The relativistically invariant expansion of a scalar function on imaginary Lobachevski space

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Using the previous analysis of Gel'fand and Graev a new relativistically invariant expansion of a scalar function on three-dimensional imaginary Lobachevski space $L_3(I)$ is given. The coordinate system used corresponds to the horospherical reduction $SO(3,1) \supset E_2 \supset SO(2)$ and covers all of $L_3(I)$.

INTRODUCTION AND SUMMARY

Explicit relativistically invariant expansions of functions defined on the three transitivity surfaces of the proper Lorentz group in Minkowksi space have been studied to varying degrees in recent years.[1-3] Of these surfaces explicit expansions on the upper sheet H_2 of the double sheeted hyperboloid[4] $[x,x]=1$ and on the cone[5] $[\xi,\xi]=0$ have been well developed.[6] (Note: x is a 4-vector in Minkowski space with $[x,x]=x_0^2-\mathbf{x}^2$ the usual scalar product.) The explicit expansions on H_2 and on the cone are based on the expansion formulas due to Gel'fand *et al.*[7] The invariant expansion of a scalar function $f(x)$ $(x \in H_2)$ is obtained by observing that H_2 corresponds to a realization of three-dimensional real Lobachevski space $L_3(R)$. An invertable horospherical integral transform then associates a function $h(\xi)$ on the cone with each $f(x)$. The invariant expansion of $f(x)$ then reduces to the invariant expansion of $h(\xi)$. The latter expansion is achieved by the decomposition of $h(\xi)$ into homogeneous components.

An analogous geometry and irreducible decomposition of a function $f(x)$ on the single sheeted hyperboloid H_1, with equation $[x,x]=-1$, has also been given in Ref. 7. The geometry of H_1 corresponds to a realization of imaginary Lobachevski space $L_3(I)$ and identifies diametrically opposed points [so that $f(x)=f(-x)$]. The irreducible decomposition on H_1 differs from that on H_2 in that it contains a discrete spectrum as well as the usual continuous spectrum.

Previously there has been (to the author's knowledge) one paper by Kuznetzov and Smorodinski[8] which has considered an explicit complete set of functions on H_1 realized as $L_3(I)$. This analysis uses the results of Ref. 7 only insofar as they consider a parametrization of $x \in H_1$ for which the discrete spectrum term is not necessary. [More specifically, they choose a coordinate system which only parametrizes points at a real distance from $x=(0,0,0,1)$.] Verdiev,[9] on the other hand, has given his attention to finding an explicit set of complete functions with spin on H_1. There are some shortcomings in Verdiev's work in that the continuous spectrum expansion functions have not been normalized and the method used to obtain the normalized discrete spectrum expansion functions needs some explanation. Zmuidzinas[2] has given a complete account of the expansion of a scalar function defined on H_1 using the eigenfunction expansion methods of Titchmarsh.[10] This analysis has been done in the canonical group reduction $SO(3,1) \supset SO(3) \supset SO(2)$ or S system. Limic *et al.*[3] have treated the general problem of the expansion of square integrable functions defined on the transitivity surfaces of $SO(p,q)$ in the canonical group reduction and hence include the results of Zmuidzinas as a special case.

In this paper we examine the expansion of a square integrable function defined on $L_3(I)$ in the noncanonical group reduction $SO(3,1) \supset E(2) \supset SO(2)$ or horospherical system. This expansion is new and serves to illustrate how the analysis of Gel'fand and Graev should be treated to yield the correct expansion formulas. There is only one other group reduction which parametrizes all of $L_3(I)$ (apart from the group reduction $SO(3,1) \supset E(2) \supset T_1 \otimes T_2$, which differs little from the horospherical system). This is the S system. We do not however give this expansion here as it differs little from the results of Zmuidzinas and Limic *et al.*

The study of the horospherical system group reduction of $SO(3,1)$ has received attention previously in application to particle physics[11,12] and is also of intrinsic group theoretical interest.

The content of this paper is arranged as follows. In Sec. 1 we collect the pertinent facts concerning the Gel'fand–Graev analysis on $L_3(I)$. In Sec. 2 we give the horospherical system expansion.

1. THE HARMONIC ANALYSIS OF A SCALAR FUNCTION ON $L_3(I)$

The central problem here is the decomposition of the representation

$$[T_g f](x) = f(xg), \quad x \in L_3(I) \tag{1.1}$$

into components which transform according to unitary irreducible representations (UIRs) of the proper Lorentz group $SO(3,1)$. The Gel'fand–Graev transform on $L_3(I)$ invertably maps $f(x)$ into a pair of functions $h(\xi)$ and $\phi(\xi,b)$. The function $h(\xi)$ gives the representation

$$[Q_g h](\xi) = h(\xi g) \tag{1.2}$$

and the functions $\phi(\xi,b)$ define the representation

$$[R_g \phi](l) = \beta^{-1}(l,g)\phi(lg), \tag{1.3}$$

where $\phi(l)=\phi(\xi,b)$ and $\beta(l,g)$ is the zeroth coordinate of ξg. This pair of functions are obtained by integration of $f(x)$ over the two distinct manifolds of horospheres on $L_3(I)$. [We assume that the reader is familiar with the rudiments of the geometry of $L_3(I)$ as found for instance in Ref. 7.] Accordingly, we have

(i) Horospheres of the first kind.

$$h(\xi) = \int f(x)\delta(|[x,\xi]|-1)dx \tag{1.4}$$

with dx the invariant measure on $L_3(I)$

$$dx = \frac{dx_1 dx_2 dx_3}{|x_0|}. \tag{1.5}$$

Here a typical horosphere of the first kind has the equation

$$|[x,\xi]| = 1. \tag{1.6}$$

(ii) Horospheres of the second kind.

In this case $\phi(\xi, b)$ is obtained by integration of $f(x)$ over the isotropic line $x = b + t\xi$ according to

$$\phi(\xi, b) = \int_{-\infty}^{\infty} f(b + t\xi)dt, \tag{1.7}$$

where

$$[b,b] = -1, \quad [b,\xi] = [\xi,\xi] = 0, \quad b_0 = 0.$$

The choice of integration over an isotropic line is more convenient than over the horosphere itself. We note that each horosphere of the second kind given by $[x,\xi] = 0$ consists of all mutually parallel isotropic lines passing through the point ξ on the cone.

$f(x)$ is given in terms of $h(\xi)$ and $\phi(\xi, b)$ by the formula

$$f(x) = \frac{1}{(4\pi)^2}\int h(\xi)\delta^{(2)}(|[x,\xi]| - 1)d\xi + \frac{1}{(2\pi)^2}\int_0^{\pi} \cot^2\theta \, d\theta \int_{\Gamma} \phi(\xi,\theta)dw, \tag{1.8}$$

where

$$d\xi = \frac{d\xi_1 d\xi_2 d\xi_3}{|\xi_0|},$$

with $\phi(\xi,\theta)$ the value of $\phi(\xi, b)$ for the isotropic line $y = b + t\xi$ lying in the $[x,y] = \cos\theta$ plane (i.e., $[x,b] = \cos\theta$). Γ is a contour on the cone intersecting each generator once and the measure dw is defined by

$$dw = |\xi_0|^{-1}(\xi_1 d\xi_2 d\xi_3 - \xi_2 d\xi_1 d\xi_3 + \xi_3 d\xi_1 d\xi_2). \tag{1.9}$$

In order to achieve the decomposition of $f(x)$ into irreducible parts it is necessary to expand the "Fourier components" $h(\xi)$ and $\phi(\xi,\theta)$ into homogeneous components. For $h(\xi)$ this is done exactly as for the case of $L_3(R)$, i.e.,

$$h(\xi) = \frac{1}{2\pi i}\int_{\delta - i\infty}^{\delta + i\infty} F(\xi;\sigma)d\sigma, \tag{1.10}$$

$$F(\xi;\sigma) = \int_0^{\infty} h(t\xi)t^{-\sigma-1}dt. \tag{1.11}$$

The expansion of $\phi(\xi,\theta)$ into irreducible (homogeneous) components is achieved by Fourier analyzing $\phi(\xi,\theta)$ with respect to the angle $\theta (0 \le \theta < \pi)$ which specifies each isotropic line in a given horosphere of the second kind. The appropriate decomposition is

$$\phi(\xi,\theta) = \frac{1}{\pi}\sum_{n=-\infty}^{\infty} \tilde{F}(\xi; x; 2n)e^{-2in\theta}. \tag{1.12}$$

The "Fourier coefficients" satisfy the homogeneity condition

$$\tilde{F}(\xi; b; 2n) = \tilde{F}(\xi; x; 2n)e^{-2in\theta}. \tag{1.13}$$

The invariant decomposition of $f(x)$ is then

$$f(x) = \frac{i}{4(2\pi)^3}\int_{\delta-i\infty}^{\delta+i\infty} \sigma(\sigma+1)\int_{\Gamma} F(\xi;\sigma)|[x,\xi]|^{-\sigma-2}d\xi d\sigma + \frac{1}{\pi^2}\sum_{n=1}^{\infty} n\int_{\Gamma}\tilde{F}(\xi;b;2n)e^{2in\theta}\delta([x,\xi])d\xi \tag{1.14}$$

and the inversion formulas are

$$F(\xi;\sigma) = \int f(x)|[x,\xi]|^{\sigma}dx, \tag{1.15}$$

$$\tilde{F}(\xi;b;2n) = \int f(x)e^{-2in\theta}\delta([x,\xi])dx. \tag{1.16}$$

Group theoretically the "Fourier coefficients" in (1.14) transform according to the irreducible representations (IRs) of $SO(3,1)$ as follows:

(i) $F(\xi;\sigma)$ transform according to the IRs

$$c = \sigma + 1 = \delta + 1 + i\rho, \quad (-\infty < \rho < \infty), \quad k_0 = 0, \tag{1.17}$$

where $[c, k_0]$ labels each IR of $SO(3,1)$. (This is the notation due to Naimark[13] that we are using here.) We obtain the unitary case (i.e., the principal series) when $\delta = -1$.

(ii) $\tilde{F}(\xi;b;2n)$ transform according to the UIRs $SO(3,1)$

$$c = 0, \quad k_0 = 2n, \quad n = 1,2,3,\cdots. \tag{1.18}$$

2. THE HOROSPHERICAL OR *Ho* SYSTEM EXPANSION ON $L_3(I)$

The *Ho* system 4-vector x on the single sheet hyperboloid H_1 is given by

$$x = (\tfrac{1}{2}[-e^{-a} + (1+r^2)e^a], re^a\cos\phi, re^a\sin\phi, \tfrac{1}{2}[-e^{-a} + (r^2-1)e^a], \tag{2.1}$$

$$-\infty < a < \infty, \quad 0 \le r < \infty, \quad 0 \le \phi < 2\pi.$$

This parametrization covers the $x_0 - x_3 \ge 0$ half of the $[x,x] = -1$ hyperboloid and so covers all of $L_3(I)$.

For the *Ho* system expansion the contour Γ is taken to be

$$\xi_0 - \xi_3 = 2 \tag{2.2}$$

and ξ is parametrized according to

$$\xi = (1 + u^2 + v^2, 2u, 2v, -1 + u^2 + v^2), \quad -\infty < u, v < \infty. \tag{2.3}$$

$F(\xi;\sigma)$ is expanded in a double Fourier series according to

$$F(\xi;\sigma) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} a_{\lambda\mu}(\sigma)e^{i\lambda u}e^{i\mu v}d\lambda d\mu \tag{2.4}$$

and the measure on the cone is

$$d\xi = 4\, du\, dv. \tag{2.5}$$

Taking $x = (\text{sh}a, 0, 0, -\text{ch}a)$ the continuous spectrum part of expansion (1.14) then reduces to the calculation of the integral

$$I^{\sigma}_{\lambda\mu}(a) = \int_{-\infty}^{\infty} du \int_{-\infty}^{\infty} dv\, |e^a(u^2+v^2) - e^{-a}|^{-\sigma-2}e^{i\lambda u}e^{i\mu v}. \tag{2.6}$$

This integral can be calculated by using the identity

$$|t|^{\beta} = t_+^{\beta} + t_-^{\beta}, \tag{2.7}$$

as well as the known Fourier transforms in two dimensions of the functions $(b^2 - u^2 - v^2)_{\pm}^{-\sigma-2}$ which are given by

$$\text{F.T.}\left[\frac{(b^2-u^2-v^2)_-^{-\sigma-2}}{\Gamma(-\sigma-1)}\right] = -i(2b)^{-\sigma-1}\left[\frac{K_{-\sigma-1}(b(Q-i0))}{(Q-i0)^{-(\sigma+1)}} - \frac{K_{-\sigma-1}(b(Q+i0))}{(Q+i0)^{-(\sigma+1)}}\right], \tag{2.8}$$

$$\text{F.T.}\left[\frac{(b^2-u^2-v^2)_+^{-\sigma-2}}{\Gamma(-\sigma-1)}\right] = -i(2b)^{-\sigma-1} \times\left[e^{i\pi(\sigma+1)}\frac{K_{-\sigma-1}(b(Q-i0))}{(Q-i0)^{-(\sigma+1)}} - e^{-i\pi(\sigma+1)}\frac{K_{-\sigma-1}(b(Q+i0))}{(Q+i0)^{-(\sigma+1)}}\right], \tag{2.9}$$

where $Q=-\lambda^2-\mu^2$.

These formulas are special cases of the general formulas for the Fourier transforms in n dimensions of the generalized functions $(b^2+P)_\pm^\lambda$ ($\lambda \neq$ integer) as given by Gel'fand and Shilov.[14] (Note: P is a general quadratic form in the n Cartesian coordinate variables expressed in canonical or diagonal form.) $I^\sigma_{\chi\mu}(a)$ is then found to be

$$I^\sigma_{\chi\mu}(a) = i\pi\left(\frac{\chi}{2}\right)^{\sigma+1}\Gamma(-\sigma-1)e^{-a}[J_{\sigma+1}(e^{-a}\chi)+J_{-\sigma-1}(e^{-a})], \tag{2.10}$$

where $\chi=(\lambda^2+\mu^2)^{1/2}$.

For the discrete part of expansion (1.14), $\tilde{F}(\xi;2n)$ is expanded according to

$$\tilde{F}(\xi;2n) = \sum_{m=-\infty}^{\infty}\int_0^\infty \chi d\chi a_m(\chi;2n)J_{2n-m}(\chi p)e^{im\psi}, \tag{2.11}$$

where $u=p\cos\psi$, $v=p\sin\psi$

The evaluation of the discrete part of (1.14) then requires the calculation of

$$I_\chi(2n) = \int J_{2n-m}(\chi p)e^{im\psi}\delta([x,\xi])d\xi. \tag{2.12}$$

This integral is readily calculated, using the identity

$$\delta(a^2-x^2) = \frac{1}{2a}[\delta(a+x)+\delta(a-x)], \tag{2.13}$$

to be

$$I_\chi(2n) = 4\pi e^{-a}J_{2n}(\chi e^{-a})J_m(\chi r)e^{im\phi}. \tag{2.14}$$

The *Ho* system expansion on $L_3(I)$ is then

$$f(x) = \sum_{m=-\infty}^{\infty}\int_0^\infty \chi d\chi e^{-a}\left(\frac{-1}{8(2\pi)^2}\int_{\delta-i\infty}^{\delta+i\infty} \times\sigma(\sigma+1)a_m(\chi,\sigma)\Gamma(-\sigma-1) \times[J_{\sigma+1}(e^{-a}\chi)+J_{-\sigma-1}(e^{-a}\chi)]d\sigma + \frac{4}{\pi}\sum_{n=1}^{\infty}na_m(\chi,2n)J_{2n}(\chi e^{-a})\right)J_m(\chi r)e^{im\phi}. \tag{2.15}$$

For the continuous part of (1.14) we have changed the expansion of $F(\xi;\sigma)$ to polar coordinates and used the identity

$$e^{i\chi r\cos(\phi-\theta)} = \sum_{m=-\infty}^{\infty} i^m J_m(\chi r)e^{im(\phi-\theta)}, \tag{2.16}$$

where $\tan\theta=\lambda/\mu$.

$a_m(\chi;\sigma)$ is then given by

$$a_m(\chi;\sigma) = i^m\int_0^{2\pi} a_{\lambda\mu}(\sigma)e^{-im\theta}d\theta. \tag{2.17}$$

The inversion formulas of (2.15) are

$$a_m(\chi;\sigma) = \frac{i}{4\pi}\Gamma(\sigma+1)\int f(x)e^{-a}[J_{\sigma+1}(e^{-a}\chi) + J_{-\sigma-1}(e^{-a}\chi)]J_m(\chi r)e^{-im\phi}dx, \tag{2.18}$$

$$a_m(\chi;2n) = \tfrac{1}{2}\int f(x)e^{-a}J_{2n}(\chi e^{-a})J_m(\chi r)e^{-im\phi}dx, \tag{2.19}$$

where

$$dx = e^{2a}da\,dr\,dr\,d\phi. \tag{2.20}$$

For the principal series $\sigma=-1+i\rho$ the continuous part of (3.15) is an expansion in terms of the functions

$$\Psi^{Ho}_{\rho\chi m}(a,r,\phi) = e^{-a}\tilde{J}_{i\rho}(\chi e^{-a})J_m(\chi r)e^{im\phi} \tag{2.21}$$

which satisfy the orthogonality relations

$$\int\Psi^{Ho}_{\rho\chi m}(a,r,\phi)\overline{\Psi^{Ho}_{\bar\rho\bar\chi\bar m}(a,r,\phi)dx} = \frac{2\,sh\,\pi\rho}{\chi\rho}\delta(\rho-\bar\rho)\delta(\chi-\bar\chi)\delta_{m\bar m}, \tag{2.22}$$

where we have put

$$\tilde{J}_{i\rho}(x) = J_{i\rho}(x)+J_{-i\rho}(x). \tag{2.23}$$

We observe that the a dependant part of (2.22) reproduces the completeness relation for the Titchmarsh integral transform,[15] i.e.,

$$\int_0^\infty \tilde{J}_{i\rho}(x)\tilde{J}_{i\bar\rho}(x)x^{-1}dx = \frac{2\,sh\,\pi\rho}{\rho}\delta(\rho-\bar\rho). \tag{2.24}$$

We also note that for the discrete spectrum expansion functions we have the orthogonality relation

$$\int_0^\infty J_{2n}(x)J_{2m}(x)x^{-1}dx = \frac{1}{4n}\delta_{nm}. \tag{2.25}$$

This is just a special case of the formula

$$\int_0^\infty J_\mu(x)J_\nu(x)x^{-1}dx = \frac{2\sin\frac{1}{2}\pi(\nu-\mu)}{\pi(\nu^2-\mu^2)}. \tag{2.26}$$

CONCLUDING REMARKS

We have given here an expansion of a function $f(x)\in L_3(I)$ in a coordinate system which is an alternative to the canonical or S system, viz., the *Ho* system. The parametrization of x we used is obtained from the corresponding coordinate system vector on H_2 via the analytic continuation $a\to a+i\pi/2$. This example illustrates not only the application of the analysis of Gel'fand and Graev in obtaining explicit expansion formulas but also that the group reduction parametrizations of $x\in H_2$ when continued in the manner above do not always cover all of H_1. The S system is the only one that covers all of H_1. It should be mentioned here that the expansion functions used for $\tilde{F}(\xi, 2n)$ in (2.11) are the natural ones in the sense that they are the basis functions for the UIR $\{o, 2n\}$ of $SO(3,1)$ when realized in a Ho system basis in the space of square-integrable functions in the plane.

In the future we intend to study all possible coordinate systems on H_1 which cover at least all of $L_3(I)$ and for which the angular part of the Laplacian Δ_L admits a separation of variables.

[1]For perhaps the most complete set of references see P. Winternitz "Two variable expansions based on the Lorentz and conformal groups," talk presented at Symposium on the de Sitter and conformal groups, Boulder, Colorado, 1970 and references contained therein.
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[15]See in this connection Eq. 5, p. 150 of *Formulas and Theorems for the Special Functions of Mathematical Physics* by W. Magnus, F. Oberhettinger, and R. P. Soni (Springer-Verlag, New York, 1966).