Vanishing of the integral of the Hurwitz zeta function

Draft 14th June 2001

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A proof is given that the improper Riemann integral of $\zeta(s,a)$ with respect to the real parameter a, taken over the interval $(0,1]$, vanishes for all complex s with $\Re(s) < 1$. The integral does not exist (as a finite real number) when $\Re(s) \geq 1$.

Key Words: Hurwitz zeta function, functional equation, improper Riemann integral.

MSC2000 11M35, 30E99.

1. INTRODUCTION

A number of authors have considered mean values of powers of the modulus of the Hurwitz zeta function $\zeta(s,a)$, see [3, 4, 5, 6, 7]. In this paper, the mean of the function itself is considered.

First a functional equation relating the Riemann zeta function to sums of the values of the Hurwitz zeta function at rational values of a is derived. This functional equation underlies the vanishing of the integral of the Hurwitz zeta function.

Consider the values of the function at negative integers:

$$\zeta(-n,a) = -\frac{B_{n+1}(a)}{n+1}, n \geq 0$$

where $B_n(a)$ is the n'th Bernoulli polynomial. The integral of the right hand side expression between 0 and 1 is zero for every n. This appears to be a side-effect of the properties of Bernoulli polynomials (namely for $n \geq 2$, $B_n(0) = B_n(1)$ and $B_n'(x) = nB_{n-1}(x)$), and nothing particularly intrinsic to the zeta function. However, as the theorem below will show, the integral vanishes at every value of the complex variable s to the left of the line $\Re(s) = 1$. The integral does not exist (as a finite real number), on or to the right of this line.

2. THE VANISHING THEOREM

The theorem is proved through developing a number of lemmas. The first is a fundamental, yet easy to derive, functional equation. See also, for example, [2].

Lemma 2.1. *For all integers $k \geq 1$ and all $s \in \mathbb{C} - \{1\}$*

$$k^s \zeta(s) = \sum_{j=1}^{k} \zeta(s, \frac{j}{k}).$$

Proof. Consider the functional equation for the Hurwitz zeta function [1]:

$$\zeta(1-s, \frac{h}{k}) = \frac{2\Gamma(s)}{(2\pi k)^s} \sum_{j=1}^{k} \cos(\frac{\pi s}{2} - \frac{2\pi j h}{k}) \zeta(s, \frac{j}{k})$$

This formula holds for all s and all integers h, k with $1 \leq h \leq k$. Set $h = k$ and obtain

$$\zeta(1-s) = \zeta(1-s, 1) = \frac{2\Gamma(s)}{(2\pi k)^s} \cos(\frac{\pi s}{2}) \sum_{j=1}^{k} \zeta(s, \frac{j}{k})$$

Using the functional equation for the zeta function to write the left hand side in terms of $\zeta(s)$:

$$2(2\pi)^{-s}\Gamma(s)\cos(\frac{\pi s}{2})\zeta(s) = \frac{2\Gamma(s)}{(2\pi k)^s} \cos(\frac{\pi s}{2}) \sum_{j=1}^{k} \zeta(s, \frac{j}{k})$$

so the formula follows for all points except zeros of $\cos(\pi s/2)$ and poles of $\Gamma(s)$. But then it must hold at these points also since each side represents an analytic function, except for $s = 1$. ∎

Corollary 2.1. *If $\zeta(s_0) = 0$ then for all integers $k \geq 1$*

$$\sum_{1 \leq j \leq k, (j,k)=1} \zeta(s_0, \frac{j}{k}) = 0.$$

Proof. Let $\zeta(s_0) = 0$. If $k = 1$ then $\zeta(s_0, 1/1) = \zeta(s_0) = 0$ so assume it is true for all $m < k$. By the Lemma

$$\sum_{j=1}^{k} \zeta(s_0, \frac{j}{k}) = 0.$$

Divide the sum on the left up into groups of terms corresponding to indices (j, k) having the same gcd. By the inductive hypothesis, each of the groups with a common gcd greater than 1 will sum to zero. Omitting these terms we obtain the result of the corollary. ▮

Observation: It follows easily from the corollary that the sums of the values of the Hurwitz zeta function over the Farey fractions of a given order, other than zero, at a zero of zeta function, are all zero.

LEMMA 2.2. *If* $\Re(s) < 1$ *then* $\lim_{n\to\infty} \sum_{j=1}^{n} \zeta(s, \frac{j}{n})\frac{1}{n} = 0$.

Proof. By Lemma 2.1

$$n^{s-1}\zeta(s) = \sum_{j=1}^{n} \zeta(s, \frac{j}{n})\frac{1}{n}.$$

Hence

$$n^{\sigma-1}|\zeta(s)| = |\sum_{j=1}^{n} \zeta(s, \frac{j}{n})\frac{1}{n}|.$$

So if $\sigma < 1$, $\lim_{n\to\infty} n^{\sigma-1}|\zeta(s)| = 0$, and the lemma follows directly. ▮

LEMMA 2.3. *Let* $f : (0, 1] \to \mathbb{R}$ *be a bounded* C^∞ *function. Extend* f *to a Riemann integrable function on* $[0, 1]$ *with* $f(0) = 0$. *If*

$$\lim_{n\to\infty} \sum_{j=1}^{n} f(\frac{j}{n})\frac{1}{n} = 0$$

then $\int_0^1 f = 0$, *because, in this case, the integral is the limit of the given Riemann sums.*

LEMMA 2.4. *If* $\sigma = \Re(s) < 0$ *there exists a positive real number* $B = B(s)$ *such that for all* $a \in (0, 1]$, $|\zeta(s, a)| \leq B(s)$.

Proof. Consider Hurwitz' formula for the zeta function in terms of the periodic zeta function [1], namely:

$$\zeta(1-s,a) = \frac{\Gamma(s)}{(2\pi)^s}\{e^{-\pi is/2}F(a,s) + e^{\pi is/2}F(-a,s)\}$$

where $0 < a \le 1$, $1 < \sigma$ and where

$$F(a,s) = \sum_{n=1}^{\infty}\frac{e^{2\pi ina}}{n^s}.$$

then

$$\zeta(s,a) = \frac{\Gamma(1-s)}{(2\pi)^{1-s}}\{e^{-\pi i(1-s)/2}F(a,1-s) + e^{\pi i(1-s)/2}F(-a,1-s)\}$$

for $\sigma < 0$. Hence

$$\begin{aligned} |\zeta(s,a)| &\le \frac{|\Gamma(1-s)|}{(2\pi)^{1-\sigma}}\{e^{-\pi t/2}|F(a,1-s)| + e^{\pi t/2}|F(-a,1-s)|\} \\ &\le \frac{|\Gamma(1-s)|}{(2\pi)^{1-\sigma}}\{e^{-\pi t/2}\sum_{n=1}^{\infty}\frac{1}{n^{1-\sigma}} + e^{\pi t/2}\sum_{n=1}^{\infty}\frac{1}{n^{1-\sigma}}\} \\ &= \frac{|\Gamma(1-s)|}{(2\pi)^{1-\sigma}}2\cosh(\frac{\pi t}{2})\zeta(1-\sigma) = B(s) \end{aligned}$$

∎

Lemma 2.5. *If $0 < \sigma < 1$, there exists a positive real number $B = B(s)$ such that for all $a \in (0,1]$,*

$$|\zeta(s,a)| \le \frac{1}{a^\sigma} + B(s).$$

Proof. Consider the following expression for the zeta function [1], valid for $0 < \sigma < 1$ and all integers $N \ge 1$, namely

$$\zeta(s,a) = \sum_{n=0}^{N}\frac{1}{(n+a)^s} + \frac{(N+a)^{1-s}}{s-1} - s\int_N^\infty \frac{x-[x]}{(x+a)^{s+1}}dx.$$

Then

$$|\zeta(s,a)| \le \sum_{n=0}^{N}\frac{1}{(n+a)^\sigma} + \frac{(N+a)^{1-\sigma}}{|s-1|} + |s|\int_N^\infty \frac{1}{(x+a)^{1+\sigma}}dx.$$

Let $N = 1$ to derive the upper bound

$$\begin{aligned} |\zeta(s,a)| &\leq \frac{1}{a^\sigma} + \frac{1}{(1+a)^\sigma} + \frac{(1+a)^{1-\sigma}}{|s-1|} + \frac{|s|}{\sigma} \\ &= \frac{1}{a^\sigma} + B(s) \end{aligned}$$

where we may take

$$B(s) = 1 + \frac{2}{|s-1|} + \frac{|s|}{\sigma}.$$

∎

Lemma 2.6. *Let $f : (0,1] \to \mathbb{R}$ be a C^∞ function. Let a positive real number M be such that, for some $\sigma \in (0,1)$*

$$|f(x)| \leq \frac{M}{x^\sigma}$$

for all x. Then f is Riemann integrable (proper if f is bounded). If $\lim_{n\to\infty} \sum_{j=1}^{n} f(\frac{j}{n})\frac{1}{n} = 0$, then $\int_{0+}^{1} f = 0$.

Proof. Let σ_1 be such that $\sigma < \sigma_1 < 1$. Then

$$\frac{|f(x)|}{1/x^{\sigma_1}} \leq x^{\sigma_1 - \sigma} M$$

so

$$\lim_{x\to 0+} \frac{|f(x)|}{1/x^{\sigma_1}} = 0.$$

It follows that f is integrable on $[0,1]$.

Let $\int_{0+}^{1} f = \alpha$ and suppose α is not zero. By replacing f with $-f$ if necessary we can assume $\alpha > 0$.

Since f is integrable there is an N_1 in $\mathbb{N}$ such that, for all $n \geq N_1$,

$$\int_{1/n}^{1} f > \frac{\alpha}{2}$$

There exists an N_2 such that for all $l \geq N_2$

$$|\sum_{j=l}^{nl} f(\frac{j}{nl})\frac{1}{nl} - \int_{1/n}^{1} f| < \frac{\alpha}{4}$$

so

$$-\frac{\alpha}{4} < \sum_{j=l}^{nl} f(\frac{j}{nl})\frac{1}{nl} - \int_{1/n}^{1} f$$

Therefore

$$\frac{\alpha}{2} < \int_{1/n}^{1} f < \frac{\alpha}{4} + \sum_{j=l}^{nl} f(\frac{j}{nl})\frac{1}{nl}$$

so

$$\frac{\alpha}{4} < \sum_{j=l}^{nl} f(\frac{j}{nl})\frac{1}{nl}.$$

By the given hypothesis

$$\lim_{n\to\infty} \sum_{j=1}^{n} f(\frac{j}{n})\frac{1}{n} = 0$$

so there is an N_3 such that for all $l \geq N_3$

$$-\frac{\alpha}{8} < \sum_{j=1}^{ln} f(\frac{j}{ln})\frac{1}{ln} < \frac{\alpha}{8}$$

Therefore

$$-\frac{\alpha}{8} < \sum_{j=1}^{l-1} f(\frac{j}{ln})\frac{1}{ln} + \sum_{j=l}^{ln} f(\frac{j}{ln})\frac{1}{ln} < \frac{\alpha}{8}$$

and so

$$\frac{\alpha}{4} < \frac{\alpha}{8} - \sum_{j=1}^{l-1} f(\frac{j}{ln})\frac{1}{ln}$$

which implies

$$\begin{aligned}
\frac{\alpha}{8} &< \sum_{j=1}^{l-1} |f(\frac{j}{ln})|\frac{1}{ln} \\
&< M\sum_{j=1}^{l} (\frac{ln}{j})^{\sigma}\frac{1}{ln} \\
&= M\frac{l^{\sigma}n^{\sigma}}{ln}\sum_{j=1}^{l}(\frac{1}{j^{\sigma}}) \\
&< 2M\frac{l^{\sigma}n^{\sigma}l^{1-\sigma}}{ln}
\end{aligned}$$

which can be made arbitarily small for n sufficiently large. This contradiction shows we must have $\alpha = 0$, so completes the proof of the Lemma. ∎

LEMMA 2.7. *If $\sigma = 0$ and $|t| \geq 1$ then*

$$|\zeta(it, a)| \leq B(t)$$

for some bound $B(t)$.

Proof. This follows directly from the inequality [1] valid for $-\delta \leq \sigma \leq \delta$ for $\delta < 1$ and $|t| \geq 1$

$$|\zeta(s, a) - a^{-s}| \leq A(\delta)|t|^{1+\delta}.$$

∎

LEMMA 2.8. *If $\sigma = 0$ and $0 \leq t \leq 1$ then*

$$|\zeta(it, a)| \leq B(t).$$

Proof. If $t = 0$, $\zeta(0, a) = 1/2 - a$ so we may assume t is not zero.

To establish a bound we use two expressions for the Hurwitz zeta function derived with Euler summation and integration by parts [1]: For $\sigma > -1$ and $N \geq 0$

$$\begin{aligned} \zeta(s, a) &= \sum_{n=0}^{N} \frac{1}{(n+a)^s} + \frac{(N+a)^{1-s}}{s-1} \\ &- \frac{s}{2!}\{\zeta(s+1, a) - \sum_{n=0}^{N} \frac{1}{(n+a)^{s+1}}\} \\ &- \frac{s(s+1)}{2!} \sum_{n=N}^{\infty} \int_0^1 \frac{u^2}{(n+a+u)^{s+2}} du \end{aligned}$$

and if $\sigma > 0$

$$\begin{aligned} \zeta(s, a) &= \sum_{n=0}^{N} \frac{1}{(n+a)^s} + \frac{(N+a)^{1-s}}{s-1} \\ &- \int_N^{\infty} \frac{x - [x]}{(x+a)^{s+1}} dx. \end{aligned}$$

Substitute $\sigma = 0$ and $N = 0$ in the first formula to obtain the equation

$$\begin{aligned}\zeta(it,a) &= \frac{1}{a^{it}} + \frac{a^{1-it}}{it-1}\\ &- \frac{it}{2!}\{\zeta(it+1,a) - \frac{1}{a^{1+it}}\}\\ &- \frac{it(it+1)}{2!}\sum_{n=1}^{\infty}\int_0^1 \frac{u^2}{(n+a+u)^{it+2}}du\end{aligned}$$

so

$$\begin{aligned}|\zeta(it,a)| &\leq 1 + \frac{1}{|it-1|} + \frac{|t|}{2!}|\zeta(it+1,a) - \frac{1}{a^{1+it}}|\\ &+ \frac{|t|(|t|+1)}{2!}\sum_{n=1}^{\infty}\int_0^1 \frac{u^2}{(n+u)^2}du\\ &\leq 1 + \frac{1}{|it-1|} + \frac{|t|(|t|+1)}{2!}(\zeta(2)+1) + \frac{|t|}{2!}|C(t,a)|\end{aligned}$$

where

$$C(t,a) = \zeta(it+1,a) - \frac{1}{a^{1+it}}.$$

In the second formula let $N = 1$ and $s = 1 + it$ so $\sigma = 1 > 0$ giving

$$C(t,a) = \frac{1}{(1+a)^{1+it}} + \frac{(1+a)^{1-(1+it)}}{1-(1+it)} - (1+it)\int_1^{\infty} \frac{x-[x]}{(x+a)^{2+it}}dx$$

so

$$|C(t,a)| \leq 1 + \frac{1}{|t|} + \sqrt{1+t^2}.$$

∎

THEOREM 2.1. *For all $s \in \mathbb{C}$ with $\Re(s) < 1$ the (improper) Riemann integral of $\zeta(s,a)$ with respect to $a \in (0,1]$ exists and*

$$\int_{0^+}^1 \zeta(s,a)da = 0.$$

Proof. The work has now been done. Simply apply the lemmas, valid in different subsets of $\sigma < 1$, to the real and imaginary parts of the integral of $\zeta(s,a)$:

If $\sigma < 0$ use Lemmas 2.2 and 2.4.

If $0 < \sigma < 1$ use 2.2, 2.5 and 2.6.
If $\sigma = 0$ and $|t| \geq 1$ use 2.2 and 2.7.
If $\sigma = 0$ and $0 \leq t \leq 1$ use 2.2 and 2.8. ▌

THEOREM 2.2. *For all $s \in \mathbb{C}$ with $\Re(s) \geq 1$ the (improper) Riemann integral of $\zeta(s, a)$ with respect to $a \in (0, 1]$ does not exist.*

Proof. For every a, $\zeta(s, a)$ has a pole at $s = 1$, so the integral makes no sense at that value of s. The rest of the proof is straight forward, based on the non existence of the improper integral of a^{-s} on $(0, 1]$ for $\sigma = \Re s \geq 1$ and $t = \Im s \neq 0$ decomposing this domain into subsets corresponding to $\sigma > 1$, $\sigma = 1$ and $|t| \geq 1$ and $\sigma = 1$ and $0 < t < 1$. ▌

ACKNOWLEDGMENT

This work was done in part while the author was on study leave at Columbia University. The support of the Department of Mathematics at Columbia University and the valuable disussions held with Patrick Gallagher are warmly acknowledged.

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