Mixed basis matrix elements for the subgroup reductions of $SO(2,1)$

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By using the irreducible decomposition on the two-dimensional light cone, the mixed basis matrix elements for the three subgroup reductions of $SO(2,1)$ are calculated. These matrix elements are calculated for the principal series only and can be expressed in terms of well-known special functions. As a consequence of appearing in this context, some new properties of these special functions are given.

INTRODUCTION

An intensive study of the representation theory of $SU(1,1)$, the covering group of $SO(2,1)$, has been carried out in recent years[1-3]. The basic motivation for such a study stems from the crossed channel partial wave expansion of the scattering amplitude in which the group $SO(2,1)$ figures as the "little group" of the spacelike momentum transfer[3]. It is also of some mathematical interest to make such a study. In this paper we are concerned with different ways of realizing a unitary irreducible representation (UIR) of $SO(2,1)$ in terms of different subgroup bases and how these realizations are related. The representation theory of $SO(2,1)$ in the compact basis corresponding to the subgroup reduction $SO(2,1) \supset SO(2)$ has been thoroughly examined by Bargmann[4]. More recently the UIR's of $SO(2,1)$ in the noncompact basis corresponding to the group reduction $SO(2,1) \supset SO(1,1)$ have been studied. Mukunda[5-7] has explicitly performed this reduction for all possible UIR's of $SO(2,1)$ and calculated the corresponding matrix elements. Macfadyen[8] has given these matrix elements in terms of known special functions, namely, the generalized Legendre functions of the second kind[9]. The only remaining subgroup basis for $SO(2,1)$ is that corresponding to the group reduction $SO(2,1) \supset T_1$. This has been partially investigated by Vilenkin,[10] who has given the matrix elements in this basis for the principal series of $SO(2,1)$.

In this paper we will show how by using the irreducible decomposition of the space of square integrable functions defined on the cone we can calculate explicit expressions for the mixed basis matrix elements in the three subgroup bases of $SO(2,1)$[11]. This method only enables us to calculate matrix elements of the single valued principal series. The explicit expressions for the matrix elements which we obtain can be expressed in terms of well-known special functions. As a consequence of appearing in this context, we use standard techniques to derive some new properties of these functions.

The content of the paper is arranged as follows; In Sec. 1 we review the irreducible decomposition on the cone and give the expansions on the cone corresponding to the three subgroup reductions of $SO(2,1)$. In Sec. 2 we carry out the explicit calculation of the mixed basis matrix elements.

1. THE IRREDUCIBLE DECOMPOSITION ON THE CONE

The problem we are concerned with here is the decomposition into irreducible components of the representation

$$U(g)\,|\xi\rangle = |\xi g\rangle \tag{1.1}$$

of functions $|\xi\rangle$ defined on the two-dimensional cone

$$[\xi,\xi] = \xi_0^2 - \xi_1^2 - \xi_2^2 = 0$$

(the reason for the notation $|\xi\rangle$ will become clear subsequently). This problem is well known[12,13] to be equivalent to the decomposition of $|\xi\rangle$ into homogeneous components. This is achieved via the formulas

$$|\xi\rangle = \frac{1}{2\pi i}\int_{\delta-i\infty}^{\delta+i\infty} |\xi;\sigma\rangle\, d\sigma, \tag{1.2}$$

$$|\xi;\sigma\rangle = \int_0^\infty |t\xi\rangle\, t^{-\sigma-1}\, dt. \tag{1.3}$$

[Comment: The notation we will use is essentially that of Vilenkin[10] with the exception that the generators of the pure Lorentz transformations along the i axis ($i = 1, 2$) are denoted by N_i and the generator of the rotation subgroup is M_3. The corresponding one-parameter subgroups are then $h_i(a) = e^{N_i a}$, $r_3(\phi) = e^{M_3\phi}$.] Group-theoretically, (1.2) is an expansion of $|\xi\rangle$ in terms of the irreducible representations

$$l = \delta - i\rho, \qquad \epsilon = 0 \qquad (-\infty < \rho < \infty) \tag{1.4}$$

of $SO(2,1)$. We recover the unitary case when $\delta = -\frac{1}{2}$. This corresponds to the single valued principal series of $SO(2,1)$. Each irreducible component as expected satisfies the homogeneity condition

$$|\xi a;\sigma\rangle = a^\sigma\,|\xi;\sigma\rangle, \qquad a \text{ real}. \tag{1.5}$$

The expansion (1.2) is made explicit by choosing a coordinate system for ξ. The three expansions are now given for the coordinate systems corresponding to the three subgroup reductions of $SO(2,1)$. (i) The spherical or S system corresponding to the subgroup reduction $SO(2,1) \supset SO(2)$. Here ξ is parametrized according to

$$\xi = \omega_S(1, \cos\phi, \sin\phi), \qquad 0 < \omega_S < \infty,\ 0 \le \phi < 2\pi. \tag{1.6}$$

From the homogeneity condition (1.6),

$$|\xi;\rho\rangle = \omega_S^{[-(1/2)+i\rho]}\,|\phi;\rho\rangle. \tag{1.7}$$

(Here we have introduced the notation $|\xi;\rho\rangle = |\xi; -\frac{1}{2} + i\rho\rangle$ etc.) By expanding $|\phi;\rho\rangle$ in a Fourier series according to

$$|\phi;\rho\rangle = \sum_{M=-\infty}^{\infty} |\rho;M\rangle\, e^{iM\phi} \tag{1.8}$$

the resulting S system expansion on the cone is

$$|\xi\rangle = \sum_{M=-\infty}^{\infty} \int_{\infty}^{\infty} d\rho\, |\rho;M\rangle\, \omega_S^{[-(1/2)+i\rho]} e^{iM\phi}. \tag{1.9}$$

(ii) The hyperbolic or H system corresponding to the subgroup reduction $SO(2,1) \supset SO(1,1)$. Here ξ is parametrized according to

$$\xi = \omega_{H\pm}(\cosh\beta_\pm, \pm 1, \sinh\beta_\pm), \quad 0 < \omega_{H\pm} < \infty, -\infty < \beta_\pm < \infty, \qquad (1.10)$$

and we define $\eta = \operatorname{sgn}\xi_2$ in the H system. To write the expansion correctly, we split $|\xi\rangle$ into two parts according to

$$|\xi\rangle = |\xi\rangle_+ + |\xi\rangle_-, \qquad |\xi\rangle_\pm = |\xi\rangle\,\theta(\pm\xi_1). \qquad (1.11)$$

Then from the homogeneity condition (1.6) we have

$$|\xi;\rho,\pm\rangle = \omega_{H\pm}{}^{[-(1/2)+i\rho]}|\beta_\pm;\rho\rangle \qquad (1.12)$$

By expanding $|\beta_\pm;\rho\rangle$ by means of a Fourier transform according to

$$|\beta_\pm;\rho\rangle = \int_{-\infty}^{\infty} |\rho;\pm,\tau\rangle e^{i\tau\beta_\pm} d\tau, \qquad (1.13)$$

the resulting H system expansion on the cone is

$$|\xi\rangle_\pm = \int_{-\infty}^{\infty} d\tau \int_{-\infty}^{\infty} d\rho\,|\rho;\pm,\tau\rangle\,\omega_{H\pm}{}^{[-(1/2)+i\rho]} e^{i\tau\beta_\pm}. \qquad (1.14)$$

(iii) The horospherical or HO system corresponding to the subgroup reduction $SO(2,1) \supset T_1$. Here T_1 is the subgroup generated by $M_3 - N_2$. ξ is parametrized according to

$$\xi = \omega_E((r^2+1),(r^2-1),2r), \quad 0 < \omega_E < \infty, \quad -\infty < r < \infty. \qquad (1.15)$$

From the homogeneity condition (1.6) we have

$$|\xi;\rho\rangle = \omega_E{}^{[-(1/2)+i\rho]}|r,\rho\rangle. \qquad (1.16)$$

By expanding $|r,\rho\rangle$ by means of a Fourier integral transform according to

$$|r,\rho\rangle = \int_{-\infty}^{\infty} ds\,|\rho,S\rangle e^{iSr}, \qquad (1.17)$$

the resulting HO system expansion is

$$|\xi\rangle = \int_{-\infty}^{\infty} dS \int_{-\infty}^{\infty} d\rho\,|\rho,S\rangle\,\omega_E{}^{[-(1/2)+i\rho]} e^{iSr}. \qquad (1.18)$$

2. CALCULATION OF THE MIXED BASIS MATRIX ELEMENTS

We give here those mixed basis matrix elements which are necessary in order to completely determine a matrix element of the form $\langle A|U(g)|B\rangle$, with g a general group element. Here $|A\rangle$ and $|B\rangle$ are basis vectors of different subgroup reductions of the same UIR of the principal series of $SO(2,1)$. The corresponding parametrization of the group element g is then of the form

$$g = g_A \delta g_B, \qquad (2.1)$$

where g_A and g_B are the two one-parameter group elements generated by the diagonalized operators in the bases A and B.

For the calculation of the $S \leftrightarrow HO$ mixed basis matrix elements the parametrization of g is

$$g = r_3(\phi)h_1(a)p_1(r), \qquad (2.2)$$

where $p_1(r) = e^{(M_3 - N_2)r}$. For the explicit calculation of the general mixed basis matrix element we rewrite (1.1) in the following form:

$$\int_{-\infty}^{\infty} U(g)|\rho,S\rangle\,\omega_E{}^{-(1/2)+i\rho} e^{iSr} dS = \sum_{M=-\infty}^{\infty} |\rho,M\rangle\,\overline{\omega}_S{}^{-(1/2)+i\rho} e^{iM\phi'} \qquad (2.3)$$

with

$$\xi g = \overline{\omega}_S(1,\cos\phi',\sin\phi').$$

This then gives the integral representation of the general matrix element

$$\langle\rho,M|U(g)|\rho,S\rangle = \frac{1}{2\pi}\int_{\infty}^{\infty}\left(\frac{\overline{\omega}_S}{\omega_E}\right)^{-(1/2)+i\rho} e^{iM\phi'} e^{-iSr}\,dr. \qquad (2.4)$$

Because of the group parametrization (2.2) we need only calculate the matrix element of $g = h_1(a)$. We then have that

$$\overline{\omega}_S/\omega_E = e^{a}(r^2 + e^{-2a}), \qquad e^{i\phi'} = (r + ie^{-a})/(r - ie^{-a})$$

and the explicit expression for the mixed basis matrix element is then

$$\langle\rho,M|h,(a)|\rho,S\rangle = \frac{(-1)^M S^{-1/2}}{\Gamma(\frac{1}{2} - i\rho - M)}(2S)^{(1/2)-i\rho} W_{-M,i\rho}(2e^{-a}S), \qquad S > 0,$$
$$= \langle\rho,-M|h_1(a)|\rho,-S\rangle, \qquad S < 0, \qquad (2.5)$$

where $W_{\mu\nu}(Z)$ is the Whittaker function as defined in Ref. 14. The standard techniques of the infinitesimal method now enable us to derive the raising and lowering operators in the index M of these functions. To do this, we use a fixed column of the mixed basis matrix element $\langle\rho,M|U(g)|\rho,S\rangle$ as an S system basis for the UIR $l = -\frac{1}{2} - i\rho$, $\epsilon = 0$ of $SO(2,1)$. In the parametrization (2.2) this basis vector has the form

$$\langle\rho,M|U(g)|\rho,S\rangle = e^{iM\phi}\langle\rho,M|h_1(a)|\rho,S\rangle e^{iSr}, \qquad (2.6)$$

and the generators of $SO(2,1)$ are expressed as differential operators in the parameters a,ϕ,r according to

$$M_3 = \frac{\partial}{\partial\phi}, \qquad N_1 \pm iN_2 = e^{\pm i\phi}\left(\frac{\partial}{\partial a} \pm i\frac{\partial}{\partial\phi} \mp ie^{-a}\frac{\partial}{\partial r}\right). \qquad (2.7)$$

Then from the formulas, for the action of the generators $N_1 \pm iN_2$, on an S system basis[9] f_M, viz.,

$$(N_1 \pm iN_2)f_M = (-\tfrac{1}{2} + i\rho \mp M) f_{M\pm1} \qquad (2.8)$$

we have on separating out the ϕ and r dependence the well-known recurrence relations for the Whittaker functions,

$$-xW_{M,i\rho}(x) + (\tfrac{1}{2}x - M)W_{M,i\rho}(x) = W_{M+1,i\rho}(x), \qquad (2.9)$$

$$xW'_{M,i\rho}(x) + (\tfrac{1}{2}x - M)W_{M,i\rho}(x) = (\tfrac{1}{2} + i\rho - M)(\tfrac{1}{2} - i\rho - M)W_{M-1,i\rho}(x). \qquad (2.10)$$

These relations are, however, known to be true for the functions $W_{\mu\nu}(Z)$ quite generally (i.e., with μ,ν,Z complex). As a further illustration of our calculation we write the identity

$$\int_{-\infty}^{\infty} ds\,\langle\rho,M|h_1(a)|\rho,S\rangle\langle\rho,S|h_1(b)|\rho,N\rangle = \langle\rho,M|h_1(a+b)|\rho,N\rangle \qquad (2.11)$$

explicitly and obtain the new identity

$$\int_0^\infty dS[\tilde{W}_{M,i\rho}(e^{-a}S)\tilde{W}_{N,-i\rho}(e^{b}S) + \tilde{W}_{-M,i\rho}(e^{-a}S)\tilde{W}_{-N,-i\rho}(e^{b}S)] = \tfrac{1}{2}(-1)^{M+N}P_{MN}^{-(1/2)+i\rho}(\cosh(a+b)) \quad (2.12)$$

where

$$\tilde{W}_{M,i\rho}(x) = W_{-M,i\rho}(x)/\Gamma(\tfrac{1}{2} - i\rho - M).$$

We note in particular that if $a = -b$, the right-hand side of this identity is $\frac{1}{2}\delta_{MN}$.

For the calculation of the $S \leftrightarrow H$ mixed basis matrix elements, the parametrization of g is

$$g = r_3(\phi)h_1(a)h_2(\beta) \quad (2.13)$$

(remember for our choice of H system coordinates on the cone we have diagonalized N_2). The explicit calculation is achieved by writing (1.1) in the form

$$\sum_\pm \int_{-\infty}^\infty d\tau\; U(g)\,|\rho;\pm,\tau\rangle\, \omega_{H\pm}^{-(1/2)+i\rho}\, e^{i\tau\beta\pm} = \sum_{M=-\infty}^{\infty} |\rho,M\rangle\, \overline{\omega}_S^{-(1/2)+i\rho}\, e^{im\phi'} \quad (2.14)$$

with $\overline{\omega}_S$ and ϕ' as in (2.3). The integral representation of the general matrix element is then

$$\langle\rho,M\,|U(g)|\,\rho;\pm,\tau\rangle = \frac{1}{2\pi}\int_{-\infty}^\infty \left(\frac{\overline{\omega}_S}{\omega_{H\pm}}\right)^{-(1/2)+i\rho} e^{iM\phi}e^{-i\tau\beta_\pm}d\beta_\pm. \quad (2.15)$$

Because of the parametrization (2.13) we need only calculate the matrix element of $g = h_1(a)$. We then have that

$$\frac{\overline{\omega}_S}{\omega_{H\pm}} = \cosh a\,\cosh\beta_\pm \pm \sinh a,$$

$$e^{i\phi'} = \frac{\sinh a\,\cosh\beta_\pm \pm \cosh a + i\sinh\beta_\pm}{\cosh a\,\cosh\beta_\pm \pm \sinh a},$$

and the explicit expression for the mixed basis matrix element is then

$$\langle\rho,M|h_1(a)\,|\rho;+,\tau\rangle = \frac{1}{\pi}\, e^{-\pi(iM+\tau)}\, \frac{\Gamma(\frac{1}{2}-i\rho-i\tau)}{\Gamma(\frac{1}{2}-i\rho-M)} \times Q_{i\tau,M}^{(1/2)+i\rho}(-i\sinh a), \quad (2.16)$$

where $Q_{\mu\nu}^{j}(Z)$ is the generalized Legendre function of the second kind as defined by Azimov[9]. The other matrix element is given by the relation

$$\langle\rho,M|h_1(a)\,|\rho;-,\tau\rangle = (-1)^{M+1}\langle\rho,M|h_1(-a)\,|\rho;+,-\tau\rangle. \quad (2.17)$$

Using the infinitesimal method we may, as we did with the Whittaker functions find the raising and lowering operators in the index M for the $Q_{\mu\nu}^{j}(Z)$ functions as they appear in (2.16). The S system basis vector is now, for the parametrization (2.13),

$$\langle\rho,M|U(g)\,|\rho;\pm,\tau\rangle = e^{iM\phi}\langle\rho,M|h_1(a)\,|\rho;\pm,\tau\rangle\, e^{i\tau\beta}, \quad (2.18)$$

and the generators $N_\pm = N_1 \pm iN_2$ have the form

$$N_\pm = e^{\pm i\phi}\left(\frac{\partial}{\partial a} \pm i\tanh a\,\frac{\partial}{\partial\phi} \pm \frac{i}{\cosh a}\,\frac{\partial}{2\beta}\right). \quad (2.19)$$

Using (2.8) and separating out the ϕ and β dependence, we get the new recurrence relations

$$\left(\frac{d}{da} - M\tanh a + \frac{\tau}{\cosh a}\right) Q_{i\tau,M}^{(1/2)+i\rho}(i\sinh a) = [(M+\tfrac{1}{2})^2+\rho^2]\,Q_{i\tau,M+1}^{(1/2)+i\rho}(i\sinh a), \quad (2.20)$$

$$\left(\frac{d}{da} + M\tanh a - \frac{\tau}{\cosh a}\right) Q_{i\tau,M}^{(1/2)+i\rho}(i\sinh a) = Q_{i\tau,M-1}^{(1/2)+i\rho}(i\sinh a). \quad (2.21)$$

The analogous identity to (2.12) for the $S \leftrightarrow H$ mixed basis matrix elements is

$$\frac{1}{\pi^2}\int_{-\infty}^\infty d\tau[\tilde{Q}_{\tau M}^{\rho}(i\sinh a)\tilde{Q}_{-\tau M}^{-\rho}(-i\sinh b) + (-1)^{M+N} \times \tilde{Q}_{-\tau M}^{\rho}(i\sinh a)\,\tilde{Q}_{\tau N}^{-\rho}(-i\sinh b)] = P_{MN}^{(1/2)+i\rho}(\cosh(a+b)), \quad (2.22)$$

where

$$\tilde{Q}_{\tau M}^{\rho}(i\sinh a) = \frac{\Gamma(\frac{1}{2}-i\rho-i\tau)}{\Gamma(\frac{1}{2}-i\rho-M)}\, Q_{i\tau,M}^{(1/2)+i\rho}(-i\sinh a)$$

Again the interesting case of this identity is when $a = -b$.

There are two group parametrizations necessary for the calculation of the $HO \leftrightarrow H$ mixed basis matrix elements, viz.,

$$g = h_2(\beta)h_1(a)p_1(r), \qquad \eta = -, \quad (2.23a)$$

$$g = h_2(\beta)h_1(a)r_3(\pi)p_1(r), \qquad \eta = +. \quad (2.23b)$$

The explicit calculation is achieved by writing (1.1) in the form

$$\sum_\pm \int_{-\infty}^\infty dr\; U(g)\,|\rho;\pm,\tau\rangle\, \omega_{H\pm}^{-(1/2)+i\rho} e^{i\tau\beta_\pm} = \int_{-\infty}^\infty ds\,|\rho,S\rangle\, \omega_E^{-(1/2)+i\rho} e^{iSr'}, \quad (2.24)$$

with

$$\xi g = \overline{\omega}_E\,(r'^2+1, r'^2-1, 2r').$$

The integral representation of the general matrix element is then

$$\langle\rho,S\,|U(g)|\,\rho;\pm,\tau\rangle = \frac{1}{2\pi}\int_{-\infty}^\infty \left(\frac{\omega_E}{\omega_{H\pm}}\right)^{-(1/2)+i\rho} e^{iSr'}e^{-i\tau\beta_\pm}d\beta_\pm. \quad (2.25)$$

Because of the parametrizations (2.23a), (2.23b) we need only calculate the matrix element of $h_1(a)$ for $\eta = -$ and $h_1(a)r_3(\pi)$ for $\eta = +$. We then have that

$$(\overline{\omega}_S/\omega_{H\pm}) = e^{-a}\cosh\tfrac{1}{2}\beta_\pm,\; r' = e^{a}\tanh\tfrac{1}{2}\beta_\pm,$$

and the explicit expression for these matrix elements is

$$\langle\rho,S|h_1(a)\,|\rho;-\tau\rangle = \langle\rho,S|h_1(a)r_3(\pi)\,|\rho;+,\tau\rangle = 1/2\pi(\tfrac{1}{4}e^{-a})^{[-(1/2)+i\rho]}B(\tfrac{1}{2}-i\rho-i\tau, \tfrac{1}{2}-i\rho+i\tau)e^{-iSe^{a}} \times {}_1F_1(\tfrac{1}{2}-i\rho-i\tau, 1-2i\rho, 2iSe^{a}), \quad (2.26)$$

where

$$B(x,y) = \Gamma(x)\Gamma(y)/\Gamma(x+y)$$

and ${}_1F_1(b;c;z)$ is the confluent hypergeometric function[15].

We also give here the expression for the matrix element $\langle \rho, S|h_1(a)|\rho; +, \tau\rangle$, it is

$$\langle \rho, S|h_1(a)|\rho; +, \tau\rangle = \frac{1}{4\pi}\left(-\tfrac{1}{4}e^{a}S^2\right)^{-(1/2)+i\rho} \times [\Gamma(\tfrac{1}{2}-i\rho+i\tau)W_{-i\tau,-i\rho}(-2iSe^{a}) + \Gamma(\tfrac{1}{2}-i\rho-i\tau)W_{i\tau,-i\rho}(2iSe^{a})]. \quad (2.27)$$

We also have directly from the integral representations the asymptotic equality

$$\langle \rho, M|h_1(a)|\rho; \pm, \tau\rangle = [\langle \rho, S|h_1(a)|\rho; \pm, \tau\rangle]_{S=M}, \quad (2.28)$$

which holds for large a. This is the direct analogy of a similar relation which is known to hold for the subgroup reductions of $SO(3,1)$[12].

CONCLUDING REMARKS

We have seen in this paper how the method on the cone can be used to directly calculate the mixed basis matrix elements for the principal series of $SO(2,1)$. The use of this method for calculating matrix elements is due to Verdiev[15] and has been extensively used for the subgroup reductions of $SO(3,1)$[12]. From our calculations we can immediately find the overlap functions by putting $a = 0$. These overlap functions can be used to factorize the overlap functions of the subgroup reductions of $SO(3,1)$. An example of this factorization is

$$\langle J, m|\pm, l, s\rangle = \langle J, m|\pm, l, m\rangle\langle m|s\rangle.$$

Here $|J,m\rangle$, $|\pm, l, m\rangle$, and $|\pm, l, s\rangle$ are basis vectors for the same UIR of $SO(3,1)$ corresponding to the group reductions $SO(3,1) \supset SO(3) \supset SO(2)$, $SO(3,1) \supset SO(2,1) \supset SO(2)$, and $SO(3,1) \supset SO(2,1) \supset T_1$, respectively. The Lorentz group labels have been suppressed in these vectors. The matrix $\langle m|s\rangle$ is then the S–HO overlap function which is given by (2.5) after putting $a = 0$.

[Note: We have assumed here that l is in the principal series $l = -\frac{1}{2} + i\rho$, $\epsilon = 0$ of $SO(2,1)$.] We intend in the near future to make a complete study of matrix elements in the subgroup reductions of $SO(2,1)$ for all possible UIR's.

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