# Misconceptions Arising From the Infinite Solenoid Magnetic Field Formula

Cite as: Phys. Teach. **59**, 213 (2021); https://doi.org/10.1119/10.0003670
Published Online: 01 March 2021

**Marcus T. Wilson**

# Misconceptions Arising From the Infinite Solenoid Magnetic Field Formula

**Marcus T. Wilson,** University of Waikato, Hamilton, New Zealand

Many high school and first-year university courses include discussion of the magnetic effect of currents. Frequently discussed textbook examples include long, straight wires, circular current loops, and solenoids, partly because these examples are tractable mathematically. The solenoid naturally leads to discussion on magnetic materials since it is readily demonstrated that a paramagnetic core significantly boosts the strength of an electromagnet. However, magnetic effects of solid and even liquid materials are subtle and confusing[1] and the mathematics is not straightforward. This leads to confusion amongst students (and their teachers), which, when taken to more advanced study, leads to significant misconceptions about the nature of magnetic properties and fields. These misconceptions can become problematic when practical (rather than stereotyped) magnetic design and analysis is required such as for transformers,[2] magnetic recording materials, geomagnetic sensors,[3] or biological stimulators[4] to name a few. In this article, I highlight examples of this confusion, in particular the failure in realistic situations of the well-quoted formula for an infinite solenoid with a paramagnetic core, and the physical interpretation of the relative permeability of a material, $\mu_r$.

## The calculation of magnetic fields due to direct current

Direct electric currents give rise to static magnetic fields. In free space, the magnetic flux density (B-field) created by a current is described completely by the Biot-Savart law,

$$dB = \frac{\mu_0 I \, dL \times \hat{r}}{4\pi r^2}, \tag{1}$$

where $\boldsymbol{dB}$ denotes the magnetic flux density due to a small length of wire $\boldsymbol{dL}$ carrying current $I$, at a distance $r$ from the element of wire in a (unit) direction $\hat{r}$, and $\mu_0$ is the permeability of free space ($1.257\times10^{-6}$ H/m). The Biot-Savart law can be integrated along the current-carrying wire to find the B-field at all points in space. Ampère's law follows from the Biot-Savart law: the line integral of the B-field around a closed loop is equal to $\mu_0$ times the current flowing through the loop. Mathematically,

$$\oint \boldsymbol{B}\cdot d\boldsymbol{l} = \mu_0 I_{\text{enc}}, \tag{2}$$

where $\boldsymbol{B}$ is the magnetic flux density, $\boldsymbol{dl}$ is an increment of distance around a loop, and $I_{\text{enc}}$ is the enclosed current. Equation (2) uniquely defines the B-field because lines of B-field must form closed loops. Applying Stokes' theorem to (2) leads to $\nabla\times\boldsymbol{B} = \mu_0\boldsymbol{J}$ where $\boldsymbol{J}$ is the current density; since $\nabla\cdot\boldsymbol{B} = 0$ (i.e. lines of B-field form closed loops), the equation has a unique solution.

## Analytically tractable examples are limited

Although Ampère's law is much presented, practical examples with analytically tractable solutions are few. The infinite solenoid in free space is one example. The B-field is uniform inside and zero outside; inside it has magnitude

$$B = \mu_0 nI, \tag{3}$$

where $n$ is the number of turns per unit length and $I$ is the current. In practice, Eq. (3) applies adequately away from the ends of a finite solenoid whose length is more than a few times its diameter. This can easily be demonstrated in a practical class, for example, passing a current through a stretched

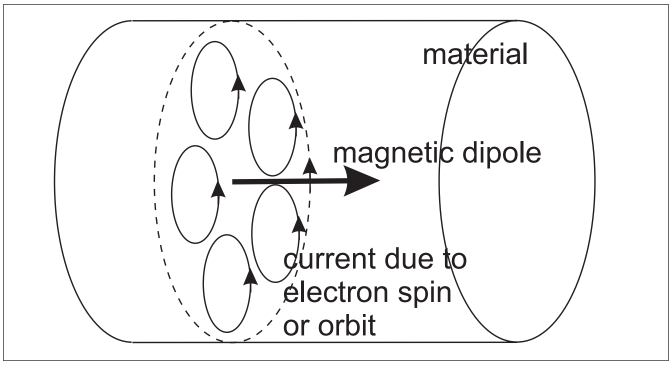


**Fig. 1. A simple picture of atomic effects producing magnetization at a macroscopic level. Electron spins and orbits create circulating charge (the small solid loops), which combine to give a net current (dotted line) around the edge of magnetic domains and thus a significant magnetic dipole (thick arrow).**

"Slinky" and measuring B-field with a Hall probe. However, when we consider a solenoid as an electromagnet, we should not jump to the conclusion that Eq. (3) is a reasonable approximation to the field. For example, often we would be interested in the B-field at the base of the solenoid, not its center, which will be, for a long solenoid in free space, exactly half that of Eq. (3). The solenoid example is frequently used to lead to discussion of electromagnets, and thus the effects of magnetic materials on B-fields.

## The magnetization of materials and the H-field

At an atomic level, magnetism arises from "spinning" or "orbiting" electrons, which can be considered simplistically as small internal loops of current, each of which generates a small magnetic field. At a microscopic level, these spins and orbits can align across magnetic domains. The magnetization $\boldsymbol{M}$ of a material is defined in terms of its total magnetic dipole moment per unit volume, i.e., a current-area per unit volume or current per unit length (see Fig. 1). In general $\boldsymbol{M}$ depends on the material, strength of applied field, and temperature. When an external field is applied, the domains can become more aligned, thus increasing $\boldsymbol{M}$. The contribution of the magnetization to the B-field is $\mu_0\boldsymbol{M}$. The extent of magnetization itself depends on the total B-field, so the calculation of the B-field must be done self-consistently and is generally non-trivial. Analytically tractable examples are confined to the simplest of geometries. In practice, computer software is used to find the fields in realistic situations.

When magnetic materials are present, Eq. (2) no longer applies unless one takes the view that $I_{\text{enc}}$ includes *both* externally applied currents *and* internal currents due to magnetism (the loops in Fig. 1), although the interpretation of the latter is moot.[5] However, mathematical convenience is restored by introducing the H-field, called magnetic field strength. By defining $\boldsymbol{H} = \boldsymbol{B}/\mu_0 - \boldsymbol{M}$ one obtains after some manipulation a circuital law for **H** in terms of externally applied currents only[6]:

$$\oint \boldsymbol{H} \cdot d\boldsymbol{l} = I_{\text{ext}}, \tag{4}$$

where $I_{ext}$ is the external current flowing through the loop. The response of a paramagnetic material is frequently described through the dimensionless parameter magnetic susceptibility $\chi$, or relative permeability $\mu_r$ (where $\mu_r = \chi + 1$). The susceptibility is defined by $\boldsymbol{M} = \chi\boldsymbol{H}$, and so $\boldsymbol{B} = \mu_r\mu_0\boldsymbol{H}$.

While the use of $\boldsymbol{H}$ makes for mathematical convenience, it is confusing to students[7] and its interpretation is hotly debated.[5] After seeing examples of the application of Eq. (2) in free space to find a B-field, some students naturally assume Eq. (4) can always be applied in the same way to find the H-field. This is incorrect; Eq. (4) *on its own* uniquely defines only $\nabla \times \boldsymbol{H}$, not $\boldsymbol{H}$ itself. Unlike the case of the B-field, we cannot generally say $\nabla \cdot \boldsymbol{H} = 0$. Many texts overlook this point. Engineering texts can skip over it since in the important but *specific* example of a transformer core, with a closed magnetic circuit, the lines of H-field do indeed form closed loops where $\nabla \cdot \boldsymbol{H} = 0$. Physics texts can get into difficulties trying to ascribe a simple physical interpretation to $\boldsymbol{H}$. For example, the well-used text of Serway[8] states:

> "The magnetic field strength is related to the magnetic field due to the conduction current in wires.…The magnetic field strength is the magnetic moment per unit volume due to currents,"

suggesting incorrectly that the H-field is independent of the magnetic properties of the medium. A student's conceptual approach to magnetic problems with external currents (e.g., a cored electromagnet) becomes incorrectly thus: (a) Use Eq. (4) to find $\boldsymbol{H}$; (b) apply $\boldsymbol{M} = \chi\boldsymbol{H}$ to find $\boldsymbol{M}$; and (c) use $\boldsymbol{B} = \mu_0(\boldsymbol{H} + \boldsymbol{M})$ to find the magnetic flux density $\boldsymbol{B}$. While this may work for very specific examples (where $\nabla \cdot \boldsymbol{H}$ is indeed zero), it is in general incorrect since Eq. (4) does not on its own define $\boldsymbol{H}$. It is not just students who fall into this trap; a teaching colleague reports, "I have been caught by a very similar misconception myself."

Physicists, but not engineers, tend to follow Lorentz and view $\boldsymbol{H}$ primarily as a mathematical construction with little physical meaning.[5,9] One might therefore be tempted to leave out discussion of the H-field altogether. However, this can also cause confusion. For example, Thornton and Rex,[10] describing magnetism without the H-field, write:

> "The magnetic susceptibility $\chi$ is defined by $\chi = \mu_0 M/B$. In other words, we may think of the magnetic susceptibility as the induced magnet moment per applied field…"

This is true and very useful if the magnetization is small. For example, one can make straightforward quantitative measurements of magnetic properties.[11,12] However, in general it is incorrect. The quoted equation implies that for a constant susceptibility, the proportionality constant between $\mu_0 M$ and $B$ is $\chi$, which is far from true for many cases including cored solenoids.

## Why the core-filled solenoid is problematic

If a magnetic core of relative permeability $\mu_r$ is inserted into an infinite solenoid (such that it fills the inside of the infinitely long solenoid), one can solve the electromagnetic equations for the B-field:

$$B = \mu_r \mu_0 nI; \tag{5}$$

in other words, $\mu_r$ times the field for the free space example of Eq. (3). For a material such as iron, with $\mu_r$ around 5000, this implies a huge boost in B-field. Unfortunately, this result is highly misleading. It is true that in an *infinite* solenoid the presence of the material boosts the B-field by a factor $\mu_r$, but only because the entire B-field is contained within the core; there is zero B-field outside. In general, including for a finite solenoid, $\mu_r$ does *not* represent the boost in B-field. However, first-year university texts are often vague about this point. For example, Giambattista[13] writes

> "The net effect is that the field inside the iron is intensified by a factor known as the *relative permeability* [emphasis included in the original],"

which might easily be interpreted incorrectly as meaning the relative permeability acts generally as a multiplicative factor on the B-field.

In fact, the boost in B-field obtained by inserting a core into a *finite* solenoid is practically far less than $\mu_r$, especially if $\mu_r$ is large. Finite element modeling[14] allows B-fields to be calculated; essentially the method solves for the field at defined grid points by minimizing the total energy in the field, subject to constraints. Figure 2 shows results obtained by finite element modeling with the software package COMSOL Multiphysics[15] for a core of $\mu_r = 20$. The field is not uniform close to the ends of the solenoid, and the field at the base is only around a third that of the center. The B-field at the center is modeled as 0.0779 T, and the field at the base 0.0214 T, considerably lower than the 0.254 T of Eq. (5).

Figure 3 shows the B-field at the base (on axis) and center of the solenoid against $\mu_r$. The result of Eq. (5) is also included for comparison. The modeled B-fields rise linearly with $\mu_r$ *only for small values of* $\mu_r$, but then the rate of rise with $\mu_r$ becomes considerably diminished.

A practical demonstration of the failure of the "multiply by $\mu_r$" strategy is not completely straightforward. Practically,

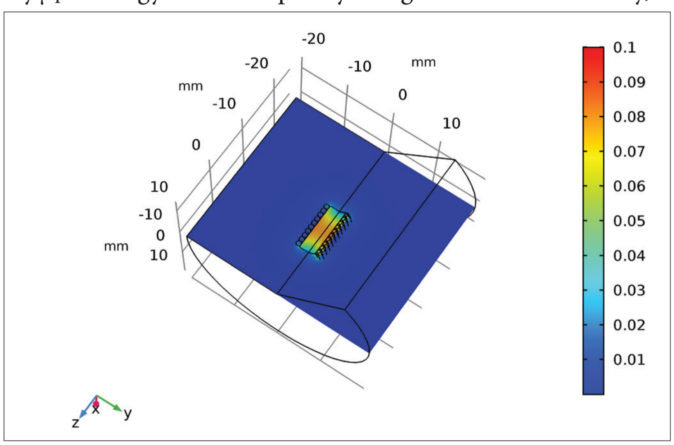


**Fig. 2. The magnitude of the B-field (in tesla) produced by a cored-solenoid of length 10 mm, with core diameter 4 mm and 10 turns of 1-mm diameter conductor carrying 10 A.**

magnetic responses are influenced by magnetic saturation, temperature, and ferromagnetism, which are not encapsulated with a single, constant value of $\mu_r$, complicating the comparison. However, one can readily demonstrate that inserting an iron core into a coil (e.g., a homemade electromagnet) does not boost the B-field by a factor anywhere close to 5000.

Another example, analytically tractable with vector calculus, is a magnetizable sphere placed into a uniform magnetic field.[16] Practically this might be a small ball bearing placed in a long solenoid. In this case, the boost in B-field at the surface of the sphere is given by $3\mu_r/(\mu_r + 2)$. In the limit of high $\mu_r$, as for iron where $\mu_r \approx 5000$, this boost becomes exactly 3. This is far less than the 5000 one might think by naively applying the result of the infinite solenoid example.

## Implications for teaching

As physicists, we are familiar with models being useful but limited descriptions of reality. Magnetic models and simplified solutions to magnetic equations give educators an opportunity to discuss with students the ways that models are and are not valid, and how they can fail. For example, the limits of applicability of Eq. (3) can be shown experimentally by simply mapping the B-field strength along a "Slinky" and the failure of Eq. (5) by adding an iron core into a solenoid. This can then lead to discussion about the implications of failure of physical models more broadly (e.g., climate modeling) and how assumptions need to be explicitly identified and their consequences evaluated.

In this article, I have highlighted three conceptual and practical misunderstandings that can be unintentionally encouraged through use of the infinite solenoid example. Specifically:

1. that the B-field for a cored infinite solenoid, Eq. (5), is a reasonable approximation for practical electromagnets;
2. that $\mu_r$ gives the boost in B-field when a magnetic material is inserted;
3. that a circuital law for $\boldsymbol{H}$, Eq. (4), can generally be used on its own to find H- and then B-fields in the presence of a paramagnetic material.

All of these can lead to considerable difficulties for students who carry on with physics at university; they do not tie-up with experiment and are not reconcilable with the more advanced mathematics of electromagnetism introduced in later courses. As teachers, we must be very careful to ensure that this incorrect understanding does not take root. Advanced electromagnetism classes have sufficient challenges of their own, involving vector calculus and the mysterious H-field; entering into this environment with incorrect understanding from junior classes does not help students. It may be best, therefore, to defer the quantitative analysis of the infinite solenoid with a core to more advanced classes.

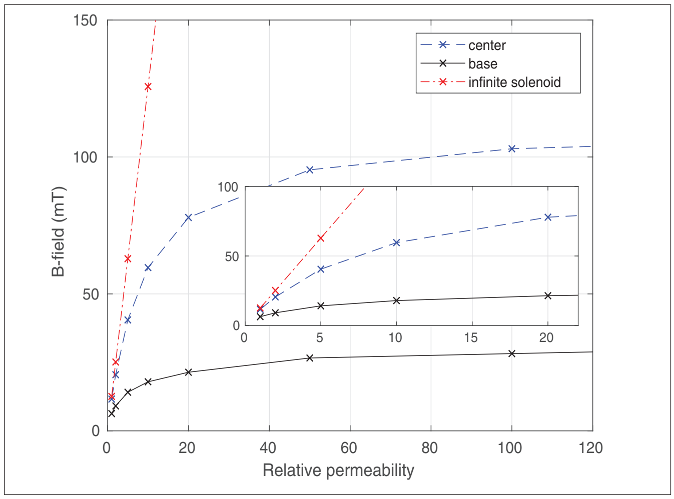


**Fig. 3. The B-field at the base of the core, on axis (black solid), and the center of the solenoid (blue dashed) as a function of core relative permeability. The prediction for an infinite solenoid with core, Eq. (5), is shown by the red dash-dot line.**

### References


1. D. Laumann, "Even liquids are magnetic: Observation of the Moses effect and the inverse Moses effect," *Phys. Teach.* **56**, 352–354 (Sept. 2018).
2. W. G. Hurley and W. H. Wölfle, *Transformers and Inductors for Power Electronics: Theory, Design and Applications* (Wiley-Blackwell, Chichester, West Sussex, UK, 2013).
3. S. Matsushita, *Principles of Geomagnetic Phenomena* (Elsevier Science, 2016).
4. M. T. Wilson, A. D. Tang, K. Iyer, H. McKee, J. Waas, and J. Rodger, "The challenges of producing effective small coils for transcranial magnetic stimulation of mice," *Biomed. Phys. Eng. Express* **4**, 037002 (2018).
5. J. J. Roche, "B and H, the intensity vectors of magnetism: A new approach to resolving a century-old controversy," *Am. J. Phys.* **68**, 438 (May 2000).
6. K. R. Demarest, *Engineering Electromagnetics* (Prentice-Hall Inc., Upper Saddle River, NJ, 1998), p. 249.
7. Z. Tanel and M. Erol, "Students' difficulties in understanding the concepts of magnetic field strength, magnetic flux density and magnetization," *Lat. Am. J. Phys. Educ.* **2** (3), 184–191 (2008).
8. R. A. Serway, *Physics for Scientists and Engineers,* 6th ed., international ed. (Brooks/Cole, Belmont, CA, 2004), p. 947.
9. J. H. Cloete, "Is B or H the fundamental magnetic field?" in *Proceedings of IEEE AFRICON '96*, Stellenbosch, South Africa (1996) pp. 354-361. DOI: 10.1109/AFRICON.1996.563137.
10. S. T. Thornton and A. Rex, *Modern Physics for Scientists and Engineers*, 3rd ed. (Brooks/Cole, Belmont, CA, 2006), p. 358.
11. D. Laumann, "Is an apple magnetic: Magnetic response of everyday materials supporting views about the nature of science," *Phys. Teach.* **55**, 142–145 (March 2017).
12. C. A. Sawicki, "Inexpensive demonstration of the magnetic properties of matter," *Phys. Teach.* **36**, 553–555 (Dec. 1998).
13. A. Giambattista, *Physics*, 5th ed. (McGraw-Hill Education, New York, 2019), p. 751.
14. Q. Wang, "Finite element analysis for the magnetostatic field," in *Practical Design of Magnetostatic Structure Using Numerical Simulation* (2013).
15. See comsol.com/acdc-module, accessed Oct. 24, 2019.
16. P. M. Pugh and P. W. Pugh, *Principles of Electricity and Magnetism*, 2nd ed. (Addison-Wesley Publishing Company, Reading, MA, 1970), p. 302.