Adversarial risk analysis for auctions using non-strategic play and level-k thinking: A general case of n bidders with regret

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ABSTRACT
First-price sealed-bid auctions have been modeled using a utility function that takes into account bidders' regret in case of winning or losing the auctioned item. However, that modeling does not consider bidders' wealth which is an important determinant of bidders' behavior in these auctions.

In this paper, we apply an adversarial risk analysis approach for these auctions and find solutions using non-strategic play and level-k thinking solution concepts assuming n bidders. We define new regret parameters and a modified utility function to incorporate the effect of bidders' wealth on their bidding behavior. In our modeling, we assume that the auctioned item is a normal item and has a reserve price. We give numerical examples to illustrate our methodology for each solution concept.

KEYWORDS
First-price sealed-bid auctions; Adversarial risk analysis; Bayesian game theory; Decision theory

1. Introduction

1.1. *Models for first-price sealed-bid auctions*

Considerable progress has been made during the past half century in modeling of various auctions formats and the bidders' behaviors. *First-price sealed-bid* (FPSB) auctions[1] are one of the commonly used auction formats. Bidders' behaviors for FPSB auctions have commonly been modeled using the *decision-theoretic* and *Bayesian game-theoretic* approaches. Using a decision-theoretic approach, the decision maker believes that all other bidders are non-strategic and finds their optimal bid by placing subjective distributions on each of the other bidders' valuations for the auctioned item. It has been argued that the bidders use decision-theoretic models for auctions since they typically assess the probability distribution of the best competitive bid and then optimize their decision about the bid against that assessed probability distribution (see e.g. Capen, Clapp, and Campbell 1971; Keefer, Smith Jr, and Back 1991; Rothkopf and Harstad 1994; Rothkopf 2007; Wang and Guo 2017, among others). However, decision theory models non-strategic adversaries, whereas bidders can often be strategic.

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[1]In FPSB auctions, bidders submit their bids in sealed envelopes to the auctioneer. Those envelopes are then opened and the bidder with the highest bid wins the auction and pays the amount equal to the bid.

A Bayesian game-theoretic model on the other hand, models bidders to be strategic adversaries. In this model, bidders draw their valuations for the auctioned item from a distribution that is considered to be commonly known to all bidders and find their optimal bids using a Bayes Nash Equilibrium (BNE) bidding strategy. Vickrey (1961) made a significant breakthrough in understanding bidders' behaviors using a Bayesian game-theoretic model. Further work using a Bayesian game-theoretic approach includes Criesmer, Levitan, and Shubikt (1967); Wilson (1969); Riley and Samuelson (1981); Cox, Smith, and Walker (1982); Cox, Roberson, and Smith (1982); Maskin and Riley (1984, 2000); Campo et al. (2011); Gentry, Li, and Lu (2015); Li and Tan (2017) among others. The assumptions of "non-strategic opponents" in decision theory and of the "common knowledge" in Bayesian game theory make these two approaches unrealistic in practical situations in which the opponents may be strategic and may wish to keep their bidding distribution a secret to gain competitive advantage. See Joshi, Aliaga, and Insua (2020) for further discussion on the limitations of a Bayesian game theoretic approach including on games with incomplete information.

Ríos Insua, Rios, and Banks (2009) introduced an approach called *adversarial risk analysis* (ARA) that is considered to be more realistic to model real life situations involving strategic adversaries than the traditional decision theory and game theoretic approaches. While ARA allows the decision maker to model strategic opponents as in game theory, it models the problem from the decision maker's point of view alone, thus eliminating the need to assume distributions that are commonly known by all players. ARA is a Bayesian approach because the decision maker uses her subjective distributions to model the unknown preferences, capabilities and beliefs of her strategic opponents. However, unlike game theory, these subjective distributions do not need to be commonly known to all players. Also, ARA allows the decision maker to model how her intelligent adversaries frame the problem which includes different solution concepts such as non-strategic play, level-k thinking, mirror equilibrium (ME) or indeed the BNE. While using a non-strategic play solution concept, the decision maker models their rivals to be non-strategic - whose bids are independent of their rivals' bids; when using a BNE solution concept, the decision maker believes that their adversaries will use a BNE strategy to find their optimal bids. In a level-k thinking solution concept, the decision maker being a level-k thinker believes that her rivals are level-$(k-1)$ thinkers and each level-$(k-1)$ thinker believes that her/his rivals are level-$(k-2)$ thinkers and so on. Thus, the decision maker always thinks one level higher than her rivals in this solution concept. While using a ME solution concept, the decision maker believes that her rivals are modeling their opponents' (including the decision maker's) actions in the same way as she is modeling their actions. All use their subjective distributions over the probabilities and utilities of their rivals and seek an equilibrium.

Banks, Rios, and Ríos Insua (2015) modeled FPSB auctions assuming that each bidder is risk-neutral and found ARA solutions for non-strategic play, minimax perspective, level-k thinking, ME and BNE solution concepts. Ejaz, Joshi, and Joe (2021) extended this work by using a more realistic utility function for these auctions and assuming that the bidders have different risk behaviors and different wealth. They derived ARA solutions using non-strategic play and the level-k thinking solution concepts assuming two bidders. Then, Ejaz, Joe, and Joshi (2021) modeled these auctions using ME and BNE solution concepts under ARA framework where they assumed n bidders, all of whom have different risk behaviors and different wealth.

In all of these models, bidders' bidding behavior is determined using a utility function that depends only upon each bidder's profit. However, there may be other potentially important components than profit such as institutional framework (Fox and Tversky

1998), misperception of probabilities of winning (Dorsey and Razzolini 2003) and bidders' regret (Engelbrecht-Wiggans 1989; Engelbrecht-Wiggans and Katok 2007) etc., that affect bidders' bidding behavior.

1.2. *Utility functions for FPSB auctions*

In FPSB auctions, suppose that the bidders are risk-neutral who draw their values independently and privately. Then, a bidding strategy that is found using a game-theoretic approach is known as a risk-neutral Nash Equilibrium (RNNE) bidding strategy (see Vickrey 1961). The FPSB auctions have extensively been analysed using experimental techniques. A consistent outcome found in the experiments is that bidders consistently bid above the RNNE bidding strategy (see e.g., Dorsey and Razzolini 2003; Engelbrecht-Wiggans 1989; Engelbrecht-Wiggans and Katok 2007, among others). Different types of explanations such as risk-aversion, interpersonal interaction, learning direction, and regret etc. have been given in economic literature for this overbidding behavior of the bidders.
Risk aversion is relevant to take into account in FPSB auctions as one explanation of bidders' overbidding behavior (see e.g., Milgrom and Weber 1982; Maskin and Riley 1984; Cox, Smith, and Walker 1988; Gentry, Li, and Lu 2015, among others). In such modeling, a constant relative risk-averse (CRRA) utility function is the most commonly used function (Holt and Laury 2002). A CRRA utility function for the ith bidder having wealth w_i is typically defined as

$$u_i(b_i, v_i, w_i) = \begin{cases} (w_i + v_i - b_i)^{r_i}, & \text{if she wins the bid,} \\ w_i^{r_i}, & \text{if she loses the bid,} \end{cases} \tag{1}$$

where, $(1 - r_i)$ is the coefficient of CRRA, v_i is the ith bidder's true value for the auctioned item and b_i is her amount of bid. However, it has been shown (Ejaz, Joshi, and Joe 2021) that the utility function (1) is unrealistic because it can yield a positive utility even when the bidder does not have the ability to buy the auctioned item. Ejaz, Joshi, and Joe (2021) assumed that the auctioned item is *normal*[2] and defined a new CRRA utility function for the ith bidder participating in FPSB auctions as

$$u(b_i, v_i, w_i) = \begin{cases} w_i + (v_i - b_i)^{a_i}, & \text{if she wins the bid,} \\ w_i, & \text{if she loses the bid,} \end{cases} \tag{2}$$

where a_i is a modified CRRA parameter that changes with the relative change in circumstances of the bidders' wealth. However, Kagel (1995, p. 525) suggested that risk aversion may not be the only factor that generates bidding above the RNNE in FPSB auctions. Inter-personal interactions and comparisons is another explanation of bidders' overbidding behavior in FPSB auctions (see e.g., Isaac and Walker 1985; Dufwenberg and Gneezy 2002; Morgan, Steiglitz, and Reis 2003). Another explanation of this overbidding behavior is using learning direction theory proposed by Selten and Stoecker (1986). Based on *feedback over time* this theory leads to the direction in which the bids are likely to be adjusted. Neugebauer and Selten (2006) found that most of the bidders adjust their bids in the same way as that of learning direction theory.

[2]An item with positive income elasticity is defined as a *normal item* in economic theory, i.e., the demand for a normal item rises with an increase in income and falls with a decrease in income (see, e.g., Fisher 1990; Goeree, Holt, and Palfrey 2002; Piros and Pinto 2013; Perloff 2015; Baisa 2017, for more details)

However, this theory does not give any explanation for bidders' initial overbidding. Engelbrecht-Wiggans (1989) gave another possible explanation of bidders' overbidding behavior and modeled their utility by a linear combination of their profit and two types of regret. They stated that in FPSB auctions, the winner pays more than the second highest bid and therefore could realize a winning regret by paying much more than the second highest bid. On the other hand, some winner's bid could be less than that of the losing bidder's valuation (willingness to pay). Thus, in this case the loser could regret bidding too low because she has missed an opportunity to win the item at a favourable price. So, the bidder could not only be sensitive to her expected profit but also to the expected amount of her winning and losing regret when deciding on her bid amount. The weight on each type of regret potentially determines her bidding behavior. Engelbrecht-Wiggans and Katok (2007) modeled the ith bidder's bidding behavior by taking into account her winning and losing regrets while assuming that she has a risk-neutral utility. They defined the ith bidder's utility function from a bid b_i while having true value v_i as

$$u_i(b_i, v_i) = \begin{cases} (v_i - b_i) - \int_{b_j : b_j \leq b_i} [\zeta_i + \eta_i(b_i - b_j)] dF_{ij}(b_j | b_j \leq b_i), & \text{if she wins the bid,} \\ - \int_{b_j : b_i < b_j \leq v_i} [\vartheta_i + \theta_i(v_i - b_j)] dF_{ij}(b_j | b_j > b_i), & \text{if she loses the bid,} \end{cases} \tag{3}$$

where b_j is the jth bidder's bid and is the maximum of the bids from the ith bidder's $(n-1)$ opponents and $F_{ij}(b_j)$ is the probability distribution on b_j that the ith bidder believes. Also, $b_i - b_j$ is the excess amount of money paid if the ith bidder wins and her utility suffers by an amount $\zeta_i + \eta_i(b_i - b_j)$ where $\eta_i \geq 0$. Larger values of η_i means a higher winning regret for the ith bidder. Negative values of ζ_i allows some pleasure to the ith bidder in case of winning. On the other hand, if the ith bidder loses and the highest bid satisfies the inequality $b_i \leq b_j \leq v_i$, then the ith bidder misses an opportunity to win at a favourable price and her utility suffers by an amount $\vartheta_i + \theta_i(v_i - b_j)$, where $\vartheta_i, \theta_i \geq 0$.
However, the model proposed by Engelbrecht-Wiggans and Katok (2007) does have a few limitations. Firstly, the utility function (3) does not take into account the bidder's wealth and therefore assumes that all bidders have zero wealth, that is $w_i = 0$, $\forall i$. It is unrealistic to assume that the ith bidder can place their optimal bid $b_i^* > 0$ found using this utility function with zero wealth. In fact, considering bidders' wealth is a significant determinant of bidders' bidding behavior in these auctions (see e.g., Gentry, Li, and Lu 2015; Ejaz, Joshi, and Joe 2021, among others). Secondly, Engelbrecht-Wiggans and Katok (2007) found their optimal bid using a decision theoretic approach, the limitations of which have been discussed above. ARA solutions for FPSB auctions using a utility function that takes into account the winning and losing regret have not been found yet. It turns out that finding ARA solutions for such a type of utility function is methodologically and computationally more challenging than finding ARA solutions using a CRRA utility function such as the one given in (2).

1.3. *Contributions in this paper*

The main contributions contained in this paper are as follows:

- We modify the utility function (3) used by Engelbrecht-Wiggans and Katok (2007) wherein, we assume that the auctioned item is normal which has a reserve

price and the bidders may have different wealth. We define new winning and losing regret parameters to incorporate the effect of increase in wealth on bidders' bidding behavior.
- We find ARA solutions for non-strategic play and level-k thinking solution concepts using the modified utility function.

1.4. *Structure of the paper*

The rest of this paper is organised as follows. In Section 2, we define new regret parameters and the modified utility function. In Section 3, we derive ARA solutions using the non-strategic play solution concept assuming n bidders participating in a FPSB auction. In Section 4, we derive ARA solutions using the level-k thinking solution concept. Finally, in Section 5, we discuss the results obtained in this paper and present some ideas for future work.

2. New regret parameters and modified utility function

The winning regret parameters ζ_i and η_i in (3) yield a linear function where ζ_i is allowed to take both the negative and positive values and $\eta_i \geq 0$. Positive values of ζ_i result in an increase in bidder's winning regret and consequently her bid would decrease. Whereas, with the negative values of ζ_i, the bidder can realize some pleasure from winning. But, with sufficiently negative values of ζ_i, the bidder could bid in excess of her true value. On the other hand, the losing regret parameters ϑ_i and θ_i also yield a linear function where ϑ_i, $\theta_i \geq 0$. The greater the values of ϑ_i and θ_i, the more the losing regret to the bidder and as a result she would bid higher.
Since in equilibrium, bidders never bid above their true values (Gentry, Li, and Lu 2015), we make a realistic assumption that the ith bidder will bid b_i such that $b_i \leq v_i \leq w_i$. Therefore, we do not allow negative values of ζ_i because negative values can result in the ith bidder bidding more than her true value. Also, with an increase in positive values of ζ_i, the ith bidder's winning regret increases which can be modeled by having larger values of η_i only in the utility function. Therefore, we model the ith bidder's winning regret only with η_i and set $\zeta_i = 0$. Similarly, the ith bidder's losing regret can be modeled by having just θ_i and therefore, we set $\vartheta_i = 0$.

When the auctioned item is normal, bidders' winning and losing regret could change with the relative change in the circumstances of their wealth. For the ith bidder, suppose her original winning and losing regret parameters are η_i and θ_i, respectively. These are fixed and we call them her baseline regret parameters. Note that η_i and θ_i are the same as defined in (3). The relative change in circumstances could occur in two possible cases; firstly, when a bidder's wealth changes and secondly, when a bidder attempts to model the winning and losing regrets of her rivals.
Firstly, we model the ith bidder's regret at an increased level of her wealth compared with her original level of wealth and thus modify her regret parameters. As mentioned above, η_i is the ith bidder's baseline winning regret parameter and θ_i is her baseline losing regret parameter, which represent her natural regret appetite at her wealth, say $w_{i(1)}$. Lets assume that the circumstances of the ith bidder change (e.g., she gains an inheritance) and her wealth is increased to $w_{i(2)}$. At this increased wealth level, we expect an increase in her losing regret and a decrease in her winning regret for the same auctioned item (since the item is assumed to be a normal item). So, we modify

her winning regret parameter having wealth $w_{i(2)}$ relative to wealth $w_{i(1)}$ as

$$\hat{\eta}_i = h \times \eta_i, \quad \eta_i > 0, \tag{4}$$

and we modify her losing regret parameter having wealth $w_{i(2)}$ relative to wealth $w_{i(1)}$ as

$$\hat{\theta}_i = \frac{1}{h} \times \theta_i, \quad \theta_i > 0, \tag{5}$$

where, we define $0 < h = w_{i(1)}/w_{i(2)} \leq 1$. Note that when $h = 1$, that means no change in the ith bidder's wealth and thus she would have $\hat{\eta}_i = \eta_i$ and $\hat{\theta}_i = \theta_i$.
Secondly, when the ith bidder is bidding against her $(n-1)$ rivals in FPSB auctions and believes that she is the wealthiest among all n bidders, then she would have more losing regret and less winning regret compared to her baseline. Suppose that w_i is the wealth of the ith bidder and w_j is the wealth of the jth bidder who the ith bidder believes is the wealthiest among the other $(n-1)$ bidders. Thus, for $w_i > w_j$, we modify the ith bidder's winning and losing regret by using (4) and (5), respectively, where $0 < h = w_j/w_i \leq 1$. In this case, she would model the other $(n-1)$ bidders' (including the jth bidder's) modified regret parameters as being equal to their baseline regret parameters, i.e., $\hat{\eta}_s = \eta_s$ and $\hat{\theta}_s = \theta_s$ for $s = 1, 2, \ldots, n$, $s \neq i$. In contrast, if the ith bidder believes that the jth bidder is the wealthiest bidder among all n bidders, she can find the jth bidder's modified regret parameters from (4) and (5) by replacing i with j and taking $0 < h = w_i/w_j \leq 1$. In this case, we assume that she would model the regret parameters of the other $(n-1)$ bidders (including herself) as being unchanged, i.e., $\hat{\eta}_s = \eta_s$ and $\hat{\theta}_s = \theta_s$ for $s = 1, 2, \ldots, n$, $s \neq j$.

In the utility function (3), $(v_i - b_j)$ yields the opportunity loss by the ith bidder at a favourable price if she loses, i.e., the extent of the difference between her true value v_i for the item and the winning bid b_j if $b_i < b_j \leq v_i$. However, the ith bidder could have regret on just the extent of the difference between her losing bid b_i and the winning bid b_j, i.e., $(b_j - b_i)$ provided that $b_i < b_j \leq v_i$ which is what tends to happen in practice. Thus, using (4) and (5), we define our modified utility function for the ith bidder as

$$u_i(b_i, v_i, w_i) = \begin{cases} (w_i + v_i - b_i) - \int_{b_j : b_j \leq b_i} \hat{\eta}_i (b_i - b_j) dF_{ij}(b_j | b_j \leq b_i), & \text{if she wins the bid,} \\ w_i - \int_{b_j : b_i < b_j \leq v_i} \hat{\theta}_i (b_j - b_i) dF_{ij}(b_j | b_j > b_i), & \text{if she loses the bid,} \end{cases} \tag{6}$$

where the effect of bidders' wealth has been incorporated in the form of modified regret parameters in (6) which has not been encountered yet for such type of utility function.

3. Non-strategic play

In this section, we assume n bidders and find ARA solutions for one of the bidders named Brenda. Using a non-strategic play solution concept, Brenda believes that all the other $(n-1)$ bidders are non-strategic, i.e., their analyses do not depend upon the situation of Brenda and the other bidders. We assume that each of Brenda's opponents will bid an amount that is independent of her bid. We assume that Brenda is the first bidder among n bidders and she bids an amount b_1, having wealth w_1 and true value v_1 for the auctioned item. We assume that the auctioned item has a reserve price τ such that $\tau < b_1 \le v_1 \le w_1$ and it is a normal item. Brenda believes that the other $(n-1)$ bidders have wealth W_i for $i = 2, \ldots, n$ that are unknown to her and therefore, she places the distributions H_{1i} with support $[\underline{w}_i, \bar{w}_i]$ on their wealth according to her belief such that $\underline{w}_i > \tau$, $i = 2, \ldots, n$. She also does not know about their true values V_i and their bids B_i for the auctioned item. Therefore, she places distributions G_{1i} with support $[\underline{v}_i, \bar{v}_i]$ on their true values according to her belief such that $\tau < \underline{v}_i \le \underline{w}_i$, $\bar{v}_i \le \bar{w}_i$, $i = 2, \ldots, n$. Then she finds the distributions F_{1i} (defined in Equation (8) later in this section) of their bids with support $[\underline{b}_i, \bar{b}_i]$ such that $\tau < \underline{b}_i \le \underline{v}_i$ and $\bar{b}_i \le \bar{v}_i$. Assuming that the bids are continuous, Brenda can find her probability of winning from a bid of amount b_1 against the ith bidder as

$$F_{1i}(b_1) = \Pr(B_i < b_1), \quad i = 2, \ldots, n,$$

where, F_{1i} is the distribution over the ith bidder's bid that Brenda believes. To obtain F_{1i}, Brenda divides her introspection into two parts as G_{1i}, the cumulative distribution function (CDF) that quantifies her uncertainty for the ith bidder's true value and T_{1i}, the CDF that quantifies her uncertainty for the fraction of the true value $p_i = b_i/v_i$ that the ith bidder bids, where b_i and v_i are the respective bids and true values of the ith bidder, $i = 2, \ldots, n$. The support for T_{1i} is $(\underline{v}_i/v_i, 1]$. She can then derive her subjective distribution function over $B_i = P_i V_i$, the amount of the ith bidder's random bid, as

$$\begin{aligned} F_{1i}(b_i) = \Pr[\underline{v}_i < P_i V_i \le b_i] = &\int_{\underline{v}_i}^{b_i} \int_{\underline{v}_i/v_i}^{1} g_{1i}(v_i) t_{1i}(p_i)\, dp_i\, dv_i \\ &+ \int_{b_i}^{\bar{v}_i} \int_{\underline{v}_i/v_i}^{b_i/v_i} g_{1i}(v_i) t_{1i}(p_i)\, dp_i\, dv_i. \end{aligned} \tag{7}$$

As $\int_{\underline{v}_i/v_i}^{1} t_{1i}(p_i) dp_i = 1$, the above equation simplifies to

$$F_{1i}(b_i) = G_{1i}(b_i) + \int_{b_i}^{\bar{v}_i} g_{1i}(v_i) T_{1i}(b_i/v_i)\, dv_i, \tag{8}$$

where, $g_{1i}(v_i)$ is the probability density function for the ith bidder's true value that Brenda elicits and $t_{1i}(p_i)$ is the probability density function for the fraction of the ith bidder's true value that Brenda believes he will bid. Equation (7) assumes that the ith bidder's true value V_i and fraction of his true value P_i are independent.

Now, based upon the assumption that the bidder having more wealth tends to bid higher than the bidder having less wealth when the auctioned item is normal, Brenda believes that the jth bidder having the highest wealth w_j among her $(n-1)$ opponents will bid b_j which is the maximum bid among the other $(n-1)$ bidders. So, Brenda will consider the jth bidder as her competitor bidder. Thus, using (4), (5) and (6), Brenda's expected utility is

$$\Psi_1 = \left[(w_1 + v_1 - b_1) - \int_{b_j : b_j \leq b_1} \hat{\eta}_1 (b_1 - b_j) dF_{1j}(b_j | b_j \leq b_1) \right] F_{1j}(b_1) + \left[w_1 - \int_{b_j : b_1 < b_j \leq v_1} \hat{\theta}_1 (b_j - b_1) dF_{1j}(b_j | b_j > b_1) \right] [1 - F_{1j}(b_1)],$$

where F_{1j} is the probability distribution on b_j that Brenda believes which can be found by using (8) and $\hat{\eta}_1$ and $\hat{\theta}_1$ are her modified winning and losing regret parameters, respectively. As, $dF_{1j}(b_j|b_j \leq b_1) = dF_{1j}(b_j)/F_{1j}(b_1)$ and $dF_{1j}(b_j|b_j > b_1) = dF_{1j}(b_j)/[1 - F_{1j}(b_1)]$, the above equation simplifies to

$$\Psi_1 = w_1 + (v_1 - b_1)F_{1j}(b_1) - \int_{b_j : b_j < b_1} \hat{\eta}_1 (b_1 - b_j) dF_{1j}(b_j) - \int_{b_j : b_1 < b_j \leq v_1} \hat{\theta}_1 (b_j - b_1) dF_{1j}(b_j).$$

Finally, Brenda can find her optimal bid b_1^* by solving the following

$$b_1^* = \arg\max_{b_1 > \tau} \left[w_1 + (v_1 - b_1)F_{1j}(b_1) - \int_{b_j : b_j < b_1} \hat{\eta}_1 (b_1 - b_j) dF_{1j}(b_j) - \int_{b_j : b_1 < b_j \leq v_1} \hat{\theta}_1 (b_j - b_1) dF_{1j}(b_j) \right]. \tag{9}$$

In fact, b_1^* is a function of $\hat{\eta}_i$ and $\hat{\theta}_i$ and therefore, a function of η_i, θ_i and w_j. As η_i and θ_i are Brenda's fixed baseline regret parameters, we can treat b_1^* as a function of w_j and so re-write (9) as

$$b_1^*(w_j) = \arg\max_{b_1 > \tau} \left[w_1 + (v_1 - b_1)F_{1j}(b_1) - \int_{b_j : b_j < b_1} \hat{\eta}_1 (b_1 - b_j) dF_{1j}(b_j) - \int_{b_j : b_1 < b_j \leq v_1} \hat{\theta}_1 (b_j - b_1) dF_{1j}(b_j) \right]. \tag{10}$$

Brenda can use (10) to find her optimal bid by taking into account her uncertainty around w_j and find the expected value of her optimal bid amount as

$$\mathrm{E}(b_1^*) = \int b_1^*(w_j) dH_{1j}(w_j), \tag{11}$$

where H_{1j} is the distribution on w_j that Brenda believes. Numerical methods may often be needed to solve (11) for $\mathrm{E}(b_1^*)$.

Example 3.1. Suppose Brenda is bidding against her opponents Alex and Charles. Brenda believes that Alex and Charles are non-strategic bidders. Let Brenda, Alex and Charles be the first, second and the third bidders, respectively. Let Brenda's true value for the item be $v_1 = \$200$ and the auctioned item have a reserve price $\tau = \$25$. Also,

- Brenda has a uniform distribution on Alex's true value with $\underline{v}_2 = \$30, \bar{v}_2 = \$200$, i.e., $G_{12} = \frac{(v_2-30)}{200-30}$.
- Brenda has a uniform distribution on Alex's wealth with $\underline{w}_2 = \$100, \bar{w}_2 = \$250$, i.e., $H_{12} = \frac{(w_2-100)}{250-100}$.
- Brenda has a uniform distribution on Charles's true value with $\underline{v}_3 = \$30, \bar{v}_3 = \$250$, i.e., $G_{13} = \frac{(v_3-30)}{250-30}$.
- Brenda has a uniform distribution on Charles's wealth with $\underline{w}_3 = \$150, \bar{w}_3 = \$300$, i.e., $H_{13} = \frac{(w_3-150)}{300-150}$.
- Brenda elicit her uncertainties around the fraction of her opponents true values that they would bid as $T_{1i} = \frac{p_i^8-(\underline{v}_i/v_i)^8}{1-(\underline{v}_i/v_i)^8}$ with support $(\underline{v}_i/v_i < p_i \leq 1]$ for $i = 2, 3$.

First, for the sake of illustration and simplicity, we assume that Brenda chooses Alex and Charles's wealth to be $w_2 = \$170$ and $w_3 = \$210$, respectively, according to her belief. Thus, she would believe Charles as her competitor as he has more wealth than Alex and will find her optimal bid against him. Thus, using (8) where b_i is replaced by b_1, she will find the distribution of her bid against Charles as

$$\begin{aligned}
F(b_1) =& \frac{b_1 - \underline{v}_3}{\bar{v}_3 - \underline{v}_3} + \frac{b_1^8 - \underline{v}_3^8}{\bar{v}_3 - \underline{v}_3}\Bigg[- \frac{\sqrt{2}}{16 \times \underline{v}_3^7}\ln\Big(\frac{\bar{v}_3^2 + \bar{v}_3\underline{v}_3\sqrt{2} + \underline{v}_3^2}{\bar{v}_3^2 - \bar{v}_3\underline{v}_3\sqrt{2} + \underline{v}_3^2}\Big) \\
&+ \frac{\sqrt{2}}{16 \times \underline{v}_3^7}\ln\Big(\frac{b_1^2 + b_1\underline{v}_3\sqrt{2} + \underline{v}_3^2}{b_1^2 - b_1\underline{v}_3\sqrt{2} + \underline{v}_3^2}\Big) - \frac{\sqrt{2}}{8 \times \underline{v}_3^7}\tan^{-1}\Big(\frac{\bar{v}_3\sqrt{2}}{\underline{v}_3} + 1\Big) \\
&+ \frac{\sqrt{2}}{8 \times \underline{v}_3^7}\tan^{-1}\Big(\frac{b_1\sqrt{2}}{\underline{v}_3} + 1\Big) - \frac{\sqrt{2}}{8 \times \underline{v}_3^7}\tan^{-1}\Big(\frac{\bar{v}_3\sqrt{2}}{\underline{v}_3} - 1\Big) \\
&+ \frac{\sqrt{2}}{8 \times \underline{v}_3^7}\tan^{-1}\Big(\frac{b_1\sqrt{2}}{\underline{v}_3} - 1\Big) - \frac{1}{4 \times \underline{v}_3^7}\tan^{-1}\Big(\frac{\bar{v}_3}{\underline{v}_3}\Big) + \frac{1}{4 \times \underline{v}_3^7}\tan^{-1}\Big(\frac{b_1}{\underline{v}_3}\Big) \\
&+ \frac{\ln(\bar{v}_3 - \underline{v}_3)}{8 \times \underline{v}_3^7} - \frac{\ln(b_1 - \underline{v}_3)}{8 \times \underline{v}_3^7} - \frac{\ln(\bar{v}_3 + \underline{v}_3)}{8 \times \underline{v}_3^7} + \frac{\ln(b_1 + \underline{v}_3)}{8 \times \underline{v}_3^7}\Bigg].
\end{aligned}$$

We assume that Brenda's baseline winning and losing regret parameters are η_1 and θ_1, respectively, and suppose that her wealth is $w_1 = \$200$. As her wealth $w_1 = \$200$ is less than that of Charles' wealth of $w_3 = \$210$, we take $\hat{\eta}_1 = \eta_1$ and $\hat{\theta}_1 = \theta_1$ as defined in Section 2. The results for this case have been presented in Table 1.

Table 1. Brenda's optimal bids, her probabilities of winning and expected utilities when $w_1 = \$200$ and $w_3 = \$210$.

$\hat{\eta}_1 = \eta_1$	$\hat{\theta}_1 = \theta_1$	0.50	1.00	2.00	3.00	4.00
	b_1^*	108.98	119.12	133.23	142.81	149.84
0.50	$F(b_1^*)$	0.43	0.48	0.54	0.59	0.62
	$\Psi_1(b_1^*)$	222.35	215.66	206.13	199.60	194.80
	b_1^*	97.39	107.63	122.34	132.57	140.21
1.00	$F(b_1^*)$	0.37	0.42	0.49	0.54	0.58
	$\Psi_1(b_1^*)$	214.78	205.93	192.92	183.70	176.77
	b_1^*	81.75	91.49	106.24	117.00	125.27
2.00	$F(b_1^*)$	0.29	0.34	0.41	0.47	0.51
	$\Psi_1(b_1^*)$	204.63	192.26	173.13	158.89	147.80
	b_1^*	71.73	80.68	94.82	105.54	114.00
3.00	$F(b_1^*)$	0.24	0.28	0.36	0.41	0.45
	$\Psi_1(b_1^*)$	198.18	183.14	159.01	140.37	125.44
	b_1^*	64.77	72.94	86.26	96.68	105.11
4.00	$F(b_1^*)$	0.20	0.24	0.31	0.36	0.41
	$\Psi_1(b_1^*)$	193.73	176.65	148.44	125.99	107.62

Now, we assume Brenda's wealth increases to $w_1 = \$250$, while Charles's wealth is unchanged at $w_3 = \$210$. In this case, we expect her to have more losing regret and less winning regret than Charles because she is able (since the item is normal) to pay more to increase her chance of winning the bid. By using (4), we model Brenda's reduced winning regret when $w_1 = \$250$ relative to $w_3 = \$210$ as $\hat{\eta}_1 = h \times \eta_1 = (210/250) \times \eta_1 = 0.84\eta_1$. Her increased losing regret is modeled by using (5) when $w_1 = \$250$ relative to $w_3 = \$210$ as $\hat{\theta}_1 = (1/h) \times \theta_1 = (250/210) \times \theta_1 = 1.19\theta_1$. We present the results for this case in Table 2.

Table 2. Brenda's optimal bids, her probabilities of winning and expected utilities when $w_1 = \$250$ and $w_3 = \$210$.

η_1	$\hat{\eta}_1$						
		θ_1	0.50	1.00	2.00	3.00	4.00
		$\hat{\theta}_1$	0.60	1.19	2.38	3.57	4.76
		b_1^*	113.34	124.44	139.21	148.84	155.73
0.50	0.42	$F(b_1^*)$	0.44	0.51	0.57	0.61	0.64
		$\Psi_1(b_1^*)$	272.46	265.45	255.95	249.72	245.29
		b_1^*	102.90	114.26	129.79	140.14	147.65
1.00	0.84	$F(b_1^*)$	0.40	0.45	0.52	0.57	0.61
		$\Psi_1(b_1^*)$	265.24	256.11	243.34	234.71	228.42
		b_1^*	88.03	99.19	115.20	126.32	134.61
2.00	1.68	$F(b_1^*)$	0.32	0.38	0.46	0.51	0.55
		$\Psi_1(b_1^*)$	255.01	242.21	223.44	210.13	200.11
		b_1^*	77.96	88.52	104.31	115.68	124.35
3.00	2.52	$F(b_1^*)$	0.27	0.32	0.40	0.46	0.50
		$\Psi_1(b_1^*)$	248.12	232.41	208.42	190.78	177.16
		b_1^*	70.71	80.57	95.83	107.16	115.97
4.00	3.36	$F(b_1^*)$	0.23	0.28	0.36	0.42	0.46
		$\Psi_1(b_1^*)$	243.19	225.12	196.68	175.13	158.13

However, in practice, Brenda may instead be uncertain about w_2 and w_3, Alex and Charles's wealth, respectively, and thus would have to take into account her uncertainty around w_2 and w_3. Under such a situation, she can elicit H_{12} and H_{13}, the distributions on Alex and Charles's wealth, respectively. In this case, she can use the expected value of her opponents' wealth as an estimate to find the wealthiest bidder among her opponents. Thus, $\max\{E(W_2), E(W_3)\} = \max\{175, 225\} = 225 = E(W_3)$. So, Brenda would believe Charles (the 3rd bidder) is her competitor because he has maximum expected wealth and thus tends to bid higher than Alex. Now, using (11) and Monte Carlo simulations, she can derive her expected optimal bid $E(b_1^*)$, her probability of winning, $F[E(b_1^*)]$ at that expected optimal bid and her expected utility, $\Psi_1[E(b_1^*)]$ for various assumed values of her winning and losing regret parameters. We assume $w_1 = \$250$ and perform $N = 500$ simulations and summarise these results in Table 3.

Table 3. Brenda's expected optimal bids, her probabilities of winning and expected utilities when $w_1 = \$250$ and $w_3 \sim H_{13}$.

η_1	$E(\hat{\eta}_1)$						
		θ_1	0.50	1.00	2.00	3.00	4.00
		$E(\hat{\theta}_1)$	0.69	1.38	2.76	4.14	5.52
		$E(b_1^*)$	113.07	124.09	138.75	148.32	155.18
0.50	0.38	$F[E(b_1^*)]$	0.45	0.50	0.57	0.61	0.64
		$\Psi_1[E(b_1^*)]$	272.23	265.16	255.56	249.28	244.80
		$E(b_1^*)$	102.56	113.82	129.20	139.47	146.95
1.00	0.75	$F[E(b_1^*)]$	0.39	0.45	0.52	0.57	0.61
		$\Psi_1[E(b_1^*)]$	264.90	255.65	242.71	233.95	227.58
		$E(b_1^*)$	87.74	98.77	114.58	125.57	133.78
2.00	1.50	$F[E(b_1^*)]$	0.32	0.38	0.45	0.51	0.55
		$\Psi_1[E(b_1^*)]$	254.56	241.59	222.50	208.96	198.76
		$E(b_1^*)$	77.76	88.21	103.76	114.96	123.52
3.00	2.26	$F[E(b_1^*)]$	0.27	0.32	0.40	0.46	0.50
		$\Psi_1[E(b_1^*)]$	247.65	231.71	207.33	189.37	175.48
		$E(b_1^*)$	70.59	80.36	95.38	106.52	115.19
4.00	3.01	$F[E(b_1^*)]$	0.23	0.28	0.36	0.41	0.46
		$\Psi_1[E(b_1^*)]$	242.71	224.39	195.50	173.57	156.24

Impact of regret parameters on bidding: Table 1 shows Brenda's optimal bids, her probabilities of winning and her expected utilities for different assumed levels of her winning and losing regret parameters when Brenda's wealth is \$200 and Charles's wealth is \$210 i.e., Brenda is less wealthier than Charles. In general, as one could expect, this table shows that an increase in losing regret parameter, $\hat{\theta}_1$ leads Brenda to a higher optimum bid, resulting in a higher probability of winning at that bid, but with a lower expected utility. Whereas, an increase in winning regret parameter, $\hat{\eta}_1$ leads Brenda to a lower optimum bid, resulting in a lower probability of winning at that bid, and a lower expected utility.
Table 2 shows how Brenda's optimal bids, her probabilities of winning and her expected utilities change with change in η_1 and θ_1 when Brenda's wealth is \$250 and Charles's wealth is \$210 i.e., Brenda is wealthier than Charles in this assumed case. Due to increase in Brenda's wealth, Table 2 shows that she bids higher than that in Table 1 for the corresponding assumed level of baseline winning and losing regret parameters. Table 3 also shows that how Brenda's optimal bid, her probabilities of winning and her expected utilities will change with the change in her winning and losing regret parameters when she takes into account the uncertainty of her opponents' wealth.

4. Level-*k* thinking

A level-k analysis is the modeling of how deeply the opponent of a decision maker (Brenda) thinks about the problem (Stahl and Wilson 1995). Thus, a level-k analysis for an n-player FPSB auction is when Brenda, a level-k thinker believes that the other $(n-1)$ bidders are level-(k-1) thinkers and each of them believes that the other $(n-1)$ bidders are level-(k-2) thinkers, and so on.
Non-strategic play which we have discussed in Section 3 is basically a level-1 analysis where Brenda believes herself to be a level-1 thinker and models the other $(n-1)$ bidders as level-0 (non-strategic) thinkers.

Here, we derive ARA solutions for Brenda for the case where $k = 2$. In a level-2 analysis, Brenda models herself as a level-2 thinker and believes that the other $(n-1)$ bidders are level-1 thinkers with each of them modeling the other $(n-1)$ bidders as level-0 thinkers. Here, again we assume that Brenda is the 1st bidder among the n bidders and believes that G_{1il} and H_{1il} are the distributions of the lth bidder's true value and wealth that the ith bidder might elicit with supports $[\underline{v}_{il}, \bar{v}_{il}]$ and $[\underline{w}_{il}, \bar{w}_{il}]$, $i = 2, \ldots, n$, $l = 1, \ldots, n$, $i \neq l$, respectively, such that $\tau < \underline{v}_{il} \leq \underline{w}_{il}$ and $\bar{v}_{il} \leq \bar{w}_{il}$. Also, Brenda believes that F_{1il} with supports $[\underline{b}_{il}, \bar{b}_{il}]$ is the distribution of the lth bidder's bid that the ith bidder might elicit such that $\tau < \underline{b}_{il} \leq \underline{v}_{il}$ and $\bar{b}_{il} \leq \bar{v}_{il}$. Brenda believes that the ith bidder will find F_{1il} using (8) as

$$F_{1il}(b_{il}) = G_{1il}(b_{il}) + \int_{b_{il}}^{\bar{v}_{il}} g_{1il}(v_{il}) T_{1il}(b_{il}/v_{il})\, dv_{il}, \tag{12}$$

where, $g_{1il}(v_{il})$ is the probability density function for the lth bidder's true value that the ith bidder might elicit that Brenda believes and $t_{1il}(p_{il})$ is Brenda's belief about the probability density function for the fraction of the lth bidder's true value this bidder will bid that the ith bidder might elicit.

Now, suppose that $w_{1i}, v_{1i}, \eta_{1i}$ and θ_{1i} are the wealth, true value, winning regret parameter and losing regret parameter, respectively, of the ith (level-1) bidder that Brenda believes. Also, let w_{1ij} be the wealth of the wealthiest bidder bidding b_{ij}. This is assumed by Brenda to be the maximum bid among the other $(n-1)$ bidders that the ith (level-1) bidder might believe. (We remark that the value of j will depend on the value of i, but for simplicity of notation, we do not formally show this dependency.) So, Brenda believes that the ith (level-1) bidder would find his optimal bid against the wealthiest among the other $(n-1)$ bidders for given $w_{1i}, w_{1ij}, v_{1i}, \eta_{1i}$ and θ_{1i} as

$$B_{1i}^*(w_{1i}, w_{1ij}, v_{1i}, \eta_{1i}, \theta_{1i}) = \underset{b_{1i}>\tau}{\arg\max}\, [W_{1i} + (V_{1i} - b_{1i})F_{1ij}(b_{1i}) \\ - \int_{b_{ij}:b_{ij}<b_{1i}} \hat{\eta}_{1i}(b_{1i} - b_{ij})dF_{1ij}(b_{ij}) - \int_{b_{ij}:b_{1i}<b_{ij}\leq V_{1i}} \hat{\theta}_{1i}(b_{ij} - b_{1i})dF_{1ij}(b_{ij})], \\ i = 2, \dots, n, \quad j \neq i,$$

where W_{1i} and V_{1i} are the ith bidder's wealth and true value, respectively, that Brenda believes. Also, F_{1ij} is the probability distribution on b_{ij} that the ith bidder can find against the jth bidder using (12) that Brenda believes. In practice, Brenda is uncertain about η_{1i} and θ_{1i} and could elicit distributions on these parameters. The distributions on η_{1i} and θ_{1i} along with the distributions on w_{1i} and w_{1ij} would allow her to derive the distributions for $\hat{\eta}_{1i}$ and $\hat{\theta}_{1i}$, respectively. We denote the distributions on $\hat{\eta}_{1i}$ and $\hat{\theta}_{1i}$ by Q_{1i} and S_{1i}, respectively. Then, she can find the ith bidder's expected optimal bid that she believe the ith bidder will derive against the wealthiest among other $(n-1)$ bidders as

$$\mathrm{E}(B_{1i}^*) = \int\int\int B_{1i}^*(w_{1i}, w_{1ij}, v_{1i}, \eta_{1i}, \theta_{1i})\, dG_{1ij}(v_{ij})\, dQ_{1i}(\hat{\eta}_{1i})\, dS_{1i}(\hat{\theta}_{1i}), \\ i = 2, \dots, n, \quad i \neq j, \tag{13}$$

where G_{1ij} is the distribution on the jth bidder's true value that the ith bidder might elicit that Brenda believes. Now, using (13), Brenda will find j such that $b_j = \max\{\mathrm{E}(B_{1i}^*), i = 2, \dots, n\}$. This bidder is the one that Brenda considers as her competitor bidder. Then, she will find F_{1j}, the distribution of her bid using the change of variable formula as

$$f_{1j} = |J_1| \times g_{1j} = \frac{1}{q_1} \times g_{1j},$$

where $q_1 = b_j/v_j$, v_j is the expected true value of the bidder bidding b_j and g_{1j} is the probability distribution of the true value of the bidder bidding b_j. Finally, after having F_{1j}, Brenda would find her optimal bid for a given value of w_j, the wealth of the bidder bidding b_j as

$$b_1^*(w_j) = \underset{b_1>\tau}{\arg\max}\, [w_1 + (v_1 - b_1)F_{1j}(b_1) - \int_{b_j:b_j<b_1} \hat{\eta}_1(b_1 - b_j)dF_{1j}(b_j) \\ - \int_{b_j:b_1<b_j\leq v_1} \hat{\theta}_1(b_j - b_1)dF_{1j}(b_j)], \tag{14}$$

where $\hat{\eta}_1$ and $\hat{\theta}_1$ are Brenda's modified winning and losing regret parameters, respectively. Brenda can use (14) to find her optimal bid by taking into account her uncertainty around w_j and can find her expected optimal bid using (11).

Example 4.1. Suppose Brenda is a level-2 thinker and believes that her opponents Alex and Charles are level-1 thinkers. Brenda believes that Alex thinks that Charles and Brenda are level-0 thinkers. Similarly, Brenda believes that Charles thinks that Alex and Brenda are level-0 thinkers. Let Brenda, Alex and Charles be the first, second and third bidders, respectively. Let Brenda's true value for the item be $v_1 = \$200$, she has her wealth $w_1 = \$250$ and the auctioned item has a reserve price $\tau = \$25$. Then Brenda believes that:

- Alex has a uniform distribution on Brenda's true value with $\underline{v}_{21} = \$30$ and $\bar{v}_{21} = \$200$, i.e., $G_{121} = \frac{(v_{21}-30)}{200-30}$.
- Alex has a uniform distribution on Brenda's wealth with $\underline{w}_{21} = \$100$ and $\bar{w}_{21} = \$250$, i.e., $H_{121} = \frac{(w_{21}-100)}{250-100}$.
- Alex has a uniform distribution on Charles's true value with $\underline{v}_{23} = \$30$ and $\bar{v}_{23} = \$250$, i.e., $G_{123} = \frac{(v_{23}-30)}{250-30}$.
- Alex has a uniform distribution on Charles's wealth with $\underline{w}_{23} = \$150$ and $\bar{w}_{23} = \$300$, i.e., $H_{123} = \frac{(w_{23}-150)}{300-150}$.
- Charles has a uniform distribution on Brenda's true value with $\underline{v}_{31} = \$30$ and $\bar{v}_{31} = \$230$, i.e., $G_{131} = \frac{(v_{31}-30)}{230-30}$.
- Charles has a uniform distribution on Brenda's wealth with $\underline{w}_{31} = \$125$ and $\bar{w}_{31} = \$275$, i.e., $H_{131} = \frac{(w_{31}-125)}{275-125}$.
- Charles has a uniform distribution on Alex's true value with $\underline{v}_{32} = \$50$ and $\bar{v}_{32} = \$220$, i.e., $G_{132} = \frac{(v_{32}-50)}{220-50}$.
- Charles has a uniform distribution on Alex's wealth with $\underline{w}_{32} = \$150$ and $\bar{w}_{32} = \$350$, i.e., $H_{132} = \frac{(w_{32}-150)}{350-150}$.
- Alex's valuation distribution is uniform with $\underline{v}_2 = \$100$ and $\bar{v}_2 = \$200$, i.e., $G_{12} = \frac{(v_2-100)}{200-100}$.
- Alex's wealth distribution is uniform with $\underline{w}_2 = \$100$ and $\bar{w}_2 = \$300$, i.e., $H_{12} = \frac{(w_2-100)}{300-100}$.
- Charles's valuation distribution is uniform with $\underline{v}_3 = \$50$ and $\bar{v}_3 = \$300$, i.e., $G_{13} = \frac{(v_3-50)}{300-50}$.
- Charles's wealth distribution is uniform with $\underline{w}_3 = \$100$ and $\bar{w}_3 = \$350$, i.e., $H_{13} = \frac{(w_3-100)}{350-100}$.
- Alex's winning regret is distributed as Gamma with shape parameter 1 and scale parameter 2 and his losing regret is also distributed as Gamma with shape parameter 3 and scale parameter 1.
- Charles's winning regret is distributed as Gamma with shape parameter 1.5 and scale parameter 1 and his losing regret is also distributed as Gamma with shape parameter 3 and scale parameter 1.5.
- Alex and Charles elicit their uncertainties around the fraction of their opponents true values that they would bid as $T_{1il} = \frac{p_{il}^8 - (\underline{v}_{il}/v_{il})^8}{1-(\underline{v}_{il}/v_{il})^8}$ with support $(\underline{v}_{il}/v_{il} < p_{il} \leq 1]$ for $i = 2, 3$ and $l = 1, 2, 3$, $i \neq l$.

Brenda believes that each of her opponents will use the expected value of their opponents' wealth as an estimate to find the wealthiest among their opponents. Thus, Brenda believes that Alex being a level-1 thinker will find his wealthiest opponent as $\max\{\mathrm{E}(W_{21}), \mathrm{E}(W_{23})\} = \max\{175, 225\} = 225 = \mathrm{E}(W_{23})$, i.e., he would believe Charles as his competitor bidder and will find his optimal bid $\mathrm{E}(B^*_{12})$ against him. Also, Brenda believes that Charles being a level-1 thinker will find his wealthiest opponent as $\max\{\mathrm{E}(W_{31}), \mathrm{E}(W_{32})\} = \max\{200, 250\} = 250 = \mathrm{E}(W_{32})$, i.e., he would believe Alex as his competitor bidder and will find his optimal bid $\mathrm{E}(B^*_{13})$ against him. Using (8), (13) and from Monte Carlo simulations ($N = 500$), she will find Alex's expected optimal bid $\mathrm{E}(B^*_{12}) = 98.58$ against Charles by taking into account Alex's winning and losing regret. Also, from Monte Carlo simulations, she will find Charles's expected optimal bid $\mathrm{E}(B^*_{13}) = 115.04$ against Alex. Then, she will find j such that $b_j = \max\{\mathrm{E}(B^*_{12}), \mathrm{E}(B^*_{13})\} = \max\{98.58, 115.04\} = 115.04 = \mathrm{E}(B^*_{13})$. So $j = 3$ and hence Charles's (3rd bidder) is her competitor bidder with expected optimal bid of 115.04. Finally, she would find

$$q_1 = \frac{115.04}{175} = 0.6573,$$

and

$$F_{1j}(b_1) = \frac{b_1 - q_1 \times \underline{v}_3}{q_1 \times (\bar{v}_3 - \underline{v}_3)} = \frac{b_1 - q_1 \times 50}{q_1 \times (300 - 50)} = \frac{b_1 - 32.87}{164.32}.$$

Now, using (11) and Monte Carlo simulations, we find Brenda's expected optimal bid $\mathrm{E}(b^*_1)$, her probability of winning $F[\mathrm{E}(b^*_1)]$ for that expected optimal bid and her expected utility $\Psi_1[\mathrm{E}(b^*_1)]$ for that expected optimal bid for various levels of her regret parameters. These are summarized in Table 4.

Table 4. Brenda's (level-2 thinker) expected optimal bids, her probabilities of winning and expected utilities when $w_1 = \$250$ and $w_3 \sim H_{13}$.

		θ_1	0.50	1.00	2.00	3.00	4.00
η_1	$\mathrm{E}(\hat{\eta}_1)$	$\mathrm{E}(\hat{\theta}_1)$	0.59	1.18	2.36	3.55	4.73
		$\mathrm{E}(b^*_1)$	120.70	133.50	149.63	159.42	166.02
0.50	0.44	$F[\mathrm{E}(b^*_1)]$	0.53	0.61	0.71	0.77	0.81
		$\Psi_1[\mathrm{E}(b^*_1)]$	270.86	261.41	249.50	242.29	237.46
		$\mathrm{E}(b^*_1)$	109.78	122.77	139.92	150.79	158.30
1.00	0.87	$F[\mathrm{E}(b^*_1)]$	0.47	0.55	0.65	0.72	0.76
		$\Psi_1[\mathrm{E}(b^*_1)]$	261.93	249.41	232.84	222.38	215.18
		$\mathrm{E}(b^*_1)$	94.54	107.09	124.80	136.74	145.39
2.00	1.74	$F[\mathrm{E}(b^*_1)]$	0.38	0.45	0.56	0.63	0.68
		$\Psi_1[\mathrm{E}(b^*_1)]$	249.44	231.75	206.66	189.70	177.48
		$\mathrm{E}(b^*_1)$	84.40	96.15	113.51	125.79	134.97
3.00	2.61	$F[\mathrm{E}(b^*_1)]$	0.31	0.38	0.49	0.56	0.62
		$\Psi_1[\mathrm{E}(b^*_1)]$	241.13	219.40	187.00	163.99	146.80
		$\mathrm{E}(b^*_1)$	77.15	88.05	104.74	116.99	126.39
4.00	3.48	$F[\mathrm{E}(b^*_1)]$	0.27	0.34	0.44	0.51	0.57
		$\Psi_1[\mathrm{E}(b^*_1)]$	235.20	210.27	171.70	143.23	121.35

Impact of regret parameters on bidding: Table 4 shows Brenda's optimal bids, her probabilities of winning and her expected utilities for different assumed levels of her winning and losing regret parameters when she takes into account the uncertainties of her opponents' wealth. As one could expect, this table also shows that Brenda's optimal bid increases with an increase in her losing regret parameter, $\hat{\theta}_1$, resulting in a higher probability of winning at that bid, but with a lower expected utility. Whereas, Brenda's optimal bid decreases with an increase in her winning regret parameter, $\hat{\eta}_1$, resulting in a lower probability of winning at that bid and a lower expected utility.
Table 4 also shows that Brenda being a level-2 thinker has higher optimal bids than those shown in Table 3 (non-strategic play) at the same corresponding assumed levels of baseline winning and losing regret parameters because now she models her opponents as level-1 thinkers (strategic).

Now, we provide a brief sketch on how to find the ARA solution when Brenda wants to perform a level-3 analysis. In this case, Brenda models herself as a level-3 thinker and believes that her $(n-1)$ opponents are level-2 thinkers. Each of these level-2 thinkers would model their $(n-1)$ opponents as level-1 thinkers and each of these level-1 thinkers would model their $(n-1)$ rivals as level-0 thinkers. To find the ARA solution in this case, Brenda would perform the level-2 analysis detailed above for each of the other $(n-1)$ bidders and will obtain their optimal bids using (14) where $b_1^*(w_i)$, b_1, w_1, v_1, F_{1j}, $\hat{\eta}_1$, and $\hat{\theta}_1$, b_j are replaced by $B_{1i}^*(w_{ij})$, b_{1i}, W_{1i}, V_{1i}, F_{1ij}, $\hat{\eta}_{1i}$, and $\hat{\theta}_{1i}$ and b_{ij} respectively and w_{ij} be wealth of the level-1 thinker bidder bidding maximum among other $(n-1)$ bidders that the ith bidder believes for $i = 2, \ldots, n$, $i \neq j$. Then, she will find b_j, the maximum bid among other $(n-1)$ bidders using (14) (with the replaced quantities) and would get her belief about F_{1j}, the distribution of the jth bidder's (level-2 thinker) bid. Then, she can obtain her optimal bid $b_1^*(w_j)$ or the expected optimal bid $\mathrm{E}(b_1^*)$ using similar process as in (10) and (11), respectively.

Using a level-k analysis, the key question is how large should the k be? Players could choose a higher k while playing games such as Chess or Go, which are highly structured games. However, Ho, Camerer, and Weigelt (1998) and Lee and Wolpert (2012) based on experimental evidences stated that, people typically do not think higher than level 2 or 3. Therefore, it makes sense to solve the level-k problem for k being 1, 2 or 3 for FPSB auctions.

5. Conclusion and future work

In this paper, we assume that a single (*normal*) item is being auctioned in a FPSB auction that has a reserve price which is typically known to each bidder in advance. We define new regret parameters to take into account the effect of bidder's wealth on their bidding behavior and modify the utility function as used by Engelbrecht-Wiggans and Katok (2007) that incorporates bidders' winning and losing regret. We find ARA solutions not only using non-strategic play but also using the level-k thinking solution concept assuming n bidders participating in these auctions. For this type of utility function, we take into account the uncertainties in bidders' winning and losing regret in addition to their valuations and wealth. We model how an increase in the decision maker's wealth will affect their winning and losing regrets. We also provide numerical examples in which we use Monte Carlo methods to illustrate our methodology to find ARA solutions for each solution concept. Finding ARA solutions for a utility function that considers winning and losing regret while taking into account the uncertainties regarding bidders' valuations, wealth and their regrets was methodologically and computationally more challenging than that for a CRRA utility function. ARA solutions for the other solution concepts such as ME and BNE for the utility function that we developed in this paper are yet to be found.

Without loss of generality, we have made an assumption that the bidder with the maximum wealth among other $(n-1)$ bidders could also have more valuation for the auctioned item (because the item is normal) and therefore would bid maximum among other $(n-1)$ bidders. Therefore, the decision maker will consider the bidder with the maximum wealth among the other $(n-1)$ bidders as her main competitor and will find her optimal bid against him. But, in practice, it could be possible that a bidder with relatively low wealth could have a higher value for the auctioned item and therefore could bid higher than the bidder having relatively higher wealth but low valuation. Thus, it is important to model the maximum bid among other $(n-1)$ bidders in a more general way that considers both the wealth and valuation of the bidders.

We have assumed that the decision maker believes that each of her rivals are of the same type, i.e., either non-strategic players or level-k thinkers. However, it might be possible that the decision maker is uncertain about her rivals' solution concepts. This is called *concept (model) uncertainty.* In this case, she needs to take into account her concept uncertainty in order to find her optimal decision. Under concept uncertainty, first she needs to find her optimal decision $\mathrm{E}(b^*|\mathcal{M})$ conditional on the given solution concept $\mathcal{M}$ (for example, non-strategic play, level-k thinking, BNE and ME etc.). Second, she needs to elicit her subjective distribution $\Pr(\mathcal{M})$ that reflects her uncertainty on $\mathcal{M}$. Then, she can find her optimal decision by taking into account concept uncertainty as

$$\mathrm{E}(b^*) = \sum_{\mathcal{M}} \mathrm{E}(b^*|\mathcal{M}) \Pr(\mathcal{M}).$$

One of the main challenges in implementing an ARA solution is the elicitation of prior distributions. The elicitation of prior distributions is considered to be a practical challenge while adopting almost any Bayesian approach, however, this challenge is further accentuated in ARA where the decision maker not only needs to elicit their own uncertainties, but also, the uncertainties of their adversaries. Analysis of the robustness of the ARA solution to the choice of the prior distributions is therefore necessary. Ríos Insua et al. (2016) provide an outline for a robustness analysis using an ARA framework. However, mathematical framework for implementing a prior robustness analysis for an ARA model has not yet been developed. This is an important direction for further work.

Further, many other types of auctions have been studied and used in practice. These include other variations on the sealed-bid auction process, such as the *second-price* (Vickrey 1961), the *third-price* (Kagel and Levin 1993), or in general, the mth-*price* (Cason 1995; Shogren et al. 2001) sealed-bid auctions have been proposed. Shogren et al. (2001) showed that a random mth-*price* sealed-bid auction[3] can induce sincere bidding in theory and practice than a second-price sealed-bid auction when bidders' values are far below or above the market-clearing price[4] of the auctioned items. ARA solutions for these variants on the sealed-bid auctions process need to be developed.

Finally, ARA methodology can further be applied to the auctions of a sequential paradigm where the decision maker and her opponents' decisions alternatively evolve over time such as in English[5] or Dutch[6] auctions etc. In such auctions, there could be short or long term interactions among the bidders over time. An approach developed

[3]In a random mth-price sealed-bid auction, each bid is rank-ordered from highest to lowest; the auctioneer selects a random number, the m in the mth-price sealed-bid auction, uniformly-distributed between 2 and n (n bidders); and the auctioneer sells one auctioned item to each of the $(m-1)$ highest bidders at the mth-price.

[4]The price at which the quantity demanded of an item or service is equal to the quantity supplied and no surplus or shortage exists in the market.

[5]The participants make increasingly higher bids and stop bidding when they are not prepared to pay more than the current highest bid. This continues until no participant is prepared to make a higher bid; the highest bidder wins the auction at the final amount bid.

[6]In Dutch auctions, the price is set by the auctioneer at a level sufficiently high to deter all bidders, and is progressively lowered until a bidder is prepared to buy at the current price auctioned item.

by González-Ortega, Insua, and Cano (2019) could be used for short term interactions with changing dynamics while a Markov decision process could be used for long term interactions, with a fixed structure as argued by Joshi, Aliaga, and Insua (2020). Thus, developing ARA solutions to sequential auctions is a challenging research problem too.

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