Critical sets of full n-Latin squares

Nicholas J. Cavenagh and Vaipuna Raass
Department of Mathematics,
The University of Waikato,
Private Bag 3105,
Hamilton 3240, New Zealand
nickc@waikato.ac.nz vraass@yahoo.com

Abstract

The full n-Latin square is the $n \times n$ array with symbols $1, 2, \dots, n$ in each cell. In a way that is analogous to critical sets of full designs, a critical set of the full n-Latin square can be used to find a defining set for any Latin square of order n. In this paper we study the size of the smallest critical set for a full n-Latin square, showing this to be somewhere between $(n^3 - 2n^2 + 2n)/2$ and $(n-1)^3 + 1$. In the case that each cell is either full or empty, we show the size of a critical set in the full n-Latin square is always equal to $n^3 - 2n^2 - n$.

1 Introduction

For convenience, we adopt the notation $N(a)$ for the set of positive integers $\{1, 2, ..., a\}$.

We introduce the combinatorial structures in this paper under the umbrella of the very general concept of a Latin array, defined as follows. Let G be a tripartite multigraph with partite sets $R = \{r_1, r_2, ..., r_m\}$, $C = \{c_1, c_2, ..., c_n\}$ and $S = N(t)$. Then, a *Latin array* $L(G)$ is any decomposition of G into triangles. We often represent L in array form, i.e. the triangle $(r, c, s) \in L$ corresponds to the occurrence of the symbol s in cell $L_{r,c}$, where $r \in R, c \in C$ and $s \in S$. If G is a simple complete tripartite graph with $m = n = t$, then L is simply a Latin square of order n. We can use the idea of a Latin array to describe the following generalizations of Latin squares.

If $m = n = t$ and there are k edges between every pair of vertices in G, then $L(G)$ is a *multi-Latin square* of order n and index k (or a *k-Latin square* of order n). Equivalently, a k-Latin square of order n is an $n \times n$ array of multisets of cardinality k from $N(n)$ with each symbol occurring exactly k times in each row and k times in each column (see [3]). The following is an example of a 3-Latin square of order 4.

1,2,4	1,2,3	2,3,4	1,3,4
1,1,3	2,2,4	1,3,4	2,3,4
2,4,3	1,3,4	1,2,2	1,3,4
4,2,3	1,3,4	1,3,4	1,2,2

Note that a 1-Latin square of order n is a Latin square. The n-Latin square of order n given by $L = \{\{r, c, s\} \mid r \in R, c \in C, s \in S\}$ is the *full n-Latin square*. That is, the full n-Latin square is the n-Latin square with each cell containing $N(n)$.

Next we say that $L(G)$ is an *(m, n, t)-balanced Latin rectangle* if G is a complete tripartite multigraph with $|R| = m$, $|C| = n$, $|S| = t$, each vertex of R connected exactly t times to each vertex of C and exactly n times to each vertex of S; and each vertex of C connected exactly m times to each vertex of S.

An *(m, n, t)-balanced Latin rectangle* is thus an $m \times n$ array of multisets of size t such that each element of $N(t)$ occurs n times in each row and m times in each column. The example below is a $(4, 5, 3)$-balanced Latin rectangle.

1,2,3	1,1,2	2,2,3	1,2,3	1,3,3
1,2,2	1,2,3	1,2,3	1,2,3	1,3,3
1,3,3	1,2,3	1,2,3	1,1,3	2,2,2
1,2,3	2,3,3	1,1,3	2,2,3	1,1,2

In the special case when each cell contains $N(t)$, the Latin array is the *full* (m, n, t)-balanced Latin rectangle. Observe that an (n, n, n)-balanced Latin rectangle is an n-Latin square of order n and that the full (n, n, n)-balanced Latin rectangle is also the full n-Latin square.

The above general notion of a Latin array allows us to succinctly define defining sets and critical sets for each of the arrays under consideration in this paper.

Given a tripartite multigraph G as above, any partial decomposition of G into triangles (that is, any set of edge-disjoint triangles in G) is a *partial*

Latin array. Thus we have defined a *partial Latin square*, a *partial k-Latin square of order n* and a *partial (m, n, t)-balanced Latin rectangle*.

For any partial Latin array, we may ask whether or not it completes to a Latin array with the same parameters. Here the graph G is always fixed, so we are in effect asking whether a partial decomposition of G into triangles may be completed to a decomposition of G into triangles. If $L(G)$ is a partial Latin array and there is an unique Latin array $L'(G)$ such that $L(G) \subseteq L'(G)$, then we say that $L(G)$ is a *defining set* of $L'(G)$. If upon removing any triangle from $L(G)$, the partial Latin array formed is no longer a defining set, then we say $L(G)$ is a *critical set* of $L'(G)$.

We emphasize again that G is fixed. Thus, a defining set of the full n-Latin square has unique completion to an n-Latin square L of order n, where L turns out to be the full n-Latin square. For example, consider the partial Latin array given below.

1,2,3,4	1,2,3,4	1,2,3,4	
1,2,3,4	1,2,3,4	1,2,3,4	
1,2,3,4	1,2,3,4	1,2,3,4	

By inspection, given that each symbol occurs 4 times in each row and column, the Latin array has unique completion to a 4-Latin square of order 4 which is the full 4-Latin square. Next, if we remove 1 from cell $(1, 1)$, the partial Latin array has two completions to 4-Latin squares of order 4:

1,2,3,4	1,2,3,4	1,2,3,4	1,2,3,4
1,2,3,4	1,2,3,4	1,2,3,4	1,2,3,4
1,2,3,4	1,2,3,4	1,2,3,4	1,2,3,4
1,2,3,4	1,2,3,4	1,2,3,4	1,2,3,4

2,2,3,4	1,2,3,4	1,2,3,4	1,1,3,4
1,2,3,4	1,2,3,4	1,2,3,4	1,2,3,4
1,2,3,4	1,2,3,4	1,2,3,4	1,2,3,4
1,1,3,4	1,2,3,4	1,2,3,4	2,2,3,4

Applying a similar argument for the removal of each element, the partial Latin array above is a critical set for the full 4-Latin square.

We identify any partial Latin array as *saturated* if each cell is either empty or contains $N(t)$. In particular, a critical set of the full n-Latin square is *saturated* if each cell is either empty or contains $N(n)$. Otherwise it is *non-saturated*. Thus the example given above is a saturated critical set. We may similarly define critical sets for the full (m, n, t)-balanced Latin rectangle as being either saturated or non-saturated.

As we are dealing with multisets we must be careful to make our notation precise. If we denote the multiplicity of an element s in a multiset A by $\nu_A(s)$, then some of the multiset notations are defined by the following *multiplicity functions*:

- $\nu_{A\cap B}(s) = \min\{\nu_A(s), \nu_B(s)\}$,
- $\nu_{A\cup B}(s) = \max\{\nu_A(s), \nu_B(s)\}$,
- $\nu_{A\setminus B}(s) = \max\{0, \nu_A(s) - \nu_B(s)\}$,
- $\nu_{A\uplus B}(s) = \nu_A(s) + \nu_B(s)$,

where $A \uplus B$ is the *multiset sum* of the multisets A and B. The *size* or the number of entries in a Latin array L, denoted by $|L|$, is the cardinality of the multiset sum of the multisets in each cell of L (i.e. the sum of multiplicities of each element over all the cells). The size of any partial Latin array $L(G)$ can also be thought of as the number of edges between R and C used in triangles. Thus the above critical set has size 36.

This paper is motivated by the analogous concept of full designs (see [1, 5, 6, 7, 9, 10]). For block size k, a full design simply consists of all the possible subsets of size k from the foundation set $N(v)$. In [6], it is shown that any minimal defining set for a design is the result of an intersection of the design with a minimal defining set of the full design of the same order.

We have the following similar result:

Theorem 1. *Let C be a defining set of the full n-Latin square and let L be any Latin square of order n. Then $L \cap C$ is a defining set for L.*

Proof. Suppose there exist two distinct Latin squares L and L' of order n such that each contain $L \cap C$. Let $T = L \setminus L'$ and $T' = L' \setminus L$. If L_n is the full n-Latin square, $(L_n \setminus T) \uplus T'$ is an n-Latin square of order n which contains C but is not equal to L_n. Thus C is not a defining set, a contradiction. □

The study of critical sets in full n-Latin squares thus has the potential to yield information on critical sets in Latin squares, which have been extensively studied (see [2, 8] for surveys).

In Section 2, our aim is to study saturated critical sets of the full n-Latin square. We investigate these under the more general guise of saturated critical sets of full balanced rectangles. We classify the exact size of such structures, showing in Theorem 6 that a saturated critical set for the full n-Latin square has size exactly equal to $n^3 - 2n^2 - n$.

In Section 3 we turn our focus on the non-saturated case. We calculate a lower bound for the smallest size of any $2 \times n$ sub-rectangle of a critical set of the full n-Latin square. Using this result, we show that a lower bound for the smallest size of a critical set of the full n-Latin square is $(n^3 - 2n^2 + 2n)/2$. Finally in Section 4 we give a construction which provides an upper bound of $(n-1)^3 + 1$ for the size of the smallest critical set of the full n-Latin square; we conjecture this is best possible.

2 Saturated critical sets of full balanced rectangles

In this section, we determine the exact size of any saturated critical set of the full (m, n, t)-balanced Latin rectangle. This in turn implies the exact size of any saturated critical set of the full n-Latin square.

We remind the reader of the following well-known results from graph theory. If T is a tree (i.e. a connected graph with no cycle) with n vertices, then T has $n-1$ edges and has at least one vertex of degree 1. A simple connected graph with n vertices and at least n edges contains a cycle.

Given a saturated partial (m, n, t)-balanced Latin rectangle, L, we define $G_e(L) = (V_1 \cup V_2, E)$ (where $V_1 = \{r_1, r_2, ..., r_m\}$ and $V_2 = \{c_1, c_2, ..., c_n\}$) to be the bipartite graph that corresponds to the empty cells of L. That is, edge $\{r_i, c_j\}$ is in $G_e(L)$ if and only if the cell $L_{i,j}$ of L is empty. We say that there is a cycle in L if and only if there is a cycle in $G_e(L)$.

First we show that any saturated partial (m, n, t)-balanced Latin rectangle is a defining set if and only if it contains no cycle of empty cells.

Theorem 2. *Let $k \geqslant 2$. A saturated partial (m, n, t)-balanced Latin rectangle is a defining set for the full (m, n, t)-balanced Latin rectangle if and only if it contains no cycle of empty cells.*

Proof. Let D be a saturated partial (m, n, t)-balanced Latin rectangle. Suppose that D contains a cycle of empty cells and let these empty cells be $D_{i(1),j(1)}, D_{i(1),j(2)}, D_{i(2),j(2)}, ..., D_{i(\mu),j(\mu)}, D_{i(\mu),j(1)}$ such that $\{r_{i(1)}, c_{j(1)}\}, \{r_{i(1)}, c_{j(2)}\}, \{r_{i(2)}, c_{j(2)}\}, ..., \{r_{i(\mu)}, c_{j(\mu)}\}, \{r_{i(\mu)}, c_{j(1)}\}$ are the edges of the corresponding cycle in $G_e(D)$. We need to show that D now also completes to a non-full (m, n, t)-balanced Latin rectangle. Such a completion can be achieved by filling $D_{i(1),j(1)}, D_{i(2),j(2)}, ..., D_{i(\mu),j(\mu)}$ with the multiset $(N(k) \setminus \{x\}) \uplus \{y\}$ and the remaining empty cells of the cycle with $(N(k) \setminus \{y\}) \uplus \{x\}$, where $x \neq y : x, y \in N(k)$. Thus D is not a defining set.

Conversely, let D be a saturated partial (m, n, t)-balanced Latin rectangle with no cycles. Then $G_e(D)$ also contains no cycles and thus has at least one vertex of degree 1. So D has a row or column with only one empty cell, and this empty cell is forced to contain $N(t)$, obtaining another saturated partial (m, n, t)-balanced Latin rectangle with no cycles and fewer empty cells. Recursively, since D is finite, D has a unique completion. □

Consequently, a saturated critical set of the full (m, n, t)-balanced Latin rectangle, being a defining set, must contain no cycle. We next show that adding an empty cell to such a critical set creates a cycle, thus increasing the number of possible completions it has.

Lemma 3. *Let C be a saturated critical set of the full (m, n, t)-Latin rectangle. Then deleting the entries of any cell which contains $N(t)$ creates a cycle in C.*

Proof. Suppose that a saturated (m, n, t)-balanced Latin rectangle, D, uniquely completes to the full (m, n, t)-balanced Latin rectangle and we can add another empty cell to D (i.e. delete the entries of a cell which contains $N(t)$) without forming a cycle. Then by Theorem 2, the new partial rectangle is also a defining set with a smaller size, so D is not a critical set by definition. □

Theorem 4. *Let C be a saturated critical set of the (m, n, t)-balanced Latin rectangle. If $|E|$ is the number of empty cells in C, then*

$$|E| = m + n - 1.$$

Proof. By Theorem 2 and Lemma 3, $G_e(C)$ is connected but has no cycle, i.e. $G_e(C)$ is a tree with $m+n$ vertices. Thus $G_e(C)$ has $m+n-1$ edges. □

Since a full n-Latin square is also an (n, n, n)-balanced Latin rectangle, the following corollary is immediate.

Corollary 5. *Let L be the full n-Latin square and let C be a saturated critical set of L. If $|E|$ is the number of empty cells in C, then*

$$|E| = 2n - 1.$$

And since an n-Latin square of order n contains n^3 entries, we can easily work out the size of any saturated critical set of the full n-Latin square from the above theorem.

Theorem 6. *A saturated critical set of the full n-Latin square has size $s = n^3 - 2n^2 - n$.*

Note that in this section we have also classified the structure of any saturated critical set of the full n-Latin square.

3 Sub-rectangles of critical sets of the full n-Latin square

We next analyse critical sets for the full n-Latin square which are not necessarily saturated. To work towards determining an upper bound for the size of such structures, we explore the properties of the sub-rectangles within these critical sets, beginning with 2×2 sub-arrays. An *intercalate* within a Latin array is a set of triangles

$$\{(r, c, s), (r, c', s'), (r', c, s'), (r', c', s)\},$$

where $r \neq r'$, $c \neq c'$ and $s \neq s'$.

Lemma 7. *Any 2×2 sub-array within a defining set of a full n-Latin square must contain no intercalates in its complement.*

Proof. If not, the intercalate may be replaced with the set of triangles

$$\{(r, c, s'), (r, c', s), (r', c, s), (r', c', s')\}$$

and more than one completion is possible. □

It is therefore necessary to avoid intercalates in the complement of any 2×2 sub-array of any defining set for a full n-Latin square. We can achieve this in two ways; either the union of the elements of at least one of the diagonally opposite pairs of cells is the set $N(n)$ or the union of the elements of the two diagonally opposite pairs of cells are equal with size $n - 1$. Thus the minimum size for one of these 2×2 sub-arrays is n. We formally present this idea in the following lemmas.

Lemma 8. *Let S be a 2×2 sub-array of the full n-Latin square with size at most $n-1$. Then the complement, $\bar{S}$, of S contains an intercalate.*

Proof. At least one symbol s appears in each cell of $\bar{S}$ and at least one symbol $s' \neq s$ appears in at least three cells of $\bar{S}$. Thus, $\bar{S}$ contains an intercalate. □

Lemma 9. *Let S be the 2×2 sub-square of a defining set of the full n-Latin square, with cells $S_{1,1}, S_{1,2}, S_{2,1}, S_{2,2}$ as shown below.*

$S_{1,1}$	$S_{1,2}$
$S_{2,1}$	$S_{2,2}$

Then $\bar{S}$ contains no intercalate if and only if at least one of the following holds:

(1) $|S_{1,1} \cup S_{2,2}| = n$,
(2) $|S_{2,1} \cup S_{1,2}| = n$,
(3) $S_{1,1} \cup S_{2,2} = S_{2,1} \cup S_{1,2}$ *and* $|S_{1,1} \cup S_{2,2}| = |S_{2,1} \cup S_{1,2}| = n-1$.

Proof. Suppose $\bar{S}$ contains an intercalate. Then $|\bar{S}_{1,1} \cap \bar{S}_{2,2}| \geqslant 1$, $|\bar{S}_{1,2} \cap \bar{S}_{2,1}| \geqslant 1$, and either $\bar{S}_{1,1} \cap \bar{S}_{2,2} \neq \bar{S}_{1,2} \cap \bar{S}_{2,1}$ or $|\bar{S}_{1,1} \cap \bar{S}_{2,2}|, |\bar{S}_{1,2} \cap \bar{S}_{2,1}| \geqslant 2$. Thus $|S_{1,1} \cup S_{2,2}| \leqslant n-1$, $|S_{1,2} \cup S_{2,1}| \leqslant n-1$ and either $S_{1,1} \cup S_{2,2} \neq S_{1,2} \cup S_{2,1}$ or at least one of these sets is of size at most $n-2$.

Conversely, suppose that conditions (1), (2) and (3) of the lemma are false. If $|S_{1,1} \cup S_{2,2}| = |S_{1,2} \cup S_{2,1}| = n-1$ then $S_{1,1} \cup S_{2,2} \neq S_{1,2} \cup S_{2,1}$. Otherwise, either $|S_{1,1} \cup S_{2,2}|$ or $|S_{1,2} \cup S_{2,1}|$ is of size of at most $n-2$. In both cases, there exists two distinct symbols $s, s' \in N(n)$ such that $s \notin S_{1,1} \cup S_{2,2}$ and $s' \notin S_{1,2} \cup S_{2,1}$; thus $\bar{S}$ contains an intercalate. □

Corollary 10. *Let S be a 2×2 sub-array of a critical set of the full n-Latin square with cells $S_{1,1}, S_{1,2}, S_{2,1}, S_{2,2}$ (as in the statement of Lemma 9) and let $S_1 = S_{1,1} \cup S_{2,1}$ and $S_2 = S_{1,2} \cup S_{2,2}$. Then either $|S_1 \cup S_2| = n$ or $|S_1 \cup S_2| = n-1$ and $|S_{1,1}| + |S_{1,2}| + |S_{2,1}| + |S_{2,2}| \geqslant 2(n-1)$.*

We now apply this result to more general $2 \times m$ sub-arrays of defining sets of full n-Latin squares. First we describe the structure of these sub-arrays in the cases when one of the columns has at most one entry. In the results below, without loss of generality, we assume the rows and columns of the sub-arrays are indexed by $N(2)$ and $N(m)$, with $S_{i,j}$ denoting the set of elements in cell (i,j). We also define $S_j = S_{1,j} \cup S_{2,j}$ ($j \in N(m)$).

Corollary 11. *Consider a $2\times m$ sub-array of a defining set for a full n-Latin square. For any $j \in N(n)$:*

(1) If $|S_j| = 0$, then either $|S_k| = n$ or $|S_{1,k}| = |S_{2,k}| = n-1$ for all $k \neq j$.
(2) If $|S_j| = 1$, then $|S_k| \geqslant n-1$ for all $k \neq j$.

We shall make use of the following technical lemma.

Lemma 12. *Let $S = \{S_1, S_2, ..., S_m\}$ be a set of subsets of $N(n)$ such that:*

$$|\bigcap_{i=1}^{m} S_i| \geqslant n - y_0;$$

for some $y_0 \in N(n)$, and

$$|S_i \cup S_j| = n$$

for all $i, j \in N(m)$ such that $i \neq j$. Then

$$\sum_{i=1}^{m} |S_i| \geqslant mn - y_0.$$

Proof. Without loss of generality, $|S_1| \leqslant |S_2| \leqslant ... \leqslant |S_m|$. For each $\mu \in N(m)$, let $T_\mu = \bigcap_{i=\mu}^{m} S_i$, and let $Y_\mu = S_\mu \setminus T_\mu$ with $|Y_\mu| = y_\mu$. Then $S_\mu = T_\mu \cup Y_\mu$ and $|S_\mu| = |T_\mu| + y_\mu$ $(*)$.

First we show that $|T_k| \geqslant n - y_{k-1}$ and $|S_k| \geqslant n - y_{k-1} + y_k$ for $1 \leqslant k \leqslant m$. For $k = 1$, observe that $|T_1| \geqslant n - y_0$ and $|S_1| \geqslant n - y_0 + y_1$. Now, let $k \geqslant 2$ and let $k \leqslant i \leqslant m$. Let $l = k-1$. Since $S_l = T_l \cup Y_l$, $T_l \subseteq S_i$, and $|S_l \cup S_i| = n$; $T_l \cup (N(n) \setminus S_l) \subseteq S_i$. So $T_l \cup (N(n) \setminus S_l) \subseteq T_k$. But $T_l \cap (N(n) \setminus S_l) = \emptyset$; thus, $|T_k| \geqslant |T_l| + |N(n) \setminus S_l| = n - y_l$ and $|S_k| \geqslant n - y_{k-1} + y_k$ (from $*$).

Finally, observe that

$$\begin{aligned} \sum_{i=1}^{m} |S_i| &\geqslant \sum_{i=1}^{m} (n - y_{i-1} + y_i) \\ &= mn - y_0 + y_m \\ &\geqslant mn - y_0. \end{aligned}$$

□

Corollary 13. *Let S be two rows of a defining set of the full n-Latin square of order $n \geqslant 2$ such that for all $j, k \in N(n)$, $|S_j \cup S_k| = n$. Then $|S| \geqslant (m-1)n$.*

Proof. Without loss of generality, let $|S_1| = x$. Then $|\bigcap_{i=2}^{m} S_i| \geqslant n - x$ so by Lemma 12, $\sum_{i=2}^{m} |S_i| \geqslant (m-1)n - x$ and thus $|S| \geqslant (m-1)n$. □

Lemma 14. *Let S be two rows of a critical set of the full n-Latin square of order $n \geqslant 2$. Then $|S| \geqslant (n-1)^2 + 1$.*

Proof. Let m be the largest integer such that there exists m columns, each with at most $n-2$ entries in S. We split our proof according to the cases $m = 0$, $m = 1$ and $2 \leqslant m \leqslant n$.

Case 1: $m = 0$. Then clearly $|S| \geqslant n(n-1) \geqslant (n-1)^2 + 1$.

Case 2: $m = 1$. Without loss of generality, let $|S_1| = x$, $0 \leqslant x \leqslant n-2$. If $x = 0$, then by Corollary 11, $|S| \geqslant n(n-1)$. Else $|S| \geqslant x + (n-1)^2 \geqslant (n-1)^2 + 1$.

Case 3: $2 \leqslant m \leqslant n$. Without loss of generality, let $|S_j| \leqslant n-2$ for all $j \in N(m)$. By Corollary 10, $|S_x \cup S_y| = n$ for all $x, y \in N(m)$, $x \neq y$. So by Corollary 13, $\sum_{i=1}^{m} |S_i| \geqslant n(m-1)$, and thus

$$\begin{aligned} |S| &= \sum_{i=1}^{m} |S_i| + \sum_{i=m+1}^{n} |S_i| \\ &\geqslant n(m-1) + (n-m)(n-1) \\ &= (n-1)^2 + m - 1 \\ &\geqslant (n-1)^2 + 1. \end{aligned}$$

□

To conclude this section, we use the above lemmas to formulate a lower bound for the size of the smallest critical set of the full n-Latin square.

Theorem 15. *Let C be a critical set of the full n-Latin square. Then*

$$|C| \geqslant \frac{n^3 - 2n^2 + 2n}{2}.$$

Proof. Being an $n \times n$ array, C has $\binom{n}{2}$ distinct pairs of rows with each row occurring in $n-1$ pairs. From the previous lemma, each pair of rows has size at least $(n-1)^2+1$. Thus

$$|C| \geqslant \frac{\binom{n}{2}((n-1)^2+1)}{n-1} = \frac{n}{2}((n-1)^2+1) = \frac{n^3-2n^2+2n}{2}.$$

□

4 An upper bound for the size of the smallest critical set

We begin this section by constructing a non-saturated critical set of the full n-Latin square of size $(n-1)^3+1$. We assume $n \geqslant 2$ throughout.

To this end, let C be the following array. The final row and column are empty except for the symbol n occurring in the cell where they intersect; any cell neither in the final row nor the final column contains $N(n-1)$.

Figure 1: The array C.

$1,2,\ldots,n-1$	$1,2,\ldots,n-1$	$\cdots$	$1,2,\ldots,n-1$	
$1,2,\ldots,n-1$	$1,2,\ldots,n-1$	$\cdots$	$1,2,\ldots,n-1$	
$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\vdots$
$1,2,\ldots,n-1$	$1,2,\ldots,n-1$	$\cdots$	$1,2,\ldots,n-1$	
		$\cdots$		n

Clearly C has size $(n-1)^3+1$. We only need to show that it is a critical set of the full n-Latin square.

Theorem 16. *The array C is a critical set of the full n-Latin square.*

Proof. We first show that C is a defining set of the full n-Latin square. Suppose we try to complete one of the first $n-1$ cells of column m, $m \in N(n-1)$, with the symbol s, $s \in N(n-1)$. Then cell $C_{n,m}$ is now forced to contain at least two copies of the symbol n and for some $k \in N(n-1)$, $k \neq m$, cell $C_{n,k}$, must not contain the symbol n. This implies that column k has at most $n-1$ occurrences of n so C cannot complete to an n-Latin square. Thus for all $i,j \in N(n-1)$, $C_{i,j}$ is forced to contain $N(n)$. Cells in the final row and column follow suit.

We next show that C is a minimal defining set. Suppose we remove the symbol x, $x \in N(n-1)$, from cell $C_{p,q}$, $p, q \in N(n-1)$. Then the 2×2 sub-square formed by the cells $C_{p,q}$, $C_{p,n}$, $C_{n,q}$, and $C_{n,n}$ does not meet the conditions stated by Corollary 10 and thus C cannot be a critical set of the full n-Latin square. On the other hand, if the symbol n is removed from $C_{n,n}$, then again, by Corollary 10, any 2×2 sub-array containing $C_{n,n}$ cannot be contained in a critical set of the full n-Latin square. □

We make the following conjecture.

Conjecture 17. *The exact size of the smallest critical set of the full n-Latin square is $(n-1)^3 + 1$.*

We finally show the above conjecture is true in the special case when a critical set of the full n-Latin square contains only $n-1$ elements of $N(n)$.

Lemma 18. *Let C be a critical set of the full n-Latin square such that a symbol from $N(n)$ does not occur in any cell. Then $|C| \geqslant n(n-1)^2$.*

Proof. Without loss of generality, suppose that the symbol 1 does not occur in C and the symbol 2 occurs at most $n(n-1)-1$ times in C. Then there exists two cells, $C_{i,j}$ and $C_{i',j'}$ of C, such that $i \neq i'$, $j \neq j'$ and neither cell contains the symbol 2. Thus the cells $C_{i,j}$, $C_{i,j'}$, $C_{i',j'}$ and $C_{i',j}$ contain an intercalate in the complement. So each symbol occurring in C (apart from 1) occurs at least $n(n-1)$ times and thus $|C| \geqslant n(n-1)^2$. □

References

[1] S. Akbari, H.R. Maimani and C.H. Maysoori, Minimal defining sets for full 2-$(v, 3, v-2)$ designs, *Australas. J. Combin.* **8** (1993), 55–73.

[2] N.J. Cavenagh, The theory and application of Latin bitrades: a survey, *Math. Slovaca* **58** (2008), 691–718.

[3] N.J. Cavenagh, C. Hamalainen, J. Lefevre and D.S. Stones, Multi-Latin squares, *Discrete Math.* **311** (2011), 1164-1171.

[4] F. Demirkale and E.S. Yazici, On the spectrum of minimal defining sets of full designs, *Graphs Combin.* **30** (2014), 141–157.

[5] D. Donovan, J. Lefevre, M. Waterhouse and E.S. Yazici, Defining sets of full designs with block size three II, *Ann. Combin.* **16** (2012), 507–515.

[6] D. Donovan, J. Lefevre, M. Waterhouse and E.S. Yazici, On defining sets of full designs with block size three, *Graphs Combin.* **25** (2009), 825–839.

[7] K. Gray, A.P. Street and R.G. Stanton, Using affine planes to partition full designs with block size three, *Ars Combin.* **97A** (2010), 383–402.

[8] A. D. Keedwell, Critical sets in Latin squares and related matters: an update, *Util. Math* **65** (2004), 97–131.

[9] E. Kolotoglu and E.S. Yazici, On minimal defining sets of full designs and self-complementary designs, and a new algorithm for finding defining sets of t-designs, *Graphs Combin.* **26** (2010), 259–281.

[10] J. Lefevre and M. Waterhouse, On defining sets of full designs, *Discrete Math.* **310** (2010), 3000-3006.