Subcubic trades in Steiner triple systems

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Abstract

We consider the problem of classifying trades in Steiner triple systems such that each block of the trade contains one of three fixed elements. We show that the fundamental building blocks for such trades are 3-regular graphs that are 1-factorisable. In the process we also generate all possible 2- and 3-way simultaneous edge colourings of graphs with maximum degree 3 using at most 3 colours, where multiple edges but not loops are allowed. Moreover, we generate all possible Latin trades within three rows.

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1 Introduction

This paper is chiefly concerned with trades in Steiner triple systems. We first recall the definitions. A *Steiner triple system* of order v, briefly STS(v), is a pair $(V, \mathbb{B})$ where V is a base set of cardinality v of *elements* (or *points*) and $\mathbb{B}$ is a collection of unordered triples, called *blocks* (or *lines*), which has the property that each pair of distinct elements of V occurs in precisely one triple. It is well known that an STS(v) exists if and only if $v \equiv 1$ or 3 (mod 6) ([10]).

An *m-line configuration* (or a *partial Steiner triple system*) is a collection of $m \geqslant 1$ triples which has the property that every pair of distinct elements occurs in at most one triple. Any subset of the blocks of a Steiner triple system forms an m-line configuration. In a configuration the *degree* of a point is the number of triples which contain it.

An *n-way trade set*, $\mathcal{T} = \{T_1, T_2, \ldots, T_n\}$, $n \geqslant 2$, is a set of pairwise disjoint m-line configurations which has the property that every pair of distinct elements occurs in precisely the same number (zero or one) of triples in each T_i, $1 \leqslant i \leqslant n$. The number of lines, m, is called the *volume* of the trade set and is denoted by vol($\mathcal{T}$). The *foundation* of the trade set, found($\mathcal{T}$), is the set of elements which occur in each T_i. A 2-way trade set $\{T_1, T_2\}$ may be called a *bitrade*. Each set T_i is called a *leg* [8], *tradeable configuration* [7], or more simply just a *trade*. We see no problem in using this latter terminology as long as the distinction between a trade and a trade set is clearly understood. Given a trade set $\mathcal{T} = \{T_1, T_2, \ldots, T_n\}$, $n \geqslant 2$, for each $T_i \in \mathcal{T}$, the trade T_j, $i \neq j$, is called a *trade mate* of T_i. A trade set $\mathcal{T} = \{T_1, T_2, \ldots, T_n\}$ is said to be *minimal* if there does not exist a trade set $\mathcal{U} = \{U_1, U_2, \ldots, U_n\}$ such that $U_i \subset T_i$ for each $1 \leqslant i \leqslant n$.

Example 1. *We exhibit an example of a minimal* 3-*way trade set of volume* 13 *with foundation set* $\{1, 2, 3, \ell, m, n, p, q, r, s, t\}$.

123	$1\ell m$	$2mn$	$3\ell n$	$3qr$	$2pq$	$1pr$	$1ns$	$2\ell s$	$3ms$	$3pt$	$2rt$	$1qt$
$12n$	23ℓ	$13m$	ℓmn	$2qr$	$1pq$	$3pr$	$3ns$	$1\ell s$	$2ms$	$2pt$	$1rt$	$3qt$
$12r$	$13q$	$23p$	pqr	$2\ell m$	$3mn$	$1\ell n$	$2ns$	$3\ell s$	$1ms$	$1pt$	$3rt$	$2qt$

We say that a trade T is *subcubic* if there exist three elements x, y, z in the foundation set such that for all $B \in T$, $x \in B$ or $y \in B$ or $z \in B$. The first trade in Example 1 is subcubic with respect to the elements 1, 2 and 3.

As a first step we classify *subquadratic trades*; i.e. trades such that there exists a subset of size 2 which intersects every block of the trade.

Lemma 2. *Let $T \in \mathcal{T}$ be a trade with trade mate T'. Suppose there exist $x, y \in \mathrm{found}(\mathcal{T})$ such that for each block $B \in T$ either $x \in B$ or $y \in B$. Then there is no block in T containing the pair $\{x, y\}$.*

Proof. To see this, observe that if there is a block $B_0 \in T$ such that $B_0 = \{x, y, z\}$, then there is a block $B_1 = \{x, z, a\} \in T'$ such that $a \neq y$. In turn there is a block $B_2 = \{z, a, b\} \in T$. However $b = x$ or $b = y$, and the pair $\{z, b\}$ occurs in two blocks of T, a contradiction. $\square$

We can thus represent any subquadratic trade by a graph G whose vertex set is $\mathrm{found}(\mathcal{T}) \setminus \{x, y\}$ and two vertices u, v are joined by an edge if the pair $\{u, v\}$ occurs in a block of T. A graph is said to have a *k-simultaneous edge colouring* if there exist k proper edge-colourings such that:

- for each vertex, the set of colours appearing on the edges incident to that vertex are the same in each colouring; and

- no edge receives the same colour in two distinct colourings.

Thus G is precisely any simple graph with a 2-simultaneous edge-colouring using 2 colours; i.e. a 2-regular bipartite graph. In this case the trade T is called a *cycle trade* and is arguably the simplest which can occur in a Steiner triple system. A cycle trade of volume $v - 1$ exists in any Steiner triple system of order $v > 3$ (simply swap any two elements, excluding the block which includes both of them).

Note that a trade may be subcubic with respect to more than one set of three elements from the foundation set. Henceforth unless otherwise stated, all trades are subcubic with respect to 1, 2 and 3. If there exists a block $B \in T$ which contains two or three of the elements 1, 2 and 3, we say that T is a subcubic trade of *Type* 2; otherwise T is of *Type* 1. Observe that the first trade in Example 1 is subcubic of Type 2.

In the process of classifying subcubic trades we in turn classify all 3-simultaneous edge colourings of graphs with maximum degree 3 using at most 3 colours, where multiple edges but not loops are allowed (see Section 2).

2   Subcubic trades of Type 1

In this section we assume that T is a subcubic trade of Type 1. Observe that if T is a subcubic trade of Type 1, then any trade mate of T is also a

subcubic trade of Type 1. We represent the trade T by a graph G whose vertex set is found$(\mathcal{T}) \setminus \{1, 2, 3\}$ and two vertices u and v are joined by an edge if $\{1, u, v\}$ or $\{2, u, v\}$ or $\{3, u, v\} \in T$. We obtain a graph, all of whose vertices are of degree 2 or 3. We will show how to construct all such graphs.

For each $i \in \{1, 2, 3\}$, we colour edge $\{u, v\}$ of G with colour i if $\{i, u, v\}$ is a block of T. Since each pair occurs in at most one triple, such a colouring is a proper edge-colouring of G with 3 colours. Since each trade mate of T yields a disjoint proper edge-colouring of G, the graph G has a k-simultaneous edge colouring for some $k \leqslant 3$. Since this process is reversible, trade sets which contain a subcubic trade of Type 1 are equivalent to k-simultaneous edge colourings of simple graphs using 3 colours.

Lemma 3. *If $T \in \mathcal{T}$ is a subcubic trade of Type 1, $\mathcal{T}$ is equivalent to a 2- or 3-simultaneous edge colouring of a simple graph with maximum degree 3 using 3 colours.*

In this section we describe a set of "moves" which construct all 2- or 3-simultaneous edge colourings of *multi*graphs with maximum degree 3 using 3 colours. Here we assume that a multigraph may contain multiple edges but not loops. From the previous lemma we in effect classify all subcubic trades of Type 1 by this process.

We will need the following result.

Lemma 4. *Let G be a graph, possibly a multigraph, which possesses a k-simultaneous edge colouring where $k \geqslant 2$. Then every edge of G belongs to an even length circuit. In particular, G is bridgeless.*

Proof. Clearly G possesses a 2-simultaneous edge-colouring; let these edge-colourings be C_1 and C_2. Let e be an edge in G coloured x in C_1 and let v_0 be a vertex incident with e. By the definition of a simultaneous edge-colouring, there exists an edge $e' = \{v_0, v_1\} \neq e$ which is coloured x in C_2. Next, there is an edge $e'' = \{v_1, v_2\} \neq e'$ which is coloured x in C_1. If $v_2 = v_0$ we have a circuit of length 2 and we are done. Otherwise, we continue this process recursively to create a sequence $v_0, v_1, \ldots$. Such a sequence is uniquely defined by our choice of v and e; thus by finiteness, $v_{2k} = v_0$ for some $k \geqslant 1$ and we are done. $\square$

We first consider the case where the graph of a subcubic trade of Type 1 is cubic (3-regular); such a trade will be called a *cubic trade*. It is worth noting that deciding whether a cubic graph has a 1-factorization is in general

NP-complete [9]. However, given the existence of a cubic graph with a proper 3-edge colouring with colours 1, 2 and 3, we can obtain two extra, disjoint colourings via applying the permutation (123). Thus we obtain in such a case a 3-way trade set with each element a trade of Type 1. The following is immediate.

Theorem 5. *Cubic trades are equivalent to* 1-*factorizations of simple cubic graphs.*

Henceforth in this section we consider the case when there exists at least one vertex v of degree 2. It is clear that in such a case a 3-way trade set is impossible, so we may assume our trade has a fixed mate T'. Either v is adjacent to another vertex of degree 2 or it is not. In either case, we show that if G has a 2-simultaneous edge colouring, G may be reduced to a graph G', also with a 2-simultaneous edge colouring. Although we are chiefly interested in the case when G is simple, our reductions may create multiple edges and thus multigraphs.

We in effect give a construction that gives any multigraph with maximum degree 3 that has a 2-simultaneous edge colouring. Our multigraphs may have multiple edges but do not have loops. Colours are specified without loss of generality.

Reduction 1. Here vertex v is adjacent to another vertex w of degree 2 and v and w do not form a dipole. If v and w are adjacent to the same vertex u, then either u, v and w induce a triangle component (contradicting the fact that each edge must belong to an even cycle) or u has degree 3 and is a bridge, contradicting Lemma 4.

Hence there is a path $[u, v, w, t]$ in G. Without loss of generality let the edge $\{v, w\}$ be coloured 1 in T and 2 in T'. Then the edges $\{u, v\}$ and $\{w, t\}$ are respectively coloured 2 in T and 1 in T'. Thus we may reduce G by replacing the path $[u, v, w, t]$ with an edge $\{u, t\}$ to form a smaller graph G'. The edge $\{u, t\}$ is coloured 2 in T and 1 in T'. All other edges of G' retain the same colourings in T and T'. It is straightforward to see this colouring is a 2-simultaneous edge colouring of G'. Note that any application of Reduction 1 reduces the size of the foundation set by 2 and the volume also by 2.

In effect, we may reduce any odd length path in G to a single edge by applying Reduction 1 recursively. Reduction 1 is illustrated in Figure 1, where x/y denotes the colouring of an edge with colours x and y in T and T', respectively.
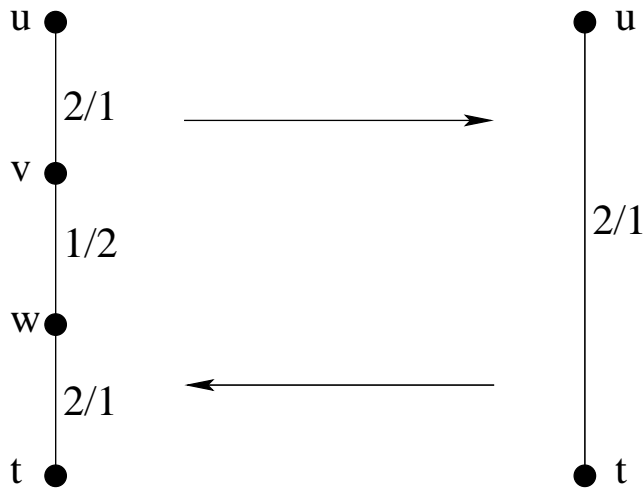
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Figure 1: Reduction 1 and Expansion 1

Reduction 2. Here the neighbours of vertex v each have degree 3. Let the neighbours of v be u and w. If $u = w$, the edge incident with u and not v forms a bridge, contradicting Lemma 4. Thus $u \neq w$; let the neighbours of u be $\{v, a, b\}$ and let the neighbours of w be $\{v, c, d\}$. Suppose that $\{u, v\}$ is coloured 1 and $\{v, w\}$ is coloured 2 in T. These colours are clearly swapped in T'. Without loss of generality: $\{u, a\}$ is coloured 2 in T and 3 in T'; $\{u, b\}$ is coloured 3 in T and 1 in T'; $\{c, w\}$ is coloured 3 in T and 2 in T' and $\{d, w\}$ is coloured 1 in T and 3 in T'. It follows that $u \neq c$, $u \neq d$, $w \neq a$, $w \neq b$, $a \neq d$ and $b \neq c$. However we ask the reader to note that $a = b$, $c = d$, $a = c$ or $b = d$ are each valid possibilities. In any case, we replace the above six edges with the following four edges using new vertices s and t: $\{a, s\}$ coloured 2 in T and 3 in T'; $\{s, c\}$ coloured 3 in T and 2 in T'; $\{b, t\}$ coloured 3 in T and 1 in T'; $\{t, d\}$ coloured 1 in T and 3 in T'. Vertices u, v and w are deleted. In the above, if $a = c$ or $b = d$ the reduction results in a dipole; if $a = c$ and $b = d$ we obtain two dipoles. Observe that any application of Reduction 2 decreases the size of the foundation set by 1 and the volume by 2.

Observe that we may always apply either Reduction 1 or 2 until we obtain a graph each component of which is either 3-regular or a dipole. Reduction 2 is illustrated in Figure 2, where x/y denotes the colouring of an edge with colours x and y in T and T', respectively.
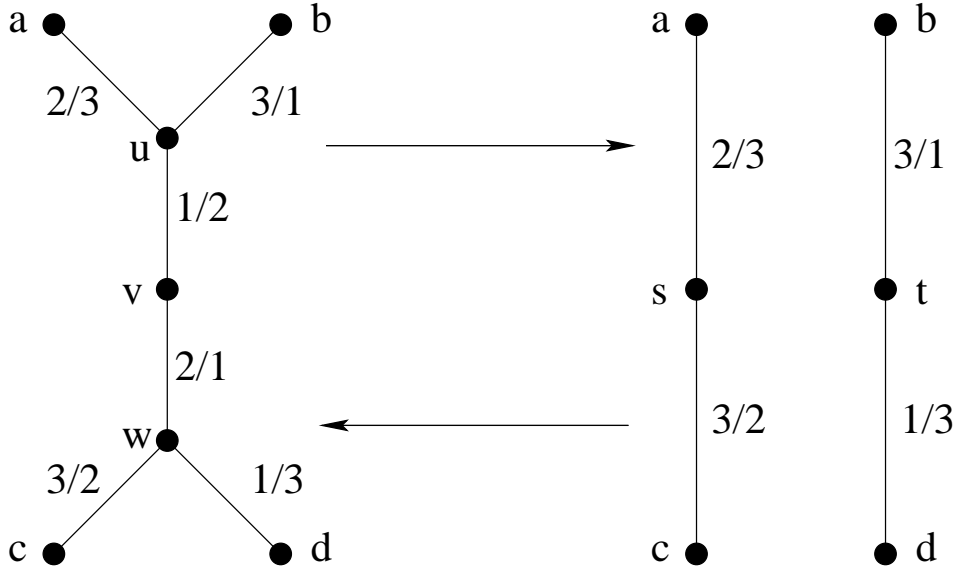
Figure 2: Reduction 2 and Expansion 2

We next describe an expansion process which is the exact inverse of the above. We assume that G is a multigraph with a 2-simultaneous edge colouring using 3 colours.

Expansion 1. Let $\{t, u\}$ be any edge in the graph G labelled 2 in T and 1 in T'. Replace $\{t, u\}$ with a path $[t, w, v, u]$, where the edge $\{w, v\}$ is coloured 1 in T and 2 in T' and the edges $\{u, v\}$ and $\{w, t\}$ are each coloured 2 in T and 1 in T'. This is clearly the exact inverse of Reduction 1. Expansion 1 is illustrated in Figure 1.

Expansion 2. Let $[a, s, c]$ and $[b, t, d]$ be two edge disjoint walks in G, such that s and t are of degree 2, $\{a, s\}$ is coloured 2 in T and 3 in T', with these colours reversed on $\{s, c\}$, and $\{b, t\}$ is coloured 3 in T and 1 in T', with these colours reversed on $\{t, d\}$. If $a = d$ or $b = c$ we do not have proper edge-colourings. Note however that other equalities are allowed. For example, if $a = c$ then $[a, s, c]$ is a dipole. If $a = b$ and $c = d$, $[a, s, c, t, a]$ is a cycle of length 4 (a subgraph of a larger component). It is both possible that $[a, s, c]$ and $[b, t, d]$ belong to distinct components or are in the same component.

We delete vertices s and t, adding vertices u, v and w and the following edges and colourings: $\{u, v\}$ is coloured 1 and $\{v, w\}$ is coloured 2 in T, these

colours are switched in T'; $\{u, a\}$ is coloured 2 in T and 3 in T'; $\{u, b\}$ is coloured 3 in T and 1 in T'; $\{c, w\}$ is coloured 3 in T and 2 in T' and $\{d, w\}$ is coloured 1 in T and 3 in T'. This is clearly the exact inverse of Reduction 2. Expansion 2 is illustrated in Figure 2.

Our results are summarized by the following theorem.

Theorem 6. *Let G_0 be a graph of which each component is either 3-regular with a 1-factorization (possibly a multigraph but without loops) or a dipole. Then G_0 clearly possesses a 2-simultaneous edge colouring using at most 3 colours. Construct G_{i+1} from G_i by applying either Expansion 1 or Expansion 2 to G_i. Whenever G_i is simple, we obtain a corresponding subcubic trade of Type 1. Moreover, any subcubic trade of Type 1 is constructed via this algorithm.*

Since any multigraph may be transformed into a simple graph by applying Expansion 1 to any repeated edges, any multigraph in the above algorithm is not a "dead end" per se, but can later generate a simple graph and thus a subcubic trade.

3   Subcubic trades of Type 2

Recall that unlike subcubic trades of Type 1, here we allow the occurrences of pairs $\{1, 2\}$, $\{2, 3\}$ or $\{1, 3\}$. Moreover, we remind the reader that trade mates of such trades are not required by definition to be subcubic. We will ultimately show that any subcubic trade of Type 2 arises from a subcubic trade of Type 1. The results in this section hold for any permutation of 1, 2 and 3.

Lemma 7. *If T is a subcubic trade of Type 2, then $\{1, 2, 3\}$ is not a block in any trade mate T' of T.*

Proof. Let T be a subcubic trade of Type 2 with trade mate T' containing the block $\{1, 2, 3\}$. Then T must contain blocks of the form $\{1, 2, \ell\}$, $\{1, 3, m\}$, $\{2, 3, n\}$ where 1, 2, 3, ℓ, m and n are distinct. Let r_i be the degree of i in T, and thus also T', for each $i \in \{1, 2, 3\}$. Since by definition either 1, 2 or 3 occurs in every block of T, the volume of T is equal to $r_1 + r_2 + r_3 - 3$. Similarly the volume of T' is at least equal to $r_1 + r_2 + r_3 - 2$, a contradiction. $\qquad\square$

Lemma 8. *Let T be a subcubic trade of Type 2 containing the block $\{1, 2, n\}$ where $n \neq 3$ and let T' be a trade mate of T. Then T' contains $\{1, 3, n\}$ or $\{2, 3, n\}$.*

Proof. Since $\{1, 2, n\} \in T$, the element n must occur in another block of T and can only do so once more in a block of the form $\{3, n, x\}$, i.e. n occurs in precisely two blocks of T. Now $\{1, 2, n\} \notin T'$ and if neither $\{1, 3, n\}$ nor $\{2, 3, n\}$ is a block of T', the pairs $\{1, n\}$, $\{2, n\}$ and $\{3, n\}$ must occur in different blocks of T', i.e. n occurs in more blocks of T' than T, which is impossible. $\square$

The next theorem quickly follows.

Theorem 9. *Let T be a subcubic trade of Type 2 containing the block $\{1, 2, n\}$ where $n \neq 3$ and let T' be a trade mate of T. Let $\alpha \notin \mathrm{found}(\{T, T'\})$. Then one of the following holds.*

1. *$\{\{1, 2, n\}, \{1, 3, m\}\} \subseteq T$, $\{\{1, 3, n\}, \{1, 2, m\}\} \subseteq T'$ where $1, 2, 3, m, n$ are distinct and*

$$(T \setminus \{\{1, 2, n\}, \{1, 3, m\}\}) \cup \{\{2, n, \alpha\}, \{3, m, \alpha\}\}$$

is a subcubic trade of Type 1 with trade mate

$$(T' \setminus \{\{1, 3, n\}, \{1, 2, m\}\}) \cup \{\{2, m, \alpha\}, \{3, n, \alpha\}\}.$$

2. *$\{\{1, 2, n\}, \{1, 3, m\}, \{2, 3, \ell\}\} \subseteq T$, $\{\{1, 3, n\}, \{1, 2, \ell\}, \{2, 3, m\}\} \subseteq T'$ where $1, 2, 3, \ell, m, n$ are distinct and*

$$(T \setminus \{\{1, 2, n\}, \{1, 3, m\}, \{2, 3, \ell\}\}) \cup \{\{2, n, \alpha\}, \{1, m, \alpha\}, \{3, \ell, \alpha\}\}$$

is a subcubic trade of Type 1 with trade mate

$$(T' \setminus \{\{1, 3, n\}, \{1, 2, \ell\}, \{2, 3, m\}\}) \cup \{\{2, m, \alpha\}, \{3, n, \alpha\}, \{1, \ell, \alpha\}\}.$$

Next we consider when a subcubic trade of Type 2 contains the triple $\{1, 2, 3\}$.

Lemma 10. *Let T be a subcubic trade of Type 2 containing the block $\{1, 2, 3\}$. Let T' be a trade mate of T. Then for some ℓ, m, n such that $1, 2, 3, \ell, m, n$ are distinct, $S' \subseteq T'$ where*

$$S' := \{\{1, 2, n\}, \{1, 3, m\}, \{2, 3, \ell\}, \{\ell, m, n\}\}.$$

Any other block of T' contains either 1, 2 or 3.

Proof. The fact that S' contains the blocks $\{1, 2, n\}$, $\{1, 3, m\}$ and $\{2, 3, \ell\}$ is immediate. Let r_i be the degree of i in T, and thus also T', for each $i \in \{1, 2, 3\}$. Then the number of blocks in T, and thus also T', is equal to $r_1 + r_2 + r_3 - 2$; it follows there exists a unique block in T' containing neither 1, 2 nor 3. We need to show that this block is $\{\ell, m, n\}$. Since $\{2, 3, \ell\} \in T'$, there are distinct blocks of T containing the pairs $\{2, \ell\}$ and $\{3, \ell\}$. Furthermore there is a block containing the pair $\{1, \ell\}$ in T if and only if there is a block containing $\{1, \ell\}$ in T'. Since the degree of ℓ is the same in T and T', it follows that ℓ must occur in the block of T' containing neither 1, 2 nor 3. The same argument applies to m and n; the result follows. $\quad\square$

Theorem 11. *Let T be a subcubic trade of Type 2 containing the block $\{1, 2, 3\}$. Let T' be a trade mate of T and let $\{T, T'\}$ be minimal. Let*

$$S' = \{\{1, 2, n\}, \{1, 3, m\}, \{2, 3, \ell\}, \{\ell, m, n\}\}.$$

Let $\alpha \notin \text{found}\{T, T'\}$. Then for some ℓ, m, n such that $1, 2, 3, \ell, m, n$ are distinct one of the following holds.

1. $T = \{\{1, 2, 3\}, \{1, m, n\}, \{2, \ell, n\}, \{3, \ell, m\}\}$ *and* $T' = S'$.

2. $S_1 := \{\{1, 2, 3\}, \{1, m, n\}, \{2, \ell, m\}, \{3, n, \ell\}\} \subseteq T$ *and*

$$(T \setminus S_1) \cup \{\{2, m, \alpha\}, \{3, n, \alpha\}\}$$

   is a subcubic trade of Type 1 with disjoint mate

$$(T' \setminus S') \cup \{\{2, n, \alpha\}, \{3, m, \alpha\}\}.$$

3. $S_2 := \{\{1, 2, 3\}, \{1, m, \ell\}, \{2, m, n\}, \{3, n, \ell\}\} \subseteq T$ *and*

$$(T \setminus S_2) \cup \{\{1, \ell, \alpha\}, \{2, m, \alpha\}, \{3, n, \alpha\}\}$$

   is a subcubic trade of Type 1 with disjoint mate

$$(T' \setminus S') \cup \{\{1, n, \alpha\}, \{2, \ell, \alpha\}, \{3, m, \alpha\}\}.$$

Proof. From the previous lemma, the pairs $\{m, n\}$, $\{\ell, m\}$ and $\{\ell, n\}$ must occur in blocks of T. The five blocks $\{1, 2, 3\} \in T$ and $\{1, 2, n\}$, $\{1, 3, m\}$, $\{2, 3, \ell\}$, $\{\ell, m, n\} \in T'$ are fixed as a set under the permutations $(mn)(23)$, $(n\ell)(13)$ and $(m\ell)(12)$. It follows without losing generality that we have three cases:

C1 : $\{1, m, n\}$, $\{2, \ell, n\}$, $\{3, \ell, m\} \in T$;

C2 : $\{1, m, n\}$, $\{2, \ell, m\}$, $\{3, n, \ell\} \in T$; or

C3 : $\{1, \ell, m\}$, $\{2, m, n\}$, $\{3, n, \ell\} \in T$.

The result follows. $\qquad\square$

We have now classified all subcubic trades of Type 2 which belong to minimal 2-way trade sets. It follows that any subcubic trade of Type 2 is the union of such a subcubic trade of Type 2 and any other subcubic trade of Type 1.

3.1 Trades with two trade mates

Finally, we classify 3-way trade sets including one trade which is subcubic of Type 2. Similarly to above, we show that such trades reduce to trades of Type 1 already classified. We also remind the reader that results hold for any permutations of 1, 2 and 3.

Henceforth in this subsection, T is a subcubic trade of Type 2 and $\{T, T', T''\}$ is a 3-way trade set. We first consider the case when T contains the block $\{1, 2, n\}$ where $n \neq 3$. The following theorem is virtually a corollary of Theorem 9.

Theorem 12. *Let* $\{T, T', T''\}$ *be a minimal* 3*-way trade set such that* $\{1, 2, n\} \in T$ *where* $n \neq 3$ *and let* $\alpha, \beta \notin \text{found}\{T, T', T''\}$ *with* $\alpha \neq \beta$. *Then*

$$
\begin{aligned}
S_1 &:= \{\{1, 2, n\}, \{2, 3, \ell\}, \{1, 3, m\}\} \subseteq T, \\
S_2 &:= \{\{3, 1, n\}, \{1, 2, \ell\}, \{3, 2, m\}\} \subseteq T' \text{ and} \\
S_3 &:= \{\{2, 3, n\}, \{3, 1, \ell\}, \{2, 1, m\}\} \subseteq T''
\end{aligned}
$$

where $1, 2, 3, \ell, m, n$ *are distinct and*

$$(T \setminus S_1) \cup \{\{1, n, \alpha\}, \{2, \ell, \alpha\}, \{3, m, \alpha\}, \{2, n, \beta\}, \{3, \ell, \beta\}, \{1, m, \beta\}\}$$

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is a cubic trade of Type 1 *with trade mates*

$$(T' \setminus S_2) \cup \{\{2, m, \alpha\}, \{3, n, \alpha\}, \{1, \ell, \alpha\}, \{1, n, \beta\}, \{2, \ell, \beta\}, \{3, m, \beta\}\}$$

and

$$(T'' \setminus S_3) \cup \{\{3, \ell, \alpha\}, \{1, m, \alpha\}, \{2, n, \alpha\}, \{3, n, \beta\}, \{1, \ell, \beta\}, \{2, m, \beta\}\}.$$

Next we consider when T contains the block $\{1, 2, 3\}$. By Lemma 10 we may assume that T' contains the blocks $\{1, 2, n\}$, $\{1, 3, m\}$, $\{2, 3, \ell\}$ and $\{\ell, m, n\}$ where $1, 2, 3, \ell, m, n$ are distinct. By Lemma 10 we may also assume that T'' contains the blocks $\{1, 2, r\}$, $\{1, 3, q\}$, $\{2, 3, p\}$ and $\{p, q, r\}$ where $1, 2, 3, p, q, r$ are distinct. Observe that Example 1 gives an example of such a trade.

Lemma 13. *Each of ℓ, m, n, p, q, r has degree 3 in T.*

Proof. Consider the point ℓ. Since T is subcubic it occurs in at most 3 blocks of T. Now suppose for the sake of contradiction that it occurs in 2 blocks of T, and thus also 2 blocks of T', and 2 blocks of T''. Then ℓ appears only in blocks with 2, 3, m and n. Thus the blocks $\{2, \ell, m\}$ and $\{3, \ell, n\}$ occur in T and $\{2, \ell, n\}$ and $\{3, \ell, m\}$ occur in T'', or vice-versa. However the pair $\{m, n\}$ occurs in a block of T'. Thus the block $\{1, m, n\}$ occurs in both T and T'', a contradiction. Similarly m, n, p, q, r each appear in 3 blocks of T. $\square$

Lemma 14. *The sets $\{\ell, m, n\}$ and $\{p, q, r\}$ are disjoint.*

Proof. If $\{\ell, m, n\} = \{p, q, r\}$, T' and T'' share a common block, a contradiction.

Suppose that $|\{\ell, m, n\} \cap \{p, q, r\}| = 2$. Without loss of generality since blocks so far specified are invariant under the permutation $(123)(\ell m n)(pqr)$, we may assume that $\{p, q\} = \{m, n\}$, $\{p, q\} = \{\ell, m\}$ or $\{p, q\} = \{\ell, n\}$.

Firstly, if $\{p, q\} = \{m, n\}$, since $\{1, 3, m\} \in T'$ and $\{1, 3, q\} \in T''$, we must have $q = n$ and $m = p$. Next, since $\{\ell, m, n\} \in T'$, the pair $\{\ell, n\}$ occurs in a block of T''; as $\{1, 3, n\} \in T''$ we must have $\{\ell, n, 2\} \in T''$. Similarly $\{\ell, m, 1\} \in T''$, $\{n, r, 3\} \in T'$ and $\{m, r, 2\} \in T'$. Next, there must be triples in T' and T'' containing the pairs $\{1, r\}$ and $\{3, r\}$, respectively. Thus we must have $\{1, r, t\} \in T'$ and $\{3, r, t\} \in T''$ for some $t \notin \{m, n\}$. Since

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the pairs $\{n,r\}$ and $\{m,r\}$ occur in triples of T we must have $\{2,r,t\} \in T$. Since $\{n,r,3\} \in T'$, it follows that $\{n,r,1\} \in T$. In turn: $\{m,r,3\} \in T$, $\{m,n,2\} \in T$, $\{\ell,n,3\} \in T$ and $\{\ell,m,1\} \in T$. But $\{\ell,m,1\} \in T''$, a contradiction.

Secondly, consider when $\{p,q\} = \{\ell,m\}$. Since $\{1,3,m\} \in T'$ and $\{1,3,q\} \in T''$, we must have $\ell = q$ and $m = p$. It follows that $\{m,r,2\}$ and $\{\ell,r,1\} \in T'$. If r occurs once more in T' and T'', then $\{3,r,t\} \in T' \cap T''$ for some t, a contradiction. Thus r occurs twice in blocks of T, contradicting Lemma 13.

Thirdly, consider the case $\{p,q\} = \{\ell,n\}$. Then since $\{2,3,\ell\} \in T'$ and $\{2,3,p\} \in T''$, we must have $q = \ell$ and $p = n$. In turn: $\{m,n,1\} \in T''$, $\{\ell,m,2\} \in T''$, $\{n,r,3\} \in T'$ and $\{\ell,r,1\} \in T'$. There must be one further occurrence of m in T' and T'', so we have $\{3,m,t\} \in T''$ and $\{2,m,t\} \in T'$ for some $t \notin \{\ell,n\}$. Thus, in turn: $\{1,m,t\} \in T$, $\{3,\ell,m\} \in T$, $\{2,m,n\} \in T$ and $\{1,n,r\} \in T$. But now no possible triple of T contains the pair $\{\ell,n\}$.

We are left with the case when $|\{\ell,m,n\} \cap \{p,q,r\}| = 1$. Since $\{1,2,n\} \in T'$ and $\{1,2,r\} \in T''$, either $r = m$ or $r = \ell$. If $r = m$, the pairs $\{m,q\}$ and $\{m,p\}$ must occur in distinct blocks of T' making the degree of m in T to be 4, a contradiction. A similar contradiction is obtained when $r = \ell$. $\square$

Lemma 15. *Either:*

$$\begin{aligned} T_1 &:= \{\{2,\ell,m\},\{3,m,n\},\{1,\ell,n\}\} \subset T \quad \text{and} \\ T_2 &:= \{\{1,\ell,m\},\{2,m,n\},\{3,\ell,n\}\} \subset T'' \end{aligned}$$

or $T_2 \subset T$ and $T_1 \subset T''$. Either

$$\begin{aligned} T_3 &:= \{\{2,p,q\},\{3,q,r\},\{1,p,r\}\} \subset T \quad \text{and} \\ T_4 &:= \{\{1,p,q\},\{2,q,r\},\{3,p,r\}\} \subset T' \end{aligned}$$

or $T_4 \subset T$ and $T_3 \subset T'$.

Proof. From Lemma 13, there are triples in T' of the form $\{1,\ell,t\}$, $\{2,m,u\}$ and $\{3,n,v\}$ where $t,u,v \notin \{2,3,m,n\}$. If $\{3,\ell,m\} \in T$, then either $\{1,\ell,n\},\{2,m,n\} \in T$ or $\{1,m,n\},\{2,\ell,n\} \in T$; in either case $\{3,n,v\} \in T$, a contradiction. Similarly $\{3,\ell,m\} \notin T''$, $\{1,m,n\} \notin T \cup T''$ and $\{2,\ell,n\} \notin T \cup T''$. The first claim of the lemma follows. The second claim follows similarly. $\square$

The next theorem now follows.

Theorem 16. *Let $\{T, T', T''\}$ be a 3-way trade set such that $\{1, 2, 3\} \in T$ where T is subcubic. Let $S = T \setminus \{1, 2, 3\}$,*

$$S' = (T' \setminus \{\{1, 2, n\}, \{2, 3, \ell\}, \{1, 3, m\}, \{\ell, m, n\}\}) \cup \{\{3, \ell, m\}, \{1, m, n\}, \{2, \ell, n\}\}$$

and

$$S'' = (T'' \setminus \{\{1, 2, r\}, \{2, 3, p\}, \{1, 3, q\}, \{p, q, r\}\}) \cup \{\{3, p, q\}, \{1, q, r\}, \{2, p, r\}\}.$$

Then $\{S, S', S''\}$ is a 3-way trade set with S cubic of Type 1.

Example 17. *We apply the previous theorem to the 3-way trade set given in Example 1 to obtain a 3-way trade set including a trade of Type 1.*

$1\ell m$	$2mn$	$3\ell n$	$3qr$	$2pq$	$1pr$	$1ns$	$2\ell s$	$3ms$	$3pt$	$2rt$	$1qt$
$3\ell m$	$1mn$	$2\ell n$	$2qr$	$1pq$	$3pr$	$3ns$	$1\ell s$	$2ms$	$2pt$	$1rt$	$3qt$
$3pq$	$2pr$	$1qr$	$2\ell m$	$3mn$	$1\ell n$	$2ns$	$3\ell s$	$1ms$	$1pt$	$3rt$	$2qt$

The final step in our classification is to show no 4-way trade sets are possible.

Theorem 18. *Let T be a subcubic trade. Then T does not belong to a 4-way trade set.*

Proof. This is clear if T is of Type 1 so suppose T is of Type 2. By Theorem 12, this is also clear if T is of Type 2 and contains a block of the form $\{1, 2, n\}$ where $n \neq 3$. Otherwise, $\{1, 2, 3\} \in T$. Then T' contains blocks $\{1, 2, n\}$, $\{1, 3, m\}$, $\{2, 3, \ell\}$ and $\{\ell, m, n\}$. Suppose for the sake of contradiction that T belongs to a 4-way trade set. Since $\{\ell, m, n\} \in T'$, we must have the blocks $\{1, m, n\}$, $\{2, m, n\}$ and $\{3, m, n\}$ in the three trade mates of T'. Thus there is a block in T' of the form $\{2, m, s\}$, where $s \notin \{1, 3, n, \ell\}$. However the pair $\{m, s\}$ must occur in each trade mate of T'. This is a contradiction as the only other blocks which may contain $\{m, s\}$ are $\{1, m, s\}$ and $\{3, m, s\}$. $\square$

4 Conclusion

In summary, we have shown that the only non-trivial building blocks for subcubic trades in Steiner triple systems are 3-regular 1-factorizable graphs. The "moves" in Section 3 which transform subcubic trades of Type 2 into trades of Type 1 are all reversible; that is, it is possible to describe a set of expansion

moves as in Section 2 that would give an algorithm that would generate all subcubic trades. However we have omitted describing such expansion moves formally to avoid tedium. There may be some room to improve the efficiency of such expansion moves if the results in this paper are applied to generate trades computationally.

As mentioned in Section 1, it is possible for a trade to be subcubic with respect to more than one set of three elements, particularly when a trade has small volume. It would certainly be possible to classify all situations when this occurs.

The results in this paper also have implications for the classification of *Latin trades*, which, analogously to trades in Steiner triple systems, describe the differences between Latin squares of the same order. We refer the reader to [2] for a survey on Latin trades and alternative definitions. For our purposes it is enough to note the following. If the foundation set of a trade T in a Steiner triple system can be partitioned into three sets R, C and S such that each block contains one element from each of these sets, then T is a *Latin trade*. The elements of R, C and S can be thought of as rows, columns and symbols so that the Latin trade takes an array structure and is possibly the subset of a Latin square. Thus if $R = \{1, 2, 3\}$ then the subcubic trades of Type 1 include all possible Latin trades within 3 rows of a Latin square.

Indeed, the subcubic trades of Type 1 which correspond to Latin trades within 3 rows of a Latin square are precisely the ones for which the graph G is bipartite. The expansion processes in Section 2 can be adapted to construct such bipartite graphs. Other recursive constructions for Latin trades using graphs or geometries may be found in [6], [5], [3] and [4] (via [1]).

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