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Anomalous Transport of Cosmic Rays in a Nonlinear Diffusion Model

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Abstract

We investigate analytically and numerically the transport of cosmic rays following their escape from a shock or another localized acceleration site. Observed cosmic-ray distributions in the vicinity of heliospheric and astrophysical shocks imply that anomalous, superdiffusive transport plays a role in the evolution of the energetic particles. Several authors have quantitatively described the anomalous diffusion scalings, implied by the data, by solutions of a formal transport equation with fractional derivatives. Yet the physical basis of the fractional diffusion model remains uncertain. We explore an alternative model of the cosmic-ray transport: a nonlinear diffusion equation that follows from a self-consistent treatment of the resonantly interacting cosmic-ray particles and their self-generated turbulence. The nonlinear model naturally leads to superdiffusive scalings. In the presence of convection, the model yields a power-law dependence of the particle density on the distance upstream of the shock. Although the results do not refute the use of a fractional advection–diffusion equation, they indicate a viable alternative to explain the anomalous diffusion scalings of cosmic-ray particles.

Key words: cosmic rays – diffusion – turbulence

1. Introduction

Observations of energetic electrons and protons in interplanetary space (e.g., Zimbardo et al. 2015) and relativistic electrons in supernova remnants (Perri et al. 2016) imply that the transport of cosmic-ray particles in the presence of turbulent scattering can be superdiffusive. In the superdiffusive regime, the mean square displacement $\langle x^2 \rangle$ of the particles increases with time as $\sim t^{\alpha}$ where $\alpha > 1$, in contrast with the familiar linear dependence $\langle x^2 \rangle \sim t$ of standard diffusion.

The observed anomalous diffusion scalings motivated the development of mathematical generalizations of Brownian motion to anomalous transport (e.g., Klafter et al. 1987). A popular recent approach invokes solutions of a formal transport equation containing fractional derivatives to describe an evolving cosmic-ray distribution (e.g., Perri & Zimbardo 2007, 2008, 2009; Litvinenko & Effenberger 2014; Perri et al. 2015; Zimbardo et al. 2015). Those studies relied on the Green's function of a fractional diffusion equation, which is equivalent to an asymptotic expression for a non-Gaussian propagator in the framework of continuous-time random walks (Chukbar 1995; Metzler & Klafter 2000). Yet the physical basis of fractional diffusion models of cosmic-ray transport remains uncertain, justifying a search for an alternative model.

For weak plasma turbulence, the quasilinear theory was argued to explain the anomalous diffusive behavior (Vanden Eijnden 1997). More generally, the cosmic-ray pressure is often comparable with the magnetic pressure in the surrounding medium, implying that the cosmic-ray particles are strongly scattered off self-generated magnetohydrodynamic waves, and necessitating a nonlinear treatment. A quantitative description of the resulting evolution of the particle distribution has obvious physical interest. Here we consider a nonlinear diffusion model of the anomalous particle transport. The idea is that the energetic particle density $f(x, t)$, where x is the distance along the mean magnetic field, is governed by a diffusion equation:

$$\frac{\partial f}{\partial t} = \frac{\partial}{\partial x}\left(D\frac{\partial f}{\partial x}\right), \tag{1}$$

where the diffusion coefficient D is determined by modeling the interaction of the energetic particles and the turbulent waves, generated by the particles themselves. Consequently D is controlled by f, which makes Equation (1) nonlinear. The one-dimensional model is justified if the perpendicular diffusion is weak and the magnetic field lines are only weakly perturbed on a relevant coherence length scale of the magnetic field and background turbulence (Nava & Gabici 2013).

Several alternative models of nonlinear cosmic-ray transport have been explored (e.g., Yan et al. 2012; Malkov et al. 2013), which generally require the solution of two coupled nonlinear equations for the resonantly interacting particles and waves: one equation describes the cosmic-ray pressure, and the other describes the energy density of the turbulence. The equations are coupled because the diffusion coefficient D depends on the energy density of the resonant waves, whereas the growth rate of the waves depends on the pressure gradient of the resonant particles (e.g., D'Angelo et al. 2016; Nava et al. 2016). Under certain simplifying assumptions for the wave generation rate, it is possible to express the turbulent energy density in terms of the particle distribution function and hence obtain an equation for the evolution of $f(x, t)$. The resulting nonlinear Equation (1) contains the diffusion coefficient D that is a function of $\partial f/\partial x$ (or ∇f in a more general three-dimensional problem). Solutions of Equation (1) may serve to interpret the cosmic-ray data and guide the development of more detailed models.

As a concrete illustration, we consider a model in which the wave generation by the streaming particles is assumed to be balanced by wave dissipation (Ptuskin et al. 2008), which yields the diffusion coefficient

$$D = D_0 \left| \frac{\partial f}{\partial x} \right|^{-\nu}, \quad D_0 = \text{const.} \tag{2}$$

Without loss of generality, below we set $D_0 = 1$, which corresponds to the change of variable $D_0 t \to t$. Concrete physical situations correspond to $\nu = 1/2$ and $\nu = 2/3$: the first case corresponds to energy dissipation by a Kolmogorov-type energy cascade (Ptuskin & Zirakashvili 2003), and the second case corresponds to wave energy transfer to the thermal ions that interact with moving magnetic mirrors formed by the waves (Zirakashvili 2000). More generally, Equation (2) with some other value of ν might approximate the nonlinear diffusion coefficient D in a certain parameter range. Mathematically, the expression for D leads to a nonlinear diffusion equation that had been termed n-diffusion (Philip 1961). Thus the model for cosmic-ray evolution which we investigate provides a concrete physical illustration of n-diffusion.

Although we assume throughout the paper that particle transport is diffusive, it is worth noting that the evolution of strongly anisotropic particle distributions on timescales that are shorter than or comparable to a characteristic scattering time is known to yield a nondiffusive behavior of the particle density even in a linear regime. More generally, both numerical (Litvinenko & Noble 2016) and analytical (Malkov 2017) solutions of the linear Fokker–Planck equation for a test-particle distribution function demonstrate the transition from a short-time ballistic propagation regime to a long-time diffusive regime. In contrast, our analysis applies to a diffusive regime only, and the nonlinear behavior is ultimately caused by the strong coupling of the particle and wave distributions.

2. Exact Solutions and Superdiffusive Scalings

We now present exact solutions for nonlinear diffusion and discuss the anomalous transport scalings that may result.

2.1. Self-similar Solutions for Nonlinear Diffusion

Consider first an initial value problem specified by

$$f(x, 0) = \delta(x). \tag{3}$$

The initial condition describes the release of particles at the origin at $t = 0$. To describe their subsequent evolution, we seek a self-similar solution that satisfies the nonlinear diffusion Equation (1) and the normalization condition

$$\int_{-\infty}^{\infty} f(x, t)dx = 1. \tag{4}$$

For simplicity, the total particle number is incorporated into the definition of D_0. As usual, an exact self-similar solution is expected to serve an intermediate asymptotic for a wider class of initial conditions, which in this case would correspond to initial distributions $f(x, 0)$, localized around $x = 0$.

Following the well-known procedure (e.g., Dresner 1983; Assis et al. 2005), we seek a solution in the form

$$f(x, t) = t^{-1/2(1-\nu)}\phi(\xi), \tag{5}$$

with the similarity variable

$$\xi = xt^{-1/2(1-\nu)} \tag{6}$$

for an arbitrary $0 < \nu < 1$. On substituting the self-similar form into Equation (1) and integrating the resulting ordinary differential equation for $\phi(\xi)$, we obtain for $\xi > 0$:

$$\phi(\xi) = \left[a_\nu + \frac{\nu}{(2-\nu)} \times (2-2\nu)^{-1/(1-\nu)}\xi^{(2-\nu)/(1-\nu)}\right]^{-(1-\nu)/\nu}, \tag{7}$$

where a_ν is an integration constant, and another constant is specified by the condition $\phi(\infty) = 0$. Symmetry dictates that we define the solution for $\xi < 0$ by $\phi(\xi) = \phi(|\xi|)$. The integration constant a_ν is defined by the normalization condition

$$\int_{-\infty}^{\infty} \phi(\xi)d\xi = 2\int_0^{\infty} \phi(\xi)d\xi = 1. \tag{8}$$

It follows that

$$a_\nu^{2(1-\nu)^2/\nu(2-\nu)} = 2\left(\frac{2-\nu}{\nu}\right)^{(1-\nu)/(2-\nu)}(2-2\nu)^{1/(2-\nu)} \times \frac{\Gamma(2-1/(2-\nu))\Gamma(2/\nu(2-\nu)-2)}{\Gamma(1/\nu-1)}, \tag{9}$$

where Γ is the gamma function. Note that $\phi \sim \xi^{-(2-\nu)/\nu}$ for $\xi \gg 1$, which ensures the convergence of the normalization integral for all $0 < \nu < 1$. For instance, if $\nu = 1/2$, we have

$$\phi(\xi) = \left[a_{1/2} + \frac{1}{3}\xi^3\right]^{-1}, \tag{10}$$

$$a_{1/2} = 8\left(\frac{\pi^6}{3^7}\right)^{1/4}. \tag{11}$$

If $\nu = 2/3$, we recover the solution derived by Ptuskin et al. (2008):

$$\phi(\xi) = \left[a_{2/3} + \frac{1}{2}\left(\frac{3}{2}\right)^3\xi^4\right]^{-1/2}, \tag{12}$$

$$a_{2/3} = \frac{1}{27\pi^2}\Gamma^8\left(\frac{1}{4}\right). \tag{13}$$

Both solutions are illustrated in Figure 1.

The essential point is that the self-similar solution for $f(x, t)$ yields an anomalous, superdiffusive scaling of the mean square displacement:

$$\langle x^2 \rangle = 2t^{1/(1-\nu)}\int_0^{\infty} \xi^2\phi(\xi)d\xi \sim t^{1/(1-\nu)}. \tag{14}$$

The convergence of the integral in Equation (14) requires that $0 < \nu < 1/2$. The limiting cases correspond to standard diffusion, $\langle x^2 \rangle \sim t$ ($\nu \to 0$), and ballistic motion, $\langle x^2 \rangle \sim t^2$ ($\nu \to 1/2$).

Physically, the divergence of the integral in Equation (14) for $\nu \geqslant 1/2$ reflects the fact that the particle flux $F \sim x^{-(2-\nu)(1-\nu)/\nu}$ falls off too slowly as $x \to \infty$, indicating the breakdown of the diffusion approximation at large x and formally leading to unphysical "superballistic" scalings for the second and higher moments of the distribution. Since the integral for $\langle x^2 \rangle$ diverges in the physically relevant case $\nu = 2/3$, another observable measure of nonlinearity is useful. Ptuskin et al. (2008) suggested to use the time $t_{\rm m}$ at which the maximum of cosmic-ray intensity is reached at a given location

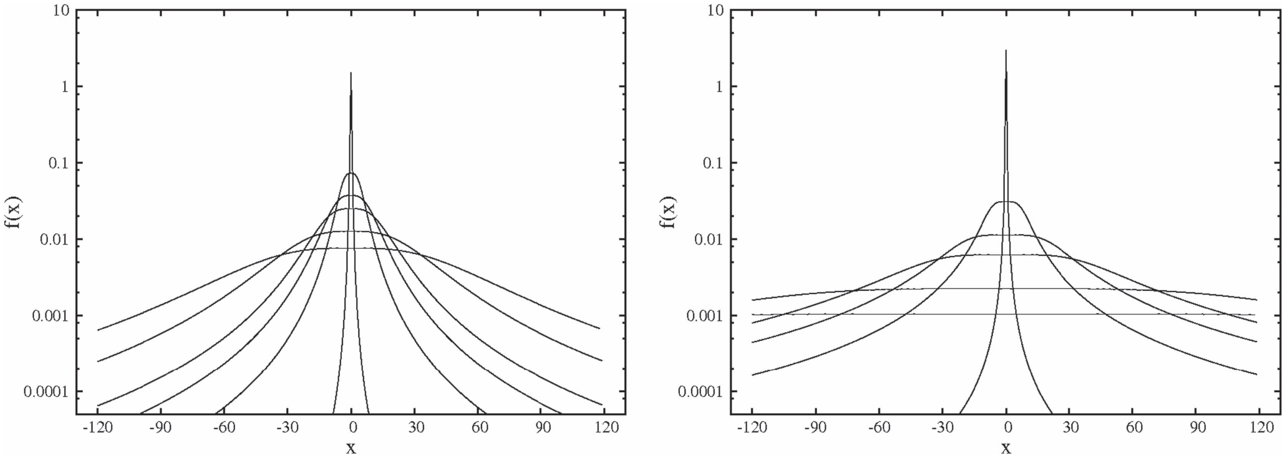

Figure 1. Solution of the diffusion Equation (1) with the diffusion coefficient (2) for $\nu = 1/2$ (Equation (10), left panel) and $\nu = 2/3$ (Equation (12), right panel) for the point source initial condition (3) at times $t = 0.001$, 2, 4, 6, 12, and 20.

$x_{\rm m}$. The above-given solution leads to the scaling:

$$x_{\rm m} \sim t_{\rm m}^{1/2(1-\nu)}, \tag{15}$$

which generalizes the scaling derived for $\nu = 2/3$ (Ptuskin et al. 2008) and reduces to the familiar linear result $x_{\rm m} \sim t_{\rm m}^{1/2}$ when $\nu = 0$.

2.2. Diffusive Escape from an Extended Reservoir

Another initial value problem with an exact solution is that of diffusive particle escape from an extended reservoir. Suppose the reservoir is large enough, so that the initial profile of the particle density near the boundary $x = 0$ can be approximated by the Heaviside step function:

$$f(x, 0) = f_0 H(x), \quad f_0 = \text{const.} \tag{16}$$

Diffusion of this initial profile will lead to $\partial f/\partial x > 0$ for $t > 0$. We seek a solution in the form

$$f(x, t) = \int_{-\infty}^{x} g(x', t) dx'. \tag{17}$$

On differentiating Equation (1) with respect to x, we obtain an equation for $g(x, t) = \partial f/\partial x$:

$$\frac{\partial g}{\partial t} = \frac{\partial^2}{\partial x^2}(g^{1-\nu}), \tag{18}$$

where

$$g(x, 0) = f_0 \delta(x). \tag{19}$$

It is well known (e.g., Barenblatt 1952; Pattle 1959) that the solution of this initial value problem has a self-similar form

$$g(x, t) = t^{-1/(2-\nu)}\psi(\zeta), \tag{20}$$

with the similarity variable

$$\zeta = x t^{-1/(2-\nu)}. \tag{21}$$

On substituting the self-similar form into Equation (18) and integrating the resulting ordinary differential equation for $\psi(\zeta)$, we obtain

$$\psi(\zeta) = \left[b_\nu + \frac{\nu}{2(1-\nu)(2-\nu)}\zeta^2\right]^{-1/\nu}. \tag{22}$$

As previously, the condition $0 < \nu < 1$ yields physically meaningful solutions for $f(x, t)$, which conserve the total number of particles. The integration constant b_ν is defined by the normalization condition

$$\int_{-\infty}^{\infty} \psi(\zeta) d\zeta = f_0. \tag{23}$$

It follows that

$$b_\nu^{1/2-1/\nu} = \left[\frac{\nu}{2(1-\nu)(2-\nu)}\right]^{1/2} \frac{\Gamma(1/\nu)}{\Gamma(1/\nu - 1/2)} \frac{f_0}{\sqrt{\pi}}. \tag{24}$$

On substituting Equation (22) into Equation (17), the solution for $f(x, t)$ can be rewritten in a compact form:

$$f(x, t) = \frac{f_0}{2}\left[1 + \text{sgn}(x)\frac{B_\eta(1/2, 1/\nu - 1/2)}{B(1/2, 1/\nu - 1/2)}\right], \tag{25}$$

where B is the beta function, B_η is the incomplete beta function, and

$$\eta = \left[1 + \frac{2(1-\nu)(2-\nu)}{\nu} b_\nu \frac{t^{2/(2-\nu)}}{x^2}\right]^{-1}. \tag{26}$$

The solution for $f(x, t)$ can be expressed in terms of elementary functions if $\nu = 2/(2 + m)$ where m is an integer, which includes the physically meaningful cases $\nu = 1/2$ and $\nu = 2/3$. If $\nu = 1/2$, we have

$$f(x, t) = \frac{f_0}{2}\left[1 + \text{sgn}(x)\frac{2}{\pi}\left(\arcsin\sqrt{\eta} + \sqrt{\eta(1-\eta)}\right)\right], \tag{27}$$

where

$$\eta = \left[1 + 3\left(\frac{\sqrt{3}\pi}{2f_0}\right)^{2/3} \frac{t^{4/3}}{x^2}\right]^{-1}. \tag{28}$$

If $\nu = 2/3$, the solution is as follows:

$$f(x, t) = \frac{f_0}{2}\left[1 + \text{sgn}(x)\left(1 + \frac{16}{3\sqrt{3}f_0}\frac{t^{3/2}}{x^2}\right)^{-1/2}\right]. \tag{29}$$

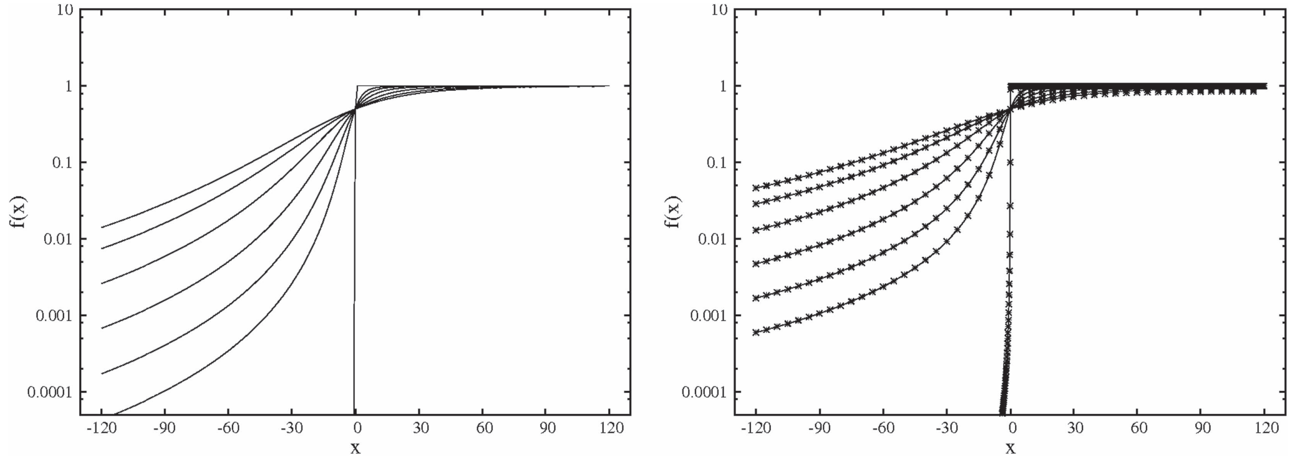

Figure 2. Solution of the diffusion Equation (1) with the diffusion coefficient (2) for $\nu = 1/2$ (Equation (27), left panel) and $\nu = 2/3$ (Equation (29), right panel) for the reservoir initial condition (16) at times t = 0.001, 5, 10, 20, 40, 70, and 100. The symbols in the right panel indicate the numerical solution and illustrate its accuracy.

Both solutions are illustrated in Figure 2.

3. Effects of Convection and a Particle Source

We may generalize the diffusive transport equation of the previous section by adding a convective term and a particle source term. That would allow us to describe, for instance, the evolution of energetic particles following their escape from a localized acceleration site, such as a heliospheric shock (e.g., Perri et al. 2015). It is worth mentioning that braided magnetic fields at a shock may also lead to anomalous spatial transport and consequently modify the spectrum of shock-accelerated particles (Duffy et al. 1995; Kirk et al. 1996), but here we only investigate the effects of nonlinear spatial diffusion in the evolution of the particle density.

Consider the following nonlinear advection–diffusion equation:

$$\frac{\partial f}{\partial t} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\left|\frac{\partial f}{\partial x}\right|^{-\nu} + Vf\right) + S\delta(x)H(t), \tag{30}$$

where V and S are constant, and $H(t)$ is the Heaviside step function. A delta-functional source $S\delta(x)$ may correspond to energetic particles injected at an interplanetary shock, and V may be interpreted as the speed of a background flow such as the solar wind. For simplicity, assume that $f(x, 0) = 0$.

3.1. Analytical Solutions for Limiting Cases

In the limit of weak diffusion ($D_0 \to 0$), the diffusive term in Equation (30) is negligible, and the density profile is given by a boxcar function:

$$f(x, t) \approx \frac{S}{V}[H(x + Vt) - H(x)], \tag{31}$$

where the solution of a first-order differential equation is specified by the particle conservation constraint $\int f dx = St$. In the opposite limit of strong diffusion or weak advection ($V \to 0$), the solution of Equation (30) has a self-similar form $f(x, t) = x\theta(xt^{-1/2})$, which leads to a nonlinear ordinary differential equation for the function θ.

For a general advection–diffusion Equation (30), a time-dependent density profile is harder to describe analytically. For instance, invariance with respect to a dilatation group no longer leads to a self-similar solution of the initial value problem. Eventually, however, diffusion should balance convection near the particle source, leading to an approximately steady solution $f(x)$ both in the range $-Vt \lesssim x \lesssim 0$ and for $x > 0$. Because $f \to 0$ as $x \to +\infty$, a steady solution for $x > 0$ corresponds to the vanishing total particle flux in that region. On setting $\partial f/\partial t = 0$ and integrating the resulting second-order ordinary differential equation, we obtain

$$f(x) \approx \begin{cases} \dfrac{S}{V}, & -Vt < x < 0, \\ \left(\dfrac{1-\nu}{\nu}\right)^{(1-\nu)/\nu} V^{-1/\nu} \\ \quad \times \left[x + \dfrac{1-\nu}{\nu}\dfrac{1}{V}S^{\nu/(\nu-1)}\right]^{(\nu-1)/\nu}, & x > 0, \end{cases} \tag{32}$$

where the integration constants are assumed to be approximately specified by the non-diffusive solution, which yields $f(0) = S/V$ and $f(\infty) = 0$. Note for clarity that dimensionally correct expressions are recovered by returning to the original dimensional variables: $V \to V/D_0$, $S \to S/D_0$. In the limit $\nu \to 0$, it is straightforward to verify that Equation (32) reduces to the familiar linear result, $f(x, \infty) = S/V$ for $x < 0$ and $f(x, \infty) = (S/V)\exp(-Vx)$ for $x > 0$, which of course would also follow directly from the solution $f(x, t)$ of Equation (30) in the linear case $\nu = 0$ when $t \to \infty$ (see, e.g., Equation (2) in Litvinenko & Effenberger 2014).

3.2. Numerical Solutions for the Fully Time-dependent Case

We employed the VLUGR3 code (Blom & Verwer 1994) to determine the time evolution of the particle density by solving Equation (30) numerically. In order to test the ability of the code to correctly solve nonlinear diffusion equations, we first numerically reproduced the full time evolution given by the analytical solution of Equation (1) for the reservoir initial condition, which is Equation (16). The symbols in the right panel of Figure 2 give representative results and demonstrate the applicability of the code.

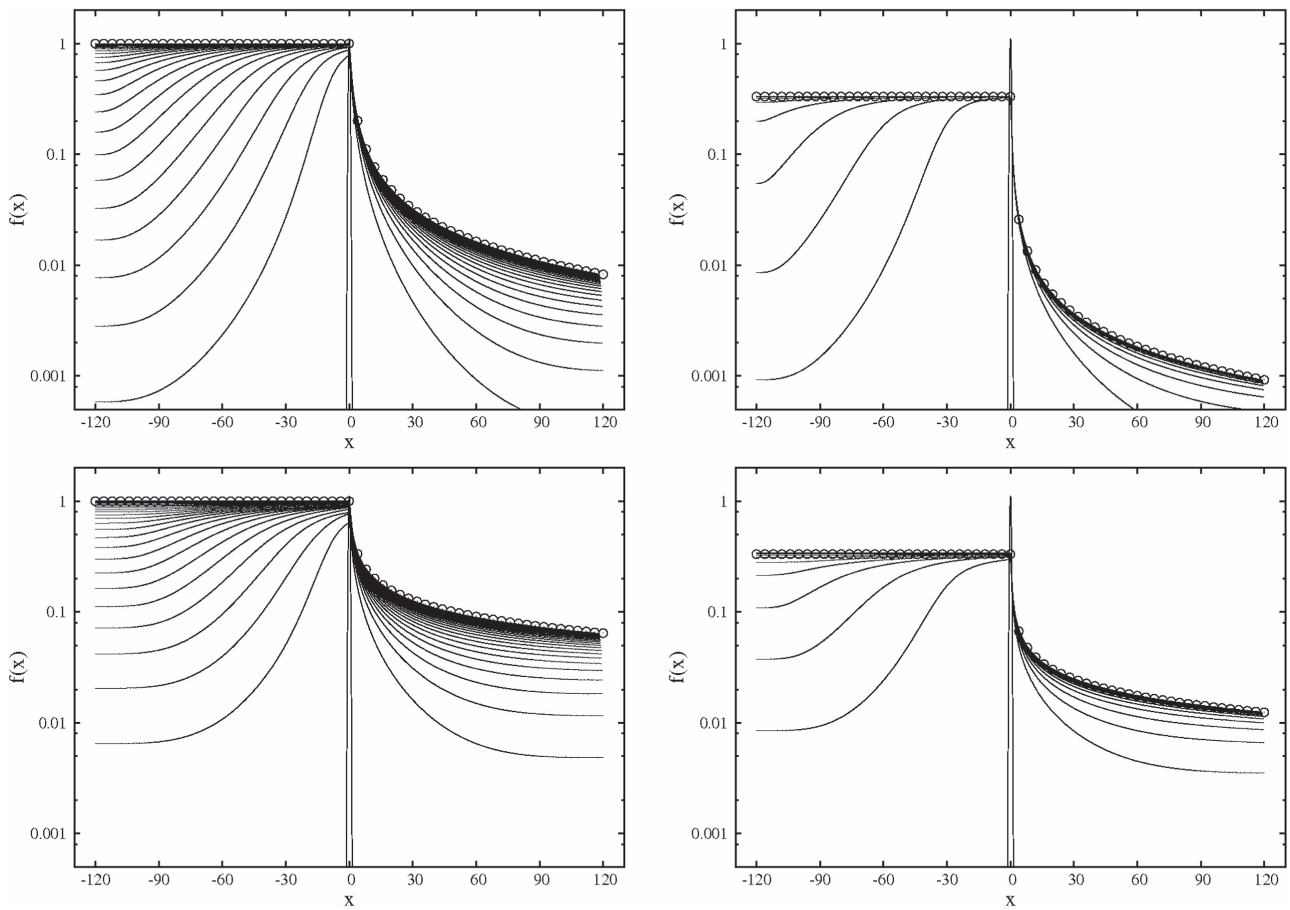

Figure 3. Solution of the nonlinear advection–diffusion Equation (30) with the source strength $S = 1$ for $\nu = 1/2$ (upper row) and $\nu = 2/3$ (lower row). The left column is for $V = 1$, and the right column is for $V = 3$. In each panel the initial condition (33) with $a = 0.5$ is shown as the narrow peak around $x = 0$. The time evolution is illustrated by the curves showing the numerical solution of Equation (30) in steps of $\Delta t = 10$ beginning with $t = 0$. The circles indicate the asymptotic steady-state solution that was derived analytically with Equation (32).

Next we solved the nonlinear advection–diffusion Equation (30) for the two physically motivated cases $\nu = 1/2$ and $\nu = 2/3$ and two dimensionless advection speeds $V = 1$ and $V = 3$. The point source at the origin was numerically represented by a Gaussian profile:

$$f(x, 0) = \frac{1}{\sqrt{\pi}a} \exp\left\{-\frac{x^2}{a^2}\right\}. \tag{33}$$

When specifying the boundary conditions in the numerical solution, we used as a guide the expected qualitative behavior of the evolving distribution in the downstream and upstream regions. Specifically, we implemented a vanishing gradient $\partial f/\partial x = 0$ at the left boundary (chosen at $x = -120$) and a vanishing flux, i.e., $\partial f/\partial x + Vf|\partial f/\partial x|^{\nu} = 0$, at the right boundary (chosen at $x = 120$). This formulation of the vanishing-flux condition allows us to avoid division by a small $\partial f/\partial x$ in the computation of the solution. We verified that the resulting numerical solution actually has $\partial f/\partial x \neq 0$ for any finite distance $x > 0$, which is consistent with the analytical steady-state solution (32).

Figure 3 displays the resulting numerical solution $f(x, t)$ in each of the four cases, presented in steps of $\Delta t = 10$ and the initial condition (33) with $a = 0.5$. Evidently, the numerical solution converges to the analytical asymptotic steady-state limit that is given by Equation (32) and represented with the symbols in each panel.

3.3. Asymptotic Power-law Behavior

The steady-state solution given by Equation (32) predicts the power-law dependence $f(x) \sim x^{(\nu-1)/\nu}$ in the upstream region $x > 0$, which is of primary interest for modeling the diffusive transport of shock-accelerated particles. Data analysis of time profiles of particles accelerated at interplanetary shocks yielded particle spectra, consistent with such power-law tails. This result was argued to be evidence of superdiffusive particle transport and formally modeled in terms of a fractional advection–diffusion equation (Perri & Zimbardo 2007; Perri et al. 2015). In the fractional diffusion model, the particle distribution near a traveling shock is obtained by a straightforward change of variables that leads to a formula for the particle density at a fixed point due to a moving source (Litvinenko & Effenberger 2014). However, we have not yet explored the effect of a moving source in the nonlinear diffusion model under consideration.

As a specific example of anomalous transport, Perri & Zimbardo (2009) analyzed the data on the energetic ions, accelerated at the termination shock of the solar wind, and interpreted the results in terms of the process of fractional diffusion, which yielded $f(x) \sim x^{\gamma}$ with $\gamma \approx -0.7$. Interestingly, this value lies between the two cases that we have

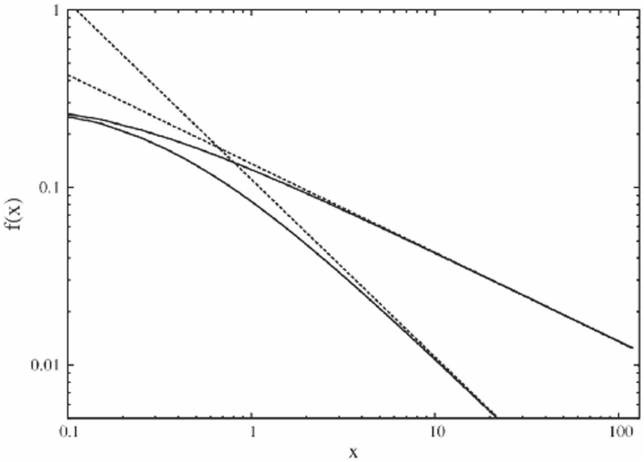

Figure 4. Asymptotic steady-state solution (32) of the nonlinear advection–diffusion Equation (30) for $\nu = 1/2$ and $\nu = 2/3$ (lower and upper solid line, respectively) and their asymptotic power-law behavior (dashed lines).

analyzed: the predicted power-law index $(\nu - 1)/\nu$ is -1 for $\nu = 1/2$ and -0.5 for $\nu = 2/3$ (see also Figure 4). Note that in our solution the advection speed V and the source strength S define the location beyond which the power law is formed but not the power-law index itself. We emphasize that the present nonlinear model, while leading to similar superdiffusive scalings, has the advantage of being based on a physical model of the interaction of the cosmic rays and their self-generated turbulence (Ptuskin et al. 2008). Finally, it is reasonable to expect that nonlinear effects become less significant in the upstream region far from the shock because of the decrease in the particle density and the corresponding weakening of the turbulent scattering.

4. Conclusion

Observations of energetic cosmic-ray particles in several astrophysical situations imply that the particle evolution can be quantified by anomalous, superdiffusive transport scalings. Understanding the mechanism responsible for superdiffusive transport has obvious physical relevance for theoretical cosmic ray studies. Motivated by these considerations, we have considered the evolution of the particle density, governed by a nonlinear diffusion equation that follows from a self-consistent treatment of the resonantly interacting particles and the turbulence that they generate. The nonlinear diffusion equation provides a concrete physical illustration of the mathematical model of n-diffusion (Philip 1961).

First, we obtained an exact self-similar solution of an initial value problem. The solution generalizes earlier results (Ptuskin et al. 2008) and yields an intermediate asymptotic for an initially localized particle distribution. The resulting time- and space-dependent solution for the particle density provides a straightforward way of explaining the observed superdiffusive transport scalings within a physically transparent model.

Second, we derived an exact solution for the problem of the nonlinear diffusive escape of cosmic rays from an extended reservoir. It is worth mentioning that cosmic-ray diffusion generally occurs both in physical space and in momentum space (e.g., Thornbury & Drury 2014), and so it would be of interest to extend our nonlinear model to describe re-acceleration of cosmic rays, which is the acceleration associated with the same turbulence that produces the spatial diffusion.

Third, we explored analytically and numerically the effects of convection and a particle source on the nonlinear diffusive transport. The results can be applied to describe the transport of cosmic rays following their escape from a shock or another localized acceleration site. The key point is that the nonlinear diffusion model naturally yields a power-law dependence of the cosmic-ray density on the distance upstream of the shock. Such a dependence was argued to be the signature of a fractional advection–diffusion equation (e.g., Perri & Zimbardo 2007, 2008, 2009; Perri et al. 2016). Yet our results clearly demonstrate that, when it comes to explaining the anomalous diffusion scalings of cosmic-ray particles, a transport equation with fractional derivatives is not the only game in town.

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