_{E,p^\infty} : \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to \mathrm{GL}_2(\mathbb{Z}_p)$ associated to the p-adic Tate module $\mathrm{Ta}_p(E) := \varprojlim_n E_{p^n}$.

Firstly (a) is true because $\rho_{E,p^\infty}\otimes\theta$ will be unramified at l, and corresponds to the Tate module of the quadratic twist $E\otimes\theta$, which has good reduction at l by the criterion of Néron, Ogg and Shafarevich; consequently $\mathrm{Sym}^2(\rho_{E,p^\infty}) \cong \mathrm{Sym}^2(\rho_{E,p^\infty}\otimes\theta)$ is also unramified at l.

To establish (b) and (c), let us now assume (i) the prime $l \geq 5$, and also (ii) $d_l := \#\Phi_l > 2$ so that $\Phi_l \in \{C_3, C_4, C_6\}$ here. Then using [11, Lemma 1.4],

$$H_l(X) \ = \ \begin{cases} 1 - lX & \text{if } \mathbb{Q}_l(E_p)/\mathbb{Q}_l \text{ is abelian} \\ 1 + lX & \text{if } \mathbb{Q}_l(E_p)/\mathbb{Q}_l \text{ is non-abelian.} \end{cases}$$

Since $\mathbb{Q}_l(E_{p^\infty})/\mathbb{Q}_l(E_p)$ is unramified, we observe that $\mathbb{Q}_l(E_p)/\mathbb{Q}_l$ is abelian if and only if $\mathbb{Q}_l(E_{p^\infty})/\mathbb{Q}_l$ is abelian.

If $\mathbb{Q}_l(E_p)/\mathbb{Q}_l$ is abelian, then Φ_l factors through the inertia subgroup inside $\mathrm{Gal}\big(\mathbb{Q}_l^{\mathrm{ab}}/\mathbb{Q}_l\big)$, and hence through $\mathrm{Gal}\big(\mathbb{Q}_l(\mu_{l^\infty})/\mathbb{Q}_l\big)$. Because $l \nmid d_l$ clearly $\mathrm{Gal}\big(\mathbb{Q}_l(\mu_{l^\infty})/\mathbb{Q}_l(\mu_l)\big)$ acts trivially on $\mathrm{Ta}_p(E)$, in which case Φ_l factors through $\mathrm{Gal}\big(\mathbb{Q}_l(\mu_l)/\mathbb{Q}_l\big)$, whence $l \equiv 1 \pmod{d_l}$.

Conversely, there exists a unique tamely ramified extension H_d of $\mathbb{Q}_p^{\mathrm{nr}}$ with degree $d > 0$. If $l \equiv 1 \pmod{d_l}$ then the action of $\Phi_l \cong \rho_{E,p^\infty}(I_l)$ on $\mathrm{Ta}_p(E)$ factors through the algebraic extension $H_{d_l} = \mathbb{Q}_l^{\mathrm{nr}}(E_{p^\infty}) \subset \mathbb{Q}_l^{\mathrm{nr}}(\mu_l)$, which is certainly an abelian extension of $\mathbb{Q}_l$.

Conclusion: The extension $\mathbb{Q}_l(E_p)/\mathbb{Q}_l$ is abelian if and only if $l \equiv 1 \pmod{d_l}$.

To complete the proof, we note that $d_l = \#\Phi_l$ can be read off from [58, p312] as follows:

- $\#\Phi_l = 3$ if and only if $\mathrm{ord}_l(\Delta_E) \equiv 4$ or 8 mod 12;
- $\#\Phi_l = 4$ if and only if $\mathrm{ord}_l(\Delta_E) \equiv 3$ or 9 mod 12;
- $\#\Phi_l = 6$ if and only if $\mathrm{ord}_l(\Delta_E) \equiv 2$ or 10 mod 12.

It is then a tedious but straightforward exercise to verify that the conditions stated in (b) correspond to $l \equiv 1 \pmod{d_l}$, while the conditions in (c) correspond to $l \not\equiv 1 \pmod{d_l}$. □

4.2 The Basic Method

Using the SAGE computer package, we implemented the method outlined in Sections 4.1.1-4.1.5 to compile tables of $\frac{\mathrm{d}}{\mathrm{d}s}\mathbf{L}_p^{\mathrm{imp}}(\mathrm{Sym}^2 E, s)\big|_{s=1}$ for all curves E of conductor $N_E \leq 300$ such that $4\big|N_E$, as well as their symmetric square $\mathcal{L}$-invariants. These numerical values are tabulated in Appendix B. Here we were mainly interested in verifying that $\mathcal{L}_p^{\mathrm{an}}(\mathrm{Sym}^2 E)$ was non-zero, rather than in computing it to a high p-adic accuracy.

4.2.1 An algorithm to compute the $\mathcal{L}$-invariant numerically

We begin with some general observations. Assume we are given an elliptic curve $E_{/\mathbb{Q}}$ with no restriction on its conductor N_E. Then $\mathcal{L}_p^{\mathrm{an}}(\mathrm{Sym}^2 E)$ depends only on the $\mathbb{Q}$-isogeny class of E. Indeed Nastasescu [52] has shown that the p-adic L-function for $\mathrm{Sym}^2 E$ uniquely determines the $\mathbb{Q}$-isogeny class of the elliptic curve E, up to a twist by a quadratic character.

Let $l \neq 2$ be a prime. We write $\omega_l : \mathbb{F}_l^\times \to \mu_{l-1}$ for the Teichmüller character modulo l. One can then define a quadratic character $\varpi_l : \mathbb{F}_l^\times \to \{\pm 1\}$ by the rule $\varpi_l(x) = \omega_l^{(l-1)/2}(x)$. However if $l = 2$, then $\varpi_2 : (\mathbb{Z}/4\mathbb{Z})^\times \to \{\pm 1\}$ denotes the quadratic character of conductor 4.

<u>*Step 1:*</u> If E has conductor N_E divisible by 4 and $2 \leq \mathrm{ord}_2(N_{E\otimes\theta}) < \mathrm{ord}_2(N_E)$ where θ is one of ϖ_2, $\left(\frac{-}{2}\right)$, and $\left(\frac{-}{-2}\right)$, then replace E with its twist $E \otimes \theta$; alternatively, if E has conductor N_E such that $\mathrm{ord}_2(N_E) \leq 1$, then replace E with its twist $E \otimes \varpi_2$ to ensure that $4|N_E$ holds.

<u>*Step 2:*</u> For our (possibly new) choice of E, let us define the set $S_1 = S_1(E)$ to be the set of primes dividing N_E for which E had additive reduction. Compute the bad Euler factors $H_l(X)$ at each prime number $l \in S_1$ as follows:

(i) If $\mathrm{ord}_l(j_E) < 0$ then $H_l(X) = 1 - X$.

(ii) If $E \otimes \theta$ has good reduction at l where $\theta \in \left\{\varpi_2, \left(\frac{-}{2}\right), \left(\frac{-}{-2}\right)\right\}$ if $l = 2$, or $\theta = \varpi_l$ if $l > 2$, then $H_l(X) = (1-\hat{\alpha}_l^2 X)(1-\hat{\beta}_l^2 X)(1-lX)$ with $\Phi_l \cong C_2$, where the Hecke polynomial $1 - a_l\big(E\otimes\theta\big)X + lX^2 = (1-\hat{\alpha}_l X)(1-\hat{\beta}_l X)$.

(iii) If $\#\Phi_l > 2$, then each factor $H_l(X)$ is determined by Proposition 4.12.

<u>*Step 3:*</u> Compute $C_{\mathrm{Sym}^2 E} = \prod_{l|N_E} l^{\mathrm{ord}_l(C_{\mathrm{Sym}^2 E})}$, where for each prime number l dividing N_E:

(i) if $\mathrm{ord}_l(j_E) < 0$ then $\mathrm{ord}_l(C_{\mathrm{Sym}^2 E}) = 2$;

(ii) if $\Phi_l \cong C_2$ then $\mathrm{ord}_l(C_{\mathrm{Sym}^2 E}) = 0$;

(iii) if $\#\Phi_l > 2$ then $\text{ord}_l(C_{\text{Sym}^2E})$ can be read off from the results in [11, pp120-121].

Step 4: Evaluate the imprimitive L-value $\xi_{\text{Sym}^2E} = \frac{D(E,1)}{\pi\langle f_E, f_E\rangle_{N_E}}$ using the formula in Lemma 4.3, which requires both $\prod_{l\in S_1} H_l(X)$ and C_{Sym^2E} from the previous steps. If $D(E,1) = 0$ then compute the primitive L-value $\frac{L_\infty(\text{Sym}^2E,1)}{\pi\langle f_E, f_E\rangle_{N_E}}$ instead.

Step 5: Find a tuple $\mathfrak{N}$ such that the coefficient matrix $M = M(\mathfrak{N})$ has $\det(M) \neq 0$. Fix the desired accuracy $m \geq 1$ and compute the vector $\underline{r_n(m,e)}$ for each $e \in (\mathbb{Z}/p^m\mathbb{Z})^\times$.

Step 6: For each $e \in (\mathbb{Z}/p^m\mathbb{Z})^\times$, compute both of the terms $\delta_1(m,e)$ and $\delta_2(m,e)$ by following the method described at the end of Corollary 4.7.

Step 7: If $D(E,1) \neq 0$ then calculate $\mathcal{L}_p^{\text{an}}(\text{Sym}^2E) \bmod p^{m+\nu_{m,p}-\epsilon_p}$ via the numerical congruences in Theorem 4.9. If however $D(E,1) = 0$, then compute $\frac{\mathrm{d}}{\mathrm{d}s}\mathbf{L}_p^{\text{aut}}(\text{Sym}^2E,s)\Big|_{s=1}$ using the congruences in Theorem 4.11 and hence $\mathcal{L}_p^{\text{an}}(\text{Sym}^2E)$ by equation 4.9.

The structure of these inertia subgroups Φ_l was worked out completely by Serre in [58, §5.6]. To summarise, if $\text{ord}_l(j_E) \geq 0$ and $l\big|N_E$ then $\Phi_l \in \{C_2, C_3, C_4, C_6\}$ provided that $l \neq 2,3$. If $l = 3$ then the semi-direct product $C_4 \rtimes C_3$ is also a possibility, while if $l = 2$ then both $\text{SL}_2(\mathbb{F}_3)$ and Q_8 (the quaternion group of size 8) can also occur as Φ_l.

Fortunately, there is an extensive table given in [11, p121] which contains the information required to pin down the structure of Φ_2 and Φ_3, as well as the 2- and 3-parts of C_{Sym^2E}. Therefore Step 3 can be fully automated.

We should also point out that the matrix $M(\mathfrak{N})$ in Step 5 need only be determined once, which is fortunate because $\mathbf{d}_{\mathcal{S}} + \mathbf{d}_{\text{Eis}}$ can typically be greater than 10^4 even if N_E is relatively small.

4.2.2 A worked example

Consider the elliptic curve

$$E = 176b1 \ : \ y^2 \ = \ x^3 + x^2 - 5x - 13$$

of conductor $N_E = 2^4 \cdot 11$. Its first few good ordinary primes are $p = 3, 5, 7, 13, \ldots$ with corresponding Hecke eigenvalues $a_3(E) = 1$, $a_5(E) = 1$, $a_7(E) = 2$, $a_{13}(E) = 4, \ldots$ respectively. The quadratic twist $E \otimes \varpi_2$ has conductor 11, and therefore will be $\mathbb{Q}$-isogenous to $X_0(11)$. One determines that $S_{1,+}(E) = S'_{1,-}(E) = \emptyset$ and $S''_{1,-}(E) = \{2\}$, with

$$\Upsilon_2(X) \ = \ \big(1 - \hat{\alpha}_2^2 X\big)\big(1 - \hat{\beta}_2^2 X\big) \ = \ 1 + 4X^2$$

where $\hat{\alpha}_2 = -1 + i$ and $\hat{\beta}_2 = -1 - i$ are the roots of $X^2 - a_2\big(X_0(11)\big)X + 2 = X^2 + 2X + 2$. Furthermore at $s = 1$, the primitive complex L-function satisfies

$$\begin{aligned} \frac{L_\infty(\mathrm{Sym}^2 E, 1)}{\pi \langle f_{E\otimes\varpi_2}, f_{E\otimes\varpi_2}\rangle_{N_{E\otimes\varpi_2}}} &= \frac{4 \cdot \sqrt{C_{\mathrm{Sym}^2 E}}}{N_{E\otimes\varpi_2}} \times \prod_{l \in S_1(E\otimes\varpi_2)} \frac{1}{H_l(l^{-2})} \\ &= \frac{4 \cdot \sqrt{121}}{11} \times 1 \ = \ 4. \end{aligned}$$

The period ratio is given by

$$\begin{aligned} &\frac{\langle f_{E\otimes\varpi_2}, f_{E\otimes\varpi_2}\rangle_{N_{E\otimes\varpi_2}}}{\langle f_E, f_E\rangle_{N_E}} \\ &= \big[\Gamma_0(N_{E\otimes\varpi_2}) : \Gamma_0(N_E)\big]^{-1} \times \mathrm{Res}_{s=2}\left(\frac{\sum_{n=1}^\infty a_n(E\otimes\varpi_2)\cdot n^{-s}}{\sum_{n=1}^\infty a_n(E)\cdot n^{-s}}\right) \\ &= \frac{N_{E\otimes\varpi_2} \cdot \prod_{l | N_{E\otimes\varpi_2}}(1 + 1/l)}{N_E \cdot \prod_{l|N_E}(1+1/l)} \times \frac{1 + 1/2}{D_2(E\otimes\varpi_2, 2)} \\ &= \frac{1}{16 \times \big((1 - 2^{1-2}) \cdot \Upsilon_2(2^{-2})\big)} \ = \ \frac{1}{10} \end{aligned}$$

in which case

$$\frac{L_\infty(\mathrm{Sym}^2 E, 1)}{\pi\langle f_E, f_E\rangle_{N_E}} = \frac{L_\infty(\mathrm{Sym}^2 E, 1)}{\pi \langle f_{E\otimes\varpi_2}, f_{E\otimes\varpi_2}\rangle_{N_{E\otimes\varpi_2}}} \times \frac{\langle f_{E\otimes\varpi_2}, f_{E\otimes\varpi_2}\rangle_{N_{E\otimes\varpi_2}}}{\langle f_E, f_E\rangle_{N_E}} = \frac{2}{5}.$$

Now for each choice of prime $p \in \{3, 5, 7, 13\}$, applying Theorem 4.11 yields the congruences

$$\mathbf{L}_p^{\mathrm{aut}}(\mathrm{Sym}^2 E, 1)' \ \equiv \ \frac{\sum_{e\in(\mathbb{Z}/p^m\mathbb{Z})^\times}\big(\log_p\langle e\rangle_p\big)^2 \cdot \mu_E^{\mathrm{imp}}(e + p^m\mathbb{Z}_p)}{2! \times \Upsilon_2(2^{-1}) \times \log_p(1/2)} \qquad \bmod p^{m+\nu_{m,p}-1}.$$

Evaluating the moments of the measure $d\mu_E^{\mathrm{imp}}$ (via Lemma 4.8) for varying $m \geq 2$, we obtain

$$\mathbf{L}_3^{\mathrm{aut}}(\mathrm{Sym}^2 E, 1)' \;=\; p + O(p^4), \qquad \mathbf{L}_5^{\mathrm{aut}}(\mathrm{Sym}^2 E, 1)' \;=\; p + O(p^2),$$
$$\mathbf{L}_7^{\mathrm{aut}}(\mathrm{Sym}^2 E, 1)' \;=\; 2p + O(p^2), \qquad \mathbf{L}_{13}^{\mathrm{aut}}(\mathrm{Sym}^2 E, 1)' \;=\; 4p + O(p^2).$$

Finally, dividing the above derivatives by $(1-\alpha_p^{-2})(1-p\alpha_p^{-2}) \times \frac{L_\infty(\mathrm{Sym}^2 E,1)}{\pi\langle f_E, f_E\rangle_{N_E}}$, we conclude that

$$\begin{aligned}
\mathcal{L}_3^{\mathrm{an}}\big(\mathrm{Sym}^2 X_0(11)\big) &= \mathcal{L}_3^{\mathrm{an}}(\mathrm{Sym}^2 E) = 1 + 2p^2 + O(p^3)\\
\mathcal{L}_5^{\mathrm{an}}\big(\mathrm{Sym}^2 X_0(11)\big) &= \mathcal{L}_5^{\mathrm{an}}(\mathrm{Sym}^2 E) = p + O(p^2)\\
\mathcal{L}_7^{\mathrm{an}}\big(\mathrm{Sym}^2 X_0(11)\big) &= \mathcal{L}_7^{\mathrm{an}}(\mathrm{Sym}^2 E) = 2p + O(p^2)\\
\mathcal{L}_{13}^{\mathrm{an}}\big(\mathrm{Sym}^2 X_0(11)\big) &= \mathcal{L}_{13}^{\mathrm{an}}(\mathrm{Sym}^2 E) = 2p + O(p^2)
\end{aligned}$$

which are all non-zero elements of $\mathbb{Z}_p$.

Remark. In fact, if one chooses $p = 11$ so that $X_0(11)$ has split multiplicative reduction at p, then it is established in [23, p51] that $\mathcal{L}_{11}^{\mathrm{an}}\big(\mathrm{Sym}^2 X_0(11)\big) = 6p + 5p^2 + 7p^3 + 7p^4 + O(p^5) \neq 0$ by using an approach based on overconvergent modular symbols[1]. It follows immediately that Conjecture 3.18 must hold for the modular elliptic curve $X_0(11)$, at all odd primes $p < 17$.

4.3 Attempts at evaluating the moments $\int x^j \cdot d\mu_E^{\mathrm{imp}}$ for $j \neq 0$?

Theoretically at least, there should be a more efficient way to compute the derivative of the imprimitive p-adic L-function at $s = 1$, which we now outline.

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[1] We also computed $\mathcal{L}_p(\mathrm{Sym}^2 E)$ for $E = 304e1$ at the good ordinary prime $p = 5$, using an identical method. In fact $\mathcal{L}_5(\mathrm{Sym}^2(304e1)) = \mathcal{L}_5(\mathrm{Sym}^2(19a1))$ because $E \otimes \varpi_2$ is $\mathbb{Q}$-isogenous to $19a1$; thankfully, the value we obtained numerically agreed with the 5-adic expansion for $\mathcal{L}_5(19a1)$ given in [23, p52], at the weight $k + 2 = 2$.

Keeping our previous notation,

$$\frac{\mathrm{d}\,\mathbf{L}_p^{\mathrm{imp}}(\mathrm{Sym}^2 E, s)}{\mathrm{d}s} = \int_{\mathbb{Z}_p^\times} \langle x\rangle_p^{s-1} \log_p\langle x\rangle_p \cdot \ \mathrm{d}\mu_E^{\mathrm{imp}}(x)$$
$$= \sum_{e=1}^{p-1}\sum_{j=0}^{\infty} \mathcal{A}_{e,j}(s) \cdot \int_{e+p\mathbb{Z}_p} x^j \cdot \mathrm{d}\mu_E^{\mathrm{imp}}(x)$$

where $\sum_{j=0}^{\infty}\mathcal{A}_{e,j}(s)x^j$ is the power series development for $\langle x\rangle_p^{s-1}\log_p\langle x\rangle_p$ along $e+p\mathbb{Z}_p$.

Question. *Is there an efficient algorithm to determine* $\int_{e+p\mathbb{Z}_p} x^j \cdot \mathrm{d}\mu_E^{\mathrm{imp}}(x)$ *when* $j \neq 0$*?*

If there is a positive answer, then one simply needs to evaluate $\sum_{j=0}^{\infty}\mathcal{A}_{e,j}(1)\cdot \int_{e+p\mathbb{Z}_p} x^j \cdot \mathrm{d}\mu_E^{\mathrm{imp}}$ to some prescribed p-adic precision, and next sum the values over the range $e = 1,\ldots,p-1$. In theory this should yield a far quicker and more accurate method than using Riemann sums, but in practice there are a number of difficulties that arise.

To better illustrate these difficulties, let us assume that $\mathcal{F}_k$ is a p-stabilised ordinary Hecke eigenform of weight $k \geq 2$ and level Np. The critical points for the L-function of the symmetric square of $\mathcal{F}_k$ are $\{1,\ldots,2k-2\}$ which, after p-adically interpolating $L^{\mathrm{imp}}\big(\mathrm{Sym}^2\mathcal{F}_k, s\big)$ at positive integer values, naturally subdivide into the disjoint subsets $\{1,\ldots,k-1\}$ and $\{k,\ldots,2k-2\}$. If $\mathrm{d}\mu^{\mathrm{imp},-}_{\mathrm{Sym}^2\mathcal{F}_k(j)}$ is the measure interpolating χ-twists of $\mathrm{Sym}^2\mathcal{F}_k(j)$ at each $j \in \{1,\ldots,k-1\}$, then the analytic methods in [13, 57] imply for some non-zero constant $c_k \in \overline{\mathbb{Q}}^\times$:

$$\int_{e+p^m\mathbb{Z}_p} x^{j-1}\cdot \mathrm{d}\mu^{\mathrm{imp},-}_{\mathrm{Sym}^2\mathcal{F}_k(1)}(x) = \mu^{\mathrm{imp},-}_{\mathrm{Sym}^2\mathcal{F}_k(j)}\big(e+p^m\mathbb{Z}_p\big)$$
$$= c_k \times \frac{\Big\langle \mathcal{F}_k^0\ ,\ \mathrm{Hol}\big(\tilde{R}_{m,e}^{(k,j)}\big)\big|U_p^{2m-1}\Big\rangle_{pN}}{\big\langle \mathcal{F}_k, \mathcal{F}_k\big\rangle_{pN}}$$

where $\tilde{R}_{m,e}^{(k,j)}$ are certain $\mathcal{C}^\infty$-modular forms exhibiting moderate growth at the cusps of $X_1(p^{2m}N)$, and 'Hol' denotes the operator of holomorphic projection, in the terminology of [33].

Remarks. (a) If $j = k-1$ then the modular forms $\tilde{R}_{m,e}^{(k,k-1)}$ are already holomorphic, and there is no need to apply the operator 'Hol' (e.g. for weight

$k = 2$, one has $R_{m,e} = \tilde{R}^{(2,1)}_{m,e}$).

(b) However if $j \in \{1, \dots, k-1\}$ and $j \neq k-1$, then $\tilde{R}^{(k,j)}_{m,e}$ is *not* a holomorphic modular form.

(c) More alarmingly, if $j \in \mathbb{Z} - \{1, \dots, k-1\}$ then $\tilde{R}^{(k,j)}_{m,e}$ no longer has moderate growth at the cusps of $X_1(p^{2m}N)$, so attempting to evaluate $\mathrm{Hol}\big(\tilde{R}^{(k,j)}_{m,e}\big)$ does not even make sense.

For each of the critical values $j \in \{1, \dots, k-1\}$, the Fourier expansion of $\mathrm{Hol}\big(\tilde{R}^{(k,j)}_{m,e}\big)\big|U_p^{2m-1}$ can be readily computed [13, pp.592-594], and is of the form

$$\mathrm{Hol}\big(\tilde{R}^{(k,j)}_{m,e}\big)\Big|U_p^{2m-1} = \sum_{n=1}^{\infty}\Bigg(\sum_{p^{2m-1}n = Nn_1^2 + n_2} C^{(k,j)}_{n_2,m,n} \times \int_{x \in e + p^m\mathbb{Z}_p} x^{j-k+1} \cdot \mathrm{d}\mu^-(x, \varepsilon_{n_2}) \Bigg) \cdot q^n \quad (4.10)$$

where the scalars $C^{(k,j)}_{n_2,m,n} \in \overline{\mathbb{Q}}$, and $\mathrm{d}\mu^-(x, \varepsilon_{n_2})$ is the twisted Kubota-Leopoldt pseudo-measure interpolating $\int_{\mathbb{Z}_p^\times} \chi x^s \cdot \mathrm{d}\mu^-(x, \varepsilon_{n_2}) = \zeta_p(s, \chi^{-1}\varepsilon_{n_2})$ at finite order characters χ, with $1 - s \in \mathbb{N}$.

In order to evaluate $\int x^j \cdot \mathrm{d}\mu^{\mathrm{imp},-}_{\mathrm{Sym}^2\mathcal{F}_k(k-1)}$, one could naively try to Tate twist the q-expansions in Equation (4.10) at integer values $j \notin [1, k-1]$, and then compute the $\mathcal{F}_k^0$-isotypic component. We attempted this for both the ranges $j > k-1$ and $j < 1$ (which lie outside the region of p-adic interpolation), but found that the corresponding q-expansions could not possibly come from modular forms of level Np. Essentially these methods fail because the operator 'Hol' cannot be extended to real analytic forms that do not exhibit moderate growth.

A possible salvage is to allow the p-stabilised eigenform $\mathcal{F}_k$ to vary in an ordinary family. For example, one could pick another weight $k' = k + t(p-1)p^r$ for some $t, r \in \mathbb{N}$, and a Hecke eigenform $\mathcal{F}_{k'} \in \mathcal{S}_{k'}\big(\Gamma_1(Np)\big)$ such that $\mathcal{F}_{k'} \equiv \mathcal{F}_k \bmod p^r$. One might expect that $\mathcal{L}_p^{\mathrm{an}}\big(\mathrm{Sym}^2(\mathcal{F}_{k'})(k'-1)\big)$ and $\mathcal{L}_p^{\mathrm{an}}\big(\mathrm{Sym}^2(\mathcal{F}_k)(k-1)\big)$ are also congruent, albeit modulo a lesser power of p.

Suppose that we want to compute the moments $\int x^j \cdot \mathrm{d}\mu^{\mathrm{imp},-}_{\mathrm{Sym}^2\mathcal{F}_{k'}(k'-1)}$ instead. Because $k' = k + t(p-1)p^r$ with the chosen $r > 1$, the strip $\{1, \ldots, k'-1\}$ is considerably larger than the strip $\{1, \ldots, k-1\}$, so the range of j's for which $\mathrm{Hol}\big(\tilde{R}^{(k',j)}_{m,e}\big)\Big|U_p^{2m-1}$ is a classical weight k' modular form is now bigger. There are also more moments $\int x^j \cdot \mathrm{d}\mu^{\mathrm{imp},-}_{\mathrm{Sym}^2\mathcal{F}_{k'}(k'-1)}$ available.

The main hindrance is that expressing $\mathrm{Hol}\big(\tilde{R}^{(k',j)}_{m,e}\big)\Big|U_p^{2m-1}$ in terms of a basis of weight k' modular forms is computationally far slower than before, as the dimension of $\mathcal{M}_{k'}\big(\Gamma_0(Np)\big)$ grows rapidly with k'. Therefore any advantage gained by calculating this larger set of moments is immediately offset by the slowness in writing each $\mathrm{Hol}\big(\tilde{R}^{(k',j)}_{m,e}\big)\Big|U_p^{2m-1}$ in terms of a $\mathbb{C}$-basis. For example, if $p = 5$, $N = 11$, $k = 2$ and $k' = 2 + (5-1)5^{10}$ then a simple SAGE calculation reveals $\dim_{\mathbb{C}}\Big(\mathcal{M}_{k'}\big(\Gamma_0(Np)\big)\Big) = 234,375,008$, which is crippling from a numerical standpoint. Nevertheless, because the subspace of p-ordinary modular forms has fixed dimension by Hida's control theory, any theoretical result which could bypass the slowness in computing a full basis for $\mathcal{M}_{k'}\big(\Gamma_0(Np)\big)$ would make the algorithm far more efficient.

Chapter 5

Computing $\mathcal{L}$-invariants for higher weight modular forms

We will further develop the methods described in Chapter 4 in order to calculate the analytic $\mathcal{L}$-invariants for symmetric squares of newforms having weight greater than two. In doing so we will establish the following result.

Theorem 5.1. *Let f be a modular form of weight $k \leq 6$ and level $C \leq 15$, or weight $k = 8$ and level $C \leq 10$, with rational coefficients and trivial nebentypus. If $p \in \{3, 5, 7\}$ such that $p \nmid C$ and f is ordinary at p then*

$$\mathcal{L}_p^{\mathrm{an}}(\mathrm{Sym}^2 f) \neq 0 \quad \textit{and} \quad \mathrm{order}_{s=k-1}\left(\mathbf{L}_p(\mathrm{Sym}^2 f, s)\right) = 1$$

with the possible exception of $f = 10.4.a.a$ [1] and $p = 3$ where we have been unable to compute $\mathcal{L}_p^{\mathrm{an}}(\mathrm{Sym}^2 f)$ to a high enough precision to prove that it is non-zero.

Before doing that, we will give an account of an alternative method for calculating symmetric square $\mathcal{L}$-invariants using p-adic families of overconvergent modular symbols, which are intimately connected to the eigencurve (a rigid analytic space parametrising all finite slope Hecke eigenforms).

[1] Throughout this chapter we employ the labelling conventions of the L-functions and modular forms database [64]

5.1 Computing $\mathcal{L}$-invariants using families of overconvergent modular symbols

The techniques described in this section are due to the work of Dummit et al. [23], which generalises the algorithms developed by Pollack and Stevens [54] for computing with overconvergent modular symbols of fixed weight. We begin with a summary of how computing families of overconvergent modular symbols allows for the computation of $\mathcal{L}$-invariants, and introducing the main objects at play. These ideas will be elaborated on in subsequent sections.

The goal here is to compute q-expansions of Hida families numerically. That is, given a newform f of weight k_0, and a Hida family $F = \sum a_n(F,k)q^n$ passing through f, we want to compute the Fourier coefficients $a_n(F,k)$ as p-adic power series in the weight variable k. Specifically, if we have an approximation of $a_p(F,k)$ then we can use the formula

$$\mathcal{L}_p(\mathrm{Sym}^2 f) = -2\frac{\mathrm{d}}{\mathrm{d}k}\log_p a_p(F,k)\Big|_{k=k_0} \tag{5.1}$$

to approximate the $\mathcal{L}$-invariant.

Let $\mathbf{A}$ be the set of $\mathbb{Q}_p$-power series that converge on the unit disc in $\mathbb{C}_p$, and $\mathbf{D} = \mathrm{Hom}_{\mathrm{cont}}(\mathbf{A}, \mathbb{Q}_p)$ be the dual space of distributions. If we define

$$\Sigma_0(p) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}_2(\mathbb{Z}_p) \;\Big|\; a \in \mathbb{Z}_p^\times \text{ and } c \in p\mathbb{Z}_p \right\}$$

then for each integer weight k we may endow $\mathbf{D}$ with a weight k action of $\Sigma_0(p)$, and we will denote $\mathbf{D}$ equipped with this action by $\mathbf{D}_k$. The space of overconvergent symbols of a fixed weight k and level $\Gamma_0 = \Gamma_0(Np)$ is denoted by $\mathrm{Symb}_{\Gamma_0}(\mathbf{D}_k)$ and will be defined later.

The space $\mathrm{Symb}_{\Gamma_0}(\mathbf{D}_k)$ can be interpolated p-adically over weight space $\mathcal{W} = \mathrm{Hom}(\mathbb{Z}_p^\times, \mathbb{C}_p^\times)$. Recalling that $\mathbb{Z}_p^\times \cong (\mathbb{Z}/p\mathbb{Z})^\times \times (1 + p\mathbb{Z}_p)$, we define $\mathcal{W}_m$ to be the subspace of $\mathcal{W}$ consisting of characters whose restriction to $(\mathbb{Z}/p\mathbb{Z})^\times$ is ω^m, where ω is the Teichmüller character. We also define $W_m \subseteq \mathcal{W}_m$ to be the subspace of characters κ that satisfy $|\kappa(\gamma) - 1|_p \leq 1/p$ for every topological generator γ of $1+p\mathbb{Z}_p^\times$. Note that every classical weight $k \in \mathbb{Z}$, when identified with the "raising to the k-th power" character, is contained in some W_m. The space $\mathbf{D}$ may also be equipped with a weight κ action of $\Sigma_0(p)$ for every p-adic weight $\kappa \in W_m$, see [23, Section 2.3].

If R is the set of convergent power series on the closed disc W_m, then the space $\mathbf{D}\hat{\otimes}R$, where $\hat{\otimes}$ denotes the completed tensor product, consists of R-valued distributions. Evaluating at a weight $\kappa \in D$ gives a specialisation $\mathbf{D}\hat{\otimes}R \to \mathbf{D}_\kappa$, so we refer to elements of $\mathbf{D}\hat{\otimes}R$ as *families* of distributions. Furthermore, $\mathbf{D}\hat{\otimes}R$ may be equipped with a $\Sigma_0(p)$-action that is simultaneously compatible with all the specialisation maps; this allows us to think of $\mathrm{Symb}_{\Gamma_0}(\mathbf{D}\hat{\otimes}R)$ as a space of p-adic families of overconvergent modular symbols of level Γ_0.

Crucially, the space $\mathrm{Symb}_{\Gamma_0}(\mathbf{D}\hat{\otimes}R)$ admits a Hecke action which allows us to define an ordinary subspace, $\mathrm{Symb}_{\Gamma_0}(\mathbf{D}\hat{\otimes}R)^{\mathrm{ord}}$, as the intersection of the images of $\mathrm{Symb}_{\Gamma_0}(\mathbf{D}\hat{\otimes}R)$ when all powers of U_p are applied. This subspace contains all the information contained in the Hida families of tame level N. Thus, by explicitly computing approximations to the characteristic polynomials of Hecke operators acting on $\mathrm{Symb}_{\Gamma_0}(\mathbf{D}\hat{\otimes}R)^{\mathrm{ord}}$, one may also extract approximations to q-expansions of Hida families as desired.

Before continuing in greater detail, we remark that we will be discussing the calculations at a theoretical level only. In order to carry out the computations numerically, one must devise a way of approximating families of overconvergent modular symbols in such a way that the assertions we make about them still hold true. The details of how the computations are implemented numerically is beyond the scope of what we wish to discuss here, and the interested reader is referred to Section 4 of [23].

5.1.1 Modular symbols

Denote by Δ_0 the set of degree zero divisors on $\mathbb{P}^1(\mathbb{Q})$ endowed with the action of $\mathrm{GL}_2(\mathbb{Q})$ by linear fractional transformations. That is, for each $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in$

$\mathrm{GL}_2(\mathbb{Z})$ and $D \in \Delta_0$ we have

$$\gamma D = \frac{aD + b}{cD + d}.$$

Definition 5.2. *Let Γ be a congruence subgroup of $\mathrm{SL}_2(\mathbb{Z})$, and V be a right Γ-module. The space of additive homomorphisms $\varphi : \Delta_0 \to V$ having the property*

$$\varphi(\gamma D) = \varphi(D) \Big| \gamma^{-1}$$

for all $\gamma \in \Gamma$ and $D \in \Delta_0$ is called the space of V-valued modular symbols of level Γ and is denoted $\mathrm{Symb}_\Gamma(V)$.

For example, $\mathrm{Symb}_\Gamma(\mathrm{Sym}^k(\mathbb{Q}_p^2))$ is the space of classical modular symbols. Here we consider mainly $\mathrm{Symb}_{\Gamma_0}(\mathbf{D}_k)$ and $\mathrm{Symb}_{\Gamma_0}(\mathbf{D}\hat{\otimes}R)$ where $\Gamma_0 = \Gamma \cap \Gamma_0(p)$. In each of these cases the action of $\Gamma_0(p)$ may be extended to the algebra

$$S_0(p) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}_2(\mathbb{Z}) \;\Big|\; \gcd(a,p) = 1 \text{ and } c \in p\mathbb{Z} \right\},$$

allowing a Hecke action to be defined on these spaces.

The space $\mathrm{Symb}_{\Gamma_0}(\mathbf{D}_k)$ consists of overconvergent modular symbols, and the Hecke eigenvalues in this space essentially match those in the corresponding space of overconvergent modular forms [55, Theorem 7.1].

For any $\kappa \in W_m$, the specialisation $\kappa : \mathbf{D}\hat{\otimes}R \to \mathbf{D}_\kappa$ induces a Hecke equivariant map

$$\mathrm{sp}_\kappa : \mathrm{Symb}_{\Gamma_0}(\mathbf{D}\hat{\otimes}R) \to \mathrm{Symb}_{\Gamma_0}(\mathbf{D}_\kappa).$$

Thus we view $\mathrm{Symb}_{\Gamma_0}(\mathbf{D}\hat{\otimes}R)$ as a family of overconvergent modular symbols, since for any Φ in this space, $\mathrm{sp}_\kappa(\Phi)$ varies in a p-adic family as κ varies p-adically.

5.1.2 Generating random families of modular symbols

In order to find a basis of the ordinary subspace $\mathrm{Symb}_{\Gamma_0}(\mathbf{D}\hat{\otimes}R)^{\mathrm{ord}}$, it is necessary to generate families of overconvergent modular symbols at random. The following result gives rise to a method for achieving this goal.

Proposition 5.3. *[23, Proposition 3.1] If V is a right Γ-module and Γ_0 is torsion free, then there exist $D_1, \dots, D_t \in \Delta_0$ and $\gamma_1, \dots, \gamma_t \in \mathrm{SL}_2(\mathbb{Z})$ such that*

$$\phi(\{0\} - \{\infty\})\Big|(T-1) = \sum_{j=1}^{t} \phi(D_j)\Big|(\gamma_j - 1)$$

for all $\phi \in \mathrm{Symb}_{\Gamma_0}(V)$, where $T = \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)$. Conversely, given any $v_1, \dots, v_t \in V$ that satisfy

$$v_\infty\Big|(T-1) = \sum_{j=1}^{t} v_j\Big|(\gamma_j - 1), \tag{5.2}$$

there is a unique $\phi \in \mathrm{Symb}_{\Gamma_0}(V)$ with $\phi(D_j) = v_j$ for each j.

Algorithms for computing the D_i and γ_i are given in [54], as are techniques for solving difference equations of the form $w\big|(T-1) = v$. The strategy therefore is to chose $v_1, \dots, v_t \in V = \mathbf{D}\hat{\otimes}R$ at random and then solve Equation (5.2) to find v_∞.

Now Proposition 5.3 implies that there is a unique family of overconvergent modular symbols Φ such that $\Phi(D_j) = v_j$. Furthermore we have $v_\infty\big|(T-1) = \Phi(\{0\}-\{\infty\})\big|(T-1)$, which by [23, Lemma 3.3] implies that $\Phi(\{0\}-\{\infty\}) = v_\infty$. Lastly, as the D_i are the $\mathbb{Z}[\Gamma]$-generators of Δ_0, we obtain a complete description of Φ.

5.1.3 The ordinary subspace

Because we are interested in families of ordinary modular forms, we will explain how to pass from families of overconvergent modular symbols to ordinary families. In this section we will consider $\mathbf{D}^0$, the unit ball of $\mathbf{D}$ under the operator norm, and R^0, the unit ball of R under the sup norm.

Recall that for a compact $\mathbb{Z}_p$-module X with a compact operator U_p, the ordinary subspace is defined to be $X^{\mathrm{ord}} = \bigcap_{n=1}^{\infty} U_p^n(X)$. If X is profinite then it canonically decomposes as $X = X^{\mathrm{ord}} \oplus X^{\mathrm{nil}}$ where U_p acts invertibly on X^{ord} and topologically nilpotently on X^{nil}.

Since R^0 is not profinite, it is not immediately clear that $\mathrm{Symb}_{\Gamma_0}(\mathbf{D}^0\hat{\otimes}R^0)$ can be decomposed with its ordinary subspace as one of the summands. However, the ring $\Lambda = \mathbb{Z}_p[\![w]\!]$ is profinite and contains R^0. If we view Λ as the ring of bounded functions on the open disc radius of $1/p$ contained in R^0, then we see that it is preserved by the action of $\Sigma_0(p)$. We thereby obtain a Hecke equivariant inclusion

$$\mathrm{Symb}_{\Gamma_0}(\mathbf{D}^0\hat{\otimes}R^0) \subseteq \mathrm{Symb}_{\Gamma_0}(\mathbf{D}^0\hat{\otimes}\Lambda).$$

This larger space does decompose into ordinary and non-ordinary parts:

$$\mathrm{Symb}_{\Gamma_0}(\mathbf{D}^0\hat{\otimes}\Lambda) \cong \mathrm{Symb}_{\Gamma_0}(\mathbf{D}^0\hat{\otimes}\Lambda)^{\mathrm{ord}} \oplus \mathrm{Symb}_{\Gamma_0}(\mathbf{D}^0\hat{\otimes}\Lambda)^{\mathrm{nil}}.$$

The rank of the ordinary subspace may be expressed in terms of the $\mathbb{Z}_p$-rank of a space of classical modular forms [23, Theorem 3.9], i.e.

$$\mathrm{rank}_\Lambda\left(\mathrm{Symb}_{\Gamma_0}(\mathbf{D}^0\hat{\otimes}\Lambda)^{\mathrm{ord}}\right) = \mathrm{rank}_{\mathbb{Z}_p}\left(\mathrm{Symb}_{\Gamma_0}(\mathrm{Sym}^k(\mathbb{Z}_p))^{\mathrm{ord}}\right) \tag{5.3}$$

which is readily computed.

5.1.4 Constructing a basis of the ordinary subspace

We now describe how to compute a Λ-basis of the ordinary subspace $X^{\mathrm{ord}} = \mathrm{Symb}_{\Gamma_0}(\mathbf{D}^0\hat{\otimes}\Lambda)^{\mathrm{ord}}$. First we need a way of determining when a set of families $\{\Phi_1,\ldots,\Phi_j\} \subseteq X^{\mathrm{ord}}$ is able to be completed to a Λ-basis of X^{ord}. To this end, suppose that $D_1,\ldots,D_t \in \Delta_0$ generate Δ_0 as a $\mathbb{Z}_p[\Gamma_0]$-module. We define the "vector of total measures" map $\alpha : X^{\mathrm{ord}} \to \Lambda^t$ by sending Φ to the vector $(\Phi(D_i)(\mathbf{1}))_{i=1}^t$. We also define a map $\alpha_k : X_k^{\mathrm{ord}} \to \mathbb{Z}_p^t$ for a fixed weight k in the same way where $X_k^{\mathrm{ord}} = \mathrm{Symb}_{\Gamma_0}(\mathbf{D}_k^0)^{\mathrm{ord}}$. Both maps α and α_k are injective, as is the induced map $\bar{\alpha} : X^{\mathrm{ord}} \otimes \Lambda/\mathfrak{m} \to (\Lambda/\mathfrak{m})^t$ where $\mathfrak{m}$ is the maximal ideal of Λ and $(\Lambda/\mathfrak{m})^t \cong \mathbb{F}_t$ [23, Proposition 3.10]. It follows that $\{\Phi_1,\ldots,\Phi_j\}$ can be completed to a Λ-basis of X^{ord} if and only if $\{\bar{\alpha}(\Phi_1),\ldots,\bar{\alpha}(\Phi_j)\}$ is a linearly independent set in $\mathbb{F}_p^t$ [23, Corollary 3.11].

Now suppose that $\mathcal{B} \subseteq X^{\mathrm{ord}}$ is a set that satisfies this criterion (but is not large enough to be a basis). We may extend $\mathcal{B}$ in the following way:

1. Produce a random $\Phi \in \mathrm{Symb}_{\Gamma_0}(\mathbf{D}^0\hat{\otimes}\Lambda)$ using the method outlined in Section 5.1.2.

2. Calculate Φ^{ord}, the projection of Φ onto X^{ord}, using the operator $e = \lim_{n\to\infty} U_p^{n!}$.

3. Check if $\mathcal{B} \cup \{\Phi^{\text{ord}}\}$ can also be completed to a basis. If it cannot, then repeat Steps (1) and (2) until a suitable Φ^{ord} is found.

Provided the method for generating Φ is sufficiently random, we will always be able to repeat this procedure in order to extend $\mathcal{B}$ to a full basis (we know when to stop because Equation (5.3) tells us what the rank of the basis must be).

5.1.5 Isolating families of congruent modular forms

Finally we explain how to isolate a component of X^{ord} containing information from a single Hida family. Let $\mathbb{T}$ be the Hecke algebra over Λ acting on X^{ord}. There is a decomposition $\mathbb{T} \cong \bigoplus_{\mathfrak{m}} \mathbb{T}_{\mathfrak{m}}$ where $\mathfrak{m}$ ranges over the maximal ideals of $\mathbb{T}$, which induces a Hecke equivariant isomorphism $X^{\text{ord}} \cong \bigoplus_{\mathfrak{m}} X^{\text{ord}}_{\mathfrak{m}}$.

For any prime l, let T be T_l or U_l, depending on whether $l \mid Np$ or $l \nmid Np$. For any pair $(\mathfrak{m}, l)$ consisting of a maximal ideal $\mathfrak{m}$ and prime number l, we define the polynomial $\bar{f}_{\mathfrak{m},l}$ over $\mathbb{F}_p$ be the characteristic polynomial of T acting on $X^{\text{ord}}/\mathfrak{m}X^{\text{ord}}$. Now for each fixed $\mathfrak{m}$, there will be some prime l such that any lift of $\bar{f}_{\mathfrak{m},l}$ to characteristic 0, say $f_{\mathfrak{m},l}$, will act invertibly on $X^{\text{ord}}_{\mathfrak{m}'}$ for all $\mathfrak{m}' \neq \mathfrak{m}$ and topologically nilpotently on $X^{\text{ord}}_{\mathfrak{m}}$. Therefore a basis of $X^{\text{ord}}_{\mathfrak{m}}$ can be constructed in the same manner as in Section 5.1.4 by forming random elements of X^{ord}, and applying the Hecke operator $\prod_{\mathfrak{m}'\neq\mathfrak{m}} f_{\mathfrak{m}',l}(T)$ to project them onto $X^{\text{ord}}_{\mathfrak{m}}$.

5.1.6 A worked example

Let $p = 5$ and consider $f = 17.2.a.a$ the unique newform of level $N = 17$ and weight $k_f = 2$ with q-expansion.

$$f(q) = q - q^2 - q^4 - 2q^5 + 4q^7 + 3q^8 - 3q^9 + O(q^{10}).$$

There are three other Galois conjugacy classes of newforms of weight 2 and level dividing $Np = 85$, they are:

(i) $h_1 = 85.2.a.a$ which has q-expansion

$$h_1(q) = q + q^2 + 2q^3 - q^4 - q^5 + 2q^6 - 2q^7 - 3q^8 + q^9 + O(q^{10});$$

(ii) $h_2 = 85.2.a.b$ which has q-expansion

$$h_2(q) = q + (-1+\eta)q^2 + (-2-\eta)q^3 + (1-2\eta)q^4 - q^5 - \eta q^6 + (-2+\eta)q^7 + (-3+\eta)q^8 + (3+4\eta)q^9 + O(q^{10})$$

where $\eta = \pm\sqrt{2}$; and

(iii) $h_3 = 85.2.a.c$ which has q-expansion

$$h_3(q) = q + \nu q^2 + (\nu+1)q^3 + q^4 + q^5 + (\nu-3)q^6 + (\nu-1)q^7 - \nu q^8 + (-2\nu+1)q^9 + O(q^{10})$$

where $\nu = \pm\sqrt{3}$.

Now the minimal polynomials for the eigenvalues $a_2(h_1)$, $a_2(h_2)$, and $a_2(h_3)$ over $\mathbb{Q}$ are $\mu_1(X) = X - 1$, $\mu_2(X) = X^2 + 2X - 1$, and $\mu_3(X) = X^2 - 3$ respectively. Therefore, even though X^{ord} is 6-dimensional, we see that if we denote by $\mathfrak{m}_i$ the maximal ideal corresponding to g_i for each $i \in \{1,2,3\}$, then the 1-dimensional space $X^{\mathrm{ord}}_{\mathfrak{m}_1}$ will be annihilated by $\mu_1(U_2) = U_2 - 1$, the 2-dimensional subspaces $X^{\mathrm{ord}}_{\mathfrak{m}_2}$ will be annihilated by $\mu_2(U_2) = U_2^2 + 2U_2 - 1$, and the 2-dimensional subspace $X^{\mathrm{ord}}_{\mathfrak{m}_3}$ will be annihilated by $\mu_3(U_2) = U_2^2 - 3$. Because each $\mu_i(a_2(f))$ is non-zero, the subspace corresponding to f will be preserved by these operators. Moreover, as this remaining space is 1-dimensional, a basis of it must be an eigensymbol. Performing this calculation yields the eigenvalue

$$a_p(k) = 3 + 2p + 4p^2 + 2p^3 + O(p^4) + \left(2p + 4p^2 + 2p^3 + O(p^4)\right)k + \left(2p^3 + O(p^4)\right)k^2 + \left(2p^3 + O(p^4)\right)k^3 + O(p^4, k^4).$$

Evaluating Equation (5.1) at $k_0 = k_f - 2 = 0$ gives

$$\mathcal{L}_5(\mathrm{Sym}^2 f) = 2p + 2p^2 + O(p^4).$$

Note the f is in fact the minimal quadratic twist of the modular form associated with the elliptic curve $E = 272b1$, fortunately the $\mathcal{L}$-invariant calculated here agrees with our calculation of $\mathcal{L}(\mathrm{Sym}^2 E)$ in Chapter 4.

Now, keeping $p = 5$, consider the newform $g = 17.4.a.a$ of level $N = 17$ and weight $k_g = 6 \equiv k_f \pmod{p-1}$ with q-expansion

$$g(q) = q - 6q^2 + 10q^3 + 4q^4 - 72q^5 - 60q^6 - 196q^7 + 168q^8 - 143q^9 + O(q^{10}).$$

We have congruences $a_n(f) \equiv a_n(g) \pmod{p}$ for all positive integers n, therefore g belongs to the same Hida family as f, and evaluating Equation (5.1) at $k_0 = k_g - 2 = 4$ with the eigenvalue above gives

$$\mathcal{L}_5(\mathrm{Sym}^2 g) = 2p + 2p^3 + O(p^4).$$

5.2 The higher weight analytic theory

Now we move on to generalise the analytic theory developed in Chapter 4 to include modular forms of weight $k > 2$. Throughout this section, we will denote by $f = \sum_{n=1}^{\infty} a_n(f) q^n$ a primitive weight k newform of level C and trivial nebentypus, and fix a prime $p \nmid C$ for which f is ordinary. We shall also assume that f is minimal in the sense that it is not a Dirichlet twist of a newform of lower level.

Recall from Definition 3.14 that the imprimitive symmetric square L-series for f twisted by χ is given by

$$D(f, \chi, s) = L_{MC}(2s - 2k + 2, \chi^2) \cdot \sum_{n=1}^{\infty} \chi(n) a_{n^2}(f) n^{-s}$$

for $s \in \mathbb{C}$, where χ is a Dirichlet character modulo M. The primitive symmetric square L-series has the form

$$L_\infty(\mathrm{Sym}^2 f \otimes \chi, s) = \prod_{q \mid C} \Upsilon_q(\chi, s)^{-1} \frac{L_{CC_\chi}(\chi^2, 2s-2k+2)}{L(\chi, s-k+1)} \times \sum_{n=1}^{\infty} \left|a_n(f)\right|^2 \chi(n) n^{-s}$$

where

$$\Upsilon_q(\chi, s) = \begin{cases} 1 - \chi(q)^2 q^{-2(s-k+1)} & \text{if } a_q(f) = 0 \text{ and } \mathrm{ord}_q(C) \text{ is even} \\ 1 - \chi(q) q^{-(s-k+1)} & \text{otherwise.} \end{cases}$$

If we define

$$\widetilde{C} = \prod_{q \mid C} q^{\mathrm{ord}_q C - m(q)}$$

with

$$m(q) = \begin{cases} \left\lfloor \frac{1}{2} \mathrm{ord}_q C \right\rfloor & \text{if } a_q(f) = 0 \\ 0 & \text{otherwise,} \end{cases}$$

then the primitive L-function $L_\infty(\mathrm{Sym}^2 f, s)$ satisfies the functional equation $\Lambda_\infty(f, s) = \Lambda_\infty(f^\rho, 2k-1-s)$ [13, Theorem 1.3.2], where

$$\Lambda_\infty(f, s) = \widetilde{C}(2\pi)^{-s}\Gamma(s)\pi^{\frac{1}{2}s}\Gamma\left(\frac{1}{2}(s-k+2)\right) L_\infty(\mathrm{Sym}^2 f, s)$$

is the completed L-function.

In parallel with our approach at weight two, we may define the analytic $\mathcal{L}$-invariant $\mathcal{L}_p^{\mathrm{an}}(\mathrm{Sym}^2 f)$ by the equation

$$\begin{aligned} \tfrac{\mathrm{d}}{\mathrm{d}s} \mathbf{L}_p(\mathrm{Sym}^2 f, s)\Big|_{s=k-1} &= \mathcal{L}_p^{\mathrm{an}}(\mathrm{Sym}^2 f) \cdot \frac{\Gamma(k-1)}{2^{2k}} \\ &\times (1 - p^{k-2}\alpha_p^{-2})(1 - p^{k-1}\alpha_p^{-2}) \cdot \frac{L_\infty(\mathrm{Sym}^2 f, k-1)}{\pi^{k-1}\langle f, f\rangle_C} \end{aligned} \tag{5.4}$$

provided the imprimitive L-function does not vanish at $s = k-1$. Here $\mathbf{L}_p(\mathrm{Sym}^2 f, s)$ is the p-adic L-function discussed in Section 3.3.3. The final term may be computed exactly as follows.

Proposition 5.4. *If f is minimal amongst its Dirichlet twists, then*

$$\xi_{\mathrm{Sym}^2 f} := \frac{L_\infty(\mathrm{Sym}^2 f, k-1)}{\pi^{k-1}\langle f, f\rangle_C} = \frac{2^{2k-2}\widetilde{C}}{(k-2)!C} \times \prod_{l \mid C} H_l(l^{-k})^{-1}$$

where

$$H_l(X) = \frac{P_l(\mathrm{Sym}^2 f, X)}{\left(1-\alpha_l^2 X\right)\left(1-\beta_l^2 X\right)\left(1-\alpha_l\beta_l X\right)}$$

and the polynomial $P_l(\mathrm{Sym}^2 f, X)$ *was defined in Section 3.3.2.*

Proof. The functions $D(f,s) := D(f,\mathbf{1},s)$ and $L_\infty(\mathrm{Sym}^2 f, s)$ differ only by Euler factors at primes dividing C, hence $D(f,s) = \prod_{l|C} H_l(l^{-s}) \cdot L_\infty(\mathrm{Sym}^2 f, s)$. The exact forms of the functions $H_l(s,f)$ are calculated in [57]. Combining the functional equation at $s = k-1$ with the result of Hida and Tilouine [41, Theorem 7.1]:

$$D(f,k) = \frac{2^{2k-1}\pi^{k+1}}{C\,\Gamma(k)}\langle f, f\rangle_C$$

yields the result. □

5.2.1 Computing the derivative of the p-adic L-function

We define the imprimitive p-adic L-function by

$$\mathbf{L}_p^{\mathrm{imp}}(\mathrm{Sym}^2 f, s) = \int_{x\in\mathbb{Z}_p^\times} \langle x\rangle_p^{s-k+1}\,\mathrm{d}D^-(x) \tag{5.5}$$

for every $s \in \mathbb{Z}_p$. Here $\mathrm{d}D^-$, defined in [13, Section 2.6], is the distribution [2] on $\mathbb{Z}_p^\times$ satisfying

$$\int_{\mathbb{Z}_p^\times} \chi\,\mathrm{d}D^- = D_{k-1}^-(\chi) \tag{5.6}$$

where for each $s \in \{1,\ldots,k-1\}$ and Dirichlet character χ of p-power conductor p^m we have

$$D_s^-(\chi) = (1-\bar{\chi}^2(2)2^{-2(s-k+1)})^\eta \cdot \frac{\tau(\chi)\Gamma(s)}{(2\pi)^s} \cdot \mathcal{E}_p(f,\bar{\chi},s) \cdot \frac{D(f,\bar{\chi},s)}{\langle f,f\rangle_C}$$

[2] The S-adic distributions in [13] are modified by the factor $(1-\chi^2(c)c^{2(s-k)})$ for a fixed c with $\gcd(c, 4Cp) = 1$. Since we are dealing only with the p-adic case, we may (and indeed do) omit this extra factor.

with

$$\mathcal{E}_p(f,\bar{\chi},s) = \begin{cases} p^{m(s-1)} \cdot \alpha_p^{-2m} & \text{if } m = 0 \\ (1-\bar{\chi}(p)p^{-(s-k+1)}) & \\ \quad \times(1-\chi(p)\alpha_p^{-2}p^{s-1}) & \text{if } m > 0 \\ \quad \times(1-\bar{\chi}(p)\beta_p^{-2}p^{-s}), & \end{cases}$$

and

$$\eta = \begin{cases} 1 & \text{if } C \text{ is odd} \\ 0 & \text{if } C \text{ is even.} \end{cases}$$

The next result relates the derivative of the imprimitive p-adic L-function to the derivative of the primitive one. For simplicity, we only consider the case where the imprimitive complex L-function does not vanish at $s = k-1$, this will be sufficient to prove Theorem 5.1.

Proposition 5.5. *Provided $D(f, k-1) \neq 0$, so that $H_l(l^{1-k})$ does not vanish,*

$$\frac{\mathrm{d}}{\mathrm{d}s}\mathbf{L}_p(\mathrm{Sym}^2 f, s)\bigg|_{s=k-1} = \frac{\frac{\mathrm{d}^{\eta+1}}{\mathrm{d}s^{\eta+1}}\mathbf{L}_p^{\mathrm{imp}}(\mathrm{Sym}^2 f, s)\Big|_{s=k-1}}{2^{k+1}\cdot H_l(l^{1-k})\cdot \left(4\log_p\langle 2\rangle_p\right)^{\eta}}.$$

Proof. Comparing the interpolation properties for the primitive and imprimitive p-adic L-functions, we see that

$$\mathbf{L}_p^{\mathrm{imp}}(\mathrm{Sym}^2 f, s) = 2^{k+1}\cdot\left(1-\langle 2\rangle^{-2(s-k+1)}\right)^{\eta}\cdot \mathfrak{I}_p(s)\cdot \mathbf{L}_p(\mathrm{Sym}^2 f, s) \tag{5.7}$$

where $\mathfrak{I}_p(s) \in \mathbb{Z}_p\langle\langle s\rangle\rangle$ is an Iwasawa function satisfying

$$\mathfrak{I}_p(k-1) = \prod_{l\mid C} H_l(l^{-(k-1)}).$$

Taking Taylor expansion about $s = k-1$ we have the following:

- $\mathbf{L}_p^{\mathrm{imp}}(\mathrm{Sym}^2 f, s) = \frac{\mathrm{d}^{\eta+1}}{\mathrm{d}s^{\eta+1}}\mathbf{L}_p^{\mathrm{imp}}(\mathrm{Sym}^2 f, s)\Big|_{s=k-1} \times \frac{(s-k+1)^{\eta+1}}{(\eta+1)!} + O\left((s-k+1)^{\eta+2}\right),$

- $\left(1-\langle 2\rangle^{-2(s-k+1)}\right)^{\eta} = \left(2\log_p\langle 2\rangle_p\cdot(s-k+1)\right)^{\eta} + O\left((s-k+1)^{\eta+1}\right),$

- $\mathfrak{I}_p(s) = \mathfrak{I}_p(k-1) + O\left(s-k+1\right)$,
- $\mathbf{L}_p(\mathrm{Sym}^2 f, s) = \frac{\mathrm{d}}{\mathrm{d}s}\mathbf{L}_p(\mathrm{Sym}^2 f, s)\Big|_{s=k-1} \cdot (s-k+1) + O\left((s-k+1)^2\right)$.

Substituting these into Equation (5.7) and equating the $(s-k+1)^{\eta+1}$ terms yields the result. □

It follows from Equation (5.5) that

$$\begin{aligned}\frac{\mathrm{d}^{\eta+1}}{\mathrm{d}s^{\eta+1}}\mathbf{L}_p^{\mathrm{imp}}(\mathrm{Sym}^2 f, s)\Big|_{s=k-1} \\ &= \int_{x\in\mathbb{Z}_p^\times} \left(\log_p\langle x\rangle_p\right)^{\eta+1} \mathrm{d}D^-(x) \\ &\approx \sum_{e\in(\mathbb{Z}/p^m\mathbb{Z})^\times} \left(\log_p\langle e\rangle_p\right)^{\eta+1} D^-(e+p^m\mathbb{Z}_p)\end{aligned} \tag{5.8}$$

so we are able to approximate $\frac{\mathrm{d}^{\eta+1}}{\mathrm{d}s^{\eta+1}}\mathbf{L}_p^{\mathrm{imp}}(\mathrm{Sym}^2 f, s)$ provided we can compute the moments $D^-(e+p^m\mathbb{Z}_p)$.

5.2.2 Computing moments of the distribution $\mathrm{d}D^-$

Now we turn to the calculation of $D^-(e+p^m\mathbb{Z}_p)$ for $e \in (\mathbb{Z}/p^m\mathbb{Z})^\times$. As usual, α_p and β_p denote the roots of the Hecke polynomial of f at p, that is

$$X^2 - a_p(f) + p^{k-1} = (X-\alpha_p)(X-\beta_p)$$

with $\alpha_p \in \mathbb{Z}_p^\times$; and $f_0(z) = f(z) - \beta_p f(pz)$ is the p-stabilisation of f.

Proposition 5.6. *The moments of the distribution D^- are given by*

$$D^-(e+p^m\mathbb{Z}_p) = \frac{2^{2k-3}p^{\frac{1}{2}k-1}C^{\frac{1}{2}k-1}}{\alpha_p^{2m}} \times \frac{\langle f_0^\rho | V_C, R_{m,e}|U_p^{2m-1}W_{4C^2p}\rangle_{4C^2p}}{\langle f, f\rangle_C}$$

where $R_{m,e}|U_p^{2m-1} = \sum_{n=1}^\infty r_{np^{2m-1}}(m,e)q^n \in \mathcal{S}_k(\Gamma_0(4Cp))$ has q-expansion

$$r_{np^{2m-1}}(m,e) = \frac{1}{\phi(p^m)} \times \sum_{\substack{\chi\in\Delta_m \\ (n_1,n_2)\in\mathcal{W}_{np^{2m-1}} \\ (a,b)\in\mathcal{V}_{n_2}}} \chi\left(\frac{e}{n_1}\right) n_1 P^-(n_2, np^{2m-1})$$
$$\times\, \mu(a)b \cdot \chi\varepsilon_{n_2}(ab^2) L_{4Cp}(0, \chi\varepsilon_{n_2}).$$

Here the notations above are precisely:

- *$L_N(s,\xi)$ denotes the ξ-twisted zeta function with Euler factors at primes dividing N removed;*

- *Δ_m is the set of **non-trivial** Dirichlet characters having conductor dividing p^m;*

- *$P^-(X,Y) \in \mathbb{Q}[X,Y]$ is the polynomial*

$$P^-(X,Y) = \sum_{j=0}^{\frac{1}{2}k-1} \frac{(-1)^j}{4^j} \times \frac{k-1}{k-2j-1} \times \binom{k-j-2}{j} \times X^{\frac{1}{2}k-j-1}Y^j;$$

- *$\mathcal{W}_n = \{(n_1,n_2) \in \mathbb{N}^2 : n = CN_1^2 + n_2 \text{ and } \gcd(n_1,p) = \gcd(n_2,p) = 1\}$;*

- *$\mathcal{V}_n = \{(a,b) \in \mathbb{N}^2 : ab \mid m \text{ and } \gcd(4Cp,a) = \gcd(4Cp,b) = 1\}$;*

- *ε_{n_2} is the character of the quadratic extension $\mathbb{Q}(\sqrt{-Cn_2})/\mathbb{Q}$.*

Proof. By the result of [13, Corollary 2.6.2], for $\chi \in \Delta_m$ we have

$$D_{k-1}^-(\chi) = \frac{2^{2k-3}p^{\frac{1}{2}k-1}C^{\frac{1}{2}k-\frac{3}{4}}}{\alpha_p^{2m}} \times \frac{\langle f_0^\rho | V_C, F^-(z,k-1,\chi)|U_p^{2m-1}W_{4C^2p}\rangle_{4C^2p}}{\langle f,f\rangle_C}$$

where $F^-(z,s,\chi) = \sum_{n=1}^\infty v^-(n,s,\chi)q^n \in \mathcal{S}_k(\Gamma_0(4Cp^{2m}))$. The Fourier coefficients of $F^-(z,s,\chi)|U_p^{2m-1}$ at $s=k-1$, given on p. 594 of *op. cit.*, are

$$v^-(np^{2m-1},k-1,\chi) = \sum_{\substack{(n_1,n_2)\in\mathcal{W}_{np^{2m-1}} \\ (a,b)\in\mathcal{V}_{n_2}}} \Big(C^{-\frac{1}{4}}\chi(n_1)n_1P^-(n_2,np^{2m-1}) \\ \times \beta(n_2,0,\varepsilon_{n_2}\chi)L_{4Cp}(0,\bar{\chi}\varepsilon_{n_2})\Big)$$

where

$$\beta(n_2,0,\varepsilon_{n_2}\chi) = \sum_{(a,b)\in\mathcal{V}_{n_2}} \mu(a)b\cdot\varepsilon_{n_2}\chi(ab^2).$$

The result follows by defining

$$R_{m,e}(z) = \frac{C^{\frac{1}{4}}}{\phi(p^m)} \sum_{\chi\in\Delta_m} \chi(e)F^-(z,k-1,\chi)$$

and noting that Equation (5.6) implies

$$D^-(e+p^m\mathbb{Z}_p) = \frac{1}{\phi(p^m)} \sum_{\chi\in\Delta_m} \chi(e)D_{k-1}^-(\bar{\chi}).$$

□

As in Chapter 4, we may write the formula for the Fourier coefficients $r_{np^{2m-1}}(m,e)$ in a form that is more practical from a computational standpoint.

Corollary 5.7. *The Fourier coefficients of* $R_{m,e}|U_p^{2m-1}$ *are given by the equivalent expression*

$$r_{np^{2m-1}}(m,e) = \frac{1}{\phi(p^m)} \sum_{\substack{(n_1,n_2)\in\mathcal{W}_{np^{2m-1}} \\ (a,b)\in\mathcal{V}_n \\ d|4Cp}} P^-(n_2, np^{2m-1}) \times \mu(ad)\varepsilon_{n_2}(ab^2d) M_m^{(n_2)}\left(\frac{ab^2de}{n_1}\right)$$

where $M_m^{(n_2)}(x)$ *is exactly the function given in Definition 4.5.*

The final step is to compute $\langle f_0^\rho | V_C, R_{m,e}|U_p^{2m-1} W_{4C^2p}\rangle_{4C^2p}$ in terms of $\langle f,f\rangle_C$, given q-expansions of f and $R_{m,e}|U_p^{2m-1}$. In order to do this, we recall some useful identities.

If g and h are two cusp forms that both exist at level $N \mid M$ then we may lower the level of the Petersson inner product according to the formula

$$\langle g,h\rangle_M = [\Gamma_0(N):\Gamma_0(M)] \times \langle g,h\rangle_N. \tag{5.9}$$

We will denote by w_f the complex number of modulus 1 such that

$$f|W_C = -w_f \cdot f.$$

We also recall the identity

$$\langle g,h\rangle_N = (-1)^k \langle g\big|W_N, h\big|W_N\rangle_N. \tag{5.10}$$

The operators V_m and W_N satisfy the compatibility relations

$$(g\big|V_m)\big|W_{Nm} = m^{-\frac{k}{2}} g\big|W_N, \tag{5.11}$$

and

$$g\big|W_{Nm} = m^{\frac{k}{2}}(g\big|W_N)\big|V_m. \tag{5.12}$$

With these identities in hand, we turn to the calculation of the inner product in Proposition 5.6.

Proposition 5.8. *If the newform f has rational coefficients and exact level C, then*

$$\langle f_0^\rho|V_C, R_{m,e}|U_p^{2m-1}W_{4C^2p}\rangle_{4C^2p} = \\ w_f C^{1-\frac{1}{2}k}\sum_{d|4p}\delta_d(m,e)\left(4^{\frac{k}{2}}p^{-\frac{k}{2}}\beta_p\langle f|V_4, f|V_d\rangle - 4^{\frac{k}{2}}p^{\frac{k}{2}}\langle f|V_{4p}, f|V_d\rangle\right)$$

where $\delta_d(m,e)$ denotes the projection of $R_{m,e}|U_p^{2m-1}$ onto $f|V_d$.

Proof. By Equations (5.10) and (5.11),

$$\begin{aligned}\langle f_0^\rho|V_C, R_{m,e}|U_p^{2m-1}W_{4C^2p}\rangle_{4C^2p} &= \langle f_0^\rho|V_C W_{4C^2p}, R_{m,e}|U_p^{2m-1}\rangle_{4C^2p} \\ &= C^{-\frac{1}{2}k}\langle f_0^\rho|W_{4Cp}, R_{m,e}|U_p^{2m-1}\rangle_{4C^2p}.\end{aligned}$$

In fact $f_0^\rho|W_{4Cp}$ and $R_{m,e}|U_p^{2m-1}$ both have level $4Cp$ so, by noting that $[\Gamma_0(4Cp):\Gamma_0(4C^2p)] = C$, we can apply Equation (5.9) to obtain

$$\langle f_0^\rho|V_C, R_{m,e}|U_p^{2m-1}W_{4C^2p}\rangle_{4C^2p} = C^{1-\frac{1}{2}k}\langle f_0^\rho|W_{4Cp}, R_{m,e}|U_p^{2m-1}\rangle_{4Cp}.$$

Since f has rational coefficients, $f_0^\rho = f - \bar{\beta}_p f|V_p$ so

$$\begin{aligned}\langle f_0^\rho|W_{4Cp}, R_{m,e}|U_p^{2m-1}\rangle_{4Cp} &= \langle f|W_{4Cp}, R_{m,e}|U_p^{2m-1}\rangle_{4Cp} \\ &\quad - \beta_p\langle f|V_p W_{4Cp}, R_{m,e}|U_p^{2m-1}\rangle_{4Cp}.\end{aligned}$$

Applying the relations in Equations (5.11) and (5.12), we have

$$f|W_{4Cp} = (4p)^{\frac{1}{2}k} f|W_C V_{4p} = -w_f(4p)^{\frac{1}{2}k} f|V_{4p},$$

and also

$$f|V_p W_{4Cp} = p^{-\frac{1}{2}k} f|W_{4C} = 4^{\frac{1}{2}k}p^{-\frac{1}{2}k} f|W_C V_4 = -w_f 4^{\frac{1}{2}k}p^{-\frac{1}{2}k} f|V_4.$$

Therefore we deduce that

$$\begin{aligned}\langle f_0^\rho|W_{4Cp}, R_{m,e}|U_p^{2m-1}\rangle_{4Cp} &= \beta_p 4^{\frac{1}{2}k}p^{-\frac{1}{2}k} w_f\langle f|V_4, R_{m,e}|U_p^{2m-1}\rangle_{4Cp} \\ &\quad - (4p)^{\frac{1}{2}k} w_f\langle f|V_{4p}, R_{m,e}|U_p^{2m-1}\rangle_{4Cp}.\end{aligned}$$

Recall that by Hecke's theory, there is a natural decomposition of cusp forms of level $4Cp$ into

$$\mathcal{S}_k(\Gamma_0(4Cp)) = \left(\bigoplus_{N|4Cp}\ \bigoplus_{d|\frac{4Cp}{N}} \mathcal{S}_k^{\mathrm{new}}(\Gamma_0(N))\Big|V_d\right).$$

If $\mathcal{B}$ is a basis of Hecke eigenforms for the subspace $\mathcal{S}_k^{\mathrm{new}}(\Gamma_0(C))$ then, under the Petersson inner product, the complement of the f-isotypic part is given by

$$\left(\bigoplus_{d|4p} \mathbb{C}\{f\}\Big|V_d\right)^{\perp} = \left(\bigoplus_{\substack{g\in\mathcal{B}\\ g\neq f}} \bigoplus_{d|4p} \mathbb{C}\{g\}\Big|V_d\right) \oplus \left(\bigoplus_{\substack{N|4Cp\\ N\neq C}} \bigoplus_{d|\frac{4Cp}{N}} \mathcal{S}_k^{\mathrm{new}}(\Gamma_0(N))\Big|V_d\right).$$

Therefore, if we write

$$R_{m,e}|U_p^{2m-1} = \sum_{d|4p} \delta_d(m,e) f\big|V_d + \widetilde{R}$$

where $\widetilde{R} \in \left(\bigoplus_{d|4p} \mathbb{C}\{f\}\big|V_d\right)^{\perp}$ is the image of $R_{m,e}|U_p^{2m-1}$ in the complementary subspace, then

$$\langle f|V_w, R_{m,e}|U_p^{2m-1}\rangle_{4Cp} = \sum_{d|4p} \delta_d(m,e)\langle f|V_w, f|V_d\rangle_{4Cp}$$

where $w \in \{4, 4p\}$. The result follows easily from these two identities. $\square$

We remark that bases of the subspaces $\mathcal{S}_k^{\mathrm{new}}(\Gamma_0(N))$ may be computed as sets of q-expansions of Hecke eigenforms in SageMath [65] using the function `Newforms(Gamma0(N), weight=k)`. Therefore the values of $\delta_d(m,e)$ are readily computed given q-expansions of $R_{m,e}|U_p^{2m-1}$ to a sufficient precision. In order to compute $D^{c-}(e+p^m\mathbb{Z}_p)$ it is sufficient to be able to compute the ratio $\frac{\langle f|V_m, f|V_n\rangle_N}{\langle f,f\rangle_C}$ for any N a multiple of C, and both m and n dividing $\frac{N}{C}$; this calculation is performed in Appendix A.

5.2.3 The basic method for computing $\mathcal{L}_p^{\mathrm{an}}(\mathrm{Sym}^2 f)$

We conclude the chapter with a summary of the method we use to compute $\mathcal{L}_p^{\mathrm{an}}(\mathrm{Sym}^2 f)$ when $D(f, k-1) \neq 0$, followed by a worked example. Tables of results of the computations we performed in order to verify Theorem 5.1 can be found in Appendix C.

<u>*Step 1*</u>. Choose a primitive newform $f \in \mathcal{S}_k(C, \mathbf{1})$ that is minimal among its quadratic twists, and has rational coefficients. Fix a prime $p \nmid C$ such that f is ordinary at p. Also fix a positive integer m which will determine the level of precision of our approximations.

<u>*Step 2*</u>. Calculate the complex L-value $\xi_{\mathrm{Sym}^2 f}$ using Proposition 5.4.

<u>*Step 3*</u>. Compute q-expansions for $R_{m,e}|U_p^{2m-1}$ for each $e \in (\mathbb{Z}/p^m\mathbb{Z})^\times$ using the formula given in Corollary 5.7.

<u>*Step 4*</u>. Find a basis of $\mathcal{S}_k(4Cp, \mathbf{1})$ consisting of Hecke eigenforms, and hence calculate $D^-(e + p^m\mathbb{Z}_p)$ for each e by expressing the $R_{m,e}|U_p^{2m-1}$'s in terms of this basis and applying Propositions 5.6 and 5.8.

<u>*Step 5*</u>. Once we have computed the moments $D^-(e + p^m\mathbb{Z}_p)$, we may use Proposition 5.5 and Equation (5.8) to obtain an approximation to the derivative $\mathbf{L}'_p(\mathrm{Sym}^2 f, k-1)$.

<u>*Step 6*</u>. Finally, we may compute an approximation to $\mathcal{L}_p^{\mathrm{an}}(\mathrm{Sym}^2 f)$ by substituting the values we have computed for $\xi_{\mathrm{Sym}^2 f}$ and $\mathbf{L}'_p(\mathrm{Sym}^2 f, k-1)$ into Equation (5.4).

Example 5.9. Consider $f = 12.4.a.a$, the unique newforms of level $C = 12$, weight $k = 4$, and trivial character. The q-expansion for f is

$$f(q) = q + 3q^3 - 18q^5 + 8q^7 + 9q^9 + O(q^{10}).$$

As $a_2(f) = 0$, we have $m(2) = \lfloor \frac{1}{2}\mathrm{ord}_2 C \rfloor = 1$, and $m(3) = 0$ since $a_3(f) \neq 0$. Therefore $\widetilde{C} = 2^{2-1} \times 3 = 6$. Furthermore, it follows from Lemmas 1.5 and 1.6 of [57] that $H_2(X) = 1 + 2^{k-1}X$, and $H_3(X) = 1$. Hence, using Proposition 5.4, we calculate the primitive L-value

$$\frac{L_\infty(\mathrm{Sym}^2 f, k-1)}{\pi^{k-1}\langle f, f\rangle_C} = \frac{2^{2\times 4-2}\cdot 6}{(4-2)!\cdot 12} \times (1 + 2^{-1})^{-1} = \frac{32}{3}.$$

Computing the moments $D^-(e + p^m\mathbb{Z}_p)$ for $p = 5$ with $m = 4$, and $p = 7$ with $m = 3$, and using the approximation in Equation (5.8) along with the result of Proposition 5.5 reveals that

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}s}\mathbf{L}_5(\mathrm{Sym}^2 f, s)\Big|_{s=3} &= 4p + 4p^2 + O(p^4), \text{ and} \\ \frac{\mathrm{d}}{\mathrm{d}s}\mathbf{L}_7(\mathrm{Sym}^2 f, s)\Big|_{s=3} &= p + 5p^2 + O(p^3).\end{aligned}$$

By substituting these values into Equation (5.4) we obtain

$$\mathcal{L}_5^{\mathrm{an}}(\mathrm{Sym}^2 f) = 3p + 2p^2 + 3p^3 + O(p^4), \text{ and}$$

$$\mathcal{L}_7^{\mathrm{an}}(\mathrm{Sym}^2 f) = 5p + 5p^2 + O(p^3).$$

Chapter 6

Congruences between double and triple product L-functions

In the next two chapters we turn our attention to studying the λ-invariants of double and triple product L-functions. Our main results are in Chapter 7, but first we must undertake a series of somewhat tedious calculations involving the Petersson inner product with a view to proving the following.

THE MAIN GOAL. *Let $\big(\mathbf{f}^{(\mathrm{I})}, \mathbf{g}^{(\mathrm{I})}, \mathbf{h}^{(\mathrm{I})}\big)$ and $\big(\mathbf{f}^{(\mathrm{II})}, \mathbf{g}^{(\mathrm{II})}, \mathbf{h}^{(\mathrm{II})}\big)$ denote triples of newforms of suitable weight, character and level. We want to prove an implication*

$$\text{“}T_p\big(M^{(\mathrm{I})}\big) \equiv T_p\big(M^{(\mathrm{II})}\big) \bmod p^{\nu} \implies \mathbf{L}_p\big(M^{(\mathrm{I})}, -, 1\big) \equiv \mathbf{L}_p\big(M^{(\mathrm{II})}, -, 1\big) \bmod p^{\nu}\text{”}$$

for the double product motives $M^{(\star)} = M\big(\mathbf{f}^{(\star)} \otimes \mathbf{g}^{(\star)}\big)$ and for the triple product motives $M^{(\star)} = M\big(\mathbf{f}^{(\star)} \otimes \mathbf{g}^{(\star)} \otimes \mathbf{h}^{(\star)}\big)$, with $T_p(-)$ denoting their p-adic realisations.

Note for $M^{(\star)} = M\big(\mathbf{f}^{(\star)}\big)$ with $\star \in \{\mathrm{I}, \mathrm{II}\}$ the above is a theorem of Vatsal [66], who established the existence of canonical periods $\Omega_{\infty}^{\pm}\big(M^{(\star)}\big) \in \mathbb{C}^{\times}$ such that if one normalises each $\mathbf{L}_p\big(M(\mathbf{f}^{(\star)}), -\big)$ using his periods, the congruences hold modulo p^{ν}. It would therefore be worthwhile to recall Vatsal's congruences in a bit more detail, but we must outline some standard definitions and terminology first.

Let $\mathbb{Q}_{\mathrm{cyc}}$ denote the cyclotomic $\mathbb{Z}_p$-extension of $\mathbb{Q}$. If one writes μ_{p^n} for the group of p^n-th roots of unity, there is a decomposition

$$G_\infty := \mathrm{Gal}\big(\mathbb{Q}(\mu_{p^\infty})/\mathbb{Q}\big) \;\cong\; \mathbb{Z}_p^\times \;\cong\; \mathbb{F}_p^\times \times \big(1+p\mathbb{Z}_p\big) \;\cong\; \Delta \times \Gamma_{\mathrm{cyc}}$$

where $\Delta := \mathrm{Gal}\big(\mathbb{Q}(\mu_p)/\mathbb{Q}\big)$, and the topological group $\Gamma_{\mathrm{cyc}} := \mathrm{Gal}\big(\mathbb{Q}_{\mathrm{cyc}}/\mathbb{Q}\big) \cong \mathbb{Z}_p$.

For a discrete valuation ring R of residue characteristic p, let us define the (cyclotomic) Iwasawa algebras

$$\Lambda_{\mathrm{cyc}} := R[[\Gamma]] = \varprojlim_{n\geq 1} R\big[\Gamma/\Gamma^{p^n}\big] \quad\text{and}\quad R[[G_\infty]] := \Lambda_{\mathrm{cyc}}[\Delta] \cong \bigoplus_{j=0}^{p-2} R[[\Gamma]]_{(\omega^j)}$$

where $\omega : \Delta \xrightarrow{\sim} \mu_{p-1}$ is obtained from the Teichmüller character modulo p via the isomorphism $\Delta \cong \mathbb{F}_p^\times$. Now fix a topological generator γ_0 of Γ. By linearity and continuity, the mapping $\gamma_0 \mapsto X+1$ induces isomorphisms

$$\Lambda_{\mathrm{cyc}} \xrightarrow{\sim} R[\![X]\!] \quad\text{and}\quad R[\![G_\infty]\!] \xrightarrow{\sim} \bigoplus_{j=0}^{p-2} R[\![X]\!]_{(\omega^j)}.$$

Definition 6.1. *Let ϖ be a uniformiser of R, and choose $\beta(X) \in R[\![X]\!][1/\pi]$.*

(i) If the power series $\beta(X) = \sum_{n=0}^{\infty} c_n(\beta)\cdot X^n$, then the integer invariant $\mu(\beta) = \mu_\varpi(\beta)$ is the largest power of ϖ such that $c_n(\beta) \in \varpi^{\mu(\beta)}\cdot R$ for all $n \geq 0$.

(ii) The non-negative integer $\lambda(\beta)$ equals the number of zeroes (counted with multiplicity) of $\beta(X)$, viewed as a function on the open p-adic unit disk inside $\mathbb{C}_p$. One can also take $\lambda(\beta) := \mathrm{rank}_{R/\varpi[\![X]\!]}\left(\frac{R[\![X]\!]}{\langle \varpi, \varpi^{-\mu(\beta)}\cdot\beta(X)\rangle}\right)$, and both are equivalent.

Suppose we are given two newforms $\mathbf{f}^{(\mathrm{I})}$ and $\mathbf{f}^{(\mathrm{II})}$ of weight $k>1$, character ψ, and of levels $N_{\mathbf{f}}^{(\mathrm{I})}$ and $N_{\mathbf{f}}^{(\mathrm{II})}$ respectively, such that their Fourier coefficients satisfy

$$a_n\big(\mathbf{f}^{(\mathrm{I})}\big) \equiv a_n\big(\mathbf{f}^{(\mathrm{II})}\big) \pmod{p^\nu} \;\text{ at each } n \in \mathbb{N} \text{ with } \gcd\big(n, N_{\mathbf{f}}^{(\mathrm{I})}N_{\mathbf{f}}^{(\mathrm{II})}\big) = 1.$$

By enlarging R if necessary, one may assume that R contains $a_n\big(\mathbf{f}^{(\star)}\big)$ for all n. The following result due to Vatsal [66, Prop 1.7] concerns congruences between the Mazur-Tate-Teitelbaum [50] p-adic L-functions $\mathbf{L}_p\big(\mathbf{f}^{(\star)}, \omega^j\big) \in \Lambda_{\mathrm{cyc}}$, and was instrumental in Greenberg and Vatsal's subsequent work on the Iwasawa Main Conjecture for elliptic curves [32].

Theorem 6.2 (Vatsal)**.** *At each ω^j-branch with $j \in \{0, \dots, p-2\}$:*

(i) $\mathbf{L}_{p,S_{\mathbf{f}}}\big(\mathbf{f}^{(\mathrm{I})}, \omega^j\big) \equiv \mathbf{L}_{p,S_{\mathbf{f}}}\big(\mathbf{f}^{(\mathrm{II})}, \omega^j\big) \mod p^{\nu} \cdot \Lambda_{\mathrm{cyc}}$*, and*

(ii) $\lambda\big(\mathbf{L}_p\big(\mathbf{f}^{(\mathrm{I})}, \omega^j\big)\big) = \lambda\big(\mathbf{L}_p\big(\mathbf{f}^{(\mathrm{II})}, \omega^j\big)\big) + \sum_{l \in S_{\mathbf{f}}} \mathbf{v}_l^{(\mathrm{II})}(\omega^j) - \mathbf{v}_l^{(\mathrm{I})}(\omega^j)$

where $S_{\mathbf{f}}$ consists of the primes dividing $N_{\mathbf{f}}^{(\mathrm{I})} \cdot N_{\mathbf{f}}^{(\mathrm{II})}$, and $\mathbf{v}_l^{(\star)}(\omega^j)$ is the λ-invariant of the power series that interpolates the Euler factor $L_l\big(\mathbf{f}^{(\star)} \otimes \omega^j, s\big)$ at a prime l.

Strictly speaking, this is not quite the statement that Vatsal proves in *op. cit.* but it is an easy exercise, involving the $S_{\mathbf{f}}$-depletions of the newforms $\mathbf{f}^{(\mathrm{I})}$ and $\mathbf{f}^{(\mathrm{II})}$, to show that it follows from his congruences (e.g. see [18, §4.1-§4.2] for a discussion). He also assumes irreducibility of the residual Galois representations $\overline{\rho}_{\mathbf{f}^{(\star)}}$ and the torsion-freeness of some H^1-groups, the details of which we ignore for brevity.

Emerton, Pollack and Weston [24] later generalised this construction to allow $\mathbf{f}$ to vary within a Hida family, and showed that the λ-invariant was stable along the branches of a certain Hecke algebra, $\mathbb{T}_{\Sigma}(\overline{\rho})$, parametrising the deformation. Recently the theory has been extended to cover anticyclotomic λ-invariants in the work of Castella, Kim and Longo [7], and also to treat non-commutative p-adic Lie extensions (with a meta-abelian structure) by Delbourgo in [16, 17]. Further generalisations of Vatsal's original ideas can be found in [8, 18, 19, 48, 59].

6.1 Statement of the main results

There are three basic approaches one can take in constructing p-adic L-functions for tensor products of modular forms:

- the *Betti realisation* approach adopted by Mazur-Tate-Teitelbaum, Vatsal, and others [50, 66, 67], which utilises modular symbols;
- the *étale realisation* approach of Perrin-Riou [14, 46], which converts Euler systems directly into p-adic L-functions; or
- the *de Rham realisation* approach of Hida and Panchishkin [37, 53], which involves both the Rankin convolution and Petersson inner product.

In the Betti approach, the two main ingredients are a 'mod p multiplicity-one' theorem and Ihara's Lemma. The multiplicity-one result is used to show that the μ-invariant is stable amongst families of p-congruent modular symbols, whilst Ihara's Lemma allows one to change between different level structures.

We follow the de Rham approach, which has the advantage of being completely explicit in nature. It also carries the disadvantage that the associated periods may not be canonical with respect to the Iwasawa Main Conjecture, hence the μ-invariants of our automorphic p-adic L-functions can sometimes be negative. Here the role of mod p multiplicity-one is played by holomorphic projection [33], while Ihara's Lemma is replaced with an explicit calculation involving depletions of χ-twisted modular forms (see Theorem 2.19 and Proposition 6.15, respectively).

6.1.1 The double product

Let $\big(\mathbf{f},\mathbf{g}^{(\mathrm{I})}\big)$ and $\big(\mathbf{f},\mathbf{g}^{(\mathrm{II})}\big)$ denote pairs of newforms of weight $(k_1,k_2)\geq\underline{1}$ with $k_1>k_2$, levels $\big(N_{\mathbf{f}},N_{\mathbf{g}}^{(\mathrm{I})}\big)$, $\big(N_{\mathbf{f}},N_{\mathbf{g}}^{(\mathrm{II})}\big)$ respectively, and nebentypes (ψ_1,ψ_2). We also assume they are p-ordinary, i.e. $a_p(\mathbf{f}),a_p\big(\mathbf{g}^{(\star)}\big)\in\mathcal{O}_{\mathbb{C}_p}^{\times}$. Using the results of Hida and Panchishkin [37, 53], for each choice of $\star\in\{\mathrm{I},\mathrm{II}\}$ there exists a p-adic L-function $\mathbf{L}_p\big(\mathbf{f}\otimes\mathbf{g}^{(\star)}\big)\in\Lambda_{\mathrm{cyc}}[\Delta][1/p]$ interpolating

$$\iota_p\circ\iota_\infty^{-1}\left(\mathcal{E}_p\big(\mathbf{f}\otimes\mathbf{g}^{(\star)},\chi^{-1},n+k_2\big)\cdot\frac{L\big(\mathbf{f}\otimes\mathbf{g}^{(\star)},n+k_2\big)}{(2\pi i)^{1-k_2}\cdot\Omega_\infty(\mathbf{f})}\right)\text{ with }\Omega_\infty(\mathbf{f})=\big\langle\mathbf{f},\mathbf{f}\big\rangle_{\mathrm{Pet}},$$

at all integers $n\in\{0,\ldots,k_1-k_2-1\}$ and special characters of the form $\chi\kappa_{\mathrm{cyc}}^n$ where χ is of finite order, and $\kappa_{\mathrm{cyc}}:G_\infty\xrightarrow{\sim}\mathbb{Z}_p^{\times}$ is the p-th cyclotomic character.

Remark. If $\mathbf{f}_E$ is the weight two newform arising from an elliptic curve $E_{/\mathbb{Q}}$, then it is an easy exercise to show that

$$\Omega_\infty(\mathbf{f}_E) \;=\; \frac{\deg\big(X_0(N_{\mathbf{f}_E}) \twoheadrightarrow E\big)}{4\pi^2\sqrt{-1}\cdot r_E^2} \times \int_{E(\mathbb{C})^+} \omega_E \cdot \int_{E(\mathbb{C})^-} \omega_E$$

where ω_E is the differential associated to a minimal Weierstrass equation for $E_{/\mathbb{Z}}$, and $r_E \in \mathbb{Q}^\times$ denotes the Manin constant for the modular parametrisation.

Let $\rho_{\mathbf{g}^{(\star)}} : G_\mathbb{Q} \to \mathrm{GL}_2\big(\overline{\mathbb{Q}}_p\big)$ be the p-adic Galois representation attached to $\mathbf{g}^{(\star)}$ by the work of Deligne if $k_2 \geq 2$, and by Deligne-Serre if $k_2 = 1$. We assume that

$$\rho_{\mathbf{g}^{(\mathrm{I})}}\Big|_{G_{\mathbb{Q}_l}} \cong \rho_{\mathbf{g}^{(\mathrm{II})}}\Big|_{G_{\mathbb{Q}_l}} \bmod p^{\nu_2} \;\text{ at all primes } l \nmid N_{\mathbf{g}}^{(\mathrm{I})} \cdot N_{\mathbf{g}}^{(\mathrm{II})},$$

which is equivalent to saying

$$a_n\big(\mathbf{g}^{(\mathrm{I})}\big) \equiv a_n\big(\mathbf{g}^{(\mathrm{II})}\big) \;\bmod\; p^{\nu_2} \;\text{ if }\; gcd\big(n, N_{\mathbf{g}}^{(\mathrm{I})} N_{\mathbf{g}}^{(\mathrm{II})}\big) = 1.$$

For technical reasons, we must also suppose that ψ_1 is trivial or a quadratic character.

Theorem 6.3. *At each branch $j \in \{0, \ldots, p-2\}$, let $\mu_{\mathrm{cyc}}^{(j)}$ denote the minimum of the μ-invariants for $\mathbf{L}_p\big(\mathbf{f}\otimes\mathbf{g}^{(\mathrm{I})}, \omega^j\big)$ and $\mathbf{L}_p\big(\mathbf{f}\otimes\mathbf{g}^{(\mathrm{II})}, \omega^j\big)$. If $p > k_1 - 2$, then*

(i) $\mathbf{L}_{p,S_\mathbf{g}}\big(\mathbf{f}\otimes\mathbf{g}^{(\mathrm{I})}, \omega^j\big) \equiv \mathbf{L}_{p,S_\mathbf{g}}\big(\mathbf{f}\otimes\mathbf{g}^{(\mathrm{II})}, \omega^j\big) \;\bmod\; p^{\mu_{\mathrm{cyc}}^{(j)}+\nu_2}\cdot\Lambda_{\mathrm{cyc}}$, *and*

(ii) $\lambda\Big(\mathbf{L}_p\big(\mathbf{f}\otimes\mathbf{g}^{(\mathrm{I})}, \omega^j\big)\Big) = \lambda\Big(\mathbf{L}_p\big(\mathbf{f}\otimes\mathbf{g}^{(\mathrm{II})}, \omega^j\big)\Big) + \sum_{l\in S_\mathbf{g}} \mathbf{e}_l^{(\mathrm{II})}(\omega^j) - \mathbf{e}_l^{(\mathrm{I})}(\omega^j)$.

where $S_\mathbf{g}$ consists of the primes dividing $N_{\mathbf{g}}^{(\mathrm{I})}\cdot N_{\mathbf{g}}^{(\mathrm{II})}$, and $\mathbf{e}_l^{(\star)}(\omega^j)$ is the λ-invariant of the power series interpolating the Euler factor $L_l\big(\mathbf{f}\otimes\mathbf{g}^{(\star)}\otimes\omega^j, s\big)$ at a prime l.

There is a nice application of this result towards the Iwasawa Main Conjecture. By the work of Kings, Loeffler and Zerbes [46, Def 3.3.2], there exist one-cocycles

$$\mathrm{Eis}_{\text{ét},b,N}^{[\mathbf{f},\mathbf{g}^{(\star)},r]} \in H^1_{\text{ét}}\Big(\mathbb{Z}[1/Np], T_p\big(\mathbf{f}\otimes\mathbf{g}^{(\star)}\big)^*\otimes\kappa_{\mathrm{cyc}}^{-r}\Big)$$

for $0 \leq r \leq k_2 - 2$, $b \in \mathbb{Z}/N\mathbb{Z}$ called Rankin-Eisenstein classes, that map to each component $\mathbf{L}_p\big(\mathbf{f} \otimes \mathbf{g}^{(\star)}, \omega^j\big)$. Applying Theorem 11.6.4 of *op. cit.* which relies on the existence of these classes outside of the critical range, one obtains a divisibility of power series

$$\mathrm{char}_{\Lambda_{\mathrm{cyc}}}\Big(\widetilde{H}^2\Big(\mathbb{Z}[1/S], T_p\big(\mathbf{f} \otimes \mathbf{g}^{(\star)}\big)^* \otimes \Lambda_\Gamma(-j); \Delta^{(\mathbf{f})}\Big)_{(\omega^j)}\Big) \;\Big|\; \mathrm{Tw}_{1+j}\Big(\mathbf{L}_p\big(\mathbf{f} \otimes \mathbf{g}^{(\star)}, \omega^j\big)\Big)$$

where the left-hand side is described fully in Proposition 11.2.9 of *op. cit.* and arises naturally from Nekovář's theory of Selmer complexes (in fact, it is helpful to think of the $\widetilde{H}^2(\cdots)$-cohomology intuitively as being a cyclotomic Selmer group).

If we now write $\lambda^{\mathrm{alg}}(\mathbf{f} \otimes \mathbf{g}^{(\star)}, \omega^j)$ for the λ-invariant of $\mathrm{char}_{\Lambda_{\mathrm{cyc}}}\big(\widetilde{H}^2(\cdots)\big)_{(\omega^j)}$ and likewise $\lambda^{\mathrm{an}}(\mathbf{f} \otimes \mathbf{g}^{(\star)}, \omega^j)$ for the λ-invariant of $\mathbf{L}_p\big(\mathbf{f} \otimes \mathbf{g}^{(\star)}, \omega^j\big)$, then their divisibility theorem implies that $\lambda^{\mathrm{alg}}(\mathbf{f} \otimes \mathbf{g}^{(\star)}, \omega^j) \leq \lambda^{\mathrm{an}}(\mathbf{f} \otimes \mathbf{g}^{(\star)}, \omega^j)$; moreover

$$\Big\{\text{zeroes of } \mathrm{char}_{\Lambda_{\mathrm{cyc}}}\big(\widetilde{H}^2(\cdots; \Delta^{(\mathbf{f})})_{(\omega^j)}\big)\Big\} \subset \Big\{\text{zeroes of } \mathrm{Tw}_{1+j}\big(\mathbf{L}_p\big(\mathbf{f} \otimes \mathbf{g}^{(\star)}, \omega^j\big)\big)\Big\}$$

for all $j \in \{0, \ldots, p-2\}$, and at either choice of $\star \in \{\mathrm{I}, \mathrm{II}\}$.

Conjecture 6.4. *At branches $j \in \{0, \ldots, p-2\}$, there is a transition formula*

$$\lambda^{\mathrm{alg}}(\mathbf{f} \otimes \mathbf{g}^{(\mathrm{I})}, \omega^j) \;=\; \lambda^{\mathrm{alg}}(\mathbf{f} \otimes \mathbf{g}^{(\mathrm{II})}, \omega^j) \;+\; \sum_{l \in S_{\mathbf{g}}} \mathbf{e}_l^{(\mathrm{II})}(\omega^j) - \mathbf{e}_l^{(\mathrm{I})}(\omega^j).$$

Assuming its validity, one can show if the Iwasawa Main Conjecture is true for one motive, $M\big(\mathbf{f} \otimes \mathbf{g}^{(\mathrm{I})}\big)$ say, it must be true for the p^{ν_2}-congruent motive $M\big(\mathbf{f} \otimes \mathbf{g}^{(\mathrm{II})}\big)$. Unfortunately we have not yet found a method to switch between two dominant weight newforms $\mathbf{f}^{(\mathrm{I})}$ and $\mathbf{f}^{(\mathrm{II})}$, if they are congruent to each other modulo p^{ν_1}.

6.1.2 The triple product

We shall now add an extra pair of forms into the discussion: let $\big(\mathbf{f}, \mathbf{g}^{(\mathrm{I})}, \mathbf{h}^{(\mathrm{I})}\big)$ and $\big(\mathbf{f}, \mathbf{g}^{(\mathrm{II})}, \mathbf{h}^{(\mathrm{II})}\big)$ denote triples of newforms of weight $\underline{k} = (k_1, k_2, k_3)$, levels

$\left(N_{\mathbf{f}}, N_{\mathbf{g}}^{(\star)}, N_{\mathbf{h}}^{(\star)}\right)$ and nebentypes (ψ_1, ψ_2, ψ_3). We further suppose that these triples are p-ordinary, so that $a_p(\mathbf{f}), a_p\big(\mathbf{g}^{(\star)}\big), a_p\big(\mathbf{h}^{(\star)}\big) \in \mathcal{O}_{\mathbb{C}_p}^{\times}$. There exist primitive Λ-adic families $\big(\mathbf{F}, \mathbf{G}^{(\star)}, \mathbf{H}^{(\star)}\big)$ passing through $\big(\mathbf{f}, \mathbf{g}^{(\star)}, \mathbf{h}^{(\star)}\big)$ at each choice of $\star \in \{\mathrm{I}, \mathrm{II}\}$. For technical reasons only, we impose the conditions:

(T1) the primitive characters satisfy $\psi_1\psi_2\psi_3 = \mathbf{1}$;

(T2) $\overline{\rho}_{\mathbf{F}_1} : G_{\mathbb{Q}} \to \mathrm{GL}_2(\overline{\mathbb{F}}_p)$ is absolutely irreducible and p-distinguished;

(T3) $\gcd\big(N_{\mathbf{f}}, N_{\mathbf{g}}^{(\star)}, N_{\mathbf{h}}^{(\star)}\big)$ is a square-free integer for both choices $\star \in \{\mathrm{I}, \mathrm{II}\}$;

(T4) $\epsilon\big(1/2, \Pi_{\underline{k},l}^{(\star)}\big) = 1$ at primes $l \big| N_{\mathbf{f}} N_{\mathbf{g}}^{(\star)} N_{\mathbf{h}}^{(\star)}$ and unbalanced $\underline{k} = (k_1, k_2, k_3)$,

where $\Pi_{\underline{k}}^{(\star)}$ is the representation attached to $\mathbf{F} \otimes \mathbf{G}^{(\star)} \otimes \mathbf{H}^{(\star)}$ at each $\underline{k}$.

Remark. To consider congruences here we will treat the following situation. Assume there exists a p-adic line $\mathcal{V}$ in the ambient weight-space for the triple $\big(\mathbf{F}, \mathbf{G}^{(\star)}, \mathbf{H}^{(\star)}\big)$, such that for all primes $l \nmid N_{\mathbf{g}}^{(\mathrm{I})} \cdot N_{\mathbf{g}}^{(\mathrm{II})} \cdot N_{\mathbf{h}}^{(\mathrm{I})} \cdot N_{\mathbf{h}}^{(\mathrm{II})}$ and unbalanced $\underline{k} = (k_1, k_2, k_3) \in \mathcal{V}$:

(i) $\rho_{\mathbf{G}_{k_2}^{(\mathrm{I})}}\Big|_{G_{\mathbb{Q}_l}} \cong \rho_{\mathbf{G}_{k_2}^{(\mathrm{II})}}\Big|_{G_{\mathbb{Q}_l}} \bmod p^{\nu_2}$, and

(ii) $\rho_{\mathbf{H}_{k_3}^{(\mathrm{I})}}\Big|_{G_{\mathbb{Q}_l}} \cong \rho_{\mathbf{H}_{k_3}^{(\mathrm{II})}}\Big|_{G_{\mathbb{Q}_l}} \bmod p^{\nu_3}$.

Whenever this line is parametrised by a finite flat extension $\mathbb{I}^{\mathcal{V}}$ of $\mathcal{O}_{K,p}[\![1+p\mathbb{Z}_p]\!]$, then we call $\mathcal{V}$ a *congruence line of type* (p^{ν_2}, p^{ν_3}) for the triples $\big(\mathbf{F}, \mathbf{G}^{(\star)}, \mathbf{H}^{(\star)}\big)$. Let $\mathbf{L}_p^{\mathcal{V}}\big(\mathbf{F} \otimes \mathbf{G}^{(\star)} \otimes \mathbf{H}^{(\star)}\big) \in \mathbb{I}^{\mathcal{V}}$ denote the restriction of the p-adic L-function to $\mathcal{V}$.

Example 6.5. Consider two modular elliptic curves $E^{(\mathrm{I})}$ and $E^{(\mathrm{II})}$ over $\mathbb{Q}$, whose p-adic Galois representations $\rho_{E^{(\star)},p} : G_{\mathbb{Q}} \to \mathrm{GL}_2(\mathbb{Z}_p)$ satisfy the congruences $\rho_{E^{(\mathrm{I})},p}\big|_{G_{\mathbb{Q}_l}} \cong \rho_{E^{(\mathrm{II})},p}\big|_{G_{\mathbb{Q}_l}} \pmod{p^{\nu_2}}$ at all prime numbers $l \nmid \mathrm{cond}_{\mathbb{Q}}(E^{(\mathrm{I})}) \cdot \mathrm{cond}_{\mathbb{Q}}(E^{(\mathrm{II})})$. Let $\mathbf{G}^{(\mathrm{I})} \in \mathbb{I}_2[\![q]\!]$ and $\mathbf{G}^{(\mathrm{II})} \in \mathbb{I}_2[\![q]\!]$ be Hida families passing through $E^{(\mathrm{I})}$ and $E^{(\mathrm{II})}$ respectively, and assume that $\mathbf{F} \in \mathbb{I}_1[\![q]\!]$ and $\mathbf{H}^{(\mathrm{I})} = \mathbf{H}^{(\mathrm{II})} \in \mathbb{I}_3[\![q]\!]$ denote arbitrary primitive $\mathbb{I}_i$-adic forms. Then we can choose our p-adic line in weight-space to be the set

$$\mathcal{V} = \Big\{ (k, 2, k-2) \;\Big|\; k \in \mathbb{D}_{\mathbf{F}} \cap \mathbb{D}_{\mathbf{H}^{(\star)}} \Big\}$$

where $\mathbb{D}_{\mathbf{F}} \subset \mathbb{Z}_p$ (resp. $\mathbb{D}_{\mathbf{H}^{(\star)}}$) is the disk of convergence for $\mathbf{F}$ (resp. $\mathbf{H}^{(\mathrm{I})} = \mathbf{H}^{(\mathrm{II})}$), and the specialisation map

$$\phi_{\mathcal{V}} : \mathbb{I}_1 \hat{\otimes}_{\mathcal{O}_{K,p}} \mathbb{I}_2 \hat{\otimes}_{\mathcal{O}_{K,p}} \mathbb{I}_3 \twoheadrightarrow \mathbb{I}^{\mathcal{V}}$$

is induced by sending $(X_1, X_2, X_3) \mapsto \left(X_{\mathcal{V}}, 0, \frac{X_{\mathcal{V}}+1}{(1+p)^2} - 1\right)$.

As it is non-standard, we should define the (weight) λ-invariant in this context. Since $\mathbb{I}^{\mathcal{V}}$ is a finite extension of $\Lambda_{\mathrm{wt}} := \mathcal{O}_{K,p}[\![1+p\mathbb{Z}_p]\!] \cong \mathcal{O}_{K,p}[\![X]\!]$, one can consider its normal closure $\mathbb{I}^{\mathcal{V},\mathrm{cl}}$ and the field of fractions $\mathcal{K}^{\mathcal{V}} = \mathrm{Frac}\left(\mathbb{I}^{\mathcal{V},\mathrm{cl}}\right)$. We then define

$$\lambda^{\mathrm{wt}}(\beta) \;:=\; \left[\mathcal{K}^{\mathcal{V}} : \mathcal{F}_{\mathrm{wt}}\right]^{-1} \times \left(\text{the number of zeroes of } \prod_{\sigma \in \mathrm{Gal}(\mathcal{K}^{\mathcal{V}}/\mathcal{F}_{\mathrm{wt}})} \beta^{\sigma}\right)$$

for each $\beta \in \mathbb{I}^{\mathcal{V}}$, where $\mathcal{F}_{\mathrm{wt}}$ is the field of fractions of Λ_{wt} (note that we have $\prod_{\sigma} \beta^{\sigma} \in \mathcal{O}_{K,p}[\![X]\!]$). Let us denote by $\mu_{\mathrm{wt}}^{(\mathcal{V})}$ the minimum value of the weight μ-invariant amongst the two p-adic L-functions, namely $\mathbf{L}_p^{\mathcal{V}}\left(\mathbf{F} \otimes \mathbf{G}^{(\mathrm{I})} \otimes \mathbf{H}^{(\mathrm{I})}\right)$ and $\mathbf{L}_p^{\mathcal{V}}\left(\mathbf{F} \otimes \mathbf{G}^{(\mathrm{II})} \otimes \mathbf{H}^{(\mathrm{II})}\right)$.

Theorem 6.6. *If the weights* $\underline{k} = (k_1, k_2, k_3)$ *satisfying* $k_1 > k_2 + k_3 - 1$ *and* $p \nmid \frac{(k_1-2)!}{\left(\frac{k_1+k_2+k_3}{2}-2\right)!}$ *are dense in* $\mathrm{Spec}(\mathbb{I}^{\mathcal{V}})$, *and if* ψ_1 *is trivial or quadratic, then*

(i)
$$\mathbf{L}_{p,S_{\mathbf{g},\mathbf{h}}}^{\mathcal{V}}\left(\mathbf{F} \otimes \mathbf{G}^{(\mathrm{I})} \otimes \mathbf{H}^{(\mathrm{I})}\right) \equiv \mathbf{L}_{p,S_{\mathbf{g},\mathbf{h}}}^{\mathcal{V}}\left(\mathbf{F} \otimes \mathbf{G}^{(\mathrm{II})} \otimes \mathbf{H}^{(\mathrm{II})}\right) \bmod p^{\mu_{\mathrm{wt}}^{(\mathcal{V})} + \min\{\nu_2, \nu_3\}},$$
and

(ii)
$$\lambda^{\mathrm{wt}}\left(\mathbf{L}_p^{\mathcal{V}}\left(\mathbf{F} \otimes \mathbf{G}^{(\mathrm{I})} \otimes \mathbf{H}^{(\mathrm{I})}\right)\right) = \lambda^{\mathrm{wt}}\left(\mathbf{L}_p^{\mathcal{V}}\left(\mathbf{F} \otimes \mathbf{G}^{(\mathrm{II})} \otimes \mathbf{H}^{(\mathrm{II})}\right)\right) + \sum_{l \in S_{\mathbf{g},\mathbf{h}}} \mathbf{w}_{l,\mathcal{V}}^{(\mathrm{II})} - \mathbf{w}_{l,\mathcal{V}}^{(\mathrm{I})}$$

where $S_{\mathbf{g},\mathbf{h}}$ *consists of those primes dividing* $N_{\mathbf{g}}^{(\mathrm{I})} \cdot N_{\mathbf{g}}^{(\mathrm{II})} \cdot N_{\mathbf{h}}^{(\mathrm{I})} \cdot N_{\mathbf{h}}^{(\mathrm{II})}$, *and* $\mathbf{w}_{l,\mathcal{V}}^{(\star)}$ *is the* λ^{wt}*-invariant for the* $\mathbb{I}^{\mathcal{V}}$*-adic factor* $L_l\left(\mathbf{F}_{k_1} \otimes \mathbf{G}_{k_2}^{(\star)} \otimes \mathbf{H}_{k_3}^{(\star)} \otimes \chi_{\underline{k}}^{-1}, \frac{k_1+k_2+k_3-2}{2}\right)\Big|_{\underline{k} \in \mathcal{V}}$.

An example of such a congruence line $\mathcal{V}$ is given by specialising $\mathbf{G}^{(\star)}$ at a fixed weight k_2 at which there exists a mod p^{ν_2} congruence between $\mathbf{G}_{k_2}^{(\mathrm{I})}$ and $\mathbf{G}_{k_2}^{(\mathrm{II})}$, and taking the weights $(k_1, k_2, k_1 - k_2)$ with k_1 the free variable: one thus obtains congruences between $\mathbf{L}_{p,S_{\mathbf{g},\mathbf{h}}}\left(\mathbf{F}_{k_1} \otimes \mathbf{G}_{k_2}^{(\mathrm{I})} \otimes \mathbf{H}_{k_1-k_2}\right)$ and $\mathbf{L}_{p,S_{\mathbf{g},\mathbf{h}}}\big(\mathbf{F}_{k_1} \otimes$

$\mathbf{G}_{k_2}^{(\mathrm{II})} \otimes \mathbf{H}_{k_1-k_2}\big)$. By symmetry, the same thing works when the roles of $\mathbf{G}^{(\star)}$ and $\mathbf{H}^{(\star)}$ are reversed.

The reader will notice that there is no cyclotomic variable appearing here, although by recent work of Hsieh and Yamana on exceptional p-adic zeroes [43], this extra variable can certainly be introduced. The techniques presented here should carry over to the four-variable (quaternionic) setting, thereby enabling us to prove transition formulae for the cyclotomic λ-invariant at balanced $(k_1, k_2, k_3) \in \mathcal{V}$.

We should also mention the results of Darmon, Rotger and others, which relate specialisations of $\mathbf{L}_p\big(\mathbf{F} \otimes \mathbf{G}^{(\star)} \otimes \mathbf{H}^{(\star)}\big)$ to generalised Kato classes [14] in global Galois cohomology with coefficients in $T_p\big(\mathbf{F}_{k_1} \otimes \mathbf{G}_{k_2}^{(\star)} \otimes \mathbf{H}_{k_3}^{(\star)}\big)$. In particular, at weight $(k_1, k_2, k_3) = (2, 1, 1)$ they obtain key information on the Birch and Swinnerton-Dyer Conjecture for elliptic curves E. Therefore given the existence of a congruence line $\mathcal{V}$ of type (p^{ν_2}, p^{ν_3}) containing $(2, 1, 1)$ as a point, one could use a balanced version of Theorem 6.6 to produce non-trivial congruences between the values of $L\big(E, \rho_2^{(\mathrm{I})} \otimes \rho_3^{(\mathrm{I})}, s\big)$ and $L\big(E, \rho_2^{(\mathrm{II})} \otimes \rho_3^{(\mathrm{II})}, s\big)$ at $s = 1$, for twists by degree four Artin representations $\rho_2^{(\star)} \otimes \rho_3^{(\star)}$ which are self-dual and congruent.

6.1.3 A brief plan

In Section 6.2 we study projections of $\mathcal{C}^\infty$-modular forms of the type $\mathbf{g} \cdot \delta_w^{(r)}(\mathbf{h})$, where the differential operator $\delta_w = \frac{1}{2\pi i}\left(\frac{w}{2iy} + \frac{\partial}{\partial z}\right)$. If $\mathbf{h}$ is an Eisenstein series, then these projections are related to double products, while if $\mathbf{h}$ is a cuspidal eigenform then they are essentially triple product L-values. In Chapter 7, by writing these critical values in terms of a linear functional $\mathcal{L}_{\mathbf{g}}^{(r,\varepsilon)}(-)$ acting on the space of nearly holomorphic forms, one can then read off congruences amongst the L-values in terms of congruences between the original modular forms. This is an ad hoc approach and we apologise in advance for the very ugly formulae!

Conventions. We employ the following terminology throughout the following two chapters.

- If $\chi : \mathbb{Z} \to \mathbb{C}$ is any Dirichlet character, then we write $\chi_{(p)}$ for its p-part and similarly we use $\chi^{(p)}$ to denote its non-p-part, so that $\chi = \chi_{(p)} \cdot \chi^{(p)}$.
- If F is a number field or local field then $\mathcal{O}_F$ will be its ring of integers, and we say that two expansions $H, H^\dagger \in \mathcal{O}_F[\![q]\!]$ are congruent modulo p^ν if their q^n-coefficients satisfy $a_n(H) \equiv a_n\big(H^\dagger\big) \bmod p^\nu$ for every $n \geq 0$.
- If $\mathbb{I}$ denotes the normal closure of $\Lambda_{\rm wt} := \mathcal{O}_K[\![1+p\mathbb{Z}_p]\!]$ inside of $\mathrm{Frac}\big(\Lambda_{\rm wt}\big)$, then we assume $K/\mathbb{Q}_p$ is chosen large enough to ensure $\mathbb{I}\cap\overline{\mathbb{Q}}_p = \mathcal{O}_K$, and that the algebraic points $\mathrm{Spec}\ \mathbb{I}\big(\mathcal{O}_K\big)^{\rm alg}$ are Zariski dense in $\mathrm{Spec}\ \mathbb{I}\big(\overline{\mathbb{Q}}_p\big)$.
- For an integer $N \geq 1$ coprime to p and a Dirichlet character χ modulo N, we use $\mathbb{T}^{\rm ord}\big(N, \chi; \mathbb{I}\big)$ to indicate the Hecke algebra acting on $\mathcal{S}^{\rm ord}\big(N, \chi; \mathbb{I}\big)$, the space of ordinary $\mathbb{I}$-adic cusp forms of tame level N and character χ.

6.2 A lowbrow study of Petersson inner products

Let F_1, G_2, G_3 be modular forms of levels N_1, N_2, N_3, weights $k_1, k_2, k_3 > 0$ and nebentypes ψ_1, ψ_2, ψ_3 respectively. We shall assume that F_1 and G_2 are cusp forms, that the primitive characters satisfy $\psi_2 \cdot \psi_3 = \psi_1^{-1}$, and that $k_1 > k_2 + k_3 - 1$. Our main goal here is to derive an explicit expression for quotients of the type

$$\frac{\big\langle F_1^\sharp, \mathrm{Tr}_{\widetilde{N}/N_0}\big(\mathrm{Hol}_\infty\big(G_2 \cdot \delta_w^{(r)}(G_3)\big)\big|_{k_1} W_{\widetilde{N}}^{\varepsilon}\big)\big\rangle_{N_0}}{\big\langle F_1, F_1\big\rangle_{N_1}}\ , \quad \varepsilon \in \{0,1\} \tag{6.1}$$

where the various operators, levels and inner products above will be defined shortly (the precise formulae for these ratios will be given in Propositions 6.16 and 6.17). We need to study these projections in some detail, as the critical values of both the double and triple product L-functions can be represented via integrals of this type.

6.2.1 Preliminaries on modular forms

For the three modular forms F_1, G_2, G_3 of levels N_1, N_2, N_3 mentioned above we will use the following notation.

Notes. (a) For each $i \in \{1,2,3\}$, we factorise the level into $N_i = p^{e_i} \cdot N_i^{(p)}$ where $e_i = \mathrm{ord}_p(N_i)$ and $N_i^{(p)}$ is the corresponding tame level.

(b) We set $\widetilde{N} := \mathrm{lcm}(N_1, N_2, N_3)$, which one decomposes into $\widetilde{N} = p^{\tilde{e}} \cdot \widetilde{N}^{(p)}$.

(c) Lastly let us choose $N_0 := p \cdot \mathrm{lcm}\big(N_1^{(p)}, N_2^{(p)}, N_3^{(p)}\big) = p^{1-\tilde{e}} \cdot \widetilde{N} \in p \cdot \mathbb{Z}_p^{\times}$.

Now F_1 belongs to $\mathcal{S}_{k_1}(N_1, \psi_1)$ with q-expansion $F_1(q) = \sum_{n=1}^{\infty} a_n(F_1) q^n$, so there exists a conjugate form $F_1^{\sharp} \in \mathcal{S}_{k_1}(N_1, \psi_1^{-1})$ with $F_1^{\sharp}(q) = \sum_{n=1}^{\infty} \overline{a_n(F_1)} q^n$. We shall further suppose that F_1 is a newform of conductor N_1, so that $F_1\Big|_{k_1} W_{N_1} = \epsilon_1 \cdot F_1^{\sharp}$ with $\epsilon_1 \in \mathbb{C}$ and $\big|\epsilon_1\big|_{\infty} = 1$. For simplicity, we will assume that $F_1^{\sharp} = F_1$ and $\psi_1^2 = \mathbf{1}$.

Lemma 6.7. *If $p \nmid N_1$ so that $e_1 = 0$, then for an arbitrary $G \in \mathcal{M}_{k_1}(\widetilde{N}, \psi_1)$,*

$$\Big\langle F_1^{\sharp}, \mathrm{Tr}_{N_0}^{\widetilde{N}}(G) \Big\rangle_{N_0} = \epsilon_1 p^{1-\frac{(k_1-2)(\tilde{e}-2)}{2}} \left(\frac{\widetilde{N}^{(p)}}{N_1}\right)^{\frac{k_1}{2}} \cdot \sum_{d \big| \frac{N_0}{N_1}} \mathfrak{c}_{d,\widetilde{N},\tilde{e}}(G) \cdot \Big\langle F_1\big|_{k_1} V_{\frac{N_0}{N_1}}, F_1\big|_{k_1} V_d \Big\rangle_{N_0}$$

where each form $G\big|_{k_1} W_{\widetilde{N}} \circ U_p^{\tilde{e}-1} \in \mathcal{M}_{k_1}(N_0, \psi_1)$ has been decomposed into a sum

$$G\big|_{k_1} W_{\widetilde{N}} \circ U_p^{\tilde{e}-1} = \sum_{d \big| \frac{N_0}{N_1}} \mathfrak{c}_{d,\widetilde{N},\tilde{e}}(G) \cdot F_1\big|_{k_1} V_d + G_{\widetilde{N},\tilde{e}}^{(\perp)} \quad \textit{for scalars } \mathfrak{c}_{d,\widetilde{N},\tilde{e}}(G) \in \mathbb{C},$$

and here the modular form $G_{\widetilde{N},\tilde{e}}^{(\perp)}$ above is obtained by projecting $G\big|_{k_1} W_{\widetilde{N}} \circ U_p^{\tilde{e}-1}$ onto the orthogonal complement of the F_1-isotypic subspace inside $\mathcal{M}_{k_1}(N_0, \psi_1)$.

Proof. As the ratio $\widetilde{N}/N_0 = p^{\tilde{e}-1}$ is a power of p and $p|N_0$, one deduces that

$$\mathrm{Tr}_{N_0}^{\widetilde{N}}(G) = p^{(1-k_1/2)(\tilde{e}-1)} \times G\big|_{k_1} W_{\widetilde{N}} \circ U_p^{\tilde{e}-1} \circ W_{N_0}.$$

Applying this standard identity to our inner product:

$$\begin{aligned}
&\left\langle F_1^\sharp, \mathrm{Tr}^{\widetilde{N}}_{N_0}(G)\right\rangle_{N_0} \\
&\quad = p^{(1-k_1/2)(\tilde{e}-1)} \times \left\langle F_1^\sharp\ ,\ G\big|_{k_1} W_{\widetilde{N}} \circ U_p^{\tilde{e}-1} \circ W_{N_0}\right\rangle_{N_0} \\
&\quad = (-1)^{k_1} p^{(1-k_1/2)(\tilde{e}-1)} \times \left\langle F_1^\sharp\big|_{k_1} W_{N_0}\ ,\ G\big|_{k_1} W_{\widetilde{N}} \circ U_p^{\tilde{e}-1}\right\rangle_{N_0} \\
&\quad = (-1)^{k_1} p^{1-\frac{(k_1-2)(\tilde{e}-2)}{2}} \left(\frac{\widetilde{N}^{(p)}}{N_1}\right)^{\frac{k_1}{2}} \times \left\langle F_1^\sharp\big|_{k_1} W_{N_1} \circ V_{p\cdot\frac{\widetilde{N}^{(p)}}{N_1}}\ ,\ G\big|_{k_1} W_{\widetilde{N}} \circ U_p^{\tilde{e}-1}\right\rangle_{N_0}
\end{aligned}$$

and the last line follows because $(-)\Big|_{k_1} W_{N_0} = \big(p\cdot\frac{\widetilde{N}^{(p)}}{N_1}\big)^{k_1/2} \cdot (-)\Big|_{k_1} W_{N_1} \circ V_{p\cdot\frac{\widetilde{N}^{(p)}}{N_1}}$. However $F_1^\sharp\big|_{k_1} W_{N_1} = \overline{\epsilon_1}\cdot(-1)^{k_1} \times F_1$ and also $p\cdot\frac{\widetilde{N}^{(p)}}{N_1} = \frac{N_0}{N_1}$, in which case

$$\left\langle F_1^\sharp, \mathrm{Tr}^{\widetilde{N}}_{N_0}(G)\right\rangle_{N_0} = \epsilon_1 p^{1-\frac{(k_1-2)(\tilde{e}-2)}{2}} \left(\frac{\widetilde{N}^{(p)}}{N_1}\right)^{\frac{k_1}{2}} \times \left\langle F_1\big|_{k_1} V_{\frac{N_0}{N_1}}\ ,\ G\big|_{k_1} W_{\widetilde{N}} \circ U_p^{\tilde{e}-1}\right\rangle_{N_0}.$$

Finally our assumption that $F_1^\sharp = F_1$ implies that the F_1-isotypic subspace inside $\mathcal{M}_{k_1}(N_0, \psi_1)$ is spanned by the normalised eigenforms $F_1\big|_{k_1} V_d$ as d runs through the divisors of N_0/N_1; we may therefore write

$$G\big|_{k_1} W_{\widetilde{N}} \circ U_p^{\tilde{e}-1} \ = \sum_{d\,\big|\,\frac{N_0}{N_1}} \mathfrak{c}_{d,\widetilde{N},\tilde{e}}(G)\cdot F_1\big|_{k_1} V_d \ + \ G^{(\perp)}_{\widetilde{N},\tilde{e}}$$

for the particular choice of scalars, $\mathfrak{c}_{d,\widetilde{N},\tilde{e}}(G)$, obtained by projecting $G\big|_{k_1} W_{\widetilde{N}} \circ U_p^{\tilde{e}-1}$ onto each basis element $F_1\big|_{k_1} V_d$. Since the modular form $G^{(\perp)}_{\widetilde{N},\tilde{e}}$ is orthogonal to $F_1\big|_{k_1} V_{\frac{N_0}{N_1}}$ under the Petersson inner product at level N_0, the result now follows. □

6.2.2 Expansions of nearly holomorphic functions

The strategy over the next two sections is to show for $G_2 \in \mathcal{S}_{k_2}(N_2, \psi_2)$ and $G_3 \in \mathcal{M}_{k_3}(N_3, \psi_3)$ as before, that the modular forms

$$\mathrm{Hol}_\infty\big(G_2 \cdot \delta^{(r)}_{k_1-k_2-2r}(G_3)\big) \quad \text{with } r = (k_1-k_2-k_3)/2 \in \mathbb{Z}_{\geq 0}$$

behave well under mod p^ν congruences, in the sense that if we replace G_2 and G_3 by p^ν-congruent forms then $\mathrm{Hol}_\infty\big((-)\cdot\delta^{(r)}_{k_1-k_2-2r}(-)\big)$ preserves these congruences.

Definition 6.8. *Let $R \subset \mathbb{C}$ be a commutative ring, and $\mathfrak{p} \lhd R$ a prime ideal.*

(i) For each $t \geq 0$ we will denote by $\mathcal{N}^{\infty,t}_{w,\mathrm{pol}}\big(\Gamma(N);R\big)$ the R-submodule of $\mathcal{M}^{\infty}_{w}\big(\Gamma(N)\big)$ consisting of $\mathcal{C}^{\infty}$-modular forms, $H(z)$, with Fourier expansions of the type

$$H(z) \;=\; \sum_{m\in N^{-1}\mathbb{Z}} e^{-2\pi m y}\cdot \mathcal{P}_H\left(\frac{1}{4\pi y}\;,\; m\right)\cdot e^{2\pi i m x}$$

where $z = x+iy \in \mathfrak{h}$ and for all $m \in N^{-1}\mathbb{Z}$, the coefficient terms $\mathcal{P}_H(X,m) \in R[X]$ satisfy $\deg(\mathcal{P}_H) \leq t$.

(ii) We similarly define $\mathcal{N}^{\infty,t}_{w,\mathrm{pol}}(N,\psi;R) := \mathcal{N}^{\infty,t}_{w,\mathrm{pol}}\big(\Gamma(N);R\big)\cap\mathcal{M}^{\infty}_{w}(N,\psi)$.

(iii) If $H(z), H^{\dagger}(z) \in \mathcal{N}^{\infty,t}_{w,\mathrm{pol}}\big(\Gamma(N);R\big)$ and there exists $\nu \geq 1$ such that

$$\mathcal{P}_H(X,m) - \mathcal{P}_{H^{\dagger}}(X,m) \in \mathfrak{p}^{\nu}\cdot R[X] \quad \textit{for every } m \in N^{-1}\mathbb{Z},$$

then we say that H is congruent to $H^{\dagger}$ modulo $\mathfrak{p}^{\nu}$, and we will write $H \equiv H^{\dagger} \pmod{\mathfrak{p}^{\nu}\cdot R}$.

For example, if $R = \mathcal{O}_K$ is the ring of integers of some number field K, and if one considers a classical form $G = \sum_{n=0}^{\infty} a_n(G)q^n \in \mathcal{M}_w(N,\psi)\cap\mathcal{O}_K[\![q]\!]$, then clearly $\mathcal{P}_G(X,m) = a_m(G)$ if $m \in \mathbb{Z}_{\geq 0}$, while $\mathcal{P}_G(X,m) = 0$ if $m \notin \mathbb{Z}_{\geq 0}$. We therefore have a natural containment $\mathcal{M}_w(N,\psi)\cap\mathcal{O}_K[\![q]\!] \subset \mathcal{N}^{\infty,0}_{w,\mathrm{pol}}(N,\psi;\mathcal{O}_K)$. Furthermore, the definition of mod $\mathfrak{p}^{\nu}$-congruent forms introduced above generalises the standard notion of modulo $\mathfrak{p}^{\nu}$ congruences used for series expansions in $\mathcal{O}_K[\![q]\!]$.

Lemma 6.9. *(a) For a commutative ring R as above, the differential operator $\delta^{(r)}_{w}$ sends the space of nearly holomorphic forms $\mathcal{N}^{\infty,t}_{w,\mathrm{pol}}\big(\Gamma(N);R\big)$ into $\mathcal{N}^{\infty,t+r}_{w+2r,\mathrm{pol}}\big(\Gamma(N);R\big)$, and by restriction sends $\mathcal{N}^{\infty,t}_{w,\mathrm{pol}}(N,\psi;R)$ into $\mathcal{N}^{\infty,t+r}_{w+2r,\mathrm{pol}}(N,\psi;R)$.*

(b) If $H(z), H^{\dagger}(z) \in \mathcal{M}_w(N,\psi)$ are $\mathfrak{p}^{\nu}$-congruent forms with R-coefficients, then one also obtains congruences

$$\delta^{(r)}_{w}(H) \equiv \delta^{(r)}_{w}(H^{\dagger}) \pmod{\mathfrak{p}^{\nu}\cdot R}$$

at all integers $r \geq 0$, in the spirit of Definition 6.8(iii).

Proof. Let us deal with part (a) first. Recall from *op. cit.* that a $\mathcal{C}^\infty$-modular form $G(z) \in \mathcal{M}_w^\infty\big(\Gamma(N)\big)$ can be always expanded as a Fourier series of the type

$$G(z) \;=\; \sum_{m\in N^{-1}\mathbb{Z}} A_G(y,m)\cdot e^{2\pi i m x} \quad \text{with } z = x+iy,$$

and each term $A_G(y,m)\in\mathcal{C}^\infty(\mathbb{R}^+)$. Applying the operator $\frac{\partial}{\partial z}$ to $G(z)$ then yields

$$\frac{\partial G(z)}{\partial z} \;=\; \sum_{m\in N^{-1}\mathbb{Z}} \left(m\pi i\cdot A_G(y,m) - \frac{i}{2}A'_G(y,m)\right)\cdot e^{2\pi i m x}$$

with $A'_G(y,m) = \frac{\mathrm{d}A_G(y,m)}{\mathrm{d}y}$, so that as an element of $\mathcal{M}_{w+2}^\infty\big(\Gamma(N)\big)$ we find that

$$\delta_w\big(G(z)\big) \;=\; \sum_{m\in N^{-1}\mathbb{Z}} \left(\left(\frac{m}{2} - \frac{w}{4\pi y}\right)\cdot A_G(y,m) - \frac{1}{4\pi}A'_G(y,m)\right)\cdot e^{2\pi i m x}.$$

In the specific situation with $G\in\mathcal{N}_{w,\mathrm{pol}}^{\infty,t}\big(\Gamma(N);R\big)$, one can further write

$$A_G(y,m) = e^{-2\pi m y}\cdot\mathcal{P}_G\left(\frac{1}{4\pi y}, m\right)$$

where $\mathcal{P}_G\left(X,m\right) = \sum_{j=0}^t \beta_j(m)\cdot X^j \in R[X]$. A straightforward calculation reveals that

$$A'_G(y,m) = -2\pi e^{-2\pi m y}\cdot\left(\sum_{j=0}^t m\beta_j(m)\cdot(4\pi y)^{-j} + 2\cdot\sum_{j=1}^t j\beta_j(m)\cdot(4\pi y)^{-j-1}\right),$$

in which case

$$\begin{aligned}
&\delta_w\big(G(z)\big)\\
&= \sum_{m\in N^{-1}\mathbb{Z}} \left[\left(\frac{m}{2}-\frac{w}{4\pi y}\right)\cdot e^{-2\pi m y}\,\mathcal{P}_G\left(\frac{1}{4\pi y}, m\right) - \frac{1}{4\pi}A'_G(y,m)\right]\cdot e^{2\pi i m x}\\
&= \sum_{m\in N^{-1}\mathbb{Z}} e^{-2\pi m y}\cdot\Bigg[m\beta_0(m)\\
&\qquad + \sum_{j=1}^t \big(m\beta_j(m) + (j-1-w)\beta_{j-1}(m)\big)\cdot(4\pi y)^{-j}\\
&\qquad\qquad + \;(t-w)\beta_t(m)\cdot(4\pi y)^{-t-1}\Bigg]\cdot e^{2\pi i m x}.
\end{aligned}$$

Consequently for every $m \in N^{-1}\mathbb{Z}$, we set $\mathcal{P}_{\delta_w(G)}(X,m)$ equal to the polynomial

$$m\beta_0(m) \;+\; \sum_{j=1}^{t} \big(m\beta_j(m) + (j-1-w)\beta_{j-1}(m)\big)\cdot X^j \;+\; (t-w)\beta_t(m)\cdot X^{t+1}$$

so in particular, $\mathcal{P}_{\delta_w(G)}(X,m) \in R[X]$ with $\deg\big(\mathcal{P}_{\delta_w(G)}\big) \le t+1$, hence

$$\delta_w\big(G(z)\big) \;=\; \sum_{m\in N^{-1}\mathbb{Z}} e^{-2\pi m y}\cdot \mathcal{P}_{\delta_w(G)}\left(\frac{1}{4\pi y}\;,\;m\right)\cdot e^{2\pi i m x} \;\in\; \mathcal{N}^{\infty,t+1}_{w+2,\mathrm{pol}}\big(\Gamma(N);R\big).$$

It follows that $\delta_w : \mathcal{N}^{\infty,t}_{w,\mathrm{pol}}(\big(\Gamma(N);R\big) \to \mathcal{N}^{\infty,t+1}_{w+2,\mathrm{pol}}\big(\Gamma(N);R\big)$, and then applying an inductive argument to $\delta^{(r)}_w = \delta_{w+2r-2}\circ\cdots\circ\delta_{w+2}\circ\delta_w$ for increasing values of $r>0$, we conclude that $\delta^{(r)}_w : \mathcal{N}^{\infty,t}_{w,\mathrm{pol}}\big(\Gamma(N);R\big) \to \mathcal{N}^{\infty,t+r}_{w+2r,\mathrm{pol}}\big(\Gamma(N);R\big)$ as asserted in (a).

To show that statement (b) is true, let us in greater generality suppose that:

$$H(z) = \sum_{m\in\mathbb{Z}} e^{-2\pi m y}\cdot \mathcal{P}_H\left(\frac{1}{4\pi y}\;,\;m\right)\cdot e^{2\pi i m x}, \qquad \mathcal{P}_H(X,m) = \sum_{j=0}^{t} \beta_j(m)\cdot X^j;$$

$$H^\dagger(z) = \sum_{m\in\mathbb{Z}} e^{-2\pi m y}\cdot \mathcal{P}_{H^\dagger}\left(\frac{1}{4\pi y}\;,\;m\right)\cdot e^{2\pi i m x}, \quad \mathcal{P}_{H^\dagger}(X,m) = \sum_{j=0}^{t} \beta^\dagger_j(m)\cdot X^j.$$

The condition $H \equiv H^\dagger \pmod{\mathfrak{p}^\nu\cdot R}$ is by definition equivalent to the family of congruences $\beta_j(m) \equiv \beta^\dagger_j(m) \pmod{\mathfrak{p}^\nu\cdot R}$ for every $m\in\mathbb{Z}$ and $j\in\{0,\ldots,t\}$. Adopting the same argument as in part (a), it directly follows that

$$\delta_w\big(H(z)\big) = \sum_{m\in\mathbb{Z}} e^{-2\pi m y}\cdot \mathcal{P}^\delta_H\left(\frac{1}{4\pi y}\;,\;m\right)\cdot e^{2\pi i m x}$$

where $\mathcal{P}^\delta_H(X,m) = \sum_{j=0}^{t+1}\beta^\delta_j(m)\cdot X^j$ and

$$\beta^\delta_j(m) = \begin{cases} (t-w)\beta_t(m) & \text{if } j = t+1\\ m\beta_j(m) + (j-1-w)\beta_{j-1}(m) & \text{if } 0<j<t+1\\ m\beta_0(m) & \text{if } j=0.\end{cases}$$

Likewise for the second Fourier expansion,

$$\delta_w\big(H^\dagger(z)\big) = \sum_{m\in\mathbb{Z}} e^{-2\pi m y}\cdot \mathcal{P}^\delta_{H^\dagger}\left(\frac{1}{4\pi y}\;,\;m\right)\cdot e^{2\pi i m x}$$

where $\mathcal{P}^{\delta}_{H^\dagger}(X,m)=\sum_{j=0}^{t+1}\beta^{\dagger,\delta}_j(m)\cdot X^j$ and

$$\beta^{\dagger,\delta}_j(m)=\begin{cases}(t-w)\beta^\dagger_t(m) & \text{if } j=t+1\\ m\beta^\dagger_j(m)+(j-1-w)\beta^\dagger_{j-1}(m) & \text{if } 0<j<t+1\\ m\beta^\dagger_0(m) & \text{if } j=0.\end{cases}$$

The implication '$\beta_j(m)\equiv\beta^\dagger_j(m) \pmod{\mathfrak{p}^\nu} \Rightarrow \beta^\delta_j(m)\equiv\beta^{\dagger,\delta}_j(m) \pmod{\mathfrak{p}^\nu}$' is now obvious since $m,j,w,t\in\mathbb{Z}$, hence $\delta_w(H)\equiv\delta_w(H^\dagger) \pmod{\mathfrak{p}^\nu\cdot R}$. Finally, recalling that $\delta^{(r)}_w=\delta_{w+2r-2}\circ\cdots\circ\delta_{w+2}\circ\delta_w$ and iterating this process above $(r-1)$-times more, one establishes that $\delta^{(r)}_w(H)\equiv\delta^{(r)}_w(H^\dagger) \pmod{\mathfrak{p}^\nu\cdot R}$. $\square$

6.2.3 Projecting Eisenstein series and cusp forms

Proceeding further with our calculation of the inner product in Equation (6.1), we shall require some background on the operator '$\mathrm{Hol}_\infty(-)$' which appears in the automorphic theory. Throughout G_2 is a cusp form of weight k_2, level N_2 and character ψ_2.

6.2.3.1 The double product case

The first case we treat relates to the double product L-function $L(F_1\otimes G_2,s)$. Consider the Eisenstein series in [60, Eqn (2.3)] of weight $w\geq 0$, character η^{-1} and level N, given by the infinite series

$$E^*_{w,N}(z,s,\eta)=\sum_{\Gamma_\infty\backslash\Gamma_0(N)}\eta(\gamma)\cdot(cz+d)^{-w}\big|cz+d\big|_\infty^{-2s},\ \gamma=\begin{pmatrix}a&b\\c&d\end{pmatrix}. \quad (6.2)$$

For technical reasons, our formulae become tidier if we renormalise these series via

$$\mathbf{E}^*_{w,N}(z,\eta)\ :=\ \frac{N^{w/2}}{2}\cdot\frac{\Gamma(w)}{(2\pi i)^w}\cdot\zeta_N(w,\eta)\times E^*_{w,N}(z,0,\eta). \quad (6.3)$$

Henceforth let us assume that $r,w\in\mathbb{Z}$ satisfy both $w=k_1-k_2-2r\geq 0$ and $r\geq 0$.

Proposition 6.10. *Setting $N = \widetilde{N}$, $\eta = \psi_3$ and $\breve{G}_3 = \mathbf{E}^*_{k_1-k_2-2r,\widetilde{N}}(z,\psi_3)$, then*

$$\mathcal{H} = \mathrm{Hol}_\infty\big(G_2 \cdot \delta^{(r)}_{k_1-k_2-2r}(\breve{G}_3)\big|_{k_1-k_2} W_{\widetilde{N}}\big) \in \mathcal{M}_{k_1}\big(\widetilde{N},\psi_2\psi_3\big)$$

has the $\mathbb{Q}\Big(a_m(G_2)\;\big|\; m\in\mathbb{N}\Big)$-rational q-expansion $\mathcal{H}(z) = \sum_{n=1}^\infty a(n,\mathcal{H})\cdot q^n$, where

$$a(n,\mathcal{H}) \;=\; \sum_{n=\xi_2+\xi_3>0} a_{\xi_2}(G_2)\cdot \sum_{\xi_3=b\cdot c} b^{k_1-k_2-2r-1}\cdot\psi_3(c)\cdot P_{-r}(\xi_3,n)$$

and for $s\in\mathbb{Z}_{\le 0}$, the rational polynomial '$P_s(-,-)$' is given by

$$P_s(X,Y) \;=\; \sum_{j=0}^{-s}(-1)^j\begin{pmatrix}-s\\ \\ j\end{pmatrix}\frac{\Gamma(k_1-k_2+s)}{\Gamma(k_1-k_2+s-j)}\frac{\Gamma(k_1-1-j)}{\Gamma(k_1-1)}\cdot X^{-s-j}Y^j.$$

Proof. Firstly applying [60, Equation (2.9)], one has the identity

$$E^*_{w+2r,\widetilde{N}}\big(z,-r,\eta\big) \;=\; \frac{\Gamma(w)}{\Gamma(w+r)}(-4\pi y)^r\cdot\delta^{(r)}_w\Big(E^*_{w,\widetilde{N}}\big(z,0,\eta\big)\Big).$$

If one has $r=0$ then $E^*_{w+2r,\widetilde{N}}\big(z,0,\eta\big)$ is of holomorphic type, while if $r>0$ then it is nearly holomorphic and has moderate growth, so that Theorem 2.19 is applicable. After rearranging the above equation, it follows directly that

$$\delta^{(r)}_w\Big(E^*_{w,\widetilde{N}}\big(z,0,\eta\big)\Big)\,\Big|W_{\widetilde{N}} = (-4\pi)^{-r}\cdot\frac{\Gamma(w+r)}{\Gamma(w)}\times\Big(y^{-r}\cdot E^*_{w+2r,\widetilde{N}}\big(z,-r,\eta\big)\Big)\,\Big|W_{\widetilde{N}}$$

and then combining it with Panchishkin's definitions [53, (4.3), (4.6) and (4.13)],

$$\Big(y^{-r}\cdot E^*_{w+2r,\widetilde{N}}\big(z,-r,\eta\big)\Big)\,\Big|_{w+2r}W_{\widetilde{N}} \;=\; \frac{2\cdot\zeta_N\big(w,\eta\big)^{-1}}{\widetilde{N}^{w/2}\cdot\Gamma(w+r)}\cdot\frac{(2\pi i)^w}{(-4\pi)^{-r}}\cdot\mathcal{E}_{w+2r}(-r,\eta).$$

Here $\mathcal{E}_{w+2r}(s,\eta)$ denotes the Eisenstein series introduced in [53, Equation (4.13)]: in particular at $s=-r$, the $\mathcal{C}^\infty$-function $\mathcal{E}_{w+2r}(-r,\eta)$ has the Fourier development

$$(4\pi y)^{-r}\cdot\sum_{\xi_3=1}^\infty\left(\sum_{\xi_3=b\cdot c} b^{w-1}\eta(c)\sum_{j=0}^r(-1)^j\begin{pmatrix}r\\ \\ j\end{pmatrix}\frac{\Gamma(w+r)}{\Gamma(w+r-j)}\cdot(4\pi\xi_3 y)^{r-j}\right)e^{2\pi i\xi_3 z}.$$

Writing everything in terms of our renormalised Eisenstein series $\mathbf{E}^*_{w,N}(z,-)$, one finds that $\delta^{(r)}_w\big(\mathbf{E}^*_{w,\widetilde{N}}(z,\eta)\big)\Big|_{w+2r}W_{\widetilde{N}}$ coincides with $\mathcal{E}_{w+2r}(-r,\eta)$, in which case

$$\mathrm{Hol}_\infty\Big(G_2\cdot\delta^{(r)}_w\Big(\mathbf{E}^*_{w,\widetilde{N}}(z,\eta)\Big)\Big|_{w+2r}W_{\widetilde{N}}\Big) \;=\; \mathrm{Hol}_\infty\big(G_2\cdot\mathcal{E}_{w+2r}(-r,\eta)\big).$$

We next apply the integral operator $\frac{(4\pi n)^{k_1-1}}{\Gamma(k_1-1)} \cdot \int_0^\infty A_H(y,n)\, e^{-2\pi n y} y^{k_1-2} \cdot \mathrm{d}y$ to the n-th Fourier coefficient of the form

$$H(z) \;=\; G_2 \cdot \mathcal{E}_{w+2r}(-r,\eta) \;=\; \sum_{m=1}^{\infty} A_H(y,m) \cdot e^{2\pi i m x}$$

and then exploit the well known identity

$$\frac{(4\pi n)^{k_1-1}}{\Gamma(k_1-1)} \cdot \int_0^\infty \Big((4\pi y)^{-j} e^{-2\pi n y} \Big) \cdot e^{-2\pi n y} y^{k_1-2} \cdot \mathrm{d}y \;=\; n^j \cdot \frac{\Gamma(k_1-j-1)}{\Gamma(k_1-1)}. \quad (6.4)$$

A tedious calculation, but essentially identical to the one given in [53, Section 5], allows us to conclude that

$$\begin{aligned} &\mathrm{Hol}_\infty\Big(G_2 \cdot \mathcal{E}_{w+2r}(-r,\eta) \Big) \\ &\qquad = \sum_{n=1}^{\infty} \left(\sum_{n=\xi_2+\xi_3>0} a_{\xi_2}(G_2) \cdot \sum_{\xi_3=b\cdot c} b^{w-1} \eta(c) \cdot P_{-r}(\xi_3,n) \right) q^n. \end{aligned}$$

The automorphy properties follow directly from Theorem 2.19 since each translate $G_2 \cdot \mathcal{E}_{w+2r}(-r,\eta)\Big|_{k_1} \gamma$ has moderate growth for $\gamma \in \mathrm{SL}_2(\mathbb{Z})$, and secondly the Fourier coefficients $A_H(y,n)$ of the form $H = G_2 \cdot \mathcal{E}_{w+2r}(-r,\eta)$ vanish at every $n \leq 0$. $\square$

Corollary 6.11. *Suppose $G_2^{(\mathrm{I})}, G_2^{(\mathrm{II})} \in \mathcal{S}_{k_2}\big(N_2,\psi_2\big)$ have expansions in $\mathcal{O}_K\big[\big[q\big]\big]$ for a given number field K, that they satisfy the p-adic congruence*

$$G_2^{(\mathrm{I})} \equiv G_2^{(\mathrm{II})} \,(\mathrm{mod}\, p^{\nu_2})$$

at some integer $\nu_2 \geq 1$, and that $\breve{G}_3 = \mathbf{E}^*_{k_1-k_2-2r,\widetilde{N}}(z,\psi_3)$. If $p > k_1 - 2$, then

$$\mathrm{Hol}_\infty\Big(G_2^{(\mathrm{I})} \cdot \delta^{(r)}_{k_1-k_2-2r}(\breve{G}_3)\big|_{k_1-k_2} W_{\widetilde{N}} \Big) \equiv \mathrm{Hol}_\infty\Big(G_2^{(\mathrm{II})} \cdot \delta^{(r)}_{k_1-k_2-2r}(\breve{G}_3)\big|_{k_1-k_2} W_{\widetilde{N}} \Big)$$

modulo $p^{\nu_2} \cdot \mathcal{O}_K\big[\big[q\big]\big]$, provided the integer r lies in the range $0 \leq r \leq \frac{1}{2}(k_1 - k_2)$.

Proof. We use the Fourier expansions given in the preceding result for both $G_2 = G_2^{(\mathrm{I})}$ and $G_2 = G_2^{(\mathrm{II})}$, and observe that $P_{-r}(X,Y) \in \mathbb{Z}_p[X,Y]$ as $p > k_1 - 2$. $\square$

6.2.3.2 The triple product case

The next case relates to $L(F_1 \otimes G_2 \otimes G_3, s)$. Here there are no Eisenstein series to contend with, and their role is replaced by the holomorphic form G_3 of weight $w = k_3$, level N_3 and nebentypus $\psi_3 = (\psi_1\psi_2)^{-1}$.

Proposition 6.12. *If $G_3 \in \mathcal{M}_w\big(N_3, \overline{\psi_1\psi_2}; R\big)$ for a given subring $R \subset \mathbb{C}$, then*

$$\mathcal{G} = \mathrm{Hol}_\infty\big(G_2 \cdot \delta_w^{(r)}(G_3)\big) \quad \text{at each } r = (k_1 - k_2 - w)/2 \in \mathbb{Z}_{\geq 0}$$

is a cusp form of weight k_1, level $\widetilde{N}$ and character $\overline{\psi_1}$; furthermore, it has the $R\Big[a_m(G_2) \mid m \in \mathbb{N}\Big]$-rational q-expansion $\mathcal{G}(z) = \sum_{n=1}^\infty a(n, \mathcal{G}) \cdot q^n$, where

$$a(n, \mathcal{G}) \;=\; \sum_{n=\xi_2+\xi_3>0} a_{\xi_2}(G_2) \cdot \sum_{j=0}^{r} \frac{\Gamma(k_1 - 1 - j)}{\Gamma(k_1 - 1)} \cdot \beta_j^{(r)}(\xi_3) \cdot n^j$$

and $\mathcal{P}_{\delta_w^{(r)}(G_3)}(X, m) \;=\; \sum_{j=0}^r \beta_j^{(r)}(m) \cdot X^j \;\in\; R[X]$ in the sense of Definition 6.8(i).

Proof. One simply points out that $G_2 \cdot \delta_w^{(r)}(G_3)$ has the Fourier expansion

$$\Big(G_2 \cdot \delta_w^{(r)}(G_3)\Big)(z) = \sum_{n=0}^{\infty} \left(\sum_{n=\xi_2+\xi_3>0} a_{\xi_2}(G_2) \cdot \sum_{j=0}^{r} \beta_j^{(r)}(\xi_3) \cdot (4\pi y)^{-j} \right) \cdot e^{2\pi i n z}$$

to which we apply the operator $\mathrm{Hol}_\infty(-)$, and then repeatedly use Equation (6.4). The property that G_2 is a cusp form directly implies $\mathcal{G}$ vanishes at cusps too. $\square$

Corollary 6.13. *If $G_2^{(\mathrm{I})}, G_2^{(\mathrm{II})} \in \mathcal{S}_{k_2}\big(N_2, \psi_2\big)$ and $G_3^{(\mathrm{I})}, G_3^{(\mathrm{II})} \in \mathcal{M}_{k_3}\big(N_3, \psi_3\big)$ have expansions in $\mathcal{O}_K\big[\![q]\!\big]$ for a given number field K, if they satisfy respectively*

$$G_2^{(\mathrm{I})} \equiv G_2^{(\mathrm{II})} \pmod{p^{\nu_2}} \quad \text{and} \quad G_3^{(\mathrm{I})} \equiv G_3^{(\mathrm{II})} \pmod{p^{\nu_3}} \quad \text{for some } \nu_2, \nu_3 \geq 1,$$

and lastly if the prime $p \nmid \frac{(k_1-2)!}{(k_1-2-r)!}$, then

$$\mathrm{Hol}_\infty\Big(G_2^{(\mathrm{I})} \cdot \delta^{(r)}_{k_1-k_2-2r}\big(G_3^{(\mathrm{I})}\big)\Big) \equiv \mathrm{Hol}_\infty\Big(G_2^{(\mathrm{II})} \cdot \delta^{(r)}_{k_1-k_2-2r}\big(G_3^{(\mathrm{II})}\big)\Big) \bmod p^{\min\{\nu_2,\nu_3\}}$$

provided again that the integer r lies inside the range $0 \le r \le \frac{1}{2}(k_1 - k_2)$.

Proof. From Lemma 6.9(b), $\delta^{(r)}_{k_1-k_2-2r}\big(G_3^{(\mathrm{I})}\big) \equiv \delta^{(r)}_{k_1-k_2-2r}\big(G_3^{(\mathrm{II})}\big) \pmod{p^{\nu_3}}$ and using the Fourier expansions which are calculated in the preceding proposition, the result follows immediately. □

6.2.4 The effect of Σ-depletion and χ-twisting

In the following discussion $\mathbf{g}^{(\mathrm{I})}$ and $\mathbf{g}^{(\mathrm{II})}$ denote primitive Hecke eigenforms of weight k, character ψ, and levels $N_{\mathbf{g}}^{(\mathrm{I})}$ and $N_{\mathbf{g}}^{(\mathrm{II})}$ respectively (note we treat both $p \nmid N_{\mathbf{g}}^{(\mathrm{I})} \cdot N_{\mathbf{g}}^{(\mathrm{II})}$ and $p \mid N_{\mathbf{g}}^{(\mathrm{I})} \cdot N_{\mathbf{g}}^{(\mathrm{II})}$). We shall further suppose the coefficients in their q-expansions satisfy:

$$a_n\big(\mathbf{g}^{(\mathrm{I})}\big) \equiv a_n\big(\mathbf{g}^{(\mathrm{II})}\big) \pmod{p^{\nu}} \tag{6.5}$$

for all $n \in \mathbb{N}$ with $\gcd\big(n, N_{\mathbf{g}}^{(\mathrm{I})} N_{\mathbf{g}}^{(\mathrm{II})}\big) = 1$. Let $\Sigma \subset \mathrm{Spec}(\mathbb{Z})$ be a finite set containing the primes dividing $N_{\mathbf{g}}^{(\mathrm{I})} N_{\mathbf{g}}^{(\mathrm{II})}$, but not p.

Definition 6.14. *(a) If $\star \in \{\mathrm{I}, \mathrm{II}\}$, then $\mathbf{g}_{\Sigma}^{(\star)}$ indicates the depleted cusp form*

$$\mathbf{g}_{\Sigma}^{(\star)}(z) = \sum_{n=1}^{\infty} a_n\big(\mathbf{g}_{\Sigma}^{(\star)}\big) \cdot q^n \in \mathcal{S}_k\big(N_{\Sigma}^{(\star)}, \psi\big), \quad N_{\Sigma}^{(\star)} = \mathrm{lcm}\Big(N_{\mathbf{g}}^{(\star)}, \prod_{l \in \Sigma} l^2\Big)$$

where $a_n\big(\mathbf{g}_{\Sigma}^{(\star)}\big) = a_n\big(\mathbf{g}^{(\star)}\big)$ if $\mathrm{supp}(n) \cap \Sigma = \emptyset$, and $a_n\big(\mathbf{g}_{\Sigma}^{(\star)}\big) = 0$ if $\mathrm{supp}(n) \cap \Sigma \neq \emptyset$.

(b) For a Dirichlet character χ of conductor $p^{n_\chi} \ge 1$, and for each choice of $\star \in \{\mathrm{I}, \mathrm{II}\}$, we define χ-twisted cusp forms by $\mathbf{g}_{\chi}^{(\star)} := \mathbf{g}^{(\star)} \otimes \chi$ and $\mathbf{g}_{\Sigma,\chi}^{(\star)} := \big(\mathbf{g}^{(\star)} \otimes \chi\big)_{\Sigma} = \mathbf{g}_{\Sigma}^{(\star)} \otimes \chi$.

If we set $\widetilde{N}_{\Sigma,\chi} := \mathrm{lcm}\big(p^{2n_\chi}, N_{\Sigma}^{(\mathrm{I})}, N_{\Sigma}^{(\mathrm{II})}\big)$ then both $\mathbf{g}_{\Sigma,\chi}^{(\mathrm{I})}$ and $\mathbf{g}_{\Sigma,\chi}^{(\mathrm{II})}$ are cuspidal Hecke eigenforms of weight k and character $\psi\chi^2$, each of whose levels divides $\widetilde{N}_{\Sigma,\chi}$. Furthermore, their q-expansions automatically satisfy

$$\mathbf{g}_{\chi}^{(\star)} = \sum_{n=1}^{\infty} \chi(n) \cdot a_n\big(\mathbf{g}^{(\star)}\big) \cdot q^n \quad \text{and} \quad \mathbf{g}_{\Sigma,\chi}^{(\star)} = \sum_{n=1}^{\infty} \chi(n) \cdot a_n\big(\mathbf{g}_{\Sigma}^{(\star)}\big) \cdot q^n$$

provided that the conductor $p^{n_\chi} \ge \max\Big\{\big|N_{\mathbf{g}}^{(\mathrm{I})}\big|_p^{-\frac{1}{2}}, \big|N_{\mathbf{g}}^{(\mathrm{II})}\big|_p^{-\frac{1}{2}}\Big\}$.

Proposition 6.15. *If $\mathbf{g}^{(\mathrm{I})}$ and $\mathbf{g}^{(\mathrm{II})}$ satisfy Equation (6.5) above, then at all characters χ of p-power conductor and for each finite set of prime numbers $\Sigma \supset \mathrm{supp}(N_{\mathbf{g}}^{(\mathrm{I})} \cdot N_{\mathbf{g}}^{(\mathrm{II})}) - \{p\}$,*

$$\mathbf{g}_{\Sigma,\chi}^{(\mathrm{I})}\Big|_k W_{\widetilde{N}} \equiv \mathbf{g}_{\Sigma,\chi}^{(\mathrm{II})}\Big|_k W_{\widetilde{N}} \pmod{p^\nu} \quad \text{if } \widetilde{N}_{\Sigma,\chi}\big|\widetilde{N} \text{ and } \mathrm{ord}_p\big(\widetilde{N}_{\Sigma,\chi}\big) = \mathrm{ord}_p\big(\widetilde{N}\big),$$

*as a congruence between p-**integral** linear sums of eigenforms* [1]*.*

Proof. For a rational prime l, if l does not divide the level we write T_l for the l-th Hecke operator, while if l does divide the level we shall use the notation U_l. For $m \in \mathbb{N}$ coprime to the level, the m-th diamond operator is denoted by $\langle m \rangle$ and for an integer $d \geq 1$, one writes V_d for the degeneracy map (as we did in Section 6.2.1). Let us begin by remarking that for each $\star \in \{\mathrm{I}, \mathrm{II}\}$,

$$\mathbf{g}_{\Sigma,\chi}^{(\star)} = \mathbf{g}_{\chi}^{(\star)}\Big|_k \prod_{\substack{l \in \Sigma, \\ l \nmid N_{\mathbf{g}}^{(\star)}}} \big(1 - T_l \cdot V_l + l^{k-1} \cdot \langle l \rangle \cdot V_{l^2}\big) \cdot \prod_{\substack{l \in \Sigma, \\ l \| N_{\mathbf{g}}^{(\star)}}} \big(1 - U_l \cdot V_l\big) \tag{6.6}$$

which gives an alternative construction of these Σ-depleted, χ-twisted cusp forms. To prove our result, it is necessary to establish that the composition of operators

$$(\,-\,)\Big|_k \prod_{\substack{l \in \Sigma, \\ l \nmid N_{\mathbf{g}}^{(\star)}}} \big(1 - T_l \cdot V_l + l^{k-1} \cdot \langle l \rangle \cdot V_{l^2}\big) \cdot \prod_{\substack{l \in \Sigma, \\ l \| N_{\mathbf{g}}^{(\star)}}} \big(1 - U_l \cdot V_l\big)\Big|_k W_{\widetilde{N}}$$

acting on newforms of weight k and character $\psi\chi^2$ preserves the integral structure.

Fix a choice of $\star \in \{\mathrm{I}, \mathrm{II}\}$. Let us assume that l is a rational prime number, and M denotes a multiple of $N_{\mathbf{g}}^{(\star)}$ such that l^2 divides M. Then for a 'weight k' action,

$$\begin{aligned}\big(1 - U_l \cdot V_l\big) \cdot W_M &= W_M - U_l \cdot V_l \cdot W_M \\ &= l^{k/2} \cdot W_{M/l} \cdot V_l - l^{-k/2} \cdot U_l \cdot W_{M/l}\end{aligned}$$

[1] By work of Vatsal [67, Prop 4.5], the canonical motivic periods associated to $\mathbf{g}_{\Sigma,\chi}^{(\star)}$ and $\mathbf{g}_{\chi}^{(\star)}$ are known to differ from each other by a p-adic unit, at least in the case where $a_p(\mathbf{g}^{(\star)}) \in \mathcal{O}_{\mathbb{C}_p}^{\times}$.

because at such a weight, we have $W_M = l^{k/2} \cdot W_{M/l} \cdot V_l$ and $V_l \cdot W_M = l^{-k/2} \cdot W_{M/l}$. One therefore deduces

$$\begin{aligned}\big(1 - U_l \cdot V_l\big) \cdot W_M \; &= \; l^{k/2} \cdot W_{M/l} \cdot V_l - l^{-k/2} \cdot W_{M/l} \cdot U_l^* \\ &= \; W_{M/l} \cdot \big(l^{k/2} \cdot V_l - l^{-k/2} \cdot U_l^*\big)\end{aligned} \tag{6.7}$$

where $(-)^*$ indicates the adjoint Hecke operator. Analogously, one calculates that

$$\begin{aligned}&\big(1 - T_l \cdot V_l + l^{k-1}\langle l\rangle \cdot V_{l^2}\big) \cdot W_M \\ &\quad = \; W_M - T_l \cdot V_l \cdot W_M + l^{k-1}\langle l\rangle \cdot V_{l^2} \cdot W_M \\ &\quad = \; l^k \cdot W_{M/l^2} \cdot V_{l^2} - l^{-k/2} \cdot T_l \cdot W_{M/l} + l^{k-1}(l^2)^{-k/2}\langle l\rangle \cdot W_{M/l^2}\end{aligned}$$

as $W_M = l^k \cdot W_{M/l^2} \cdot V_{l^2}, \; V_l \cdot W_M = l^{-k/2} \cdot W_{M/l}$ and $V_{l^2} \cdot W_M = (l^2)^{-k/2} W_{M/l^2}$. We then obtain a string of equalities

$$\begin{aligned}&\big(1 - T_l \cdot V_l + l^{k-1} \cdot \langle l\rangle \cdot V_{l^2}\big) \cdot W_M \\ &\quad = \; l^k \cdot W_{M/l^2} \cdot V_{l^2} - T_l \cdot W_{M/l^2} \cdot V_l + l^{-1} \cdot \langle l\rangle \cdot W_{M/l^2} \\ &\quad = \; l^k \cdot W_{M/l^2} \cdot V_{l^2} - W_{M/l^2} \cdot T_l^* \cdot V_l + l^{-1} \cdot W_{M/l^2} \cdot \langle l\rangle^* \\ &\quad = \; W_{M/l^2} \cdot \Big(l^k \cdot V_{l^2} - T_l^* \cdot V_l + l^{-1} \cdot \langle l^{-1}\rangle\Big)\end{aligned} \tag{6.8}$$

and these three lines follow from the respective identities: $l^{-k/2} \cdot W_{M/l} = W_{M/l^2} \cdot V_l$, $T_l \cdot W_{M/l^2} = W_{M/l^2} \cdot T_l^*$ and $\langle l\rangle^* = \langle l^{-1}\rangle$, applied in consecutive order.

Returning to the description in (6.6), our calculations in Equations (6.7-6.8) imply via an inductive argument that

$$\begin{aligned}&\mathbf{g}_\chi^{(\star)}\Big|_k\Bigg(\prod_{\substack{l \in \Sigma, \\ l \nmid N_{\mathbf{g}}^{(\star)}}} \big(1 - T_l \cdot V_l + l^{k-1}\langle l\rangle \cdot V_{l^2}\big) \\ &\qquad\qquad \times \prod_{\substack{l \in \Sigma, \\ l \| N_{\mathbf{g}}^{(\star)}}} \big(1 - U_l \cdot V_l\big)\Bigg)\Bigg|_k W_{\widetilde{N}_{\Sigma,\chi}} \\ &= \mathbf{g}_\chi^{(\star)}\Big|_k W_{\widetilde{M}_{\Sigma,\chi}} \cdot \prod_{\substack{l \in \Sigma, \\ l \nmid N_{\mathbf{g}}^{(\star)}}} \Big(l^k \cdot V_{l^2} - T_l^* \cdot V_l + l^{-1} \cdot \langle l^{-1}\rangle\Big) \\ &\qquad\qquad \times \prod_{\substack{l \in \Sigma, \\ l \| N_{\mathbf{g}}^{(\star)}}} \big(l^{k/2} \cdot V_l - l^{-k/2} \cdot U_l^*\big)\end{aligned} \tag{6.9}$$

with the level of the W-operator being decreased to

$$\widetilde{M}_{\Sigma,\chi} := \widetilde{N}_{\Sigma,\chi} \cdot \prod_{\substack{l \in \Sigma, \\ l \nmid N_{\mathbf{g}}^{(\star)}}} l^{-2} \cdot \prod_{\substack{l \in \Sigma, \\ l \| N_{\mathbf{g}}^{(\star)}}} l^{-1} = N_{\mathbf{g}^{(\star)}\otimes\chi} \times M_{\Sigma,\mathbf{g}}^{(\star)}$$

for some $M_{\Sigma,\mathbf{g}}^{(\star)} \in \mathbb{N} \cap \mathbb{Z}_p^\times$. Under this weight k action, we may factorise

$$W_{\widetilde{M}_{\Sigma,\chi}} = \left(M_{\Sigma,\mathbf{g}}^{(\star)}\right)^{k/2} \cdot W_{N_{\mathbf{g}^{(\star)}\otimes\chi}} \cdot V_{M_{\Sigma,\mathbf{g}}^{(\star)}}$$

and one readily deduces that

$$\begin{aligned} \mathbf{g}_\chi^{(\star)}\Big|_k W_{\widetilde{M}_{\Sigma,\chi}} &= \left(M_{\Sigma,\mathbf{g}}^{(\star)}\right)^{k/2} \cdot \left(\mathbf{g}^{(\star)} \otimes \chi\Big|_k W_{N_{\mathbf{g}^{(\star)}\otimes\chi}}\right)\Big|_k V_{M_{\Sigma,\mathbf{g}}^{(\star)}} \\ &= \left(M_{\Sigma,\mathbf{g}}^{(\star)}\right)^{k/2} \cdot \Bigg(\psi\big(p^{2n_\chi}\big)\chi\big(N_{\mathbf{g}}^{(\star)}\big)\frac{\tau(\chi)^2}{p^{n_\chi}} \\ &\qquad \times \epsilon_{\mathbf{g}}^{(\star)}\left(\mathbf{g}^{(\star),\sharp} \otimes \chi^{-1}\right)\Bigg)\Big|_k V_{M_{\Sigma,\mathbf{g}}^{(\star)}} \end{aligned} \tag{6.10}$$

where $\epsilon_{\mathbf{g}}^{(\star)} \in \mathbb{C}$, $\left|\epsilon_{\mathbf{g}}^{(\star)}\right|_\infty = 1$ satisfies $\mathbf{g}^{(\star)}\big|_k W_{N_{\mathbf{g}}^{(\star)}} = \epsilon_{\mathbf{g}}^{(\star)} \cdot \mathbf{g}^{(\star),\sharp}$ (see [53, Eqn (1.24)]). If we define the algebraic number

$$\mathcal{Z}_{\Sigma,\chi}^{(\star)} := \left(M_{\Sigma,\mathbf{g}}^{(\star)}\right)^{k/2} \cdot \psi\big(p^{2n_\chi}\big)\chi\big(N_{\mathbf{g}}^{(\star)}\big)\frac{\tau(\chi)^2}{p^{n_\chi}}\ \epsilon_{\mathbf{g}}^{(\star)}$$

which is a p-adic unit as $\frac{\tau(\chi)^2}{p^{n_\chi}}, \epsilon_{\mathbf{g}}^{(\star)} \in \mathcal{O}_{\mathbb{C}_p}^\times$, Equations (6.7) and (6.9-6.10) imply

$$\begin{aligned} \mathbf{g}_{\Sigma,\chi}^{(\star)}\Big|_k W_{\widetilde{N}_{\Sigma,\chi}} = \mathcal{Z}_{\Sigma,\chi}^{(\star)} \cdot \left(\mathbf{g}^{(\star),\sharp} \otimes \chi^{-1}\right)\Big|_k & V_{M_{\Sigma,\mathbf{g}}^{(\star)}} \\ \times & \prod_{\substack{l \in \Sigma, \\ l \nmid N_{\mathbf{g}}^{(\star)}}} \left(l^k \cdot V_{l^2} - T_l^* \cdot V_l + l^{-1} \cdot \langle l^{-1}\rangle\right) \\ \times & \prod_{\substack{l \in \Sigma, \\ l \| N_{\mathbf{g}}^{(\star)}}} \left(l^{k/2} \cdot V_l - l^{-k/2} \cdot U_l^*\right). \end{aligned}$$

The right-hand side of the above equation is clearly a p-integral combination of eigenforms with algebraic integer q-expansions, therefore the left-hand side is too. To pass from $\mathbf{g}_{\Sigma,\chi}^{(\star)}\big|_k W_{\widetilde{N}_{\Sigma,\chi}}$ to the cusp form $\mathbf{g}_{\Sigma,\chi}^{(\star)}\big|_k W_{\widetilde{N}}$, one employs the identity

$$\mathbf{g}_{\Sigma,\chi}^{(\star)}\Big|_k W_{\widetilde{N}} = \left(\widetilde{N}/\widetilde{N}_{\Sigma,\chi}\right)^{k/2} \cdot \left(\mathbf{g}_{\Sigma,\chi}^{(\star)}\Big|_k W_{\widetilde{N}_{\Sigma,\chi}}\right)\Big|_k V_{\widetilde{N}/\widetilde{N}_{\Sigma,\chi}}$$

and observes that the quotient $\widetilde{N}/\widetilde{N}_{\Sigma,\chi} \in \mathbb{N}\cap\mathbb{Z}_p^\times$ since $\mathrm{ord}_p\big(\widetilde{N}_{\Sigma,\chi}\big) = \mathrm{ord}_p\big(\widetilde{N}\big)$.

Finally, those congruences asserted in the statement of the proposition now follow from the system of congruences

$$\chi^{-1}(n)\cdot\overline{a_n\big(\mathbf{g}_\Sigma^{(\mathrm{I})}\big)} \;\equiv\; \chi^{-1}(n)\cdot\overline{a_n\big(\mathbf{g}_\Sigma^{(\mathrm{II})}\big)} \pmod{p^\nu}$$

which hold at integers $n\geq 1$ by Equation (6.5), and the proof is complete. □

6.2.5 Finishing off the inner product calculation

Let us return to our earlier computation of the numerator from Equation (6.1), namely we must evaluate

$$\Big\langle F_1^\sharp\,,\mathrm{Tr}_{\widetilde{N}/N_0}\big(\mathrm{Hol}_\infty\big(G_2\cdot\delta_w^{(r)}(G_3)\big)\big|_{k_1}W_{\widetilde{N}}^\varepsilon\big)\Big\rangle_{N_0}\,,\quad \varepsilon\in\{0,1\}$$

for forms F_1,G_2,G_3 of level N_1,N_2,N_3, weight k_1,k_2,k_3 and nebentypus ψ_1, ψ_2, ψ_3 with $\psi_2\cdot\psi_3=\psi_1^{-1}$. Throughout we will again suppose that $F_1^\sharp=F_1$ and $\psi_1^2=\mathbf{1}$.

In particular, after dividing through by the period $\langle F_1,F_1\rangle_{N_1}$, one wants to see how this quantity varies when we replace G_2 and G_3 with p^ν-congruent forms. We shall treat the same two cases as in Section 6.2.3, corresponding to the double product $L(F_1\otimes G_2,s)$ and the triple product $L(F_1\otimes G_2\otimes G_3,s)$, respectively.

6.2.5.1 The double product case

Assume we are given newforms $\mathbf{g}^{(\mathrm{I})}$ and $\mathbf{g}^{(\mathrm{II})}$ of common weight $k=k_2>0$, common character ψ, and conductors $N_{\mathbf{g}}^{(\mathrm{I})}$ and $N_{\mathbf{g}}^{(\mathrm{II})}$. Let us further suppose Equation (6.5) holds for their q-expansions with $\nu=\nu_2$, i.e.

$$a_n\big(\mathbf{g}^{(\mathrm{I})}\big)\equiv a_n\big(\mathbf{g}^{(\mathrm{II})}\big) \pmod{p^{\nu_2}} \text{ for all } n\in\mathbb{N} \text{ with } \gcd\big(n,N_{\mathbf{g}}^{(\mathrm{I})}N_{\mathbf{g}}^{(\mathrm{II})}\big)=1.$$

We shall carefully select the subset $\Sigma\subset\mathrm{Spec}(\mathbb{Z})$ of primes in order to satisfy the three conditions: (i) $\mathrm{supp}\big(N_{\mathbf{g}}^{(\mathrm{I})}N_{\mathbf{g}}^{(\mathrm{II})}\big)-\{p\}\subset\Sigma$, (ii) $\#\Sigma<\infty$, and (iii) $p\notin\Sigma$.

Let χ denote a character of conductor $p^{n_\chi}\geq 1$. If we set $\widetilde{N}=\mathrm{lcm}\big(N_1,\widetilde{N}_{\Sigma,\chi}\big)$ and $\psi_2=\overline{\psi}\chi^{-2}$, one may consider $\mathbf{g}_{\Sigma,\chi}^{(\mathrm{I})}\big|_{k_2}W_{\widetilde{N}}$ and $\mathbf{g}_{\Sigma,\chi}^{(\mathrm{II})}\big|_{k_2}W_{\widetilde{N}}$ as belonging to the vector space $\mathcal{S}_{k_2}\big(\widetilde{N},\psi_2\big)$; they have p-integral q-expansions by Proposition 6.15, and their Fourier coefficients lie in some finite algebraic extension of $\mathbb{Q}$.

Now for any integer r in the range $0\le 2r\le k_1-k_2$, just as in Equation (6.3) one can define

$$\breve{G}_3(z) \;:=\; \mathbf{E}^{*}_{k_1-k_2-2r,\widetilde{N}}\big(z,\psi_3\big)$$

where $\psi_3=(\psi_1\psi_2)^{-1}=\psi_1\cdot\psi\cdot\chi^2$, and the level of the Eisenstein series equals $\widetilde{N}$. It follows for each choice of $\star\in\{\mathrm{I},\mathrm{II}\}$, the product of the two modular forms

$$G^{(\star)} = \mathbf{g}^{(\star)}_{\Sigma,\chi}\cdot\delta^{(r)}_{k_1-k_2-2r}(\breve{G}_3) \;\in\; \mathcal{M}^{\infty}_{k_1}\big(\widetilde{N},(\psi_2\psi_3)^{-1}\big)$$

is such that $G^{(\star)}\big|_{k_1}\gamma$ has moderate growth at every $\gamma\in\mathrm{SL}_2(\mathbb{Z})$, in which case

$$\mathcal{H}^{(\star)} := \mathrm{Hol}_\infty\big(G^{(\star)}\big)\Big|_{k_1}W_{\widetilde{N}} = \mathrm{Hol}_\infty\left(\mathbf{g}^{(\star)}_{\Sigma,\chi}\Big|_{k_2}W_{\widetilde{N}}\cdot\delta^{(r)}_{k_1-k_2-2r}(\breve{G}_3)\Big|_{k_1-k_2}W_{\widetilde{N}}\right)$$

is an element of $\mathcal{M}_{k_1}\big(\widetilde{N},\psi_2\psi_3\big)$.

Let $\mathcal{O}_{K,\chi}$ denote the integral extension of $\mathbb{Z}$ generated by the Fourier coefficients $a_n\big(\mathbf{g}^{(\star)}_{\Sigma}\big)$ and the character values $\chi(n)$, for all positive integers n and $\star\in\{\mathrm{I},\mathrm{II}\}$. Note that in the context of Lemma 6.7, each of the holomorphic modular forms

$$\mathcal{H}^{(\star)}\Big|_{k_1}U_p^{\tilde{e}-1} = \mathrm{Hol}_\infty\big(G^{(\star)}\big)\Big|_{k_1}W_{\widetilde{N}}\circ U_p^{\tilde{e}-1}\in\mathcal{M}_{k_1}\big(N_0,\psi_2\psi_3\big)\cap\mathcal{O}_{K,\chi}[[q]]$$

can be decomposed into its F_1-isotypic and non-F_1-isotypic components via

$$\mathcal{H}^{(\star)}\Big|_{k_1}U_p^{\tilde{e}-1} \;=\; \sum_{d\,\big|\,\frac{N_0}{N_1}}\mathfrak{c}^{(\star)}_{d,\widetilde{N},\tilde{e}}(\mathcal{H})\cdot F_1\big|_{k_1}V_d \;+\; \mathcal{H}^{(\star),(\perp)}_{\widetilde{N},\tilde{e}}$$

for scalars $\mathfrak{c}^{(\star)}_{d,\widetilde{N},\tilde{e}}(\mathcal{H})\in\mathcal{O}_{K,\chi}$. If we define $\widetilde{M}:=\widetilde{N}/\widetilde{N}_{\Sigma,\chi}\in\mathbb{N}\cap\mathbb{Z}_p^\times$, using Proposition 6.15 one finds that

$$\mathbf{g}^{(\mathrm{I})}_{\Sigma,\chi}\Big|_{k_2}W_{\widetilde{N}} \;\equiv\; \mathbf{g}^{(\mathrm{II})}_{\Sigma,\chi}\Big|_{k_2}W_{\widetilde{N}} \;\;\big(\mathrm{mod}\,p^{\nu_2}\big)$$

and moreover, if the prime $p>k_2-1$, then Corollary 6.11 implies

$$\mathcal{H}^{(\mathrm{I})}\equiv\mathcal{H}^{(\mathrm{II})}\;\;\big(\mathrm{mod}\,p^{\nu_2}\big). \tag{6.11}$$

We next apply the results in Section 6.2.1 to this pair of congruent modular forms.

Proposition 6.16. *If $\varepsilon = 0$ and $G^{(\star)} = \mathbf{g}^{(\star)}_{\Sigma,\chi} \cdot \delta^{(r)}_{k_1-k_2-2r}\big(\mathbf{E}^*_{k_1-k_2-2r,\widetilde{N}}(z,\psi_3)\big)$ as above for either $\star \in \{\mathrm{I},\mathrm{II}\}$ with the prime $p \notin \Sigma$, $p > k_2 - 1$ and $p \nmid N_1$, then*

$$\frac{\big\langle F_1^\sharp, \mathrm{Tr}^{\widetilde{N}}_{N_0}\big(\mathrm{Hol}_\infty(G^{(\star)})\big|_{k_1} W^{\varepsilon}_{\widetilde{N}}\big)\big\rangle_{N_0}}{\big\langle F_1, F_1\big\rangle_{N_1}} = \epsilon_1 \cdot p^{1-\frac{(k_1-2)(\tilde{e}-2)}{2}} \cdot \left(\frac{\widetilde{N}^{(p)}}{N_1}\right)^{\frac{k_1}{2}} \times \sum_{d \,\big|\, \frac{N_0}{N_1}} \mathfrak{c}^{(\star)}_{d,\widetilde{N},\tilde{e}}(\mathcal{H}) \cdot \frac{\big\langle F_1\big|_{k_1} V_{\frac{N_0}{N_1}}, F_1\big|_{k_1} V_d\big\rangle_{N_0}}{\big\langle F_1, F_1\big\rangle_{N_1}} \tag{6.12}$$

where $\widetilde{N} = \mathrm{lcm}\big(N_1, p^{2n_\chi}, N^{(\mathrm{I})}_\Sigma, N^{(\mathrm{II})}_\Sigma\big)$, $\widetilde{N}^{(p)} = \big|\widetilde{N}\big|_p \cdot \widetilde{N}$ and lastly $N_0 = p \cdot \widetilde{N}^{(p)}$. Moreover the congruences $\mathfrak{c}^{(\mathrm{I})}_{d,\widetilde{N},\tilde{e}}(\mathcal{H}) \equiv \mathfrak{c}^{(\mathrm{II})}_{d,\widetilde{N},\tilde{e}}(\mathcal{H}) \pmod{p^{\nu_2}}$ hold at integers $d\big|\frac{N_0}{N_1}$.

Proof. Most of these assertions follow upon applying Lemma 6.7 directly to the forms $G = \mathrm{Hol}_\infty\Big(\mathbf{g}^{(\mathrm{I})}_{\Sigma,\chi} \cdot \delta^{(r)}_{k_1-k_2-2r}(\breve{G}_3)\Big)$ and $G = \mathrm{Hol}_\infty\Big(\mathbf{g}^{(\mathrm{II})}_{\Sigma,\chi} \cdot \delta^{(r)}_{k_1-k_2-2r}(\breve{G}_3)\Big)$. The levels $\widetilde{N}$, $\widetilde{N}^{(p)}$ and N_0 are easily determined from their descriptions in Section 6.2.1. We should point out that the q-expansions of $\mathcal{H}^{(\mathrm{I})}$ and $\mathcal{H}^{(\mathrm{II})}$ take values in $\mathcal{O}_{K,\chi}$ by Propositions 6.10 and 6.15, hence so do the q-expansions of the N_0-level modular forms $\mathcal{H}^{(\mathrm{I})}\big|_{k_1} U_p^{\tilde{e}-1}$ and $\mathcal{H}^{(\mathrm{II})}\big|_{k_1} U_p^{\tilde{e}-1}$. Finally, one may combine Equation (6.11) together with the implication

$$\mathcal{H}^{(\mathrm{I})} \equiv \mathcal{H}^{(\mathrm{II})} \pmod{p^{\nu_2}} \implies \mathcal{H}^{(\mathrm{I})}\big|_{k_1} U_p^{\tilde{e}-1} \equiv \mathcal{H}^{(\mathrm{II})}\big|_{k_1} U_p^{\tilde{e}-1} \pmod{p^{\nu_2}}$$

to conclude that the F_1-isotypic parts of $\mathcal{H}^{(\mathrm{I})}\big|_{k_1} U_p^{\tilde{e}-1}$ and $\mathcal{H}^{(\mathrm{II})}\big|_{k_1} U_p^{\tilde{e}-1}$ are similarly congruent modulo $p^{\nu_2} \cdot \mathcal{O}_{K,\chi}[\![q]\!]$, whence $\mathfrak{c}^{(\mathrm{I})}_{d,\widetilde{N},\tilde{e}}(\mathcal{H}) \equiv \mathfrak{c}^{(\mathrm{II})}_{d,\widetilde{N},\tilde{e}}(\mathcal{H}) \pmod{p^{\nu_2}}$. □

6.2.5.2 The triple product case

Alternatively, suppose one is given cusp forms $\mathbf{g}^{(\mathrm{I})}, \mathbf{g}^{(\mathrm{II})}$ of weight k_2, character ψ_2, and that their respective levels are $N^{(\mathrm{I})}_{\mathbf{g}}, N^{(\mathrm{II})}_{\mathbf{g}}$. In addition, we suppose that $\mathbf{h}^{(\mathrm{I})}, \mathbf{h}^{(\mathrm{II})}$ are modular forms of weight $k_3 = k_1 - k_2 - 2r$, character $\psi_3 = \overline{\psi_1\psi_2}$, with levels $N^{(\mathrm{I})}_{\mathbf{h}}$ and $N^{(\mathrm{II})}_{\mathbf{h}}$ respectively. One further assumes:

$$a_n\big(\mathbf{g}^{(\mathrm{I})}\big) \equiv a_n\big(\mathbf{g}^{(\mathrm{II})}\big) \pmod{p^{\nu_2}} \quad \text{if } \gcd\big(n, N^{(\mathrm{I})}_{\mathbf{g}} N^{(\mathrm{II})}_{\mathbf{g}}\big) = 1, \text{ and} \tag{6.13}$$

$$a_n\big(\mathbf{h}^{(\mathrm{I})}\big) \equiv a_n\big(\mathbf{h}^{(\mathrm{II})}\big) \pmod{p^{\nu_3}} \quad \text{if } \gcd\big(n, N^{(\mathrm{I})}_{\mathbf{h}} N^{(\mathrm{II})}_{\mathbf{h}}\big) = 1. \tag{6.14}$$

We shall now choose the set of rational primes Σ to satisfy the three modified conditions: (i) $\mathrm{supp}\big(N_{\mathbf{g}}^{(\mathrm{I})}N_{\mathbf{g}}^{(\mathrm{II})}N_{\mathbf{h}}^{(\mathrm{I})}N_{\mathbf{h}}^{(\mathrm{II})}\big) - \{p\} \subset \Sigma$, (ii) $\#\Sigma < \infty$ and (iii) $p \notin \Sigma$.

Notes. (a) If we construct a 'suitably large enough' level by taking

$$\widetilde{N} \;:=\; \mathrm{lcm}\left(N_1\,,\; N_{\mathbf{g}}^{(\mathrm{I})}\,,\; N_{\mathbf{g}}^{(\mathrm{II})}\,,\; N_{\mathbf{h}}^{(\mathrm{I})}\,,\; N_{\mathbf{h}}^{(\mathrm{II})}\,,\; \prod_{l\in\Sigma} l^2\right)$$

then the Σ-depleted forms $\mathbf{g}_\Sigma^{(\mathrm{I})}, \mathbf{g}_\Sigma^{(\mathrm{II})}, \mathbf{h}_\Sigma^{(\mathrm{I})}, \mathbf{h}_\Sigma^{(\mathrm{II})}$ will each exist at this top level $\widetilde{N}$.

(b) Let $K = K\big(\mathbf{g}_\Sigma, \mathbf{h}_\Sigma\big)$ denote the number field generated by the q-coefficients of the depleted modular forms $\mathbf{g}_\Sigma^{(\mathrm{I})}, \mathbf{g}_\Sigma^{(\mathrm{II})}, \mathbf{h}_\Sigma^{(\mathrm{I})}$ and $\mathbf{h}_\Sigma^{(\mathrm{II})}$.

(c) We shall write $\mathcal{O}_K = \mathcal{O}_K\big(\mathbf{g}_\Sigma, \mathbf{h}_\Sigma\big)$ for the ring of integers of $K\big(\mathbf{g}_\Sigma, \mathbf{h}_\Sigma\big)$.

Proposition 6.17. *If $\varepsilon = 1$ and $G^{(\star)} = \mathbf{g}_\Sigma^{(\star)} \cdot \delta_{k_1-k_2-2r}^{(r)}\big(\mathbf{h}_\Sigma^{(\star)}\big)$ for $\star \in \{\mathrm{I}, \mathrm{II}\}$ with $p \notin \Sigma$, $p \nmid \frac{(k_1-2)!}{(k_1-2-r)!}$ and $p \nmid N_1$, then $G^{(\mathrm{I})}$ and $G^{(\mathrm{II})}$ belong to $\mathcal{N}_{k_1,\mathrm{pol}}^{\infty,r}\big(\widetilde{N}, \psi_1^{-1}; \mathcal{O}_K\big)$ and they both satisfy Equation (6.12), where $\mathcal{H}^{(\star)} = \mathrm{Hol}_\infty\big(G^{(\star)}\big)\Big|_{k_1} W_{\widetilde{N}}^2$ and*

$$\mathcal{H}^{(\star)}\Big|_{k_1} U_p^{\tilde{e}-1} = \sum_{d\left|\frac{N_0}{N_1}\right.} \mathfrak{c}_{d,\widetilde{N},\tilde{e}}^{(\star)}(\mathcal{H}) \cdot F_1\big|_{k_1} V_d \;+\; \mathcal{H}_{\widetilde{N},\tilde{e}}^{(\star),(\perp)},$$

with $\mathfrak{c}_{d,\widetilde{N},\tilde{e}}^{(\star)}(\mathcal{H}) \in \mathcal{O}_K\big(\mathbf{g}_\Sigma, \mathbf{h}_\Sigma\big)$. Moreover the congruences $\mathfrak{c}_{d,\widetilde{N},\tilde{e}}^{(\mathrm{I})}(\mathcal{H}) \equiv \mathfrak{c}_{d,\widetilde{N},\tilde{e}}^{(\mathrm{II})}(\mathcal{H}) \;\big(\bmod\, p^{\min\{\nu_2,\nu_3\}}\big)$ hold for $d\big|\frac{N_0}{N_1}$.

Proof. The forms above satisfy $\mathbf{h}_\Sigma^{(\star)} \in \mathcal{M}_{k_3}\big(\widetilde{N}, \psi_3; \mathcal{O}_K\big) \subset \mathcal{N}_{k_3,\mathrm{pol}}^{\infty,0}\big(\widetilde{N}, \psi_3; \mathcal{O}_K\big)$ so that $\delta_{k_1-k_2-2r}^{(r)}\big(\mathbf{h}_\Sigma^{(\star)}\big) \in \mathcal{N}_{k_1-k_2,\mathrm{pol}}^{\infty,r}\big(\widetilde{N}, \psi_3; \mathcal{O}_K\big)$ by Lemma 6.9(a); consequently

$$G^{(\star)} = \mathbf{g}_\Sigma^{(\star)} \cdot \delta_{k_1-k_2-2r}^{(r)}\big(\mathbf{h}_\Sigma^{(\star)}\big) \in \mathcal{N}_{k_1,\mathrm{pol}}^{\infty,r}\big(\widetilde{N}, \psi_2\psi_3; \mathcal{O}_K\big),$$

and combining Equations (6.13) and (6.14) with Lemma 6.9(b) implies the congruence $G^{(\mathrm{I})} \equiv G^{(\mathrm{II})} \bmod p^{\min\{\nu_2,\nu_3\}}$. From Corollary 6.13 with $G_2^{(\star)} = \mathbf{g}_\Sigma^{(\star)}$ and $G_3^{(\star)} = \mathbf{h}_\Sigma^{(\star)}$, it follows directly that

$$\mathrm{Hol}_\infty\big(G^{(\mathrm{I})}\big) \equiv \mathrm{Hol}_\infty\big(G^{(\mathrm{II})}\big) \bmod p^{\min\{\nu_2,\nu_3\}} \cdot \mathcal{O}_K[\![q]\!].$$

Next we apply Lemma 6.7 to the pair of cusp forms $G = \mathrm{Hol}_\infty\Big(G^{(\mathrm{I})}\Big)\,\big|_{k_1} W_{\widetilde{N}}$ and $G = \mathrm{Hol}_\infty\Big(G^{(\mathrm{II})}\Big)\,\big|_{k_1} W_{\widetilde{N}}$. By copying the same argument as in the previous proof, the required congruences are a consequence of the implication

$$\mathcal{H}^{(\mathrm{I})} \equiv \mathcal{H}^{(\mathrm{II})} \bmod p^{\min\{\nu_2,\nu_3\}} \implies \mathcal{H}^{(\mathrm{I})}\Big|_{k_1} U_p^{\tilde{e}-1} \equiv \mathcal{H}^{(\mathrm{II})}\Big|_{k_1} U_p^{\tilde{e}-1} \bmod p^{\min\{\nu_2,\nu_3\}}$$

and the property that taking the F_1-isotypic projection will respect congruences (because the module $\mathcal{M}_{k_1}\left(N_0, \psi_1^{-1}\right) \cap \mathcal{O}_K[\![q]\!]$ contains a basis consisting of Hecke eigenforms whose q-expansion coefficients also lie in the ring of integers $\mathcal{O}_K$). □

Chapter 7

Variation between the analytic λ-invariants

With the technical calculations in Chapter 6 complete, we now use the established formulae to study the λ-invariant for both the double and triple product p-adic L-functions. A nice feature of our inner product expression is that the special values of both types of p-adic L-function can be treated on an equal footing, using the same ideas. However let us begin by streamlining the existing notation to avoid clutter later.

Definition 7.1. *(a) For $\varepsilon \in \{0,1\}$ and an integer $r \in \{0,\dots,\lfloor k_1/2\rfloor\}$, one defines a linear functional*

$$\mathcal{L}_{F_1}^{(r,\varepsilon)} = \mathcal{L}_{F_1}^{(r,\varepsilon)}(p, N_0, N_1, \widetilde{N}) : \mathcal{N}_{k_1,\mathrm{pol}}^{\infty,r}\big(\widetilde{N}, \psi_1^{-1}\big) \longrightarrow \mathbb{C}$$

by

$$\mathcal{L}_{F_1}^{(r,\varepsilon)}(H) \;:=\; \epsilon_1^{-1} p^{\frac{(k_1-2)(\tilde{e}-2)}{2}-1} \left(\frac{\widetilde{N}^{(p)}}{N_1}\right)^{-\frac{k_1}{2}} \left(\frac{N_0}{N_1}\right)^{k_1} \times \frac{\big\langle F_1^{\sharp}, \mathrm{Tr}_{N_0}^{\widetilde{N}}\big(\mathrm{Hol}_\infty(H)\big|_{k_1} W_{\widetilde{N}}^{\varepsilon}\big)\big\rangle_{N_0}}{\big\langle F_1, F_1\big\rangle_{N_1}}$$

where $F_1\big|_{k_1} W_{N_1} = \epsilon_1 \cdot F_1^{\sharp}$, and the levels $\widetilde{N} = p^{\tilde{e}} \cdot \widetilde{N}^{(p)}$, $N_0 = p \cdot \widetilde{N}^{(p)}$ are as before.

(b) At each positive divisor d of N_0/N_1, we introduce the algebraic number

$$\begin{aligned}\mathbf{X}_d(N_0,N_1) &:= \\ &\prod_{l|N_1} l^{\mathrm{ord}_l(N_0)-\mathrm{ord}_l(N_1)} \times \prod_{l|N_0,\, l\nmid N_1} (l+1)\cdot l^{\mathrm{ord}_l(N_0)-1} \\ &\times \prod_{\substack{l|\frac{N_0}{dN_1} \\ l|N_1}} a_{l^{t_{l,d}}}(f) \times \prod_{\substack{l\|\frac{N_0}{dN_1} \\ l\nmid N_1}} \frac{a_l(F_1)}{1+\psi_1(l)\cdot l^{-1}} \\ &\times \prod_{\substack{l^2|\frac{N_0}{dN_1} \\ l\nmid N_1}} \frac{a_{l^{t_{l,d}}}(F_1) - l^{k_1-2} a_{l^{t_{l,d}-2}}(F_1)}{1+\psi_1(l)\cdot l^{-1}}\end{aligned}$$

with the exponent $t_{l,d} := \mathrm{ord}_l(N_0) - \mathrm{ord}_l(dN_1)$.

For instance, using these definitions above along with Corollary A.2, one may repackage Equation (6.12) into the more succinct form

$$\mathcal{L}_{F_1}^{(r,\varepsilon)}\big(G^{(\star)}\big) \;=\; \sum_{d\,\big|\,\frac{N_0}{N_1}} \mathfrak{c}^{(\star)}_{d,\widetilde{N},\tilde{e}}(\mathcal{H}) \cdot \mathbf{X}_d(N_0,N_1) \tag{7.1}$$

where $\mathcal{H}^{(\star)} = \mathrm{Hol}_\infty\big(G^{(\star)}\big)\Big|_{k_1} W_{\widetilde{N}}^{1+\varepsilon}$ at either choice of $\star \in \{\mathrm{I}, \mathrm{II}\}$.

The $\mathbf{X}_d(N_0,N_1)$'s each have bounded denominators, and are independent of $G^{(\star)}$. Furthermore, if $G^{(\star)} = \mathbf{g}^{(\star)}_{\Sigma,\chi} \cdot \delta^{(r)}_{k_1-k_2-2r}\big(\mathbf{E}^*_{k_1-k_2-2r,\widetilde{N}}(z,\psi_3)\big)$ or if $G^{(\star)} = \big(\mathbf{g}^{(\star)}_\Sigma \cdot \delta^{(r)}_{k_1-k_2-2r}\big(\mathbf{h}^{(\star)}_\Sigma\big)\big)\Big|_{k_1} W_{\widetilde{N}}$, corresponding to the double product and triple product cases respectively, then the scalars $\mathfrak{c}^{(\star)}_{d,\widetilde{N},\tilde{e}}(\mathcal{H})$ are algebraic integers which are congruent to each other as one switches between $\star = \mathrm{I}$ and $\star = \mathrm{II}$.

Although we shall treat the double and triple product separately, the underlying methods are basically the same. In both situations $F_1 = \mathbf{f}$ will be a weight k_1 newform of level N_1, $p \nmid N_1$ and nebentypus ψ_1, where $\mathbf{f}^\sharp = \mathbf{f}$ and $\psi_1^2 = \mathbf{1}$. In addition, it is now necessary to assume that the cusp form $\mathbf{f}$ is ordinary at p.

7.1 The double product p-adic L-function

For two eigenforms F and G of weights $k_1 > k_2$ and characters η_1, η_2, the L-function attached to $F \otimes G$ equals

$$\Psi(s, F, G) \;:=\; \frac{\Gamma(s)\Gamma(s+1-k_2)}{(2\pi)^{2s}} \times \zeta(2s+2-k_1-k_2, \eta_1\eta_2) \cdot D(s, F, G) \tag{7.2}$$

with $\mathrm{Re}(s) \gg 0$, and this admits an analytic continuation to the complex plane. We write $\Psi_\Sigma(s, F, G)$ for the L-function stripped of Euler factors at primes $l \in \Sigma$.

Throughout assume we are given newforms $\mathbf{g}^{(\mathrm{I})}, \mathbf{g}^{(\mathrm{II})}$ of weight k_2, character ψ, with conductors $N_{\mathbf{g}}^{(\mathrm{I})}, N_{\mathbf{g}}^{(\mathrm{II})}$ respectively, and which satisfy:

$$a_n\big(\mathbf{g}^{(\mathrm{I})}\big) \equiv a_n\big(\mathbf{g}^{(\mathrm{II})}\big) \pmod{p^{\nu_2}} \;\text{ for all } n \in \mathbb{N} \text{ with } \gcd\big(n, N_{\mathbf{g}}^{(\mathrm{I})} N_{\mathbf{g}}^{(\mathrm{II})}\big) = 1.$$

We again choose the set Σ so that $\mathrm{supp}\big(N_{\mathbf{g}}^{(\mathrm{I})} N_{\mathbf{g}}^{(\mathrm{II})}\big) - \{p\} \subset \Sigma$, $\#\Sigma < \infty$ and $p \notin \Sigma$.

Proposition 7.2. *If χ has conductor $p^{n_\chi} \geq \max\Big\{\big|N_{\mathbf{g}}^{(\mathrm{I})}\big|_p^{-\frac{1}{2}}, \big|N_{\mathbf{g}}^{(\mathrm{II})}\big|_p^{-\frac{1}{2}}\Big\}$, then*

$$\mathcal{L}_{\mathbf{f}}^{(r,0)}\Bigg(\mathbf{g}_{\Sigma,\chi}^{(\star)} \cdot \delta_{k_1-k_2-2r}^{(r)}\Big(\mathbf{E}^*_{k_1-k_2-2r,\widetilde{N}}(z, \overline{\psi}_1\psi\chi^2)\Big)\Bigg) = \frac{\big(\widetilde{N}^{(p)}\big)^{k_1-k_2/2-r} N_1^{-k_1/2}}{\epsilon_1 \cdot 2 \cdot (2i)^{k_1-1}} \times p^{n_\chi(2k_1-k_2-2r-2)+1} \times \frac{\Psi_\Sigma\big(k_1-1-r, \mathbf{f}, \mathbf{g}^{(\star)} \otimes \chi\big)}{(2\pi i)^{1-k_2} \cdot \big\langle \mathbf{f}, \mathbf{f}\big\rangle_{N_1}}$$

at each integer r in the range $0 \leq 2r < k_1 - k_2$, and for either choice of $\star \in \{\mathrm{I}, \mathrm{II}\}$.

Proof. Recall that $\psi_3 = \overline{\psi}_1 \cdot \psi \cdot \chi^2$ and also $\widetilde{N} = \mathrm{lcm}\big(N_1, \widetilde{N}_{\Sigma,\chi}\big) = p^{\tilde{e}} \cdot \widetilde{N}^{(p)}$. An essential starting point is the following formula [1] of Shimura [60, Theorem 2],

$$D\big(k_1-1-r, \mathbf{f}, \mathbf{g}_{\Sigma,\chi}^{(\star)}\big) \;=\; \frac{(-1)^r(4\pi)^{k_1-1} \cdot \Gamma(k_1-k_2-2r)}{\Gamma(k_1-1-r) \cdot \Gamma(k_1-k_2-r)} \times \Big\langle \mathbf{f}^\sharp,\; \mathbf{g}_{\Sigma,\chi}^{(\star)} \cdot \delta_{k_1-k_2-2r}^{(r)}\big(E^*_{k_1-k_2-2r,\widetilde{N}}(z, \psi_3)\big)\Big\rangle_{\widetilde{N}}$$

[1] His normalisation of the Petersson inner product differs from ours by $\mathbf{vol}\big(\Gamma_1(\widetilde{N})\backslash\mathfrak{h}\big)^{-1}$.

where $E^*_{k_1-k_2-2r,\widetilde{N}}(z,\eta)$ denotes the $\mathcal{C}^\infty$-modular form defined in Equation (6.2), and $D\big(s,\mathbf{f},\mathbf{g}^{(\star)}_{\Sigma,\chi}\big)$ coincides with the Σ-depleted convolution L-function

$$D_\Sigma\big(s,\mathbf{f},\mathbf{g}^{(\star)}_\chi\big) \;=\; \sum_{\substack{n\,=\,1,\\ \mathrm{supp}(n)\cap\Sigma=\emptyset}}^{\infty} a_n(\mathbf{f})a_n\big(\mathbf{g}^{(\star)}\big)\chi(n)\cdot n^{-s}, \quad \mathrm{Re}(s)\gg 0.$$

Reconciling the different normalisation of Eisenstein series in Equations (6.2-6.3), one may rephrase Shimura's identity above into an equivalent form

$$\begin{aligned}
&\Big\langle \mathbf{f}^\sharp,\ \mathbf{g}^{(\star)}_{\Sigma,\chi}\cdot\delta^{(r)}_{k_1-k_2-2r}\big(\mathbf{E}^*_{k_1-k_2-2r,\widetilde{N}}(z,\psi_3)\big)\Big\rangle_{\widetilde{N}}\\
&\qquad= \frac{(-1)^r}{(4\pi)^{k_1-1}}\cdot\frac{\widetilde{N}^{\frac{k_1-k_2-2r}{2}}}{2(2\pi i)^{k_1-k_2-2r}}\\
&\qquad\qquad\times\ \Gamma(k_1-1-r)\cdot\Gamma(k_1-k_2-r)\\
&\qquad\qquad\times\zeta_{\widetilde{N}}(k_1-k_2-2r,\psi_3)\cdot D_\Sigma\big(k_1-1-r,\mathbf{f},\mathbf{g}^{(\star)}_\chi\big)\\
&\qquad= \big(4\pi^2\big)^{k_1-1-r}\cdot\frac{(-1)^r}{(4\pi)^{k_1-1}}\cdot\frac{\widetilde{N}^{\frac{k_1-k_2-2r}{2}}}{2(2\pi i)^{k_1-k_2-2r}}\times\Psi_\Sigma\big(k_1-1-r,\mathbf{f},\mathbf{g}^{(\star)}_\chi\big).
\end{aligned}$$

In fact, the terms directly before $\Psi_\Sigma(\cdots)$ can be simplified to $(2i)^{k_2-k_1}\cdot\frac{\widetilde{N}^{\frac{k_1-k_2-2r}{2}}}{2\pi^{1-k_2}}$, which means that if $G^{(\star)}=\mathbf{g}^{(\star)}_{\Sigma,\chi}\cdot\delta^{(r)}_{k_1-k_2-2r}\big(\mathbf{E}^*_{k_1-k_2-2r,\widetilde{N}}(z,\psi_3)\big)$ then

$$\frac{\big\langle\mathbf{f}^\sharp,G^{(\star)}\big\rangle_{\widetilde{N}}}{\big\langle\mathbf{f},\mathbf{f}\big\rangle_{N_1}} \;=\; \frac{\widetilde{N}^{\frac{k_1-k_2-2r}{2}}}{2(2i)^{k_1-k_2}}\times\frac{\Psi_\Sigma\big(k_1-1-r,\mathbf{f},\mathbf{g}^{(\star)}_\chi\big)}{\pi^{1-k_2}\cdot\big\langle\mathbf{f},\mathbf{f}\big\rangle_{N_1}}\,.$$

Focussing on the left-hand side, since $G^{(\star)}\big|_{k_1}\gamma$ has moderate growth for all $\gamma\in\mathrm{SL}_2(\mathbb{Z})$ it follows from Theorem 2.19 that

$$\frac{\big\langle\mathbf{f}^\sharp,G^{(\star)}\big\rangle_{\widetilde{N}}}{\big\langle\mathbf{f},\mathbf{f}\big\rangle_{N_1}} \;=\; \frac{\big\langle\mathbf{f}^\sharp,\mathrm{Hol}_\infty\big(G^{(\star)}\big)\big\rangle_{\widetilde{N}}}{\big\langle\mathbf{f},\mathbf{f}\big\rangle_{N_1}} \;=\; \frac{\big\langle\mathbf{f}^\sharp,\mathrm{Tr}_{\widetilde{N}/N_0}\big(\mathrm{Hol}_\infty\big(G^{(\star)}\big)\big)\big\rangle_{N_0}}{\big\langle\mathbf{f},\mathbf{f}\big\rangle_{N_1}}$$

and so by Definition 7.1(a),

$$\begin{aligned}
\mathcal{L}^{(r,0)}_{\mathbf{f}}\big(G^{(\star)}\big) \;&=\; \epsilon_1^{-1}\cdot p^{\frac{(k_1-2)(\tilde{e}-2)}{2}-1}\cdot\left(\frac{\widetilde{N}^{(p)}}{N_1}\right)^{-\frac{k_1}{2}}\cdot\left(\frac{N_0}{N_1}\right)^{k_1}\times\frac{\big\langle\mathbf{f}^\sharp,G^{(\star)}\big\rangle_{\widetilde{N}}}{\big\langle\mathbf{f},\mathbf{f}\big\rangle_{N_1}}\\
&=\; \epsilon_1^{-1}\cdot p^{\frac{(k_1-2)(\tilde{e}-2)}{2}-1}\cdot\left(\frac{\widetilde{N}^{(p)}}{N_1}\right)^{-\frac{k_1}{2}}\cdot\left(\frac{N_0}{N_1}\right)^{k_1}\\
&\qquad\qquad\times\frac{\widetilde{N}^{\frac{k_1-k_2-2r}{2}}}{2(2i)^{k_1-k_2}}\cdot\frac{\Psi_\Sigma\big(k_1-1-r,\mathbf{f},\mathbf{g}^{(\star)}_\chi\big)}{\pi^{1-k_2}\cdot\big\langle\mathbf{f},\mathbf{f}\big\rangle_{N_1}}.
\end{aligned}$$

Provided that $p^{2n_\chi}\geq\max\Big\{\big|N^{(\mathrm{I})}_{\mathbf{g}}\big|_p^{-1},\big|N^{(\mathrm{II})}_{\mathbf{g}}\big|_p^{-1}\Big\}$, the p-part of the level of both cusp forms $\mathbf{g}^{(\mathrm{I})}_{\Sigma,\chi}$ and $\mathbf{g}^{(\mathrm{II})}_{\Sigma,\chi}$ equals p^{2n_χ}: thus $\tilde{e}=2n_\chi$, $\widetilde{N}=p^{2n_\chi}\cdot\widetilde{N}^{(p)}$ and

$N_0 = p \cdot \widetilde{N}^{(p)}$. Substituting these values into our formula, the result follows after a clean-up. $\square$

Let K be the number field generated by the Fourier coefficients of $\mathbf{f}, \mathbf{g}^{(\mathrm{I})}, \mathbf{g}^{(\mathrm{II})}$. Since the newform $\mathbf{f}$ is p-ordinary, we can factorise its Hecke polynomial at p into

$$X^2 - a_p(\mathbf{f})X + \psi_1(p) \cdot p^{k_1-1} \;=\; \big(X - \alpha_p\big)\big(X - \alpha'_p\big)$$

where $\left|\alpha_p\right|_p = 1$ and $\left|\alpha'_p\right|_p = p^{1-k_1} < 1$. Now applying the results of Hida and Panchishkin [37, 53], for each choice of $\star \in \{\mathrm{I}, \mathrm{II}\}$ there exists a p-adic L-function $\mathbf{L}_p\big(\mathbf{f} \otimes \mathbf{g}_\Sigma^{(\star)}\big) \in \mathcal{O}_{K,p}\big[\big[\mathbb{Z}_p^\times\big]\big][1/p]$ interpolating

$$\chi x_p^s \left(\mathbf{L}_p\big(\mathbf{f} \otimes \mathbf{g}_\Sigma^{(\star)}\big)\right) \;=\; \psi(p)^{n_\chi} \cdot \frac{\tau(\overline{\chi})^2 \cdot p^{n_\chi(k_2+2s-1)}}{(-1)^s \cdot \alpha_p^{2n_\chi}} \times \mathcal{A}(s, \overline{\chi}) \cdot \frac{\Psi\big(k_2 + s, \mathbf{f}, \mathbf{g}_{\Sigma,\chi}^{(\star)}\big)}{(2\pi i)^{1-k_2} \cdot \big\langle \mathbf{f}, \mathbf{f}\big\rangle_{N_1}}$$

at all integers $s \in \{0, \ldots, k_1 - k_2 - 1\}$. Here $\tau(\chi) = \sum_{j=1}^{p^{n_\chi}} \chi(j) e^{2\pi i j/p^{n_\chi}}$ denotes a Gauss sum for χ, and the p-Euler factor term $\mathcal{A}(s, \overline{\chi})$ is equal to 1 whenever $\chi \neq \mathbf{1}$.

Remarks. (i) If one changes variable by instead setting $s = k_1 - k_2 - r - 1$, then for $\chi \neq \mathbf{1}$ the above becomes

$$\chi x_p^s \left(\mathbf{L}_p\big(\mathbf{f} \otimes \mathbf{g}_\Sigma^{(\star)}\big)\right) \;=\; \psi(p)^{n_\chi} \cdot \frac{\tau(\overline{\chi})^2 \cdot p^{n_\chi(2k_1-k_2-2r-3)}}{(-1)^{k_1-k_2-r-1} \cdot \alpha_p^{2n_\chi}} \times \frac{\Psi\big(k_1 - 1 - r, \mathbf{f}, \mathbf{g}_{\Sigma,\chi}^{(\star)}\big)}{(2\pi i)^{1-k_2} \cdot \big\langle \mathbf{f}, \mathbf{f}\big\rangle_{N_1}}.$$

(ii) The formula in Proposition 7.2 can similarly be expressed in the form

$$\mathcal{L}_{\mathbf{f}}^{(r,0)}\big(G^{(\star)}\big) = \frac{\big(\widetilde{N}^{(p)}\big)^{k_1-k_2/2-r} N_1^{-k_1/2}}{\epsilon_1 \cdot 2 \cdot (2i)^{k_1-1}} \cdot p^{n_\chi(2k_1-k_2-2r-2)+1} \times \frac{\Psi\big(k_1 - 1 - r, \mathbf{f}, \mathbf{g}_{\Sigma,\chi}^{(\star)}\big)}{(2\pi i)^{1-k_2} \cdot \big\langle \mathbf{f}, \mathbf{f}\big\rangle_{N_1}}.$$

(iii) Consequently, $(-1)^s \cdot \chi x_p^s\Big(\mathbf{L}_p\big(\mathbf{f} \otimes \mathbf{g}_\Sigma^{(\star)}\big)\Big) = p^{-1} \cdot \Xi_{r,\chi} \times \mathcal{L}_{\mathbf{f}}^{(r,0)}\Big(G^{(\star)}\Big)$ where

$$\Xi_{r,\chi} := \left(\frac{\psi(p)}{\alpha_p^2}\right)^{n_\chi} \cdot \frac{\tau(\overline{\chi})^2}{p^{n_\chi}} \times \frac{\epsilon_1 \cdot 2 \cdot (2i)^{k_1-1}}{\big(\widetilde{N}^{(p)}\big)^{k_1-k_2/2-r} N_1^{-k_1/2}}$$

is actually a p-adic unit.

One can split the Iwasawa algebra up into $\mathbb{F}_p^\times$-eigenfactors

$$\mathcal{O}_{K,p}\big[\big[\mathbb{Z}_p^\times\big]\big] \;\cong\; \bigoplus_{j=0}^{p-2} \mathcal{O}_{K,p}\big[\big[1+p\mathbb{Z}_p\big]\big]_{(\omega^j)} \;\xrightarrow{\sim}\; \bigoplus_{j=0}^{p-2} \mathcal{O}_{K,p}\big[\big[X\big]\big]_{(\omega^j)}$$

where the last isomorphism arises by sending $1+p\in\mathbb{Z}_p^\times$ to the polynomial $X+1$. For each $j\in\mathbb{Z}$ and $\star\in\{\mathrm{I},\mathrm{II}\}$, we will write $\mathbf{L}_p\big(\mathbf{f}\otimes\mathbf{g}_\Sigma^{(\star)},\omega^j\big)$ for the image of the Hida-Panchishkin p-adic L-function inside the ω^j-eigenspace $\mathcal{O}_{K,p}\big[\big[X\big]\big][1/p]_{(\omega^j)}$. Let us also choose a local parameter, ϖ, for the discrete valuation ring $\mathcal{O}_{K,p}$.

Theorem 7.3. *At each $j\in\{0,\ldots,p-2\}$, let us define $\mu_{\mathrm{I},\mathrm{II}}^{(j)}$ to be the minimum of $\mu_\varpi\big(\mathbf{L}_p\big(\mathbf{f}\otimes\mathbf{g}_\Sigma^{(\mathrm{I})},\omega^j\big)\big)$ and $\mu_\varpi\big(\mathbf{L}_p\big(\mathbf{f}\otimes\mathbf{g}_\Sigma^{(\mathrm{II})},\omega^j\big)\big)$. If the prime $p>k_1-2$, then one obtains a congruence of Σ-imprimitive p-adic L-functions*

$$\mathbf{L}_p\big(\mathbf{f}\otimes\mathbf{g}_\Sigma^{(\mathrm{I})},\omega^j\big)\equiv\mathbf{L}_p\big(\mathbf{f}\otimes\mathbf{g}_\Sigma^{(\mathrm{II})},\omega^j\big)\bmod \varpi^{\mathfrak{e}_p\nu_2+\mu_{\mathrm{I},\mathrm{II}}^{(j)}}\cdot\mathcal{O}_{K,p}\big[\big[X\big]\big]_{(\omega^j)}$$

where the ramification index $\mathfrak{e}_p\in\mathbb{N}$ satisfies $\langle\varpi\rangle^{\mathfrak{e}_p}=p\cdot\mathcal{O}_{K,p}$.

Proof. We first pick an integer $s=k_1-k_2-r-1\geq 0$ to Tate twist by. Consider the $\mathcal{O}_{\mathbb{C}_p}$-module, $\mathbb{L}^{(j,r)}$, generated by the special values $\mathcal{L}_{\mathbf{f}}^{(r,0)}\big(G_\chi^{(\star)}\big)$ where for any non-trivial character χ conductor $p^{n_\chi}\geq\max\Big\{\big|N_{\mathbf{g}}^{(\mathrm{I})}\big|_p^{-\frac{1}{2}},\big|N_{\mathbf{g}}^{(\mathrm{II})}\big|_p^{-\frac{1}{2}}\Big\}$ such that $\chi\big|_{\mathbb{F}_p^\times}=\omega^j$ we define

$$G_\chi^{(\star)}:=\mathbf{g}_{\Sigma,\chi}^{(\star)}\cdot\delta_{k_1-k_2-2r}^{(r)}\big(\mathbf{E}^*_{k_1-k_2-2r,\widetilde{N}}(z,\overline{\psi}_1\psi\chi^2)\big)\;\in\;\mathcal{M}_{k_1}^{\infty}\big(\widetilde{N},\psi_1\big).$$

Using the identity $\chi x_p^s\Big(\mathbf{L}_p\big(\mathbf{f}\otimes\mathbf{g}_\Sigma^{(\star)}\big)\Big)=\pm p^{-1}\Xi_{r,\chi}\cdot\mathcal{L}_{\mathbf{f}}^{(r,0)}\Big(G^{(\star)}\Big)$ in Remark (iii), and also because $\big|\Xi_{r,\chi}\big|_p^{-1}=1$, it follows that $\mathbb{L}^{(j,r)}=\varpi^{\mathfrak{e}_p+\mu_{\mathrm{I},\mathrm{II}}^{(j)}}\cdot\mathcal{O}_{\mathbb{C}_p}$ where

$$\mu_{\mathrm{I},\mathrm{II}}^{(j)}=\min_{\star\in\{\mathrm{I},\mathrm{II}\}}\Big\{\mu_\varpi\big(\mathbf{L}_p\big(\mathbf{f}\otimes\mathbf{g}_\Sigma^{(\star)},\omega^j\big)\big)\Big\}\in\mathbb{Z}\cup\big\{\pm\infty\big\}.$$

From a naive perspective only three possibilities can ever happen:

(a) $\mathbb{L}^{(j,r)}=\{0\}$,

(b) $\mathbb{L}^{(j,r)}=\;\varpi^{\mathfrak{e}_p+\mu_{\mathrm{I},\mathrm{II}}^{(j)}},\cdot\mathcal{O}_{\mathbb{C}_p}$ with $\mu_{\mathrm{I},\mathrm{II}}^{(j)}\neq\pm\infty$, or

(c) $\mathbb{L}^{(j,r)}=\mathbb{C}_p$

In case (a) one has $\mathbf{L}_p\big(\mathbf{f}\otimes\mathbf{g}_\Sigma^{(\mathrm{I})},\omega^j\big) = \mathbf{L}_p\big(\mathbf{f}\otimes\mathbf{g}_\Sigma^{(\mathrm{II})},\omega^j\big) = 0$ and therefore $\mu_\varpi\big(\mathbf{L}_p(\mathbf{f}\otimes\mathbf{g}_\Sigma^{(\star)},\omega^j)\big) = +\infty$, so the congruence is vacuously true and content-free. On the other hand, if we are in case (c) then $\mu_\varpi\big(\mathbf{L}_p(\mathbf{f}\otimes\mathbf{g}_\Sigma^{(\star)},\omega^j)\big) = -\infty$, which would then imply that the ω^j-branches of $\mathbf{L}_p(\mathbf{f}\otimes\mathbf{g}_\Sigma^{(\star)})$ arise from an unbounded p-adic measure – this directly contradicts the work in [37, 53] and so never occurs!

This leaves us to deal with the interesting case (b). Recall from Equation (7.1) that the linear functional degenerates into a finite sum

$$\mathcal{L}_{F_1}^{(r,0)}\big(G_\chi^{(\star)}\big) \;=\; \sum_{d\left|\frac{N_0}{N_1}\right.} \mathfrak{c}^{(\star)}_{d,\widetilde{N},\tilde{e}}(\mathcal{H}_\chi)\cdot \mathbf{X}_d(N_0,N_1)$$

where $\mathcal{H}_\chi^{(\star)} = \mathrm{Hol}_\infty\big(G_\chi^{(\star)}\big)\big|_{k_1} W_{\widetilde{N}}$, and the $\mathbf{X}_d(N_0,N_1)$'s are independent of $G_\chi^{(\star)}$. Applying Proposition 6.16, one has congruences

$$\mathfrak{c}^{(\mathrm{I})}_{d,\widetilde{N},\tilde{e}}(\mathcal{H}_\chi) \equiv \mathfrak{c}^{(\mathrm{II})}_{d,\widetilde{N},\tilde{e}}(\mathcal{H}_\chi) \;\big(\mathrm{mod}\, p^{\nu_2}\big)$$

at every $d\big|\frac{N_0}{N_1}$ and finite order character χ on $\mathbb{Z}_p^\times$. As an immediate consequence

$$\mathcal{L}_{F_1}^{(r,0)}\big(G_\chi^{(\mathrm{I})}\big) - \mathcal{L}_{F_1}^{(r,0)}\big(G_\chi^{(\mathrm{II})}\big) \;\in\; \varpi^{\mathfrak{e}_p+\mu^{(j)}_{\mathrm{I},\mathrm{II}}}\cdot p^{\nu_2}\cdot \mathcal{O}_{\mathbb{C}_p},$$

i.e. $\chi x_p^s\Big(\mathbf{L}_p\big(\mathbf{f}\otimes\mathbf{g}_\Sigma^{(\mathrm{I})}\big) - \mathbf{L}_p\big(\mathbf{f}\otimes\mathbf{g}_\Sigma^{(\mathrm{II})}\big)\Big) \in \varpi^{\mathfrak{e}_p\nu_2+\mu^{(j)}_{\mathrm{I},\mathrm{II}}}\cdot\mathcal{O}_{\mathbb{C}_p}$ at almost all characters[2] $\chi:\mathbb{Z}_p^\times \to \overline{\mathbb{Q}}_p^\times$ such that $\chi\big|_{\mathbb{F}_p^\times} = \omega^j$. The rest now follows by p-adic continuity.

□

Let us instead consider primitive versions of these double product L-functions, namely $\mathbf{L}_p\big(\mathbf{f}\otimes\mathbf{g}^{(\mathrm{I})},\omega^j\big)$ and $\mathbf{L}_p\big(\mathbf{f}\otimes\mathbf{g}^{(\mathrm{II})},\omega^j\big)$ which belong to $\mathcal{O}_{K,p}[[X]][1/p]_{(\omega^j)}$. For either choice of $\star\in\{\mathrm{I},\mathrm{II}\}$, they are related to their Σ-imprimitive cousins via

$$\mathbf{L}_p\big(\mathbf{f}\otimes\mathbf{g}_\Sigma^{(\star)},\omega^j\big) \;=\; \mathbf{L}_p\big(\mathbf{f}\otimes\mathbf{g}^{(\star)},\omega^j\big)\times\prod_{l\in\Sigma} E_l(\mathbf{f}\otimes\mathbf{g}^{(\star)},\omega^j) \qquad (7.3)$$

where each $E_l(\mathbf{f}\otimes\mathbf{g}^{(\star)},\omega^j)\in\mathcal{O}_{K,p}[[X]]$ p-adically interpolates the Euler factor $L_l\big(\mathbf{f}\otimes\mathbf{g}^{(\star)}\otimes\chi\omega^j,s\big)$ as χ ranges over finite order characters on $1+p\mathbb{Z}_p\subset\mathbb{Z}_p^\times$.

[2]This containment is also true for the missing characters, which can be seen by exploiting the p-adic density of finite order characters χ with $\chi\big|_{\mathbb{F}_p^\times} = \omega^j$ inside the parameter space $1+p\mathbb{Z}_p$.

Definition 7.4. *At each prime $l \in \mathrm{Spec}(\mathbb{Z})$ and any branch $j \in \{0, \ldots, p-2\}$, let us define the non-negative integer $\mathbf{e}_l^{(\star)}(\omega^j)$ to be the λ-invariant of the Euler factor $E_l(\mathbf{f} \otimes \mathbf{g}^{(\star)}, \omega^j)$.*

Theorem 7.5. *If the prime $p > k_1 - 2$, then*

$$\lambda\big(\mathbf{L}_p\big(\mathbf{f} \otimes \mathbf{g}^{(\mathrm{I})}, \omega^j\big)\big) \;=\; \lambda\big(\mathbf{L}_p\big(\mathbf{f} \otimes \mathbf{g}^{(\mathrm{II})}, \omega^j\big)\big) \;+\; \sum_{l \mid N_{\mathbf{g}}^{(\mathrm{I})} N_{\mathbf{g}}^{(\mathrm{II})}} \mathbf{e}_l^{(\mathrm{II})}(\omega^j) - \mathbf{e}_l^{(\mathrm{I})}(\omega^j).$$

Proof. Firstly, we note that the Euler factors $E_l(\mathbf{f} \otimes \mathbf{g}^{(\star)}, \omega^j)$ in Equation (7.3) for primes $l \in \Sigma$ each have unit content, and therefore possess a trivial μ-invariant. If $\mu_{\mathrm{I},\mathrm{II}}^{(j)} \in \mathbb{Z} \cup \{+\infty\}$ denotes the minimum of the μ-invariants for $\mathbf{L}_p\big(\mathbf{f} \otimes \mathbf{g}^{(\mathrm{I})}, \omega^j\big)$ and $\mathbf{L}_p\big(\mathbf{f} \otimes \mathbf{g}^{(\mathrm{II})}, \omega^j\big)$, then by Theorem 7.3 one has

$$\varpi^{-\mu_{\mathrm{I},\mathrm{II}}^{(j)}} \cdot \mathbf{L}_p\big(\mathbf{f} \otimes \mathbf{g}_{\Sigma}^{(\mathrm{I})}, \omega^j\big) \equiv \varpi^{-\mu_{\mathrm{I},\mathrm{II}}^{(j)}} \cdot \mathbf{L}_p\big(\mathbf{f} \otimes \mathbf{g}_{\Sigma}^{(\mathrm{II})}, \omega^j\big) \bmod \varpi^{\mathfrak{c}_p \cdot \nu_2} \cdot \mathcal{O}_{K,p}[[X]].$$

Moreover as $\mathfrak{c}_p \cdot \nu_2 \geq 1$, we can then deduce that

$$\begin{aligned}
&\lambda\big(\mathbf{L}_p\big(\mathbf{f} \otimes \mathbf{g}_{\Sigma}^{(\mathrm{I})}, \omega^j\big)\big) \\
&\quad = \mathrm{rank}_{\mathbb{F}[\![X]\!]} \left(\mathcal{O}_{K,p}[[X]] \Big/ \big\langle \varpi, \varpi^{-\mu_{\mathrm{I},\mathrm{II}}^{(j)}} \cdot \mathbf{L}_p\big(\mathbf{f} \otimes \mathbf{g}_{\Sigma}^{(\mathrm{I})}, \omega^j\big) \big\rangle \right) \\
&\quad = \mathrm{rank}_{\mathbb{F}[\![X]\!]} \left(\mathcal{O}_{K,p}[[X]] \Big/ \big\langle \varpi, \varpi^{-\mu_{\mathrm{I},\mathrm{II}}^{(j)}} \cdot \mathbf{L}_p\big(\mathbf{f} \otimes \mathbf{g}_{\Sigma}^{(\mathrm{II})}, \omega^j\big) \big\rangle \right) \\
&\quad = \lambda\big(\mathbf{L}_p\big(\mathbf{f} \otimes \mathbf{g}_{\Sigma}^{(\mathrm{II})}, \omega^j\big)\big)
\end{aligned}$$

where $\mathbb{F} = \mathcal{O}_{K,p}/\langle \varpi \rangle$ indicates the residue field. Finally, using Equation (7.3) in tandem with the additivity of the λ-invariant, clearly one has a relation

$$\lambda\big(\mathbf{L}_p\big(\mathbf{f} \otimes \mathbf{g}_{\Sigma}^{(\star)}, \omega^j\big)\big) \;=\; \lambda\big(\mathbf{L}_p\big(\mathbf{f} \otimes \mathbf{g}^{(\star)}, \omega^j\big)\big) \;+\; \mathbf{e}_l^{(\star)}(\omega^j).$$

The result follows upon observing that $\mathbf{e}_l^{(\mathrm{I})}(\omega^j) = \mathbf{e}_l^{(\mathrm{II})}(\omega^j)$ at any prime $l \in \Sigma$ such that $l \nmid N_{\mathbf{g}}^{(\mathrm{I})} N_{\mathbf{g}}^{(\mathrm{II})}$, because here $E_l(\mathbf{f} \otimes \mathbf{g}^{(\mathrm{I})}, \omega^j) \equiv E_l(\mathbf{f} \otimes \mathbf{g}^{(\mathrm{II})}, \omega^j) \bmod \varpi^{\mathfrak{c}_p \cdot \nu_2}$.

□

7.2 The triple product p-adic L-function

At the risk of bombarding the reader with too many superscripts, suppose that we are given two primitive $\mathbb{I}_i$-adic triples

$$\big(\mathbf{F}_1, \mathbf{G}^{(2),(\mathrm{I})}, \mathbf{G}^{(3),(\mathrm{I})}\big) \quad \text{and} \quad \big(\mathbf{F}_1, \mathbf{G}^{(2),(\mathrm{II})}, \mathbf{G}^{(3),(\mathrm{II})}\big)$$

where $\mathbf{F}_1$ has level $N_1 = C_1$, and the families $\mathbf{G}^{(i),(\star)}$ have level equal to $C_i^{(\star)}$. Assume there exists a one-dimensional subset (i.e. line) $\mathcal{V} \subset \mathfrak{X}_1 \times \mathfrak{X}_2 \times \mathfrak{X}_3$ in the parameter space, such that for all unbalanced points $\underline{\mathcal{Q}} \in \mathcal{V} \cap \mathfrak{X}_{\mathcal{R}}^{\mathbf{F}_1}$:

$$\underline{\mathcal{Q}}\left(a_n\big(\mathbf{G}^{(2),(\mathrm{I})}\big)\right) \equiv \underline{\mathcal{Q}}\left(a_n\big(\mathbf{G}^{(2),(\mathrm{II})}\big)\right) \pmod{p^{\nu_2}} \text{ if } \gcd\left(n, C_2^{(\mathrm{I})}C_2^{(\mathrm{II})}\right) = 1, \quad (7.4)$$

$$\underline{\mathcal{Q}}\left(a_n\big(\mathbf{G}^{(3),(\mathrm{I})}\big)\right) \equiv \underline{\mathcal{Q}}\left(a_n\big(\mathbf{G}^{(3),(\mathrm{II})}\big)\right) \pmod{p^{\nu_3}} \text{ if } \gcd\left(n, C_3^{(\mathrm{I})}C_3^{(\mathrm{II})}\right) = 1. \quad (7.5)$$

We also suppose the image of the specialisations $\phi_{\mathcal{V}} : \mathcal{R} \to \bigoplus_{\underline{\mathcal{Q}} \in \mathcal{V} \cap \mathfrak{X}_{\mathcal{R}}^{\mathbf{F}_1}} \underline{\mathcal{Q}}(\mathcal{R})$ glues into a one-parameter algebra, $\mathbb{I}^{\mathcal{V}} \cong \overline{\phi_{\mathcal{V}}(\mathcal{R})}$, of finite-type over Λ_{wt}.

Let us write $\mu_{\mathrm{wt}}^{(\mathcal{V})} \in \mathbb{Z} \cup \{-\infty, +\infty\}$ for the minimum of the (weight) μ-invariants associated to $\phi_{\mathcal{V}}\left(\mathbf{L}_p\big(\mathbf{F}_1, \mathbf{G}^{(2),(\star)}, \mathbf{G}^{(3),(\star)}\big)\right) \in \mathbb{I}^{\mathcal{V}}$ over both choices of $\star \in \{\mathrm{I}, \mathrm{II}\}$. The theorem immediately below is the primary technical result in this section.

Theorem 7.6. *If both triples* $\left(\mathbf{F}_1, \mathbf{G}^{(2),(\mathrm{I})}, \mathbf{G}^{(3),(\mathrm{I})}\right)$ *and* $\left(\mathbf{F}_1, \mathbf{G}^{(2),(\mathrm{II})}, \mathbf{G}^{(3),(\mathrm{II})}\right)$ *satisfy Hypotheses* **(T1)**–**(T4)***, the congruences (7.4)-(7.5) hold for* $\nu_2, \nu_3 \geq 1$*, the points* $\underline{\mathcal{Q}} \in \mathfrak{X}_{\mathcal{R}}^{\mathbf{F}_1}$ *with* $p \nmid \frac{(k_1-2)!}{(k_1-2-r)!}$ *are dense in* $\mathrm{Spec}(\mathbb{I}^{\mathcal{V}})$*, and if* $\psi_1^2 = \mathbf{1}$*, then*

$$\phi_{\mathcal{V}}\left(\mathbf{L}_{p,\Sigma}\big(\mathbf{F}_1, \mathbf{G}^{(2),(\mathrm{I})}, \mathbf{G}^{(3),(\mathrm{I})}\big)\right) \equiv \phi_{\mathcal{V}}\left(\mathbf{L}_{p,\Sigma}\big(\mathbf{F}_1, \mathbf{G}^{(2),(\mathrm{II})}, \mathbf{G}^{(3),(\mathrm{II})}\big)\right)$$

modulo $p^{\mu_{\mathrm{wt}}^{(\mathcal{V})} + \min\{\nu_2, \nu_3\}} \cdot \mathbb{I}^{\mathcal{V}}$*, where the finite set* $\Sigma := \mathrm{supp}\big(C_2^{(\mathrm{I})}C_2^{(\mathrm{II})}C_3^{(\mathrm{I})}C_3^{(\mathrm{II})}\big)$.

In particular, this is equivalent to Theorem 6.6(i) stated in the Introduction. Moreover let us recall that the Σ-imprimitive p-adic L-function factorises into

$$\mathbf{L}_{p,\Sigma}\big(\mathbf{F}_1, \mathbf{G}^{(2),(\star)}, \mathbf{G}^{(3),(\star)}\big) = \mathbf{L}_p\big(\mathbf{F}_1, \mathbf{G}^{(2),(\star)}, \mathbf{G}^{(3),(\star)}\big) \times \prod_{l \in \Sigma} E_l^{(\star)}\big(\mathbf{F}_1, \mathbf{G}^{(2)}, \mathbf{G}^{(3)}\big)$$

where $E_l^{(\star)}(-)$ interpolates $L_l\big(\mathbf{F}_1(m) \otimes \mathbf{G}^{(2),(\star)}(m) \otimes \mathbf{G}^{(3),(\star)}(m) \otimes \chi_{\underline{\mathcal{Q}}}, \frac{w}{2}\big)$ on $\mathfrak{X}_{\mathcal{R}}^{\mathbf{F}_1}$. Applying an identical argument to that used in the proof of Theorem

7.5,

$$\begin{aligned}
&\lambda^{\mathrm{wt}} \circ \phi_{\mathcal{V}} \left(\mathbf{L}_p\big(\mathbf{F}_1, \mathbf{G}^{(2),(\mathrm{I})}, \mathbf{G}^{(3),(\mathrm{I})}\big)\right) \\
&\quad + \sum_{l\in\Sigma} \lambda^{\mathrm{wt}} \circ \phi_{\mathcal{V}} \left(E_l^{(\mathrm{I})}\big(\mathbf{F}_1, \mathbf{G}^{(2)}, \mathbf{G}^{(3)}\big)\right) \\
&\qquad\qquad = \lambda^{\mathrm{wt}} \circ \phi_{\mathcal{V}} \left(\mathbf{L}_{p,\Sigma}\big(\mathbf{F}_1, \mathbf{G}^{(2),(\mathrm{I})}, \mathbf{G}^{(3),(\mathrm{I})}\big)\right) \\
&\qquad\qquad \overset{\text{by } 7.5}{=} \lambda^{\mathrm{wt}} \circ \phi_{\mathcal{V}} \left(\mathbf{L}_{p,\Sigma}\big(\mathbf{F}_1, \mathbf{G}^{(2),(\mathrm{II})}, \mathbf{G}^{(3),(\mathrm{II})}\big)\right) \\
&\qquad\qquad = \lambda^{\mathrm{wt}} \circ \phi_{\mathcal{V}} \left(\mathbf{L}_p\big(\mathbf{F}_1, \mathbf{G}^{(2),(\mathrm{II})}, \mathbf{G}^{(3),(\mathrm{II})}\big)\right) \\
&\qquad\qquad\qquad + \sum_{l\in\Sigma} \lambda^{\mathrm{wt}} \circ \phi_{\mathcal{V}} \left(E_l^{(\mathrm{II})}\big(\mathbf{F}_1, \mathbf{G}^{(2)}, \mathbf{G}^{(3)}\big)\right)
\end{aligned}$$

and Theorem 6.6(ii) now follows as an immediate corollary.

Remarks. The strategy we adopt to establish Theorem 7.6 has three steps:

(1) At each point $\underline{\mathcal{Q}} \in \mathfrak{X}_{\mathcal{R}}^{\mathbf{F}_1}$ and $\star \in \{\mathrm{I}, \mathrm{II}\}$, we will express the special value $\underline{\mathcal{Q}}\Big(L^{\mathbf{F}_1}_{\mathbf{G}_\Sigma^{(2),(\star)}, \mathbf{G}_\Sigma^{(3),(\star)}}\Big)$ in terms of $\underline{\mathcal{Q}}\left(a_1\Big(\eta_{\mathbf{F}_1} \cdot 1_{\mathbf{F}_1} \cdot \mathrm{Tr}_{\widetilde{N}/C_1}\big(\mathbf{H}_\Sigma^{\mathrm{aux},(\star)}\big)\Big)\right)$. Note that by construction, both of the Σ-depleted families

$$\mathbf{H}_\Sigma^{\mathrm{aux},(\star)} \in \mathcal{S}^{\mathrm{ord}}\big(\widetilde{N}, \psi_{1,(p)}\overline{\psi}_1^{(p)}; \mathbb{I}_1\big) \otimes_{\mathbb{I}_1} \mathcal{R}$$

exist at the top-most level

$$\widetilde{N} := \mathrm{lcm}\left(C_1 C_2^{(\mathrm{I})} C_3^{(\mathrm{I})}, C_1 C_2^{(\mathrm{II})} C_3^{(\mathrm{II})}, \prod_{l\in\Sigma} l^2 \right).$$

(2) By replacing the original triple $\big(\mathbf{F}_1, \mathbf{G}_\Sigma^{(2),(\star)}, \mathbf{G}_\Sigma^{(3),(\star)}\big)$ with the twisted triple

$$\Big(\mathbf{F}_1 \otimes \big(\omega^{-k^{(1)}(m)}\epsilon_m^{(1)}\big)^{-1/2}, \mathbf{G}_\Sigma^{(2),(\star)} \otimes \big(\omega^{-k^{(1)}(m)}\epsilon_m^{(1)}\big)^{1/2}, \mathbf{G}_\Sigma^{(3),(\star)}\Big),$$

we relate $\underline{\mathcal{Q}}\left(a_1\Big(\eta_{\mathbf{F}_1} \cdot 1_{\mathbf{F}_1} \cdot \mathrm{Tr}_{\widetilde{N}/C_1}\big(\mathbf{H}_\Sigma^{\mathrm{aux},(\star)}\big)\Big)\right)$ to the special value of our functional $\mathcal{L}_{F_1}^{(r,1)}\Big(\underline{\mathcal{Q}}\big(\mathbf{G}_\Sigma^{(2),(\star)}\big) \cdot \delta_{k_3}^{(r)}\Big(\underline{\mathcal{Q}}\big(\mathbf{G}_\Sigma^{(3),(\star)}\big)\Big)\Big|\mathbf{??}\Big)$ with $F_1^\alpha = \underline{\mathcal{Q}}(\mathbf{F}_1) \otimes \big(\omega^{-k^{(1)}(m)}\epsilon_m^{(1)}\big)^{-1/2}$, $\underline{k} = (k_1, k_2, k_3)$, $r = (k_1 - k_2 - k_3)/2$, and '**??**' a combination of Hecke operators.

(3) Finally, upon exploiting the congruence preserving properties of the linear functionals $\mathcal{L}_{F_1}^{(r,1)}\big(-\,\big|\mathbf{??}\big)$ and the Zariski density of $\mathcal{V} \cap \mathfrak{X}_{\mathcal{R}}^{\mathbf{F}_1}$ inside

of $\mathrm{Spec}(\mathbb{I}^{\mathcal{V}})$, the mod $p^{\min\{\nu_2,\nu_3\}}$-congruences between $\underline{\mathcal{Q}}\big(\mathbf{H}_{\Sigma}^{\mathrm{aux},(\mathrm{I})}\big)$ and $\underline{\mathcal{Q}}\big(\mathbf{H}_{\Sigma}^{\mathrm{aux},(\mathrm{II})}\big)$ will produce mod $p^{\mu_{\mathrm{wt}}^{(\mathcal{V})}+\min\{\nu_2,\nu_3\}}$-congruences between the respective triple product L-values.

Step 1: Expressing the special value $\underline{\mathcal{Q}}\Big(L^{\mathbf{F}_1}_{\mathbf{G}_{\Sigma}^{(2),(\star)},\mathbf{G}_{\Sigma}^{(3),(\star)}}\Big)$ in terms of $\underline{\mathcal{Q}}\left(a_1\Big(\eta_{\mathbf{F}_1}\cdot 1_{\mathbf{F}_1}\cdot \mathrm{Tr}_{\widetilde{N}/C_1}\big(\mathbf{H}_{\Sigma}^{\mathrm{aux},(\star)}\big)\Big)\right)$

Let us begin by reviewing the important properties of $\mathbf{H}^{\mathrm{aux},(\star)}$. In fact this family is obtained from a secondary $\mathcal{R}$-adic family, $\mathbf{H}^{\mathrm{ord},(\star)}$, through

$$\mathbf{H}^{\mathrm{aux},(\star)} = \sum_{I\subset \Sigma_{1,0}^{\mathrm{IIb}}} (-1)^{\#I}\cdot \frac{\psi_{1,(p)(n_I/d_1)}\langle n_I/d_1\rangle_{\mathbb{I}_1} d_1}{\beta_I(\mathbf{F}_1)\cdot n_I}\circ \mathbf{H}^{\mathrm{ord},(\star)}\Big| U_{d_1/n_I}$$

where definitions of the sets $I, \Sigma_{1,0}^{\mathrm{IIb}}$ and the positive integers n_I, d_1 can be found in [28, Sect 4]. Each $\beta_I(\mathbf{F}_1)\in \mathbb{I}_1^{\times}$ is a distinguished root of the polynomial $X^2 - a_l(\mathbf{F}_1)X + \psi_1(l)\cdot l^{-1}\langle l\rangle_{\mathbb{I}_1}$ at the primes $l\big|C_1C_2^{(\star)}C_3^{(\star)}$, in which case the denominator $\beta_I(\mathbf{F}_1)\cdot n_I$ must be a unit.

Definition 7.7. *The operator* $\Upsilon^{\mathrm{aux}}_{N,\mathbf{F}_1}\in \mathrm{End}_{\mathbb{I}_1}\big(\mathcal{S}^{\mathrm{ord}}\big(N, \psi_{1,(p)}\overline{\psi}_1^{(p)};\mathbb{I}_1\big)\otimes_{\mathbb{I}_1}\mathcal{R}\big)$ *is obtained via the formula*

$$\mathbf{H}\,\Big|\,\Upsilon^{\mathrm{aux}}_{N,\mathbf{F}_1} := \sum_{I\subset \Sigma_{1,0}^{\mathrm{IIb}}} (-1)^{\#I}\cdot \frac{\psi_{1,(p)}(n_I/d_1)\langle n_I/d_1\rangle_{\mathbb{I}_1} d_1}{\beta_I(\mathbf{F}_1)\cdot n_I}\circ \mathbf{H}\Big| U_{d_1/n_I}.$$

If we instead deplete our families by omitting the q^n-coefficients involving those integers n such that $\mathrm{supp}(n)\cap\Sigma\neq\emptyset$, then analogously $\mathbf{H}_{\Sigma}^{\mathrm{aux},(\star)} = \mathbf{H}_{\Sigma}^{\mathrm{ord},(\star)}\big|\,\Upsilon^{\mathrm{aux}}_{\widetilde{N},\mathbf{F}_1}$. Now by its very definition,

$$L^{\mathbf{F}_1}_{\mathbf{G}_{\Sigma}^{(2),(\star)},\mathbf{G}_{\Sigma}^{(3),(\star)}} := a_1\Big(\eta_{\mathbf{F}_1}\cdot 1_{\mathbf{F}_1}\cdot \mathrm{Tr}_{\widetilde{N}/C_1}\big(\mathbf{H}_{\Sigma}^{\mathrm{aux},(\star)}\big)\Big)$$

(e.g. see [28, §4.2.5]) where $\eta_{\mathbf{F}_1}\in\mathbb{I}_1$ generates the annihilator of the congruence module attached to $\mathbf{F}_1$, while $1_{\mathbf{F}_1}\in \mathbb{T}^{\mathrm{ord}}\big(C_1,\psi_1;\mathbb{I}_1\big)_{\mathbf{m}_{\mathbf{F}_1}}\otimes_{\mathbb{I}_1}\mathrm{Frac}(\mathbb{I}_1)$ is the idempotent element[3] which cuts the $\mathbf{F}_1$-isotypic part out from $\mathcal{S}^{\mathrm{ord}}\big(C_1,\psi_{1,(p)}\overline{\psi}_1^{(p)};\mathbb{I}_1\big)$.

[3]Hsieh and Fukunaga consider $\eta_{\check{\mathbf{F}}_1}$ and $1_{\check{\mathbf{F}}_1}$ where $\check{\mathbf{F}}_1 := \mathbf{F}_1\Big|\big[\overline{\psi_1^{(p)}}\big]$; however our condition $\psi_1^2=\mathbf{1}$ implies $\mathbf{F}_1$ and $\check{\mathbf{F}}_1$ share the same character, so we suppress notation and ignore this switch.

Therefore at every $\underline{\mathcal{Q}} \in \mathfrak{X}_{\mathcal{R}}^{\mathbf{F}_1}$,

$$\underline{\mathcal{Q}}\Big(L^{\mathbf{F}_1}_{\mathbf{G}_\Sigma^{(2),(\star)},\mathbf{G}_\Sigma^{(3),(\star)}}\Big) \\ = \mathcal{Q}^{(1)}_{m_1}(\eta_{\mathbf{F}_1}) \times \underline{\mathcal{Q}}\left(a_1\left(1_{\mathbf{F}_1}\cdot \mathrm{Tr}_{\widetilde{N}/C_1}\Big(\mathbf{H}_\Sigma^{\mathrm{ord},(\star)}\Big|\ \Upsilon^{\mathrm{aux}}_{\widetilde{N},\mathbf{F}_1}\Big)\right)\right) \tag{7.6}$$

and the next stage is to relate the right-hand side of this to our functional.

Step 2: Relating $\underline{\mathcal{Q}}\left(a_1\left(\eta_{\mathbf{F}_1}\cdot 1_{\mathbf{F}_1}\cdot \mathrm{Tr}_{\widetilde{N}/C_1}\big(\mathbf{H}_\Sigma^{\mathrm{aux},(\star)}\big)\right)\right)$ to $\mathcal{L}_{F_1}^{(r,1)}$

Before we can proceed further, a word of caution: for a fixed unbalanced point $\underline{\mathcal{Q}} \in \mathfrak{X}_{\mathcal{R}}^{\mathbf{F}_1}$, the specialisation $\underline{\mathcal{Q}}(\mathbf{F}_1) = \mathcal{Q}^{(1)}_{m_1}(\mathbf{F}_1)$ has the character $\psi_1\omega^{-k^{(1)}(m)}\epsilon_m^{(1)}$, which in general is *not* quadratic. Consequently the theory we developed in Section 6.2 cannot be directly applied to the classical eigenform $\underline{\mathcal{Q}}(\mathbf{F}_1)$.

To salvage the argument, we replace the triple $\big(\mathbf{F}_1, \mathbf{G}_\Sigma^{(2),(\star)}, \mathbf{G}_\Sigma^{(3),(\star)}\big)$ with its modified version $\Big(\mathbf{F}_1 \otimes \big(\omega^{-k^{(1)}(m)}\epsilon_m^{(1)}\big)^{-1/2}, \mathbf{G}_\Sigma^{(2),(\star)} \otimes \big(\omega^{-k^{(1)}(m)}\epsilon_m^{(1)}\big)^{1/2}, \mathbf{G}_\Sigma^{(3),(\star)}\Big)$, which works fine for **even** $k^{(1)}(m)$. If the original triple satisfies **(T1)**–**(T4)**, it is easy to check the modified version does too. Furthermore, it follows readily that

$$F_1^\alpha \ := \ \underline{\mathcal{Q}}\Big(\mathbf{F}_1 \otimes \big(\omega^{-k^{(1)}(m)}\epsilon_m^{(1)}\big)^{-1/2}\Big) \ \in \ \mathcal{S}_{k^{(1)}(m)}\big(pC_1, \psi_1; \mathcal{O}_{K,\epsilon_m^{(1)}}\big)$$

must be an ordinary p-stabilised newform. If $k^{(1)}(m) > 2$ then we can assume it is principal series at p, in which case $F_1^\alpha(z) = F_1(z) - \psi_1(p)p^{k^{(1)}(m)-1}\alpha^{-1}\cdot F_1(pz)$ where the underlying newform $F_1 \in \mathcal{S}_{k^{(1)}(m)}\big(C_1, \psi_1\big)$ is exactly as in Section 6.2.

Remarks. (a) If $k^{(1)}(m) = 2$ and F_1^α is Steinberg at p, then $F_1^\alpha = F_1$ is already a newform of level pC_1, and we cannot apply the calculations in Section 6.2 to it.

(b) Replacing $\big(\mathbf{F}_1, \mathbf{G}_\Sigma^{(2),(\star)}, \mathbf{G}_\Sigma^{(3),(\star)}\big)$ by the modified (twisted) triple above has no effect on the triple product L-function as the Galois representation is unchanged, however $L^{\mathbf{F}_1}_{\mathbf{G}_\Sigma^{(2),(\star)},\mathbf{G}_\Sigma^{(3),(\star)}}$ is essentially a square-root so it might flip its sign around.

By the previous discussion, after first modifying $\big(\mathbf{F}_1, \mathbf{G}_\Sigma^{(2),(\star)}, \mathbf{G}_\Sigma^{(3),(\star)}\big)$ one may then assume $F_1^\alpha = \underline{\mathcal{Q}}\big(\mathbf{F}_1\big)$ has exact level pC_1 and character ψ_1, such that $\psi_1^2 = \mathbf{1}$. To simplify the notation suppose that we have fixed a point $\underline{\mathcal{Q}} \in \mathfrak{X}_{\mathcal{R}}^{\mathbf{F}_1}$, and define $(k_1, k_2, k_3) = \big(k^{(1)}(m), k^{(2)}(m), k^{(3)}(m)\big)$, $N_1 = C_1$, and $N_i = p^{e^{(i)}(m)} C_i$ for $i = 2, 3$. We shall also require the depleted Hecke eigenforms

$$\mathbf{g}_\Sigma^{(\star)} := \underline{\mathcal{Q}}_{m_2}^{(2)}\big(\mathbf{G}_\Sigma^{(2),(\star)}\big) \quad \text{and} \quad \mathbf{h}_\Sigma^{(\star)} := \underline{\mathcal{Q}}_{m_3}^{(3)}\big(\mathbf{G}_\Sigma^{(3),(\star)}\big)\Big|\, \Theta_{\underline{\mathcal{Q}}}$$

in the context of Section 6.2.5, where $\Theta_{\underline{\mathcal{Q}}} = \psi_{1,(p)} \cdot \omega^{-(k_1-k_2-k_3)/2} \cdot \Big(\epsilon_m^{(1)} \overline{\epsilon_m^{(2)} \epsilon_m^{(3)}}\Big)^{1/2}$ and the twisting operation '$-\,\big|\,\Theta_{\underline{\mathcal{Q}}}$' sends $\sum_{n=1}^\infty c_n \cdot q^n \mapsto \sum_{n=1}^\infty c_n \Theta_{\underline{\mathcal{Q}}}(n) \cdot q^n$.

Lemma 7.8. *If $\underline{\mathcal{Q}}$ is unbalanced of weight (k_1, k_2, k_3) and $k_1 \in 2 \cdot \mathbb{Z}_{\geq 2}$, then*

$$\underline{\mathcal{Q}}\Big(a_1\big(1_{\mathbf{F}_1} \cdot \mathrm{Tr}_{\widetilde{N}/N_1}\big(\mathbf{H}_\Sigma^{\mathrm{ord},(\star)}\big)\big)\Big) = \mathfrak{u}_{\underline{\mathcal{Q}}} \cdot \mathcal{L}_{F_1}^{(r,1)}\left(\mathbf{g}_\Sigma^{(\star)} \cdot \delta_{k_3}^{(r)}\big(\mathbf{h}_\Sigma^{(\star)}\big)\Big|_{k_1}\left(\frac{1}{p} \cdot \mathrm{id} - \frac{\psi_1(p)}{p^2 \alpha} \cdot U_p^*\right)\right)$$

with $\mathfrak{u}_{\underline{\mathcal{Q}}} \in \mathcal{O}_{\mathbb{C}_p}^\times$ independent of $\star \in \{\mathrm{I}, \mathrm{II}\}$, and U_p^* the adjoint of U_p at level $\widetilde{N}$.

Proof. We start by using a convenient formula of Hida in [37, Lemma 9.1], which implies that the specialised coefficient

$$\underline{\mathcal{Q}}\Big(a_1\big(1_{\mathbf{F}_1} \cdot \mathrm{Tr}_{\widetilde{N}/N_1}\big(\mathbf{H}_\Sigma^{\mathrm{ord},(\star)}\big)\big)\Big) = \frac{\Big\langle \underline{\mathcal{Q}}(\mathbf{F}_1)^\sharp, e^{\mathrm{ord}} \cdot \underline{\mathcal{Q}}\big(\mathbf{H}_\Sigma^{\mathrm{ord},(\star)}\big)\big|_{k_1} W_{\widetilde{N}} \Big\rangle_{\widetilde{N}}}{\big\langle \underline{\mathcal{Q}}(\mathbf{F}_1)^\sharp, \underline{\mathcal{Q}}(\mathbf{F}_1)\big|_{k_1} W_{\widetilde{N}} \big\rangle_{\widetilde{N}}}.$$

Here the idempotent $e^{\mathrm{ord}} = \lim_{n\to\infty} U_p^{n!}$ and $\underline{\mathcal{Q}}(\mathbf{F}_1) = F_1^\alpha$ as before, while from [28, Lemma 4.2.3] we know that $\underline{\mathcal{Q}}\big(\mathbf{H}_\Sigma^{\mathrm{ord},(\star)}\big)$ coincides with

$$e^{\mathrm{ord}} \cdot \mathrm{Hol}_\infty\left(\underline{\mathcal{Q}}_{m_2}^{(2)}\big(\mathbf{G}_\Sigma^{(2),(\star)}\big) \cdot \delta_{k_3}^{(r_{\underline{\mathcal{Q}}})} \underline{\mathcal{Q}}_{m_3}^{(3)}\big(\mathbf{G}_\Sigma^{(3),(\star)}\big)\Big|\, \Theta_{\underline{\mathcal{Q}}}\right) = e^{\mathrm{ord}} \cdot \mathrm{Hol}_\infty\Big(\mathbf{g}_\Sigma^{(\star)} \cdot \delta_{k_3}^{(r)}\big(\mathbf{h}_\Sigma^{(\star)}\big)\Big)$$

with $r = r_{\underline{\mathcal{Q}}} = (k_1 - k_2 - k_3)/2$.

As an immediate consequence, one deduces that

$$\underline{\mathcal{Q}}\Big(a_1\big(1_{\mathbf{F}_1} \cdot \mathrm{Tr}_{\widetilde{N}/N_1}\big(\mathbf{H}_\Sigma^{\mathrm{ord},(\star)}\big)\big)\Big) = \frac{\Big\langle (F_1^\alpha)^\sharp, e^{\mathrm{ord}} \cdot \mathrm{Hol}_\infty\Big(\mathbf{g}_\Sigma^{(\star)} \cdot \delta_{k_3}^{(r)}\big(\mathbf{h}_\Sigma^{(\star)}\big)\Big)\Big|_{k_1} W_{\widetilde{N}} \Big\rangle_{\widetilde{N}}}{\big\langle (F_1^\alpha)^\sharp, F_1^\alpha\big|_{k_1} W_{\widetilde{N}} \big\rangle_{\widetilde{N}}}.$$

To deal with the denominator first, applying [37, Lemma 5.3(vi)] it can be shown

$$\left\langle (F_1^\alpha)^\sharp, F_1^\alpha\big|_{k_1} W_{\widetilde{N}}\right\rangle_{\widetilde{N}} = (-1)^{k_1}\left\langle (F_1^\alpha)^\sharp\big|_{k_1} W_{\widetilde{N}}, F_1^\alpha\right\rangle_{\widetilde{N}}$$
$$= (-1)^{k_1} p^{\left(\frac{2-k_1}{2}\right)\tilde{e}} \mathfrak{u}_\dagger \cdot \left\langle F_1, F_1\right\rangle_{N_1}$$

where the term $\mathfrak{u}_\dagger$ is composed of Euler factors/Gauss sums[4], and is a p-adic unit.

To study the numerator term, if we write '$\mathbf{gh}$' as shorthand for $\mathbf{g}_\Sigma^{(\star)} \cdot \delta_{k_3}^{(r)}\left(\mathbf{h}_\Sigma^{(\star)}\right)$ then because the p-stabilised newform F_1^α is p-ordinary,

$$\begin{aligned}
&\left\langle (F_1^\alpha)^\sharp, e^{\mathrm{ord}} \cdot \mathrm{Hol}_\infty(\mathbf{gh})\big|_{k_1} W_{\widetilde{N}}\right\rangle_{\widetilde{N}} \\
&\quad = \left\langle (F_1^\alpha)^\sharp,\ \mathrm{Hol}_\infty(\mathbf{gh})\big|_{k_1} W_{\widetilde{N}}\right\rangle_{\widetilde{N}} \\
&\quad \overset{\text{by } 2.19}{=} \left\langle (F_1^\alpha)^\sharp, \mathbf{gh}\big|_{k_1} W_{\widetilde{N}}\right\rangle_{\widetilde{N}} \\
&\quad = \left\langle F_1, \mathbf{gh}\big|_{k_1} W_{\widetilde{N}}\right\rangle_{\widetilde{N}} - \frac{\psi_1(p)p^{k_1-1}}{\alpha}\left\langle F_1|_{k_1}V_p, \mathbf{gh}\big|_{k_1} W_{\widetilde{N}}\right\rangle_{\widetilde{N}}
\end{aligned}$$

and the last equality follows since $(F_1^\alpha)^\sharp(q) = F_1(q) - \overline{\frac{\psi_1(p)p^{k_1-1}}{\alpha}} \cdot F_1(q^p)$ if $k_1 > 2$. Now $\left\langle F_1|_{k_1}V_p, \mathbf{gh}\big|_{k_1} W_{\widetilde{N}}\right\rangle_{\widetilde{N}} = p^{-k_1}\left\langle F_1, \mathbf{gh}\big|_{k_1} W_{\widetilde{N}}\circ U_p\right\rangle_{\widetilde{N}}$ while $W_{\widetilde{N}}\circ U_p = U_p^* \circ W_{\widetilde{N}}$, in which case

$$\left\langle (F_1^\alpha)^\sharp, e^{\mathrm{ord}} \cdot \mathrm{Hol}_\infty(\mathbf{gh})\big|_{k_1} W_{\widetilde{N}}\right\rangle_{\widetilde{N}}$$
$$= \left\langle F_1, \mathbf{gh}\Big|_{k_1}\left(\mathrm{id} - \frac{\psi_1(p)}{p\alpha}\cdot U_p^*\right)\circ W_{\widetilde{N}}\right\rangle_{\widetilde{N}}.$$

Therefore, combining together the numerator and denominator calculations:

$$\underline{\mathcal{Q}}\Big(a_1\big(1_{\mathbf{F}_1}\cdot \mathrm{Tr}_{\widetilde{N}/N_1}\big(\mathbf{H}_\Sigma^{\mathrm{ord},(\star)}\big)\big)\Big)$$
$$= \frac{p^{\left(\frac{k_1-2}{2}\right)\tilde{e}}}{(-1)^{k_1}\mathfrak{u}_\dagger} \cdot \frac{\left\langle F_1, \mathbf{gh}\Big|_{k_1}\left(\mathrm{id} - \frac{\psi_1(p)}{p\alpha}\cdot U_p^*\right)\circ W_{\widetilde{N}}\right\rangle_{\widetilde{N}}}{\left\langle F_1, F_1\right\rangle_{N_1}}.$$

On the other hand, carefully rearranging the factors in Definition 7.1(a) one

[4]In fact, the term $\mathfrak{u}_\dagger = \eta(p)^{\tilde{e}} \cdot \psi_\infty(-1)\cdot W'(F_1^\alpha)\cdot S(P)\cdot \prod_{\mathfrak{q}\in\Sigma_1}\tau(\eta'^{-1}\psi'^{-1})\cdot\prod_{\mathfrak{v}\in\Sigma}\frac{\eta\eta'(d_{\mathfrak{v}})}{|\eta\eta'(d_{\mathfrak{v}})|}$ in the notation of [37, Section 5]; one then carefully checks each individual term is a unit of $\mathcal{O}_{\mathbb{C}_p}$.

finds

$$\mathcal{L}_{F_1}^{(r,1)}\left(\mathbf{gh}\Big|_{k_1}\left(\mathrm{id}-\frac{\psi_1(p)}{p\alpha}\cdot U_p^*\right)\right) \;=\; \epsilon_1^{-1}\cdot p^{1+\left(\frac{k_1-2}{2}\right)\tilde{e}}\cdot\left(\frac{N_0^{(p)}}{N_1}\right)^{\frac{k_1}{2}}$$
$$\times\;\frac{\left\langle F_1,\mathbf{gh}\Big|_{k_1}\left(\mathrm{id}-\frac{\psi_1(p)}{p\alpha}\cdot U_p^*\right)\circ W_{\widetilde{N}}\right\rangle_{\widetilde{N}}}{\left\langle F_1,F_1\right\rangle_{N_1}}$$

and then setting $\mathfrak{u}_{\underline{\mathcal{Q}}}:=\epsilon_1\cdot\left(\frac{N_0^{(p)}}{N_1}\right)^{-\frac{k_1}{2}}\cdot(-1)^{k_1}\cdot\mathfrak{u}_\dagger^{-1}\in\mathcal{O}_{\mathbb{C}_p}^{\times}$, the result is proven. $\square$

Of course, we want the value of $a_1\big(\eta_{\mathbf{F}_1}\cdot 1_{\mathbf{F}_1}\cdot\mathrm{Tr}_{\widetilde{N}/N_1}\big(\mathbf{H}_\Sigma^{\mathrm{aux},(\star)}\big)\big)$ at a point $\underline{\mathcal{Q}}$ not the value of $a_1\big(1_{\mathbf{F}_1}\cdot\mathrm{Tr}_{\widetilde{N}/N_1}\big(\mathbf{H}_\Sigma^{\mathrm{ord},(\star)}\big)\big)$ at $\underline{\mathcal{Q}}$, but they are closely connected. Comparing the preceding lemma with Definition 7.7, at even weight $k_1>2$

$$\underline{\mathcal{Q}}\Big(a_1\big(\eta_{\mathbf{F}_1}\cdot 1_{\mathbf{F}_1}\cdot\mathrm{Tr}_{\widetilde{N}/N_1}\big(\mathbf{H}_\Sigma^{\mathrm{aux},(\star)}\big)\big)\Big) \;=\; \mathfrak{u}_{\underline{\mathcal{Q}}}\cdot\mathcal{Q}_{m_1}^{(1)}\big(\eta_{\mathbf{F}_1}\big)$$
$$\times\,\mathcal{L}_{F_1}^{(r,1)}\left(\mathbf{g}_\Sigma^{(\star)}\cdot\delta_{k_3}^{(r)}\big(\mathbf{h}_\Sigma^{(\star)}\big)\Big|_{k_1}\underline{\mathcal{Q}}\big(\Upsilon_{\widetilde{N},\mathbf{F}_1}^{\mathrm{aux}}\big)\circ\left(\frac{1}{p}\cdot\mathrm{id}-\frac{\psi_1(p)}{p^2\alpha}\cdot U_p^*\right)\right).$$

Moreover by its construction $L_{\mathbf{G}_\Sigma^{(2),(\star)},\mathbf{G}_\Sigma^{(3),(\star)}}^{\mathbf{F}_1}=a_1\Big(\eta_{\mathbf{F}_1}\cdot 1_{\mathbf{F}_1}\cdot\mathrm{Tr}_{\widetilde{N}/N_1}\big(\mathbf{H}_\Sigma^{\mathrm{aux},(\star)}\big)\Big)$, and so we may summarise the various calculations of *Step (2)* in the following way.

Corollary 7.9. *If $\underline{\mathcal{Q}}\in\mathfrak{X}_{\mathcal{R}}^{\mathbf{F}_1}$ has weight $\underline{k}=(k_1,k_2,k_3)$ and $k_1\in 2\cdot\mathbb{Z}_{\geq 2}$, then the special value of $L_{\mathbf{G}_\Sigma^{(2),(\star)},\mathbf{G}_\Sigma^{(3),(\star)}}^{\mathbf{F}_1}$ at the unbalanced point $\underline{\mathcal{Q}}$ is equal to*

$$p^{-2}\cdot\mathfrak{u}_{\underline{\mathcal{Q}}}\cdot\mathcal{Q}_{m_1}^{(1)}\big(\eta_{\mathbf{F}_1}\big)\times\mathcal{L}_{F_1}^{(r,1)}\left(\mathbf{g}_\Sigma^{(\star)}\cdot\delta_{k_3}^{(r)}\big(\mathbf{h}_\Sigma^{(\star)}\big)\Big|_{k_1}\underline{\mathcal{Q}}\big(\Upsilon_{\widetilde{N},\mathbf{F}_1}^{\mathrm{aux}}\big)\circ\left(p\cdot\mathrm{id}-\frac{\psi_1(p)}{\alpha}\cdot U_p^*\right)\right).$$

*N.B. The operator $\underline{\mathcal{Q}}\big(\Upsilon_{\widetilde{N},\mathbf{F}_1}^{\mathrm{aux}}\big)\circ\big(p\cdot\mathrm{id}-\frac{\psi_1(p)}{\alpha}\cdot U_p^*\big)$ is '**??**' mentioned in the remarks after Theorem 7.6.*

Step 3: Proving the congruences

The final task is to prove the congruences for $\mathcal{L}_{\mathbf{G}_\Sigma^{(2),(\star)},\mathbf{G}_\Sigma^{(3),(\star)}}^{\mathbf{F}_1}$ by reading them off at enough unbalanced specialisations $\underline{\mathcal{Q}}$ which are Zariski dense. An important initial observation is that

$$\mathcal{L}_{\mathbf{G}_\Sigma^{(2),(\star)},\mathbf{G}_\Sigma^{(3),(\star)}}^{\mathbf{F}_1} \;=\; \big(-\psi_{1,(p)}(-1)\big)^{-1/2}\cdot L_{\mathbf{G}_\Sigma^{(2),(\star)},\mathbf{G}_\Sigma^{(3),(\star)}}^{\mathbf{F}_1}\times\prod_{l|N}\mathfrak{f}_l^{-1/2}$$

where the factors $\mathfrak{f}_l \in \mathcal{R}^\times$ are given in [28, Prop 5.1.4], but are not required here. Thus to prove a congruence for the $\mathcal{L}^{\mathbf{F}_1}_{\mathbf{G}_\Sigma^{(2),(\star)},\mathbf{G}_\Sigma^{(3),(\star)}}$'s over the one-dimensional set $\mathcal{V}$, it is necessary and sufficient to show the same congruence for the $L^{\mathbf{F}_1}_{\mathbf{G}_\Sigma^{(2),(\star)},\mathbf{G}_\Sigma^{(3),(\star)}}$'s. Because each $\mathcal{L}^{\mathbf{F}_1}_{\mathbf{G}_\Sigma^{(2),(\star)},\mathbf{G}_\Sigma^{(3),(\star)}}$ is a square-root, one has an equality of μ-invariants

$$\mu \circ \phi_{\mathcal{V}}\left(\mathbf{L}_p\big(\mathbf{F}_1, \mathbf{G}_\Sigma^{(2),(\star)}, \mathbf{G}_\Sigma^{(3),(\star)}\big)\right) = 2 \cdot \mu \circ \phi_{\mathcal{V}}\Big(L^{\mathbf{F}_1}_{\mathbf{G}_\Sigma^{(2),(\star)},\mathbf{G}_\Sigma^{(3),(\star)}}\Big)$$

at either $\star \in \{\mathrm{I}, \mathrm{II}\}$, which means $\underline{\mathcal{Q}}\big(L^{\mathbf{F}_1}_{\mathbf{G}_\Sigma^{(2),(\star)},\mathbf{G}_\Sigma^{(3),(\star)}}\big)$ takes values in $p^{\mu_{\mathrm{wt}}^{(\mathcal{V})}/2} \cdot \mathcal{O}_{\mathbb{C}_p}$ for all $\underline{\mathcal{Q}} \in \mathcal{V} \cap \mathfrak{X}^{\mathbf{F}_1}_{\mathcal{R}}$. It follows directly from Corollary 7.9 that for each $\star \in \{\mathrm{I}, \mathrm{II}\}$,

$$\mathcal{L}^{(r,1)}_{F_1}\left(\mathbf{g}_\Sigma^{(\star)} \cdot \delta_{k_3}^{(r)}\big(\mathbf{h}_\Sigma^{(\star)}\big)\Big|_{k_1} \underline{\mathcal{Q}}\big(\Upsilon^{\mathrm{aux}}_{\widetilde{N},\mathbf{F}_1}\big) \circ \left(p \cdot \mathrm{id} - \frac{\psi_1(p)}{\alpha} \cdot U_p^*\right)\right)$$

lies inside $\mathcal{Q}^{(1)}_{m_1}\big(\eta_{\mathbf{F}_1}\big)^{-1} p^{2+\mu_{\mathrm{wt}}^{(\mathcal{V})}/2} \cdot \mathcal{O}_{\mathbb{C}_p}$, provided that $\underline{\mathcal{Q}} \in \mathcal{V} \cap \mathfrak{X}^{\mathbf{F}_1}_{\mathcal{R}}$ with $k_1 \in 2 \cdot \mathbb{Z}_{\geq 2}$.

Remarks. (i) By Equation (7.1), the functional values below degenerate into

$$\mathcal{L}^{(r,1)}_{F_1}\Big(\mathbf{g}_\Sigma^{(\star)} \cdot \delta_{k_3}^{(r)}\big(\mathbf{h}_\Sigma^{(\star)}\big)\Big) \;=\; \sum_{d \left| \frac{N_0}{N_1}\right.} \mathfrak{c}^{(\star)}_{d,\widetilde{N},\tilde{e}}\big(\mathcal{H}_\Sigma\big) \cdot \mathbf{X}_d(N_0, N_1)$$

where $\mathcal{H}_\Sigma^{(\star)} = \mathrm{Hol}_\infty\big(\mathbf{g}_\Sigma^{(\star)} \cdot \delta_{k_3}^{(r)}\big(\mathbf{h}_\Sigma^{(\star)}\big)\big)\big|_{k_1} W_{\widetilde{N}}^2 = (-1)^{k_1} \cdot \mathrm{Hol}_\infty\big(\mathbf{g}_\Sigma^{(\star)} \cdot \delta_{k_3}^{(r)}\big(\mathbf{h}_\Sigma^{(\star)}\big)\big)$.

(ii) Applying Proposition 6.17 at divisors $d \big| \frac{N_0}{N_1}$ and if $p \nmid \frac{(k_1-2)!}{(k_1-2-r)!}$, one has

$$\mathfrak{c}^{(\mathrm{I})}_{d,\widetilde{N},\tilde{e}}\big(\mathcal{H}_\Sigma\big) \equiv \mathfrak{c}^{(\mathrm{II})}_{d,\widetilde{N},\tilde{e}}\big(\mathcal{H}_\Sigma\big) \ \ \big(\mathrm{mod}\, p^{\min\{\nu_2,\nu_3\}}\big).$$

Since the composition of operators $\mathfrak{R}_{\underline{\mathcal{Q}}} := \underline{\mathcal{Q}}\big(\Upsilon^{\mathrm{aux}}_{\widetilde{N},\mathbf{F}_1}\big) \circ \big(p \cdot \mathrm{id} - \frac{\psi_1(p)}{\alpha} \cdot U_p^*\big)$ does not introduce any new denominators involving p, it follows from these remarks that

$$\mathcal{L}^{(r,1)}_{F_1}\Big(\mathbf{g}_\Sigma^{(\mathrm{I})} \cdot \delta_{k_3}^{(r)}\big(\mathbf{h}_\Sigma^{(\star)}\big)\Big|_{k_1} \mathfrak{R}_{\underline{\mathcal{Q}}}\Big) - \mathcal{L}^{(r,1)}_{F_1}\Big(\mathbf{g}_\Sigma^{(\mathrm{II})} \cdot \delta_{k_3}^{(r)}\big(\mathbf{h}_\Sigma^{(\star)}\big)\Big|_{k_1} \mathfrak{R}_{\underline{\mathcal{Q}}}\Big)$$

belongs to $\mathcal{Q}^{(1)}_{m_1}\big(\eta_{\mathbf{F}_1}\big)^{-1} p^{2+\min\{\nu_2,\nu_3\}+\mu_{\mathrm{wt}}^{(\mathcal{V})}/2} \cdot \mathcal{O}_{\mathbb{C}_p}$ at all the points $\underline{\mathcal{Q}} \in \mathcal{V} \cap \mathfrak{X}^{\mathbf{F}_1}_{\mathcal{R}}$ satisfying $k_1 \in 2 \cdot \mathbb{Z}_{\geq 2}$ and $p \nmid \frac{(k_1-2)!}{(k_1-2-r_{\underline{\mathcal{Q}}})!}$. Reversing the previous chain of reasoning,

$$\underline{\mathcal{Q}}\big(L^{\mathbf{F}_1}_{\mathbf{G}_\Sigma^{(2),(\mathrm{I})},\mathbf{G}_\Sigma^{(3),(\star)}}\big) - \underline{\mathcal{Q}}\big(L^{\mathbf{F}_1}_{\mathbf{G}_\Sigma^{(2),(\mathrm{II})},\mathbf{G}_\Sigma^{(3),(\star)}}\big) \;\in\; p^{\min\{\nu_2,\nu_3\}+\mu_{\mathrm{wt}}^{(\mathcal{V})}/2} \cdot \mathcal{O}_{\mathbb{C}_p}$$

hence both $\underline{\mathcal{Q}}\big(\mathbf{L}_p\big(\mathbf{F}_1 \otimes \mathbf{G}_{\Sigma}^{(2),(\mathrm{I})} \otimes \mathbf{G}_{\Sigma}^{(3),(\mathrm{I})}\big)\big)$ and $\underline{\mathcal{Q}}\big(\mathbf{L}_p\big(\mathbf{F}_1 \otimes \mathbf{G}_{\Sigma}^{(2),(\mathrm{II})} \otimes \mathbf{G}_{\Sigma}^{(3),(\mathrm{II})}\big)\big)$ are congruent to each other modulo $p^{\mu_{\mathrm{wt}}^{(\mathcal{V})}+\min\{\nu_2,\nu_3\}}$.

Lastly as $p \neq 2$, we use the density of those $\underline{\mathcal{Q}} \in \mathcal{V} \cap \mathfrak{X}_{\mathcal{R}}^{\mathbf{F}_1}$ with $p \nmid \frac{(k_1-2)!}{(k_1-2-r_{\underline{\mathcal{Q}}})!}$ and $2 \big| k_1$ inside $\mathrm{Spec}(\mathbb{I}^{\mathcal{V}})$ to obtain the full congruence, and Theorem 7.6 is proved.

Appendix A

Determining the ratio $\frac{\langle f|V_m, f|V_n\rangle}{\langle f,f\rangle}$ explicitly

We will derive a useful technical result relating the value of $\langle f|V_m, f|V_n\rangle_M$ to $\langle f,f\rangle_N$ where $f \in \mathcal{S}_k(N,\psi)$, m and n are positive integers, and M is a multiple of $\mathrm{lcm}(Mm, Mn)$. We assume that f has rational coefficients so that $f^\rho = f$.

Applying [60, Lemma 1],

$$\frac{\langle f\big|V_m, f\big|V_n\rangle_M}{\langle f,f\rangle_M} \;=\; \mathrm{Res}_{s=k}\left(\frac{D\big(s, f^\rho\big|V_m, f\big|V_n\big)}{D(s, f^\sharp, f)}\right)$$

where the L-series $D(s,\mathcal{F},\mathcal{G}) := \sum_{n=1}^{\infty} a_n(\mathcal{F})a_n(\mathcal{G}) \cdot n^{-s}$ for $\mathrm{Re}(s) \gg 0$. Because $f^\rho = f$, we may factorise the ratio of L-functions above into

$$\frac{D\big(s, f^\rho\big|V_m, f\big|V_n\big)}{D(s, f^\sharp, f)} \;=\; \left(\frac{mn}{\gcd(m,n)}\right)^{-s} \cdot \prod_{l\big|m'n'} \frac{\sum_{j=0}^{\infty} a_{l^j}(f)\; a_{l^{j+t_l}}(f)\cdot l^{-js}}{\sum_{j=0}^{\infty} a_{l^j}(f)^2 \cdot l^{-js}}$$

where m' and n' denote the positive integers satisfying $m = m'\gcd(m,n)$ and $n = n'\gcd(m,n)$, and the integer exponent $t_l := \mathrm{ord}_l(m'n') \geq 1$.

Lemma A.1. *If the prime l divides into $m'n'$, then*

$$\frac{\sum_{j=0}^{\infty} a_{l^j}(F_1)\; a_{l^{j+t_l}}(f)\cdot l^{-jk}}{\sum_{j=0}^{\infty} a_{l^j}(f)^2\cdot l^{-jk}} \;=\; \begin{cases} a_{l^{t_l}}(f) & \textit{if } l \mid N \\ \frac{a_{l^{t_l}}(f) - l^{k-2}a_{l^{t_l-2}}(f)}{1+\psi(l)\cdot l^{-1}} & \textit{if } l \nmid N \textit{ and } t_l \geq 2 \\ \frac{a_l(f)}{1+\psi(l)\cdot l^{-1}} & \textit{if } l \nmid N \textit{ and } t_l = 1. \end{cases}$$

Proof. At each prime l, let us factorise the Hecke polynomial for f into $X^2 - a_l(f)X + \psi(l)\cdot l^{k-1} = \big(X - \alpha_l\big)\big(X - \alpha_l'\big)$ where we choose $\alpha_l' = 0$ if $l\big|N$. Quoting verbatim from Equation (3.1) of *op. cit.*, for any integer $t \geq 0$:

$$Y_l(s) \times \sum_{j=0}^{\infty} a_{l^j}(f)\; a_{l^{j+t}}(f) \cdot l^{-js}$$
$$= \begin{cases} a_{l^t}(f) - a_{l^{t-1}}(f)a_l(f)\alpha_l\alpha_l' \cdot l^{-s} + a_{l^{t-2}}(f)\big(\alpha_l\alpha_l'\big)^3 \cdot l^{-2s} & \text{if } \ t \geq 2 \\ a_l(f) - a_l(f)\alpha_l\alpha_l' \cdot l^{-s} & \text{if } \ t = 1 \\ 1 - \big(\alpha_l\alpha_l'\big)^2 \cdot l^{-2s} & \text{if } \ t = 0, \end{cases}$$

and the Euler factor[1] here is defined by

$$Y_l(s) \;:=\; \big(1 - \alpha_l^2 \cdot l^{-s}\big)\big(1 - \alpha_l'^2 \cdot l^{-s}\big)\big(1 - \alpha_l\alpha_l' \cdot l^{-s}\big)^2.$$

Putting $s = k$ and utilising the identities $\alpha_l + \alpha_l' = a_l(f)$ and $\alpha_l\alpha_l' = \psi(l)\cdot l^{k_1-1}$, the required quotient can be readily computed from this expression at $t = t_l$. We will leave these details as an exercise for the reader. □

Corollary A.2. *For any positive integers m and n, and M a multiple of* $\mathrm{lcm}(mN, nN)$*, one has the identity*

$$\frac{\big\langle f\big|V_m, f\big|V_n\big\rangle_M}{\big\langle f, f\big\rangle_N} = \left(\frac{mn}{\gcd(m,n)}\right)^{-k}$$
$$\times \prod_{l|N} l^{\mathrm{ord}_l(M)-\mathrm{ord}_l(N)} \times \prod_{l|M,\, l\nmid N} (l+1)\cdot l^{\mathrm{ord}_l(M)-1}$$
$$\times \prod_{\substack{l|m'n' \\ l|N}} a_{l^{t_l}}(f) \times \prod_{\substack{l\|m'n' \\ l\nmid N}} \frac{a_l(f)}{1+\psi(l)\cdot l^{-1}}$$
$$\times \prod_{\substack{l^2|m'n' \\ l\nmid N}} \frac{a_{l^{t_l}}(f) - l^{k-2}a_{l^{t_l-2}}(f)}{1+\psi(l)\cdot l^{-1}}$$

[1]In general, given two distinct cusp forms $\mathcal{F} = \sum_{n=1}^{\infty} a_n(\mathcal{F})\cdot q^n$ and $\mathcal{G} = \sum_{n=1}^{\infty} a_n(\mathcal{G})\cdot q^n$, the Euler factor $Y_l(s) = (1-\alpha_l\beta_l\cdot l^{-s})(1-\alpha_l\beta_l'\cdot l^{-s})(1-\alpha_l'\beta_l\cdot l^{-s})(1-\alpha_l'\beta_l'\cdot l^{-s})$ where α_l, α_l' (resp. β_l, β_l') denote the Weil numbers of $\mathcal{F}$ (resp. $\mathcal{G}$); moreover the actual formula for $\sum_{j=0}^{\infty} a_{l^j}(\mathcal{F})\; a_{l^{j+t}}(\mathcal{G})\cdot l^{-js}$ involves $\alpha_l, \alpha_l', \beta_l, \beta_l'$, and only simplifies to the above when $\mathcal{F} = \mathcal{G}$.

Proof. The result follows upon splitting up the quotient into a product

$$\frac{\langle f|V_m, f|V_n\rangle_M}{\langle f, f\rangle_N} = \frac{\langle f|V_m, f|V_n\rangle_M}{\langle f, f\rangle_M} \times \frac{\langle f, f\rangle_M}{\langle f, f\rangle_N}$$

and using the above lemma to compute the first ratio, whilst it is well known that

$$\frac{\langle f, f\rangle_M}{\langle f, f\rangle_N} = \left[\Gamma_0(N) : \Gamma_0(M)\right] = \frac{\prod_{l|M} l^{\mathrm{ord}_l(M)} + l^{\mathrm{ord}_l(M)-1}}{\prod_{l|N} l^{\mathrm{ord}_l(N)} + l^{\mathrm{ord}_l(N)-1}} .$$

□

Appendix B

Tables of $\mathcal{L}$-invariants for elliptic curves

We have only considered elliptic curves $E_{/\mathbb{Q}}$ whose conductors are divisible by 4. We first treat the curves with $D(E,1) \neq 0$, and then the six exceptional curves with $D(E,1) = 0$.

B.1 Tables of $\mathcal{L}$-invariants for elliptic curves E with $D(E,1) \neq 0$

Tabulated below are the values we computed for both the derivative of the imprimitive p-adic L-function $\mathbf{L}_p^{\mathrm{imp}}(\mathrm{Sym}^2 E, s)$ at $s = 1$, together with the corresponding $\mathcal{L}$-invariant term, for the elliptic curves E with $D(E,1) \neq 0$. If the elliptic curve E is already a quadratic twist of another (earlier) elliptic curve listed in our tables, then we omit the $\mathcal{L}$-invariant data for E completely.

The reader will notice for the elliptic curves of conductor 32 and 36, which have complex multiplication by $\mathbb{Q}(\sqrt{-1})$ and $\mathbb{Q}(\sqrt{-3})$ respectively, that $\mathcal{L}_p(\mathrm{Sym}^2 E)$ coincides with $\log_p(\alpha_p^{-2})$ in agreement with the Ferrero-Greenberg formula. However if E has no complex multiplication, this identity no longer appears to hold in general.

$E = 20\mathrm{a}1$, $C_{\mathrm{Sym}^2 E} = 10^2$, $S_1 = \{2\}$, $\xi_{\mathrm{Sym}^2 E} = \frac{8}{3}$

p	$a_p(E)$	$\mathbf{L}_p^{\mathrm{imp}}(\mathrm{Sym}^2 E, 1)'$	$\mathcal{L}_p^{\mathrm{an}}(\mathrm{Sym}^2 E)$
3	-2	$p^2 + p^3 + 2p^4 + 2p^5 + 2p^6 + O(p^7)$	$p^2 + 2p^4 + O(p^7)$
7	2	$2p + 2p^2 + p^3 + O(p^4)$	$p + 2p^2 + p^3 + O(p^4)$
13	2	$11p + 2p^2 + O(p^3)$	$12p + 12p^2 + O(p^3)$

$E = 24\mathrm{a}1$, $C_{\mathrm{Sym}^2 E} = 12^2$, $S_1 = \{2\}$, $\xi_{\mathrm{Sym}^2 E} = 2$

p	$a_p(E)$	$\mathbf{L}_p^{\mathrm{imp}}(\mathrm{Sym}^2 E, 1)'$	$\mathcal{L}_p^{\mathrm{an}}(\mathrm{Sym}^2 E)$
5	-2	$2p^2 + p^3 + p^4 + O(p^5)$	$3p^2 + 2p^3 + O(p^5)$
11	4	$6p + 8p^2 + O(p^3)$	$p + O(p^3)$
13	-2	$7p + 9p^2 + O(p^3)$	$9p + 4p^2 + O(p^3)$

$E = 32\mathrm{a}1$, $C_{\mathrm{Sym}^2 E} = 8^2$, $S_1 = \{2\}$, $\xi_{\mathrm{Sym}^2 E} = 1$

p	$a_p(E)$	$\mathbf{L}_p^{\mathrm{imp}}(\mathrm{Sym}^2 E, 1)'$	$\mathcal{L}_p^{\mathrm{an}}(\mathrm{Sym}^2 E)$
5	-2	$3p + p^2 + 2p^3 + 4p^4 + O(p^5)$	$4p + 3p^2 + 3p^3 + 3p^4 + O(p^5)$
13	6	$p + 9p^2 + O(p^3)$	$4p + 7p^2 + O(p^3)$

$E = 36\mathrm{a}1$, $C_{\mathrm{Sym}^2 E} = 6^2$, $S_1 = \{2, 3\}$, $\xi_{\mathrm{Sym}^2 E} = \frac{4}{3}$

p	$a_p(E)$	$\mathbf{L}_p^{\mathrm{imp}}(\mathrm{Sym}^2 E, 1)'$	$\mathcal{L}_p^{\mathrm{an}}(\mathrm{Sym}^2 E)$
7	-4	$6p + 3p^2 + 2p^3 + O(p^4)$	$2p + 3p^2 + 2p^3 + O(p^4)$
13	2	$p^2 + O(p^3)$	$p^2 + O(p^3)$

$E = 40\mathrm{a}1$, $C_{\mathrm{Sym}^2 E} = 20^2$, $S_1 = \{2\}$, $\xi_{\mathrm{Sym}^2 E} = 2$

p	$a_p(E)$	$\mathbf{L}_p^{\mathrm{imp}}(\mathrm{Sym}^2 E, 1)'$	$\mathcal{L}_p^{\mathrm{an}}(\mathrm{Sym}^2 E)$
7	-4	$p + 6p^3 + O(p^4)$	$p + 4p^2 + 6p^3 + O(p^4)$
11	4	$5p + 4p^2 + O(p^3)$	$10p + 3p^2 + O(p^3)$
13	-2	$10p + 2p^2 + O(p^3)$	$11p + 11p^2 + O(p^3)$

$E = 44\mathrm{a}1$, $C_{\mathrm{Sym}^2 E} = 22^2$, $S_1 = \{2\}$, $\xi_{\mathrm{Sym}^2 E} = \frac{8}{3}$

p	$a_p(E)$	$\mathbf{L}_p^{\mathrm{imp}}(\mathrm{Sym}^2 E, 1)'$	$\mathcal{L}_p^{\mathrm{an}}(\mathrm{Sym}^2 E)$
3	1	$2p^3 + O(p^6)$	$p^3 + p^5 + O(p^6)$
5	-3	$4p + 2p^2 + p^4 + O(p^5)$	$2p + 3p^3 + 3p^4 + O(p^5)$
7	2	$3p + 5p^3 + O(p^4)$	$5p + p^2 + O(p^4)$
13	-4	$3p + p^2 + O(p^3)$	$9p + 11p^2 + O(p^3)$

$E = 52\text{a}1$, $C_{\mathrm{Sym}^2 E} = 26^2$, $S_1 = \{2\}$, $\xi_{\mathrm{Sym}^2 E} = \frac{8}{3}$

p	$a_p(E)$	$\mathbf{L}_p^{\mathrm{imp}}(\mathrm{Sym}^2 E, 1)'$	$\mathcal{L}_p^{\mathrm{an}}(\mathrm{Sym}^2 E)$
5	2	$4p + 2p^4 + O(p^5)$	$2p + 2p^3 + 3p^4 + O(p^5)$
7	-2	$p^3 + O(p^4)$	$4p^3 + O(p^4)$
11	-2	$10p + p^2 + O(p^3)$	$5p + 9p^2 + O(p^3)$

$E = 56\text{a}1$, $C_{\mathrm{Sym}^2 E} = 28^2$, $S_1 = \{2\}$, $\xi_{\mathrm{Sym}^2 E} = 2$

p	$a_p(E)$	$\mathbf{L}_p^{\mathrm{imp}}(\mathrm{Sym}^2 E, 1)'$	$\mathcal{L}_p^{\mathrm{an}}(\mathrm{Sym}^2 E)$
5	2	$p + p^2 + 2p^3 + O(p^4)$	$4p + 2p^3 + O(p^4)$
11	-4	$p + O(p^3)$	$2p + 10p^2 + O(p^3)$
13	2	$9p + O(p^2)$	$6p + O(p^2)$

$E = 56\text{b}1$, $C_{\mathrm{Sym}^2 E} = 28^2$, $S_1 = \{2\}$, $\xi_{\mathrm{Sym}^2 E} = 2$

p	$a_p(E)$	$\mathbf{L}_p^{\mathrm{imp}}(\mathrm{Sym}^2 E, 1)'$	$\mathcal{L}_p^{\mathrm{an}}(\mathrm{Sym}^2 E)$
3	2	$2p + p^4 + O(p^5)$	$2 + p^3 + O(p^4)$
5	-4	$3p + p^2 + O(p^3)$	$4 + 2p + O(p^2)$

$E = 76\text{a}1$, $C_{\mathrm{Sym}^2 E} = 38^2$, $S_1 = \{2\}$, $\xi_{\mathrm{Sym}^2 E} = \frac{8}{3}$

p	$a_p(E)$	$\mathbf{L}_p^{\mathrm{imp}}(\mathrm{Sym}^2 E, 1)'$	$\mathcal{L}_p^{\mathrm{an}}(\mathrm{Sym}^2 E)$
3	2	$1 + 2p + O(p^4)$	$1 + p + 2p^2 + O(p^4)$
5	-1	$4p + 3p^2 + O(p^3)$	$3 + 2p + O(p^2)$
7	-3	$p^2 + O(p^3)$	$6p^2 + O(p^3)$
11	5	$5p + 4p^2 + O(p^3)$	$9p + 4p^2 + O(p^3)$
13	-4	$4p + 9p^2 + O(p^3)$	$12p + 3p^2 + O(p^3)$

$E = 84\text{a}1$, $C_{\mathrm{Sym}^2 E} = 42^2$, $S_1 = \{2\}$, $\xi_{\mathrm{Sym}^2 E} = \frac{8}{3}$

p	$a_p(E)$	$\mathbf{L}_p^{\mathrm{imp}}(\mathrm{Sym}^2 E, 1)'$	$\mathcal{L}_p^{\mathrm{an}}(\mathrm{Sym}^2 E)$
11	-6	$5p + 3p^2 + O(p^3)$	$9p + 4p^2 + O(p^3)$
13	2	$10p + 10p^2 + O(p^3)$	$5p + 6p^2 + O(p^3)$

$E = 84\text{b}1$, $C_{\mathrm{Sym}^2 E} = 42^2$, $S_1 = \{2\}$, $\xi_{\mathrm{Sym}^2 E} = \frac{8}{3}$

p	$a_p(E)$	$\mathbf{L}_p^{\mathrm{imp}}(\mathrm{Sym}^2 E, 1)'$	$\mathcal{L}_p^{\mathrm{an}}(\mathrm{Sym}^2 E)$
5	4	$p + 2p^2 + O(p^3)$	$1 + 3p + O(p^2)$
11	2	$5p + 4p^2 + O(p^3)$	$8p + 3p^2 + O(p^3)$
13	-6	$7p + 3p^2 + O(p^3)$	$4p + 2p^2 + O(p^3)$

$E = 88\text{a}1$, $C_{\mathrm{Sym}^2 E} = 44^2$, $S_1 = \{2\}$, $\xi_{\mathrm{Sym}^2 E} = 2$

p	$a_p(E)$	$\mathbf{L}_p^{\mathrm{imp}}(\mathrm{Sym}^2 E, 1)'$	$\mathcal{L}_p^{\mathrm{an}}(\mathrm{Sym}^2 E)$
5	-3	$4p + p^3 + O(p^4)$	$p + 4p^2 + p^3 + O(p^4)$
7	-2	$6p + 5p^2 + O(p^3)$	$4p + 5p^2 + O(p^3)$

$E = 92\text{a}1$, $C_{\mathrm{Sym}^2 E} = 46^2$, $S_1 = \{2\}$, $\xi_{\mathrm{Sym}^2 E} = \frac{8}{3}$

p	$a_p(E)$	$\mathbf{L}_p^{\mathrm{imp}}(\mathrm{Sym}^2 E, 1)'$	$\mathcal{L}_p^{\mathrm{an}}(\mathrm{Sym}^2 E)$
3	1	$p^4 + 2p^5 + O(p^6)$	$2p^4 + 2p^5 + O(p^6)$
7	2	$6p + 2p^2 + O(p^3)$	$3p + 4p^2 + O(p^3)$
13	-1	$11p + O(p^2)$	$2 + O(p)$

$E = 92\text{b}1$, $C_{\mathrm{Sym}^2 E} = 46^2$, $S_1 = \{2\}$, $\xi_{\mathrm{Sym}^2 E} = \frac{8}{3}$

p	$a_p(E)$	$\mathbf{L}_p^{\mathrm{imp}}(\mathrm{Sym}^2 E, 1)'$	$\mathcal{L}_p^{\mathrm{an}}(\mathrm{Sym}^2 E)$
5	-2	$4p + p^2 + 2p^3 + O(p^4)$	$2p + 3p^2 + O(p^4)$
7	-4	$3p + 3p^2 + O(p^3)$	$4p + 4p^2 + O(p^3)$
11	2	$10p + O(p^2)$	$5p + O(p^2)$
13	-5	$12p + O(p^2)$	$12p + O(p^2)$

$E = 96\text{a}1$, $C_{\mathrm{Sym}^2 E} = 24^2$, $S_1 = \{2\}$, $\xi_{\mathrm{Sym}^2 E} = 1$

p	$a_p(E)$	$\mathbf{L}_p^{\mathrm{imp}}(\mathrm{Sym}^2 E, 1)'$	$\mathcal{L}_p^{\mathrm{an}}(\mathrm{Sym}^2 E)$
5	2	$p^2 + 2p^3 + O(p^4)$	$3p^2 + 4p^3 + O(p^4)$
7	-4	$4p + 2p^2 + O(p^3)$	$p + 2p^2 + O(p^3)$
11	4	$9p + O(p^2)$	$3p + O(p^2)$
13	-2	$3p + O(p^2)$	$4p + O(p^2)$

$E = 104\text{a}1$, $C_{\mathrm{Sym}^2 E} = 52^2$, $S_1 = \{2\}$, $\xi_{\mathrm{Sym}^2 E} = 2$

p	$a_p(E)$	$\mathbf{L}_p^{\mathrm{imp}}(\mathrm{Sym}^2 E, 1)'$	$\mathcal{L}_p^{\mathrm{an}}(\mathrm{Sym}^2 E)$
3	1	$p + p^2 + 2p^3 + p^4 + O(p^5)$	$2 + 2p + 2p^3 + O(p^4)$
5	-1	$2p^2 + O(p^3)$	$2p + O(p^2)$
7	5	$p + O(p^3)$	$3p + 6p^2 + O(p^3)$
11	-2	$9p + O(p^2)$	$6p + O(p^2)$

$E = 108\text{a}1$, $C_{\mathrm{Sym}^2 E} = 18^2$, $S_1 = \{2, 3\}$, $\xi_{\mathrm{Sym}^2 E} = \frac{8}{9}$

p	$a_p(E)$	$\mathbf{L}_p^{\mathrm{imp}}(\mathrm{Sym}^2 E, 1)'$	$\mathcal{L}_p^{\mathrm{an}}(\mathrm{Sym}^2 E)$
7	5	$6p + 5p^2 + O(p^3)$	$2p + 3p^2 + O(p^3)$
13	-7	$6p^2 + O(p^3)$	$p^2 + O(p^3)$

$E = 112\text{c1}$, $C_{\mathrm{Sym}^2 E} = 14^2$, $S_1 = \{2\}$, $\xi_{\mathrm{Sym}^2 E} = \frac{1}{3}$

p	$a_p(E)$	$\mathbf{L}_p^{\mathrm{imp}}(\mathrm{Sym}^2 E, 1)'$	$\mathcal{L}_p^{\mathrm{an}}(\mathrm{Sym}^2 E)$
3	2	$p + 2p^2 + O(p^5)$	$2p + p^2 + p^3 + O(p^5)$
13	-4	$9p + O(p^2)$	$8p + O(p^2)$

$E = 116\text{a1}$, $C_{\mathrm{Sym}^2 E} = 58^2$, $S_1 = \{2\}$, $\xi_{\mathrm{Sym}^2 E} = \frac{8}{3}$

p	$a_p(E)$	$\mathbf{L}_p^{\mathrm{imp}}(\mathrm{Sym}^2 E, 1)'$	$\mathcal{L}_p^{\mathrm{an}}(\mathrm{Sym}^2 E)$
5	3	$p + 3p^2 + O(p^3)$	$3p + 2p^2 + O(p^3)$
7	4	$4p + 3p^2 + O(p^3)$	$3p + 2p^2 + O(p^3)$
11	-1	$O(p^2)$	$O(p)$
13	-3	$12p + O(p^2)$	$p + O(p^2)$

$E = 116\text{b1}$, $C_{\mathrm{Sym}^2 E} = 58^2$, $S_1 = \{2\}$, $\xi_{\mathrm{Sym}^2 E} = \frac{8}{3}$

p	$a_p(E)$	$\mathbf{L}_p^{\mathrm{imp}}(\mathrm{Sym}^2 E, 1)'$	$\mathcal{L}_p^{\mathrm{an}}(\mathrm{Sym}^2 E)$
3	1	$2p^2 + 2p^3 + O(p^5)$	$p^2 + p^3 + p^4 + O(p^5)$
5	3	$3p + 4p^2 + 4p^3 + O(p^4)$	$4p + 2p^2 + 4p^3 + O(p^4)$
7	-4	$6p + 4p^2 + O(p^3)$	$p + 4p^2 + O(p^3)$
11	3	$5p + O(p^2)$	$4p + O(p^2)$
13	5	$9p + O(p^2)$	$9p + O(p^2)$

$E = 116\text{c1}$, $C_{\mathrm{Sym}^2 E} = 58^2$, $S_1 = \{2\}$, $\xi_{\mathrm{Sym}^2 E} = \frac{8}{3}$

p	$a_p(E)$	$\mathbf{L}_p^{\mathrm{imp}}(\mathrm{Sym}^2 E, 1)'$	$\mathcal{L}_p^{\mathrm{an}}(\mathrm{Sym}^2 E)$
3	2	$1 + 2p + p^2 + 2p^3 + O(p^4)$	$1 + p + 2p^3 + O(p^4)$
5	-2	$p + 4p^2 + O(p^3)$	$3p + 4p^2 + O(p^3)$
7	4	$3p + 4p^2 + O(p^3)$	$4p + 3p^2 + O(p^3)$
11	-6	$2p + O(p^2)$	$8p + O(p^2)$
13	2	$8p + O(p^2)$	$4p + O(p^2)$

$E = 120\text{a1}$, $C_{\mathrm{Sym}^2 E} = 60^2$, $S_1 = \{2\}$, $\xi_{\mathrm{Sym}^2 E} = 2$

p	$a_p(E)$	$\mathbf{L}_p^{\mathrm{imp}}(\mathrm{Sym}^2 E, 1)'$	$\mathcal{L}_p^{\mathrm{an}}(\mathrm{Sym}^2 E)$
11	-4	$6p + O(p^2)$	$p + O(p^2)$
13	6	$11p + O(p^2)$	$9p + O(p^2)$

$E = 120\text{b1}$, $C_{\mathrm{Sym}^2 E} = 60^2$, $S_1 = \{2\}$, $\xi_{\mathrm{Sym}^2 E} = 2$

p	$a_p(E)$	$\mathbf{L}_p^{\mathrm{imp}}(\mathrm{Sym}^2 E, 1)'$	$\mathcal{L}_p^{\mathrm{an}}(\mathrm{Sym}^2 E)$
7	4	$5p + 3p^2 + O(p^3)$	$5p + 2p^2 + O(p^3)$
13	-6	$6p + O(p^2)$	$12p + O(p^2)$

$E = 124\text{a}1$, $C_{\mathrm{Sym}^2 E} = 62^2$, $S_1 = \{2\}$, $\xi_{\mathrm{Sym}^2 E} = \frac{8}{3}$

p	$a_p(E)$	$\mathbf{L}_p^{\mathrm{imp}}(\mathrm{Sym}^2 E, 1)'$	$\mathcal{L}_p^{\mathrm{an}}(\mathrm{Sym}^2 E)$
3	-2	$2 + 2p + p^2 + O(p^4)$	$2 + p^2 + 2p^3 + O(p^4)$
5	-3	$2p + 3p^2 + 2p^3 + O(p^4)$	$p + p^2 + 2p^3 + O(p^4)$
7	-1	$2p^2 + O(p^3)$	$4p + O(p^2)$
11	-6	$7p^2 + O(p^3)$	$6p^2 + O(p^3)$
13	2	$4p + O(p^2)$	$2p + O(p^2)$

$E = 124\text{b}1$, $C_{\mathrm{Sym}^2 E} = 62^2$, $S_1 = \{2\}$, $\xi_{\mathrm{Sym}^2 E} = \frac{8}{3}$

p	$a_p(E)$	$\mathbf{L}_p^{\mathrm{imp}}(\mathrm{Sym}^2 E, 1)'$	$\mathcal{L}_p^{\mathrm{an}}(\mathrm{Sym}^2 E)$
5	1	$p + 3p^2 + O(p^3)$	$2 + p + O(p^2)$
7	3	$5p + 6p^2 + O(p^3)$	$2p + 5p^2 + O(p^3)$
11	6	$O(p^3)$	$O(p^3)$
13	-4	$11p + O(p^2)$	$7p + O(p^2)$

$E = 128\text{a}1$, $C_{\mathrm{Sym}^2 E} = 16^2$, $S_1 = \{2\}$, $\xi_{\mathrm{Sym}^2 E} = \frac{1}{2}$

p	$a_p(E)$	$\mathbf{L}_p^{\mathrm{imp}}(\mathrm{Sym}^2 E, 1)'$	$\mathcal{L}_p^{\mathrm{an}}(\mathrm{Sym}^2 E)$
3	-2	$2p + 2p^2 + p^3 + O(p^5)$	$2 + p + p^2 + 2p^3 + O(p^4)$
5	-2	$4p + 2p^2 + p^3 + O(p^4)$	$4p + 3p^3 + O(p^4)$
7	-4	$3p + 5p^2 + O(p^3)$	$5p + 6p^2 + O(p^3)$
11	2	$7p + O(p^2)$	$4p + O(p^2)$
13	-2	$5p + O(p^2)$	$9p + O(p^2)$

$E = 132\text{a}1$, $C_{\mathrm{Sym}^2 E} = 66^2$, $S_1 = \{2\}$, $\xi_{\mathrm{Sym}^2 E} = \frac{8}{3}$

p	$a_p(E)$	$\mathbf{L}_p^{\mathrm{imp}}(\mathrm{Sym}^2 E, 1)'$	$\mathcal{L}_p^{\mathrm{an}}(\mathrm{Sym}^2 E)$
5	2	$4p + p^2 + p^3 + O(p^4)$	$2p + 3p^2 + 2p^3 + O(p^4)$
7	-2	$p + O(p^3)$	$4p + O(p^3)$
13	-2	$12p + O(p^2)$	$6p + O(p^2)$

$E = 132\text{b}1$, $C_{\mathrm{Sym}^2 E} = 66^2$, $S_1 = \{2\}$, $\xi_{\mathrm{Sym}^2 E} = \frac{8}{3}$

p	$a_p(E)$	$\mathbf{L}_p^{\mathrm{imp}}(\mathrm{Sym}^2 E, 1)'$	$\mathcal{L}_p^{\mathrm{an}}(\mathrm{Sym}^2 E)$
5	2	$2p + 3p^2 + O(p^3)$	$p + 4p^2 + O(p^3)$
7	2	$3p + p^2 + O(p^3)$	$5p + 5p^2 + O(p^3)$
13	6	$3p + O(p^2)$	$11p + O(p^2)$

$E = 136\text{a}1$, $C_{\mathrm{Sym}^2E} = 68^2$, $S_1 = \{2\}$, $\xi_{\mathrm{Sym}^2E} = 2$

p	$a_p(E)$	$\mathbf{L}_p^{\mathrm{imp}}(\mathrm{Sym}^2E, 1)'$	$\mathcal{L}_p^{\mathrm{an}}(\mathrm{Sym}^2E)$
3	-2	$2p^2 + 2p^3 + p^4 + O(p^5)$	$2p + 2p^2 + p^3 + O(p^4)$
5	-2	$3p + 3p^2 + 3p^3 + O(p^4)$	$2p + 2p^2 + 4p^3 + O(p^4)$
7	-2	$2p + 2p^2 + O(p^3)$	$6p + 2p^2 + O(p^3)$
11	-6	$10p + O(p^2)$	$2p + O(p^2)$
13	2	$9p + O(p^2)$	$6p + O(p^2)$

$E = 136\text{b}1$, $C_{\mathrm{Sym}^2E} = 68^2$, $S_1 = \{2\}$, $\xi_{\mathrm{Sym}^2E} = 2$

p	$a_p(E)$	$\mathbf{L}_p^{\mathrm{imp}}(\mathrm{Sym}^2E, 1)'$	$\mathcal{L}_p^{\mathrm{an}}(\mathrm{Sym}^2E)$
3	2	$2p + 2p^2 + 2p^3 + 2p^4 + O(p^5)$	$2 + 2p + 2p^2 + 2p^3 + O(p^4)$
11	2	$3p + O(p^2)$	$2p + O(p^2)$
13	-6	$p + O(p^2)$	$2p + O(p^2)$

$E = 140\text{a}1$, $C_{\mathrm{Sym}^2E} = 70^2$, $S_1 = \{2\}$, $\xi_{\mathrm{Sym}^2E} = \frac{8}{3}$

p	$a_p(E)$	$\mathbf{L}_p^{\mathrm{imp}}(\mathrm{Sym}^2E, 1)'$	$\mathcal{L}_p^{\mathrm{an}}(\mathrm{Sym}^2E)$
3	1	$2p + p^2 + O(p^4)$	$p + 2p^2 + 2p^3 + O(p^4)$
11	3	$p + O(p^2)$	$3p + O(p^2)$
13	-1	$O(p^2)$	$O(p)$

$E = 140\text{b}1$, $C_{\mathrm{Sym}^2E} = 70^2$, $S_1 = \{2\}$, $\xi_{\mathrm{Sym}^2E} = \frac{8}{3}$

p	$a_p(E)$	$\mathbf{L}_p^{\mathrm{imp}}(\mathrm{Sym}^2E, 1)'$	$\mathcal{L}_p^{\mathrm{an}}(\mathrm{Sym}^2E)$
11	-5	$2p + O(p^2)$	$8p + O(p^2)$
13	-3	$8p + O(p^2)$	$5p + O(p^2)$

$E = 148\text{a}1$, $C_{\mathrm{Sym}^2E} = 74^2$, $S_1 = \{2\}$, $\xi_{\mathrm{Sym}^2E} = \frac{8}{3}$

p	$a_p(E)$	$\mathbf{L}_p^{\mathrm{imp}}(\mathrm{Sym}^2E, 1)'$	$\mathcal{L}_p^{\mathrm{an}}(\mathrm{Sym}^2E)$
3	-1	$2p + p^2 + O(p^4)$	$p + 2p^2 + 2p^3 + O(p^4)$
5	-4	$3p + 4p^2 + O(p^3)$	$3 + 2p + O(p^2)$
7	-3	$p + 3p^2 + O(p^3)$	$6p + 4p^2 + O(p^3)$
11	5	$8p + O(p^2)$	$10p + O(p^2)$

$E = 152a1$, $C_{\mathrm{Sym}^2 E} = 76^2$, $S_1 = \{2\}$, $\xi_{\mathrm{Sym}^2 E} = 2$

p	$a_p(E)$	$\mathbf{L}_p^{\mathrm{imp}}(\mathrm{Sym}^2 E, 1)'$	$\mathcal{L}_p^{\mathrm{an}}(\mathrm{Sym}^2 E)$
3	-2	$2p + O(p^5)$	$2 + O(p^4)$
5	-1	$p + 2p^2 + O(p^3)$	$1 + 4p + O(p^2)$
7	-3	$p + 6p^2 + O(p^3)$	$p + O(p^3)$
11	-3	$3p + O(p^2)$	$p + O(p^2)$
13	-4	$4p + O(p^2)$	$3p + O(p^2)$

$E = 152b1$, $C_{\mathrm{Sym}^2 E} = 76^2$, $S_1 = \{2\}$, $\xi_{\mathrm{Sym}^2 E} = 2$

p	$a_p(E)$	$\mathbf{L}_p^{\mathrm{imp}}(\mathrm{Sym}^2 E, 1)'$	$\mathcal{L}_p^{\mathrm{an}}(\mathrm{Sym}^2 E)$
3	1	$2p + 2p^2 + 2p^3 + 2p^4 + O(p^5)$	$1 + 2p + O(p^4)$
7	3	$p + O(p^3)$	$p + p^2 + O(p^3)$
11	2	$8p + O(p^2)$	$9p + O(p^2)$
13	1	$7p + O(p^2)$	$8 + O(p)$

$E = 156a1$, $C_{\mathrm{Sym}^2 E} = 78^2$, $S_1 = \{2\}$, $\xi_{\mathrm{Sym}^2 E} = \frac{8}{3}$

p	$a_p(E)$	$\mathbf{L}_p^{\mathrm{imp}}(\mathrm{Sym}^2 E, 1)'$	$\mathcal{L}_p^{\mathrm{an}}(\mathrm{Sym}^2 E)$
5	-4	$2p + p^2 + O(p^3)$	$2 + 3p + O(p^2)$
7	-2	$4p + O(p^3)$	$2p + 2p^2 + O(p^3)$
11	-4	$7p + O(p^2)$	$5p + O(p^2)$

$E = 156b1$, $C_{\mathrm{Sym}^2 E} = 78^2$, $S_1 = \{2\}$, $\xi_{\mathrm{Sym}^2 E} = \frac{8}{3}$

p	$a_p(E)$	$\mathbf{L}_p^{\mathrm{imp}}(\mathrm{Sym}^2 E, 1)'$	$\mathcal{L}_p^{\mathrm{an}}(\mathrm{Sym}^2 E)$
7	2	$6p + O(p^3)$	$3p + 3p^2 + O(p^3)$

$E = 160a1$, $C_{\mathrm{Sym}^2 E} = 40^2$, $S_1 = \{2\}$, $\xi_{\mathrm{Sym}^2 E} = 1$

p	$a_p(E)$	$\mathbf{L}_p^{\mathrm{imp}}(\mathrm{Sym}^2 E, 1)'$	$\mathcal{L}_p^{\mathrm{an}}(\mathrm{Sym}^2 E)$
3	-2	$2p + 2p^2 + p^3 + p^4 + O(p^5)$	$1 + 2p + O(p^4)$
7	-2	$p + 3p^2 + O(p^3)$	$6p + O(p^3)$
11	-4	$2p + O(p^2)$	$8p + O(p^2)$
13	-6	$10p + O(p^2)$	$p + O(p^2)$

$E = 168a1$, $C_{\mathrm{Sym}^2 E} = 84^2$, $S_1 = \{2\}$, $\xi_{\mathrm{Sym}^2 E} = 2$

p	$a_p(E)$	$\mathbf{L}_p^{\mathrm{imp}}(\mathrm{Sym}^2 E, 1)'$	$\mathcal{L}_p^{\mathrm{an}}(\mathrm{Sym}^2 E)$
5	2	$p + p^2 + 3p^3 + O(p^4)$	$4p + p^3 + O(p^4)$
13	-2	$3p + O(p^2)$	$2p + O(p^2)$

$E = 168\text{b}1$, $C_{\mathrm{Sym}^2 E} = 84^2$, $S_1 = \{2\}$, $\xi_{\mathrm{Sym}^2 E} = 2$

p	$a_p(E)$	$\mathbf{L}_p^{\mathrm{imp}}(\mathrm{Sym}^2 E, 1)'$	$\mathcal{L}_p^{\mathrm{an}}(\mathrm{Sym}^2 E)$
5	2	$4p^2 + 2p^3 + O(p^4)$	$p^2 + O(p^4)$
13	6	$10p + O(p^2)$	$7p + O(p^2)$

$E = 172\text{a}1$, $C_{\mathrm{Sym}^2 E} = 86^2$, $S_1 = \{2\}$, $\xi_{\mathrm{Sym}^2 E} = \frac{8}{3}$

p	$a_p(E)$	$\mathbf{L}_p^{\mathrm{imp}}(\mathrm{Sym}^2 E, 1)'$	$\mathcal{L}_p^{\mathrm{an}}(\mathrm{Sym}^2 E)$
3	-2	$2 + p + O(p^4)$	$2 + 2p + 2p^3 + O(p^4)$
7	-4	$4p + 2p^2 + O(p^3)$	$3p + 3p^2 + O(p^3)$
11	-3	$3p + O(p^2)$	$9p + O(p^2)$
13	-1	$9p + O(p^2)$	$4 + O(p)$

$E = 184\text{a}1$, $C_{\mathrm{Sym}^2 E} = 92^2$, $S_1 = \{2\}$, $\xi_{\mathrm{Sym}^2 E} = 2$

p	$a_p(E)$	$\mathbf{L}_p^{\mathrm{imp}}(\mathrm{Sym}^2 E, 1)'$	$\mathcal{L}_p^{\mathrm{an}}(\mathrm{Sym}^2 E)$
3	-1	$2p^2 + p^3 + p^4 + O(p^5)$	$p + p^3 + O(p^4)$
5	-4	$2p^2 + O(p^3)$	$p + O(p^2)$
7	2	$p + 6p^2 + O(p^3)$	$3p + 2p^2 + O(p^3)$
11	-4	$6p + O(p^2)$	$p + O(p^2)$
13	-5	$3p + O(p^2)$	$4p + O(p^2)$

$E = 184\text{b}1$, $C_{\mathrm{Sym}^2 E} = 92^2$, $S_1 = \{2\}$, $\xi_{\mathrm{Sym}^2 E} = 2$

p	$a_p(E)$	$\mathbf{L}_p^{\mathrm{imp}}(\mathrm{Sym}^2 E, 1)'$	$\mathcal{L}_p^{\mathrm{an}}(\mathrm{Sym}^2 E)$
3	-1	$2p^2 + 2p^3 + O(p^5)$	$p + 2p^2 + 2p^3 + O(p^4)$
5	-2	$p + 3p^2 + O(p^4)$	$4p + 3p^2 + 2p^3 + O(p^4)$
7	-4	$3p + p^2 + O(p^3)$	$3p + 6p^2 + O(p^3)$
11	-2	$4p + O(p^2)$	$10p + O(p^2)$
13	7	$3p + O(p^2)$	$6p + O(p^2)$

$E = 184\text{c}1$, $C_{\mathrm{Sym}^2 E} = 92^2$, $S_1 = \{2\}$, $\xi_{\mathrm{Sym}^2 E} = 2$

p	$a_p(E)$	$\mathbf{L}_p^{\mathrm{imp}}(\mathrm{Sym}^2 E, 1)'$	$\mathcal{L}_p^{\mathrm{an}}(\mathrm{Sym}^2 E)$
7	4	$5p + 5p^2 + O(p^3)$	$5p + 4p^2 + O(p^3)$
11	6	$8p + O(p^2)$	$6p + O(p^2)$
13	-2	$7p + O(p^2)$	$9p + O(p^2)$

$E = 184\text{d}1$, $C_{\text{Sym}^2E} = 92^2$, $S_1 = \{2\}$, $\xi_{\text{Sym}^2E} = 2$

p	$a_p(E)$	$\mathbf{L}_p^{\text{imp}}(\text{Sym}^2E, 1)'$	$\mathcal{L}_p^{\text{an}}(\text{Sym}^2E)$
7	-2	$p + O(p^3)$	$3p + 5p^2 + O(p^3)$
13	-5	$p + O(p^2)$	$10p + O(p^2)$

$E = 200\text{a}1$, $C_{\text{Sym}^2E} = 20^2$, $S_1 = \{2, 5\}$, $\xi_{\text{Sym}^2E} = \frac{2}{3}$

p	$a_p(E)$	$\mathbf{L}_p^{\text{imp}}(\text{Sym}^2E, 1)'$	$\mathcal{L}_p^{\text{an}}(\text{Sym}^2E)$
7	2	$5p + 6p^2 + O(p^3)$	$3p + 2p^2 + O(p^3)$
11	1	$O(p^2)$	$O(p)$
13	4	$6p + O(p^2)$	$7p + O(p^2)$

$E = 204\text{a}1$, $C_{\text{Sym}^2E} = 102^2$, $S_1 = \{2\}$, $\xi_{\text{Sym}^2E} = \frac{8}{3}$

p	$a_p(E)$	$\mathbf{L}_p^{\text{imp}}(\text{Sym}^2E, 1)'$	$\mathcal{L}_p^{\text{an}}(\text{Sym}^2E)$
5	-1	$2p + p^2 + O(p^3)$	$4 + 2p + O(p^2)$
7	4	$6p^3 + O(p^4)$	$p^3 + O(p^4)$
11	3	$8p + O(p^2)$	$2p + O(p^2)$
13	3	$3p + O(p^2)$	$10p + O(p^2)$

$E = 204\text{b}1$, $C_{\text{Sym}^2E} = 102^2$, $S_1 = \{2\}$, $\xi_{\text{Sym}^2E} = \frac{8}{3}$

p	$a_p(E)$	$\mathbf{L}_p^{\text{imp}}(\text{Sym}^2E, 1)'$	$\mathcal{L}_p^{\text{an}}(\text{Sym}^2E)$
5	1	$2p^2 + O(p^3)$	$4p + O(p^2)$
11	5	$10p + O(p^2)$	$7p + O(p^2)$
13	-5	$8p + O(p^2)$	$8p + O(p^2)$

$E = 208\text{a}1$, $C_{\text{Sym}^2E} = 26^2$, $S_1 = \{2\}$, $\xi_{\text{Sym}^2E} = \frac{1}{3}$

p	$a_p(E)$	$\mathbf{L}_p^{\text{imp}}(\text{Sym}^2E, 1)'$	$\mathcal{L}_p^{\text{an}}(\text{Sym}^2E)$
3	-1	$p + 2p^2 + 2p^4 + O(p^5)$	$p + p^3 + p^4 + O(p^5)$
5	-3	$2p + 3p^2 + 4p^3 + O(p^4)$	$3p + 4p^2 + O(p^4)$
7	1	$6p + O(p^2)$	$5 + O(p)$
11	-6	$7p + O(p^2)$	$4p + O(p^2)$

$E = 208\text{d}1$, $C_{\text{Sym}^2E} = 26^2$, $S_1 = \{2\}$, $\xi_{\text{Sym}^2E} = \frac{1}{3}$

p	$a_p(E)$	$\mathbf{L}_p^{\text{imp}}(\text{Sym}^2E, 1)'$	$\mathcal{L}_p^{\text{an}}(\text{Sym}^2E)$
5	-1	$4p + 4p^2 + O(p^3)$	$4 + p + O(p^2)$
7	-1	$3p^2 + O(p^3)$	$6p + O(p^2)$
11	2	$6p + O(p^2)$	$2p + O(p^2)$

$E = 212\text{a}1$, $C_{\mathrm{Sym}^2E} = 106^2$, $S_1 = \{2\}$, $\xi_{\mathrm{Sym}^2E} = \frac{8}{3}$

p	$a_p(E)$	$\mathbf{L}_p^{\mathrm{imp}}(\mathrm{Sym}^2E,1)'$	$\mathcal{L}_p^{\mathrm{an}}(\mathrm{Sym}^2E)$
3	-1	$2+2p+p^3+O(p^4)$	$1+p+p^2+p^3+O(p^4)$
5	-2	$p^2+2p^3+O(p^4)$	$3p^2+3p^3+O(p^4)$
7	-2	$3p+4p^2+O(p^3)$	$5p+3p^2+O(p^3)$
11	2	$8p+O(p^2)$	$4p+O(p^2)$
13	-7	$11p+O(p^2)$	$10p+O(p^2)$

$E = 212\text{b}1$, $C_{\mathrm{Sym}^2E} = 106^2$, $S_1 = \{2\}$, $\xi_{\mathrm{Sym}^2E} = \frac{8}{3}$

p	$a_p(E)$	$\mathbf{L}_p^{\mathrm{imp}}(\mathrm{Sym}^2E,1)'$	$\mathcal{L}_p^{\mathrm{an}}(\mathrm{Sym}^2E)$
3	2	$1+p+p^2+2p^3+O(p^4)$	$1+p^2+p^3+O(p^4)$
5	2	$4p+p^2+3p^3+O(p^4)$	$2p+3p^2+3p^3+O(p^4)$
11	-4	$10p+O(p^2)$	$4p+O(p^2)$
13	-2	$3p+O(p^2)$	$8p+O(p^2)$

$E = 216\text{a}1$, $C_{\mathrm{Sym}^2E} = 36^2$, $S_1 = \{2,3\}$, $\xi_{\mathrm{Sym}^2E} = \frac{2}{3}$

p	$a_p(E)$	$\mathbf{L}_p^{\mathrm{imp}}(\mathrm{Sym}^2E,1)'$	$\mathcal{L}_p^{\mathrm{an}}(\mathrm{Sym}^2E)$
5	-4	$2p+3p^2+O(p^3)$	$3+p+O(p^2)$
7	-3	$p+4p^2+O(p^3)$	$3p+p^2+O(p^3)$
11	-4	$6p+O(p^2)$	$3p+O(p^2)$
13	1	$4p+O(p^2)$	$10+O(p)$

$E = 216\text{c}1$, $C_{\mathrm{Sym}^2E} = 36^2$, $S_1 = \{2,3\}$, $\xi_{\mathrm{Sym}^2E} = \frac{2}{3}$

p	$a_p(E)$	$\mathbf{L}_p^{\mathrm{imp}}(\mathrm{Sym}^2E,1)'$	$\mathcal{L}_p^{\mathrm{an}}(\mathrm{Sym}^2E)$
5	1	$2p+O(p^2)$	$1+O(p)$
7	3	$4p+2p^2+O(p^3)$	$5p+5p^2+O(p^3)$
11	-5	$10p+O(p^2)$	$6p+O(p^2)$
13	4	$2p+O(p^2)$	$11p+O(p^2)$

$E = 220\text{a}1$, $C_{\mathrm{Sym}^2E} = 110^2$, $S_1 = \{2\}$, $\xi_{\mathrm{Sym}^2E} = \frac{8}{3}$

p	$a_p(E)$	$\mathbf{L}_p^{\mathrm{imp}}(\mathrm{Sym}^2E,1)'$	$\mathcal{L}_p^{\mathrm{an}}(\mathrm{Sym}^2E)$
3	-2	$2+O(p^2)$	$2+p+O(p^2)$
7	-4	$p+O(p^3)$	$6p+4p^2+O(p^3)$
13	-4	$2p+O(p^2)$	$6p+O(p^2)$

$E = 220\text{b1}$, $C_{\mathrm{Sym}^2 E} = 110^2$, $S_1 = \{2\}$, $\xi_{\mathrm{Sym}^2 E} = \frac{8}{3}$

p	$a_p(E)$	$\mathbf{L}_p^{\mathrm{imp}}(\mathrm{Sym}^2 E, 1)'$	$\mathcal{L}_p^{\mathrm{an}}(\mathrm{Sym}^2 E)$
3	2	$1 + p^2 + O(p^3)$	$1 + 2p + p^2 + O(p^3)$

$E = 224\text{a1}$, $C_{\mathrm{Sym}^2 E} = 56^2$, $S_1 = \{2\}$, $\xi_{\mathrm{Sym}^2 E} = 1$

p	$a_p(E)$	$\mathbf{L}_p^{\mathrm{imp}}(\mathrm{Sym}^2 E, 1)'$	$\mathcal{L}_p^{\mathrm{an}}(\mathrm{Sym}^2 E)$
3	-2	$p^3 + O(p^4)$	$2p^2 + O(p^3)$
11	-4	$10p + O(p^2)$	$7p + O(p^2)$
13	-4	$8p + O(p^2)$	$12p + O(p^2)$

$E = 228\text{a1}$, $C_{\mathrm{Sym}^2 E} = 114^2$, $S_1 = \{2\}$, $\xi_{\mathrm{Sym}^2 E} = \frac{8}{3}$

p	$a_p(E)$	$\mathbf{L}_p^{\mathrm{imp}}(\mathrm{Sym}^2 E, 1)'$	$\mathcal{L}_p^{\mathrm{an}}(\mathrm{Sym}^2 E)$
5	2	$p + O(p^3)$	$3p + 2p^2 + O(p^3)$
11	2	$9p + O(p^2)$	$10p + O(p^2)$
13	2	$11p + O(p^2)$	$12p + O(p^2)$

$E = 228\text{b1}$, $C_{\mathrm{Sym}^2 E} = 114^2$, $S_1 = \{2\}$, $\xi_{\mathrm{Sym}^2 E} = \frac{8}{3}$

p	$a_p(E)$	$\mathbf{L}_p^{\mathrm{imp}}(\mathrm{Sym}^2 E, 1)'$	$\mathcal{L}_p^{\mathrm{an}}(\mathrm{Sym}^2 E)$
5	-3	$2p + 4p^2 + O(p^3)$	$p + 4p^2 + O(p^3)$
7	1	$6p + O(p^2)$	$5 + O(p)$
11	-5	$2p + O(p^2)$	$8p + O(p^2)$
13	-6	$p + O(p^2)$	$8p + O(p^2)$

$E = 232\text{a1}$, $C_{\mathrm{Sym}^2 E} = 116^2$, $S_1 = \{2\}$, $\xi_{\mathrm{Sym}^2 E} = 2$

p	$a_p(E)$	$\mathbf{L}_p^{\mathrm{imp}}(\mathrm{Sym}^2 E, 1)'$	$\mathcal{L}_p^{\mathrm{an}}(\mathrm{Sym}^2 E)$
3	-1	$p + O(p^4)$	$2 + 2p^2 + O(p^3)$
5	-3	$p + 3p^2 + O(p^3)$	$4p + p^2 + O(p^3)$
7	2	$p + O(p^3)$	$3p + 5p^2 + O(p^3)$
11	-3	$3p + O(p^2)$	$p + O(p^2)$
13	-5	$2p + O(p^2)$	$7p + O(p^2)$

$E = 232\text{b}1$, $C_{\mathrm{Sym}^2 E} = 116^2$, $S_1 = \{2\}$, $\xi_{\mathrm{Sym}^2 E} = 2$

p	$a_p(E)$	$\mathbf{L}_p^{\mathrm{imp}}(\mathrm{Sym}^2 E, 1)'$	$\mathcal{L}_p^{\mathrm{an}}(\mathrm{Sym}^2 E)$
3	1	$p + 2p^3 + O(p^4)$	$2 + O(p^3)$
5	1	$3p + O(p^2)$	$3 + O(p)$
7	2	$2p + O(p^3)$	$6p + 3p^2 + O(p^3)$
11	3	$6p + O(p^2)$	$2p + O(p^2)$
13	-1	$O(p)$	$O(1)$

$E = 236\text{a}1$, $C_{\mathrm{Sym}^2 E} = 118^2$, $S_1 = \{2\}$, $\xi_{\mathrm{Sym}^2 E} = \frac{8}{3}$

p	$a_p(E)$	$\mathbf{L}_p^{\mathrm{imp}}(\mathrm{Sym}^2 E, 1)'$	$\mathcal{L}_p^{\mathrm{an}}(\mathrm{Sym}^2 E)$
3	-1	$2p^2 + O(p^3)$	$p^2 + O(p^3)$
5	-1	$4p + O(p^2)$	$3 + O(p)$
7	-3	$6p + O(p^3)$	$p + 5p^2 + O(p^3)$
11	-2	$3p + O(p^2)$	$7p + O(p^2)$

$E = 236\text{b}1$, $C_{\mathrm{Sym}^2 E} = 118^2$, $S_1 = \{2\}$, $\xi_{\mathrm{Sym}^2 E} = \frac{8}{3}$

p	$a_p(E)$	$\mathbf{L}_p^{\mathrm{imp}}(\mathrm{Sym}^2 E, 1)'$	$\mathcal{L}_p^{\mathrm{an}}(\mathrm{Sym}^2 E)$
3	1	$2p^2 + O(p^4)$	$p^2 + O(p^4)$
5	3	$2p^3 + O(p^4)$	$p^3 + O(p^4)$
7	-1	$6 + O(p)$	$5p^{-1} + O(1)$
11	6	$5p + O(p^2)$	$9p + O(p^2)$
13	-4	$7p + O(p^2)$	$8p + O(p^2)$

$E = 240\text{b}1$, $C_{\mathrm{Sym}^2 E} = 30^2$, $S_1 = \{2\}$, $\xi_{\mathrm{Sym}^2 E} = \frac{1}{3}$

p	$a_p(E)$	$\mathbf{L}_p^{\mathrm{imp}}(\mathrm{Sym}^2 E, 1)'$	$\mathcal{L}_p^{\mathrm{an}}(\mathrm{Sym}^2 E)$
7	4	$p + 2p^2 + O(p^3)$	$6p + p^2 + O(p^3)$
13	2	$10p + O(p^2)$	$p + O(p^2)$

$E = 244\text{a}1$, $C_{\mathrm{Sym}^2 E} = 122^2$, $S_1 = \{2\}$, $\xi_{\mathrm{Sym}^2 E} = \frac{8}{3}$

p	$a_p(E)$	$\mathbf{L}_p^{\mathrm{imp}}(\mathrm{Sym}^2 E, 1)'$	$\mathcal{L}_p^{\mathrm{an}}(\mathrm{Sym}^2 E)$
5	-3	$p + 4p^2 + O(p^3)$	$3p + O(p^3)$
7	-3	$3p + 2p^2 + O(p^3)$	$4p + O(p^3)$
11	-1	$4p + O(p^2)$	$2 + O(p)$
13	1	$O(p^2)$	$O(p)$

$E = 248\text{a}1$, $C_{\text{Sym}^2 E} = 124^2$, $S_1 = \{2\}$, $\xi_{\text{Sym}^2 E} = 2$

p	$a_p(E)$	$\mathbf{L}_p^{\text{imp}}(\text{Sym}^2 E, 1)'$	$\mathcal{L}_p^{\text{an}}(\text{Sym}^2 E)$
3	-2	$p^2 + O(p^4)$	$p + O(p^3)$
5	1	$3p + O(p^2)$	$3 + O(p)$
7	-3	$2p + 2p^2 + O(p^3)$	$2p + 4p^2 + O(p^3)$
11	-2	$7p^2 + O(p^3)$	$p^2 + O(p^3)$
13	-2	$6p + O(p^2)$	$4p + O(p^2)$

$E = 248\text{b}1$, $C_{\text{Sym}^2 E} = 124^2$, $S_1 = \{2\}$, $\xi_{\text{Sym}^2 E} = 2$

p	$a_p(E)$	$\mathbf{L}_p^{\text{imp}}(\text{Sym}^2 E, 1)'$	$\mathcal{L}_p^{\text{an}}(\text{Sym}^2 E)$
3	-2	$2p^3 + O(p^4)$	$2p^2 + O(p^3)$
5	2	$4p + O(p^3)$	$p + 2p^2 + O(p^3)$
11	2	$5p + 5p^2 + O(p^3)$	$7p + 5p^2 + O(p^3)$
13	4	$12p + O(p^2)$	$9p + O(p^2)$

$E = 248\text{c}1$, $C_{\text{Sym}^2 E} = 124^2$, $S_1 = \{2\}$, $\xi_{\text{Sym}^2 E} = 2$

p	$a_p(E)$	$\mathbf{L}_p^{\text{imp}}(\text{Sym}^2 E, 1)'$	$\mathcal{L}_p^{\text{an}}(\text{Sym}^2 E)$
5	-3	$p^3 + O(p^4)$	$4p^3 + O(p^4)$
7	-3	$6p + O(p^3)$	$6p + 6p^2 + O(p^3)$
11	2	$3p^2 + O(p^3)$	$2p^2 + O(p^3)$
13	-4	$7p + O(p^2)$	$2p + O(p^2)$

$E = 256\text{a}1$, $C_{\text{Sym}^2 E} = 8^2$, $S_1 = \{2\}$, $\xi_{\text{Sym}^2 E} = \frac{1}{6}$

p	$a_p(E)$	$\mathbf{L}_p^{\text{imp}}(\text{Sym}^2 E, 1)'$	$\mathcal{L}_p^{\text{an}}(\text{Sym}^2 E)$
3	-2	$p + p^3 + O(p^4)$	$p + p^2 + p^3 + O(p^4)$
11	-6	$2p + O(p^2)$	$7p + O(p^2)$

$E = 256\text{b}1$, $C_{\text{Sym}^2 E} = 8^2$, $S_1 = \{2\}$, $\xi_{\text{Sym}^2 E} = \frac{1}{6}$

p	$a_p(E)$	$\mathbf{L}_p^{\text{imp}}(\text{Sym}^2 E, 1)'$	$\mathcal{L}_p^{\text{an}}(\text{Sym}^2 E)$
5	-4	$p^2 + O(p^3)$	$p + O(p^2)$
13	-4	$5p + O(p^2)$	$6p + O(p^2)$

$E = 260\text{a}1$, $C_{\text{Sym}^2 E} = 130^2$, $S_1 = \{2\}$, $\xi_{\text{Sym}^2 E} = \frac{8}{3}$

p	$a_p(E)$	$\mathbf{L}_p^{\text{imp}}(\text{Sym}^2 E, 1)'$	$\mathcal{L}_p^{\text{an}}(\text{Sym}^2 E)$
3	2	$2 + p^2 + O(p^3)$	$2 + p + 2p^2 + O(p^3)$
7	2	$p^2 + O(p^3)$	$4p^2 + O(p^3)$
11	4	$4p + O(p^2)$	$6p + O(p^2)$

$E = 264\text{a1}$, $C_{\mathrm{Sym}^2E} = 132^2$, $S_1 = \{2\}$, $\xi_{\mathrm{Sym}^2E} = 2$

p	$a_p(E)$	$\mathbf{L}_p^{\mathrm{imp}}(\mathrm{Sym}^2E,1)'$	$\mathcal{L}_p^{\mathrm{an}}(\mathrm{Sym}^2E)$
7	2	$3p + p^2 + O(p^3)$	$2p + 5p^2 + O(p^3)$

$E = 264\text{b1}$, $C_{\mathrm{Sym}^2E} = 132^2$, $S_1 = \{2\}$, $\xi_{\mathrm{Sym}^2E} = 2$

p	$a_p(E)$	$\mathbf{L}_p^{\mathrm{imp}}(\mathrm{Sym}^2E,1)'$	$\mathcal{L}_p^{\mathrm{an}}(\mathrm{Sym}^2E)$
5	2	$p + 2p^2 + O(p^3)$	$4p + 4p^2 + O(p^3)$
13	2	$12p + O(p^2)$	$8p + O(p^2)$

$E = 264\text{c1}$, $C_{\mathrm{Sym}^2E} = 132^2$, $S_1 = \{2\}$, $\xi_{\mathrm{Sym}^2E} = 2$

p	$a_p(E)$	$\mathbf{L}_p^{\mathrm{imp}}(\mathrm{Sym}^2E,1)'$	$\mathcal{L}_p^{\mathrm{an}}(\mathrm{Sym}^2E)$
5	-2	$p^2 + O(p^3)$	$4p^2 + O(p^3)$
7	4	$5p + 6p^2 + O(p^3)$	$5p + 5p^2 + O(p^3)$
13	6	$5p + O(p^2)$	$10p + O(p^2)$

$E = 264\text{d1}$, $C_{\mathrm{Sym}^2E} = 132^2$, $S_1 = \{2\}$, $\xi_{\mathrm{Sym}^2E} = 2$

p	$a_p(E)$	$\mathbf{L}_p^{\mathrm{imp}}(\mathrm{Sym}^2E,1)'$	$\mathcal{L}_p^{\mathrm{an}}(\mathrm{Sym}^2E)$
5	4	$3p + O(p^2)$	$4 + O(p)$
7	-2	$6p + O(p^2)$	$4p + O(p^2)$

$E = 268\text{a1}$, $C_{\mathrm{Sym}^2E} = 134^2$, $S_1 = \{2\}$, $\xi_{\mathrm{Sym}^2E} = \frac{8}{3}$

p	$a_p(E)$	$\mathbf{L}_p^{\mathrm{imp}}(\mathrm{Sym}^2E,1)'$	$\mathcal{L}_p^{\mathrm{an}}(\mathrm{Sym}^2E)$
3	2	$p^{-1} + 2 + p + O(p^2)$	$p^{-1} + 1 + O(p^2)$
5	2	$4p + 3p^2 + O(p^3)$	$2p + 4p^2 + O(p^3)$
7	2	$p + 6p^2 + O(p^3)$	$4p + 3p^2 + O(p^3)$
11	-4	$3p + O(p^2)$	$10p + O(p^2)$
13	-6	$6p + O(p^2)$	$9p + O(p^2)$

$E = 272\text{d1}$, $C_{\mathrm{Sym}^2E} = 34^2$, $S_1 = \{2\}$, $\xi_{\mathrm{Sym}^2E} = \frac{1}{3}$

p	$a_p(E)$	$\mathbf{L}_p^{\mathrm{imp}}(\mathrm{Sym}^2E,1)'$	$\mathcal{L}_p^{\mathrm{an}}(\mathrm{Sym}^2E)$
3	2	$p + p^2 + p^3 + O(p^4)$	$2p + 2p^2 + 2p^3 + O(p^4)$
7	4	$3p + 2p^2 + O(p^3)$	$4p + 2p^2 + O(p^3)$
11	-6	$p + O(p^2)$	$10p + O(p^2)$
13	2	$7p + O(p^2)$	$2p + O(p^2)$

$E = 280\text{a}1$, $C_{\mathrm{Sym}^2 E} = 140^2$, $S_1 = \{2\}$, $\xi_{\mathrm{Sym}^2 E} = 2$

p	$a_p(E)$	$\mathbf{L}_p^{\mathrm{imp}}(\mathrm{Sym}^2 E, 1)'$	$\mathcal{L}_p^{\mathrm{an}}(\mathrm{Sym}^2 E)$
3	-1	$p + 2p^2 + 2p^3 + O(p^4)$	$2 + p + p^2 + O(p^3)$
11	-5	$7p + O(p^2)$	$8p + O(p^2)$
13	1	$O(p)$	$O(1)$

$E = 280\text{b}1$, $C_{\mathrm{Sym}^2 E} = 140^2$, $S_1 = \{2\}$, $\xi_{\mathrm{Sym}^2 E} = 2$

p	$a_p(E)$	$\mathbf{L}_p^{\mathrm{imp}}(\mathrm{Sym}^2 E, 1)'$	$\mathcal{L}_p^{\mathrm{an}}(\mathrm{Sym}^2 E)$
11	-5	$4p + O(p^2)$	$3p + O(p^2)$
13	-5	$10p + O(p^2)$	$9p + O(p^2)$

$E = 288\text{a}1$, $C_{\mathrm{Sym}^2 E} = 24^2$, $S_1 = \{2, 3\}$, $\xi_{\mathrm{Sym}^2 E} = \frac{1}{2}$

p	$a_p(E)$	$\mathbf{L}_p^{\mathrm{imp}}(\mathrm{Sym}^2 E, 1)'$	$\mathcal{L}_p^{\mathrm{an}}(\mathrm{Sym}^2 E)$
5	-4	$2p^2 + O(p^3)$	$4p + O(p^2)$
13	-6	$7p + O(p^2)$	$4p + O(p^2)$

$E = 296\text{a}1$, $C_{\mathrm{Sym}^2 E} = 148^2$, $S_1 = \{2\}$, $\xi_{\mathrm{Sym}^2 E} = 2$

p	$a_p(E)$	$\mathbf{L}_p^{\mathrm{imp}}(\mathrm{Sym}^2 E, 1)'$	$\mathcal{L}_p^{\mathrm{an}}(\mathrm{Sym}^2 E)$
3	-1	$p^2 + O(p^4)$	$2p + O(p^3)$
5	-2	$4p + p^2 + O(p^3)$	$p + p^2 + O(p^3)$
7	1	$4p + O(p^2)$	$6 + O(p)$
11	1	$O(p)$	$O(1)$
13	-6	$2p + O(p^2)$	$4p + O(p^2)$

$E = 296\text{b}1$, $C_{\mathrm{Sym}^2 E} = 148^2$, $S_1 = \{2\}$, $\xi_{\mathrm{Sym}^2 E} = 2$

p	$a_p(E)$	$\mathbf{L}_p^{\mathrm{imp}}(\mathrm{Sym}^2 E, 1)'$	$\mathcal{L}_p^{\mathrm{an}}(\mathrm{Sym}^2 E)$
3	-1	$2p + 2p^2 + p^3 + O(p^4)$	$1 + 2p + p^2 + O(p^3)$
7	-3	$2p + 3p^2 + O(p^3)$	$2p + 5p^2 + O(p^3)$
11	-3	$8p + O(p^2)$	$10p + O(p^2)$

$E = 300\text{a}1$, $C_{\mathrm{Sym}^2 E} = 30^2$, $S_1 = \{2, 5\}$, $\xi_{\mathrm{Sym}^2 E} = \frac{8}{9}$

p	$a_p(E)$	$\mathbf{L}_p^{\mathrm{imp}}(\mathrm{Sym}^2 E, 1)'$	$\mathcal{L}_p^{\mathrm{an}}(\mathrm{Sym}^2 E)$
7	1	$p + O(p^2)$	$6 + O(p)$
11	6	$5p^2 + O(p^3)$	$5p^2 + O(p^3)$
13	-5	$3p + O(p^2)$	$9p + O(p^2)$

B.2 Tables of $\mathcal{L}$-invariants for elliptic curves E with $D(E,1) = 0$

Included below are the values we computed for both the derivative of the automorphic p-adic L-function $\mathbf{L}_p^{\mathrm{aut}}(\mathrm{Sym}^2 E, s)$ at $s = 1$ and the corresponding $\mathcal{L}$-invariant term, for the six exceptional elliptic curves with $D(E,1) = 0$ (we omitted these specimens from Section B.1 as $\mathbf{L}_p^{\mathrm{imp}}(\mathrm{Sym}^2 E, 1)' = 0$ for each of these six curves). To calculate these p-adic numbers, we used the generalised congruences given in Theorem 4.11.

$E = 176\mathrm{b}1$, $C_{\mathrm{Sym}^2 E} = 11^2$, $S_1 = \{2\}$, $\xi_{\mathrm{Sym}^2 E} = \frac{2}{5}$

p	$a_p(E)$	$\mathbf{L}_p^{\mathrm{imp}}(\mathrm{Sym}^2 E, 1)'$	$\mathcal{L}_p^{\mathrm{an}}(\mathrm{Sym}^2 E)$
3	1	$p + O(p^4)$	$1 + 2p^2 + O(p^3)$
5	1	$p + O(p^2)$	$p + O(p^2)$
7	2	$2p + O(p^2)$	$2p + O(p^2)$
13	4	$4p + O(p^2)$	$2p + O(p^2)$

$E = 196\mathrm{a}1$, $C_{\mathrm{Sym}^2 E} = 14^2$, $S_1 = \{2, 7\}$, $\xi_{\mathrm{Sym}^2 E} = \frac{2}{9}$

p	$a_p(E)$	$\mathbf{L}_p^{\mathrm{imp}}(\mathrm{Sym}^2 E, 1)'$	$\mathcal{L}_p^{\mathrm{an}}(\mathrm{Sym}^2 E)$
3	-1	$1 + 2p + p^2 + O(p^3)$	$2p + p^2 + 2p^3 + O(p^4)$
5	-3	$3p + O(p^2)$	$3p + O(p^2)$
11	-3	$p + O(p^2)$	$3p + O(p^2)$
13	-2	$10p + O(p^2)$	$8p + O(p^2)$

$E = 200\mathrm{b}1$, $C_{\mathrm{Sym}^2 E} = 20^2$, $S_1 = \{2, 5\}$, $\xi_{\mathrm{Sym}^2 E} = \frac{1}{2}$

p	$a_p(E)$	$\mathbf{L}_p^{\mathrm{imp}}(\mathrm{Sym}^2 E, 1)'$	$\mathcal{L}_p^{\mathrm{an}}(\mathrm{Sym}^2 E)$
3	-2	$p + p^3 + O(p^4)$	$1 + p + p^2 + O(p^3)$
7	-2	$4p + O(p^2)$	$6p + O(p^2)$
11	-4	$7p + O(p^2)$	$p + O(p^2)$
13	-4	$O(p^2)$	$O(p^2)$

$E = 240\mathrm{d}1$, $C_{\mathrm{Sym}^2 E} = 15^2$, $S_1 = \{2\}$, $\xi_{\mathrm{Sym}^2 E} = \frac{1}{4}$

p	$a_p(E)$	$\mathbf{L}_p^{\mathrm{imp}}(\mathrm{Sym}^2 E, 1)'$	$\mathcal{L}_p^{\mathrm{an}}(\mathrm{Sym}^2 E)$
11	4	$10p + O(p^2)$	$6p + O(p^2)$
13	-2	$3p + O(p^2)$	$3p + O(p^2)$

$E = 272\text{b1}$, $C_{\mathrm{Sym}^2 E} = 17^2$, $S_1 = \{2\}$, $\xi_{\mathrm{Sym}^2 E} = \frac{1}{4}$

p	$a_p(E)$	$\mathbf{L}_p^{\mathrm{imp}}(\mathrm{Sym}^2 E, 1)'$	$\mathcal{L}_p^{\mathrm{an}}(\mathrm{Sym}^2 E)$
5	-2	$p + 4p^2 + O(p^3)$	$2p + 2p^2 + O(p^3)$
7	-4	$2p + O(p^2)$	$2p + O(p^2)$
13	-2	$O(p)$	$O(p)$

$E = 300\text{c1}$, $C_{\mathrm{Sym}^2 E} = 30^2$, $S_1 = \{2, 5\}$, $\xi_{\mathrm{Sym}^2 E} = \frac{1}{3}$

p	$a_p(E)$	$\mathbf{L}_p^{\mathrm{imp}}(\mathrm{Sym}^2 E, 1)'$	$\mathcal{L}_p^{\mathrm{an}}(\mathrm{Sym}^2 E)$
7	4	$2p + O(p^2)$	$5p + O(p^2)$
11	-4	$4p + O(p^2)$	$4p + O(p^2)$

Appendix C

Tables of $\mathcal{L}$-invariants for higher weight modular forms

Below are tables of $\mathcal{L}$-invariants and derivatives of the primitive p-adic L-function for the symmetric squares of modular forms of weight $k > 2$. Labelling of modular forms follows the conventions of the "L-functions and Modular Forms Database" [64]. For example $f = 5.4.a.a$ is the unique cusp form of level $N = 5$, weight $k = 4$ with trivial character.

C.1 Weight $k = 4$

$f = 5.4.a.a$, $\xi_{\mathrm{Sym}^2 f} = 32$

p	$a_p(f)$	$L'_p(\mathrm{Sym}^2 f, k-1)$	$\mathcal{L}_p(\mathrm{Sym}^2 f)$
3	2	$p + 2p^2 + 2p^3 + p^4 + O(p^6)$	$p + 2p^4 + 2p^5 + O(p^6)$
7	6	$p + 2p^2 + O(p^3)$	$4p + p^2 + O(p^3)$

$f = 6.4.a.a$, $\xi_{\mathrm{Sym}^2 f} = 32$

p	$a_p(f)$	$L'_p(\mathrm{Sym}^2 f, k-1)$	$\mathcal{L}_p(\mathrm{Sym}^2 f)$
5	6	$3p + 2p^2 + O(p^4)$	$2p + 4p^3 + O(p^4)$
7	-16	$p^2 + O(p^3)$	$4p^2 + O(p^3)$

$f = 7.4.a.a$, $\xi_{\mathrm{Sym}^2 f} = 32$

p	$a_p(f)$	$L'_p(\mathrm{Sym}^2 f, k-1)$	$\mathcal{L}_p(\mathrm{Sym}^2 f)$
3	-2	$p + 2p^4 + O(p^6)$	$p + p^2 + p^3 + p^5 + O(p^6)$
5	16	$3p^2 + 3p^3 + O(p^4)$	$2p^2 + 4p^3 + O(p^4)$

$f = 8.4.a.a$, $\xi_{\mathrm{Sym}^2 f} = 16$

p	$a_p(f)$	$L_p'(\mathrm{Sym}^2 f, k-1)$	$\mathcal{L}_p(\mathrm{Sym}^2 f)$
3	-4	$p + p^3 + O(p^6)$	$2p + 2p^2 + p^3 + 2p^5 + O(p^6)$
5	-2	$2p + p^3 + O(p^4)$	$p + 3p^2 + 2p^3 + O(p^4)$
7	24	$4p + O(p^3)$	$4p + 4p^2 + O(p^3)$

$f = 9.4.a.a$, $\xi_{\mathrm{Sym}^2 f} = 8$

p	$a_p(f)$	$L_p'(\mathrm{Sym}^2 f, k-1)$	$\mathcal{L}_p(\mathrm{Sym}^2 f)$
7	20	$p + 4p^2 + O(p^3)$	$2p + 3p^2 + O(p^3)$

$f = 10.4.a.a$, $\xi_{\mathrm{Sym}^2 f} = 32$

p	$a_p(f)$	$L_p'(\mathrm{Sym}^2 f, k-1)$	$\mathcal{L}_p(\mathrm{Sym}^2 f)$
3	-8	$O(p^6)$	$O(p^6)$
7	-4	$6p + O(p^3)$	$3p + 3p^2 + O(p^3)$

$f = 12.4.a.a$, $\xi_{\mathrm{Sym}^2 f} = \frac{32}{3}$

p	$a_p(f)$	$L_p'(\mathrm{Sym}^2 f, k-1)$	$\mathcal{L}_p(\mathrm{Sym}^2 f)$
5	-18	$4p + 4p^2 + O(p^4)$	$3p + 2p^2 + 3p^3 + O(p^4)$
7	8	$p + 5p^2 + O(p^3)$	$5p + 5p^2 + O(p^3)$

$f = 13.4.a.a$, $\xi_{\mathrm{Sym}^2 f} = 32$

p	$a_p(f)$	$L_p'(\mathrm{Sym}^2 f, k-1)$	$\mathcal{L}_p(\mathrm{Sym}^2 f)$
3	-7	$p + 2p^3 + 2p^4 + O(p^5)$	$p + p^2 + O(p^5)$
5	-7	$3p + p^2 + O(p^4)$	$2p + p^2 + 4p^3 + O(p^4)$
7	-13	$2p + 5p^2 + O(p^3)$	$p + O(p^3)$

$f = 14.4.a.a$, $\xi_{\mathrm{Sym}^2 f} = 32$

p	$a_p(f)$	$L_p'(\mathrm{Sym}^2 f, k-1)$	$\mathcal{L}_p(\mathrm{Sym}^2 f)$
3	8	$2p^2 + p^3 + p^4 + O(p^5)$	$2p^2 + 2p^4 + O(p^5)$
5	-14	$p + 4p^3 + O(p^4)$	$4p + O(p^4)$

$f = 14.4.a.b$, $\xi_{\mathrm{Sym}^2 f} = 32$

p	$a_p(f)$	$L_p'(\mathrm{Sym}^2 f, k-1)$	$\mathcal{L}_p(\mathrm{Sym}^2 f)$
3	-2	$2p + p^2 + p^3 + p^4 + O(p^5)$	$2p + 2p^3 + O(p^5)$
5	-12	$2p + 2p^2 + 2p^3 + O(p^4)$	$3p + 4p^2 + p^3 + O(p^4)$

$f = 15.4.a.a$, $\xi_{\mathrm{Sym}^2 f} = 32$

p	$a_p(f)$	$L_p'(\mathrm{Sym}^2 f, k-1)$	$\mathcal{L}_p(\mathrm{Sym}^2 f)$
7	-24	$2p + 4p^2 + O(p^3)$	$p + 3p^2 + O(p^3)$

$f = 15.4.a.b$, $\xi_{\mathrm{Sym}^2 f} = 32$

p	$a_p(f)$	$L'_p(\mathrm{Sym}^2 f, k-1)$	$\mathcal{L}_p(\mathrm{Sym}^2 f)$
7	20	$4p + O(p^3)$	$2p + 2p^2 + O(p^3)$

C.2 Weight $k = 6$

$f = 3.6.a.a$, $\xi_{\mathrm{Sym}^2 f} = \frac{128}{3}$

p	$a_p(f)$	$L'_p(\mathrm{Sym}^2 f, k-1)$	$\mathcal{L}_p(\mathrm{Sym}^2 f)$
5	6	$2p + 4p^2 + O(p^4)$	$2p + 2p^2 + p^3 + O(p^4)$
7	-40	$3p + 4p^2 + O(p^3)$	$2p + 3p^2 + O(p^3)$

$f = 4.6.a.a$, $\xi_{\mathrm{Sym}^2 f} = \frac{128}{9}$

p	$a_p(f)$	$L'_p(\mathrm{Sym}^2 f, k-1)$	$\mathcal{L}_p(\mathrm{Sym}^2 f)$
5	54	$p + p^3 + O(p^4)$	$3p + 2p^2 + 2p^3 + O(p^4)$
7	-88	$6p^2 + O(p^3)$	$5p^2 + O(p^3)$

$f = 5.6.a.a$, $\xi_{\mathrm{Sym}^2 f} = \frac{128}{3}$

p	$a_p(f)$	$L'_p(\mathrm{Sym}^2 f, k-1)$	$\mathcal{L}_p(\mathrm{Sym}^2 f)$
3	-4	$2p + p^3 + p^4 + O(p^6)$	$p + p^3 + 2p^5 + O(p^6)$
7	192	$3p^2 + O(p^3)$	$2p^2 + O(p^3)$

$f = 6.6.a.a$, $\xi_{\mathrm{Sym}^2 f} = \frac{128}{3}$

p	$a_p(f)$	$L'_p(\mathrm{Sym}^2 f, k-1)$	$\mathcal{L}_p(\mathrm{Sym}^2 f)$
5	-66	$p + p^2 + 2p^3 + O(p^4)$	$p + p^3 + O(p^4)$
7	176	$5p + O(p^3)$	$p + 4p^2 + O(p^3)$

$f = 7.6.a.a$, $\xi_{\mathrm{Sym}^2 f} = \frac{128}{3}$

p	$a_p(f)$	$L'_p(\mathrm{Sym}^2 f, k-1)$	$\mathcal{L}_p(\mathrm{Sym}^2 f)$
3	-14	$2p + 2p^3 + 2p^4 + 2p^5 + O(p^6)$	$p + p^4 + p^5 + O(p^6)$
5	-56	$2p + 3p^2 + 4p^3 + O(p^4)$	$2p + p^2 + p^3 + O(p^4)$

$f = 8.6.a.a$, $\xi_{\mathrm{Sym}^2 f} = \frac{64}{3}$

p	$a_p(f)$	$L'_p(\mathrm{Sym}^2 f, k-1)$	$\mathcal{L}_p(\mathrm{Sym}^2 f)$
3	20	$2p + 2p^3 + 2p^5 + O(p^6)$	$2p + 2p^5 + O(p^6)$
5	-74	$p^2 + O(p^4)$	$2p^2 + 3p^3 + O(p^4)$
7	-24	$5p + 4p^2 + O(p^3)$	$2p + 4p^2 + O(p^3)$

$f = 10.6.a.a$, $\xi_{\mathrm{Sym}^2 f} = \frac{128}{3}$

p	$a_p(f)$	$L'_p(\mathrm{Sym}^2 f, k-1)$	$\mathcal{L}_p(\mathrm{Sym}^2 f)$
3	-26	$2p^2 + p^3 + p^4 + O(p^5)$	$p^2 + 2p^3 + 2p^4 + O(p^5)$
7	-22	$4p + 4p^2 + O(p^3)$	$5p + 2p^2 + O(p^3)$

$f = 10.6.a.b$, $\xi_{\mathrm{Sym}^2 f} = \frac{128}{3}$

p	$a_p(f)$	$L'_p(\mathrm{Sym}^2 f, k-1)$	$\mathcal{L}_p(\mathrm{Sym}^2 f)$
7	-172	$3p + 2p^2 + O(p^3)$	$2p + 4p^2 + O(p^3)$

$f = 10.6.a.c$, $\xi_{\mathrm{Sym}^2 f} = \frac{128}{3}$

p	$a_p(f)$	$L'_p(\mathrm{Sym}^2 f, k-1)$	$\mathcal{L}_p(\mathrm{Sym}^2 f)$
7	-118	$5p + 4p^2 + O(p^3)$	$p + 2p^2 + O(p^3)$

$f = 11.6.a.a$, $\xi_{\mathrm{Sym}^2 f} = \frac{128}{3}$

p	$a_p(f)$	$L'_p(\mathrm{Sym}^2 f, k-1)$	$\mathcal{L}_p(\mathrm{Sym}^2 f)$
5	-19	$2p + 2p^2 + O(p^4)$	$2p + 3p^3 + O(p^4)$
7	10	$4p + p^2 + O(p^3)$	$5p + O(p^3)$

$f = 14.6.a.a$, $\xi_{\mathrm{Sym}^2 f} = \frac{128}{3}$

p	$a_p(f)$	$L'_p(\mathrm{Sym}^2 f, k-1)$	$\mathcal{L}_p(\mathrm{Sym}^2 f)$
3	10	$p + 2p^2 + 2p^4 + O(p^5)$	$2p + 2p^2 + 2p^3 + 2p^4 + O(p^5)$
5	84	$p + 2p^2 + O(p^3)$	$p + p^2 + O(p^3)$

$f = 14.6.a.b$, $\xi_{\mathrm{Sym}^2 f} = \frac{128}{3}$

p	$a_p(f)$	$L'_p(\mathrm{Sym}^2 f, k-1)$	$\mathcal{L}_p(\mathrm{Sym}^2 f)$
3	8	$2p^2 + 2p^4 + O(p^5)$	$p^2 + O(p^5)$

$f = 15.6.a.a$, $\xi_{\mathrm{Sym}^2 f} = \frac{128}{3}$

p	$a_p(f)$	$L'_p(\mathrm{Sym}^2 f, k-1)$	$\mathcal{L}_p(\mathrm{Sym}^2 f)$
7	-132	$6p + 2p^2 + O(p^3)$	$4p + 2p^2 + O(p^3)$

$f = 15.6.a.b$, $\xi_{\mathrm{Sym}^2 f} = \frac{128}{3}$

p	$a_p(f)$	$L'_p(\mathrm{Sym}^2 f, k-1)$	$\mathcal{L}_p(\mathrm{Sym}^2 f)$
7	12	$5p + O(p^3)$	$p + 4p^2 + O(p^3)$

C.3 Weight $k = 8$

$f = 2.8.a.a$, $\xi_{\mathrm{Sym}^2 f} = \frac{1024}{45}$

p	$a_p(f)$	$L_p'(\mathrm{Sym}^2 f, k-1)$	$\mathcal{L}_p(\mathrm{Sym}^2 f)$
7	1016	$6p + 2p^2 + 4p^3 + O(p^4)$	$3p + 4p^2 + 3p^3 + O(p^4)$

$f = 3.8.a.a$, $\xi_{\mathrm{Sym}^2 f} = \frac{1024}{45}$

p	$a_p(f)$	$L_p'(\mathrm{Sym}^2 f, k-1)$	$\mathcal{L}_p(\mathrm{Sym}^2 f)$
7	-64	$p + O(p^3)$	$4p + O(p^3)$

$f = 5.8.a.a$, $\xi_{\mathrm{Sym}^2 f} = \frac{1024}{45}$

p	$a_p(f)$	$L_p'(\mathrm{Sym}^2 f, k-1)$	$\mathcal{L}_p(\mathrm{Sym}^2 f)$
7	-1644	$5p + 2p^2 + O(p^3)$	$6p + 3p^2 + O(p^3)$

$f = 6.8.a.a$, $\xi_{\mathrm{Sym}^2 f} = \frac{1024}{45}$

p	$a_p(f)$	$L_p'(\mathrm{Sym}^2 f, k-1)$	$\mathcal{L}_p(\mathrm{Sym}^2 f)$
5	-114	$3p + 4p^2 + 3p^3 + O(p^4)$	$2p + 3p^2 + O(p^4)$
7	-1576	$4p + p^2 + O(p^3)$	$2p + 6p^2 + O(p^3)$

$f = 7.8.a.a$, $\xi_{\mathrm{Sym}^2 f} = \frac{1024}{45}$

p	$a_p(f)$	$L_p'(\mathrm{Sym}^2 f, k-1)$	$\mathcal{L}_p(\mathrm{Sym}^2 f)$
5	-84	$3p^2 + 4p^3 + O(p^4)$	$2p^2 + 3p^3 + O(p^4)$

$f = 8.8.a.a$, $\xi_{\mathrm{Sym}^2 f} = \frac{512}{45}$

p	$a_p(f)$	$L_p'(\mathrm{Sym}^2 f, k-1)$	$\mathcal{L}_p(\mathrm{Sym}^2 f)$
5	-82	$2p + 3p^2 + O(p^4)$	$p + 2p^2 + O(p^4)$
7	-456	$4p + 4p^2 + O(p^3)$	$4p + p^2 + O(p^3)$

$f = 8.8.a.b$, $\xi_{\mathrm{Sym}^2 f} = \frac{512}{45}$

p	$a_p(f)$	$L_p'(\mathrm{Sym}^2 f, k-1)$	$\mathcal{L}_p(\mathrm{Sym}^2 f)$
3	44	$p + p^2 + 2p^3 + 2p^4 + p^5 + O(p^6)$	$2p + p^2 + p^3 + p^4 + O(p^6)$
7	-1224	$6p^2 + O(p^3)$	$6p^2 + O(p^3)$

$f = 10.8.a.a$, $\xi_{\mathrm{Sym}^2 f} = \frac{1024}{45}$

p	$a_p(f)$	$L_p'(\mathrm{Sym}^2 f, k-1)$	$\mathcal{L}_p(\mathrm{Sym}^2 f)$
3	28	$2p^2 + 2p^3 + p^4 + O(p^5)$	$2p^2 + p^3 + p^4 + O(p^5)$
7	104	$3p + 5p^2 + O(p^3)$	$5p + O(p^3)$

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