Row-column Block Designs:

Efficiency and Structure

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Abstract

A row-column design is any rearrangement of the blocks of a combinatorial design into a rectangular array. A row-column block design is a row-column design in which the blocks form a balanced incomplete block design; that is, each pair of elements occurs in a constant number of blocks. In this thesis we study the efficiency and structure of row-column block designs. In particular, we use solutions to Heffter's difference problem to give construct row-column block designs with 3 elements per cell with optimal regularity in rows and columns.

In Section 1 we review definitions and theorems related to Latin squares. We introduce related concepts of balanced incomplete block designs and incidence matrix and concurrence matrix of designs. In Section 2 we give our main results of row-column block designs with block size 3. In Section 3 we use Heffter's difference problem to give some solutions. In Section 4 we explain efficiency measures for block designs. In Section 5 we introduce Trojan semi-Latin squares. We give efficiency measures for Trojan semi-Latin squares. In Section 6 we show the applications to experimental design. Original results are given in the following theorems: Theorem 2.9, Theorem 3.1, Theorem 3.3, Theorem 3.4, Theorem 4.2, Theorem 5.2, Theorem 5.12.

Results from Section 2 and Section 3 have been submitted as a manuscript for publication [17] (Xiao Nie, Row-column block designs with blocks of size three, Ars Combinatoria, submitted).

Keywords: Latin Squares, orthogonal Latin square, mutually orthogonal Latin square, semi-Latin square, uniform semi-Latin square, (v, b, r, k)-design, row-column design, row-column block design, balanced incomplete block design (BIBD), scaled information matrix.

Contents

1 Literature Review

In this section we survey the relevant concepts and theorems that need to be used in this thesis in order to understand the research in the following sections.

1.1 Latin squares and mutually orthogonal Latin squares

A **Latin square** from a set of size n is an $n \times n$ array, with the entries from a set of size n arranged in such a way that no row or column contains the same entry twice.

Examples of Latin Squares

Table 1: A Latin square of order 3

0	1	2
1	2	0
2	0	1

Table 2: A Latin Square of order 4

1	2	3	4
3	4	1	2
4	3	2	1
2	1	4	3

However, the following is not a Latin square of 3 because the size of set $\{1, 2, 3, 4, 5\}$ is 5.

1	2	3
2	3	4
3	4	5

A **Graeco-Latin square** is a type of experimental design used in statistics and combinatorics. It is formed by overlapping two Latin squares of the same size. The resulting square has the property that each ordered pair of symbols appears exactly once.

Table 3: Graeco-Latin square of order 3

(1,0)	(2,2)	(0,1)
(0,2)	(1,1)	(2,0)
(2,1)	(0,0)	(1,2)

A set of Latin squares of the same order such that every pair of squares are orthogonal (that is, form a Graeco-Latin square) is called a set of **mutually orthogonal Latin squares** (or pairwise orthogonal Latin squares).

Theorem 1.1. *If $n = p^c$ for some prime n and $c > 0$, then there are $n - 1$ mutually orthogonal $n \times n$ Latin squares.*

Proof. The proof given as [20]. Let $(F, +, \cdot)$ be a finite field of order n, where $F =$

Table 4: Three mutually orthogonal Latin squares of order 4

(0,0,0)	(0,1,1)	(0,2,2)	(0,3,3)
(1,0,1)	(1,1,3)	(1,2,0)	(1,3,2)
(2,0,2)	(2,1,0)	(2,2,3)	(2,3,1)
(3,0,3)	(3,1,2)	(3,2,1)	(3,3,0)

$\{1, 2, 3, \ldots, n = 0\}$. For $k = 1, 2, \ldots, n-1$, let L_k be the $n \times n$ array $(a_{i,j}^{(k)})$, where

$$a_{i,j}^{(k)} = i \cdot k + j.$$

for all $1 \leq i, j \leq n$.

Firstly, we will start from the entries in column j of L_k. If $a_{r,j}^{(k)} = a_{s,j}^{(k)}$ then $k \cdot r + j = k \cdot s + j$ so $k \cdot r = k \cdot s$, and so we can know $k \neq 0$ since $k < n$, so we obtain $r = s$. Hence if $r \neq s$, then $a_{r,j}^{(k)} \neq a_{s,j}^{(k)}$. Thus all entries in each column are distinct. Simiarly, the entries in each row are distinct. Then by the definition of Latin squares, obviously L_k is a Latin square.

Secondly we need to prove any two distinct L_k's are orthogonal. So suppose $1 \leq k < l < n$; then we only need to show L_k and L_l are orthogonal.

Suppose two of the entries in $L_k \circ L_l$ are equal, then

$$(a_{i,j}^{(k)}, a_{i,j}^{(l)}) = (a_{u,v}^{(k)}, a_{u,v}^{(l)}).$$

Then the entries are equal, so $a_{i,j}^{(k)} = a_{u,v}^{(k)}$ and $a_{i,j}^{(l)} = a_{u,v}^{(l)}$, so

$$k \cdot i + j = k \cdot u + v$$

and

$$l \cdot i + j = l \cdot u + v.$$

Hence

$$k \cdot (i - u) = v - j.$$

and

$$l \cdot (i - u) = v - j.$$

So

$$\begin{aligned}(k - l) \cdot (i - u) &= k \cdot (i - u) - l \cdot (i - u) \\ &= (v - j) - (v - j) \\ &= 0.\end{aligned}$$

But $k \neq l$, that is $i - u = 0$, at last we obtain $i = u$. According to the equation $k \cdot i + j = k \cdot u + v$, hence $j = v$, so the two entries in $L_k \circ L_l$ are in the same position. Therefore L_k and L_l are orthogonal. By these two steps, the theorem is proved. □

Example : Let us find four mutually orthogonal Latin squares of size 5. Here, $n = 5 = 5^1$, so the previous theorem applies. We use $\mathbb{Z}_5 = \{1, 2, 3, 4, 0\}$. We define $f_1 = 1, f_2 = 2, f_3 = 3, f_4 = 4, f_5 = 0$.

we get four mutually orthogonal Latin squares :

Table 5: L_1

2	3	4	0	1
3	4	0	1	2
4	0	1	2	3
0	1	2	3	4
1	2	3	4	0

Table 6: L_2

3	4	0	1	2
0	1	2	3	4
2	3	4	0	1
4	0	1	2	3
1	2	3	4	0

Table 7: L_3

4	0	1	2	3
2	3	4	0	1
0	1	2	3	4
3	4	0	1	2
1	2	3	4	0

Table 8: L_4

0	1	2	3	4
4	0	1	2	3
3	4	0	1	2
2	3	4	0	1
1	2	3	4	0

A Latin square is in **standard form** if both its first row and its first column are in increasing order.

1	2	3
2	3	1
3	1	2

Theorem 1.2. *If $\{L_1, L_2, \ldots, L_t\}$ is a set of mutually orthogonal Latin squares of order n (MOLS(n)), then it has the size $t \leq n-1$.*

Proof. The proof given is based on [23]: Let $\{L_1, L_2, \ldots, L_t\}$ be a set of MOLS(n). We assume that $n \geq 2$, then we can consider the symbols in cell $(2,1)$ of $L_1, \ldots, L_t$ to determine the upper bound of t. The symbol 1 occurs in cell $(1,1)$ of each of these Latin squares. If we consider $(2,1)$, the cell in each of them must be occupied by a symbol from the set $\{2, 3, \ldots, n\}$. In addition, because they are in standard form, any two of these Latin squares overlap, so the first line gives the ordered pairs $(1,1), (2,2), (3,3), (4,4), ..., (n,n)$. Therefore the cell $(2,1)$ cannot be occupied by the same symbol x in different squares. If not the pair of (x,x) would occur twice when they are superimposed. Thus the symbols in cell $(2,1)$ of $L_1, \ldots, L_t$ are all distinct and belong to the set of $\{2, 3, \ldots, n\}$. Hence, if $n \geq 2$ then $t \leq n-1$. □

1.2 Semi-Latin Squares

A **semi-Latin square** of order n and index k (or a semi-k-Latin square of n) is a $n \times n$ array such that each element from a set of size nk occurs once per row and once per column and each cell contains a set of size k. We sometimes call this an $(n \times n)/k$ semi-Latin square. The elements are sometimes called treatments.

Example of a semi-Latin square

Table 9: A semi-Latin square of order 3

1,2	3,4	5,6
3,5	2,6	1,4
4,6	1,5	2,3

A semi-Latin square is called **simple** if no pair of letters occurs together in more than one cell.

We now have another definition of orthogonal. Two Latin squares of order n are said to be orthogonal if when overlapped by different pair of symbol sets, we obtain a $n \times n$ simple semi-Latin square of order n and index 2.

Table 10: A semi-3-Latin Square of order 4

1,3,5	2,4,6	7,9,11	8,10,12
2,4,6	7,9,11	8,10,12	1,3,5
7,9,11	8,10,12	1,3,5	2.4.6
8,10,12	1,3,5	2,4,6	7,9,11

Table 11: Overlap Latin square of the same order 3 on different symbol sets

(1,A)	(2,B)	(3,C)
(2,B)	(3,C)	(1,A)
(3,C)	(1,A)	(2,B)

In Table 12, we give an example of a simple semi-Latin Square.

Table 12: A simple semi-Latin Square

1,2	4,5	3,6
4,6	1,3	2,5
3,5	2,6	1,4

1.3 Uniform semi-Latin squares

A semi-Latin square S is **uniform** if every pair of blocks, not in the same row or column, intersect in the same positive number of treatments. We denote this constant intersection number by $\mu(S)$.

Table 13: A uniform semi-Latin square with $\mu(S) = 2$.

1,4,7,10	2,5,8,11	3,6,9,12
3,6,8,11	1,4,9,12	2,5,7,10
2,5,9,12	3,6,7,10	1,4,8,11

Lemma 1.3. [15] *If S is a uniform $(n \times n)/k$ semi-Latin square then $\mu(S) = k/(n-1)$, and in particular, $n-1$ divides k.*

Proof. Let S is a uniform $(n \times n)/k$ semi-Latin square, and $i, j \in \{1, 2, \ldots, n\}$. So we can count in two ways the number of triples (i', j', α), such that $i', j' \in \{1, 2, \ldots, n\}$, $i' \neq i$ and $j' \neq j$, and $\alpha \in S(i, j) \cap S(i', j')$. The first way to count is: each block has k elements, and each elements from $1, 2, \ldots, k$ occurs $n-1$ times. So the sum is $k(n-1)$. Another way is the intersection α of each group of numbers is multiplied by blocks, remove the non-duplicate elements in the first row and column, there are $(n-1)^2$ blocks. Then the sum is $\mu(n-1)^2$, where μ is the number of elements of α. Therefore we get the equation: $k(n-1) = \mu(n-1)^2$. Solving the equation we get: $\mu = \frac{k}{n-1}$. The lemma is proved. □

Table 14: A uniform $(3 \times 3)/4$ semi-Latin square.

1,4,7,10	2,5,8,11	3,6,9,12
3,6,8,11	1,4,9,12	2,5,7,10
2,5,9,12	3,6,7,10	1,4,8,11

In Table 14, we show a uniform $(3 \times 3)/4$ semi-Latin square. We can count the sum by first way $k = 4$ and $n = 3$, than $k(n-1) = 4 \times (3-1) = 8$. By the second way:

because $\{1,4,7,10\} \cap \{2,5,7,10\} = \{7,10\}$, so the number of $\{7,10\}$ is 2, $\mu = 2$. Then $\mu(n-1)^2 = 2 \times (3-1)^2 = 8$. The final result is verified to be indeed equal.

1.4 Block Designs, BIBDs and Incidence Matrices

A **(v, b, r, k)-design** is a set v of elements and blocks of size k (considered as k-subsets of the set of elements), such that each element is in exactly r subsets.

Lemma 1.4. *If exist a semi-Latin square S of order n and index k, than there exists a (nk, n^2, n, k)-design $\Lambda(S)$.*

Proof. First consider a semi-Latin square of order n and index k. For each cell create a block containing entries from that cell.
Let B be set of all blocks, so that there are $|B| = b = n^2$ blocks. Then let V be set of elements, so that the number of the elements is $|V| = n \times k = nk$. Each treatment is in exactly r subsets, so $r = n$. Each subset has size k. So, there exists a (nk, n^2, n, k)-design. □

We give an example of Lemma 1.4 as follows:
The semi-Latin square in Table 15
corresponds to a (v, b, r, k)-design [2], with $v = 6, b = 9, r = 3, k = 2$. To see this, write the subsets in the cells as a set of blocks:

$$\{\{1,2\},\{4,6\},\{3,5\},\{4,5\},\{1,3\},\{2,6\},\{3,6\},\{2,5\},\{1,4\}\}.$$

This can be generalized as follows.

Table 15: Example of a (v, b, r, k)-design

1,2	4,5	3,6
4,6	1,3	2,5
3,5	2,6	1,4

Definition 1.5. *Let $\Delta = (V, \mathcal{B})$ be a (v, b, r, k)-design, define a $v \times b$ 0-1 matrix $A = (a_{i,j})$, whose rows are indexed by the points $x_1, x_2, \cdots, x_v$ and columns are indexed by the block $B_1, B_2, \cdots, B_b$ by*

$$a_{ij} = \begin{cases} 1 & x_i \in B_j \\ 0 & otherwise \end{cases}$$

Then A is called incidence matrix of the block design.

Definition 1.6. *The* **incidence matrix** *for a $(n \times n)/k$ semi-latin square S is the incidence matrix for $\Lambda(S)$. Equivalently, it's on $(nk) \times (n^2)$ matrix such that the rows are the nk elements and the columns are the n^2 blocks. Then put a 1 in row i and column j if and only if element i is in block j, otherwise are filled with 0.*

Continuing the example above, we obtain the incidence matrix as follows:

Table 16: Incidence matrix

	X_1	X_2	X_3	X_4	X_5	X_6	X_7	X_8	X_9
1	1	0	0	0	1	0	0	0	1
2	1	0	0	0	0	1	0	1	0
3	0	0	1	0	1	0	1	0	0
4	0	1	0	1	0	0	0	0	1
5	0	1	0	0	0	1	1	0	0
6	0	0	1	1	0	0	0	1	0

$$I = \begin{pmatrix} 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}$$

Definition 1.7. A balanced incomplete block design (BIBD) *is a* (v, b, r, k)*-design such that every pair of elements occurs together in* λ *of the* b *blocks.*

Example of BIBD In the design (v, b, r, k, λ): consider $b = 10, v = 6, k = 3, r = 5, \lambda = 2$

$B = \{\{1, 2, 5\}, \{1, 2, 6\}, \{1, 3, 4\}, \{1, 3, 6\}, \{1, 4, 5\}, \{2, 3, 4\}, \{2, 3, 5\}, \{2, 4, 6\},$
$\{3, 5, 6\}, \{4, 5, 6\}\}.$

	1	2	3	4	5	6
X_1	1	1	0	0	0	0
X_2	0	0	0	1	1	0
X_3	0	0	1	0	0	1
X_4	0	0	0	1	0	1
X_5	1	0	1	0	0	0
X_6	0	1	0	0	1	0
X_7	0	0	1	0	1	0
X_8	0	1	0	0	0	1
X_9	1	0	0	1	0	0

There are 6 “points” 1,2,3,4,5,6, each subset of a set of size v has 3 points, we call these “blocks”, and any pair of points occurs in exactly 2 blocks.

Theorem 1.8. [7] *Let $\Delta = (V, \mathcal{B})$ be a $(v, b, r, k, \lambda) - \mathrm{BIBD}$, and $A = (a_{i,j})$ is a $v \times b$ 0-1 maxtrix, then A the incidence matrix of Δ if and only if both*

$$AA^T = \lambda J + (r - \lambda)I_v$$

and

$$1_v A = k 1_b$$

hold, where A^T denotes the transpose of A, $r = \frac{\lambda(v-1)}{k-1}$, J_v and I_v are the $v \times v$ all 1’s matrix and the identity matrix,respectively, and 1_v and 1_b are the v-dimensional and b-dimensional all 1 row vectors, respectively.

Table 17: Example of a $(7,3,1)$-BIBD index 3

1, 3, 4	2, 4, 5	3, 5, 6	4, 6, 7	5, 7, 1	6, 1, 2	7, 2, 3

Then we consider the incidence matrix:

$$I = \begin{pmatrix} 1 & 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 1 \end{pmatrix}$$

$$I \cdot I^T = \begin{pmatrix} 1 & 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 & 1 & 0 & 1 \\ 1 & 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 3 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 3 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 3 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 3 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 3 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 3 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 3 \end{pmatrix}$$

$$= \lambda J + (\frac{\lambda(v-1)}{k-1} - \lambda)I_v = \lambda J + (\frac{6\lambda}{2} - \lambda)I_v = \lambda J + 2\lambda I_v$$

Therefore, there is a $(7, 3, 1)$-BIBD.

Let $\Delta = (V, \mathcal{B})$ be a (v, b, r, k)-design. The **concurrence matrix** Λ of Δ is the $v \times v$ matrix whose rows and columns are indexed by the treatments of Δ, and whose (α, β)-entry is the number of blocks containing both α and β (this entry is the **concurrence** of treatments α and β).

Lemma 1.9. *Let Δ be a (v, b, r, k)-design, if the incidence matrix of Δ is I, then the concurrence matrix $\Lambda = I \cdot I^T$.*

Proof. Let $\Lambda = (a_{ij})_{v\times v}$. Then, observe that $a_{ij} = |B_i \cap B_j|$. Next, the elements of the main diagonal are $a_{11}, a_{22}, \ldots, a_{vv}$. Then $|B_1 \cap B_1| = |B_2 \cap B_2| = \cdots = |B_b \cap B_b| = k$, since each block has size k, so the main diagonal of Δ is k. □

We show an example of Lemma 1.9 in Table 17.

2 Row-column block designs: main results

A **row-column design**, in its most general sense, is any rearrangement of the blocks of a combinatorial design into a rectangular array. Thus, a row-column design admits two partitions of blocks, given by the rows and columns. Most of the row-column designs developed in the literature have one unit corresponding to the intersection of row and column. Examples of such row-column designs are: Latin squares, Youden squares and generalized Youden designs [19]. Row-column designs with more than one unit per cell include: semi-Latin squares [22] and Trojan squares [9]. Such designs are used when the number of elements is substantially large with a limited number of replicates [16].

In this thesis, we say that an $m \times n$ **row-column block design** is any rearrangement of the blocks of a Balanced Incomplete Block Design BIBD (v, k, λ) into an $m \times n$ array. The **index** k is equal to the number of elements (or, in the context of experimental design, **treatments**) in each block. The property of a row-column design being a row-column **block** design is referred to in [7] as **structurally complete**. This is desirable for experimental design applications as each pair of treatments occurs a constant number of times within the design.

We say that a row or column is **regular**, if each element occurs the same number of times in that row or column. Regularity is desirable in the context of experimental design because this ensures that a row or column effect is not "confounded" with the effect of an individual treatment (see [3] for more information). We say that a row or column is **near-regular** if it is not regular but there is an integer x such that every entry occurs either x or $x + 1$ times.

We assume the set of treatments in BIBD (v, k, λ) is given by $\mathbb{Z}_v$. A row-column block design is in said to be **row-cyclic** if whenever B is a block in cell (i, j),

$$B' = \{b + 1 \pmod{v} \mid b \in B\}$$

is the block in cell $(i, j + 1 \pmod{n})$). We define **column-cyclic** similarly.

The above definitions are demonstrated in the example given in Table 18.

Table 18: A row-cyclic 5×7 array row-column block design of index 3, where each row is regular and each column is near-regular.

1, 2, 3	2, 3, 4	3, 4, 5	4, 5, 6	5, 6, 0	6, 0, 1	0, 1, 2
4, 5, 0	5, 6, 1	6, 0, 2	0, 1, 3	1, 2, 4	2, 3, 5	3, 4, 6
5, 6, 2	6, 0, 3	0, 1, 4	1, 2, 5	2, 3, 6	3, 4, 0	4, 5, 1
6, 0, 4	0, 1, 5	1, 2, 6	2, 3, 0	3, 4, 1	4, 5, 2	5, 6, 3
6, 1, 3	0, 2, 4	1, 3, 5	2, 4, 6	3, 5, 0	4, 6, 1	5, 0, 2

The problem of constructing row-column block designs with index 2 was considered in [7]:

Theorem 2.1. [7]

For any t, there exists:

(a) *a row-cyclic $t \times (2t + 1)$ row column block design of index 2, where each row is regular and each column is near-regular.*

(b) *a row-cyclic $2t \times 2t$ row column block design of index 2, where each row is near-regular and each column is regular.*

We describe their construction in detail in Section 3.

In this thesis, we generalize the previous theorem to index 3. Our main theorem is as follows.

Theorem 2.2. *For any* $v \equiv 1 \bmod 6$ *there exists a row-cyclic row-column block design with* $(v-1)/6$ *rows,* v *columns and index* 3 *such that each row is regular and each column is near-regular.*

Our proof of the above theorem in Section 3 will make use of *Heffter difference sets*. We also find more specific results for $v = 7$ and $v = 13$ in Subsection 2.3. These are summarized in Theorem 3.3.

2.1 Row-column block designs with block size 2.

In this section we dscribe in detail the construction given in [7].

This thesis [8] introduced some related concepts about row-column design. In the first section they introduce their research background, inspired by the interconnection between design and Trojan squares. This gives a construction of row-column design using two elements per cell. And in the section 2 they give the calculation method of the information matrix corresponding to the row-column design.

Then, in the section 3 they give two methods of constructing row-column designs with two units per cell. The first method is for an odd number of treatments and the second is for an even number.

The main result of that section is Theorem 2.1.

The construction uses the classic Walecki cyclic one-factorization of the complete graph [11].

Odd number case: Let $v = 2t + 1$ obtain the following initial column having two units per cell:

$1,\ 2t+1$
$2,\ 2t$
$3,\ 2t-1$
$\vdots, \vdots$
$t-1,\ t+3$
$t,\ t+2$

Then developing this column incrementing by 1 *modulo* v would get a $m \times n$ row-column design with two elements per cell.

Lemma 2.3. [6] *Let G and H be graphs. Suppose there exists some subgraphs of G, each isomorphic to H, such that each edges of these subgraphs partition the edges of G. Then graph G is said to decompose into graph H.*

Lemma 2.4. [6] *Let the vertices of K_n be labeled with the elements of $\mathbb{Z}_n$. Given a subgraph H of K_n, and any integer i, let $H+i$ be the graph created by replacing each vertex v of H with $(v+i)$ mod n. Then a cyclic decomposition is one in which whenever H is a subgraph, $H+1$ is also a subgraph.*

Label the vertices of K_n with the elements of $\mathbb{Z}_n$. We define the "difference" of edge a, b to be the minimum value of $a - b \pmod{n})$ and $b - a \pmod{n})$.

Theorem 2.5. [6] *Let the vertices of K_n be labeled with the elements of $\mathbb{Z}_n$. If H is a subgraph of K_n with n odd, and the edges of H have the differences $1, 2, ...(n-1)/2$, then there is a cyclic decomposition of K_n into copies of H.*

Proof. Let H be a graph with at most n vertices and $(n-1)/2$ edges, where n is odd. Then form n copies of H by incrementing the vertices modulo n. Since the edges of H cover each possible difference, the K_n decomposes into H. So there is a cyclic decomposition of K_n into copies of H. □

Using Theorem 2.5, we consider the odd number case above. Assume that the elements in the table are the vertices of graph H with the different elements of $\mathbb{Z}_n$, where n is an odd number. Observe that all the differences are covered. Then by Theorem 2.5, there is a cyclic decomposition. In turn, in the row-column design, each pair is covered exactly once.

Even number case: Let $v = 2t$ obtain the following initial column having two elements per cell.

$1,\ 2t$
$2t,\ 2$
$2,\ 2t-1$
$2t-1,\ 3$
$\vdots,\ \vdots$
$t+2,\ t$
$t,\ t+1$

Then we use this column incrementing by 1 mod v would get a $m \times n$ row-column design with two elements per cell.

We consider the even number case above. Assume that the elements in the table are the vertices of graph H with the different elements of $\mathbb{Z}_n$, where n is even number. Observe that all the differences are occurring twice. Thus, there is also a cyclic decomposition. In turn, in the row-column design, each pair is covered twice.

2.2 Another approach for index 3

In the previous sections, our constructions only work when the number of columns is a particular integer. In this section, we give a different construction where the number of columns can be any integer, but we relax the condition of being a row-column block design.

These constructions are inspired by the paper by Anindita, Seema, et.al [8] in Section 2.

Case 1:

When $v = 3t$ $(t > 1)$ then according to the following initial column having three units per cell:

$$B_1 = \begin{bmatrix} U_1 \\ U_2 \\ U_3 \\ \vdots \\ \vdots \\ U_t \end{bmatrix}$$

where $|U_i| = 3, i = 1, 2, \cdots, t$.

We take $U_i, i = 1, 2, \cdots, t$. as follows:

Develop $(t-1)$ more columns horizontally from the initial column by adding $1, 2, 3, \ldots, t-1$ consecutively reducing mod v. The resulting design is a balanced semi-Latin rectangles with 3 units per cell incomplete rows and complete columns.

$1,\ 2t,\ 2t+1$
$2,\ 2t-1,\ 2t+2$
$3,\ 2t-2,\ 2t+3$
$\vdots, \vdots, \vdots$
$t-1,\ t+2,\ 3t-1$
$t,\ t+1,\ 3t$

B_1	B_2	B_3	$\cdots$	B_t

Then $\mathcal{B} = \{B_1, B_2, B_3 ,\cdots ,B_v\}$ is an incomplete Row-Column Design.

Example:

Let $v = 12 = 3t, t = 4, t-1 = 3$. The contents of the initial column are obtained as follows :

1, 8, 9
2, 7, 10
3, 6, 11
4, 5, 12

Then developing this column by adding $1, 2, 3, \ldots t-1 = 3$ reducing mod 12 would get result as follows:

1, 8, 9	2, 9, 10	3, 10, 11	4, 11, 12
2, 7, 10	3, 8, 11	4, 9, 12	5, 10, 1
3, 6, 11	4, 7, 12	5, 8, 1	6, 9, 2
4, 5, 12	5, 6, 1	6, 7, 2	7, 8, 3

Lemma 2.6. *The resulting design is a $t \times t$ row-column design with index* 3*, incomplete rows and complete columns.*

Proof. In $\mathcal{B} = \{B_1, B_2, B_3, \cdots, B_v\}$, it is not difficult to see from the final row-column design table that the elements from 1 to v in each column appear once times in B_i, so it is complete columns. Then some of the elements in each row appear once, some appear twice, and some do not appear, so it is incomplete. Thus, it is a incomplete rows and complete columns design. □

Case 2:

When $v = 3t + 1$ $(t > 1)$ then according to the following initial column having three units per cell:

$$B_1 = \begin{bmatrix} U_1 \\ U_2 \\ U_3 \\ \vdots \\ \vdots \\ U_t \end{bmatrix}$$

where $|U_i| = 3, i = 1, 2, \cdots, t$.

We take $U_i, i = 1, 2, \cdots, t.$ as follows:

$1,\ 2t+1,\ 2t+2$
$2,\ 2t,\ 2t+3$
$3,\ 2t-1,\ 2t+4$
$\vdots, \vdots, \vdots$
$t-1,\ t+3,\ 3t$
$t,\ t+2,\ 3t+1$

Develop $(3t+1)$ more columns horizontally from the initial column by adding $1, 2, 3, \ldots, 3t+1$ consecutively reducing mod v. The resulting design is a row-column design with 3 units per cell incomplete rows and complete columns.

B_1	B_2	B_3	$\cdots$	B_v

Then $\mathcal{B} = \{B_1, B_2, B_3\ ,\cdots\ ,B_v\}$ is a incomplete Row-Column Design.

Example:

Let $v = 10 = 3t + 1, t = 3$. The contents of the initial column are obtained as follows :

1, 7, 8
2, 6, 9
3, 5, 10

Then developing this column by adding $1, 2, 3, \ldots 3t + 1 = 10$ reducing mod 10 would get result as follows:

1, 7, 8	2, 8, 9	3, 9, 10	4, 10, 1	5, 1, 2
2, 6, 9	3, 7, 10	4, 8, 1	5, 9, 2	6, 10, 3
3, 5, 10	4, 6, 1	5, 7, 2	6, 8, 3	7, 9, 4
6, 2, 3	7, 3, 4	8, 4, 5	9, 5, 6	10, 6, 7
7, 1, 4	8, 2, 5	9, 3, 6	10, 4, 7	1, 5, 8
8, 10, 5	9, 1, 6	10, 2, 7	1, 3, 8	2, 4, 9

Lemma 2.7. *The resulting design is a $t \times v$ row-column design with index 3, regular rows and near-regular columns.*

Proof. In $\mathcal{B} = \{B_1, B_2, B_3\ , \cdots\ , B_v\}$, according to the final row-column design table, the elements from 1 to v in each column does not appear once times. Because the first column lost element $t + 1$, the second column lost $t + 2$ and so on in each B_i. So it is incomplete columns. And the elements from 1 to v appear three times each row, so it is complete rows. Therefore, it is a complete rows and incomplete columns design. □

Case 3:

When $v = 3t + 2$ $(t > 1)$ then according to the following initial column having three units per cell:

$$B_1 = \begin{bmatrix} U_1 \\ U_2 \\ U_3 \\ \vdots \\ \vdots \\ U_t \end{bmatrix}$$

where $|U_i| = 3, i = 1, 2, \cdots, t$.

Take $U_i, i = 1, 2, \cdots, t$. as follows:

$t,\ 2t,\ 3t$
$t-1,\ 2t-1,\ 3t-1$
$\vdots, \vdots, \vdots$
$3,\ t+3,\ 2t+3$
$2,\ t+2,\ 2t+2$
$1,\ 3t+1,\ 3t+2$

Develop this columns horizontally from the initial column by adding $1, 2, 3, \ldots, v-1$ consecutively reducing mod v. The resulting design is a row-column design with 3 units per cell. The design is complete row-wise and incomplete column-wise.

B_1	B_2	B_3	$\cdots$	B_v

Then $\mathcal{B} = \{B_1, B_2, B_3, \cdots, B_v\}$ is an incomplete Row-Column Design.

Example:

Let $v = 11 = 3t + 2, t = 3$. The contents of the initial column are obtained as follows :

3, 6, 9
2, 5, 8
1, 10, 11

Then developing this column incrementing by 1 mod 11 would get result as follows.

3, 6, 9	4, 7, 10	5, 8, 11	6, 9, 1	7, 10, 2	8, 11, 3
2, 5, 8	3, 6, 9	4, 7, 10	5, 8, 11	6, 9, 1	7, 10, 2
1, 10, 11	2, 11, 1	3, 1, 2	4, 2, 3	5, 3, 4	6, 4, 5

9, 1, 4	10, 2, 5	11, 3, 6	1, 4, 7	2, 5, 8
8, 11, 3	9, 1, 4	10, 2, 5	11, 3, 6	1, 4, 7
7, 5, 6	8, 6, 7	9, 7, 8	10, 8, 9	11, 9, 10

Lemma 2.8. *The resulting design is a $t \times v$ row-column design with index* 3*, complete rows and incomplete columns.*

Proof. In $\mathcal{B} = \{B_1, B_2, B_3\ , \cdots\ , B_v\}$, according to the final row-column design table, the elements from 1 to v in each column does not appear once times. Because the first column lost element $t+1, 2t+1$, the second column lost element $t+2, 2t+2$ and so on in each B_i. So it is incomplete columns. Then the elements from 1 to v appear three times each row, so it is complete rows. Thus, it is a near-regular block design. □

2.3 Row-column block designs for 7 and 13 columns

In this section we give specific results for a small number of columns.

Theorem 2.9. *For any $m \geq 1$, there exists a row-cyclic $m \times 7$ row-column block design of index 3 where each row is regular and each column is regular (if m is divisible by 7) or near-regular (if m is not divisible by 7).*

Proof. We first show the above theorem is true for the cases $1 \leq m \leq 7$. If $m > 7$, let $m = 7a + m'$. A solution is then formed by adjoining a copies of the solution for 7 rows with one copy of the solution for m' rows. □

Lemma 2.10. *If $n \in \{2, 4\}$, there exists a row-cyclic $n \times 13$ row-column block design of index 3 where each row is regular and each column is regular (if n is divisible by 13) or near-regular (if n is not divisible by 13).*

Solution for $v = 7$:

This gives a cyclic decomposition of K_7 into triangles with the difference of $\{1, 2, 3\}$.
In turn, we get a 1×7 array in which each cell has 3 entries, each column is formed from the previous one incrementally mod 7 and each pair is covered exactly once! Rows are complete, columns are not.

Table 19: A row-cyclic 1×7 row-column block design of index 3, where each row is regular and each column is near-regular.

1, 3, 4	2, 4, 5	3, 5, 6	4, 6, 7	5, 7, 1	6, 1, 2	7, 2, 3

Table 20: A row-cyclic 2×7 row-column block design of index 3, where each row is regular and each column is near-regular.

1, 3, 4	2, 4, 5	3, 5, 6	4, 6, 7	5, 7, 1	6, 1, 2	7, 2, 3
6, 7, 2	7, 1, 3	1, 2, 4	2, 3, 5	3, 4, 6	4, 5, 7	5, 6, 1

Table 21: A row-cyclic 3×7 row-column block design of index 3, where each row is regular and each column is near-regular.

1, 3, 4	2, 4, 5	3, 5, 6	4, 6, 7	5, 7, 1	6, 1, 2	7, 2, 3
6, 7, 2	7, 1, 3	1, 2, 4	2, 3, 5	3, 4, 6	4, 5, 7	5, 6, 1
5, 1, 7	6, 2, 1	7, 3, 2	1, 4, 3	2, 5, 4	3, 6, 5	4, 7, 6

The array above is not column-complete but every entry occurs either 1 or 2 times. By definition, the array is near-complete.

Table 22: A row-cyclic 4×7 row-column block design of index 3, where each row is regular and each column is near-regular.

1, 3, 4	2, 4, 5	3, 5, 6	4, 6, 7	5, 7, 1	6, 1, 2	7, 2, 3
6, 7, 2	7, 1, 3	1, 2, 4	2, 3, 5	3, 4, 6	4, 5, 7	5, 6, 1
5, 1, 7	6, 2, 1	7, 3, 2	1, 4, 3	2, 5, 4	3, 6, 5	4, 7, 6
2, 3, 5	3, 4, 6	4, 5, 7	5, 6, 1	6, 7, 2	7, 1, 3	1, 2, 4

Table 23: A row-cyclic 5 × 7 row-column block design of index 3, where each row is regular and each column is near-regular.

1, 3, 4	2, 4, 5	3, 5, 6	4, 6, 7	5, 7, 1	6, 1, 2	7, 2, 3
6, 7, 2	7, 1, 3	1, 2, 4	2, 3, 5	3, 4, 6	4, 5, 7	5, 6, 1
5, 1, 7	6, 2, 1	7, 3, 2	1, 4, 3	2, 5, 4	3, 6, 5	4, 7, 6
2, 3, 5	3, 4, 6	4, 5, 7	5, 6, 1	6, 7, 2	7, 1, 3	1, 2, 4
4, 6, 7	5, 7, 1	6, 1, 2	7, 2, 3	1, 3, 4	2, 4, 5	3, 5, 6

Table 24: 5×7 array without repeating triples

1, 2, 3	1, 6, 7	2, 4, 7	1, 2, 5	3, 4, 5	5, 6, 7	3, 4, 6
2, 4, 6	2, 3, 4	1, 3, 6	4, 5, 7	1, 5, 7	2, 5, 6	1, 3, 7
1, 4, 7	2, 5, 7	1, 5, 6	2, 3, 6	3, 6, 7	1, 4, 3	2, 4, 5
3, 5, 7	4, 6, 7	2, 3, 5	1, 4, 6	1, 2, 4	1, 2, 7	3, 5, 6
4, 5, 6	1, 3, 5	3, 4, 7	2, 3, 7	1, 2, 6	1, 4, 5	2, 6, 7

The 5×7 array without repeating triples below:

Row1, Row2, Row3, Row4, Row5 are all complete, since all the numbers from 1 to 7 occur three times in each row.

Column1 to Column5 are all near-complete, since all the numbers from 1 to 7 occur twice or three times in each column.

We try to choose 3 different numbers from 7 numbers. There are $C_7^3 = \frac{7!}{3!(7-3)!} = \frac{7\times6\times5}{3\times2\times1} =$ 35 options. It is less than 49 cells, so it's not possible to make such an array without repeating

Table 25: a cyclic solution for the 5×7 array

2, 3, 4	3, 4, 5	4, 5, 6	5, 6, 7	6, 7, 1	7, 1, 2	1, 2, 3
5, 6, 1	6, 7, 2	7, 1, 3	1, 2, 4	2, 3, 5	3, 4, 6	4, 5, 7
6, 7, 3	7, 1, 4	1, 2, 5	2, 3, 6	3, 4, 7	4, 5, 1	5, 6, 2
7, 1, 5	1, 2, 6	2, 3, 7	3, 4, 1	4, 5, 2	5, 6, 3	6, 7, 4
7, 2, 4	1, 3, 5	2, 4, 6	3, 5, 7	4, 6, 1	5, 7, 2	6, 1, 3

Table 26: A row-cyclic 6×7 row-column block design of index 3, where each row is regular and each column is near-regular.

1, 3, 4	2, 4, 5	3, 5, 6	4, 6, 7	5, 7, 1	6, 1, 2	7, 2, 3
6, 7, 2	7, 1, 3	1, 2, 4	2, 3, 5	3, 4, 6	4, 5, 7	5, 6, 1
5, 1, 7	6, 2, 1	7, 3, 2	1, 4, 3	2, 5, 4	3, 6, 5	4, 7, 6
2, 3, 5	3, 4, 6	4, 5, 7	5, 6, 1	6, 7, 2	7, 1, 3	1, 2, 4
4, 6, 7	5, 7, 1	6, 1, 2	7, 2, 3	1, 3, 4	2, 4, 5	3, 5, 6
1, 2, 4	2, 3, 5	3, 4, 6	4, 5, 7	5, 6, 1	6, 7, 2	7, 1, 3

triples.

Solution for $v = 13$:

This gives a cyclic decomposition of K_{13} into triangles with the difference of $\{1, 3, 4\}$ and $\{2, 5, 6\}$.

In turn, we get a 2×13 array in which each cell has 3 entries, each column is formed from the previous one incrementally mod 13 and each pair is covered exactly once! Rows are

Table 27: A row-cyclic and column-cyclic 7×7 row-column block design of index 3, where each row is regular and each column is also regular.

1, 3, 4	2, 4, 5	3, 5, 6	4, 6, 7	5, 7, 1	6, 1, 2	7, 2, 3
6, 7, 2	7, 1, 3	1, 2, 4	2, 3, 5	3, 4, 6	4, 5, 7	5, 6, 1
5, 1, 7	6, 2, 1	7, 3, 2	1, 4, 3	2, 5, 4	3, 6, 5	4, 7, 6
2, 3, 5	3, 4, 6	4, 5, 7	5, 6, 1	6, 7, 2	7, 1, 3	1, 2, 4
4, 6, 7	5, 7, 1	6, 1, 2	7, 2, 3	1, 3, 4	2, 4, 5	3, 5, 6
1, 2, 4	2, 3, 5	3, 4, 6	4, 5, 7	5, 6, 1	6, 7, 2	7, 1, 3
3, 6, 5	4, 7, 6	5, 1, 7	6, 2, 1	7, 3, 2	1, 4, 3	2, 5, 4

1, 3, 4	2, 4, 5	3, 5, 6	4, 6, 7	5, 7, 1	6, 1, 2	7, 2, 3
2, 4, 5	3, 5, 6	4, 6, 7	5, 7, 1	6, 1, 2	7, 2, 3	1, 3, 4
3, 5, 6	4, 6, 7	5, 7, 1	6, 1, 2	7, 2, 3	1, 3, 4	2, 4, 5
4, 6, 7	5, 7, 1	6, 1, 2	7, 2, 3	1, 3, 4	2, 4, 5	3, 5, 6
5, 7, 1	6, 1, 2	7, 2, 3	1, 3, 4	2, 4, 5	3, 5, 6	4, 6, 7
6, 1, 2	7, 2, 3	1, 3, 4	2, 4, 5	3, 5, 6	4, 6, 7	5, 7, 1
7, 2, 3	1, 3, 4	2, 4, 5	3, 5, 6	4, 6, 7	5, 7, 1	6, 1, 2

complete, columns are not.

Solution for $v = 15$:

This gives a cyclic decomposition of K_{15} into triangles with the difference of $\{1,3,4\}$, $\{2,6,7\}$ and $\{5\}$.

Table 28: A row-cyclic and column-cyclic 2×13 row-column block design of index 3, where each row is regular and each column is near-regular.

1, 2, 5	2, 3, 6	3, 4, 7	4, 5, 8	5, 6, 9	6, 7, 10	7, 8, 11
4, 6, 11	5, 7, 12	6, 8, 13	7, 9, 1	8, 10, 2	9, 11, 3	10, 12, 4

8, 9, 12	9, 10, 13	10, 11, 1	11, 12, 2	12, 13, 3	13, 1, 4
11, 13, 5	12, 1, 6	13, 2, 7	1, 3, 8	2, 4, 9	3, 5, 10

Table 29: A row-cyclic and column-cyclic 4×13 row-column block design of index 3, where each row is regular and each column is near-regular.

1, 2, 5	2, 3, 6	3, 4, 7	4, 5, 8	5, 6, 9	6, 7, 10	7, 8, 11
4, 6, 11	5, 7, 12	6, 8, 13	7, 9, 1	8, 10, 2	9, 11, 3	10, 12, 4
3, 12, 13	9, 10, 13	10, 11, 1	11, 12, 2	12, 13, 3	13, 1, 4	1, 2, 5
7, 8, 10	4, 8, 11	5, 9, 12	6, 10, 13	7, 11, 1	8, 12, 2	9, 13, 3

8, 9, 12	9, 10, 13	10, 11, 1	11, 12, 2	12, 13, 3	13, 1, 4
11, 13, 5	12, 1, 6	13, 2, 7	1, 3, 8	2, 4, 9	3, 5, 10
2, 3, 6	3, 4, 7	4, 5, 8	5, 6, 9	6, 7, 10	7, 8, 11
10, 1, 4	11, 2, 5	12, 3, 6	13, 4, 7	1, 5, 8	2, 6, 9

In turn, we get a 3×15 array in which each cell has 3 entries, each column is formed from the previous one incrementally mod 15 and each pair is covered exactly once! Rows are complete, columns are not.

Table 30: A row-cyclic and column-cyclic 2×15 row-column block design of index 3, where each row is regular and each column is near-regular.

1, 2, 5	2, 3, 6	3, 4, 7	4, 5, 8	5, 6, 9	6, 7, 10	7, 8, 11	8, 9, 12
4, 6, 13	5, 7, 14	6, 8, 15	7, 9, 1	8, 10, 2	9, 11, 3	10, 12, 4	11, 13, 5
5, 10, 15	6, 11, 1	7, 12, 2	8, 13, 3	9, 14, 4	10, 15, 5	11, 1, 6	12, 2, 7

9, 10, 13	10, 11, 14	11, 12, 15	12, 13, 1	13, 14, 2	14, 15, 3	15, 1, 4
12, 14, 6	13, 15, 7	14, 1, 8	15, 2, 9	1, 3, 10	2, 4, 11	3, 5, 12
13, 3, 8	14, 4, 9	15, 5, 10	1, 6, 11	2, 7, 12	3, 8, 13	4, 9, 14

3 Row-column block designs from solutions to Heffter's Difference Problem

Consider Heffter's First and Second Difference Problems:

(1) Let $v = 6n + 1$. Is it possible to partition the set $\{1, 2, \ldots, 3n\}$ into difference triples?

(2) Let $v = 6n+3$. Is it possible to partition the set $\{1, 2, \ldots, 3n+1\} \backslash \{2n+1\}$ into difference triples?

Heffter's First Difference Problem deals with partitioning the set $\{1, 2, \ldots, (v-1)/2\}$, and Heffter's Second Difference Problem deals with partitioning the set $\{1, 2, \ldots, (v-1)/2\} \backslash \{v/3\}$.

A **difference triple** $\{a, b, c\}$ is a set of three different elements from $\{1, 2, \ldots, v-1\}$, whose sum modulo v equals zero ($a + b + c = 0 \pmod{v}$) or for which one element mod v equals the sum of the other two ($a + b = c \pmod{v}$).

Heffter proposed the following problems:

(1) First Difference Problem of Heffter:

Let $v = 6m + 1$. Is there a partition of the set $\{1, 2, \ldots, 3m\}$ in difference triples.

(2) Second Difference Problem of Heffter:

Let $v = 6m+3$. Is there a partition of the set $\{1, 2, \ldots, 2m\} \cup \{2m+2, 2m+3, \ldots, 3m+1\}$ in difference triples?

For example, $\{\{1, 5, 6\}, \{2, 8, 10\}, \{3, 4, 7\}\}$ is a solution to Heffter's first difference problem for $v = 19$.

Solutions to Heffter difference problems are known for all cases except $v = 9$:

Theorem 3.1. (Peltesohn, [18]) *There exists a solution to Heffter's first and second difference problem for each $m \geq 1$ with the exception of the case $v = 9$.*

Any solution of the first difference problem of Heffter also gives a construction of a BIBD$(v, 3, 1)$ (also known as a **Steiner triple system**) which has the extra property of being cyclically generated.

We explain this process as follows. Given two elements a and b of $\mathbb{Z}_v$, the **difference** of $\{a, b\}$, denoted by $d(a, b)$, is defined to be the minimum value of $a - b \pmod v$ and $b - a \pmod v$. Given a subset T of $\mathbb{Z}_v$, we define $d(T)$ to be the set of **differences** that arise from S. That is,

$$d(T) = \{d(a, b) \mid \{a, b\} \subseteq T\}.$$

In turn, given S, a set of subsets of $\mathbb{Z}_v$, we define

$$\Delta(S) = \bigcup_{T \in S} d(T).$$

Now, for any difference triple B, we can easily construct a set T such that $d(T) = B$. If B has the form $\{a, b, a+b\}$, then let $T = \{0, a, a+b\}$. Otherwise, if B has the form $\{a, b, c\}$

where $a+b+c=0 \pmod v$, then similarly let $T=\{0,a,a+b\}$. By cycling such a set of blocks modulo v, we create a BIBD$(v,3,1)$.

For example, consider the solution $B=\{\{1,5,6\},\{2,8,10\},\{3,4,7\}\}$ to Heffter's first difference problem for $v=19$, given above. Then, $T=\{\{0,1,6\},\{0,2,10\},\{0,3,7\}\}$ as above. Observe that $\Delta(T)=\{1,2,\ldots,9\}$, so Table 31 is a BIBD$(19,3,1)$.

Table 31: A cyclic BIBD$(19,3,1)$

0, 1, 6	1, 2, 7	2, 3, 8	$\cdots$	16, 17, 3	17, 18, 4	18, 0, 5
0, 2, 10	1, 3, 11	2, 4, 12	$\cdots$	16, 18, 7	17, 0, 8	18, 1, 9
0, 3, 7	1, 4, 8	2, 5, 9	$\cdots$	16, 0, 4	17, 1, 5	18, 2, 6

Now we are ready to see the connection between solutions to Heffter's first difference problem and row-column block designs with block size 3.

Theorem 3.2. [23] *Suppose there exists a set S of $(v-1)/6$ triples from $\mathbb{Z}_v$ such that:*
(1) *$\Delta(S)$ is the set $\{1,2,\ldots,(v-1)/2\}$;*
(2) *The set S is near-regular.*
Then, there exists a row-cyclic row-column block design with $(v-1)/6$ rows, v columns and index 3 such that each row is regular and each column is near-regular.

For example, for $v=19$, we can replace T from above with a set T' such that $\Delta(T)=\Delta(T')$ but T' is near-regular. These triples form the first column of the 3×19 row-cyclic row-column block design given in Table 47, which is also row-regular and column near-regular.

We prove the following theorem in the next section. As demonstrated in the above example, together with the above theorem, this implies Theorem 2.2.

Theorem 3.3. *For any* $v \equiv 1 \bmod 6$ *there exists a set* S *of* $(v-1)/6$ *triples such that:*

(1) *The triples from* S *form a solution to Heffter's first difference problem;*

(2) *The set* S *is near-regular.*

We also prove an equivalent version of the above theorem in the context of Heffter's second difference problem in Section 3.2. In this case, the row-column designs created are not row-column block designs; in particular pairs of treatments of difference $v/3$ are never included.

Theorem 3.4. *For any* $v \equiv 3 \bmod 6$ *there exists a set* S *of* $(v-3)/6$ *triples such that:*

(1) *The triples from* S *form a solution to Heffter's second difference problem;*

(2) *The set* S *is near-regular.*

We will make use of the following solutions to Heffter's difference set problem (originally in [18], see also in Subsections 3.1 and Subsection 3.2.

Case 1:

Suppose $v \equiv 1 \ (mod\ 18)$ and $v \geq 37$. Say $v = 18s+1$ where $s \geq 2$. Notice that $(3r+1)+(4s-r+1) = (4s+2r+2)$, $(3r+2)+(8s-r) = (8s+2r+2)$, $(3r+3)+(6s-2r-1) = (6s+r+2)$ and $(3s)+(3s+1) = (6s+1)$, so each of the triples given in the first column of the following table is, in fact, a difference triple.

Lemma 3.5. [18] *Let* $v \equiv 1 \pmod{18}$ *and* $v \geq 19$. *Then the following is a solution to Heffter's first difference problem, where* $v = 18s+1$.

$$\{\{3r+1, 4s-r+1, 4s+2r+2\} \mid 0 \leq r \leq s-1\};$$
$$\{\{3r+2, 8s-r, 8s+2r+2\} \mid 0 \leq r \leq s-1\};$$
$$\{\{3r+3, 6s-2r-1, 6s+r+2\} \mid 0 \leq r \leq s-2\};$$
$$\{\{3s, 3s+1, 6s+1\}\}.$$

Case 2:

Suppose $v \equiv 3$ $(mod\ 18)$ and $v \geq 21$. Say $v = 18s+3$ where $s \geq 1$. Notice that $(3r+1)+(8s-r+1) = (8s+2r+2)$, $(3r+2)+(4s-r) = (4s+2r+2)$, and $(3r+3)+(6s-2r-1) = (6s+r+2)$ so each of the triples given in the first column of the following table is, in fact, a difference triple.

Lemma 3.6. [18] *Let* $v \equiv 3 \pmod{18}$ *and* $v \geq 21$*. Then the following is a solution to Heffter's second difference problem, where* $v = 18s+3$*.*

$$\{\{3r+1, 8s-r+1, 8s+2r+2\} \mid 0 \leq r \leq s-1\};$$
$$\{\{3r+2, 4s-r, 4s+2r+2\} \mid 0 \leq r \leq s-1\};$$
$$\{\{3r+3, 6s-2r-1, 6s+r+2\} \mid 0 \leq r \leq s-1\}.$$

Case 3:

Suppose $v \equiv 7$ $(mod\ 18)$ and $v \geq 25$. Say $v = 18s+7$ where $sge1$. Notice that $(3r+1)+(8s-r+3) = (8s+2r+4)$, $(3r+2)+(6s-2r+1) = (6s+r+3)$, $(3r+3)+(4s-r+1) = (4s+2r+4)$, and $(3s+1)+(4s+2) = (7s+3)$ so each of the triples given in the first column of the following table is a difference triple.

Lemma 3.7. [18] *Let* $v \equiv 7 \pmod{18}$ *and* $v \geq 25$*. Then the following is a solution to*

Heffter's first difference problem, where $v = 18s + 7$.

$$\{\{3r+1, 8s-r+3, 8s+2r+4\} \mid 0 \le r \le s-1\};$$
$$\{\{3r+2, 6s-2r+1, 6s+r+3\} \mid 0 \le r \le s-1\};$$
$$\{\{3r+3, 4s-r+1, 4s+2r+4\} \mid 0 \le r \le s-1\};$$
$$\{\{3s+1, 4s+2, 7s+3\}\}.$$

Case 4:

Suppose $v \equiv 9$ $(mod\ 18)$ and $v \ge 81$. Say $v = 18s+9$ where $s \ge 4$. Notice that $(3r+1)+(4s-r+3) = (4s+2r+4)$, $(3r+2)+(8s-r+2) = (8s+2r+4)$, and $(3r+3)+(6s-2r+1) = (6s+r+4)$, $2+(8s+3) = (8s+5)$, $3+(8s+1) = (8s+4)$, $5+(8s+2) = (8s+7)$, $(3s-1)+(3s+2) = (6s+1)$ and $3s+(7s+3) \equiv -(8s+6) \equiv 10s+3$ $(mod\ 18s+9)$, so each of the triples given in the first column of the following table is, in fact, a difference triple.

Lemma 3.8. [18] *Let $v \equiv 9 \pmod{18}$ and $v \ge 81$. Then the following is a solution to Heffter's second difference problem, where $v = 18s + 9$.*

$$\{\{3r+1, 4s-r+3, 4s+2r+4\} \mid 0 \le r \le s\};$$
$$\{\{3r+2, 8s-r+2, 8s+2r+4\} \mid 0 \le r \le s-2\};$$
$$\{\{3r+3, 6s-2r+1, 6s+r+4\} \mid 0 \le r \le s-2\};$$
$$\{\{2, 8s+3, 8s+5\}, \{3, 8s+1, 8s+4\}, \{5,\ 8s+2, 8s+7\},$$
$$\{3s-1, 3s+2, 6s+1\}, \{3s, 7s+3, 8s+6\}\}.$$

Case 5:

Suppose $v \equiv 13$ $(mod\ 18)$ and $v \ge 31$. Say $v = 18s+13$ where $s \ge 1$. Notice that

$(3r+2)+(6s-2r+3) = (6s+r+5)$, $(3r+3)+(8s-r+5) = (8s+2r+8)$, $(3r+1)+(4s-r+3) = (4s+2r+4)$, and $(3s+2)+(7s+5) \equiv -(8s+6) \equiv 10s+7$ $(mod\ 18s+13)$ so each of the triples given in the first column of the following table is, in fact, a difference triple.

Lemma 3.9. [18] *Let $v \equiv 13 \pmod{18}$ and $v \geq 31$. Then the following is a solution to Heffter's first difference problem, where $v = 18s+13$.*

$$\{\{3r+1, 4s-r+3, 4s+2r+4\} \mid 0 \leq r \leq s\};$$
$$\{\{3r+2, 6s-2r+3, 6s+r+5\} \mid 0 \leq r \leq s-1\};$$
$$\{\{3r+3, 8s-r+5, 8s+2r+8\} \mid 0 \leq r \leq s-1\};$$
$$\{\{3s+2, 7s+5, 8s+6\}\}.$$

Case 6:

Suppose $v \equiv 15$ $(mod\ 18)$ and $v \geq 33$. Say $v = 18s+15$ where $s \geq 1$. Notice that $(3r+1)+(4s-r+3) = (4s+2r+4)$, $(3r+2)+(8s-r+6) = (8s+2r+8)$, and $(3r+3)+(6s-2r+3) = (6s+r+6)$ so each of the triples given in the first column of the following table is, in fact, a difference triple.

Lemma 3.10. [18] *Let $v \equiv 15 \pmod{18}$ and $v \geq 33$. Then the following is a solution to Heffter's second difference problem, where $v = 18s+15$.*

$$\{\{3r+1, 4s-r+3, 4s+2r+4\} \mid 0 \leq r \leq s\};$$
$$\{\{3r+2, 8s-r+6, 8s+2r+8\} \mid 0 \leq r \leq s\};$$
$$\{\{3r+3, 6s-2r+3, 6s+r+6\} \mid 0 \leq r \leq s-1\}.$$

3.1 Solutions from Heffter's first difference problem

Here we give a solution to Theorem 3.3, splitting into the following cases:

Case 1: $v \equiv 1 \pmod{18}$.

Case 2: $v \equiv 7 \pmod{18}$.

Case 3: $v \equiv 13 \pmod{18}$.

Lemma 3.11. *Let $v \equiv 1 \pmod{18}$. Then there exists a set S of $(v-1)/6$ triples such that:*

(1) $\Delta(S)$ *is the set* $\{1, 2, \ldots, (v-1)/2\}$*;*

(2) *The set S is near-regular.*

Proof. Let $v = 18s+1$. First consider the case $s = 1$. Let $S = \{\{1, 6, 7\}, \{0, 2, 9\}, \{3, 6, 10\}\}$. Then $\Delta(S) = \{1, 5, 6\} \cup \{2, 8, 9\} \cup \{3, 4, 7\}$.

Otherwise, $s \geq 2$. Let

$$S_1 = \{\{1-r, 2+2r, 4s+3+r\} \mid 0 \leq r \leq s-1\};$$

$$S_2 = \{\{6s+2-r, 6s+4+2r, 14s+4+r\} \mid 0 \leq r \leq s-1\};$$

$$S_3 = \{\{2s-1-2r, 2s+2+r, 8s+1-r\} \mid 0 \leq r \leq s-2\};$$

$$S_4 = \{\{3s+3, 6s+3, 9s+4\}\}.$$

We define $S = S_1 \cup S_2 \cup S_3 \cup S_4$.

Since S_1, S_2, S_3 and S_4 are pairwise disjoint, Condition (2) is satisfied. This can be more clearly seen in Table 37 in Appendix A.

Then, observe that:

$$\Delta(S_1) = \{\{3r+1, 4s-r+1, 4s+2r+2\} \mid 0 \le r \le s-1\};$$
$$\Delta(S_2) = \{\{3r+2, 8s-r, 8s+2r+2\} \mid 0 \le r \le s-1\};$$
$$\Delta(S_3) = \{\{3r+3, 6s-2r-1, 6s+r+2\} \mid 0 \le r \le s-2\};$$
$$\Delta(S_4) = \{\{3s, 3s+1, 6s+1\}\}.$$

From Lemma 3.5, Condition (1) is also satisfied. □

Lemma 3.12. *Let* $v \equiv 7 \pmod{18}$. *Then there exists a set* S *of* $(v-1)/6$ *triples such that:*

(1) $\Delta(S)$ *is the set* $\{1, 2, \ldots, (v-1)/2\}$*;*

(2) *The set* S *is near-regular.*

Proof. Let $v = 18s+7$. First consider the case $s = 1$. Let $S = \{\{1, 2, 13\}, \{3, 5, 12\}, \{6, 9, 14\}, \{0, 4, 10\}\}$.

Then $\Delta(S) = \{1, 11, 12\} \cup \{2, 7, 9\} \cup \{3, 5, 8\} \cup \{4, 6, 10\}$.

Otherwise, $s \ge 2$.

Let

$$S_1 = \{\{1-r, 2+2r, 8s+5+r\} \mid 0 \le r \le s-1\};$$
$$S_2 = \{\{4s+1-2r, 4s+3+r, 10s+4-r\} \mid 0 \le r \le s-1\};$$
$$S_3 = \{\{11s+4-r, 11s+7+2r, 16s+3+r\} \mid 0 \le r \le s-1\};$$
$$S_4 = \{\{4s+2, 7s+3, 12s\}\}.$$

We define $S = S_1 \cup S_2 \cup S_3 \cup S_4$.

Since S_1, S_2, S_3 and S_4 are pairwise disjoint, Condition (2) is satisfied. This can be more clearly seen in Table 47 in Appendix A.

Then, observe that:

$$\Delta(S_1) = \{\{3r+1, 8s-r+3, 8s+2r+4\} \mid 0 \le r \le s-1\};$$
$$\Delta(S_2) = \{\{3r+2, 6s-2r+1, 6s+r+3\} \mid 0 \le r \le s-1\};$$
$$\Delta(S_3) = \{\{3r+3, 4s-r+1, 4s+2r+4\} \mid 0 \le r \le s-1\};$$
$$\Delta(S_4) = \{\{3s+1, 4s+2, 7s+3\}\}.$$

From Lemma 3.7 , condition (1) is also satisfied. □

Lemma 3.13. *Let* $v \equiv 13 \pmod{18}$. *Then there exists a set* S *of* $(v-1)/6$ *triples such that:*

(1) $\Delta(S)$ *is the set* $\{1, 2, \ldots, (v-1)/2\}$*;*

(2) *The set* S *is near-regular.*

Proof. Let $v = 18s+13$. First consider the case $s = 1$. Let $S = \{1, 8, 9\}, \{0, 4, 10\}, \{3, 5, 14\}, \{6, 9, 22\}, \{7, 12, 24\}$.

Then $\Delta(S) = \{1, 7, 8\} \cup \{4, 6, 10\} \cup \{2, 9, 11\} \cup \{3, 13, 15\} \cup \{5, 12, 14\}$.

Otherwise, $s \ge 2$.

Let

$$S_1 = \{\{1-r, 2+2r, 4s+5+r\} \mid 0 \le r \le s\};$$
$$S_2 = \{\{10s+8-2r, 10s+10+r, 16s+13-r\} \mid 0 \le r \le s-1\};$$
$$S_3 = \{\{6s+5-r, 6s+8+2r, 14s+13+r\} \mid 0 \le r \le s-1\};$$
$$S_4 = \{\{3s+4, 5s+6, 14s+6\}\}.$$

We define $S = S_1 \cup S_2 \cup S_3 \cup S_4$.

Since S_1, S_2, S_3 and S_4 are pairwise disjoint, Condition (2) is satisfied. This can be more clearly seen in Table 57 in Appendix A.

Then, observe that:

$$\Delta(S_1) = \{\{3r+1, 4s-r+3, 4s+2r+4\} \mid 0 \le r \le s\};$$
$$\Delta(S_2) = \{\{3r+2, 6s-2r+3, 6s+r+5\} \mid 0 \le r \le s-1\};$$
$$\Delta(S_3) = \{\{3r+3, 8s-r+5, 8s+2r+8\} \mid 0 \le r \le s-1\};$$
$$\Delta(S_4) = \{\{3s+2, 7s+5, 8s+6\}\}.$$

From Lemma 3.9, Condition (1) is also satisfied. □

3.2 Solutions from Heffter's second difference problem

Here we give a solution to Theorem 3.4, splitting into the following cases:

Case 4: $v \equiv 3 \pmod{18}$.

Case 5: $v \equiv 9 \pmod{18}$.

Case 6: $v \equiv 15 \pmod{18}$.

Lemma 3.14. *Let $v \equiv 3 \pmod{18}$. Then there exists a set S of $(v-3)/6$ triples such that:*

(1) $\Delta(S)$ is the set $\{1, 2, \ldots, (v-1)/2\}$;

(2) The set S is near-regular.

Proof. Let $v = 18s+3$. First consider the case $s = 1$.

Let $S = \{\{1, 2, 11\}, \{4, 6, 10\}, \{0, 3, 8\}, \{5, 12, 19\}\}$. Then $\Delta(S) = \{1, 9, 10\} \cup \{2, 4, 6\} \cup \{3, 5, 8\} \cup \{7\}$.

Otherwise, $s \ge 2$.

Let

$$S_1 = \{\{1-r, 2+2r, 8s+3+r\} \mid 0 \le r \le s-1\};$$

$$S_2 = \{\{5s+1-r, 5s+3+2r, 9s+3+r\} \mid 0 \le r \le s-1\};$$

$$S_3 = \{\{7s-2r, 7s+3+r, 13s+2-r\} \mid 0 \le r \le s-1\};$$

$$S_4 = \{\{8s+3, 14s+4, 20s+5\}\}.$$

We define $S = S_1 \cup S_2 \cup S_3 \cup S_4$.

Since S_1, S_2, S_3 and S_4 are pairwise disjoint, Condition (2) is satisfied. This can be more clearly seen in Table 42 in Appendix A.

Then, observe that:

$$\Delta(S_1) = \{\{3r+1, 8s-r+1, 8s+2r+2\} \mid 0 \le r \le s-1\};$$

$$\Delta(S_2) = \{\{3r+2, 4s-r, 4s+2r+2\} \mid 0 \le r \le s-1\};$$

$$\Delta(S_3) = \{\{3r+3, 6s-2r-1, 6s+r+2\} \mid 0 \le r \le s-1\};$$

$$\Delta(S_4) = \{\{6s+1\}\}.$$

From Lemma 3.6, condition (1) is also satisfied. □

Lemma 3.15. *Let $v \equiv 9 \pmod{18}$ and $v \neq 9$. Then there exists a set S of $(v-3)/6$ triples such that:*

(1) *$\Delta(S)$ is the set $\{1, 2, \ldots, (v-1)/2\}$;*

(2) *The set S is near-regular.*

Proof. Let $v = 18s + 9$.

Do the cases $s = 1$, 2 and 3.

Suppose first that $s = 1$. Let

$$S = \{\{1, 13, 14\}, \{0, 2, 7\}, \{5, 8, 16\}, \{11, 15, 21\}, \{10, 19, 28\}\}.$$

Then

$$\Delta(S) = \{1, 12, 13\} \cup \{2, 5, 7\} \cup \{3, 8, 11\} \cup \{4, 6, 10\} \cup \{9\}.$$

Suppose that $s = 2$. Let

$$S = \{\{1, 12, 13\}, \{0, 2, 19\}, \{3, 6, 26\}, \{4, 8, 18\}, \{10, 15, 23\}, \{11, 17, 35\}, \{9, 16, 25\}, \{7, 22, 37\}\}.$$

Then

$$\Delta(S) = \{1, 11, 12\} \cup \{2, 17, 19\} \cup \{3, 20, 22\} \cup \{4, 10, 14\} \cup \{5, 8, 13\} \cup \{6, 18, 21\}$$
$$\cup \{7, 9, 16\} \cup \{15\}.$$

Suppose that $s = 3$. Let

$$S = \{\{1, 16, 17\}, \{0, 2, 29\}, \{3, 6, 31\}, \{4, 8, 22\}, \{5, 10, 36\}, \{7, 13, 30\}, \{11, 18, 31\}, \{14, 23, 34\},$$
$$\{12, 21, 45\}, \{15, 25, 37\}, \{19, 40, 61\}\}.$$

Then

$$\Delta(S) = \{1, 15, 16\} \cup \{2, 27, 29\} \cup \{3, 25, 28\} \cup \{4, 14, 18\} \cup \{5, 26, 31\} \cup \{6, 17, 23\} \cup \{7, 13, 20\}$$
$$\cup \{8, 11, 19\} \cup \{9, 24, 30\} \cup \{10, 12, 22\} \cup \{21\}.$$

Otherwise, $s \geq 4$.

$S_1 = \{\{1-r, 2+2r, 4s+5+r\} \mid 0 \le r \le s\};$

$S_2 = \{\{5s+12-2r, 8s+2+2r, 16s+2+2r\} \mid 0 \le r \le s-2\};$

$S_3 = \{\{10s+6-2r, 10s+10-r, 16s+11-r\} \mid 0 \le r \le s-2\};$

$S_4 = \{\{3s+4, 5s+6, 14s+6\}\}.$

We define $S = S_1 \cup S_2 \cup S_3 \cup S_4$.

Since S_1, S_2, S_3 and S_4 are pairwise disjoint, Condition (2) is satisfied. This can be more clearly seen in Table 51 in Appendix A.

Then, observe that:

$$\Delta(S_1) = \{\{3r+1, 4s-r+3, 4s+2r+4\} \mid 0 \le r \le s\};$$
$$\Delta(S_2) = \{\{3r+2, 8s-r+2, 8s+2r+4\} \mid 0 \le r \le s-2\};$$
$$\Delta(S_3) = \{\{3r+3, 6s-2r+1, 6s+r+4\} \mid 0 \le r \le s-2\};$$
$$\Delta(S_4) = \{\{3s, 7s+3, 8s+6\}\}.$$

From Lemma 3.8, Condition (1) is also satisfied. □

Lemma 3.16. *Let* $v \equiv 15 \pmod{18}$. *Then there exists a set* S *of* $(v-3)/6$ *triples such that:*

(1) $\Delta(S)$ *is the set* $\{1, 2, \ldots, (v-1)/2\}$*;*

(2) *The set* S *is near-regular.*

Proof. Let $v = 18s + 15$.

First consider the case $s = 1$. Let $S = \{\{1,2,9\}, \{0,4,10\}, \{3,5,19\}, \{6,11,24\}, \{9,12,21\}, \{7,18,29\}\}$.

Then $\Delta(S) = \{1,7,8\} \cup \{4,6,10\} \cup \{2,14,16\} \cup \{5,13,15\} \cup \{3,9,12\} \cup \{11\}$.

Otherwise, $s \ge 2$.

Let

$$S_1 = \{\{1 - r, 2 + 2r, 4s + 5 + r\} \mid 0 \le r \le s\};$$

$$S_2 = \{\{6s + 6 - r, 6s + 8 + 2r, 14s + 14 + r\} \mid 0 \le r \le s\};$$

$$S_3 = \{\{10s + 8 - 2r, 10s + 11 + r, 16s + 14 - r\} \mid 0 \le r \le s - 1\};$$

$$S_4 = \{\{6s + 7, 12s + 12, 18s + 13\}\}.$$

We define $S = S_1 \cup S_2 \cup S_3 \cup S_4$.

Since S_1, S_2, S_3 and S_4 are pairwise disjoint, Condition (2) is satisfied. This can be more clearly seen in Table 62 in Appendix A.

Then, observe that:

$$\Delta(S_1) = \{\{3r + 1, 4s - r + 3, 4s + 2r + 4\} \mid 0 \le r \le s\};$$

$$\Delta(S_2) = \{\{3r + 2, 8s - r + 6, 8s + 2r + 8\} \mid 0 \le r \le s\};$$

$$\Delta(S_3) = \{\{3r + 3, 6s - 2r + 3, 6s + r + 6\} \mid 0 \le r \le s - 1\};$$

$$\Delta(S_4) = \{\{6s + 5\}\}.$$

From Lemma 3.10, Condition (1) is also satisfied. □

4 Efficiency Measures for Block Designs

In this section we consider efficiency measures used in experimental design. We first define the eigenvalue and eigenvector of a matrix. Let A be a $n \times n$ matrix and let $\mathbf{v}$ be an $n \times 1$ vector. Then the the scale factor λ is called an **eigenvalue** of A and v is an **eigenvector** of A which corresponding to that eigenvalue λ. The equation $Av = \lambda v$ is called the eigenvalue equation for the matrix A.

Example

$$A = \begin{pmatrix} 1 & -\frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & 1 & -\frac{1}{2} \\ -\frac{1}{2} & -\frac{1}{2} & 1 \end{pmatrix}, \; v = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

we have

$$Av = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

then scale $\lambda_1 = 0$ is the eigenvalue of A and the

$$v = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

is an eigenvector of A which corresponding to that eigenvalue $\lambda_1 = 0$.

The $\lambda_2 = \lambda_3 = \frac{3}{2}$ are the other eigenvalues of A, and

$$v_2 = \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}, \; v_3 = \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix}$$

are eigenvectors of A those corresponding to that eigenvalues $\lambda_2 = \lambda_3 = \frac{3}{2}$.

The following matrix is an important measure of the efficiency of a block design in experimental design [11].

Lemma 4.1. *The* **scaled information matrix** *of a concurrence matrix* Λ *for a block design with parameters* (v, b, r, k) *is*

$$\mathrm{F}(\Lambda) = \mathrm{I_v} - (rk)^{-1}\Lambda$$

where I_v *denotes the* $v \times v$ *identity matrix.*

The eigenvalues of $F(\Lambda)$ are all real and lie strictly between 0 and 1 [11]. The remaining eigenvalues (counting repeats) are called the **canonical efficiency factors** of Λ.

Example

Table 32: Example of a $(6, 9, 3, 2)$-design

1,2	4,5	3,6
4,6	1,3	2,5
3,5	2,6	1,4

The concurrence matrix of (6,9,3,2)-design

$$\Lambda = \begin{pmatrix} 3 & 1 & 1 & 1 & 0 & 0 \\ 1 & 3 & 0 & 0 & 1 & 1 \\ 1 & 0 & 3 & 0 & 1 & 1 \\ 1 & 0 & 0 & 3 & 1 & 1 \\ 0 & 1 & 1 & 1 & 3 & 0 \\ 0 & 1 & 1 & 1 & 0 & 3 \end{pmatrix}$$

Then the scaled information matrix of Λ is

$$\mathrm{F}(\Lambda) = \mathrm{I_v} - (rk)^{-1}\Lambda = \begin{pmatrix} \frac{1}{2} & -\frac{1}{6} & -\frac{1}{6} & -\frac{1}{6} & 0 & 0 \\ -\frac{1}{6} & \frac{1}{2} & 0 & 0 & -\frac{1}{6} & -\frac{1}{6} \\ -\frac{1}{6} & 0 & \frac{1}{2} & 0 & -\frac{1}{6} & -\frac{1}{6} \\ -\frac{1}{6} & 0 & 0 & \frac{1}{2} & -\frac{1}{6} & -\frac{1}{6} \\ 0 & -\frac{1}{6} & -\frac{1}{6} & -\frac{1}{6} & \frac{1}{2} & 0 \\ 0 & -\frac{1}{6} & -\frac{1}{6} & -\frac{1}{6} & 0 & \frac{1}{2} \end{pmatrix}$$

then scale $\lambda_1 = 0$ is the eigenvalue of A and the

$$v_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$$

is an eigenvector of A which corresponding to that eigenvalue $\lambda_1 = 0$.

The $\lambda_2 = \lambda_3 = \lambda_4 = \lambda_5 = \frac{1}{2}$ and $\lambda_6 = 1$ are the other eigenvalues of A.

Theorem 4.2. [1] *Let* Δ *be the concurrence matrix for a* (nk, n^2, n, k)*-design, then* 0 *is an eigenvalue for* $F(\Delta)$ *with eigenvector* $v = \underbrace{(1, 1, \ldots, 1, 1)}_{n}$.

Proof. Consider a concurrence matrix Δ for a (nk, n^2, n, k)-design. Since $v = nk$, there are nk rows by definition. Since $r = n$, the elements on the main diagonal of the matrix are all n. And y appears with $(k-1)$ times each row. Then we can see the product of $(nk)^{-1}\Delta = 1$ and $r = n$. Hence, $(rk)^{-1}\Delta = 1$. And it is easy to deduce that the difference between the identity matrix and it is 0. That is $I - (rk)^{-1}\Delta = 0$. □

Let I be the information matrix of a block design (v, b, r, k).

$$C(I) = I - (nk)^{-1} N N^T.$$

Let $\delta_0 \leq \delta_1 \leq \cdots \leq \delta_{v-1}$ be the eigenvalues of $C(I)$.
We say that I is **Schur-optimal** if for any block design Γ with the same parameters as I, if $\gamma_0 \leq \gamma_1 \leq \cdots \leq \gamma_{v-1}$ be the eigenvalues of $C(\Gamma)$, then

$$\sum_{i=0}^{l} \delta_i \geq \sum_{i=0}^{l} \gamma_i, \text{ for } l = 0, 1, \ldots, v-1.$$

The follows definition come from [24]. If the harmonic mean of the canonical efficiency factors of a design is at least as large as that of any other design with the same value of t, b, r, k, then the design is said to be **A-optimal**.

The harmonic mean H of the positive real numbers $x_1, x_2, \cdots, x_n$ is defined as :

$$H(x_1, x_2, \ldots, x_n) = \frac{n}{\frac{1}{x_1} + \frac{1}{x_2} + \cdots + \frac{1}{x_n}} = \frac{n}{\sum_{i=1}^{n} \frac{1}{x_i}}.$$

A design whose geometric mean of the canonical efficiency factors is at least as large as that of any other design with the same values of t, b, r, k is said to be **D-optimal**.

The geometric mean is a average that indicates the central tendency of a set of finite real values by using their product as a product, the geometric mean is defined as the nth root of the product of n numbers. e.g. For a set of numbers $a_1, a_2, \cdots, a_n$, the geometric mean G is defined :

$$G = \sqrt[n]{a_1 a_2 \cdots a_n} = (\prod_{i=1}^{n} a_i)^{\frac{1}{n}}.$$

A design whose smallest canonical efficiency factor is at least as large as that of any other design with the same values of t, b, r, k is said to be **E-optimal**.

Let I be a scaled information matrix. Then $A(I)$ is the harmonic mean of the eigenvalues of the information matrix I, $D(I)$ is the geometric mean of the eigenvalues of the information matrix I and $E(I)$ is the minimum eigenvalue of the information matrix I.

Example

Consider the following semi-Latin square.

1,2	4,5	3,6
4,6	1,3	2,5
3,5	2,6	1,4

The blocks in the underlying block design are given by: $X_1 = \{1, 2\}, X_2 = \{4, 5\}, X_3 =$

$\{3,6\}, X_4 = \{4,6\}, X_5 = \{1,3\}, X_6 = \{2,5\}, X_7 = \{3,5\}, X_8 = \{2,6\}, X_9 = \{1,4\}$.

Then the canonical efficiency factors of design are the eigenvalues of the information matrix I as follow by using matlab :

$$\text{Eigenvalues of } I = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 3 \\ 3 \\ 3 \\ 3 \\ 6 \end{pmatrix}$$

$$\begin{aligned} A(I) &= \frac{9}{\frac{1}{3}+\frac{1}{3}+\frac{1}{3}+\frac{1}{3}+\frac{1}{6}} \approx 0.1667 \\ D(I) &= \sqrt[9]{3\times 3\times 3\times 3\times 6} \approx 1.985 \\ E(I) &= \min\ (3,3,3,3,6) = 3 \end{aligned}$$

5 Trojan semi-Latin squares

A **Trojan semi-Latin square** of order n and index k is a $(n \times n)/k$ semi-Latin square formed by overlapping a set of mutually orthogonal Latin squares of order n, where each Latin square is on a distinct set of symbols.

In this section we study Trojan semi-Latin squares and construct Semi-Latin Squares which are not Trojan. We also measure efficiency for Trojan semi-Latin squares.

Table 33: Trojan Semi-Latin Square of Order 5, Index 2

1 B	2 C	3 D	4 E	5 A
5 C	1 D	2 E	3 A	4 B
4 D	5 E	1 A	2 B	3 C
3 E	4 A	5 B	1 C	2 D
2 A	3 B	4 C	5 D	1 E

5.1 An equivalence relation for Trojan semi-Latin squares

In this subsection we show that a uniform $(n \times n)/(n-1)$ semi-Latin square is Trojan.

Lemma 5.1. [1] *Let S be a uniform $(n \times n)/(n-1)$ semi-Latin square. Let R be a relation on the elements of S, such that xRy if and only if the pair (x, y) does not occur in a cell. Then R is an equivalence relation on the set of elements, where each equivalence class is size n.*

Proof. From Lemma 1.3, $\mu(S) = 1$.

Let x be any element. Obviously the pair (x, x) does not occur in the same block, so xRx, reflexivity is satisfied.

Let x, y be two distinct elements. If xRy, then the pair of (x, y) does not occur in the same block. Then (y, x) also does not occur in a block. So we obtain yRx, symmetry is satisfied.

Let x, y, z be distinct elements and xRy, yRz. Then pairs $(x, y), (y, z)$ are not in the same block. Assume (x, z) is in the same block. Because the concurrence of each pair less than 1, each cell of X (the square which removed the blocks in the first row and column) contains one element from $1, 2, 3, \ldots, n-1$. Now y occurs in $n-2$ rows or columns of X. So y is paired with $n-2$ elements, therefore y is paired with at least one of x, z. This contradicts the premise, so (x, z) not in the same block, thus xRz. Hence transitivity is satisfied

Therefore R is an equivalence relation.

The number of elements is equal to $n \times (n-1)$, and each element is involved with $n \times (n-2)$ pairs. So element 1 is related to $n(n-1) - n(n-2) - 1 = n-1$ other elements, but 1 is not related to itself. Therefore each equivalence class has size n. □

Example: The relation R yields the following $(5 \times 5)/4$ semi-Latin square. The equivalence classes are $\{1, 2, 3, 4, 5\}$, $\{A, B, C, D, E\}$, $\{a, b, c, d, e\}$, $\{1, 2, 3, \ldots, n-1\}$, $\{\alpha, \beta, \gamma, \mu, \theta\}$.

2 D e α	3 E a β	4 A b γ	0 B c μ	1 C d θ
3 A c θ	4 B d α	0 C e β	1 D a γ	2 E b μ
4 C a μ	0 D b θ	1 E c α	2 A d β	3 B e γ
0 E d γ	1 A e μ	2 B a θ	3 C b α	4 D c β
1 B b β	2 C c γ	3 D d μ	4 E e θ	0 A a α

Theorem 5.2. *Let S be a uniform $(n \times n)/(n-1)$ semi-Latin square. Then S is a Trojan semi-Latin square.*

Proof. Equivalence classes make the Latin squares $L_1, \ldots, L_{n-1}$, and each equivalence class has size n. As $\mu(s) = 1$, therefore there is no pairs repeated. Thus each pair of Latin squares is orthogonal. Then $L_1, \ldots, L_{n-1}$ are mutually orthogonal. So S is a Trojan semi-Latin square. □

5.2 Semi-Latin Squares which are not Trojan

In this subsection we establish the existence of semi-Latin squares which are not Trojan.

Table 34: Trojan Semi-Latin Square of Order 7, Index 2

1 B	2 C	3 D	4 E	5 F	6 G	7 A
7 C	1 D	2 E	3 F	4 G	5 A	6 B
6 D	7 E	1 F	2 G	3 A	4 B	5 C
5 E	6 F	7 G	1 A	2 B	3 C	4 D
4 F	5 G	6 A	7 B	1 C	2 D	3 E
3 G	4 A	5 B	6 C	7 D	1 E	2 F
2 A	3 B	4 C	5 D	6 E	7 F	1 G

The relation R on the transformed Trojan semi-Latin square (Table 35) is not an equivalence relation. This is because 1 and D do not occur in the same block and then A and D also do not occur in the same block. But 1 and A occur in the same block in the Latin square. Therefore it is not satisfied the transitivity. So it is not an equivalence relation.

Table 35: Simple but not Trojan Semi-Latin Square of Order 7, Index 2

C B	1 2	3 D	4 E	5 F	6 G	7 A
1 7	C D	2 E	3 F	4 G	5 A	6 B
6 D	7 E	1 F	2 G	3 A	4 B	5 C
5 E	6 F	7 G	1 A	2 B	3 C	4 D
4 F	5 G	6 A	7 B	1 C	2 D	3 E
3 G	4 A	5 B	6 C	7 D	1 E	2 F
2 A	3 B	4 C	5 D	6 E	7 F	1 G

Theorem 5.3. *For each odd $n \geq 5$, there exists simple $(n \times n)/2$ semi -Latin square that is not Trojan.*

Proof. We can overlap two Latin squares with numbers and roman letters to get a Trojan semi-Latin square. According to Theorem 3.8, we can easily find that when $n = 5$, we can successfully overlap the semi-Latin square using L_4 and L_1, then if we change the number with the roman letters in the first four blocks, we obtain a simple semi -Latin square but not Trojan. When $n = 7$, we can also use L_6 and L_1 to overlap a Trojan semi-Latin square, then change the number with the English letters in the first four blocks, we also get a simple semi -Latin square but not Trojan. In general, we can find that when $n \geq 5$ and is an odd number, we can overlap L_{n-1} and L_1 to get a Trojan semi-Latin square, then change the number with the roman letters in the first four blocks. Therefore first square has the same element in cells $(1, 1)$ and $(2, 2)$. The second square has the same element in cells $(1, 2)$ and $(2, 1)$. So there always exists simple semi -Latin square that is not Trojan. □

5.3 Efficiency measures for Trojan semi-Latin squares

As the following lemma indicates, Trojan semi-Latin squares achieve all the desirable properties of optimality.

Lemma 5.4. [21] *Every Trojan square is A-optimal, D-optimal, and E-optimal.*

We demonstrate this in examples below, by comparing optimality statistics for Trojan and non-Trojan semi-Latin squares.

Below is the concurrence matrix Λ_1 of the Trojan semi-Latin square given in Table 33 above.

$$\Lambda_1 = \begin{pmatrix} 5 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 \\ 0 & 5 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 5 & 0 & 0 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 5 & 0 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 5 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 5 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 & 0 & 5 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 & 0 & 0 & 5 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 5 & 0 \\ 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 5 \end{pmatrix}$$

Then the scaled information matrix of Λ_1 is

$$F(\Lambda_1) = I_v - (rk)^{-1}\Lambda_1 = \begin{pmatrix} \frac{1}{2} & 0 & 0 & 0 & 0 & -\frac{1}{10} & -\frac{1}{10} & -\frac{1}{10} & -\frac{1}{10} & -\frac{1}{10} \\ 0 & \frac{1}{2} & 0 & 0 & 0 & -\frac{1}{10} & -\frac{1}{10} & -\frac{1}{10} & -\frac{1}{10} & -\frac{1}{10} \\ 0 & 0 & \frac{1}{2} & 0 & 0 & -\frac{1}{10} & -\frac{1}{10} & -\frac{1}{10} & -\frac{1}{10} & -\frac{1}{10} \\ 0 & 0 & 0 & \frac{1}{2} & 0 & -\frac{1}{10} & -\frac{1}{10} & -\frac{1}{10} & -\frac{1}{10} & -\frac{1}{10} \\ 0 & 0 & 0 & 0 & \frac{1}{2} & -\frac{1}{10} & -\frac{1}{10} & -\frac{1}{10} & -\frac{1}{10} & -\frac{1}{10} \\ -\frac{1}{10} & -\frac{1}{10} & -\frac{1}{10} & -\frac{1}{10} & -\frac{1}{10} & \frac{1}{2} & 0 & 0 & 0 & 0 \\ -\frac{1}{10} & -\frac{1}{10} & -\frac{1}{10} & -\frac{1}{10} & -\frac{1}{10} & 0 & \frac{1}{2} & 0 & 0 & 0 \\ -\frac{1}{10} & -\frac{1}{10} & -\frac{1}{10} & -\frac{1}{10} & -\frac{1}{10} & 0 & 0 & \frac{1}{2} & 0 & 0 \\ -\frac{1}{10} & -\frac{1}{10} & -\frac{1}{10} & -\frac{1}{10} & -\frac{1}{10} & 0 & 0 & 0 & \frac{1}{2} & 0 \\ -\frac{1}{10} & -\frac{1}{10} & -\frac{1}{10} & -\frac{1}{10} & -\frac{1}{10} & 0 & 0 & 0 & 0 & \frac{1}{2} \end{pmatrix}$$

By using matlab, then scale $\lambda_1 = 0$ is the eigenvalue of the A and the

$$v_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$$

is an eigenvector of A which corresponding to that eigenvalue $\lambda_1 = 0$.

The $\lambda_2 = \lambda_3 = \lambda_4 = \lambda_5 = \lambda_6 = \lambda_7 = \lambda_8 = \lambda_9 = \frac{1}{2}$ and $\lambda_{10} = 1$ are the other eigenvalues of the matrix A.

Next, we compare with a non-Trojan semi-Latin square with the same paramaters.

Table 36: Simple but not trojan Semi-Latin Square of Order 5, Index 2

C B	1 2	3 D	4 E	5 A
1 5	C D	2 E	3 A	4 B
4 D	5 E	1 A	2 B	3 C
3 E	4 A	5 B	1 C	2 D
2 A	3 B	4 C	5 D	1 E

The concurrence matrix Λ_2 of this transformed Semi-Latin Square is given below.

$$\Lambda_2 = \begin{pmatrix} 5 & 1 & 0 & 0 & 1 & 1 & 0 & 1 & 0 & 1 \\ 1 & 5 & 0 & 0 & 0 & 1 & 1 & 0 & 1 & 1 \\ 0 & 0 & 5 & 0 & 0 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 5 & 0 & 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 & 5 & 1 & 1 & 0 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 5 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 1 & 0 & 5 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 & 0 & 1 & 5 & 1 & 0 \\ 0 & 1 & 1 & 1 & 1 & 0 & 0 & 1 & 5 & 0 \\ 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 5 \end{pmatrix}$$

Then the scaled information matrix of Λ_2 is

$$\mathrm{F}(\Lambda_2) = \mathrm{I_v} - (rk)^{-1}\Lambda_2 = \begin{pmatrix} \frac{1}{2} & -\frac{1}{10} & 0 & 0 & -\frac{1}{10} & -\frac{1}{10} & 0 & -\frac{1}{10} & 0 & -\frac{1}{10} \\ -\frac{1}{10} & \frac{1}{2} & 0 & 0 & 0 & -\frac{1}{10} & -\frac{1}{10} & 0 & -\frac{1}{10} & -\frac{1}{10} \\ 0 & 0 & \frac{1}{2} & 0 & 0 & -\frac{1}{10} & -\frac{1}{10} & -\frac{1}{10} & -\frac{1}{10} & -\frac{1}{10} \\ 0 & 0 & 0 & \frac{1}{2} & 0 & -\frac{1}{10} & -\frac{1}{10} & -\frac{1}{10} & -\frac{1}{10} & -\frac{1}{10} \\ -\frac{1}{10} & 0 & 0 & 0 & \frac{1}{2} & -\frac{1}{10} & -\frac{1}{10} & 0 & -\frac{1}{10} & -\frac{1}{10} \\ -\frac{1}{10} & -\frac{1}{10} & -\frac{1}{10} & -\frac{1}{10} & -\frac{1}{10} & \frac{1}{2} & 0 & 0 & 0 & 0 \\ 0 & -\frac{1}{10} & -\frac{1}{10} & -\frac{1}{10} & -\frac{1}{10} & 0 & \frac{1}{2} & -\frac{1}{10} & 0 & 0 \\ -\frac{1}{10} & 0 & -\frac{1}{10} & -\frac{1}{10} & 0 & 0 & -\frac{1}{10} & \frac{1}{2} & -\frac{1}{10} & 0 \\ 0 & -\frac{1}{10} & -\frac{1}{10} & -\frac{1}{10} & -\frac{1}{10} & 0 & 0 & -\frac{1}{10} & \frac{1}{2} & 0 \\ -\frac{1}{10} & -\frac{1}{10} & -\frac{1}{10} & -\frac{1}{10} & -\frac{1}{10} & 0 & 0 & 0 & 0 & \frac{1}{2} \end{pmatrix}$$

By using matlab, then $\lambda_1 = 0$ is the eigenvalue of A and the

$$v_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$$

is an eigenvector of A which corresponding to that eigenvalue $\lambda_1 = 0$.

The $\lambda_2 = 0.3438$ and $\lambda_3 = \lambda_4 = \lambda_5 = \lambda_6 = \lambda_7 = \lambda_8 = \lambda_9 = \frac{1}{2}$ and $\lambda_{10} = 0.9$ are the other eigenvalues of the A.

$$\text{Eigenvalues of } \Lambda_1 = \begin{pmatrix} 0 \\ \frac{1}{2} \\ \frac{1}{2} \\ \frac{1}{2} \\ \frac{1}{2} \\ \frac{1}{2} \\ \frac{1}{2} \\ \frac{1}{2} \\ \frac{1}{2} \\ 1 \end{pmatrix}$$

$$\begin{aligned} A(\Delta_1) &= \frac{10}{2+2+2+2+2+2+2+2+1} \approx 0.5882 \\ D(\Delta_1) &= \sqrt[10]{\frac{1}{2} \times \frac{1}{2} \times \frac{1}{2} \times \frac{1}{2} \times \frac{1}{2} \times \frac{1}{2} \times \frac{1}{2} \times \frac{1}{2} \times 1} \approx 0.5743 \\ E(\Delta_1) &= \min\,(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, 1) = \frac{1}{2} \end{aligned}$$

$$\text{Eigenvalues of } \Lambda_2 = \begin{pmatrix} 0 \\ 0.3438 \\ \frac{1}{2} \\ \frac{1}{2} \\ \frac{1}{2} \\ \frac{1}{2} \\ \frac{1}{2} \\ \frac{1}{2} \\ \frac{1}{2} \\ 0.9 \end{pmatrix}$$

$$A(\Lambda_2) \ = \frac{10}{2.9090 + 2 + 2 + 2 + 2 + 2 + 2 + 2 + 1.1111} \approx 0.5549$$

$$D(\Lambda_2) \ = \sqrt[10]{0.3438 \times \frac{1}{2} \times \frac{1}{2} \times \frac{1}{2} \times \frac{1}{2} \times \frac{1}{2} \times \frac{1}{2} \times \frac{1}{2} \times 0.9} \approx 0.5474$$

$$E(\Lambda_2) \ = \min\ (0.3438, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, 0.9) = 0.3438$$

$$\Rightarrow A(\Lambda_1) > A(\Lambda_2),\ D(\Lambda_1) > D(\Lambda_2),\ E(\Lambda_1) > E(\Lambda_2).$$

5.4 Efficiency of Trojan Matrices

In this section, we explore in a deeper fashion the theory behind Lemma 5.4.

In particular, we define Λ_n to be the concurrence matrix for an $(n \times n)/(n-1)$ Trojan semi-Latin square. The aim of this section is to prove Theorem 5.10.

By Theorem 5.2, we may infer that the the $(3 \times 3)/2$ Trojan semi-Latin square formed by matrices $2I$ on the main diagonal and matrices N_3 on other places.

This is the concurrence matrix for a $(3 \times 3)/2$ Trojan semi-Latin square:

$$\Lambda_3 = \begin{pmatrix} 2 & 0 & 0 & 0 & 1 & 1 & 0 & 1 & 1 \\ 0 & 2 & 0 & 1 & 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 2 & 1 & 1 & 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 2 & 0 & 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 & 2 & 0 & 1 & 0 & 1 \\ 1 & 1 & 0 & 0 & 0 & 2 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 & 1 & 1 & 2 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 & 1 & 0 & 2 & 0 \\ 1 & 1 & 0 & 1 & 1 & 0 & 0 & 0 & 2 \end{pmatrix}$$

$$= \begin{pmatrix} 2I & N_3 & N_3 \\ N_3 & 2I & N_3 \\ N_3 & N_3 & 2I \end{pmatrix}$$

Next, we find the concurrence matrix for a $(4 \times 4)/3$ Trojan semi-Latin square:

$$\Lambda_4 = \begin{pmatrix} 3&0&0&0&0&1&1&1&0&1&1&1&0&1&1&1 \\ 0&3&0&0&1&0&1&1&1&0&1&1&1&0&1&1 \\ 0&0&3&0&1&1&0&1&1&1&0&1&1&1&0&1 \\ 0&0&0&3&1&1&1&0&1&1&1&0&1&1&1&0 \\ 0&1&1&1&3&0&0&0&0&1&1&1&0&1&1&1 \\ 1&0&1&1&0&3&0&0&1&0&1&1&1&0&1&1 \\ 1&1&0&1&0&0&3&0&1&1&0&1&1&1&0&1 \\ 1&1&1&0&0&0&0&3&1&1&1&0&1&1&1&0 \\ 0&1&1&1&0&1&1&1&3&0&0&0&0&1&1&1 \\ 1&0&1&1&1&0&1&1&0&3&0&0&1&0&1&1 \\ 1&1&0&1&1&1&0&1&0&0&3&0&1&1&0&1 \\ 1&1&1&0&1&1&1&0&0&0&0&3&1&1&1&0 \\ 0&1&1&1&0&1&1&1&0&1&1&1&3&0&0&0 \\ 1&0&1&1&1&0&1&1&1&0&1&1&0&3&0&0 \\ 1&1&0&1&1&1&0&1&1&1&0&1&0&0&3&0 \\ 1&1&1&0&1&1&1&0&1&1&1&0&0&0&0&3 \end{pmatrix}$$

$$= \begin{pmatrix} 3I & N_4 & N_4 & N_4 \\ N_4 & 3I & N_4 & N_4 \\ N_4 & N_4 & 3I & N_4 \\ N_4 & N_4 & N_4 & 3I \end{pmatrix}$$

Theorem 5.5. *The concurrence matrix for an $(n \times n)/(n-1)$ Trojan semi-Latin square is* *given by:*

$$\Lambda_n = \begin{pmatrix} (n-1)I & N_n & N_n & N_n & \cdots & N_n \\ N_n & (n-1)I & N_n & N_n & \cdots & N_n \\ N_n & N_n & (n-1)I & N_n & \cdots & N_n \\ N_n & N_n & N_n & (n-1)I & \cdots & N_n \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ N_n & N_n & N_n & N_n & \cdots & (n-1)I \end{pmatrix}$$

The rest of this subsection is devoted to finding the eigenvalues of Λ_n, as in [7]. To this end, we define M_K and N_k as follows : Let M_k be a $k \times k$ matrix such that the first row and column are all 1. Start from the second row and second column, the elements on the main diagonal are negative numbers in the order $-1, -2, \ldots, -(k-1)$, and all other elements above the diagonal are 0, and all elements below are 1.

Let N_k be a $k \times k$ matrix such that the elements on the main diagonal are all 0, and the other elements are all 1.

$$M_k = \begin{pmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & -1 & 0 & \cdots & 0 \\ 1 & 1 & -2 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & 1 & \cdots & -(k-1) \end{pmatrix} \quad N_k = \begin{pmatrix} 0 & 1 & 1 & \cdots & 1 \\ 1 & 0 & 1 & \cdots & 1 \\ 1 & 1 & 0 & \cdots & 1 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & 1 & \cdots & 0 \end{pmatrix}$$

Let

$$M_2 = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \quad N_2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$$\Rightarrow M_2 \times N_2 = \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}$$

Let

$$M_3 = \begin{pmatrix} 1 & 1 & 1 \\ 1 & -1 & 0 \\ 1 & 1 & -2 \end{pmatrix} \quad N_3 = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}$$

$$\Rightarrow M_3 \times N_3 = \begin{pmatrix} 2 & 2 & 2 \\ -1 & 1 & 0 \\ -1 & -1 & 2 \end{pmatrix}$$

Let

$$M_4 = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 0 & 0 \\ 1 & 1 & -2 & 0 \\ 1 & 1 & 1 & -3 \end{pmatrix} \quad N_4 = \begin{pmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{pmatrix}$$

$$\Rightarrow M_4 \times N_4 = \begin{pmatrix} 3 & 3 & 3 & 3 \\ -1 & 1 & 0 & 0 \\ -1 & -1 & 2 & 0 \\ -1 & -1 & -1 & 3 \end{pmatrix}$$

$\Rightarrow M_k \times N_k$ will look like :

$$M_k \times N_k = \begin{pmatrix} k-1 & k-1 & k-1 & k-1 & \cdots & k-1 \\ -1 & 1 & 0 & 0 & \cdots & 0 \\ -1 & -1 & 2 & 0 & \cdots & 0 \\ -1 & -1 & -1 & 3 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ -1 & -1 & -1 & -1 & \cdots & k-1 \end{pmatrix}$$

Lemma 5.6. *The matrix $M_k \times N_k$ has the following form: all elements in the first row are $k-1$. The elements on the main diagonal are $k-1, 1, 2, 3, \ldots, k-1$. All other elements above the main diagonal are all 0, and the elements below are all -1.*

Proof. Let $B_k = M_k \times N_k$. Cell $(1,1)$ of A_k is the dot product of the first row of M_k and the first column of N_k. Since the elements in first row of M_k are $(1,1,1,\ldots,1)$ and the elements in the first column of N_k are $(0,1,1,\ldots,1)$. Thus, the entry in cell $(1,1)$ of A_k is $k-1$.
Then we consider cell (i,j) in different cases:
Case 1: cell (i,j) of A_k where $i<j$. The dot product of i^{th} row of M_k with the j^{th} column of N_k is $(1,1,\ldots,1,-i,0,\ldots,0)\cdot(1,1,\ldots,1,0,1,\ldots,1)$, where $-i$ in the $(i+1)^{th}$ position of first vector. This is equal to the sum of the entries in the i^{th} row of M_k which is zero.
Case 2: cell (i,j) of A_k where $i=j$. The dot product of i^{th} row of M_k with the j^{th} column of N_k is $(1,1,\ldots,1,-i,0,\ldots,0)\cdot(1,1,\ldots,1,0,1,\ldots,1)$, where there are i numbers of the sum of 1 equal to i, and adding others are zeros. Thus, the number of main diagonal is equal to $i-1$ for $i \geq 2$.
Case 3: cell (i,j) of A_k where $i>j$. The dot product of i^{th} row of M_k with the j^{th} column of N_k is $(1,1,\ldots,1,-i,0,\ldots,0)\cdot(1,1,\ldots,1,0,1,\ldots,1)$, where $-i$ is in the $(j+1)^{th}$ position

of first vector. We obtain the sum of the entries in the j^{th} row of M_k which is -1. Therefore, the matrix $M_k \times N_k$ has the form above. $\square$

Next, consider $M_k \times N_k \times M_k^T$:

$$(M_2 \times N_2) \times M_2^T = \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} = \begin{pmatrix} 2 & 0 \\ 0 & -2 \end{pmatrix}$$

$$(M_3 \times N_3) \times M_3^T = \begin{pmatrix} 2 & 2 & 2 \\ -1 & 1 & 0 \\ -1 & -1 & 2 \end{pmatrix} \begin{pmatrix} 1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 0 & -2 \end{pmatrix} = \begin{pmatrix} 6 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -6 \end{pmatrix}$$

$$(M_4 \times N_4) \times M_4^T = \begin{pmatrix} 3 & 3 & 3 & 3 \\ -1 & 1 & 0 & 0 \\ -1 & -1 & 2 & 0 \\ -1 & -1 & -1 & 3 \end{pmatrix} \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & 1 \\ 1 & 0 & -2 & 1 \\ 1 & 0 & 0 & -3 \end{pmatrix}$$
$$= \begin{pmatrix} 12 & 0 & 0 & 0 \\ 0 & -2 & 0 & 0 \\ 0 & 0 & -6 & 0 \\ 0 & 0 & 0 & -12 \end{pmatrix}$$

In general,

$$\Rightarrow (M_k \times N_k) \times M_k^T = \begin{pmatrix} k(k-1) & 0 & 0 & 0 & \cdots & 0 \\ 0 & -1\times 2 & 0 & 0 & \cdots & 0 \\ 0 & 0 & -2\times 3 & 0 & \cdots & 0 \\ 0 & 0 & 0 & -3\times 4 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & -(k-1)k \end{pmatrix}$$

Lemma 5.7. *$M_k \times N_k \times M_k^T$ is a diagonal matrix such that cell $(1,1)$ contains $k(k-1)$ and cell (i,i) contains $-1\times 2,\ -2\times 3,\ \ldots,\ -k(k-1)$, for each $i \geq 2$. Then all other elements are* 0.

Proof. Let $A_k = (M_k \times N_k) \times M_k^T$. This is the dot product: Cell $(1,1)$ of A_k is the dot product of first row of $M_k \times N_k$ and first column of M_k^T. From Lemma 6.7, all the elements in the first row of $M_k \times N_k$ are $(k-1, k-1, \ldots, k-1)$. The elements of the first column of M_k^T are $(1,1,1,\ldots,1)$. Thus, the entry in cell $(1,1)$ of A_k is $k(k-1)$.

Then we consider cell (i,j) in different cases:

Case 1: cell (i,j) of A_k where $i < j$. The dot product of i^{th} row of $(M_k \times N_k)$ with the j^{th} column of M_k^T is $(-1,-1,\ldots,-1,i,0,\ldots,0)\cdot(1,1,\ldots,1,-j,0,\ldots,0)$, where i is in the $(i+1)^{th}$ position of first vector. This is equal to the sum of the entries in the i^{th} row of $(M_k \times N_k)$ which is zero.

Case 2: cell (i,j) of A_k where $i = j$. The dot product of i^{th} row of $(M_k \times N_k)$ with the j^{th} column of M_k^T is $(-1,-1,\ldots,-1,i,0,\ldots,0)\cdot(1,1,\ldots,1,-j,0,\ldots,0)$, where there are i numbers of the sum of -1 equal to $-i$, and adding $i \times -j$ equal to $-i^2$. Thus, the number of main diagonal is equal to $-i \times (i+1)$.

Case 3: cell (i, j) of A_k where $i > j$. The dot product of i^{th} row of $(M_k \times N_k)$ with the j^{th} column of M_k^T is $(-1, -1, \ldots, -1, i, 0, \ldots, 0) \cdot (1, 1, \ldots, 1, -j, 0, \ldots, 0)$, where j is in the $(j+1)^{th}$ position of first vector. This is equal to the sum of the entries in the j^{th} row of $(M_k \times N_k)$ which is zero.

Therefore, only the elements on the main diagonal are not 0, and the others are zeros. And the elements on the main diagonal are all negative numbers start from the entry in cell $(2, 2)$ of A_k. Therefore it has the form. □

Next, consider $M_k \times M_k^T$:

$$M_2 \times M_2^T = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$$

$$M_3 \times M_3^T = \begin{pmatrix} 1 & 1 & 1 \\ 1 & -1 & 0 \\ 1 & 1 & -2 \end{pmatrix} \begin{pmatrix} 1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 0 & -2 \end{pmatrix} = \begin{pmatrix} 3 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 6 \end{pmatrix}$$

$$M_4 \times M_4^T = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 0 & 0 \\ 1 & 1 & -2 & 0 \\ 1 & 1 & 1 & -3 \end{pmatrix} \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & 1 \\ 1 & 0 & -2 & 1 \\ 1 & 0 & 0 & -3 \end{pmatrix} = \begin{pmatrix} 4 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 6 & 0 \\ 0 & 0 & 0 & 12 \end{pmatrix}$$

Then

$$\Rightarrow M_k \times M_k^T = \begin{pmatrix} k & 0 & 0 & \cdots & 0 \\ 0 & 2 & 0 & \cdots & 0 \\ 0 & 0 & 6 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & (k-1)k \end{pmatrix}$$

Lemma 5.8. *$M_k \times M_k^T$ is a diagonal matrix such that cell $(1,1)$ contains k and cell (i,i) contains $1 \times 2, 2 \times 3, \ldots, (k-1)k$, for each $i \geq 2$.*

Proof. Let $A_k = M_k \times M_k^T$. Cell $(1,1)$ of A_k is the dot product of the first row of M_k and the first column of M_k^T. Since the elements in first row of M_k and first column of M_k^T are all $(1,1,1,\ldots,1)$. Thus, the entry in cell $(1,1)$ of A_k is k. Then we consider cell (i,j) in different cases:

Case 1: cell (i,j) of A_k where $i \neq j$.

Since two matrices are transpose matrices of each other, the rows and columns are also in the same form, so the dot product can be expressed as $(1,1,\ldots,-i,0,\ldots,0)\cdot(1,1,\ldots,-j,0,\ldots,0)$, where $i \neq j$, and the sum of 1s in row equal to i, in column equal to j. So no matter which one of i,j is bigger or smaller, the sum of the number of 1s and $-i$ or $-j$ is zero. Thus, the elements outside the main diagonal are all zero.

Case 2: cell (i,j) of A_k where $i = j$.

Since two matrices are transpose matrices of each other, the dot product of row and column on main diagonal is $(1,1,\ldots,-i,0,\ldots,0) \cdot (1,1,\ldots,-j,0,\ldots,0)$, where $i = j$. It is easy to obtain the result is $i + i^2 = i(i+1)$. Because i start from the second row, so $i = k-1$. The elements equal to $(k-1)k$.

Therefore, A_k become such a diagonal matrix. □

Then

$$B_3 \times (\Lambda_3 - \lambda I) \times B_3^T = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & -1 & 0 & 1 & -1 & 0 & 1 & -1 & 0 \\ 1 & 1 & -2 & 1 & 1 & -2 & 1 & 1 & -2 \\ 1 & 1 & 1 & -1 & -1 & -1 & 0 & 0 & 0 \\ 1 & -1 & 0 & -1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & -2 & -1 & -1 & 2 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 & -2 & -2 & -2 \\ 1 & -1 & 0 & 1 & -1 & 0 & -2 & 2 & 0 \\ 1 & 1 & -2 & 1 & 1 & -2 & -2 & -2 & 4 \end{pmatrix}$$

$$\times \begin{pmatrix} 2-\lambda & 0 & 0 & 0 & 1 & 1 & 0 & 1 & 1 \\ 0 & 2-\lambda & 0 & 1 & 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 2-\lambda & 1 & 1 & 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 2-\lambda & 0 & 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 & 2-\lambda & 0 & 1 & 0 & 1 \\ 1 & 1 & 0 & 0 & 0 & 2-\lambda & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 & 1 & 1 & 2-\lambda & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 & 1 & 0 & 2-\lambda & 0 \\ 1 & 1 & 0 & 1 & 1 & 0 & 0 & 0 & 2-\lambda \end{pmatrix}$$

$$\times \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & 1 & -1 & 1 & 1 & -1 & 1 \\ 1 & 0 & -2 & 1 & 0 & -2 & 1 & 0 & -2 \\ 1 & 1 & 1 & -1 & -1 & -1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 \\ 1 & 0 & -2 & -1 & 0 & 2 & 1 & 0 & -2 \\ 1 & 1 & 1 & 0 & 0 & 0 & -2 & -2 & -2 \\ 1 & -1 & 1 & 0 & 0 & 0 & -2 & 2 & -2 \\ 1 & 0 & -2 & 0 & 0 & 0 & -2 & 0 & 4 \end{pmatrix}$$

$$= \begin{pmatrix} M_3 & M_3 & M_3 \\ M_3 & -M_3 & 0 \\ M_3 & M_3 & -2M_3 \end{pmatrix} \begin{pmatrix} (2-\lambda)I & N_3 & N_3 \\ N_3 & (2-\lambda)I & N_3 \\ N_3 & N_3 & (2-\lambda)I \end{pmatrix} \begin{pmatrix} M_3^T & M_3^T & M_3^T \\ M_3^T & -M_3^T & M_3^T \\ M_3^T & 0 & -2M_3^T \end{pmatrix}$$

$$= \begin{pmatrix} M_3(2-\lambda)I + 2M_3N_3 & M_3(2-\lambda)I + 2M_3N_3 & M_3(2-\lambda)I + 2M_3N_3 \\ M_3(2-\lambda)I - M_3N_3 & -[M_3(2-\lambda)I - M_3N_3] & 0 \\ M_3(2-\lambda)I - M_3N_3 & M_3(2-\lambda)I - M_3N_3 & -2[M_3(2-\lambda)I - M_3N_3] \end{pmatrix}$$

$$\times \begin{pmatrix} M_3^T & M_3^T & M_3^T \\ M_3^T & -M_3^T & M_3^T \\ M_3^T & 0 & -2M_3^T \end{pmatrix}$$

$$= \begin{pmatrix} 3[M_3(2-\lambda)I + 2M_3N_3] \cdot M_3^T & 0 & 0 \\ 0 & 2[M_3(2-\lambda)I - M_3N_3] \cdot M_3^T & 0 \\ 0 & 0 & 6[M_3(2-\lambda)I - M_3N_3] \cdot M_3^T \end{pmatrix}$$

The Kronecker product of two matrices is a special operation resulting two matrices in a larger block matrix.

Definition 5.9. *The Kronecker product of two matrices is a special operation resulting two matrices in a larger block matrix. Its operation form is that each element of the first matrix is multiplied by the second matrix to obtain a larger matrix.*

$$A \otimes B = \begin{pmatrix} a_{11}B & a_{12}B & \cdots & a_{1n}B \\ a_{21}B & a_{22}B & \cdots & a_{2n}B \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1}B & a_{m2}B & \cdots & a_{mn}B \end{pmatrix}$$

Example :

$$B_3 = M_3 \otimes M_3 = \begin{pmatrix} 1 & 1 & 1 \\ 1 & -1 & 0 \\ 1 & 1 & -2 \end{pmatrix} \begin{pmatrix} 1 & 1 & 1 \\ 1 & -1 & 0 \\ 1 & 1 & -2 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & -1 & 0 & 1 & -1 & 0 & 1 & -1 & 0 \\ 1 & 1 & -2 & 1 & 1 & -2 & 1 & 1 & -2 \\ 1 & 1 & 1 & -1 & -1 & -1 & 0 & 0 & 0 \\ 1 & -1 & 0 & -1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & -2 & -1 & -1 & 2 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 & -2 & -2 & -2 \\ 1 & -1 & 0 & 1 & -1 & 0 & -2 & 2 & 0 \\ 1 & 1 & -2 & 1 & 1 & -2 & -2 & -2 & 4 \end{pmatrix}$$

In general, we define B_k to be $M_k \otimes M_k$.

Then recall that:

$$\Lambda_3 = \begin{pmatrix} 2 & 0 & 0 & 0 & 1 & 1 & 0 & 1 & 1 \\ 0 & 2 & 0 & 1 & 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 2 & 1 & 1 & 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 2 & 0 & 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 & 2 & 0 & 1 & 0 & 1 \\ 1 & 1 & 0 & 0 & 0 & 2 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 & 1 & 1 & 2 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 & 1 & 0 & 2 & 0 \\ 1 & 1 & 0 & 1 & 1 & 0 & 0 & 0 & 2 \end{pmatrix}$$

According to the Lemma 6.8 $M_k \times N_k \times M_k^T$ is a diagonal matrix and Lemma 6.9 $M_k \times M_k^T$ is a diagonal matrix. Therefore $B \times \Lambda \times B^T$ is also a diagonal matrix.

Theorem 5.10. *Let A be an $n \times n$ matrix and let B be an $m \times m$ matrix. Then the determinant of $A \otimes B$ is given by:*

$$|A \otimes B| = |A|^m |B|^n$$

Let

$$|B_2| = |M_2 \otimes M_2| = |M_2|^2 \cdot |M_2|^2 = \begin{vmatrix} 1 & 1 \\ 1 & -1 \end{vmatrix}^2 \cdot \begin{vmatrix} 1 & 1 \\ 1 & -1 \end{vmatrix}^2 = (-2)^2 \times (-2)^2 = 16$$

$$|B_3| = |M_3 \otimes M_3| = |M_3|^3 \cdot |M_3|^3 = \begin{vmatrix} 1 & 1 & 1 \\ 1 & -1 & 0 \\ 1 & 1 & -2 \end{vmatrix}^3 \cdot \begin{vmatrix} 1 & 1 & 1 \\ 1 & -1 & 0 \\ 1 & 1 & -2 \end{vmatrix}^3 = \begin{vmatrix} 1 & 1 & 1 \\ 0 & -2 & -1 \\ 0 & 0 & -3 \end{vmatrix}^3 \cdot \begin{vmatrix} 1 & 1 & 1 \\ 0 & -2 & -1 \\ 0 & 0 & -3 \end{vmatrix}^3$$

$$= 6^3 \times 6^3 = 46656$$

$$|B_4| = |M_4 \otimes M_4| = |M_4|^4 \cdot |M_4|^4 = \begin{vmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 0 & 0 \\ 1 & 1 & -2 & 0 \\ 1 & 1 & 1 & -3 \end{vmatrix}^4 \cdot \begin{vmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 0 & 0 \\ 1 & 1 & -2 & 0 \\ 1 & 1 & 1 & -3 \end{vmatrix}^4$$

$$= \begin{vmatrix} 1 & 1 & 1 & 1 \\ 0 & -2 & -1 & -1 \\ 0 & 0 & -3 & -1 \\ 0 & 0 & 0 & -4 \end{vmatrix}^4 \cdot \begin{vmatrix} 1 & 1 & 1 & 1 \\ 0 & -2 & -1 & -1 \\ 0 & 0 & -3 & -1 \\ 0 & 0 & 0 & -4 \end{vmatrix}^4 = 24^4 \times 24^4 = 24^8$$

$$\Rightarrow |B_k| = |M_k \otimes M_k| = |M_k|^k \cdot |M_k|^k = [1 \times (-2) \times (-3) \times \cdots \times (-k)]^{2k} = (k!)^{2k}$$

Lemma 5.11. *The determinant of* $M_k = 1 \times (-2) \times (-3) \times \cdots \times (-k) = k! \times (-1)^{k-1}$.

Proof. Let $\det(M_k)$ be the determinant of M_k. Since, the first row of M_k is $(1, 1, \ldots, 1)$, all the elements are 1's. Subtract row 1 from every other row. Then we obtain a determinant of upper diagonal matrix with entries: $(1, -2, -3, \ldots, -k)$ on the main diagonal. According to the properties of the upper diagonal determinant, the value of the determinant is equal to the product of the elements on the main diagonal. $\square$

Consider

$$\det|B_3 \times (\Lambda_3 - \lambda I) \times B_3^T| =$$

$$\begin{vmatrix} 3[M_3(2-\lambda)I + 2M_3N_3] \cdot M_3^T & 0 & 0 \\ 0 & 2[M_3(2-\lambda)I - M_3N_3] \cdot M_3^T & 0 \\ 0 & 0 & 6[M_3(2-\lambda)I - M_3N_3] \cdot M_3^T \end{vmatrix}$$

$$= 36^3[(2-\lambda)|M_3M_3^T| + 2|M_3N_3M_3^T|] \cdot [(2-\lambda)|M_3M_3^T| - |M_3N_3M_3^T|]^2$$

$$= 36^3[(2-\lambda)\begin{vmatrix} 3 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 6 \end{vmatrix} + 2\begin{vmatrix} 6 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -6 \end{vmatrix}] \cdot [(2-\lambda)\begin{vmatrix} 3 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 6 \end{vmatrix} - \begin{vmatrix} 6 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -6 \end{vmatrix}]^2$$

$$= 36^3[(18-3\lambda)(-2\lambda)(-6\lambda)] \cdot [(-3\lambda)(6-2\lambda)(6-6\lambda)]^2$$

$$= 36^3 \times 12 \times 9\lambda^4(18-3\lambda)[(6-2\lambda)(6-6\lambda)]^2$$

Since

$$\det|B_3| = \det|B_3^T| = 6^3 \times 6^3 = 36^3$$

Thus

$$\det|\Lambda_3 - \lambda I| = 36^3 \times 12 \times 9\lambda^4(18-3\lambda)[(6-2\lambda)(6-6\lambda)]^2/36^6$$

$$B_4 = M_4 \otimes M_4 = \begin{pmatrix} M_4 & M_4 & M_4 & M_4 \\ M_4 & -M_4 & 0 & 0 \\ M_4 & M_4 & -2M_4 & 0 \\ M_4 & M_4 & M_4 & -3M_4 \end{pmatrix}$$

$$B_4 \times (\Lambda_4 - \lambda I) = \begin{pmatrix} M_4 & M_4 & M_4 & M_4 \\ M_4 & -M_4 & 0 & 0 \\ M_4 & M_4 & -2M_4 & 0 \\ M_4 & M_4 & M_4 & -3M_4 \end{pmatrix} \begin{pmatrix} (3-\lambda)I & N_4 & N_4 & N_4 \\ N_4 & (3-\lambda)I & N_4 & N_4 \\ N_4 & N_4 & (3-\lambda)I & N_4 \\ N_4 & N_4 & N_4 & (3-\lambda)I \end{pmatrix}$$

$$= \begin{pmatrix} (3-\lambda)M_4 + 3M_4N_4 & (3-\lambda)M_4 + 3M_4N_4 & (3-\lambda)M_4 + 3M_4N_4 & (3-\lambda)M_4 + 3M_4N_4 \\ (3-\lambda)M_4 - M_4N_4 & -(3-\lambda)M_4 + M_4N_4 & 0 & 0 \\ (3-\lambda)M_4 - M_4N_4 & (3-\lambda)M_4 - M_4N_4 & -2(3-\lambda)M_4 + 2M_4N_4 & 0 \\ (3-\lambda)M_4 - M_4N_4 & (3-\lambda)M_4 - M_4N_4 & (3-\lambda)M_4 - M_4N_4 & -3(3-\lambda)M_4 + 3M_4N_4 \end{pmatrix}$$

Then consider

$$B_4 \times (\Lambda_4 - \lambda I) \times B_4^T = B_4 \times (\Lambda_4 - \lambda I) \times \begin{pmatrix} M_4^T & M_4^T & M_4^T & M_4^T \\ M_4^T & -M_4^T & M_4^T & M_4^T \\ M_4^T & 0 & -2M_4^T & M_4^T \\ M_4^T & 0 & 0 & -3M_4^T \end{pmatrix}$$

Ten we get a diagonal matrix, the elements on the main diagonal are: $4(3-\lambda)M_4M_4^T + 12M_4N_4M_4^T$, $2(3-\lambda)M_4M_4^T - 2M_4N_4M_4^T$, $6(3-\lambda)M_4M_4^T - 6M_4N_4M_4^T$, $12(3-\lambda)M_4M_4^T -$

$12M_4N_4M_4^T$.

Then generalizing to a $(n \times n)/(n-1)$ Trojan semi-Latin square:

$$\Lambda_n = \begin{pmatrix} (n-1)I & N_n & N_n & N_n & \cdots & N_n \\ N_n & (n-1)I & N_n & N_n & \cdots & N_n \\ N_n & N_n & (n-1)I & N_n & \cdots & N_n \\ N_n & N_n & N_n & (n-1)I & \cdots & N_n \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ N_n & N_n & N_n & N_n & \cdots & (n-1)I \end{pmatrix}$$

$$B_n = M_n \otimes M_n = \begin{pmatrix} M_n & M_n & M_n & \cdots & M_n \\ M_n & -M_n & 0 & \cdots & 0 \\ M_n & M_n & -2M_n & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & 0 \\ M_n & M_n & \cdots & M_n & -(n-1)M_n \end{pmatrix}$$

Thus

$$B_n \times (\Lambda_n - \lambda I) = \begin{pmatrix} M_n & M_n & M_n & \cdots & M_n \\ M_n & -M_n & 0 & \cdots & 0 \\ M_n & M_n & -2M_n & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & 0 \\ M_n & M_n & \cdots & M_n & -(n-1)M_n \end{pmatrix}$$

$$\times \begin{pmatrix} (n-1-\lambda)I & N_n & N_n & N_n & \cdots & N_n \\ N_n & (n-1-\lambda)I & N_n & N_n & \cdots & N_n \\ N_n & N_n & (n-1-\lambda)I & N_n & \cdots & N_n \\ N_n & N_n & N_n & (n-1-\lambda)I & \cdots & N_n \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ N_n & N_n & N_n & N_n & \cdots & (n-1-\lambda)I \end{pmatrix}$$

$$= \begin{pmatrix} (n-1-\lambda)M_n + (n-1)M_nN_n & \cdots & \cdots & (n-1-\lambda)M_n + (n-1)M_nN_n \\ (n-1-\lambda)M_n - M_nN_n & -(n-1-\lambda)M_n + M_nN_n & 0 & \vdots \\ \vdots & \vdots & \ddots & 0 \\ (n-1-\lambda)M_n - M_nN_n & (n-1-\lambda)M_n - M_nN_n & \cdots & -(n-1)(n-1-\lambda)M_n - M_nN_n \end{pmatrix}$$

We describe the matrix $B_n \times (\Lambda_n - \lambda I)$. The first row contains only the entry $(n-1-\lambda)M_n + (n-1)M_nN_n$. Apart from the first row, entries above the diagonal are 0. Entries below diagonal are $(n-1-\lambda)M_n - M_nN_n$. Finally, on the main diagonal, cell (i,i) contains $-(i-1)(n-1-\lambda)M_n - M_nN_n$. See the figure above.

Observe that $B_n \times (\Lambda_n - \lambda I) \times B_n^T$ is a diagonal matrix, therefore we obtain the elements on the main diagonal is:
Cell $(1,1)$ contains $n(n-1-\lambda)M_nM_n^T + n(n-1)M_nN_nM_n^T$.
Cell (i,i), $i \geq 2$ contains $(i-1)i[(n-1-\lambda)M_nM_n^T - M_nN_nM_n^T]$.

According to Lemma 6.8 and Lemma 6.9 $M_n \times M_n^T$ and $(M_n \times N_n) \times M_n^T$ are diagonal matrices. Then simplifying $M_n \times M_n^T$ and $(M_n \times N_n) \times M_n^T$, we obtain:

Cell (1,1) is a diagonal matrix as follows

$$\begin{pmatrix} n^2(n-1-\lambda) & 0 & 0 & \cdots & 0 \\ 0 & 2n(n-1-\lambda) & 0 & \cdots & 0 \\ 0 & 0 & 6n(n-1-\lambda) & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & 0 \\ 0 & 0 & \cdots & 0 & n^2(n-1)(n-1-\lambda) \end{pmatrix}$$

$$+\begin{pmatrix} n^2(n-1)^2 & 0 & 0 & \cdots & 0 \\ 0 & -2n(n-1) & 0 & \cdots & 0 \\ 0 & 0 & -6n(n-1) & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & 0 \\ 0 & 0 & \cdots & 0 & -n^2(n-1)^2 \end{pmatrix}$$

$$=\begin{pmatrix} n^2[(n-1)^2+(n-1-\lambda)] & 0 & 0 & \cdots & 0 \\ 0 & -2n\lambda & 0 & \cdots & 0 \\ 0 & 0 & -6n\lambda & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & 0 \\ 0 & 0 & \cdots & 0 & -(n-1)n^2\lambda \end{pmatrix}$$

Cell (i,i), $i \geq 2$ contains as follows

$$i(i-1)[\begin{pmatrix} (n-1-\lambda)n & 0 & 0 & \cdots & 0 \\ 0 & 2(n-1-\lambda) & 0 & \cdots & 0 \\ 0 & 0 & 6(n-1-\lambda) & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & 0 \\ 0 & 0 & \cdots & 0 & n(n-1)(n-1-\lambda) \end{pmatrix}$$

$$
- \begin{pmatrix} n(n-1) & 0 & 0 & \cdots & 0 \\ 0 & -2 & 0 & \cdots & 0 \\ 0 & 0 & -6 & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & 0 \\ 0 & 0 & \cdots & 0 & -n(n-1) \end{pmatrix}]
$$

$$
= i(i-1) \begin{pmatrix} -n\lambda & 0 & 0 & \cdots & 0 \\ 0 & 2(n-\lambda) & 0 & \cdots & 0 \\ 0 & 0 & 6(n-\lambda) & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & 0 \\ 0 & 0 & \cdots & 0 & n(n-1)(n-\lambda) \end{pmatrix}
$$

The multiplicities of each eigenvalue:

0 (multiplicity $= 2(n-1)$),

n (multiplicity $= (n-1)^2$),

$n^2 - n$ (multiplicity $= 1$).

Theorem 5.12. *The eigenvalues of Λ_n (in descending order) are: 0 (multiplicity is $2(n-1)$), n (multiplicity is $(n-1)^2$), n^2-n (multiplicity is 1).*

Lemma 5.13. *Let A be a square matrix. Then AA^T and A^TA have the same set of eigenvalues.*

Proof. Let v be an eigenvector of AA^T with eigenvalue k. By definition:

$$
AA^Tv = kv
$$

So

$$A^T A A^T v = k A^T v$$

Let

$$w = A^T v$$

Then

$$A^T A w = k w$$

Therefore k is also an eigenvalue for $A^T A$ (but with a different eigenvector). □

Lemma 5.14. *The eigenvalues for the information matrix*

$$nI - (n-1)^{-1} N N^T$$

are: $0, n - n/(n-1)$ *(with multiplicty* $(n-1)^2$*) and* n *with multiplicty* $2n-2$*.*

6 Applications to experimental design

In this section we introduce experimental design and the use of symbols and knowledge of Trojan Square from previous section to illustrate the application of experimental design. Then we give an example to explain the composition.

Experimental design in research is a process of planning and conducting scientific experiments to investigate a hypothesis or research question [5]. It involves carefully designing an experiment that can test the hypothesis, and controlling for other variables that may influence the results [13].

Treatments [2]: In an experimental design, treatments are variables that the researchers control. They are the primary independent variables of interest. Researchers control the treatment to the subjects or items in the experiment and want to know whether it causes changes in the consequence.

Effects [2]: We try to study in some way and measure the effect of that change on some measure of performance. In experimental design talk, measures of performance are called **responses**. **Factors** are effects that we can change or control.

Example [2]: In experiments with greenhouse crops, there are generally row and column effects. We assume that the greenhouse is rectangular. Since the distance change in the north-south direction is usually greater than that in the east-west direction, we divide the greenhouse into n rows and n columns. If there are nk treatments, they can be applied to the symbols of a semi-Latin square. The Trojan square as follows is suitable for 15 varieties of crops in a glasshouse which is 15 east-west by 5 plots north-south:

A 1 a	B 2 b	C 3 c	D 4 d	E 5 e
E 4 c	A 5 d	B 1 e	C 2 a	D 3 b
D 2 e	E 3 a	A 4 b	B 5 c	C 1 d
C 5 b	D 1 c	E 2 d	A 3 e	B 4 a
B 3 d	C 4 e	D 5 a	E 1 b	A 2 c

Further results can be found in [14], [10], [4] and [12].

7 Conclusion

For future work, we make the following comment. For experimental design applications, it may also be desirable that block repetition is minimized. In fact, this can be achieved nicely in the case of the 5×7 row column design in Table 57, where we observe that each of the possible 35 triples from $\mathbb{Z}_7$ occurs exactly once. An open problem to consider is the following: For each $v \equiv 1 \pmod 6$, does there exist a $(v-1)(v-2)/6 \times v$ row-column block design of index 3, where each row is regular and each column is near-regular *such that* each triple from $\mathbb{Z}_v$ occurs exactly once?

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A Tables for Heffter's Difference Problem

Table 37: General solution for $v = 18s + 1$ from Lemma 3.5.

$S_1 : r = 0$	$1,\ 2,\ 4s+3$
$r = 1$	$0,\ 4,\ 4s+4$
$\vdots$	$\vdots$
$r = s-1$	$-s+2,\ 2s,\ 5s+2$
$S_2 : r = 0$	$6s+2,\ 6s+4,\ 14s+4$
$r = 1$	$6s+1,\ 6s+6,\ 14s+5$
$\vdots$	$\vdots$
$r = s-2$	$5s+4,\ 8s,\ 15s+2$
$r = s-1$	$5s+3,\ 8s+2,\ 15s+3$
$S_3 : r = 0$	$2s-1,\ 2s+2,\ 8s+1$
$\vdots$	$\vdots$
$r = s-3$	$5,\ 3s-1,\ 7s+4$
$r = s-2$	$3,\ 3s,\ 7s+3$
$3s,\ 3s+1,\ 6s+1$	$3s+3,\ 6s+3,\ 9s+4$

Table 38: $v = 37,\ s = 2$ from Lemma 3.5.

difference	$\overline{V}$
1, 9, 10	1, 2, 11
4, 8, 12	0, 4, 12
2, 16, 18	14, 16, 32
5, 15, 20	13, 18, 30
3, 11, 14	5, 8, 19
6, 7, 13	9, 15, 22

Table 39: $v = 55$, $s = 3$ from Lemma 3.5.

difference	$\overline{V}$
1, 13, 14	1, 2, 15
4, 12, 16	0, 4, 16
7, 11, 18	54, 6, 17
2, 24, 26	20, 22, 46
5, 23, 28	19, 24, 47
8, 22, 30	18, 26, 48
3, 17, 20	9, 12, 29
6, 15, 21	7, 13, 28
9, 10, 19	14, 23, 33

Table 40: $v = 73$, $s = 4$ from Lemma 3.5.

difference	$\overline{V}$
1, 17, 18	1, 2, 19
4, 16, 20	0, 4, 20
7, 15, 22	72, 6, 21
10, 14, 24	71, 8, 22
2, 32, 34	26, 28, 60
5, 31, 36	25, 30, 61
8, 30, 38	24, 32, 62
11, 29, 40	23, 34, 63
3, 23, 26	13, 16, 39
6, 21, 27	11, 17, 38
9, 19, 28	9, 18, 37
12, 13, 25	15, 27, 40

Table 41: $v = 91$, $s = 5$ from Lemma 3.5.

difference	$\overline{V}$
1, 21, 22	1, 2, 23
4, 20, 24	0, 4, 24
7, 19, 26	90, 6, 25
10, 18, 28	89, 8, 26
13, 17, 30	88, 10, 27
2, 40, 42	32, 34, 74
5, 39, 44	31, 36, 75
8, 38, 46	30, 38, 76
11, 37, 48	29, 40, 77
14, 36, 50	28, 42, 78
3, 29, 32	17, 20, 49
6, 27, 33	15, 21, 48
9, 25, 34	13, 22, 47
12, 23, 35	11, 23, 46
15, 16, 31	24, 39, 55

Table 42: General solution for $v = 18s + 3$ from Lemma 3.6.

$S_1 : r = 0$	$1,\ 2,\ 8s+3$
$r = 1$	$0,\ 4,\ 8s+4$
$\vdots$	$\vdots$
$r = s-1$	$-s+2,\ 2s,\ 9s+2$
$S_2 : r = 0$	$5s+1,\ 5s+3,\ 9s+3$
$r = 1$	$5s,\ 5s+5,\ 9s+4$
$\vdots$	$\vdots$
$r = s-2$	$4s+3,\ 7s-1,\ 10s+1$
$r = s-1$	$4s+2,\ 7s+1,\ 10s+2$
$S_3 : r = 0$	$7s,\ 7s+3,\ 13s+2$
$\vdots$	$\vdots$
$r = s-3$	$5s+4,\ 8s+1,\ 12s+4$
$r = s-2$	$5s+2,\ 8s+2,\ 12s+3$
$6s+1$	$8s+3,\ 14s+4,\ 20s+5$

Table 43: $v = 39$, $s = 2$ from Lemma 3.6.

difference	$\overline{V}$
1, 17, 18	1, 2, 19
4, 16, 20	0, 4, 20
2, 8, 10	12, 14, 22
5, 7, 12	11, 16, 23
3, 11, 14	7, 10, 21
6, 9, 15	9, 15, 24
13	18, 31, 5

Table 44: $v = 57$, $s = 3$ from Lemma 3.6.

difference	$\overline{V}$
1, 25, 26	1, 2, 27
4, 24, 28	0, 4, 28
7, 23, 30	56, 6, 29
2, 12, 14	20, 22, 34
5, 11, 16	19, 24, 35
8, 10, 18	18, 26, 36
3, 17, 20	12, 15, 32
6, 15, 21	10, 16, 31
9, 13, 22	8, 17, 30
19	14, 33, 52

Table 45: $v = 75$, $s = 4$ from Lemma 3.6.

difference	$\overline{V}$
1, 33, 34	1, 2, 35
4, 32, 36	0, 4, 36
7, 31, 38	74, 6, 37
10, 30, 40	73, 8, 38
2, 16, 18	21, 23, 39
5, 15, 20	20, 25, 40
8, 14, 22	19, 27, 41
11, 13, 24	18, 29, 42
3, 17, 20	28, 31, 48
6, 15, 21	26, 32, 47
9, 13, 22	24, 33, 46
12, 11, 23	22, 34, 45
25	30, 55, 5

Table 46: $v = 93$, $s = 5$ from Lemma 3.6.

difference	$\overline{V}$
1, 41, 42	1, 2, 43
4, 40, 44	0, 4, 44
7, 39, 46	92, 6, 45
10, 38, 48	91, 8, 46
13, 37, 50	90, 10, 47
2, 20, 22	26, 28, 48
5, 19, 24	25, 30, 49
8, 18, 26	24, 32, 50
11, 17, 28	23, 34, 51
14, 16, 30	22, 36, 52
3, 29, 32	35, 38, 67
6, 27, 33	33, 39, 66
9, 25, 34	31, 40, 65
12, 23, 35	29, 41, 64
15, 21, 36	27, 42, 63
31	43, 74, 12

Table 47: General solution of $v = 18s + 7$ from Lemma 3.7.

$S_1 : r = 0$	$1,\ 2,\ 8s+5$
$r = 1$	$0,\ 4,\ 8s+6$
$\vdots$	$\vdots$
$r = s-1$	$-s+2,\ 2s,\ 9s+4$
$S_2 : r = 0$	$4s+1,\ 4s+3,\ 10s+4$
$r = 1$	$4s-1,\ 4s+4,\ 10s+3$
$\vdots$	$\vdots$
$r = s-2$	$2s+1,\ 5s+1,\ 9s+6$
$r = s-1$	$2s+3,\ 5s+2,\ 9s+5$
$S_3 : r = 0$	$11s+4,\ 11s+7,\ 16s+3$
$\vdots$	$\vdots$
$r = s-2$	$10s+6,\ 12s+8,\ 16s+6$
$r = s-1$	$10s+5,\ 12s+10,\ 16s+7$
$3s+1,\ 4s+2,\ 7s+3$	$4s+2,\ 7s+3,\ 12s$

Table 48: $v = 61,\ s = 3$ from Lemma 3.7.

difference	$\overline{V}$
1, 27, 28	1, 2, 29
4, 26, 30	0, 4, 30
7, 25, 32	60, 6, 31
2, 23, 21	14, 16, 35
5, 21, 22	12, 17, 34
8, 10, 23	10, 18, 33
3, 13, 16	39, 42, 55
6, 12, 18	38, 44, 56
9, 11, 20	37, 46, 57
10, 14, 24	26, 36, 50

Table 49: $v = 79$, $s = 4$ from Lemma 3.7.

difference	$\overline{V}$
1, 35, 36	1, 2, 37
4, 34, 38	0, 4, 38
7, 33, 40	78, 6, 39
10, 32, 42	77, 8, 40
2, 19, 21	24, 26, 45
5, 17, 22	22, 27, 44
8, 15, 23	20, 28, 43
11, 13, 24	18, 29, 42
3, 13, 16	49, 52, 65
6, 12, 18	48, 54, 66
9, 11, 20	47, 56, 67
12, 10, 22	46, 58, 68
13, 18, 31	10, 23, 41

Table 50: $v = 97$, $s = 5$ from Lemma 3.7.

difference	$\overline{V}$
1, 43, 44	1, 2, 45
4, 42, 46	0, 4, 46
7, 41, 48	96, 6, 47
10, 40, 50	95, 8, 48
13, 39, 52	94, 10, 49
2, 31, 33	21, 23, 54
5, 29, 34	19, 24, 53
8, 27, 35	17, 25, 52
11, 25, 36	15, 26, 51
14, 23, 37	13, 27, 50
3, 21, 24	59, 62, 83
6, 20, 26	58, 64, 84
9, 19, 28	57, 66, 85
12, 18, 30	56, 68, 86
15, 17, 32	55, 70, 87
16, 22, 38	22, 38, 60

Table 51: General solution of $v = 18s + 9$ from Lemma 3.8.

$S_1 : r = 0$	$1,\ 2,\ 4s+5$
$r = 1$	$0,\ 4,\ 4s+6$
$\vdots$	$\vdots$
$r = s$	$-s+1,\ 2s+2,\ 5s+5$
$S_2 : r = 0$	$8s+9,\ 8s+11,\ 16s+14$
$r = 1$	$8s+8,\ 8s+13,\ 16s+15$
$r = 2$	$5s+12,\ 8s+2,\ 16s+2$
$\vdots$	$\vdots$
$r = s-3$	$5s+10,\ 8s+6,\ 16s+4$
$r = s-2$	$8s+7,\ 11s+6,\ 14s+8$
$S_3 : r = 0$	$10s+8,\ 10s+11,\ 18s+12$
$r = 1$	$10s+6,\ 10s+10,\ 16s+11$
$\vdots$	$\vdots$
$r = s-3$	$10s+2,\ 11s+8,\ 16s+8$
$r = s-2$	$10s,\ 11s+9,\ 16s+7$
$3s,\ 7s+3,\ 8s+6$	$7,\ 3s+7,\ 8s+13$
$6s+3$	$4s,\ 10s+3,\ 16s+6$

Table 52: $v = 45$, $s = 2$ from Lemma 3.8.

difference	$\overline{V}$
1, 11, 12	1, 2, 13
4, 10, 14	0, 4, 14
7, 9, 16	44, 6, 15
2, 19, 21	18, 20, 39
5, 18, 23	17, 22, 40
3, 15, 18	26, 29, 44
6, 17, 22	24, 30, 46
15	12, 27, 42

Table 53: $v = 63$, $s = 3$ from Lemma 3.8.

difference	$\overline{V}$
1, 15, 16	1, 2, 17
4, 14, 18	0, 4, 18
7, 13, 20	62, 6, 19
10, 12, 22	61, 8, 20
2, 27, 29	31, 33, 60
5, 26, 31	20, 25, 51
8, 11, 19	29, 37, 48
3, 25, 28	38, 41, 3
6, 17, 23	36, 42, 59
9, 24, 30	7, 16, 37
21	11, 32, 53

Table 54: $v = 81$, $s = 4$ from Lemma 3.8.

difference	$\overline{V}$
1, 19, 20	1, 2, 21
4, 18, 22	0, 4, 22
7, 17, 24	80, 6, 23
10, 16, 26	79, 8, 24
13, 15, 28	78, 10, 25
2, 35, 37	40, 42, 77
5, 34, 39	30, 35, 69
8, 32, 40	28, 36, 68
11, 14, 25	39, 50, 64
3, 33, 36	48, 51, 3
6, 23, 29	46, 52, 75
9, 21, 30	44, 53, 74
12, 31, 38	7, 19, 45
27	18, 45, 72

Table 55: $v = 99$, $s = 5$ from Lemma 3.8.

difference	$\overline{V}$
1, 23, 24	1, 2, 25
4, 22, 26	0, 4, 26
7, 21, 28	98, 6, 27
10, 20, 30	97, 8, 28
13, 19, 32	96, 10, 29
16, 18, 34	95, 12, 30
2, 43, 45	47, 49, 92
5, 42, 47	46, 51, 93
8, 40, 48	32, 40, 80
11, 39, 50	31, 42, 81
14, 17, 31	45, 59, 76
3, 41, 44	58, 61, 3
6, 29, 35	56, 62, 91
9, 27, 36	54, 63, 90
12, 25, 37	52, 64, 89
15, 38, 46	7, 22, 53
33	20, 53, 86

Table 56: $v = 117$, $s = 6$ from Lemma 3.8.

difference	$\overline{V}$
1, 27, 28	1, 2, 29
4, 26, 30	0, 4, 30
7, 25, 32	116, 6, 31
10, 24, 34	115, 8, 32
13, 23, 36	114, 10, 33
16, 22, 38	113, 12, 34
19, 21, 40	112, 14, 35
2, 51, 53	57, 59, 110
5, 50, 55	56, 61, 111
8, 48, 56	42, 50, 98
11, 47, 58	41, 52, 99
14, 46, 60	40, 54, 100
17, 20, 37	55, 72, 92
3, 49, 52	68, 71, 3
6, 37, 41	66, 72, 107
9, 35, 42	64, 73, 106
12, 33, 43	62, 74, 105
15, 31, 44	60, 75, 104
18, 45, 54	7, 25, 61
39	24, 63, 102

Table 57: General solution of $v = 18s + 13$ from Lemma 3.9.

$S_1 : r = 0$	$1, 2, 4s + 5$
$r = 1$	$0, 4, 4s + 6$
$\vdots$	$\vdots$
$r = s - 1$	$-s + 1, 2s + 2, 5s + 5$
$S_2 : r = 0$	$10s + 8, 10s + 10, 16s + 13$
$r = 1$	$10s + 6, 10s + 11, 16s + 12$
$\vdots$	$\vdots$
$r = s - 2$	$8s + 12, 11s + 8, 15s + 15$
$r = s - 1$	$8s + 10, 11s + 9, 15s + 14$
$S_3 : r = 0$	$6s + 5, 6s + 8, 14s + 13$
$\vdots$	$\vdots$
$r = s - 2$	$5s + 7, 8s + 4, 15s + 11$
$r = s - 1$	$5s + 6, 8s + 6, 15s + 12$
$3s + 2, 7s + 5, 8s + 6$	$3s + 4, 5s + 6, 14s + 6$

Table 58: $v = 49$, $s = 2$ from Lemma 3.9.

difference	$\overline{V}$
1, 11, 12	1, 2, 13
4, 10, 14	0, 4, 14
7, 9, 16	48, 6, 15
2, 15, 17	28, 30, 45
5, 13, 18	26, 31, 44
3, 21, 24	17, 20, 41
6, 20, 26	16, 22, 42
8, 19, 22	3, 11, 25

Table 59: $v = 67$, $s = 3$ from Lemma 3.9.

difference	$\overline{V}$
1, 15, 16	1, 2, 17
4, 14, 18	0, 4, 18
7, 13, 20	66, 6, 19
10, 12, 22	65, 8, 20
2, 21, 23	38, 40, 61
5, 19, 24	36, 41, 60
8, 17, 25	34, 42, 59
3, 29, 32	23, 26, 55
6, 28, 34	22, 28, 56
9, 27, 36	21, 30, 57
11, 26, 30	3, 14, 33

Table 60: $v = 85$, $s = 4$ from Lemma 3.9.

difference	$\overline{V}$
1, 19, 20	1, 2, 21
4, 18, 22	0, 4, 22
7, 17, 24	84, 6, 23
10, 16, 26	83, 8, 24
13, 15, 28	82, 10, 25
2, 27, 29	48, 50, 77
5, 25, 30	46, 51, 76
8, 23, 31	44, 52, 75
11, 21, 32	42, 53, 74
3, 37, 40	29, 32, 69
6, 36, 42	28, 34, 70
9, 35, 44	27, 36, 71
12, 34, 46	26, 38, 72
14, 33, 38	3, 17, 50

Table 61: $v = 103$, $s = 5$ from Lemma 3.9.

difference	$\overline{V}$
1, 23, 24	1, 2, 25
4, 22, 26	0, 4, 26
7, 21, 28	102, 6, 27
10, 20, 30	101, 8, 28
13, 19, 32	100, 10, 29
16, 18, 34	99, 12, 30
2, 33, 35	58, 60, 93
5, 31, 36	56, 61, 92
8, 29, 37	54, 62, 91
11, 27, 38	52, 63, 90
14, 25, 39	50, 64, 89
3, 45, 48	35, 38, 83
6, 44, 50	34, 40, 84
9, 43, 52	33, 42, 85
12, 42, 54	32, 44, 86
15, 41, 56	31, 46, 87
17, 40, 46	19, 36, 76

Table 62: General solution of $v = 18s + 15$ from Lemma 3.10.

$S_1 : r = 0$	$1,\ 2,\ 4s + 5$
$r = 1$	$0,\ 4,\ 4s + 6$
$\vdots$	$\vdots$
$r = s$	$-s + 1,\ 2s + 2,\ 5s + 5$
$S_2 : r = 0$	$6s + 6,\ 6s + 8,\ 14s + 14$
$r = 1$	$6s + 5,\ 6s + 10,\ 14s + 15$
$\vdots$	$\vdots$
$r = s - 1$	$5s + 7,\ 8s + 6,\ 15s + 13$
$r = s$	$5s + 6,\ 8s + 8,\ 15s + 14$
$S_3 : r = 0$	$10s + 8,\ 10s + 11,\ 16s + 14$
$\vdots$	$\vdots$
$r = s - 2$	$8s + 12,\ 11s + 9,\ 15s + 16$
$r = s - 1$	$8s + 10,\ 11s + 10,\ 15s + 15$
$6s + 5$	$6s + 7,\ 12s + 12,\ 18s + 13$

Table 63: $v = 51,\ s = 2$ from Lemma 3.10.

difference	$\overline{V}$
1, 11, 12	1, 2, 13
4, 10, 14	0, 4, 14
7, 9, 16	50, 6, 15
2, 22, 24	18, 20, 42
5, 21, 26	17, 22, 43
8, 20, 28	16, 24, 44
3, 15, 18	28, 31, 46
6, 13, 19	26, 32, 45
17	13, 30, 47

Table 64: $v = 69$, $s = 3$ from Lemma 3.10.

difference	$\overline{V}$
1, 15, 16	1, 2, 17
4, 14, 18	0, 4, 18
7, 13, 20	68, 6, 19
10, 12, 22	67, 8, 20
2, 30, 32	24, 26, 56
5, 29, 34	23, 28, 57
8, 28, 36	22, 30, 58
11, 27, 38	21, 32, 59
3, 21, 24	38, 41, 62
6, 19, 25	36, 42, 61
9, 17, 26	34, 43, 60
23	25, 48, 71

Table 65: $v = 87$, $s = 4$ from Lemma 3.10.

difference	$\overline{V}$
1, 19, 20	1, 2, 21
4, 18, 22	0, 4, 22
7, 17, 24	86, 6, 23
10, 16, 26	85, 8, 24
13, 15, 28	84, 10, 25
2, 38, 40	30, 32, 70
5, 37, 42	29, 34, 71
8, 36, 44	28, 36, 72
11, 35, 46	27, 38, 73
14, 34, 48	26, 40, 74
3, 27, 30	48, 51, 78
6, 25, 31	46, 52, 77
9, 23, 32	44, 53, 76
12, 21, 33	42, 54, 75
29	31, 60, 89

Table 66: $v = 105$, $s = 5$ from Lemma 3.10.

difference	$\overline{V}$
1, 23, 24	1, 2, 25
4, 22, 26	0, 4, 26
7, 21, 28	104, 6, 27
10, 20, 30	103, 8, 28
13, 19, 32	102, 10, 29
16, 18, 34	101, 12, 30
2, 46, 48	36, 38, 84
5, 45, 50	35, 40, 85
8, 44, 52	34, 42, 86
11, 43, 54	33, 44, 87
14, 42, 56	32, 46, 88
17, 41, 58	31, 48, 89
3, 33, 36	58, 61, 94
6, 31, 37	56, 62, 93
9, 29, 38	54, 63, 92
12, 27, 39	52, 64, 91
15, 25, 40	50, 65, 90
35	37, 72, 109